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Data-Driven System Level Synthesis

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Abstract

We establish data-driven versions of the System Level Synthesis (SLS) parameterization of stabilizing controllers for linear-time-invariant systems. Inspired by recent work in data-driven control that leverages tools from behavioral theory, we show that optimization problems over system-responses can be posed using only libraries of past system trajectories, without explicitly identifying a system model. We first consider the idealized setting of noise free trajectories, and show an exact equivalence between traditional and data-driven SLS. We then show that in the case of a system driven by process noise, tools from robust SLS can be used to characterize the effects of noise on closed-loop performance. We then draw on tools from matrix concentration to show that a simple trajectory averaging technique can be used to mitigate these effects. We end with numerical experiments showing the soundness of our methods.

Keywords: Data-driven control, model-free control, system level synthesis

As the systems we control become increasingly complex, dynamic, heterogeneous, and difficult to model, the intricate and detailed system models needed by traditional tools from robust and optimal control, derived either from first principles or expensive and time-consuming system identification methods, will no longer be available. Fortunately, contemporary systems are also inherently *data-rich*, allowing for data-driven control algorithms to be deployed, wherein techniques rooted in machine learning take the place of the traditional system identification step.

Focusing on the control of an unknown linear system, approaches ranging from identify-then-control (Dean et al., 2019; Mania et al., 2019), to adaptive methods based on robust control (Dean et al., 2018) and online-learning (Simchowitz et al., 2020; Hazan et al., 2020), and more closely related to this paper, to those based on purely data-driven methods (De Persis and Tesi, 2019; Coulson et al., 2019a) rooted in behavioral theory (Willems, 1986; Willems and Polderman, 1997; Willems et al., 2005) have been explored. For a more exhaustive overview of recent developments please see Matni et al. (2019) and Recht (2019) for tutorials aimed at audiences from control-theoretic and machine learning backgrounds, respectively.

This paper is motivated by the results presented in (De Persis and Tesi, 2019; Coulson et al., 2019a, 2020, 2019b; van Waarde et al., 2020; Rotulo et al., 2019). Broadly, these papers leverage the behavioral framework of Willems and Polderman (1997), which allows for the achievable input/output behavior of a system to be characterized in terms of a library of past trajectories, assuming certain *persistence of excitation* conditions are satisfied. For example, in (Coulson et al., 2019a), the authors show how trajectory tracking in output-feedback based model-predictive-control (MPC) (Garcia et al., 1989; Borrelli et al., 2017) can be posed as an optimization problem over a library of past system trajectories, with follow up work establishing connections to distributionally robust programming (Coulson et al., 2020) and allowing for real-time implementations (Coulson et al., 2019b). Similarly, in (Rotulo et al., 2019; De Persis and Tesi, 2019), it is shown that data-driven synthesis of linear quadratic regulators can be achieved through the solution of semidefinite programs without identifying an explicit system model. We note that to the best of our knowledge, no characterization of the effects of noise in the data on the performance achieved by the data-driven controllers is provided in the aforementioned papers.

Contributions: In this paper, we establish *data-driven* versions of the System Level Synthesis (SLS) (Anderson et al., 2019) parameterization of stabilizing controllers for linear-time-invariant systems. SLS has been central to breakthroughs in distributed optimal control (Wang et al., 2019b), robust and distributed MPC (Amo Alonso and Matni, 2019; Wang et al., 2019a), and learning-enabled control (Dean et al., 2019, 2018): our goal in this work is to take a first step towards extending its advantages to the purely data-driven, model-free setting. In particular, we show that optimization problems over system-responses can be posed using only libraries of past system trajectories. We first consider the idealized setting of noise free trajectories, and show an exact equivalence. We then show that in the case of a system driven by process noise, tools from robust SLS (Matni et al., 2017) can be used to characterize the effects of using "noisy" trajectories to synthesize data-driven controllers on closed-loop performance. We further draw on tools from matrix concentration (Tropp, 2012) to show that a trajectory averaging can be used to mitigate these effects.

Notation: We use $x_{[i,j]}$ as shorthand for the signal $[x^\top(i)\; x^\top(i+1)\; \cdots\; x^\top(j)]^\top$. Define

$$\mathcal{H}_L(\sigma_{[0,T-1]}) = \begin{bmatrix} \sigma(0) & \sigma(1) & \cdots & \sigma(T-L) \\ \sigma(1) & \sigma(2) & \cdots & \sigma(T-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma(L-1) & \sigma(L) & \cdots & \sigma(T-1) \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} R(0:0) & & & \\ R(1:1) & R(1:0) & & \\ \vdots & \ddots & \ddots & \\ R(T:T) & \cdots & R(T:1) & R(T:0). \end{bmatrix}$$

For a signal $\sigma_{[0,T-1]}$, we denote the above Hankel matrix of order L by $\mathcal{H}_L(\sigma_{[0,T-1]})$. A linear, causal operator $\mathbf{R}$ defined over a horizon of T has matrix representation, as shown above: here $R(i:j) \in \mathbb{R}^{p\times q}$ is a matrix of compatible dimension. We denote the set of such linear causal operators by $\mathcal{L}_{TV}^{T,p\times q}$ and drop the superscript $T, p\times q$ when it is clear. $\mathbf{R}(i,:)$ denotes the i-th block row of $\mathbf{R}$ and $\mathbf{R}(:,j)$ denotes the j-th block column of $\mathbf{R}$, both indexing from 0. All omitted proofs and supporting results can be found in the Appendix.

1. Problem Statement

We consider finite-time optimal state-feedback control of the discrete time linear-time-invariant (LTI) dynamical system

$$x(t+1) = Ax(t) + Bu(t) + w(t), \text{ for } t = 0, 1, \ldots, L-1, \tag{1.1}$$

where $L > 0$ is the control horizon, $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, and $w \in \mathbb{R}^n$ is the disturbance. We assume that the pair (A, B) is controllable. In order to simplify notation going forward, we adopt the convention of embedding the initial condition of system (1.1) in the disturbance signal as $w(-1) = x(0)$.

When the system model (A, B) is known, the above problem can be efficiently solved for many cases of interest by making suitable assumptions on the noise signal $w_{[-1,L-2]}$ and control objective. This paper focuses instead on solving an optimal control problem when the model describing system (1.1) is unknown, but when a collection of state and input trajectories (over a slightly longer horizon $T > L$ to be specified later) $\{x^{(i)}_{[0,T-1]}, u^{(i)}_{[0,T-1]}\}_{i=1}^N$ are available. Moreover, our goal is to solve this task *without explicitly estimating* the system model.

In particular, to make the discussion concrete, we focus on finite-horizon Linear Quadratic Gaussian (LQG) control, wherein the disturbances are assumed to be independently and identically distributed as $w(t) \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2 I)$, the control policy at time t is given by a linear-time-varying function of past states, i.e., $u(t) = K_t(x_{[0,t]})$, and the cost function to be minimized is given by:

$$\mathbb{E}\left[\sum_{t=0}^{L-2} x^\top(t)Qx(t) + u^\top(t)Ru(t) + x^\top(L-1)Q_F x(L-1)\right]. \tag{1.2}$$

We note however that much of our analysis extends to other cost functions in a natural way.

2. Data-Driven System Level Synthesis

We begin by considering the simplified setting in which there is no driving noise in system (1.1), i.e., $w(-1) = x(0)$ and $w(t) = 0$ for all $t \geq 0$. Our approach is to connect tools from behavioral control theory, namely Willems' Fundamental Lemma (Willems and Polderman, 1997; Markovsky and Rapisarda, 2008), with the *System Level Synthesis* (SLS) (Anderson et al., 2019) parameterization of closed loop controllers. While such a connection offers no immediate benefits in this simplified setting of noise-free centralized control, it establishes the tools needed to tackle the general problem of interest. We also comment at the end of this section as to future work in distributed data-driven control that this parameterization may enable.

Willems' Fundemtal Lemma: Tools from behavioral system theory provide a natural way of characterizing the behavior of a dynamical system in terms of its input/output signals. For our purposes, we rely on recent specializations of *Willems' Fundaemental Lemma* to state-space realizations of LTI systems (De Persis and Tesi, 2019). Central to Willems' fundamental lemma is *persistence of excitation*, which is specified in terms of a rank condition on a Hankel matrx constructed from the of the control input signal u.

Definition 1 *Let $\sigma : \mathbb{Z} \to \mathbb{R}^p$ be a signal. We say that its finite-horizon restriction $\sigma_{[0,T-1]}$ is* persistently exciting *(PE) of order L if the Hankel matrix $\mathcal{H}_L(\sigma_{[0,T-1]})$ has full row rank.*

The rank condition implies that $T \geq (p+1)L - 1$ is a lower-bound on the horizon T – in what follows, we assume that the order L and data horizon T are chosen such that this bound is satisfied.

Lemma 2 (Willems et al. (2005); De Persis and Tesi (2019)) *Consider the system (1.1) with (A, B) controllable, and assume that there is no driving noise. Let $\{x_{[0,T-1]}, u_{[0,T-1]}\}$ be the state and input signals generated by the system. Then if $u_{[0,T-1]}$ is PE of order $n + L$, the signals $x^\star_{[0,L-1]}$ and $u^\star_{[0,L-1]}$ are valid trajectories of length L of system (1.1) if and only if*

$$\begin{bmatrix} x^\star_{[0,L-1]} \\ u^\star_{[0,L-1]} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_L(x_{[0,T-1]}) \\ \mathcal{H}_L(u_{[0,T-1]}) \end{bmatrix} g, \textit{ for some } g \in \mathbb{R}^{T-L+1}$$

Lemma 2 states that if the underlying system is controllable, and $\operatorname{rank} \mathcal{H}_{n+L}(u) = n + L$, then: (1) all initial conditions and inputs are parameterizable from observed signal data; and (2) all valid trajectories lie in the linear span of a suitable Hankel matrix constructed from the system trajectories. Our goal is to exploit this relationship to characterize valid *closed loop system responses* of the unknown system (1.1) by establishing a connection to the SLS parameterization.

