Haotian Chu z5188746
Daifei Zhang z5185668
Shuxiang Zou z5187969
Xudong Liu z5193103
Pan Luo z5192086

MATH 5905 Statistical Inference

19T2 Assignment2

Group work

Submitted: Aug 2019

Haotian Chu z5188746
Daifei Zhang z5185668
Shuxiang Zou z5187969
Xudong Liu z5193103
Pan Luo z5192086

1) a) $\because f(x,\theta) = \begin{cases} \frac{1}{\theta} & \text{if } 0<x<\theta \\ 0 & \text{else.} \end{cases}$

$\therefore f(x,\theta) = \frac{1}{\theta} \cdot I_{(0,\theta)}(x)$

$\therefore L(x,\theta) = \frac{1}{\theta^n} \cdot I_{(0,\theta)}(x_1) \cdot I_{(0,\theta)}(x_2) \cdots I_{(0,\theta)}(x_n)$

$= \frac{1}{\theta^n} \cdot I_{(X_{(n)},\infty)}(\theta)$

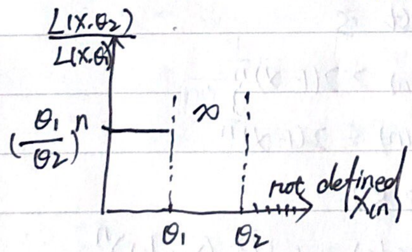
Assume that $0<\theta_1<\theta_2$.

$\therefore L(X,\theta_2) = \frac{1}{\theta_2^n} \cdot I_{(X_{(n)},\infty)}(\theta_2)$

$L(X,\theta_1) = \frac{1}{\theta_1^n} \cdot I_{(X_{(n)},\infty)}(\theta_1)$

$\therefore \frac{L(X,\theta_2)}{L(X,\theta_1)} = \frac{\theta_1^n}{\theta_2^n} \cdot \frac{I_{(X_{(n)},\infty)}(\theta_2)}{I_{(X_{(n)},\infty)}(\theta_1)}$

$\therefore$ the graph of $\frac{L(X,\theta_2)}{L(X,\theta_1)}$ is



$\therefore$ for $\theta>0$, the family has a monotone likelihood ration in $X_{(n)}$.

b) ~~A~~ According to $H_0: \theta \le 2$ versus $H_1: \theta > 2$, and density function of X in a family has MLR. $\therefore$ the UMP α-test is

Haotian Chu z5188746
Daifei Zhang z5185668
Shuxiang Zou z5187969
Xudong Liu z5193103
Pan Luo z5192086

$$\varphi^*(X) = \begin{cases} 1, & X_{(n)} > k \\ 0, & X_{(n)} \le k \end{cases}, \quad E_{\theta_0}(\varphi^*) = \alpha$$

$$\therefore E_{\theta_0}(\varphi^*) = 1 \cdot P_{\theta_0}(X_{(n)} > k)$$
$$= 1 - P_{\theta_0}(X_{(n)} \le k)$$
$$= 1 - P_{\theta_0}(X_1 \le k) \cdot P_{\theta_0}(X_2 \le k) \cdots P_{\theta_0}(X_n \le k)$$
$$= 1 - (P_{\theta_0}(X_1 \le k))^n$$

$$\because P_{\theta_0}(X_1 \le k) = \int_0^k p_{\theta_0}(x)\,dx$$
$$= \int_0^k \frac{1}{\theta_0}\,dx = \frac{x}{\theta_0}\Big|_0^k = \frac{k}{\theta_0}$$

$$\therefore E_{\theta_0}(\varphi^*) = 1 - \left(\frac{k}{\theta_0}\right)^n = \alpha$$
$$k = \theta_0 \cdot (1-\alpha)^{\frac{1}{n}} = 2 \cdot (1-\alpha)^{\frac{1}{n}}$$

$\therefore$ Hence the UMP α-test is

$$\varphi^* = \begin{cases} 1 & \text{if } X_{(n)} > 2(1-\alpha)^{\frac{1}{n}}, \\ 0 & \text{if } X_{(n)} \le 2(1-\alpha)^{\frac{1}{n}} \end{cases}$$

