Competitive Programming and Mathematics Society

Mathematics Workshop

Geometry Fundamentals

CPMsoc Maths

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CPMSOC

Welcome

We would like to thank everyone for coming, even if its just for the pizza :D

We are looking forward to expanding our activities from here onwards, if you have any ideas for what you think we can do to satisfy your interests, please let us know!!

Attendance form

Clarifications and Assumptions

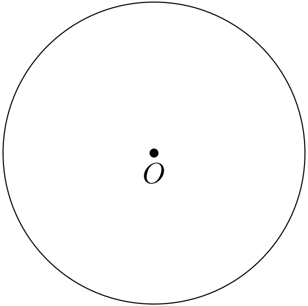
- This will cover Geometry on the Euclidean Plane. We will not use the coordinate nor complex plane.
- We will mainly discuss angle relations. This tends to be easier for beginners.
- As the slides have said, this is a beginner workshop - do not expect to solve hard questions just from this. Skill come from practice.
- We assume you are familiar with the following:
  - Angles of a triangle add to $180°$.
  - Equal angles that arise from parallel lines.
  - Vertically opposite angles.
  - Similar and congruent triangles.

Notation

- AB refers to the line passing through points A, B. This is used interchangeably with AB as the *length* of the line segment AB. What it will refer to is based on context, but $\overline{AB}$ and $|AB|$ may be used to denote the line and length respectively to resolve ambiguity.
- Unless otherwise mentioned, $\angle BAC$ will refer to the non-reflex angle (the angle less than $180°$.
- $A - B - C$ means that points A, B, C are colinear.
- (ABC) refers to the circle passing through points A, B, C. (AB) refers to the circle with diameter AB.
- $AB \cap CD = E$ means that E is the intersection of lines AB and CD. $AB \cap (XYZ) = \{P, Q\}$ means that line AB and circle (XYZ) intersect at points P and Q.
- The foot of the perpendicular of A to BC is the point D on line BC such that $AD \perp BC$.

What is a circle

A circle is the set of points that are a fixed distance from a certain point (called the "centre").

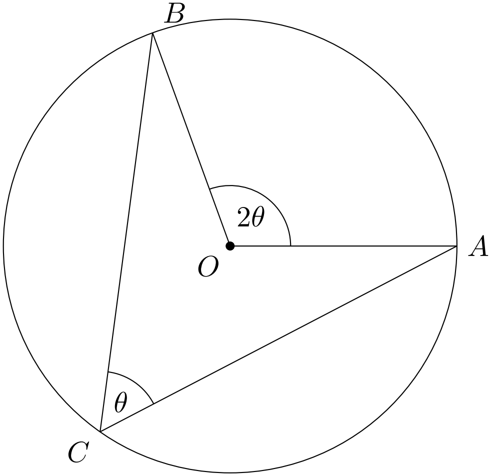


We usually denote the centre of circles as O.

Inscribed Angle Theorem

Theorem: (Inscribed Angle Theorem)

Let A, B, C be points on a circle with centre O. Then $2\angle ACB = \angle AOB$



Inscribed Angle Theorem

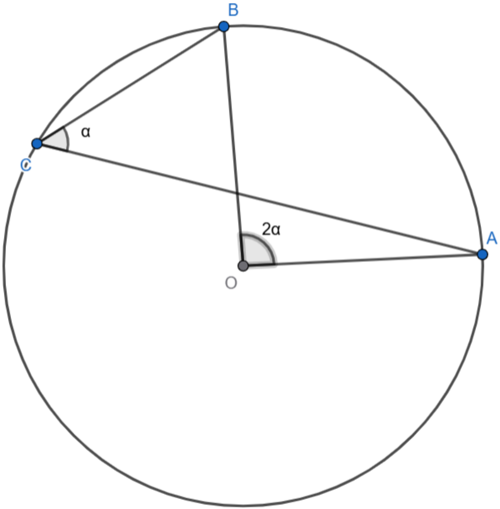
Let $\angle ACO = \alpha$ and $\angle BCO = \beta$.
Then, we find that $\angle CAO = \alpha$ and $\angle BCO = \beta$ as $\triangle AOC$ and $\triangle BOC$ are isosceles triangles. Thus

$$\begin{aligned}\angle AOB &= 360^\circ - \angle AOC - \angle BOC \\ &= 360^\circ - (180^\circ - 2\alpha) - (180^\circ - 2\beta) \\ &= 360^\circ - 180^\circ + 2\alpha - 180^\circ + 2\beta \\ &= 2\alpha + 2\beta \\ &= 2\angle ACB.\end{aligned}$$

since $\angle ABC = \angle ACO + \angle BCO = \alpha + \beta$.