System Level Synthesis: Consider an L-length trajectory from system (1.1) expressed as block matrix operations

$$x_{[0,L-1]} = \mathcal{Z}\mathcal{A}x_{[0,L-1]} + \mathcal{Z}\mathcal{B}u_{[0,L-1]} + w_{[-1,L-2]} \tag{2.1}$$

where $\mathcal{A} = I_L \otimes A$, and $\mathcal{B} = I_L \otimes B$, and where $\mathcal{Z}$ is the block-downshift operator. If it is also the case that the system (2.1) satisfies the linear feedback control law $u_{[0,L-1]} = \mathcal{K}x_{[0,L-1]}$ for a causal linear-time-varying state-feedback control policy $\mathcal{K} \in \mathcal{L}_{TV}^{L,m\times n}$, then rewriting (2.1) we arrive at

$$\begin{aligned} x_{[0,L-1]} &= (I - \mathcal{Z}(\mathcal{A} + \mathcal{B}\mathcal{K}))^{-1}w_{[-1,L-2]} = \mathbf{\Phi}_x w_{[-1,L-2]} \\ u_{[0,L-1]} &= \mathcal{K}(I - \mathcal{Z}(\mathcal{A} + \mathcal{B}\mathcal{K}))^{-1}w_{[-1,L-2]} = \mathbf{\Phi}_u w_{[-1,L-2]} \end{aligned} \tag{2.2}$$

which captures how the process noise w maps to the state x and control u. We refer to the causal linear operators $\mathbf{\Phi}_x \in \mathcal{L}_{TV}^{L,n\times n}$ and $\mathbf{\Phi}_u \in \mathcal{L}_{TV}^{L,m\times n}$ as the *system responses*, which characterize the closed-loop system behavior from noise to state and control input, respectively.

Theorem 3 (Theorem 2.1, (Anderson et al., 2019)) *For a system* (1.1) *with state-feedback control law $\mathcal{K} \in \mathcal{L}_{TV}^{L,m\times n}$, i.e., $u_{[0,L-1]} = \mathcal{K}x_{[0,L-1]}$, the following are true*

1. *The affine subspace defined by*

$$\begin{bmatrix}(I - \mathcal{Z}\mathcal{A}) & -\mathcal{Z}\mathcal{B}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_x \\ \mathbf{\Phi}_u\end{bmatrix} = I, \quad \mathbf{\Phi}_x \in \mathcal{L}_{TV}^{L,n\times n}, \mathbf{\Phi}_u \in \mathcal{L}_{TV}^{L,m\times n} \tag{2.3}$$

parameterizes all possible system responses from $w_{[-1,L-2]} \to (x_{[0,L-1]}, u_{[0,L-1]})$.

2. *For any causal linear operators $\mathbf{\Phi}_x, \mathbf{\Phi}_u$ satisfying* (2.3)*, the controller $\mathcal{K} = \mathbf{\Phi}_u\mathbf{\Phi}_x^{-1} \in \mathcal{L}_{TV}^{L,m\times n}$ achieves the desired closed-loop responses* (2.2).

Theorem 3 allows for the problem of controller synthesis to be equivalently posed as a search over the affine space of system responses characterized by constraint (2.3) by setting $x_{[0,L-1]} = \mathbf{\Phi}_x w_{[-1,L-2]}$ and $u_{[0,L-1]} = \mathbf{\Phi}_u w_{[-1,L-2]}$. In particular, the LQG problem posed in Section 1 can be recast as a search over system responses (see Section 2.2 of Anderson et al. (2019)) as:[1]

$$\underset{\mathbf{\Phi}_x,\mathbf{\Phi}_u}{\text{minimize}} \quad \left\| \begin{bmatrix}\mathcal{Q}^{1/2} & \\ & \mathcal{R}^{1/2}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_x \\ \mathbf{\Phi}_u\end{bmatrix}\right\|_F \qquad \text{subject to (2.3)}, \tag{2.4}$$

where $\mathcal{Q} = I_L \otimes Q$ and $\mathcal{R} = I_L \otimes R$. In the noise free setting, i.e., when the initial condition $w(-1) = x(0)$ is known, and $w(t) = 0$ for $t \geq 0$, the objective function of this problem instead simplifies to

$$\underset{\mathbf{\Phi}_x(:,0),\mathbf{\Phi}_u(:,0)}{\text{minimize}} \quad \left\| \begin{bmatrix}\mathcal{Q}^{1/2} & \\ & \mathcal{R}^{1/2}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_x(:,0) \\ \mathbf{\Phi}_u(:,0)\end{bmatrix} x(0)\right\|_F, \tag{2.5}$$

and similarly, because only the initial condition is nonzero, the affine constraint (2.3) reduces to

$$\begin{bmatrix}(I - \mathcal{Z}\mathcal{A}) & -\mathcal{Z}\mathcal{B}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_x(:,0) \\ \mathbf{\Phi}_u(:,0)\end{bmatrix} = I(:,0). \tag{2.6}$$

A Data-Driven Formulation: We now show how the simplified achievability constraints (2.6) can be replaced by a data-driven representation through the use of Lemma 2. Our key insight is to recognize that the i-th column of $\mathbf{\Phi}_x(:,0)$ and $\mathbf{\Phi}_u(:,0)$ are the impulse response of the state and control

1. We drop both the squaring of the objective function, and the scaling factor σ^2, for brevity of notation going forward, as neither affect the optimal solution, or the order-wise scaling of the derived bounds.

input, respectively, to the i-th disturbance channel, which are themselves, valid system trajectories that can be characterized using Willems' fundamental lemma.

Theorem 4 *Consider the system (1.1) with (A, B) controllable, and assume that there is no driving noise. Suppose that a state/input signal pair $\{x_{[0,T-1]}, u_{[0,T-1]}\}$ is collected, and assume that $u_{[0,T-1]}$ is PE of order at least $n + L$. We then have that the set of feasible solutions to constraint (2.6) defined over a time horizon $t = 0, 1, \ldots, L-1$ can be equivalently characterized as:*

$$\begin{bmatrix} \mathcal{H}_L(x) \\ \mathcal{H}_L(u) \end{bmatrix} G, \quad \textit{for all } G \in \Gamma(x) := \{G : \mathcal{H}_1(x)G = I\}. \tag{2.7}$$

Proof [sketch] We prove the following relationship $\big\{\{\mathbf{\Phi}_x(:,0), \mathbf{\Phi}_u(:,0)\}$ satisfying (2.6)$\big\} = \big\{\{\mathcal{H}_L(x)G, \mathcal{H}_L(u)G\} : G \in \Gamma(x)\big\}$. To alleviate notational burden, denote the left and right-hand sets as LHS and RHS respectively.
($\subseteq$) Consider some $\mathbf{\Phi}_x(:,0), \mathbf{\Phi}_u(:,0) \in$ LHS. By Theorem 3, we then have that for any initial condition $x(0)$, that $\{x_{[0,L-1]}, u_{[0,L-1]}\} = \{\mathbf{\Phi}_x(:,0)x(0), \mathbf{\Phi}_u(:,0)x(0)\}$ is a valid system trajectory. By Lemma 2, we then have that for any $x(0)$, there exists a $g \in \mathbb{R}^{T-L+1}$ such that $\{\mathbf{\Phi}_x(:,0)x(0), \mathbf{\Phi}_u(:,0)x(0)\} = \{\mathcal{H}_L(x)g, \mathcal{H}_L(u)g\}$. Let $x^{(i)}(0) = e_i$, $i = 1, \ldots, n$ be the standard basis element, and let g_i be the corresponding vector such that equation (A.1) holds. Concatenating the resulting expressions row-wise, we obtain $\{\mathbf{\Phi}_x(:,0), \mathbf{\Phi}_u(:,0)\} = \{\mathcal{H}_L(x)G, \mathcal{H}_L(u)G\}$ for $G = [g_1, \ldots, g_n]$. Further, this implies that $\mathcal{H}_1(x)G = \Phi_x(0,0) = I$, and thus $G \in \Gamma(x)$.
($\supseteq$) Consider a $\{\mathcal{H}_L(x)G, \mathcal{H}_L(u)G\} \in$ RHS. Substituting these into constraint (2.6), we obtain $(I - \mathcal{Z}\mathcal{A})\mathcal{H}_L(x)G - \mathcal{Z}\mathcal{B}\mathcal{H}_L(u)G = [(\mathcal{H}_1(x)G)^\top\ 0]^\top = [I\ 0]^\top$, where we use that $G \in \Gamma(x)$. ■

Thus, if a state/input pair $\{x_{[0,T-1]}, u_{[0,T-1]}\}$ is generated by a PE input signal of order at least $n + L$, Theorem 4 gives conditions under which $\{\mathcal{H}_L(x), \mathcal{H}_L(u)\}$ can be used to parameterize achievable system responses for system (1.1) under no driving noise. In particular, one can then reformulate the deterministic optimal control problem formulated in equations (2.5) and (2.6) as

$$\underset{G \in \Gamma(x)}{\text{minimize}} \quad \left\| \begin{bmatrix} \mathcal{Q}^{1/2} & \\ & \mathcal{R}^{1/2} \end{bmatrix} \begin{bmatrix} \mathcal{H}_L(x) \\ \mathcal{H}_L(u) \end{bmatrix} G x(0) \right\|_F. \tag{2.8}$$

3. Robust Data-Driven System Level Synthesis

We now turn our attention to the original stochastic LQG optimal control problem (2.4), with driving noise $w(t) \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2 I)$ for $t = -1, 0, \ldots, T-2$. To differentiate between the state of the noise free and noisy system, we will denote the state signal by $\tilde{x}_{[0,T-1]}$ when driving noise is present. This additional *unmeasurable input* means that valid system trajectories can no longer be solely characterized in terms of the Hankel matrices $\mathcal{H}_L(\tilde{x})$ and $\mathcal{H}_L(u)$, as the effect of the process noise, as captured by a corresponding Hankel matrix $\mathcal{H}_L(w)$, must also be accounted for. To address this challenge, we relate the state-trajectories of system (1.1) under driving noise to those of system (1.1) under no driving noise, and use this relationship to construct *approximate system responses* that lie a bounded distance from the affine subspace defined in (2.3). We then leverage a robust SLS parameterization to bound the effects of this approximation error on the closed loop behavior.