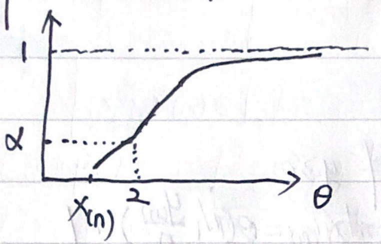
c) $\because E_\theta(\varphi^*) = 1 \cdot P_\theta \cdot (X_{(n)} > k)$
$$= 1 - P_\theta \cdot (X_{(n)} \le k) = 1 - (P_\theta(X_1 \le k))^n$$

$$\because P_\theta(X_1 \le k) = \int_0^k p_\theta(x)\,dx = \int_0^k \frac{1}{\theta}\,dx = \frac{x}{\theta}\Big|_0^k = \frac{x}{\theta}$$

$$\therefore E_\theta(\varphi^*) = 1 - \left(\frac{k}{\theta}\right)^n = 1 - \left(\frac{2\cdot(1-\alpha)^{\frac{1}{n}}}{\theta}\right)^n = 1 - \frac{2^n}{\theta^n}\cdot(1-\alpha)$$

Haotian Chu z5188746
Daifei Zhang z5185668
Shuxiang Zou z5187969
Xudong Liu z5193103
Pan Luo z5192086

$\because$ if $\theta = 2$, $E_\theta(\varphi^*) = \alpha$. The graph of $E_\theta(\varphi^*)$ is



d) $\because$ From part c, cdf of $X_{(n)}$ should be

$F_{X_{(n)}}(x) = \left(\frac{x}{\theta}\right)^n$ if $0 \le x < \theta$.

$$F_{Y_{(n)}}(y) = P(Y_{(n)} \le y) = P\left(n\cdot\left(1 - \frac{X_{(n)}}{\theta}\right) \le y\right)$$
$$= P\left(1 - \frac{X_{(n)}}{\theta} < \frac{y}{n}\right)$$
$$= P\left(\frac{X_{(n)}}{\theta} > 1 - \frac{y}{n}\right)$$
$$= P\left(X_{(n)} > \theta\left(1 - \frac{y}{n}\right)\right) = 1 - P\left(X_{(n)} \le \theta\cdot\left(1-\frac{y}{n}\right)\right)$$
$$= 1 - P\left(X_1 \le \theta\cdot\left(1-\frac{y}{n}\right)\right)\cdot P\left(X_2 \le \theta\left(1-\frac{y}{n}\right)\right)\cdots P\left(X_n \le \theta\left(1-\frac{y}{n}\right)\right)$$
$$= 1 - \left[P\left(X_1 \le \theta\cdot\left(1-\frac{y}{n}\right)\right)\right]^n$$
$$= 1 - \left[F_{X_{(n)}}\left(\theta\cdot\left(1-\frac{y}{n}\right)\right)\right]^n$$

$$\therefore F_{Y_{(n)}}\, y = 1 - \frac{\left(\theta\cdot\left(1-\frac{y}{n}\right)\right)^n}{\theta^n} = 1 - \left(1 - \frac{y}{n}\right)^n$$

$$\because \lim_{n\to\infty}\left(1-\frac{y}{n}\right)^n = \lim_{n\to\infty} e^{n\cdot\log\left(1-\frac{y}{n}\right)}$$

consider $Z(y_{(n)}) = \log\left(1 - \frac{y}{n}\right)$

$$Z(0) = 0 \qquad Z(y_{(n)})' = \frac{1}{1-\frac{y_{(n)}}{n}}\cdot\left(-\frac{1}{n}\right) = \frac{-1}{n - y_{(n)}}$$
$$Z(0)' = \frac{-1}{n} \qquad Z(0)'' = \frac{1}{n^2}$$

Haotian Chu z5188746
Daifei Zhang z5185668
Shuxiang Zou z5187969
Xudong Liu z5193103
Pan Luo z5192086

$$\lim_{n\to\infty} \log(1-\frac{y}{n}) = Z_0 + Z'(0)\cdot y + \frac{Z''(0)}{2!}(y)^2 + \cdots$$
$$= -\frac{y}{n}$$

$\therefore \lim_{n\to\infty} e^{n\log(1-\frac{y}{n})} = e^{-y}$

$\therefore \lim_{n\to\infty} F_{y_{(n)}}(y) = 1-e^{-y}$, if $y \geq 0$.

