Two-Sided Matchings and Non-Cooperative Game Theory

COMP 4418 – Assignment 2

Due 31st Oct. 2025, 17:00
Total Marks: 100
Late Penalty: 20 marks per day

Worth: 15% of the course

Note: For ALL the questions, a complete solution must be provided. Proofs or justifying calculations are required to obtain full marks. Answers stated without adequate explanation will receive at most half credit.

Question 1 (15 marks) Consider a Student Proposing Deferred Acceptance (SPDA) algorithm on an one-one matching instance $\langle S, C \prec \rangle$ where $|S| = |C| = n$. For each of the following statements, please decide whether it is **true** or **false**. Support your answer with an example or a proof, as needed.

a) For any instance, at least one student always makes multiple proposals. **(5 marks)**

b) There is an instance where the number of proposals made is $\frac{n(n+1)}{2}$. **(5 marks)** **Hint.** Recall that $\frac{n(n+1)}{2} = \sum_{i=1}^{n} i$.

c) For any fixed college $c \in C$, the optimal manipulation for c will always match c to its optimal partner under any stable matching. **(5 marks)**

Note: For this and all other assignment questions, to show that a statement is true for *one* instance, it is enough to give one such example. To show that a statement is true for *all* one-one matching instances, a proof is required. To show that a statement is not true for *all* instances, it is sufficient to give one instance where it does not hold.

Question 2 (10 marks) Consider the following one-one matching instance with $n = 5$.

$$
\begin{array}{ll}
s_1 : c_1 \succ c_2 \succ c_3 \succ c_4 \succ c_5 & \qquad c_1 : s_5 \succ s_1 \succ s_3 \succ s_4 \succ s_2 \\
s_2 : c_3 \succ c_4 \succ c_5 \succ c_1 \succ c_2 & \qquad c_2 : s_1 \succ s_5 \succ s_2 \succ s_3 \succ s_4 \\
s_3 : c_4 \succ c_5 \succ c_3 \succ c_2 \succ c_1 & \qquad c_3 : s_3 \succ s_4 \succ s_5 \succ s_2 \succ s_1 \\
s_4 : c_5 \succ c_3 \succ c_4 \succ c_1 \succ c_2 & \qquad c_4 : s_4 \succ s_5 \succ s_3 \succ s_1 \succ s_2 \\
s_5 : c_2 \succ c_1 \succ c_4 \succ c_5 \succ c_3 & \qquad c_5 : s_1 \succ s_5 \succ s_2 \succ s_3 \succ s_4
\end{array}
$$

Identify all colleges who can manipulate under the SPDA and explain why. Give an inconspicuous optimal manipulation for each, if any. Analogously, identify all students who can manipulate under the CPDA and give an inconspicuous optimal manipulation for each, if any.

Question 3 (10 marks) Give a polynomial time algorithm to check if a given many-to-one matching instance has multiple stable matchings. Prove its correctness and running time.

Note. You can take any algorithm discussed in class/slides as a known subroutine that you do not need to redefine.

Question 4 (20 marks) Consider the following normal-form game, which is parameterized by a value $\alpha \in \mathbb{R}$.

	x	y
a	-2, 2	α, -1
b	3, -3	-3, 3

a) For which value of α is the game zero-sum? (**3 marks**)

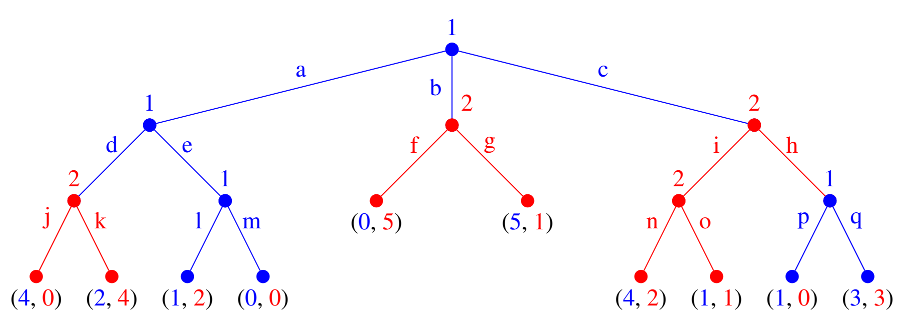
b) For which values of α is the outcome $(\alpha, -1)$ Pareto-optimal? (**3 marks**)

c) For which values of α can the game be solved by iterated strict dominance? (**4 marks**)

d) For which value of α is it the maximin strategy of the row player to play a with probability $\frac{1}{2}$? (**5 marks**)

e) For which value of α will the column player play y with probability $\frac{3}{4}$ in a Nash equilibrium? (**5 marks**)

