

UNSW Sydney  
Department of Statistics  
Term 2, 2019  
MATH5905 - Statistical Inference  
Assignment 1

For this assignment, you are allowed to work in groups of 1 or 2 or 3 or 4 or 5 (max).

You must write all members of your group with their student numbers on each page at the top right hand corner of the assignment.

You must submit one pdf file via Moodle Turnitin. The University requires each student to submit the assignment in **Moodle Turnitin** even though this could be a group assignment.

This assignment must be submitted no later than noon on Friday, 28th June 2019 on Moodle. Your submission on **Moodle Turnitin** will be your declaration that the assignment is your own work, except where acknowledged and that you understood the University Rules on plagiarism.

**Maximal number of pages: 8**

1. Consider a decision problem with parameter space  $\Theta = \{\theta_1, \theta_2\}$  and a set of non randomized decisions  $D = \{d_i, 1 \leq i \leq 6\}$  with risk points  $\{R(\theta_1, d_i), R(\theta_2, d_i)\}$  as follows:

$i$	1	2	3	4	5	6
$R(\theta_1, d_i)$	0	1	3	6	7	6
$R(\theta_2, d_i)$	7	5	5	2	5	8

- Find the minimax rule(s) amongst the **nonrandomized** rules in  $D$ ;
- Plot the risk set of all **randomized** rules  $\mathcal{D}$  generated by the set of rules in  $D$ .
- Find the risk point of the minimax rule in  $\mathcal{D}$  and determine its minimax risk.
- Define the minimax rule in the set  $\mathcal{D}$  in terms of rules in  $D$ .
- For which prior on  $\{\theta_1, \theta_2\}$  is the minimax rule in the set  $\mathcal{D}$  also a Bayes rule?
- Determine the Bayes rule and the Bayes risk for the prior  $\left(\frac{5}{6}, \frac{1}{6}\right)$  on  $\{\theta_1, \theta_2\}$ .
- For a small positive  $\epsilon = \frac{1}{2}$ , illustrate on the risk set the risk points of all rules which are  $\epsilon$ -minimax.

2. In a Bayesian estimation problem, we sample  $n$  i.i.d. observations  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  from a population with conditional distribution of each single observation being the geometric distribution

$$f_{X_1|\Theta}(x|\theta) = \theta^x(1 - \theta), x = 0, 1, 2, \dots; 0 < \theta < 1.$$

The parameter  $\theta$  is considered as random in the interval  $\Theta = (0, 1)$ .

i) If the prior on  $\Theta$  is given by  $\tau(\theta) = 3\theta^2, 0 < \theta < 1$ , show that the posterior distribution  $h(\theta|\mathbf{X} = (x_1, x_2, \dots, x_n))$  is in the Beta family. Hence determine the Bayes estimator of  $\theta$  with respect to quadratic loss.

**Hint:** For  $\alpha > 0$  and  $\beta > 0$  the beta function  $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$  satisfies  $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$  where  $\Gamma(\alpha) = \int_0^\infty \exp(-x)x^{\alpha-1}dx$ . A Beta  $(\alpha, \beta)$  distributed random variable  $X$  has a density  $f(x) = \frac{1}{B(\alpha, \beta)}x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1$ , with  $E(X) = \alpha/(\alpha + \beta)$ .

ii) Five observations from this distribution were observed: 2, 4, 7, 1, 3. Using zero-one loss, what is your decision when testing  $H_0 : \theta \leq 0.75$  against  $H_1 : \theta > 0.75$ . (You may use the `integrate` function in R or another numerical integration routine from your favourite programming package to answer the question.)

3. Let  $X_1, X_2, \dots, X_n$  be i.i.d. uniform in  $(0, \theta)$  and let the prior on  $\theta$  be the Pareto prior given by  $\tau(\theta) = \beta\alpha^\beta\theta^{-(\beta+1)}, \theta > \alpha$ . (Here  $\alpha > 0$  and  $\beta > 0$  are assumed to be known constants). Show that the Bayes estimator with respect to quadratic loss is given by  $\hat{\theta}_{Bayes} = \max(\alpha, x_{(n)})\frac{n+\beta}{n+\beta-1}$ . Justify all steps in the derivation.

4. At a critical stage in the development of a new aeroplane in the UK, a decision must be taken to continue or to abandon the project. The financial viability of the project can be measured by a parameter  $\theta \in (0, 1)$ , the project being profitable if  $\theta > \frac{1}{2}$ . Data  $x$  provide information about  $\theta$ . If  $\theta < 1/2$ , the cost to the taxpayer of continuing the project is  $(\frac{1}{2} - \theta)$  (in units of \$ billion) whereas if  $\theta > 1/2$  it is zero (since the project will be privatised if profitable). If  $\theta > \frac{1}{2}$  the cost of abandoning the project is  $(\theta - \frac{1}{2})$  (due to contractual arrangements for purchasing the aeroplane from the French), whereas if  $\theta < \frac{1}{2}$  it is zero. Two actions are on the table:  $a_0$  : continue the project and  $a_1$  : abandon it.

Derive the Bayesian decision rule in terms of the posterior mean of  $\theta$  given  $x$ . The Minister of Aviation has prior density  $6\theta(1 - \theta)$  for  $\theta$ . The prime minister has prior density  $4\theta^3$ . The prototype aeroplane is subjected to trials, each independently having probability  $\theta$  of success, and the data  $x$  consists of the total number of trials required for the first successful result to be obtained (i.e., one realization from a geometric distribution). For which values of  $x$  will there be most serious ministerial disagreement?