$\therefore Y_{(n)}$ is exponential distribution with mean 1

$$E(X_{(n)}) = E(\lim_{n\to\infty} \theta(1-\frac{y_{(n)}}{n})) \qquad y_{(n)} = n(1-\frac{X_{(n)}}{\theta})$$
$$X_{(n)} = \theta(1-\frac{y_{(n)}}{n})$$
$$= \theta \cdot E(\lim_{n\to\infty}(1-\frac{y_{(n)}}{n}))$$
$$= \theta - \theta \cdot E(\lim_{n\to\infty}\frac{y_{(n)}}{n})$$
$$= \theta$$

$\therefore X_{(n)}$ is a consistent estimator of θ.

2) a) the likelihood function for one observation is

$$L(X,\theta) = f(x,\theta) = \frac{2}{\theta}x e^{-\frac{x^2}{\theta}}$$
$$\log L(X,\theta) = \log(\frac{2}{\theta}x e^{-\frac{x^2}{\theta}}) = \log 2 - \log\theta + \log X + (-\frac{X^2}{\theta})$$
$$\frac{\partial(\log L(X,\theta))}{\partial\theta} = -\frac{1}{\theta} + \frac{X^2}{\theta^2}$$
$$\frac{\partial^2 \log L(X,\theta)}{\partial\theta^2} = \frac{1}{\theta^2} - \frac{2X^2}{\theta^3}$$
$$\therefore I_1(\theta) = -E(\frac{\partial^2 \log(L(X,\theta))}{\partial\theta^2}) = -\frac{1}{\theta^2} + \frac{2}{\theta^3}E(X^2)$$

We set $Y = X^2$, so $X = \sqrt{y}$. We could use the transformation technique.

$$f_Y(y) = f_X(X(y))\left|\frac{dx}{dy}\right| \qquad \frac{dx}{dy} = \frac{\sqrt{y}}{dy} = \frac{1}{2\sqrt{y}}$$
$$= f_X(\sqrt{y})\left|\frac{dx}{dy}\right|$$
$$= \frac{2\sqrt{y}}{\theta}e^{-y/\theta}\cdot\frac{1}{2\sqrt{y}} = \frac{1}{\theta}e^{-y/\theta}$$

$\therefore y$ is the density function of Gamma$(x|1,\theta)$ distribution.

$\therefore E(y) = E(X^2) = \theta$

$\therefore I_1(\theta) = -\frac{1}{\theta^2} + \frac{2\theta}{\theta^3} = \frac{1}{\theta^2}$

$\therefore I_n(\theta) = n\cdot I_1(\theta) = \frac{n}{\theta^2}$

b) the likelihood function for the sample

$$L(X,\theta) = \prod_{i=1}^{n} \frac{2X_i}{\theta}e^{-\frac{X_i^2}{\theta}} = (\frac{2}{\theta})^n \prod_{i=1}^{n} X_i e^{-\frac{1}{\theta}\sum_{i=1}^{n}X_i^2}$$
$$\log(L(X,\theta)) = \log((\frac{2}{\theta})^n \prod_{i=1}^{n} X_i e^{-\frac{1}{\theta}\sum_{i=1}^{n}X_i^2})$$
$$= \log 2^n - \log\theta^n + \sum_{i=1}^{n}\log X_i + (-\frac{1}{\theta}\sum_{i=1}^{n}X_i^2)$$
$$\frac{\partial \log(L(X,\theta))}{\partial\theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n}X_i^2}{\theta^2}$$
$$\frac{\partial \log(L(X,\theta))}{\partial\theta} = 0 \qquad -\frac{n}{\theta} + \frac{\sum_{i=1}^{n}X_i^2}{\theta^2} = 0 \qquad \theta = \frac{\sum_{i=1}^{n}X_i^2}{n}$$