Question 5 (10 marks) Consider the following extensive-form game.



a) Compute the subgame-perfect Nash equilibrium. Specify the strategy of each player in each subtree of the game.**(5 marks)**

b) Is there a pure Nash equilibrium where player 2 has a utility of 4? Explain your answer! **(5 marks)**

Question 6 (20 marks) A picture is sold in an auction, which is attended by three potential buyers: Aaron (A), Ben (B), and Charles (C). Each of the buyers has a personal valuation for the picture: Aaron values the picture at $v_A = 90$, Ben at $v_B = 60$, and Charles at $v_C = 40$. The rules of the auction are simple: each agent simultaneously submits an integer bid $b_x \in \{0,1,2,\dots,100\}$, the agent submitting the highest bid wins the picture and pays his bid b_x. The other agents leave empty-handed and do not pay anything. Ties are broken lexicographically, so Aaron wins the picture if he submits the highest bid even if another agent submits the same bid; further, if Ben and Charles tie for the highest bid, Ben wins. We assume that the utility of each agent is quasi-linear: the utility of the agent who wins the picture is his valuation v_x minus his payment b_x, and the utility of the other agents is 0.

Example: Suppose Aaron bids 40, Ben 50, and Charles 30. Ben wins the picture and pays his bid of 50. The utility of Ben is thus $v_B - b_B = 60 - 50 = 10$. The utility of Aaron and Charles is 0.

a) Treat the above situation as a normal-form game, where the action space of each agent is his bid $b_x \in \{0,\dots,100\}$. Identify a pure Nash equilibrium (NE) of this game and justify why it is a NE, or explain why no pure NE exists. **(10 marks)**

b) We modify the rules of the auction by introducing a participation fee: each agent who reports a non-zero bid has to pay a fee of $c = 1$. Hence, the utility of the agent who gets the picture is his valuation v_x minus his bid b_x minus the participation cost c. The utility of every other agent is $-c$ if they report a non-zero bid, and 0 if they bid 0.

   Example: Suppose Aaron bids 40, Ben 50, and Charles 0. Ben wins the picture, pays his bid of 50 and a participation cost of 1. The utility of Ben is thus $v_B - b_B - c = 60 - 50 - 1 = 9$. The utility of Aaron is -1 since he reports a non-zero bid and thus pays the participation fee, whereas the utility of Charles is 0.

   Identify a pure Nash equilibrium (NE) of this modified normal-form game and justify why it is a NE, or explain why no pure NE exists. **(10 marks)**

Question 7 (15 marks) Prove the following statements.

a) Let $G_1 = (\{1,2\},(A_i^1)_{i\in\{1,2\}},(u_i^1)_{i\in\{1,2\}})$, $G_2 = (\{1,2\},(A_i^2)_{i\in\{1,2\}},(u_i^2)_{i\in\{1,2\}})$, and $G_3 = (\{1,2\},(A_i^3)_{i\in\{1,2\}},(u_i^3)_{i\in\{1,2\}})$ denote three two-player normal-form games such that $A_i^1 = A_i^2 = A_i^3$ for $i \in \{1,2\}$ and $u_i^3(a) = \frac{1}{2}(u_i^1(a) + u_i^2(a))$ for both players $i \in \{1,2\}$ and all action profiles $a \in A$. Show that, if a strategy profile s is a Nash equilibrium for G_1 and G_2, then it is a Nash equilibrium for G_3. **(5 marks)**

b) Let $G = (\{1,2\},(A_i)_{i\in\{1,2\}},(u_i)_{i\in\{1,2\}})$ denote a two-player zero-sum game such that $A_1 = A_2 = \{a_1,\dots,a_k\}$ for $k > 1$ and $u_1(a_i,a_j) = u_2(a_j,a_i)$ for all $a_i,a_j \in A$. Show that the value of G (i.e., the security level of player 1) is 0. **(10 marks)**
   Hint: Consider any mixed strategy of player 2. Show that if player 1 uses the same mixed strategy, his expected utility is 0. Conclude that player 1 can always secure 0.

SUBMISSION

- Submit your solution directly via Moodle in the assessment hub at the end of the Moodle page. Please make sure that your manuscript contains your name and zID.

- Your answers are to be submitted in a single PDF file. We will not accept any other file formats. Please make sure that your solutions are clearly readable.
- The deadline for this submission is 31st October 2025, 17:00.

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