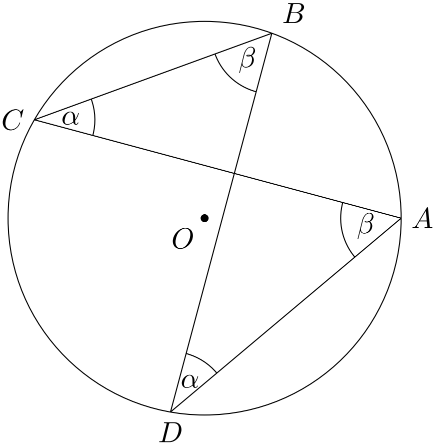
Inscribed Angle Theorem

What are we missing?



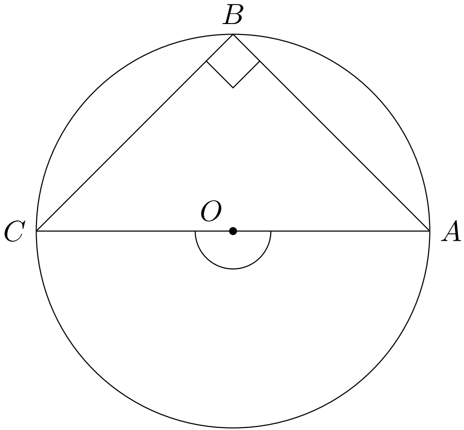
Bowtie Theorem

Theorem: (Bowtie Theorem)
Let points A, B, C, D lie on a circle such that C and D are on the same side of line AB.
Then $\angle ACB = \angle ADB$.



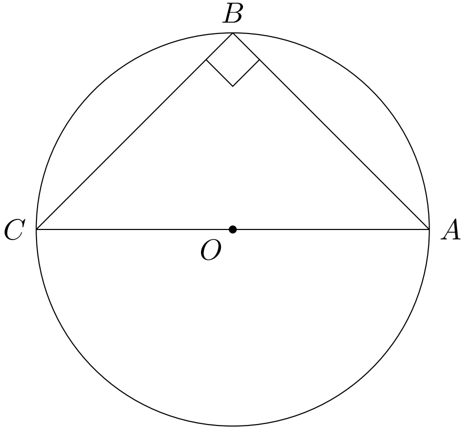
Inscribing Right Angles

What happens if $\angle ACB = 90°$?

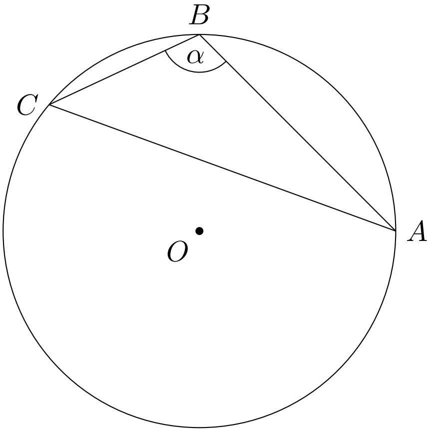


Thales' Theorem

Theorem: (Thales' Theorem)
Let AB be a diameter of a circle Γ. Then for any point C that lies on the circle, $\angle ACB = 90°$.