Robust SLS: We begin with a robust variant of Theorem 3 that characterizes the behavior achieved by a controller constructed from system responses lying near the affine subspace characterized by constraint (2.3).

Theorem 5 (Theorem 2.2, Anderson et al. (2019)) *Let $\mathbf{\Delta}$ be a strictly causal linear operator (i.e., its matrix representation is strictly block-lower-triangular), and suppose that $\{\hat{\mathbf{\Phi}}_x, \hat{\mathbf{\Phi}}_u\}$ satisfy*

$$\begin{bmatrix}(I - \mathcal{Z}\mathcal{A}) & -\mathcal{Z}\mathcal{B}\end{bmatrix} \begin{bmatrix}\hat{\mathbf{\Phi}}_x \\ \hat{\mathbf{\Phi}}_u\end{bmatrix} = I + \mathbf{\Delta}, \quad \hat{\mathbf{\Phi}}_x \in \mathcal{L}_{TV}^{L,n\times n}, \hat{\mathbf{\Phi}}_u \in \mathcal{L}_{TV}^{L,m\times n} \tag{3.1}$$

Then the controller $\hat{\mathcal{K}} = \hat{\mathbf{\Phi}}_u \hat{\mathbf{\Phi}}_x^{-1}$ achieves the system responses

$$\begin{bmatrix}x_{[0,L-1]} \\ u_{[0,L-1]}\end{bmatrix} = \begin{bmatrix}\hat{\mathbf{\Phi}}_x \\ \hat{\mathbf{\Phi}}_u\end{bmatrix} (I + \mathbf{\Delta})^{-1} w_{[-1,L-2]} \tag{3.2}$$

Equation (3.2) shows that the effect of the error term $\mathbf{\Delta}$ in the approximate achievability constraint (3.1) is to map the original disturbance signal to as $w_{[-1,L-2]} \to \tilde{w}_{[-1,L-2]} := (I + \mathbf{\Delta})^{-1} w_{[-1,L-2]}$. In particular, this makes clear that we must design the full system responses $\{\mathbf{\Phi}_x, \mathbf{\Phi}_u\}$, and not just their first block-columns as in the idealized setting considered in the previous section, as $\tilde{w}_{[-1,L-2]}$ will have full support even if $w_{[-1,L-2]}$ is only nonzero for $w(-1) = x(0)$.

A Robust Data-Driven Formulation: Thus our challenge is to construct causal linear operators $\{\hat{\mathbf{\Phi}}_x, \hat{\mathbf{\Phi}}_u\}$ from noisy system data $\{\tilde{x}_{[0,T-1]}, u_{[0,T-1]}\}$ that satisfy equation (3.1) with as small a perturbation term $\mathbf{\Delta}$ as possible. Our approach is to construct each block-column of the approximate system responses individually, and then suitably concatenating them to construct a feasible solution to constraint (3.1), allowing us to explicitly characeterize the effect of the driving noise $w_{[-1,T-2]}$ on the the perturbation term $\mathbf{\Delta}$. We emphasize that we have access to only $\{\tilde{x}_{[0,T-1]}, u_{[0,T-1]}\}$, but it is instructive to also consider $w_{[-1,T-2]}$ in our analysis. To begin, as each column of $\mathcal{H}_L(\tilde{x}), \mathcal{H}_L(u), \mathcal{H}_L(w)$ satisfies (2.1), it follows that

$$\begin{bmatrix}(I - \mathcal{Z}\mathcal{A}) & -\mathcal{Z}\mathcal{B}\end{bmatrix} \begin{bmatrix}\mathcal{H}_L(\tilde{x}) \\ \mathcal{H}_L(u)\end{bmatrix} = \begin{bmatrix}\mathcal{H}_1(\tilde{x}) \\ 0\end{bmatrix} + \mathcal{Z}\mathcal{H}_L(w) \tag{3.3}$$

Then, fix a $\hat{G} \in \Gamma(\tilde{x})$ and let $\hat{\mathbf{\Phi}}_x(:,0) = \mathcal{H}_L(\tilde{x})\hat{G}$ and $\hat{\mathbf{\Phi}}_u(:,0) = \mathcal{H}_L(u)\hat{G}$ as in the proof of Theorem 4,

$$\begin{bmatrix}(I - \mathcal{Z}\mathcal{A}) & -\mathcal{Z}\mathcal{B}\end{bmatrix} \begin{bmatrix}\hat{\mathbf{\Phi}}_x(:,0) \\ \hat{\mathbf{\Phi}}_u(:,0)\end{bmatrix} = I(:,0) + \underbrace{\mathcal{Z}\mathcal{H}_L(w)\hat{G}}_{\mathbf{\Delta}(:,0)} \tag{3.4}$$

Modulo down-shifting, (3.4) demonstrates the construction of a single block-column of $\{\hat{\mathbf{\Phi}}_x, \hat{\mathbf{\Phi}}_u\}$ and $\mathbf{\Delta}$. The construction of the other columns is similar; in general consider $\hat{G}_0, \ldots, \hat{G}_{L-1} \in \Gamma(\tilde{x})$, where each $\hat{G}_{k-1}$ is used to construct the kth column of $\hat{\mathbf{\Phi}}_x, \hat{\mathbf{\Phi}}_u$. Note that $\mathcal{Z}$ commutes with block-diagonal matrices with identical block-diagonal entries (adjusting for dimensions): this can be seen by observing that left-multiplication by $\mathcal{Z}$ (down-shifting) is equivalent to right-multiplication by $\mathcal{Z}$ (left-shifting). Thus, we can construct down-shifted block-columns of the form (3.4) as follows

$$\begin{bmatrix}(I - \mathcal{Z}\mathcal{A}) & -\mathcal{Z}\mathcal{B}\end{bmatrix} \begin{bmatrix}\mathcal{Z}^{k-1}\mathcal{H}_L(x) \\ \mathcal{Z}^{k-1}\mathcal{H}_L(u)\end{bmatrix} \hat{G}_{k-1} = \mathcal{Z}^{k-1} I(:,0) + \mathcal{Z}^k \mathcal{H}_L(w)\hat{G}_{k-1}, \tag{3.5}$$

from which full arpproximate system responses can be constructed, as formalized in the following.

Theorem 6 *For system (1.1) with (A, B) controllable, and control input $u_{[0,T-1]}$ and disturbance process $w_{[-1,T-2]}$ PE of order $n + L$, the approximate system response matrices $\{\hat{\mathbf{\Phi}}_x, \hat{\mathbf{\Phi}}_u\}$ and perturbation term $\mathbf{\Delta}$ are defined as*

$$\hat{\mathbf{\Phi}}_x = \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(\tilde{x}))\hat{\mathcal{G}}, \ \hat{\mathbf{\Phi}}_u = \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(u))\hat{\mathcal{G}}, \ \mathbf{\Delta} = \mathcal{Z}_L(I_L \otimes \mathcal{Z}\mathcal{H}_L(w))\hat{\mathcal{G}} \tag{3.6}$$

and satisfy the approximate achievability constraint (3.1)*, where* $\mathcal{Z}_L = \begin{bmatrix} I & \mathcal{Z} & \cdots & \mathcal{Z}^{L-1} \end{bmatrix}$ *and* $\hat{\mathcal{G}} \in \mathcal{L}_{TV}$ *with block-diagonal elements* $\hat{G}(i:i) \in \Gamma(\tilde{x})$ *for* $i = 0, 1, \ldots, L-1$*, and off-diagonal blocks* $\mathcal{H}_1(\tilde{x})\hat{G}(i,j) = 0$ *for* $i \neq j$*.*

Theorem 6 thus allows us to apply the robust SLS parameterization of Theorem 5 to characterize the closed-loop behavior (3.2) achieved by a controller constructed from the data-driven approximate system responses $\{\hat{\mathbf{\Phi}}_x, \hat{\mathbf{\Phi}}_u\}$ in terms of the perturbation term $\mathbf{\Delta}$, as described in equation (3.6). In particular, if we assume that $\|\mathcal{H}(w)\|_2 \leq \varepsilon$, but is otherwise acting adversarially, we can pose the following robust LQG problem:

$$\begin{aligned} \underset{\hat{\mathcal{G}}}{\text{minimize}} \ \max_{\|\mathcal{H}_L(w)\| \leq \varepsilon} \ & \left\| \begin{bmatrix} \mathcal{Q}^{1/2} & \\ & \mathcal{R}^{1/2} \end{bmatrix} \begin{bmatrix} \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(\tilde{x})) \\ \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(u)) \end{bmatrix} \hat{\mathcal{G}}(I + (I_L \otimes \mathcal{Z}\mathcal{H}_L(w))\hat{\mathcal{G}})^{-1} \right\|_F \\ \text{subject to} \quad & \hat{G}(i,i) \in \Gamma(\tilde{x}) \text{ for } i = 0, 1, \ldots, L-1, \mathcal{H}_1(x)\hat{G}(i,j) = 0 \text{ for all } i \neq j. \end{aligned} \tag{3.7}$$

Note that although we assume that the disturbance process is drawn as $w(t) \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2 I)$, we conservatively treat the effects of the unknown Hankel matrix $\mathcal{H}_L(w)$ on the estimated system responses as adversarial in our analysis. This approach also allows our method to generalize naturally to other optimal control settings, such as those with $\mathcal{H}_\infty$ and $\mathcal{L}_1$ cost functions.