$\therefore$ MLE of θ, $\hat{\theta} = \frac{\sum_{i=1}^{n}X_i^2}{n}$

Haotian Chu z5188746
Daifei Zhang z5185668
Shuxiang Zou z5187969
Xudong Liu z5193103
Pan Luo z5192086

$$E(\hat{\theta}) = E\left(\frac{\sum_{i=1}^{n} x_i^2}{n}\right) = \frac{\sum_{i=1}^{n} E(x_i^2)}{n}$$

According to part a. $E(x_i^2) = \theta$. $V(x_i^2) = \theta^2$

$\therefore E(\hat{\theta}) = \frac{n \cdot \theta}{n} = \theta$

$\therefore$ MLE of θ is unbiased.

$$V[\hat{\theta}] = V\left(\frac{\sum_{i=1}^{n} x_i^2}{n}\right) = \frac{1}{n^2} \cdot \sum_{i=1}^{n} V(x_i^2) = \frac{n\theta^2}{n^2} = \frac{\theta^2}{n}$$

Haotian Chu z5188746
Daifei Zhang z5185668
Shuxiang Zou z5187969
Xudong Liu z5193103
Pan Luo z5192086

Cramer-Rao bound is $\dfrac{\frac{\partial^2 \tau(\theta)}{\partial \theta^2}}{I_n(\theta)} = \dfrac{1}{n \cdot \frac{1}{\theta^2}} = \dfrac{\theta^2}{n}$

$\therefore$ MLE of $\hat{\theta}$ attains the Cramer-Rao Bound.

c> the asymptotic distribution of $\hat{\theta}$ is

$$\sqrt{n}(\hat{\theta} - \theta) \to N\left(0, \frac{1}{I_1(\theta)}\right)$$
$$(\hat{\theta} - \theta) \to \frac{1}{\sqrt{n}} N\left(0, \frac{1}{\frac{1}{\theta^2}}\right)$$

$$(\hat{\theta} - \theta) \to N\left(0, \frac{\theta^2}{n}\right)$$
$$\hat{\theta} \to N\left(\hat{\theta}, \frac{\theta^2}{n}\right)$$

d> $f(x,\theta)$ is one-parameter exponential family

$f(x,\theta) = \frac{2}{\theta} x e^{-\frac{x^2}{\theta}}$

$a(\theta) = \frac{2}{\theta}$, $b(x) = x$, $e^{c(\theta)} = e^{-\frac{1}{\theta}}$, $e^{d(x)} = e^{x^2}$

$\therefore d(x) = x^2$

this family has MLR is $T(x) = \sum_{i=1}^{n} x_i^2$, because $c(\theta)$ is increasing in θ. $T(x) = \sum_{i=1}^{n} x_i^2$ is minimal sufficient.

e> $H_0: \theta \le \theta_0$. vs $H_1: \theta > \theta_0$

According to 6.6.2, $T(x) = \sum_{i=1}^{n} x_i^2$ is a sufficent statistic for θ and the family is an MLR family. the UMP α-test is

$$\varphi^*(x) \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i^2 \ge k, \\ 0 & \text{otherwise.} \end{cases}$$

Haotian Chu z5188746
Daifei Zhang z5185668
Shuxiang Zou z5187969
Xudong Liu z5193103
Pan Luo z5192086
Haotian Chu z5188746
Daifei Zhang z5185668
Shuxiang Zou z5187969
Xudong Liu z5193103
Pan Luo z5192086

∵ $E_{\theta_0}(\varphi^*) = \alpha$

∴ $E_{\theta_0}(\varphi^*) = 1 \cdot P\left(\sum_{i=1}^{n} x_i^2 \geq k\right)$

$= \alpha$

$P_{\theta_0}\left(\sum_{i=1}^{n} x_i^2 \geq k\right) = P_{\theta_0}\left(\frac{\sum_{i=1}^{n} x_i^2}{n} \geq \frac{k}{n}\right)$

∵ $\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2}{n} \approx N\left(\theta, \frac{\theta^2}{n}\right)$ ∴ $\frac{\frac{\sum_{i=1}^{n} x_i^2}{n} - \theta}{\frac{\theta}{\sqrt{n}}} \approx N(0,1)$