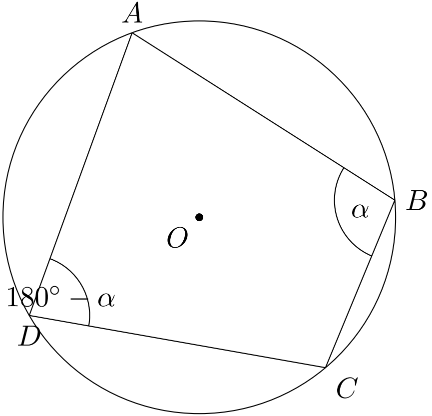
Inscribing Obtuse Angles



Angles on Opposite Arcs

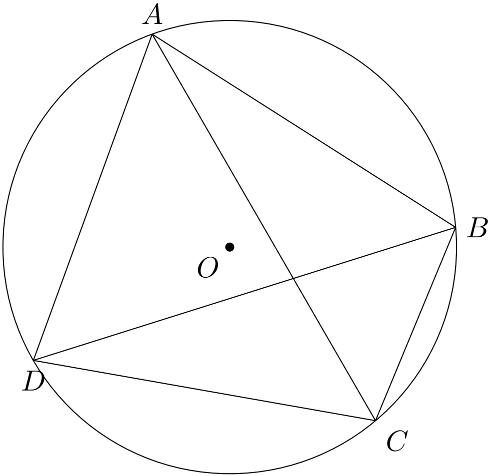
Lemma:

Let A, B, C, D be points on a circle where C and D are on opposite sides of line AB. Then $\angle ACB + \angle ADC = 180°$.



Cyclic Quadrilaterals

A cyclic quadrilateral is a quadrilateral whose vertices all lie on a circle.

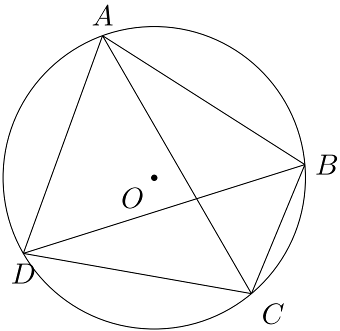


Cyclic Quadrilaterals

Lemma: (Cyclic Quads)

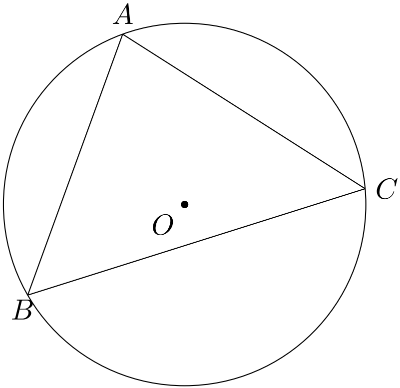
Let $ABCD$ be a quadrilateral, then the following are equivalent:

- $\angle ACB = \angle ADB$
- $\angle ABC + \angle ADC = 180°$
- $ABCD$ is a cyclic quadrilateral.



What about cyclic triangles?

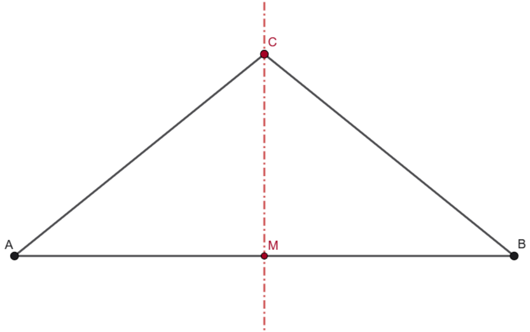
Do the vertices of a triangle always lie on a circle?



Yes they do!

What about cyclic triangles?

We can look to find the centre of this circle for each triangle.
The perpendicular bisector of a line segment AB as the line ℓ that passes through the midpoint of AB, and is perpendicular to AB.



We can see $\triangle ACM \equiv \triangle BCM$ and so $AC = BC$.