The objective function of optimization problem (3.7) is a non-convex, but its structure allows for a transparent and data-independent quasi-convex upper-bound to be derived. First, we observe that we can upper bound $\|\mathbf{\Delta}\|_2$ as given in equation (3.6), by

$$\|\mathbf{\Delta}\|_2 = \|\mathcal{Z}_L(I_L \otimes \mathcal{Z}\mathcal{H}_L(w))\hat{\mathcal{G}}\|_2 \leq \|\mathcal{Z}_L\|_2 \|\mathcal{H}_L(w)\|_2 \|\hat{\mathcal{G}}\|_2 \leq \sqrt{L}\varepsilon\|\hat{\mathcal{G}}\|_2, \tag{3.8}$$

from which it follows immediately that if $\sqrt{L}\varepsilon\|\hat{\mathcal{G}}\|_2 < 1$, we have the following upper bound $\|(I + \mathbf{\Delta})^{-1}\|_2 \leq 1/(1 - \sqrt{L}\varepsilon\|\hat{\mathcal{G}}\|_2)$. This observation allows us to follow a similar argument as in Dean et al. (2019), to derive the following quasi-convex upper bound to problem (3.7):

$$\begin{aligned} \underset{\gamma \in [0,1),\, \hat{\mathcal{G}}}{\text{minimize}} \quad & \frac{1}{1-\gamma} \left\| \begin{bmatrix} \mathcal{Q}^{1/2} & \\ & \mathcal{R}^{1/2} \end{bmatrix} \begin{bmatrix} \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(\tilde{x})) \\ \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(u)) \end{bmatrix} \hat{\mathcal{G}} \right\|_F \\ \text{subject to} \quad & \|\hat{\mathcal{G}}\|_2 \leq \frac{\gamma}{\sqrt{L}\varepsilon}, \ \hat{G}(i,i) \in \Gamma(\tilde{x}) \text{ for } i = 0, 1, \ldots, L-1, \mathcal{H}_1(\tilde{x})\hat{G}(i,j) = 0 \text{ for } i \neq j. \end{aligned} \tag{3.9}$$

which is quasi-convex in γ and convex in $\hat{\mathcal{G}}$, allowing for an efficient solution via bisection.

4. Sub-optimality Analysis

In this section, we prove the following suboptimality result, which relates the performance $\hat{J}$ achieved by the controller synthesized via the robust synthesis problem (3.9) to the optimal performance $J^\star$ achieved by the optimal LQG controller.

Theorem 7 *Let* $(\hat{\mathcal{G}}, \hat{\gamma})$ *be the optimal solution to* (3.9)*, let* $\hat{J}$ *be the LQG cost that the controller* $\hat{\mathcal{K}} = \hat{\mathbf{\Phi}}_u \hat{\mathbf{\Phi}}_x^{-1}$ *constructed from the system responses* (3.6) *achieves on system* (1.1)*. Assume that* $T \geq 2L + 1$*, that* $\|\mathcal{H}_L(w)\|_2 \leq \varepsilon$*, and that* ε *satisfies the bounds* (4.1)*. Let* $\mathcal{G}^\star$ *be the optimal parameter such that in Theorem 4* $\{\mathbf{\Phi}_x^\star, \mathbf{\Phi}_u^\star\} = \{\mathcal{H}_L(x)\mathcal{G}^\star, \mathcal{H}_L(u)\mathcal{G}^\star\}$*, for* $\{\mathbf{\Phi}_x^\star, \mathbf{\Phi}_u^\star\}$ *corresponding to the optimal LQG system responses to cost* (2.4)*. Letting* $J^\star$ *be the optimal LQG cost achieved by the optimal controller* $\mathcal{K}_\star = \mathbf{\Phi}_u^\star(\mathbf{\Phi}_x^\star)^{-1}$*, we then have that*

$$\frac{\hat{J} - J^\star}{J^\star} \leq 3\|\mathcal{G}^\star\|_2 \varepsilon \left(2\sqrt{L} + \|\mathcal{T}_{T-L+1}(I)\|_2 + (1 + \|\mathcal{O}_L(A)\|_2)\|\mathcal{Q}^{1/2}\|_2 \|\mathcal{T}_{T-L+1}(I)\|_2 / J^\star \right)$$

Our proof strategy is to construct a feasible solution to problem (3.9) using the optimal $\mathcal{G}^\star$ that corresponds to the optimal system responses $\{\mathbf{\Phi}_x^\star, \mathbf{\Phi}_u^\star\}$ induced by the optimal LQG controller $\mathcal{K}_\star$, such that $\{\mathbf{\Phi}_x^\star, \mathbf{\Phi}_u^\star\} = \{\mathcal{H}_L(x)\mathcal{G}^\star, \mathcal{H}_L(u)\mathcal{G}^\star\}$ for data $\{x_{[0,T-1]}, u_{[0,T-1]}\}$ generated by system (1.1) with no driving noise.[2] First, we introduce the following technical lemma relating Hankel matrices constructed from state trajectories of system (1.1) with and without driving noise.

Lemma 8 *Let $x_{[0,T-1]}$ and $\tilde{x}_{[0,T-1]}$ be the state signals for system (1.1), driven by $u_{[0,T-1]}$ and $\{u_{[0,T-1]}, w_{[-1,T-2]}\}$, respectively, and suppose that $\{u_{[0,T-1]}, w_{[-1,T-2]}\}$ are PE of order $n+L$. We then have that $\mathcal{H}_L(\tilde{x}) = \mathcal{H}_L(x) + \mathcal{T}_L(I)\mathcal{H}_L(w) + \mathcal{O}_L(A)\mathcal{W}_{[0,T-L]}$ where $\mathcal{W}(t) = \sum_{k=0}^{t-1} A^{t-1-k}w(k)$ are columns of $\mathcal{W}_{[0,T-L]} = \begin{bmatrix}\mathcal{W}(0) & \cdots & \mathcal{W}(T-L)\end{bmatrix}$,[3] and*

$$\mathcal{O}_L(A) = \begin{bmatrix} I \\ A \\ \vdots \\ A^{L-1} \end{bmatrix}, \qquad \mathcal{T}_L(X) = \begin{bmatrix} 0 & & & & \\ X & 0 & & & \\ AX & X & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ A^{L-2} & A^{L-3} & \cdots & X & 0 \end{bmatrix}$$

Proof Let $x_{[t,t-L+1]}$ and $\tilde{x}_{[t,t+L-1]}$ be an arbitrary pair of columns of $\mathcal{H}_L(z)$ and $\mathcal{H}_L(\tilde{x})$. Then, by the dynamics of system (1.1), we have that $x_{[t,t+L-1]} = \mathcal{O}_L(A)z(t) + \mathcal{T}_L(B)u_{[t,t+L-1]}$, and $\tilde{x}_{[t,t+L-1]} = \mathcal{O}_L(A)\tilde{x}(t) + \mathcal{T}_L(B)u_{[t,t+L-1]} + \mathcal{T}_L(I)w_{[t,t+L-1]}$.. Since $\tilde{x}(0) = x(0)$, using the same control sequence $u_{[0,T-1]}$ on both systems reveals the perturbation factor $\tilde{x}(t) = x(t) + \mathcal{W}(t)$. Consequently, $\tilde{x}_{[t,t+L-1]} = \mathcal{O}_L(A)(x(t)+\mathcal{W}(t)) + \mathcal{T}_L(B)u_{[t,t+L-1]} + \mathcal{T}_L(I)w_{[t,t+L-1]}$. Extending this to all columns of the Hankel matrices completes the proof. ■

We now use Lemma 8 to construct a feasible solution to the robust optimization problem (3.9) using the optimal solution $\mathcal{G}^\star$, which is subsequently used to prove the main result of this section.