∴ $P_{\theta_0}\left(\frac{\frac{\sum_{i=1}^{n} x_i^2}{n} - \theta}{\theta/\sqrt{n}} \geq \frac{\frac{k}{n} - \theta}{\theta/\sqrt{n}}\right) = \alpha$

$1 - P_{\theta_0}\left(\frac{\frac{\sum_{i=1}^{n} x_i^2}{n} - \theta}{\theta/\sqrt{n}} < \frac{\frac{k}{n} - \theta}{\theta/\sqrt{n}}\right) = \alpha$

$P_{\theta_0}\left(\frac{\frac{\sum_{i=1}^{n} x_i^2}{n} - \theta}{\theta/\sqrt{n}} < \frac{\frac{k}{n} - \theta}{\theta/\sqrt{n}}\right) = 1 - \alpha$

∴ $\Phi = \left[\frac{\frac{\sum_{i=1}^{n} x_i^2}{n} - \theta_0}{\theta/\sqrt{n}}\right] = 1 - \alpha \to \left(\frac{\frac{\sum_{i=1}^{n} x_i^2}{n} - \theta_0}{\theta/\sqrt{n}}\right) = z_a$

$z_a = \frac{\frac{k}{n} - \theta_0}{\theta/\sqrt{n}} \Rightarrow k = n\theta_0 + \sqrt{n}\, z_a \theta_0$

UMP α-size test is

$$\varphi^*(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i^2 \geq n\theta_0 + \sqrt{n}\, z_a \theta_0 \\ 0 & \text{otherwise.} \end{cases}$$

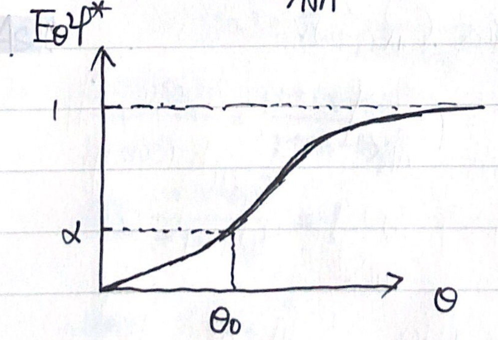
f) $E_\theta \varphi^* = 1 \cdot P_\theta\left(\sum_{i=1}^{n} x_i^2 \geq n\theta_0 + \sqrt{n}\, z_a \theta_0\right)$

∴ $= P_\theta\left(\frac{\frac{\sum_{i=1}^{n} x_i^2}{n} - \theta}{\theta/\sqrt{n}} > \frac{\theta_0 + \frac{z_a\theta_0}{\sqrt{n}} - \theta}{\theta/\sqrt{n}}\right)$

$P_\theta\left(\frac{\frac{\sum_{i=1}^{n} x_i^2}{n} - \theta}{\theta/\sqrt{n}} > \frac{\theta_0 + \frac{z_a\theta_0}{\sqrt{n}} - \theta}{\theta/\sqrt{n}}\right)$

$= 1 - P_\theta\left(\frac{\frac{\sum_{i=1}^{n} x_i^2}{n} - \theta}{\theta/\sqrt{n}} \leq \frac{\theta_0 + \frac{z_a\theta_0}{\sqrt{n}} - \theta}{\theta/\sqrt{n}}\right)$

$= 1 - \Phi\left(\frac{\theta_0 + \frac{z_a\theta_0}{\sqrt{n}} - \theta}{\theta/\sqrt{n}}\right)$



Haotian Chu z5188746
Daifei Zhang z5185668
Shuxiang Zou z5187969
Xudong Liu z5193103
Pan Luo z5192086

3.a) $\because$ X from Cauchy distribution.