A Centre of a Triangle

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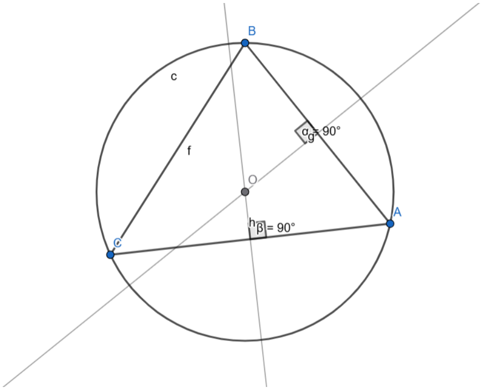
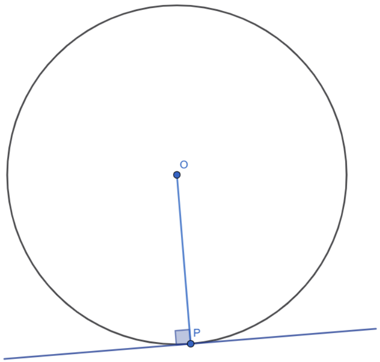
Let the perpendicular bisectors of AB and AC meet at some point P.



Then we can see $AP = BP$, and $AP = CP$ and so the circle with centre P with radius $r = AP$ would pass through our 3 vertices.

Tangents

A tangent ℓ to a circle Γ is a line that intersects it at only one point. We can alternatively say that ℓ is tangent to Γ.

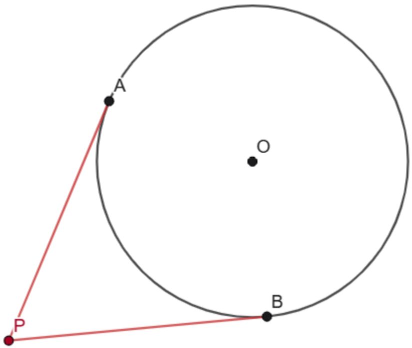


If they intersect at a point P, then OP is perpendicular to line ℓ.

Ice Cream Cone Lemma

Lemma: (Ice Cream Cone)

Let A and B be points on a circle. Then if PA and PB are tangents, we get that $|PA| = |PB|$.



Ice Cream Cone Lemma

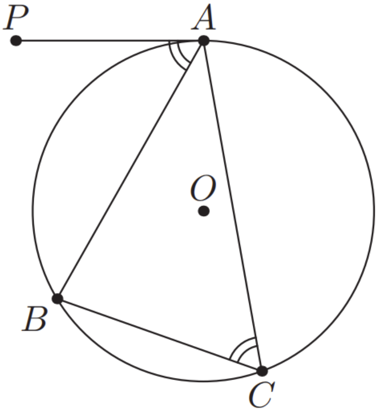
Let O be the centre of the circle. We can see that for triangles $\triangle AOP$ and $\triangle BOP$:

- OP is shared.
- $AO = BO$ as they are both radii
- $\angle OAP = \angle OBP = 90^\circ$

and so they are congruent from RHS. Thus, we find that $AP = BP$ as they are corresponding sides.

Alternate Segment Theorem

Theorem: (Alternate Segment Theorem)
Let A, B, C be points on a circle Γ, and AP be a tangent. Then $\angle PAB = \angle ACB$.



Alternate Segment Theorem

We find that $\angle BAO = 90^\circ - \angle PAB$. Thus, we find that

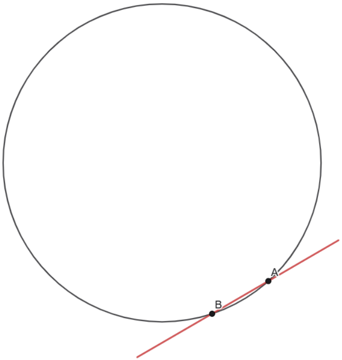
$$
\begin{aligned}
\angle ACB &= \frac{1}{2}\angle AOB \\
&= \frac{1}{2}\left(180^\circ - 2\angle BAO\right) \\
&= \frac{1}{2}\left(180^\circ - 2(90^\circ - \angle PAB)\right) \\
&= \frac{1}{2}\left(180^\circ - 180^\circ + 2\angle PAB)\right) \\
&= \angle PAB.
\end{aligned}
$$

as desired.

What does AA mean?

Sometimes you will see AA being used in a question or solution, and it will refer to the tangent at A. Why is this?

Essentially, this is because the tangent is the line AB for points A, B on a circle, where B limits to point A.