Lemma 9 *Let $x_{[0,T-1]}$ and $\tilde{x}_{[0,T-1]}$ be the state signals for system (1.1), driven by $u_{[0,T-1]}$ and $\{u_{[0,T-1]}, w_{[-1,T-2]}\}$, respectively, and suppose that $\{u_{[0,T-1]}, w_{[-1,T-2]}\}$ are PE of order $n+L$. Let $\mathcal{G}^\star = \mathrm{blkdiag}(G_0^\star, \ldots, G_{L-1}^\star)$, for $G_0^\star, \ldots, G_{L-1}^\star \in \Gamma(x)$ be the parameter to the optimal LQG system responses in Theorem 4 such that $\{\mathbf{\Phi}_x^\star, \mathbf{\Phi}_u^\star\} = \{\mathcal{H}_L(x)\mathcal{G}^\star, \mathcal{H}_L(u)\mathcal{G}^\star\}$. Then, if*

$$\varepsilon \leq \min\left\{\tfrac{1}{3\sqrt{L}\|\mathcal{G}^\star\|_2}, \ \tfrac{1}{2\|\mathcal{G}^\star\|_2 \cdot \|\mathcal{T}_{T-L+1}(I)\|_2}\right\}, \tag{4.1}$$

the pair $\{\mathcal{G}^0 = \mathcal{G}^\star(I+\mathcal{D})^{-1}, \ \gamma^0 = 2\varepsilon\|\mathcal{G}^\star\|_2\sqrt{L}\}$ is a feasible solution to (3.9) where

$$\mathcal{D} = \mathrm{blkdiag}(D_0, \ldots, D_{L-1}), \qquad D_k = \mathcal{W}_{[0,T-L]}G_k^\star, \quad \textit{for} \ \ 0 \leq k \leq L-1.$$

Proof [Proof of Theorem 7 (sketch)] The cost $\hat{J}$ achieved by the optimal solutions to (3.9) $\{\hat{\mathbf{\Phi}}_x, \hat{\mathbf{\Phi}}_u\} = \{\mathcal{H}_L(\tilde{x})\hat{\mathcal{G}}, \mathcal{H}_L(u)\hat{\mathcal{G}}\}$ on the true dynamics is upper-bounded via

$$\hat{J} \leq \frac{1}{1-\hat{\gamma}}\left\|\begin{bmatrix}\mathcal{Q}^{1/2} & \\ & \mathcal{R}^{1/2}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{\Phi}}_x \\ \hat{\mathbf{\Phi}}_u\end{bmatrix}\right\|_F \leq \frac{1}{1-\gamma^0}\left\|\begin{bmatrix}\mathcal{Q}^{1/2} & \\ & \mathcal{R}^{1/2}\end{bmatrix}\begin{bmatrix}\mathcal{Z}_L(I_L \otimes \mathcal{H}_L(\tilde{x})) \\ \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(u))\end{bmatrix}\mathcal{G}_0\right\|_F$$

where the first inequality follows from the derivation of (3.7), and the second from the optimality of $\{\hat{\mathbf{\Phi}}_x, \hat{\mathbf{\Phi}}_u \, \hat{\gamma}\}$ and Lemma 9. Letting $\alpha(T) = \|\mathcal{T}_{T-L+1}(I)\|_2\|\mathcal{G}^\star\|_2$, we apply Lemma 8, leverage

2. Such a $\mathcal{G}^\star$ exists by Theorem 4, and we select the minimum norm $\mathcal{G}^\star$ satisfying the desired relationship. Future work will seek explicit relationships between the norms of the system responses, data matrices, and $\mathcal{G}^\star$.

3. We let $\mathcal{W}(0) = 0$ by convention.

that $\|\mathcal{T}_L\|_2 \leq \|\mathcal{T}_{T-L+1}\|_2$ if $L \leq T-L+1$, which is assumed in the Theorem, and the relations

$$\mathcal{G}_0 = \mathcal{G}^\star(I+\mathcal{D})^{-1}, \quad \begin{bmatrix}\mathbf{\Phi}_x^\star \\ \mathbf{\Phi}_u^\star\end{bmatrix} = \begin{bmatrix}\mathcal{Z}_L(I_L \otimes \mathcal{H}_L(x)) \\ \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(u))\end{bmatrix}\mathcal{G}^\star, \quad \left\|(I+\mathcal{D})^{-1}\right\|_2 \leq \frac{1}{1-\alpha(T)\varepsilon}$$

to conclude that

$$\hat{J} \leq \frac{J^\star}{(1-\gamma^0)(1-\alpha(T)\varepsilon)} + \frac{(1+\|\mathcal{O}_L(A)\|_2)\|\mathcal{Q}^{1/2}\|_2\|\mathcal{G}^\star\|_2\|\mathcal{T}_{T-L+1}(I)\|_2\varepsilon}{(1-\gamma^0)(1-\alpha(T)\varepsilon)}$$

Simplifying and rearranging the above yields

$$\frac{J-J^\star}{J^\star} \leq \frac{\gamma^0+\alpha(T)\varepsilon}{(1-\gamma^0)(1-\alpha(T)\varepsilon)} + \frac{(1+\|\mathcal{O}_L(A)\|_2)\|\mathcal{Q}^{1/2}\|_2\|\mathcal{G}^\star\|_2\|\mathcal{T}_{T-L+1}(I)\|_2\varepsilon}{(1-\gamma^0)(1-\alpha(T)\varepsilon)J^\star}$$

Recall that $\gamma^0 = 2\varepsilon\|\mathcal{G}^\star\|_2\sqrt{L}$, and by the assumptions of the Theorem we have that $\alpha(T)\varepsilon \leq 1/2$ and $\gamma^0 \leq 2/3$; thus $\frac{\gamma^0+\alpha(T)\varepsilon}{(1-\gamma^0)(1-\alpha(T)\varepsilon)} \leq 3\varepsilon(2\|\mathcal{G}^\star\|_2\sqrt{L}+\alpha(T))$. Similarly, the second term is bounded by $3(1+\|\mathcal{O}_L(A)\|_2)\varepsilon\|\mathcal{Q}^{1/2}\|_2\|\mathcal{G}^\star\|_2\|\mathcal{T}_{T-L+1}(I)\|_2/J^\star$ ■

4.1. Sample Complexity

We now show *trajectory averaging* can be used to mitigate the effects of noise on the data $\mathcal{H}_L(\tilde{x})$: as system (1.1) is LTI, given a collection of independently collected signals $\{\tilde{x}^{(i)}_{[0,T-1]}, u^{(i)}_{[0,T-1]}, w^{(i)}_{[-1,T-2]}\}_{i=1}^N$, the average trajectories $\bar{x}_{[0,T-1]} = \frac{1}{N}\sum_{i=1}^N \tilde{x}^{(i)}_{[0,T-1]}$, $\bar{u}_{[0,T-1]} = \frac{1}{N}\sum_{i=1}^N u^{(i)}_{[0,T-1]}$, $\bar{w}_{[-1,T-2]} = \frac{1}{N}\sum_{i=1}^N w^{(i)}_{[-1,T-2]}$, are themselves valid. Importantly, due to the effects of averaging, we now have that the elements $\bar{w}(t) \overset{\text{iid}}{\sim} \mathcal{N}(0,\sigma^2/N)$. We then have that,

$$\mathbb{P}\left[\|\mathcal{H}_L(\bar{w})\|_2 \geq t\right] \leq 2nTe^{-\frac{t^2N}{2\sigma^2 nT}} \tag{4.2}$$

for all $t \geq 0$, which follows from matrix Gaussian series concentration inequalities (Tropp, 2012, Theorem 4.1.1) (see Lemma 11 in the Appendix of the extended version).

Corollary 10 *If $N \geq 2\sigma^2 nT\log(2nT/\delta)\max\left\{9L\|\mathcal{G}^\star\|_2^2, 4\|\mathcal{G}^\star\|_2^2\|\mathcal{T}_{T-L+1}(I)\|_2^2\right\}$, then, with probability at least $1-\delta$, we have that*

$$\frac{\hat{J}-J^\star}{J^\star} \leq 3\|\mathcal{G}^\star\|_2\sqrt{\tfrac{2\sigma^2 nT}{N}\log\left(\tfrac{2nT}{\delta}\right)}\left(2\sqrt{L} + \|\mathcal{T}_{T-L+1}(I)\|_2\left(1+\tfrac{(1+\|\mathcal{O}_L(A)\|_2)\|\mathcal{Q}^{1/2}\|_2}{J^\star}\right)\right)$$

Proof [sketch]. Follows from inverting probability bound (4.2) and recognizing that bounds (4.1) are satisfied under the assumptions of the Corollary. ■

5. Experiments

We present experiments on the system from (Dean et al., 2019)

$$A = \begin{bmatrix}1.01 & 0.01 & 0.00 \\ 0.01 & 1.01 & 0.01 \\ 0.00 & 0.01 & 1.01\end{bmatrix}, \quad \begin{matrix} B = I, & \sigma^2 = 0.1, \\ Q = 10^{-3}I, & R = I, \end{matrix}$$

which corresponds to a slightly unstable graph Laplacian system with input significantly more penalized than the output. All experiments were done in Julia v1.3.1 using the JuMP v0.21.2 library with MOSEK v9.2.9 as a backend. We found it effective to solve the dual problem using JuMP's `dual_optimizer` function.[4]

4. All code is open source and available at https://github.com/unstable-zeros/data-driven-sls.

Bootstrap Estimation of Noise: We use a vanilla bootstrap method (Chernick et al., 2011) to empirically estimate confidence bounds on $\|\mathcal{H}_L(\bar{w})\|_2$, for $L = 10$ and $T = 45$, as a function of the number of trajectory samples N: Fig. 1 shows the bootstrap estimated 95-th percentile bound compared to true 95-th percentile over 1000 trials.

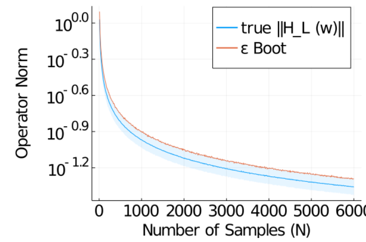

Fig. 1: Bootstrap Estimation of ε: The solid line is the 95-th percentile bound on the bootstrapped estimate of $\|\mathcal{H}_L(\bar{w})\|_2$ over $M = 1000$ trials. In blue are the median, 5-th, and 95-th percentiles of $\|\mathcal{H}_L(\bar{w})\|_2$ computed across an additional $M = 1000$ independent trials.

