Competitive Programming and Mathematics Society

Mathematics Workshop

Combinatorial Game Theory

CPMsoc Maths

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Welcome

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- We would like to thank everyone for coming, even if its just for the pizza :D
- We are looking forward to expanding our activities from here onwards, if you have any ideas for what you think we can do to satisfy your interests, please let us know!!
- We do have a lot more planned for TERM 2! More Competitive Mathematics heading your way...

Attendance form

What are Combinatorial Games?

Combinatorial games are a type of game where:

- There are two players who take turns to move.
  - We call these players *Left* and *Right*
- There is no hidden information or chance.
- The game ends in finitely many moves, in which either one person wins and the other loses.

Theorem: (Fundamental Theorem of Combinatorial Games)
In any combinatorial game, given a starting position and starting player, exactly one player can always force a win.

Game Positions and Values

Game positions describe the state of a game at a certain point

- Determined by positions available for Left and Right to move into.

Positions have game value falls into one of four **outcome classes**.

- Left win
- Right win
- First to move win
- Second to move win

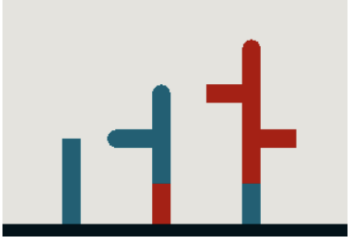
A common condition is **standard play**:

- In standard play, the loser is determined based on whoever runs out of moves first.
- *Misère play* is where the winner is whoever runs out of moves first.

Example: Hackenbush

A very simple game which illustrates positions is **Hackenbush**.

- Players take turns chopping branches off of trees.
- Red and blue branches are only choppable by their respective players, green branches are choppable by both.
- A chopped branch removes all other branches which are now disconnected from the ground.
- The player who cannot chop any more loses.



Impartial Games

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An *impartial game* is one where the moves and payoffs (values) don't depend on the player, only the position

- Both players have the same move opportunities
- Only difference in who wins is who goes first or second

Dealing with impartial games tends to be easier since we only have to consider the position, rather than the particular moves of the players.

A classic example is **Nim**:

- Players remove objects from distinct piles.
  - They must remove at least 1, and all from the same pile.
- The last player to remove an object wins.

Nim is impartial since both players have the same set of actions (removing stones), and are thus interchangeable other than turn order.

Mirroring

A simple strategy for some game positions is **mirroring.**

- If the position can be split into 2 components where players can copy the rival's moves in one component to the other

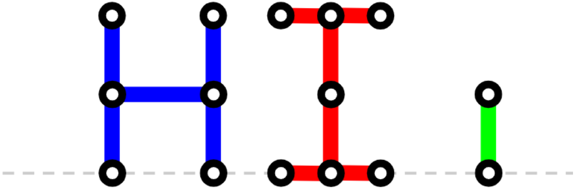

Figure: Who wins in this hackenbush position?

More Mirroring

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Example: Mortimer and Susan play **Ukodus**, on a 4×4 square grid divided into 2×2 subgrids (*blocks*). They take turns writing numbers from $1, 2, \ldots, 4$ in squares, such that each row, column and block must not have duplicates of the same number. The loser is the person who can't write a number. Mortimer starts first. Who has the winning strategy?

More Mirroring

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Example: Mortimer and Susan play **Ukodus**, on a 16×16 square grid divided into 2×2 subgrids (*blocks*). They take turns writing numbers from $1, 2, \ldots, 16$ in squares, such that each row, column and block must not have duplicates of the same number.
The loser is the person who can't write a number (if they complete a valid Sudoku board they both win!). Mortimer starts first. Is there a winning strategy for either player?

Solution: Susan has a winning strategy by a mirroring argument:

- Whenever Mortimer places a number, Susan places the same number on the square opposite across the centre of the board
- This play is legal, only if Mortimer's play was also legal.

Strategy-Stealing Arguments

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A proof by contradiction to show that a winning strategy does not exist for the second player. Most famously used on tic-tac-toe to show the 2nd player has no winning strategy:

- If they did, the 1st player can utilise this strategy after the 2nd player's first move, modifying it with a random move if the strategy requires they move in the position they occupied on the first move.

Works for **maker-maker games** where:

- The aim is to make a certain winning configuration.
- There is never a disadvantage of having free moves (no matter the move).

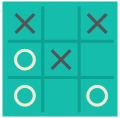


Figure: Why does the 1st player win in tic-tac-toe?

Winning and Losing Positions

We can classify positions as winning and losing depending on the following:

- A *winning* position for a particular player A is one where player A can make a move, such that the new position will be a losing position for player B. Winning positions also include all positions defined as 'won' by the game.
- A *losing* position for a particular player A is one where regardless of what move player A makes, the resulting position is a winning position for player B. Losing positions also include all positions defined as 'lost' by the game.