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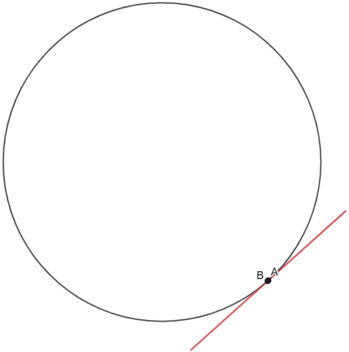
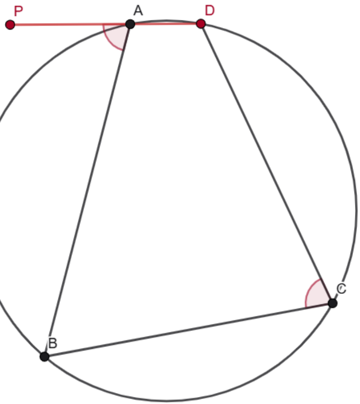


Alternate Segment Theorem

Another way to think about the alternate segment theorem is through a degenerate "cyclic quad", with two of the vertices being the same.



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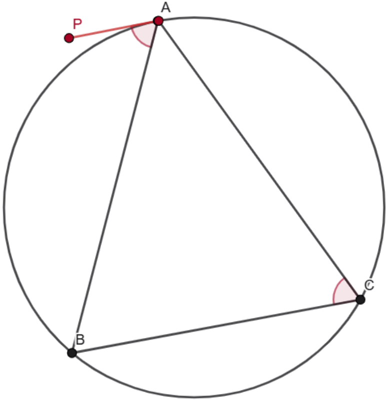

Power of a Point Pt. 1

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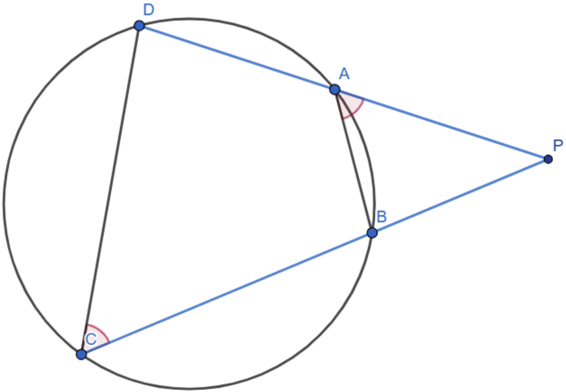
Lemma: (Power of A Point)
Let $ABCD$ be a cyclic quadrilateral, and lines $AD \cap BC = P$. Then $PA \times PD = PB \times PC$.



Power of a Point Pt. 1

As $ABCD$ is cyclic, we have:

- $\angle PAB = 180^\circ - \angle DAB = \angle PCD$
- $\angle PBA = 180^\circ - \angle ABC = \angle PDC$

and so we can see $\triangle PAB \simeq \triangle PCD$ from AA similarity. Thus, we get that

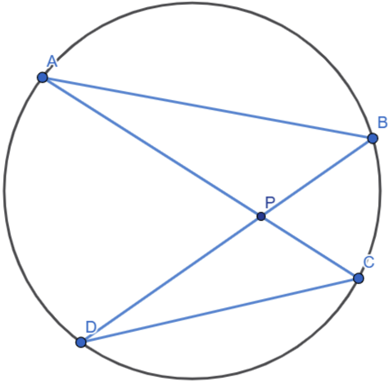
$$\frac{PA}{PB} = \frac{PC}{PD}$$
$$PA \times PD = PB \times PC$$

as desired.

Power of a Point Pt. 2

Lemma: (Power of A Point)

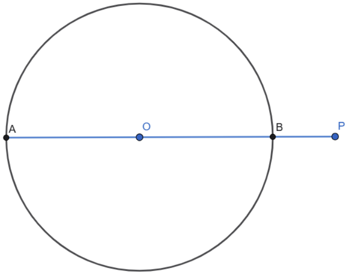
Let $ABCD$ be a cyclic quadrilateral, and lines $AC \cap BD = P$. Then $PA \times PC = PB \times PD$.