Controller Performance in MPC Loop: We consider the performance of four types of unconstrained MPC controllers based on the following finite-horizon LTV feedback gains: $\mathcal{K}^\star$ the optimal LQG controller synthesized with noise-free data; $\mathcal{K}_B$ and $\mathcal{K}_T$ the robust controllers synthesized using the bootstrap value of ε_B and true $\varepsilon = \|\mathcal{H}_L(w)\|_2$ respectively in problem (3.9); and $\mathcal{K}_N$ the naive controller is synthesized by dropping the robustness constraint in problem (3.9). For the selected values of $N = 10, 20, 40, \ldots, 5120$, random $T = 45$ length system trajectories $\{\tilde{x}^{(i)}_{[0,T-1]}, u^{(i)}_{[0,T-1]}, w^{(i)}_{[-1,T-2]}\}$ is generated and used to form the appropriate Hankel matrices $\mathcal{H}_L(\bar{\tilde{x}}), \mathcal{H}_L(\bar{u}), \mathcal{H}_L(\bar{w})$, by averaging trajectories. Each finite-horizon controller is then synthesized with the running cost matrices (Q, R) specified above, and terminal cost $Q_L = 400Q \succ P_\star$, for $P_\star$ the solution to the discrete algebraic Riccati equation for the infinite horizon LQG problem specified in terms of (A, B, Q, R), thus ensuring that $\mathcal{K}^\star$ is stabilizing. An MPC loop is then implemented over a horizon of $H = 100$ starting from an initial state $x(0) = 0$, resulting in an effective feedback control policy of $u(t) = \mathcal{K}(0:0)x(t)$, and we apply driving noise $w(t) \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2 I)$.

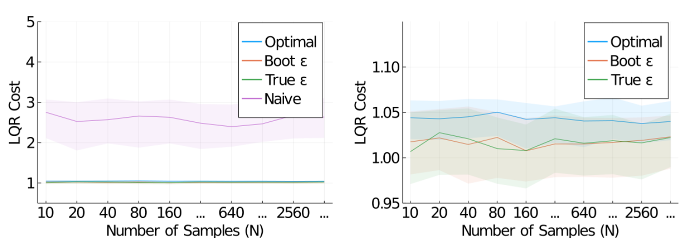

Fig. 2: (Left) Median and quartiles of performance of MPC controllers, with infeasibility assumed to be $+\infty$ in cost. (Right) Same plot without naive controller.

We evaluate $M = 150$ trials at each value of N, with results of the MPC loop performance shown in Figs. 2. We see that the robustness constraints lead to significant performance gains over the naive controller, which suffers significant degradation in performance, even for large values of N. We further see that it even leads to a slight performance gain over the noise-free controller $\mathcal{K}^\star$: we conjecture that this is due to the regularizing effect of the robustness constraint in problem (3.9) leading to an *effective terminal cost* closer to that of the optimal $P_\star$.

6. Conclusion

We defined and analyzed data-driven SLS parameterizations of stabilizing controllers for LTI systems. We showed that when given noise free trajectories there exists an exact equivalence between traditional and data-driven SLS. We then showed that when given noisy system trajectories, tools from robust SLS and matrix concentration theory can be used to characterize the sample-complexity needed to mitigate the effects of noise on closed-loop performance. Future work will look to extend these results to the infinite horizon, distributed & robust MPC, and output-feedback settings.

References

Carmen Amo Alonso and Nikolai Matni. Distributed and localized model predictive control via system level synthesis. *arXiv preprint arXiv:1909.10074*, 2019.

James Anderson, John C Doyle, Steven H Low, and Nikolai Matni. System level synthesis. *Annual Reviews in Control*, 2019.

Francesco Borrelli, Alberto Bemporad, and Manfred Morari. *Predictive control for linear and hybrid systems*. Cambridge University Press, 2017.

Michael R Chernick, Wenceslao González-Manteiga, Rosa M Crujeiras, and Erniel B Barrios. Bootstrap methods, 2011.

Jeremy Coulson, John Lygeros, and Florian Dörfler. Data-enabled predictive control: In the shallows of the deepc. In *2019 18th European Control Conference (ECC)*, pages 307–312. IEEE, 2019a.

Jeremy Coulson, John Lygeros, and Florian Dörfler. Regularized and distributionally robust data-enabled predictive control. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 2696–2701. IEEE, 2019b.

Jeremy Coulson, John Lygeros, and Florian Dörfler. Distributionally robust chance constrained data-enabled predictive control. *arXiv preprint arXiv:2006.01702*, 2020.

Claudio De Persis and Pietro Tesi. On persistency of excitation and formulas for data-driven control. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 873–878. IEEE, 2019.

Sarah Dean, Horia Mania, Nikolai Matni, Benjamin Recht, and Stephen Tu. Regret bounds for robust adaptive control of the linear quadratic regulator. In *Advances in Neural Information Processing Systems*, pages 4188–4197, 2018.

Sarah Dean, Horia Mania, Nikolai Matni, Benjamin Recht, and Stephen Tu. On the sample complexity of the linear quadratic regulator. *Foundations of Computational Mathematics*, pages 1–47, 2019.

Carlos E Garcia, David M Prett, and Manfred Morari. Model predictive control: theory and practice—a survey. *Automatica*, 25(3):335–348, 1989.

Elad Hazan, Sham Kakade, and Karan Singh. The nonstochastic control problem. In *Algorithmic Learning Theory*, pages 408–421. PMLR, 2020.

Horia Mania, Stephen Tu, and Benjamin Recht. Certainty equivalence is efficient for linear quadratic control. In *Advances in Neural Information Processing Systems*, pages 10154–10164, 2019.

Ivan Markovsky and Paolo Rapisarda. Data-driven simulation and control. *International Journal of Control*, 81(12):1946–1959, 2008.

Nikolai Matni, Yuh-Shyang Wang, and James Anderson. Scalable system level synthesis for virtually localizable systems. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 3473–3480. IEEE, 2017.

Nikolai Matni, Alexandre Proutiere, Anders Rantzer, and Stephen Tu. From self-tuning regulators to reinforcement learning and back again. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 3724–3740. IEEE, 2019.

Benjamin Recht. A tour of reinforcement learning: The view from continuous control. *Annual Review of Control, Robotics, and Autonomous Systems*, 2:253–279, 2019.

Monica Rotulo, Claudio De Persis, and Pietro Tesi. Data-driven linear quadratic regulation via semidefinite programming. *arXiv preprint arXiv:1911.07767*, 2019.

Max Simchowitz, Karan Singh, and Elad Hazan. Improper learning for non-stochastic control. *arXiv preprint arXiv:2001.09254*, 2020.

Joel A Tropp. User-friendly tail bounds for sums of random matrices. *Foundations of computational mathematics*, 12(4):389–434, 2012.

Henk J van Waarde, Claudio De Persis, M Kanat Camlibel, and Pietro Tesi. Willems' fundamental lemma for state-space systems and its extension to multiple datasets. *IEEE Control Systems Letters*, 4(3):602–607, 2020.

Han Wang, Shaoru Chen, Victor M Preciado, and Nikolai Matni. Robust model predictive control via system level synthesis. *arXiv*, pages arXiv–1911, 2019a.

Yuh-Shyang Wang, Nikolai Matni, and John C Doyle. A system-level approach to controller synthesis. *IEEE Transactions on Automatic Control*, 64(10):4079–4093, 2019b.

J. Willems. From time series to linear system - part i. finite dimensional linear time invariant systems. *Autom.*, 22:561–580, 1986.

Jan C Willems, Paolo Rapisarda, Ivan Markovsky, and Bart LM De Moor. A note on persistency of excitation. *Systems & Control Letters*, 54(4):325–329, 2005.

J.C. Willems and J.W. Polderman. *Introduction to Mathematical Systems Theory: A Behavioral Approach*. Texts in Applied Mathematics. Springer New York, 1997. ISBN 9780387982663. URL https://books.google.com/books?id=AbmTA1HbhKsC.

Appendix A. Proofs and Intermediate Results

Proof [Proof of Theorem 4] Our goal is to prove the following relationship

$$\left\{ \begin{bmatrix} \mathbf{\Phi}_x(:,0) \\ \mathbf{\Phi}_u(:,0) \end{bmatrix} \text{ satisfying (2.6)} \right\} = \left\{ \begin{bmatrix} \mathcal{H}_L(x) \\ \mathcal{H}_L(u) \end{bmatrix} G : G \in \Gamma(x) \right\}.$$

To alleviate notational burden, denote the left and right-hand sets as LHS and RHS respectively.

($\subseteq$) Consider some $\mathbf{\Phi}_x(:,0), \mathbf{\Phi}_u(:,0) \in$ LHS. By Theorem 3, we then have that for any initial condition $x(0)$, that $\{x_{[0,L-1]}, u_{[0,L-1]}\} = \{\mathbf{\Phi}_x(:,0)x(0), \mathbf{\Phi}_u(:,0)x(0)\}$ is a valid system trajectory. By Lemma 2, we then have that for any $x(0)$, there exists a $g \in \mathbb{R}^{T-L+1}$ such that

$$\begin{bmatrix} \mathbf{\Phi}_x(:,0) \\ \mathbf{\Phi}_u(:,0) \end{bmatrix} x(0) = \begin{bmatrix} \mathcal{H}_L(x) \\ \mathcal{H}_L(u) \end{bmatrix} g. \tag{A.1}$$

Let $x^{(i)}(0) = e_i$, $i = 1, \ldots, n$ be the standard basis element, and let g_i be the corresponding vector such that equation (A.1) holds. Concatenating the resulting expressions row-wise, we obtain the following expression:

$$\begin{bmatrix} \mathbf{\Phi}_x(:,0) \\ \mathbf{\Phi}_u(:,0) \end{bmatrix} \begin{bmatrix} | & | & \cdots & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_x(:,0) \\ \mathbf{\Phi}_u(:,0) \end{bmatrix} = \begin{bmatrix} \mathcal{H}_L(x) \\ \mathcal{H}_L(u) \end{bmatrix} \begin{bmatrix} | & | & \cdots & | \\ g_1 & g_2 & \cdots & g_n \\ | & | & \cdots & | \end{bmatrix} =: \begin{bmatrix} \mathcal{H}_L(x) \\ \mathcal{H}_L(u) \end{bmatrix} G \tag{A.2}$$

Furthermore, from equation (A.2), we have that $\mathcal{H}_1(x)G = \Phi_x(0,0) = I$, as $\mathcal{H}_1$ is the first block row of $\mathcal{H}_L(x)$. It therefore follows that $G \in \Gamma(x)$, proving that LHS $\subseteq$ RHS.