If the game includes a draw, we can add an additional 'drawn' position:

- A *drawn* position for a particular player A is one where player A cannot send player B to a losing position, but can send them to another 'drawn' position. Drawn positions also include all positions defined as 'drawn' by the game.

Reversible and Dominated Moves

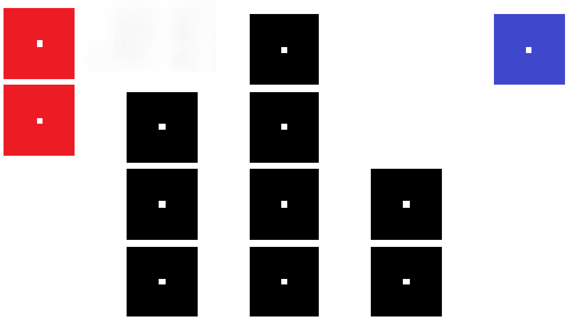
Often we ignore sub-optimal moves to simplify a game:

- A move is **dominated** by another move if it's resultant position has less value than the other option. For example, promoting a pawn to a bishop is dominated by promoting to a queen (if it doesn't cause stalemate).
- A move is **reversible** if the opponent can return the position to one that was at least as good for them.

Example: Poker Nim

Poker Nim plays out the same as Nim, except players can choose to create/add onto piles with the objects they have removed from other piles.

Adding onto a pile is a reversible move since the opponent can remove the same number of objects from the pile. This is then better for the opponent, since they now have more objects to place than before.



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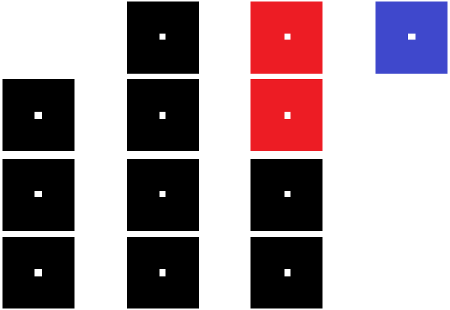

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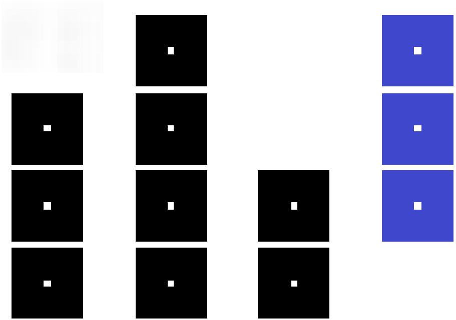

Invariants and co. in Games

Often there may be an invariant quantity in the winning sequence of moves which guarantees one player's success

- The winning player **can** keep the quantity invariant
- The losing player **cannot** break the invariant for the winning player
  - From any of the losing player's move, the winning player can always **restore** the invariant for themselves

Example: Alice and Bob play the game of **Impartial Rook.** They take turns moving a rook leftwards or downwards on a chessboard. Whoever moves the rook to the bottom-left corner wins. If Alice starts first, which starting coordinates (m, n) give her the win?

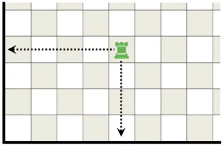

Figure: Impartial Rook

Invariants and co. in Games

Solution: Consider the quantity $I = m - n$ (which represents which diagonal the rook is on).

- The winning position $(0, 0)$ has $I = 0$.
- Any move from a position (j, j) which has $I = 0$, will change I.
- You can always move from any position with $I \neq 0$ to (j, j) which has $I = 0$.
  - These ensure that the winning player can maintain $I = 0$ for themselves.

So Alice wins if she is able to move to a position with $I = 0$. This is possible whenever the starting coordinates are not equal: $m \neq n$

Equivalences

Often we can rethink one game to an **equivalent** game to gain some insight.

- Equivalent games share the same abstract positions

Example: The game of **Impartial Rook** is equivalent to the game of **Nim**.

- The strategy preserving the invariant $I = m - n$ in Impartial Rook was secretly just the Mirroring strategy for 2-pile Nim (mirroring keeps the pile sizes equal).

Nimbers

There is a brilliant example invariant in Nim, called the nimber sum:

- If the number of powers of 2 in the binary expansion of each pile size is even for all digits, the next player to move loses.
    - Equivalent to if the bitwise XOR of the pile sizes (called the grundy nimber) is 0
- The end position is a zero-value position and the next player has already lost
- All moves from a zero-value position go to a nonzero-value position
    - Any move takes away an odd number of 1s digits
- There is always a move from a nonzero nim-value to a zero one
    - Take away the number you get by adding all the unpaired powers of 2

Sprague-Grundy Theorem

All impartial games are equivalent to playing Nim

- Hence all impartial positions can be assigned nimber values to better understand.

Additionally **all Nim positions are equivalent to 1-pile Nim, whose pile size is it's nimber value.** *"One Game to Rule them all"*

Attendance form :D

Further events

Please join us for:

- Upcoming Event