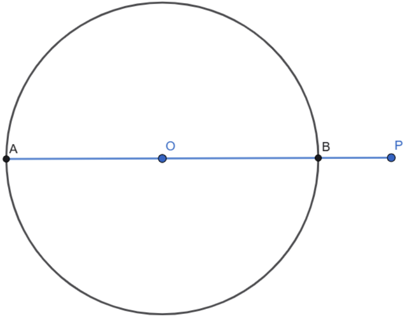
Power of a Point

We can see that no matter which line we pick through our point P, the product will be fixed. We define the power of a point with respect to a circle to be this value if P is outside the circle, and the negative of this value if P is inside.

What is this value though?
We can pick the line through P and the centre O of the circle and let it intersect at points A and B.



Power of a Point

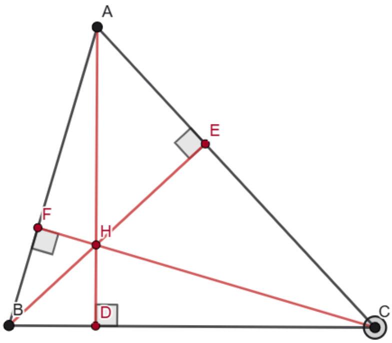


Now we see that if r is the radius of the circle,

$$\begin{aligned} PA \times PB &= (PO + OA) \times (PO - OB) \\ &= (PO + r) \times (PO - r) \\ &= PO^2 - r^2 \end{aligned}$$

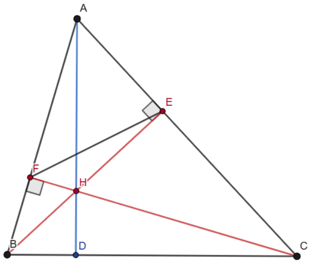
The Orthocentre

We define the orthocentre as the intersection of the altitudes of the triangle. How do we know it exists?



The Orthocentre

Let altitudes CF and BE intersect at H, and AH meet BC at D. We want to show that AD is an altitude. We can see that since $\angle BFC = \angle BEC = 90^\circ$ and $\angle AFH + \angle AEH = 180^\circ$, $AFHE$ and $BFEC$ are cyclic quadrilaterals.



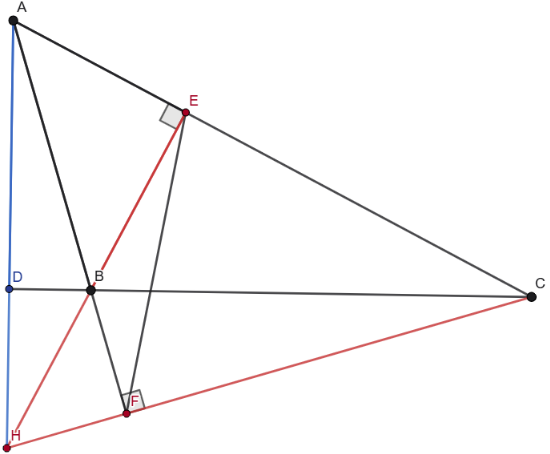
Thus from bowtie,

$$\angle HAE = \angle HFE = \angle CFE = \angle CBE$$

and so $AEDB$ is also a cyclic quad. We thus get $\angle ADB = \angle AEH = 90^\circ$ as desired.

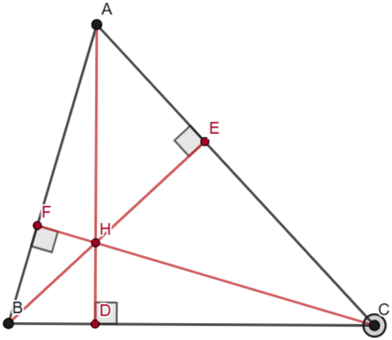
The Diagram

The proof from before doesn't always work! This is because it relies on H being inside our triangle. If $\angle ABC$ our logic doesn't necessarily follow.



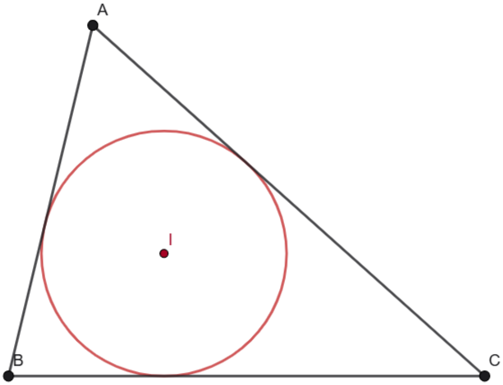
Lots of Cyclic Quads

We can see that there are a lot of cyclic quadrilaterals in our diagram. For example, $AEHF$ and $BFEC$ are (from before), but also $BFHD$, $CDHE$, $AFDC$, and $AEDB$ are.