($\supseteq$) Consider a $\{\mathcal{H}_L(x)G, \mathcal{H}_L(u)G\} \in$ RHS. Substituting these into constraint (2.6), we obtain

$$\begin{bmatrix} (I - \mathcal{Z}\mathcal{A}) & -\mathcal{Z}\mathcal{B} \end{bmatrix} \begin{bmatrix} \mathcal{H}_L(x) \\ \mathcal{H}_L(u) \end{bmatrix} G = \begin{bmatrix} \mathcal{H}_1(x) \\ 0 \end{bmatrix} G$$

where the equality is a direct consequence of the stacked system dynamics (2.1) with no driving noise. As $G \in \Gamma(x)$ by assumption, we have that $\mathcal{H}_1(x)G = I$, and thus $\{\mathcal{H}_L(x)G, \mathcal{H}_L(u)G\}$ define valid system responses, from which the desired result follows. ■

Proof [Proof of Theorem 6] First, we see that by construction, $\hat{\mathbf{\Phi}}_x \in \mathcal{L}_{TV}^{L,n\times n}$, $\hat{\mathbf{\Phi}}_u \in \mathcal{L}_{TV}^{L,m\times n}$, and $\mathbf{\Delta} \in \mathcal{L}_{TV}^{L,n\times n}$ is strictly block-lower-triangular, and thus it suffices to verify that they satisfy the approximate achievability constraint (3.1). As $\mathcal{Z}_L$ commutes with $\mathcal{A}$ and $\mathcal{B}$, it follows from equation (3.3) that

$$\begin{bmatrix} (I - \mathcal{Z}\mathcal{A}) & -\mathcal{Z}\mathcal{B} \end{bmatrix} \begin{bmatrix} \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(\tilde{x})) \\ \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(u)) \end{bmatrix} = \mathcal{Z}_L\left(I_L \otimes \left(\begin{bmatrix} \mathcal{H}_1(\tilde{x}) \\ 0 \end{bmatrix} + \mathcal{Z}\mathcal{H}_L(w) \right)\right) \tag{A.3}$$

Consider a $\hat{\mathcal{G}} \in \mathcal{L}_{TV}$ constructed as described in the Theorem statement. Right-multiplying the left-hand-side of equation (A.3) by $\hat{\mathcal{G}}$ yields the desired left-hand-side of constraint (3.1). Further, notice that from the definition of $\Gamma(x)$ and the construction of $\hat{\mathcal{G}}$, we have that

$$\mathcal{Z}_L\left(I_L \otimes \begin{bmatrix} \mathcal{H}_1(\tilde{x}) \\ 0 \end{bmatrix} \right)\hat{\mathcal{G}} + \mathcal{Z}_L(I_L \otimes \mathcal{Z}\mathcal{H}_L(w))\hat{\mathcal{G}} = I_{nL} + \mathbf{\Delta},$$

concluding the proof. ■

Proof [Proof of Lemma 9] We first show that the candidate system responses $\mathcal{G}^0$ satisfies the algebraic constraints of robust optimization problem (3.7) defined in terms of the noisy data $\{\tilde{x}_{[0,T-1]}, u_{[0,T-1]}\}$.

First, note that since $\mathcal{G}^0$ is block-diagonal, we trivially have that $\mathcal{H}_1(\tilde{x})G^0(i,j) = 0$ for $i \neq j$. Thus it suffices to verify that $G_i^0 = G_i^\star(I + D_k)^{-1} \in \Gamma(\tilde{x})$, i.e., that it satisfies $\mathcal{H}_1(\tilde{x})G_i^0 = I$.

First, the relation $\mathcal{H}_1(\tilde{x}) = \mathcal{H}_1(x) + \mathcal{W}_{[0,T-L]}$ follows from Lemma 8 by examining the first block row of

$$\mathcal{H}_L(\tilde{x}) = \mathcal{H}_L(x) + \mathcal{T}_L(I)\mathcal{H}_L(w) + \mathcal{O}_L(A)\mathcal{W}_{[0,T-L]}.$$

The first block row of $\mathcal{T}_L(I)\mathcal{H}_L(w)$ is zero and the top block of $\mathcal{O}_L(A)$ is I. With this identity, right multiplying $\mathcal{H}_1(\tilde{x})$ by any $G_k^\star \in \Gamma(x)$ yields

$$\mathcal{H}_1(\tilde{x})G_k^\star = \mathcal{H}_1(x)G_k^\star + \mathcal{W}_{[0,T-L]}G_k^\star = I + D_k$$

This implies that $G_k^\star(I + D_k)^{-1} \in \Gamma(\tilde{x})$ provided the inverse exists. A sufficient condition for $(I + \mathcal{D})^{-1}$ to exist is that $\|\mathcal{D}\|_2 < 1$ – we will prove this fact, and the additional norm constraint in the robust optimization problem (3.7), next.

First, the condition $\gamma^0 = 2\varepsilon\|\mathcal{G}^\star\|_2\sqrt{L} \leq 2/3 < 1$ is satisfied by the assumption (4.1). Next, observe that

$$\begin{aligned}\|D_k\|_2 &= \left\|\mathcal{W}_{[0,T-L]}G_k^\star\right\|_2 \leq \left\|\mathcal{W}_{[0,T-L]}\right\|_2 \cdot \|G_k^\star\|_2 = \left\|\mathrm{vec}(W_{[0,T-L]})\right\|_2 \cdot \|G_k^\star\|_2 \\ &= \left\|\mathcal{T}_{T-L+1}(I)\mathcal{H}_{T-L}^\dagger(\mathcal{H}_{T-L}(w_{[0,T-L]}))\right\|_2 \cdot \|G_k^\star\|_2 \leq \|\mathcal{T}_{T-L+1}(I)\|_2\|\mathcal{H}_L^\dagger\|_2\varepsilon\|G_k^\star\|_2 \leq 1/2,\end{aligned}$$

where we exploited the identity $\mathrm{vec}(W_{[0,T-L}) = \mathcal{T}_{T-L+1}(I)w_{[0,T-L]}$ in the second equality, that the linear map $w_{[0,T-L]} \to \mathcal{H}_{T-L}(w_{[0,T-L]})$ has a left-inverse $H_{T-L}^\dagger$ with spectral norm bounded by one (this follows from the definition of the Hankel matrix, and by noting that $H_{T-L}^\dagger$ extracts unique sub-elements of $H_{T-L}(w_{[0,T-L]})$ to reconstruct $w_{[0,T-L]}$), the norm bound assumption on $H_{T-L}(w_{[0,T-L]}$, and assumption (4.1) in the final inequalities.

Thus each $\|D_k\|_2 \leq 1/2$, and consequently $\|(I + \mathcal{D})^{-1}\|_2 \leq 2$, proving that the candidate $\mathcal{G}^0 \in \Gamma(\tilde{x})$. Finally we show that $\left\|\mathcal{G}^0\right\|_2 \leq \gamma^0/\varepsilon\sqrt{L}$ is satisfied.

Observe that,

$$\|\mathcal{G}^0\|_2 = \|\mathcal{G}^\star(I + \mathcal{D})^{-1}\|_2 \leq \frac{\|\mathcal{G}^\star\|_2}{1 - \|\mathcal{D}\|_2} \leq 2\|\mathcal{G}^\star\|_2 = \frac{2\|\mathcal{G}^\star\|_2\gamma^0}{2\|\mathcal{G}^\star\|_2\varepsilon\sqrt{L}} = \frac{\gamma^0}{\varepsilon\sqrt{L}},$$

where the last inequality follows from the previously derived bound that $\|\mathcal{D}\|_2 \leq 1/2$, and the final equality from the definition of γ^0. ■

