Competitive Programming and Mathematics Society

Mathematics Workshop

Integration

CPMsoc Maths

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Welcome

- We would like to thank everyone for coming, even if its just for the pizza :D
- We are looking forward to expanding our activities from here onwards, if you have any ideas for what you think we can do to satisfy your interests, please let us know!!
- We do have a lot more planned for TERM 2! More Competitive Mathematics heading your way...
- We may run an integration bee sometime in the future. Full disclaimer, this is nothing official, but there have been discussions.

Attendance form

About today

Today's workshop is all about integration, specifically the type that is featured in integration bees / competitions.

- We will be dealing with only real valued functions.
- The vast majority of the workshop is about single value functions. However, we will touch on multivariate calculus at the end, but it will not be complicated.
- This is not a rigorous treatment of integral calculus. We are just solving problems, and some formalities will be brushed aside.

Assumed knowledge

This is an introductory workshop, so there is minimal assumed knowledge:

- You can use properties of and take derivatives of all elementary functions, e.g.
  - Know $x^{a+b} = x^a x^b, \ln(uv) = \ln(u) + \ln(v)$[1].
  - Find $\frac{dy}{dx}$ with $y = x^n, y = \sin^{-1} x, y = e^{\sin x} \ln(x^2 + 1)$.
  - Know the chain, product, and quotient rules.
- You know what integration is!
  - Know that the anti-derivative of $f(x)$ is $F(x)$ with $\frac{d}{dx} F(x) = f(x)$.
  - Familiar with most standard integrals, such those you would find on a 1131 / 1141 formula sheet.
  - E.g. $\int x^n \, dx, \int \sin x \, dx, \int e^x \, dx$.
  - Familiar with basic integration rules.
  - E.g. $\int \lambda f(x) \, dx = \lambda \int f(x) \, dx, \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$.

[1] $u, v > 0$

Notation

Pretty standard math things.

- $\int f$ represents the antiderivative of a function f. Usually, we notate this as F, with $F' = f$. This is not usually used in our context.
- $\int f(x)\,dx$ represents the antiderivative of f with respect to x. This is a family of functions which, when we take the derivative wrt. x, yield $f(x)$. Once we have found a particular function with this property, say F_p, since the derivative of any constant is zero, all solutions to $\frac{d}{dx}F(x) = f(x)$ are $F(x) = F_p(x) + C$, where $C \in \mathbb{R}$.

  Conventionally, we skip all of this and just write

$$\int f(x)\,dx = F(x) + C.$$

Notation

- $\displaystyle\int_a^b f(x)\,dx$ represents the definite integral of f from a to b, provided $f(x)$ is integrable on (a, b). If F is any antiderivative from the previous slide, then we have

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This represents the signed area of the graph $y = f(x)$ from a to b, if $b \geq a$.

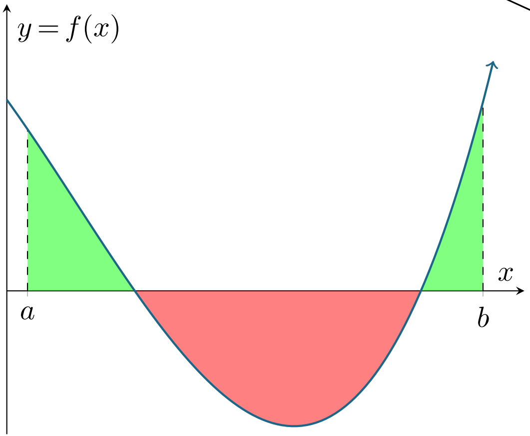
Notation

When we say signed area, we mean that area underneath the x-axis is counted as having negative contribution.

E.g. for the figure on the right,

$$\int_a^b f(x)\,dx = A_{\color{lightgreen}\blacksquare} - A_{\color{salmon}\blacksquare}.$$

This has many useful applications, which we will cover.



Notation

- $\int_a^b f(x)\,dx$ represents the definite integral of f from a to b, provided $f(x)$ is integrable on (a,b). If F is any antiderivative from the previous slide, then we have

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Also, note that the variable x we integrate wrt. to is arbitrary. Often, we write

$$\int_a^x f(t)\,dt = F(x) - F(a),$$

and this is another way of expressing the indefinite integral of f explicitly as a function. The variable t is what we call a "dummy variable", acting as a placeholder and replaceable with anything.

u substitution

Suppose that $F(x)$ is an antiderivative of $f(x)$, and we are interested in finding

$$\int f(g(x))g'(x)\,dx,$$

where $g(x)$ is some differentiable function. Recall the chain rule when taking derivatives.