$\therefore$ density of x is

$$f(x|\theta) = \frac{1}{\pi}\left(\frac{1}{1+(x-\theta)^2}\right)$$

$\because$ X is one observation

$$\therefore\ L(x|\theta) = \frac{1}{\pi}\cdot\left(\frac{1}{1+(x-\theta)^2}\right)$$

we assume $0<\theta_1<\theta_2$

$$\therefore\ L(x|\theta_1) = \frac{1}{\pi}\left(\frac{1}{1+(x-\theta_1)^2}\right)$$

$$L(x|\theta_2) = \frac{1}{\pi}\cdot\left(\frac{1}{1+(x-\theta_2)^2}\right)$$

$$\therefore\ \frac{L(x|\theta_2)}{L(x|\theta_1)} = \frac{1+(x-\theta_1)^2}{1+(x-\theta_2)^2}$$

$$\lim_{x\to\infty}\frac{1+(x-\theta_1)^2}{1+(x-\theta_2)^2} = 1$$

when $x=\theta_2$

$$\frac{1+(x-\theta_1)^2}{1+(x-\theta_2)^2} = \frac{1+(\theta_2-\theta_1)^2}{1} > 1$$

But when $x=\theta_1$

$$\frac{1+(x-\theta_1)^2}{1+(x-\theta_2)^2} = \frac{1}{1+(\theta_1-\theta_2)^2} < 1$$

$\therefore$ the likelihood ratio is not monotone as increase of x.

b) $H_0:\theta=0$ versus $H_1:\theta=1$

According to Lemma 6.1

for every $\alpha\in(0,1)$, there exists a constant C and test.

$$\varphi^* \begin{cases} 1 & \text{if } x\in\left\{x \mid \frac{f(x,\theta_1)}{f(x,\theta_0)} > C\right\} \\ 0 & \text{otherwise.} \end{cases}$$

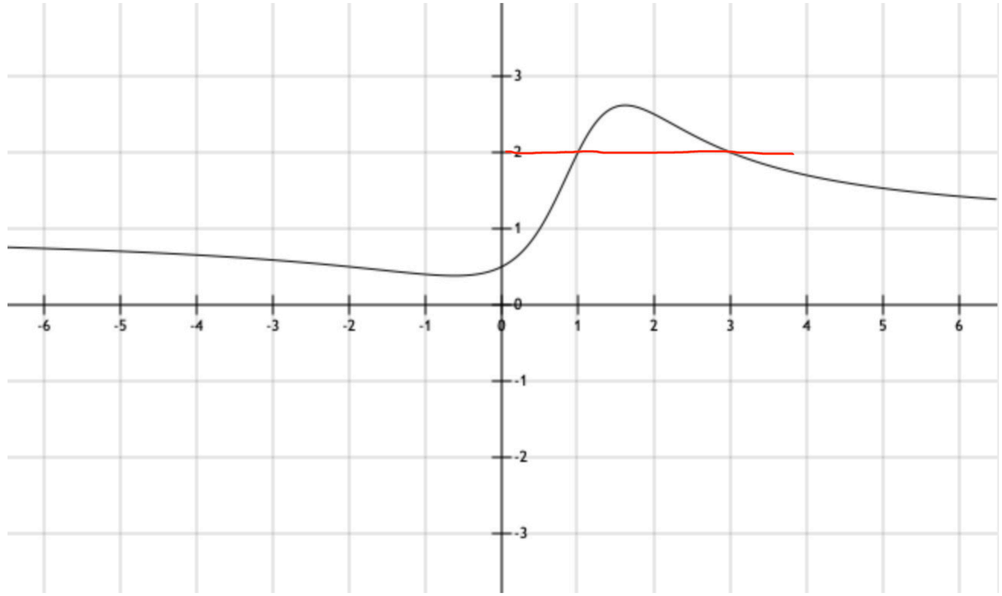
with $E_{\theta_0}\varphi^* = \alpha$

$$\because\ \frac{f(x,\theta_1)}{f(x,\theta_0)} = \frac{1+x^2}{1+(x-1)^2}$$

we assume $y=\frac{1+x^2}{1+(x-1)^2}$, and the graph of y is

Haotian Chu z5188746
Daifei Zhang z5185668
Shuxiang Zou z5187969
Xudong Liu z5193103
Pan Luo z5192086

Haotian Chu z5188746
Daifei Zhang z5185668
Shuxiang Zou z5187969
Xudong Liu z5193103
Pan Luo z5192086



