Math 5905 Assignment 1

Group work

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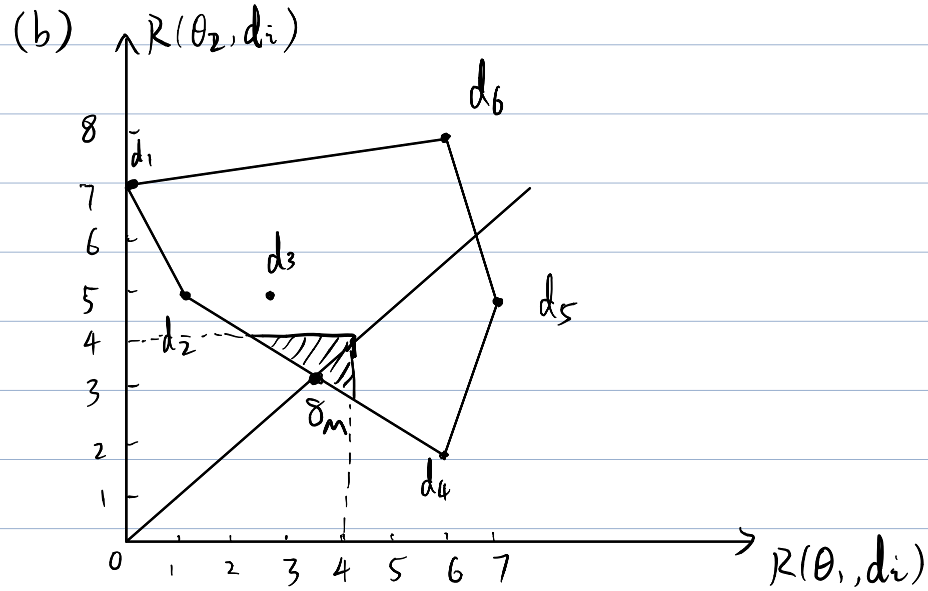
1.(a) minimax rule is: $\inf_{d_i} \sup_{\theta} R(\theta, d)$

and $\sup_{\theta} R(\theta, d) = \{\overset{d_1}{7}, \overset{d_2}{5}, \overset{d_3}{5}, \overset{d_4}{6}, \overset{d_5}{7}, \overset{d_6}{8}\}$

So both d_2 and d_3 are minimax rules among the nonrandomized rules in D.

(b)



As is shown above, the pentagon is randomized rules D

(c) The point δ_M in the graph is the minimax rule in D.

We have $\begin{cases} y = x \\ y-5 = \frac{2-5}{6-1}(x-1) \end{cases}$ and we can get $\begin{cases} x = 3.5 \\ y = 3.5 \end{cases}$

So δ_M is the minimax rule in D and the risk is 3.5

(d) We are looking for α such that

$\alpha * 1 + (1-\alpha) * 6 = 3.5$, so $\alpha = \frac{1}{2}$

This means we have 50% probability of choosing d_2 and 50% probability of choosing d_4 in terms of δ_M as the minimax rule.

(e) If the prior is $(p, 1-p)$, then the line with a slope of $\frac{-p}{1-p}$ and this slope should be the same with $\overline{d_2 d_4}$.

So $\frac{-p}{1-p} = -\frac{3}{5}$

$p = \frac{3}{8}$

So $(\frac{3}{8}, \frac{5}{8})$ is also a Bayes rule in D

(f) The slope of the prior $(\frac{5}{6}, \frac{1}{6})$ is $\frac{-\frac{5}{6}}{1-\frac{5}{6}} = -5$

This slope corresponds to a southwest line. Move it until it intersects with d_1, which is the Bayes rule for prior $(\frac{5}{6}, \frac{1}{6})$.

(g) As is shown above, the shadow area is the risk set.

2 (i) Because X is iid, the conditional distribution of

X and θ $\quad f(X|\theta) = \prod_{i=1}^{n} f_{X_i}(x_i|\theta)$

$$= \prod_{i=1}^{n} \theta^{x_i} (1-\theta)$$

$$= \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^n$$

$$h(\theta|X) = \frac{f(X,\theta)}{g(X)} = \frac{f(X|\theta)\cdot\pi(\theta)}{g(X)}$$

$$= \frac{\theta^{\sum_{i=1}^{n} x_i} (1-\theta)^n \cdot 3\theta^2}{\int_\theta f(X|\theta)\cdot\pi(\theta)\,d\theta}$$

The density of Beta family is

$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}$, here $\frac{1}{B(\alpha,\beta)} = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx$

which is a constant.

And $h(\theta|X) = \frac{3}{\int_\theta f(X|\theta)\cdot\pi(\theta)d\theta} \cdot \theta^{\sum_{i=1}^{n} x_i+2} \cdot (1-\theta)^n$

The first part of $h(\theta|X)$ is also a constant.

