The University of New South Wales

Department of Statistics

Term 2, 2019

MATH5905 - Statistical Inference

Assignment 2

For this assignment, you are allowed to work in groups of 1 or 2 or 3 or 4 or 5 (max).

You must write all members of your group with their student numbers on each page at the top right hand corner of the assignment.

You must submit one pdf file via Moodle Turnitin. The University requires each student to submit the assignment in **Moodle Turnitin** even though this could be a group assignment.

This assignment must be submitted no later than **noon on Friday, 9th June 2019** on Moodle. Your submission on **Moodle Turnitin** will be your declaration that the assignment is your own work, except where acknowledged and that you understood the University Rules on plagiarism.

Maximal number of pages: 8

1) Let $X = (X_1, X_2, \ldots, X_n)$ be a sample of n observations each with a uniform in $[0, \theta)$ density

$$f(x, \theta) = \begin{cases} \frac{1}{\theta} \text{ if } 0 < x < \theta \\ 0 \text{ else} \end{cases}$$

where $\theta > 0$ is an unknown parameter. Denote the joint density by $L(X, \theta)$.

a) Show that the family $\{L(X, \theta)\},\ \theta > 0$ has a monotone likelihood ratio in $X_{(n)}$.

b) Show that the uniformly most powerful α-size test of $H_0 : \theta \leq 2$ versus $H_1 : \theta > 2$ is given by

$$\varphi^*(\mathbf{X}) = \begin{cases} 1 \text{ if } X_{(n)} > 2(1-\alpha)^{\frac{1}{n}} \\ 0 \text{ if } X_{(n)} \leq 2(1-\alpha)^{\frac{1}{n}} \end{cases}$$

c) Find the power function of the test and sketch the graph of $E_\theta \varphi^*$ as accurately as possible.

d) Show that the random variable $Y_n = n(1 - \frac{X_{(n)}}{\theta})$ converges in distribution to the exponential distribution with mean 1 as $n \to \infty$. Hence justify that $X_{(n)}$ is a consistent estimator of θ.

2) Let $\mathbf{X} = (\mathbf{X_1}, \mathbf{X_2}, \ldots, \mathbf{X_n})$ be i.i.d. random variables, each with a density

$$f(x, \theta) = \begin{cases} \frac{2}{\theta} x e^{-\frac{x^2}{\theta}}, x > 0 \\ 0 \text{ elsewhere} \end{cases}$$

where $\theta > 0$ is a parameter. (This is called the Raleigh-distribution.)

a) Find the Fisher information about θ in one observation and in the sample of n observations.

b) Find the MLE of θ. Is it unbiased? If YES, does its variance attain the Cramer Rao bound?

c) What is the asymptotic distribution of the MLE of θ?

d) Does that the family $L(\mathbf{X}, \theta)$ has a monotone likelihood ratio? If YES, in which statistic?

e) Does a UMP α test of $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$ exist? If YES, outline its structure. Also, using asymptotic arguments (e.g., from c)), find the threshold constant in the definition of the test.

f) Calculate (asymptotic approximation to) the power function $E_\theta \varphi^*$ and sketch a graph.

3. Let X be one observation from a Cauchy (θ) distribution.

a) Show that this family does not have a MLR.

b) Show that the test

$$\varphi(x) = \begin{cases} 1 & \text{if } 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

is most powerful of its size for testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. Calculate the Type 1 and Type II Error probabilities.

c) Prove or disprove: The test in part b) is UMP for testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$. What can be said about UMP tests in general for the Cauchy location family?

d) Suppose that X comes from the Cauchy scale pdf

$$f(x;\theta) = \frac{\theta}{\pi} \frac{1}{\theta^2 + x^2}, \quad -\infty < x < \infty, \quad \theta > 0$$

If X is one observation from $f(x;\theta)$, show that $|X|$ is sufficient for θ and that distribution of $|X|$ does have a MLR.