Proof [Proof of Theorem 7] From the derivation of (3.9), the cost $\hat{J}$ achieved by the system responses $\{\hat{\mathbf{\Phi}}_x, \hat{\mathbf{\Phi}}_u\} = \{\mathcal{H}_L(\tilde{x})\hat{\mathcal{G}}, \mathcal{H}_L(u)\hat{\mathcal{G}}\}$ on the true dynamics is upper-bounded via

$$\hat{J} = \left\|\begin{bmatrix}\mathcal{Q}^{1/2} & \\ & \mathcal{R}^{1/2}\end{bmatrix}\begin{bmatrix}\hat{\Phi}_x \\ \hat{\Phi}_u\end{bmatrix}(I + \Delta)^{-1}\right\|_F \leq \frac{1}{1-\hat{\gamma}}\left\|\begin{bmatrix}\mathcal{Q}^{1/2} & \\ & \mathcal{R}^{1/2}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{\Phi}}_x \\ \hat{\mathbf{\Phi}}_u\end{bmatrix}\right\|_F \tag{A.4}$$

where the RHS is the optimal value of (3.9). Letting $\alpha(T) = \|\mathcal{T}_{T-L+1}(I)\|_2 \|\mathcal{G}^\star\|_2$, it then follows by optimality of $\hat{\gamma}, \hat{\mathcal{G}}$ and by Lemma 9 that

$$\begin{aligned}
\hat{J} &\leq \frac{1}{1-\gamma^0} \left\| \begin{bmatrix} \mathcal{Q}^{1/2} & \\ & \mathcal{R}^{1/2} \end{bmatrix} \begin{bmatrix} \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(\tilde{x})) \\ \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(u)) \end{bmatrix} \mathcal{G}_0 \right\|_F \\
&\leq \frac{1}{1-\gamma^0} \left\| \begin{bmatrix} \mathcal{Q}^{1/2} & \\ & \mathcal{R}^{1/2} \end{bmatrix} \begin{bmatrix} \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(x)) \\ \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(u)) \end{bmatrix} \mathcal{G}^\star (I+\mathcal{D})^{-1} \right\|_F \\
&\quad + \frac{1}{1-\gamma^0} \left\| \mathcal{Q}^{1/2} (\mathcal{T}_L(I)\mathcal{H}_L(w) + \mathcal{O}_L(A)\mathcal{W}_{[0,T-L]}) \mathcal{G}^\star (I+\mathcal{D})^{-1} \right\|_F \\
&\leq \frac{1}{1-\gamma^0} \left\| \begin{bmatrix} \mathcal{Q}^{1/2} & \\ & \mathcal{R}^{1/2} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_x^\star \\ \mathbf{\Phi}_u^\star \end{bmatrix} (I+\mathcal{D})^{-1} \right\|_F + \frac{\|\mathcal{Q}^{1/2}\|_2 \|\mathcal{G}^\star\|_2 (\|\mathcal{T}_L(I)\|_2 \varepsilon + \|\mathcal{T}_{T-L+1}(I)\|_2 \|\mathcal{O}_L(A)\|_2 \varepsilon)}{(1-\gamma^0)(1-\alpha(T)\varepsilon)} \\
&\leq \frac{1}{(1-\gamma^0)(1-\alpha(T)\varepsilon)} \left\| \begin{bmatrix} \mathcal{Q}^{1/2} & \\ & \mathcal{R}^{1/2} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_x^\star \\ \mathbf{\Phi}_u^\star \end{bmatrix} \right\|_F + \frac{(1+\|\mathcal{O}_L(A)\|_2) \|\mathcal{Q}^{1/2}\|_2 \|\mathcal{G}^\star\|_2 \|\mathcal{T}_{T-L+1}(I)\|_2 \varepsilon}{(1-\gamma^0)(1-\alpha(T)\varepsilon)} \\
&= \frac{J^\star}{(1-\gamma^0)(1-\alpha(T)\varepsilon)} + \frac{(1+\|\mathcal{O}_L(A)\|_2) \|\mathcal{Q}^{1/2}\|_2 \|\mathcal{G}^\star\|_2 \|\mathcal{T}_{T-L+1}(I)\|_2 \varepsilon}{(1-\gamma^0)(1-\alpha(T)\varepsilon)}
\end{aligned}$$

where the second inequality follows from Lemma 8, $\|\mathcal{T}_L\|_2 \leq \|\mathcal{T}_{T-L+1}\|_2$ if $L \leq T-L+1$, which is assumed in the Theorem, and the relations

$$\mathcal{G}_0 = \mathcal{G}^\star (I+\mathcal{D})^{-1}, \quad \begin{bmatrix} \mathbf{\Phi}_x^\star \\ \mathbf{\Phi}_u^\star \end{bmatrix} = \begin{bmatrix} \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(x)) \\ \mathcal{Z}_L(I_L \otimes \mathcal{H}_L(u)) \end{bmatrix} \mathcal{G}^\star, \quad \left\|(I+\mathcal{D})^{-1}\right\|_2 \leq \frac{1}{1-\alpha(T)\varepsilon}$$

Simplifying the and rearranging the above yields

$$\frac{J - J^\star}{J^\star} \leq \frac{\gamma^0 + \alpha(T)\varepsilon}{(1-\gamma^0)(1-\alpha(T)\varepsilon)} + \frac{(1+\|\mathcal{O}_L(A)\|_2) \|\mathcal{Q}^{1/2}\|_2 \|\mathcal{G}^\star\|_2 \|\mathcal{T}_{T-L+1}(I)\|_2 \varepsilon}{(1-\gamma^0)(1-\alpha(T)\varepsilon) J^\star}$$

Recall that $\gamma^0 = 2\varepsilon \|\mathcal{G}^\star\|_2 \sqrt{L}$, and by the assumptions of the Theorem we have that $\alpha(T)\varepsilon \leq 1/2$ and $\gamma^0 \leq 2/3$; thus

$$\frac{\gamma^0 + \alpha(T)\varepsilon}{(1-\gamma^0)(1-\alpha(T)\varepsilon)} \leq 3\varepsilon(2\|\mathcal{G}^\star\|_2 \sqrt{L} + \alpha(T)).$$

Similarly, the second term can be bound as $3(1+\|\mathcal{O}_L(A)\|_2)\varepsilon \|\mathcal{Q}^{1/2}\|_2 \|\mathcal{G}^\star\|_2 \|\mathcal{T}_{T-L+1}(I)\|_2 / J^\star$ ■

Lemma 11 *Let* $\bar{w}_{[-1,T-2]} = \frac{1}{N}\sum_{i=1}^N w_{[-1,T-2]}^{(i)}$. *Then*

$$\mathbb{P}\left[\|\mathcal{H}_L(\bar{w})\|_2 \geq t\right] \leq 2nTe^{-\frac{t^2 N}{2\sigma^2 nT}}$$

for all $t \geq 0$.

Proof [Proof of Lemma 11] To simplify the proof, we bound the spectral norm of $\mathcal{H}_T(\bar{w})$, because as $\mathcal{H}_L(\bar{w})$ is a sub-matrix of $\mathcal{H}_T(\bar{w})$, it follows that $\|\mathcal{H}_L(\bar{w})\|_2 \leq \|\mathcal{H}_T(\bar{w})\|_2$.

We write $\mathcal{H}_T(\bar{w}) = \sum_k v_k B_k$, where the $v_k \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$, and the $\{B_k\}$ are a finite sequence of fixed matrices of dimension $nT \times nT$. In particular, letting $v_{[-1,T-2]} \overset{\text{dist}}{=} \frac{\sqrt{N}}{\sigma}\bar{w}_{[-1,T-2]}$, $v_{i,j} \overset{\text{iid}}{\sim}$

$\mathcal{N}(0,1)$ denote the jth component of $v(j)$, and

$$H = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathbb{R}^{T\times T},\ Z = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \in \mathbb{R}^{nT\times nT},\ H_i = \frac{\sigma}{\sqrt{N}} H \otimes e_i,$$

where $e_i \in \mathbb{R}^n$, $i = 1, \dots, n$ is the standard basis element. One can check that we then have that

$$\mathcal{H}_T(\bar{w}) \overset{\text{dist}}{=} \sum_{k=1}^{n} v_{0,k} H_k + \sum_{i=1}^{T-1}\sum_{k=1}^{n} v_{i,k} Z^i H_k. \tag{A.5}$$

By Theorem 4.1.1 of Tropp (2012), we have that

$$\mathbb{P}\left[\|\mathcal{H}_L(\bar{w})\|_2 \geq t\right] \leq 2nTe^{-\frac{t^2}{2\nu(\mathcal{H}_T(\bar{w}))}},$$

where $\nu(\mathcal{H}_T(\bar{w})) := \max\left\{\mathbb{E}\|\mathcal{H}_T^\top(\bar{w})\mathcal{H}_T(\bar{w})\|_2, E\|\mathcal{H}_T(\bar{w})\mathcal{H}_T^\top(\bar{w})\|_2\right\}$. Thus it suffices to show that $\nu(\mathcal{H}_T(\bar{w})) \leq nT\sigma^2/N$ and the result follows immediately. Using the following identities,

$$H_k^\top Z^i (Z^\top)^i H_k = \sum_{j=1}^{nT-i} e_j e_j^\top,\ H_k^\top (Z^\top)^i Z^i H_k = \sum_{j=i+1}^{nT} e_j e_j^\top,$$

$$(Z^\top)^i H_k H_k^\top Z^i = \left(\sum_{j=1}^{nT-i} e_j e_j^\top\right) \otimes e_k e_k^\top,\ Z^i H_k H_k^\top (Z^\top)^i = \left(\sum_{j=i+1}^{nT} e_j e_j^\top\right) \otimes e_k e_k^\top,$$

the decomposition (A.5), and a careful counting argument, one can check that

$$\mathbb{E}\|\mathcal{H}_T^\top(\bar{w})\mathcal{H}_T(\bar{w})\|_2 = \|\frac{\sigma^2}{N} T I_{nT}\|_2 = \frac{\sigma^2 T}{N}, \quad E\|\mathcal{H}_T(\bar{w})\mathcal{H}_T^\top(\bar{w})\|_2 = \|nTI_T\|_2 = \frac{\sigma^2 nT}{N},$$

from which the result follows. ■

Proof [Proof of Corollary 10]. Inverting the probability bound (4.2), we have that with probability at least $1-\delta$ that$\|\mathcal{H}_L(\bar{w})\|_2 \leq \sqrt{\frac{2\sigma^2 nT}{N} \log\left(\frac{2nT}{\delta}\right)}$. Thus, we have that the bounds (4.1) are satisfied under the assumptions of the Corollary, proving the result by combining the above bound with Theorem 7. ■