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x),$$

but $F'(x) = f(x)$ by definition! Thus,

$$\frac{d}{dx}F(g(x)) = f(g(x))g'(x) \iff \int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Why is this useful?

u substitution

Why is this useful? If we have a function which can be expressed in the form $f(g(x))g'(x)$ for some f and g, finding the antiderivative of $f(g(x))g'(x)$ reduces to the problem of finding the antiderivative of $f(x)$, which can be much easier. This technique is called integration by substitution.

The name u substitution comes from an equivalent formulation:

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du,$$

where $u = g(x)$. E.g. with $u = 1 + x^2$,

$$\begin{aligned}\int x(1+x^2)^\pi\,dx = \frac{1}{2}\int u^\pi \frac{du}{dx}\,dx = \frac{1}{2}\int u^\pi\,du \\ = \frac{1}{2(\pi+1)}u^{\pi+1} + C.\end{aligned}$$

u substitution

We can also do things the other way. Suppose, instead, that $x = g(u)$, where g is an invertible function. Then,

$$\int f(x)\,dx = \int f(g(u))g'(u)\,du.$$

This looks like we are complicating the integral, however if $f(g(u))$ simplifies nicely, for example $f(x) = \sqrt{1-x^2}$ and $g(u) = \sin u$, and we know a way to integrate functions in the family of $g(u)$ and $g'(u)$ (trig!), then this allows us to evaluate integrals we otherwise cannot.

Since our answer will be in terms of u, g must be invertible or we will have no way to express our answer in terms of x. This is particularly important for definite integrals.

u substitution

u substitution can also be used for definite integrals. With $u = g(x)$,

$$\int_a^b f(u)\frac{du}{dx}\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Likewise, if $x = g(u)$ where g is invertible,

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(x)\frac{dx}{du}\,du.$$

This may lead you to think the second type of u substitution is more finicky with invertibility. This is not true, because *if* an inverse exists for g (we can pick any branch) with $g^{-1}(x) = u$ for all $x \in [a, b]$, then the theorem will always hold. The first theorem is often more troublesome.

u substitution

For example, with the substitution $u = x^2$, on the integral

$$\int_{-1}^{1} x^2 \, dx,$$

an unaware student may write $x^2 = \frac{\sqrt{x^2}}{2} \cdot 2x = \frac{\sqrt{u}}{2} \frac{du}{dx}$. Then, using the formula presented above,

$$\int_{-1}^{1} x^2 \, dx = \int_{-1}^{1} \frac{\sqrt{u}}{2} \frac{du}{dx} \, dx = \int_{(-1)^2}^{1^2} \frac{\sqrt{u}}{2} \, du.$$

Note that our bounds are now from 1 to 1, and thus the integral must be zero!

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Obviously not. $\int_{-1}^{1} x^2 \, dx = \left[\frac{x^3}{3} \right]_{-1}^{1} = \frac{2}{3}$.

u substitution

Turns out, there exists no function $f(u)$ which satisfies $f(u)\dfrac{du}{dx} = x^2$, as this would require $f(x^2) = \dfrac{x}{2}$, and f must be a many to one relation. Indeed, we simplified $\sqrt{x^2} = x$, when it should be $|x|$. Instead, if we split the integral up into

$$\int_{-1}^{0} x^2\,dx + \int_{0}^{1} x^2\,dx,$$

we can define $f(x) = -\dfrac{\sqrt{x}}{2}$ for the first integral, and $f(x) = \dfrac{\sqrt{x}}{2}$ for the second. Then, all is fine.

$$\int_{-1}^{0} x^2\,dx = \int_{1}^{0} -\frac{\sqrt{u}}{2}\,du = \frac{1}{3}, \int_{-1}^{0} x^2\,dx = \int_{0}^{1} \frac{\sqrt{u}}{2}\,du = \frac{1}{3}.$$

The reason why this happened is that the sub $u = x^2$ is secretly $x = \pm\sqrt{u}$, and this is not invertible for u unless we restrict the domain of x.

u substitution

When we carry out u substitution, it is bothersome to write $f(u)\dfrac{du}{dx}\,dx$ in full and nobody does it. Notice that we effectively replace $\dfrac{du}{dx}\,dx$ with du, so in a sense, if $u = g(x)$, we can write $du = g'(x)\,dx$, and directly substitute this into the formula.

We can go even further and skip the notation of u all together, and just write $d(g(x)) = g'(x)\,dx$. E.g, with the integral earlier,

$$\begin{aligned}\int x(1+x^2)^\pi\,dx &= \frac{1}{2}\int (1+x^2)^\pi d(1+x^2)\\ &= \frac{(1+x^2)^{\pi+1}}{2(\pi+1)} + C.\end{aligned}$$

This is called implicit integration.

