Solutions to selected problems from Part Two.

Q2 $\qquad X_i \sim \text{Bernoulli}(p)$

$S = X_1 + X_2 + X_3$. - This was done in last week. We know that S is sufficient.

$T = X_1 X_2 + X_3$. We need to show that that T is not sufficient. It is enough to show that

$$f_{X_1, X_2, X_3 | T=1}(0,0,1|1)$$

does not depend on p.

$$f_{X_1,X_2,X_3|T=1}(0,0,1|1) = \frac{\mathbb{P}(X_1 = 0 \cap X_2 = 0 \cap X_3 = 1 \wedge T = 1)}{\mathbb{P}(T=1)}$$

$$= \frac{(1-p)^2 p}{3p^2(1-p) + p(1-p)^2} = \frac{\cancel{p}\cancel{(1-p)}(1-p)}{3p^{\cancel{2}}\cancel{(1-p)} + \cancel{p}(1-p)^{\cancel{2}}}$$

$$= \frac{(1-p)}{3p + 1 - p} = \frac{1-p}{1+2p}.$$

So T is not sufficient because it still depends p.

Q3 We need to show that either $T = X_1 + X_2$ or $S = X_1 X_2$ sufficient statistic with the given joint distribution of (x_1, x_2).

First check:

- $P(X_1 = 0 \cap X_2 = 0 \mid X_1 + X_2 = 0) = 1$
- $\mathbb{P}(X_1 = 1 \cap X_2 = 0 \mid X_1 + X_2 = 0) = 0$

$= \mathbb{P}(X_1 = 0 \cap X_2 = 1 \mid X_1 + X_2 = 0)$.

- $\mathbb{P}(X_1 = 1 \cap X_2 = 0 \mid X_1 + X_2 = 1)$

$$= \frac{\frac{\theta}{12}(4-\theta)}{\frac{\theta}{12}(4-\theta) + \frac{\theta}{12}(4-\theta)} = \frac{1}{2}$$

$= \mathbb{P}(X_1 = 0 \cap X_2 = 1 \mid X_1 + X_2 = 1)$.

- $\mathbb{P}(X_1 = 0 \cap X_2 = 0 \mid X_1 + X_2 = 1) = 0$.
- $\mathbb{P}(X_1 = 1 \cap X_2 = 1 \mid X_1 + X_2 = 0) = 0$.
- $\mathbb{P}(X_1 = 1 \cap X_2 = 1 \mid X_1 + X_2 = 2) = \underline{1}$.
- $\mathbb{P}(X_1 = 0 \cap X_2 = 0 \mid X_1 + X_2 = 2) = 0$.

We see that all these conditional probabilities do not depend on θ. So $T = X_1 + X_2$ is a sufficient statistic.

Now check $S = X_1 X_2$.

③

$$P(X_1 = 1 \cap X_2 = 0 \mid X_1 X_2 = 0) = \frac{\frac{\theta}{12}(4-\theta)}{\frac{\theta}{12}(4-\theta) + \frac{\theta}{12}(4-\theta) + \frac{1}{12}(12-7\theta+\theta^2)}$$

$$= \frac{4\theta - \theta^2}{\theta - \theta^2 + 12}.$$

This will definitely depends on θ. Thus $S = X_1 X_2$ is not sufficient.

Q8 b $\quad X_1, X_2, \ldots, X_n$ iid. with density

$$f(x;\theta) = \frac{1}{6\theta^4} x^3 e^{-x/\theta}$$

$$L(\underline{x};\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \frac{1}{6\theta^4} x_i^3 e^{-x_i/\theta}.$$

$$= \frac{1}{(6\theta^4)^n} \prod_{i=1}^{n} x_i^3 \, e^{-\sum_{i=1}^{n} x_i/\theta}$$

Here $h(\underline{x}) = \prod_{i=1}^{n} x_i^3$

$$g(t,\theta) = \frac{1}{(6\theta^4)^n} e^{-\sum_{i=1}^{n} x_i/\theta} \qquad t = \sum x_i$$

$$= \frac{1}{(6\theta^4)^n} e^{-t/\theta} \quad \text{where } t = \sum_{i=1}^{n} x_i$$