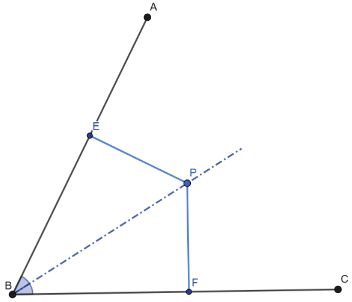
The Incentre

We know that there's a circle that passes through the vertices of any triangle.
Does there exist a circle which is tangent to all of the sides of a triangle?



The Incentre

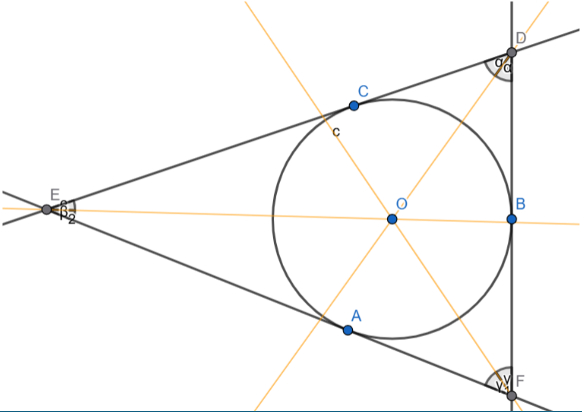
The *angle bisector* of an angle $\angle ABC$ is a line that splits that angle into two equal angles. Let P be a point on this line, and E, F be the feet of the perpendicular from P to AB, CB



Then we can see that $\triangle BEP \equiv \triangle BFP$ from AAS congruency.

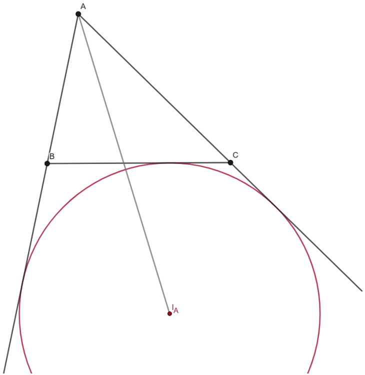
The Incentre

Let the angle bisectors of $\angle BAC$ and $\angle ABC$ of triangle ABC meet at a point I. Then consider the feet of the perpendiculars D, E, F from I to BC, AC, and AB respectively. We then see that $ID = IE$ and $ID = IF$, and so the circle with centre I and radius $r = ID$ is going to be tangent to all 3 sides.



The Excentre

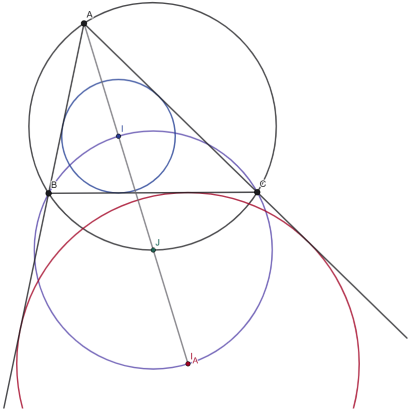
The $A-$excentre of a triangle ABC is the circle that is tangent to side BC, ray AB at a point beyond B and AC at a point beyond C. We can see it is the intersection of the external angle bisectors of $\angle ABC$ and $\angle ACB$ along with the internal angle bisector of $\angle BAC$.



Incentre/Excentre Lemma

Lemma: (Incentre/Excentre Lemma)

Let I and I_A are the incentre and $A-$excentre of triangle ABC. Then I, I_A, B, C lie on a circle with diameter II_A, with its centre on (ABC).



Attendance form :D

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Further events

Please join us for:

- Rookie Code Rumble (Wed 11th June)
- Chicken Contest Debrief (Thur June 19th)