So the most powerful of its size for testing $H_0: \theta=0$ vs $H_1: \theta=1$

is $\varphi^* = \begin{cases} 1 & \text{if } 1<x<3 \\ 0 & \text{otherwise.} \end{cases}$

type I error: $P(1<x<3 \mid \theta=0) = \int_1^3 \frac{1}{\pi} \cdot \frac{1}{1+x^2} dx = \frac{1}{\pi} \arctan x \Big|_1^3$

$= 0.148$

type II error: $P(x<1 \text{ or } x>3 \mid \theta=1) = 1 - P(1<x<3 \mid \theta=1)$

$= 1 - \int_1^3 \frac{1}{\pi} \frac{1}{1+(x-1)^2} dx$

$= 1 - \frac{1}{\pi} \arctan(x-1) \Big|_1^3$

$= 1 - 0.352 = 0.648$

Haotian Chu z5188746
Daifei Zhang z5185668
Shuxiang Zou z5187969
Xudong Liu z5193103
Pan Luo z5192086

c) Cause the cauchy distribution doesn't have MLR, the test in part b) is not UMP. According to 6.6. In general, for such type of family doesn't have UMP α-size text.

d) Assume $y = |x|$.

$$\therefore f(|x|, \theta) = \begin{cases} \frac{\theta}{\pi} \cdot \frac{1}{\theta^2 + x^2} & x \geq 0 \\ \frac{\theta}{\pi} \cdot \frac{1}{\theta^2 + (-x)^2}, & x < 0 \end{cases}$$

$$\therefore f(y, \theta) = \begin{cases} \frac{\theta}{\pi} \cdot \frac{1}{\theta^2 + y^2}, & y \geq 0 \\ \frac{\theta}{\pi} \cdot \frac{1}{\theta^2 + y^2}. & y < 0 \end{cases}$$

$$f(y, \theta) = \frac{\theta}{\pi} \cdot \frac{1}{\theta^2 + y^2}$$

we assume that $0 < \theta_1 < \theta_2$.

$$\therefore \text{Likelihood ratio} = \frac{f(y|\theta_2)}{f(y|\theta_1)} = \frac{\theta_2 \cdot (\theta_1^2 + y^2)}{\theta_1 \cdot (\theta_2^2 + y^2)}$$

$$\frac{\partial \left(\frac{f(y|\theta_2)}{f(y|\theta_1)} \right)}{\partial y} = \frac{\partial \left(\frac{\theta_2 \cdot (\theta_1^2 + y^2)}{\theta_1 \cdot (\theta_2^2 + y^2)} \right)}{\partial y} = \frac{2y \cdot (\theta_2^2 + y^2) - (\theta_1^2 + y^2) \cdot 2y}{(\theta_2^2 + y^2)^2} \cdot \frac{\theta_2}{\theta_1}$$

$$= \frac{2y(\theta_2^2 - \theta_1^2)}{(\theta_2^2 + y^2)^2} \cdot \frac{\theta_2}{\theta_1}$$

$\because \frac{\theta_2}{\theta_1} > 0$, and for $\forall\, \theta_1 < \theta_2$, exists $\frac{2y(\theta_2^2 - \theta_1^2)}{(\theta_2^2 + y^2)^2} > 0$ $\therefore$ Likelihood ratio is increasing as the increase of y.

Haotian Chu z5188746
Daifei Zhang z5185668
Shuxiang Zou z5187969
Xudong Liu z5193103
Pan Luo z5192086

$\therefore$ distribution of $|x|$ have a MLR.

And

$$L(x, \theta) = \frac{\theta}{\pi} \cdot \frac{1}{\theta^2 + x^2}$$

$$L(y, \theta) = \frac{\theta}{\pi} \cdot \frac{1}{\theta^2 + y^2}$$

$$\therefore \frac{L(x, \theta)}{L(y, \theta)} = \frac{\theta^2 + y^2}{\theta^2 + x^2}$$

This is independent of θ iff $x^2 = y^2 \Rightarrow |x| = |y|$.

Thus $T = |x|$ is sufficient.