So $h(\theta|X)$ is belong to Beta family and

$$\begin{cases} \sum_{i=1}^{n} x_i + 2 = \alpha - 1 \\ n = \beta - 1 \end{cases} \rightarrow \begin{cases} \alpha = \sum_{i=1}^{n} x_i + 3 \\ \beta = n+1 \end{cases}$$

So $h(\theta|X)$ belongs to $\text{Beta}(\sum_{i=1}^{n} x_i + 3, n+1)$

$$\hat{\theta}_{Bayes} = \frac{\sum_{i=1}^{n} x_i + 3}{\sum_{i=1}^{n} x_i + n + 4}$$

(ii) We know that $n=5$ and $\sum_{i=1}^{n} x_i = 20$

So $Pr(\theta<0.75|X) = \int_0^{0.75} \frac{1}{B(20,6)} x^{19} (1-x)^5 dx$

$= 0.378 < 0.5$

Then H_0 should be rejected.

3. Because $X_1, X_2 \cdots X_n$ are i.i.d, the conditional distribution of X is $f(x_i|\theta) = \frac{1}{\theta} I_{(x_i, +\infty)}(\theta)$

The corresponding joint distribution is

$f(X|\theta) = \frac{1}{\theta^n} I_{(x_{(n)}, +\infty)}(\theta)$

We know that $\pi(\theta) = \beta \alpha^\beta \theta^{-(\beta+1)} I_{(\alpha, +\infty)}(\theta)$

So we can get $f(x,\theta) = f(x|\theta) \cdot \pi(\theta)$

$= \beta \alpha^\beta \theta^{-(n+\beta+1)} I_{(\max(\alpha, x_{(n)}), +\infty)}$

$h(\theta|X) = \frac{f(x,\theta)}{g(x)}$ and $\hat{\theta}_{Bayes} = E(\theta|X)$

So $\hat{\theta}_{Bayes} = \frac{\int \beta \alpha^\beta \theta^{-(n+\beta)} d\theta}{\int \beta \alpha^\beta \theta^{-(n+\beta+1)} d\theta} = \frac{(n+\beta)\theta^{-n-\beta-1}}{(n+\beta-1)\theta^{-n-\beta}} \Bigg|_{\max(\alpha, X_{(n)})}^{\infty}$

$= \frac{n+\beta}{n+\beta-1} \max(\alpha, X_{(n)})$

4. The joint probability distribution of the Minister is $f(x,\theta) = f(x|\theta) \cdot \pi(\theta)$

$= \theta(1-\theta)^{x-1} \cdot 6\theta(1-\theta)$

$= 6\theta^2 (1-\theta)^x$

The corresponding $h \propto \theta^2 (1-\theta)^x$ and it belongs to Beta$(3, x+1)$

Similarly, for prime minister, the joint distribution is $f(x,\theta) = f(x|\theta)\cdot\pi(\theta)$

$$= \theta(1-\theta)^{x-1}\cdot 4\theta^3$$

$$= 4\theta^4(1-\theta)^{x-1}$$

The corresponding $h_1 \propto \theta^4(1-\theta)^{x-1}$ and it belongs to Beta$(5, x)$

Notice that we have 2 actions: a_0 for continuing and a_1 for abandoning. The loss of these actions are:

$$L_0(\theta, a_0) = \begin{cases} \frac{1}{2}-\theta & \theta < \frac{1}{2} \\ 0 & \theta \geq \frac{1}{2} \end{cases}$$

$$L_1(\theta, a_1) = \begin{cases} 0 & \theta < \frac{1}{2} \\ \theta - \frac{1}{2} & \theta \geq \frac{1}{2} \end{cases}$$

$$Q_0(x, a_0) = \int_0^{\frac{1}{2}} (\tfrac{1}{2}-\theta)h(\theta)\,d\theta = \frac{1}{2}\int_0^{\frac{1}{2}} h(\theta)d(\theta) - \int_0^{\frac{1}{2}}\theta h(\theta)d(\theta)$$

$$Q_1(x, a_1) = \int_{\frac{1}{2}}^1 (\theta - \tfrac{1}{2})h(\theta)d\theta = \int_{\frac{1}{2}}^1 \theta h(\theta)d\theta - \frac{1}{2}\int_{\frac{1}{2}}^1 h(\theta)d(\theta)$$

If $Q_0 > Q_1$, we will do a_1

If $Q_0 < Q_1$, we will do a_0

If $Q_0 = Q_1$, a_0 and a_1 are equal favoriable

For $Q_0 > Q_1$, $\frac{1}{2}\int_0^{\frac{1}{2}} h(\theta)d(\theta) - \int_0^{\frac{1}{2}}\theta h(\theta)d\theta > \int_{\frac{1}{2}}^1 \theta h(\theta)d\theta - \frac{1}{2}\int_{\frac{1}{2}}^1 h(\theta)d(\theta)$

$$\frac{1}{2} > \int_0^1 \theta h(\theta)\,d\theta = E(\theta|x)$$

$$= \frac{3}{x+4}$$

so $x > 2$

For $Q_0 < Q_1$, $\frac{1}{2}\int_0^{\frac{1}{2}} h(\theta)d(\theta) - \frac{1}{2}\int_0^{\frac{1}{2}}\theta h(\theta)d(\theta) < \int_{\frac{1}{2}}^1 \theta h(\theta)d(\theta) - \frac{1}{2}\int_{\frac{1}{2}}^1 h(\theta)d(\theta)$

$$\frac{1}{2} < \int_0^1 \theta h(\theta) d(\theta) = E(\theta \mid x) = \frac{5}{x+5}$$

so $x < 5$

So when $2 < x < 5$, the Minister and prime minister have disagreement.