Using u substitution

Enough pedanticism, we are here to evaluate some integrals.

$$\int \frac{dx}{\sqrt{\sqrt{x}+1}}$$

Using u substitution

$$\int \frac{dx}{x - x^{\frac{3}{5}}}$$

Basic trig sub

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Here are some common trig substitutions that are useful.

$$I = \int \frac{dx}{x^2\sqrt{x^2-1}}.$$

I see a $\sqrt{x^2-1}$, I sub $x = \sec u, dx = \sec x \tan x \, du$.

$$\begin{aligned} I &= \int \frac{\sec u \tan u}{\sec^2 u \sqrt{\sec^2 u - 1}} \, du \\ &= \int \frac{du}{\sec u} = \sin u + C \\ &= \frac{\sqrt{\sec^2 u - 1}}{\sec u} + C = \frac{\sqrt{x^2-1}}{x} + C. \end{aligned}$$

Basic trig sub

Of course, trig sub goes the other way too. Consider

$$I = \int \frac{\cos^2 x}{1+\sin^2 x}\,dx.$$

The substitution $t = \tan x$ works for all integrals with even powers of $\sin$ and $\cos$, but in this case it gets us to

$$I = \int \frac{dt}{(t^2+1)(2t^2+1)} = \int \left(\frac{2}{2t^2+1} - \frac{1}{t^2+1}\right) dt.$$

but this is slow. Instead, write

$$\begin{aligned} I &= \int \frac{1-\sin^2 x}{1+\sin^2 x}\,dx = \int \frac{2-(1+\sin^2 x)}{1+\sin^2 x}\,dx \\ &= 2\int \frac{\sec^2 x}{\sec^2 x + \tan^2 x}\,dx - x = 2\int \frac{d(\tan x)}{2\tan^2 x + 1} - x. \end{aligned}$$

This is effectively the same thing we did with the t sub, but it is much easier to see.

Basic trig sub

$$\int \frac{dx}{\cos x\sqrt{\sin 2x}}\,dx,\ \int \frac{dx}{\sin x\sqrt{\sin 2x}}\,dx$$

Basic trig sub

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$$\int \sqrt{1+\sin x}\,\cot x\,dx$$

Basic trig sub

Being able to trig sub requires a solid foundation of trig. E.g.,

$$\int \sqrt{1+\sin x}\,dx$$

Advanced trig sub

Now, for the integral

$$I = \int \frac{\tan x}{\sqrt{1+\sin^2 x}}\, dx,$$

it would be unwise to do the sub $u = \sin x$, as if we do,

$$I = \int \frac{\sin x}{\cos x\sqrt{1+\sin^2 x}} = \int \frac{\sin x \cos x}{\cos^2 x\sqrt{1-\sin^2 x}}\, dx = \int \frac{u}{(1-u^2)\sqrt{1+u^2}}\, du,$$

and we would need more substitutions to proceed. Instead, you could try $u = \cos x$, so

$$I \int \frac{\sin x}{\cos x\sqrt{2-\cos^2 x}}\, dx = -\int \frac{du}{u\sqrt{2-u^2}},$$

and this seems much simpler. The next step is to let $u = \sqrt{2}\sin v$ for the radical, but we might as well skip this step because $u = \cos x$, so $\cos = \sqrt{2}\sin v$.

Advanced trig sub

If we directly substitute $\cos x = \sqrt{2}\sin u$, we get $-\sin x\, dx = \sqrt{2}\cos u\, du$, so

$$I = -\int \frac{\sqrt{2}\cos u}{\sqrt{2}\sin u\sqrt{2 - 2\sin^2 u}}\, du = -\frac{1}{\sqrt{2}}\int \csc u\, du,$$

and this is easy. Using the integral of $\csc$, which you should memorise,

$$I = -\frac{1}{\sqrt{2}}\ln\left|\csc u - \cot u\right| + C = \frac{1}{\sqrt{2}}\ln\left|\frac{\sin u}{1 - \cos u}\right| + C,$$

and after a little bit of simplification we get

$$I = \frac{1}{\sqrt{2}}\ln\left|\frac{\cos x}{\sqrt{2} - \sqrt{1 + \sin^2 x}}\right| + C.$$

Advanced trig sub

Of course, there are always better ways to do integrals. If we let $u = \sec x$ in I, we have

$$I = \int \frac{\sec x \tan x}{\sqrt{\sec^2 x + \tan^2 x}}\, dx = \int \frac{d(\sec x)}{\sqrt{2\sec^2 x - 1}}.$$

The formula $\displaystyle\int \frac{dx}{\sqrt{a^2x^2 \pm b^2}} = \frac{1}{a}\ln\left|ax + \sqrt{a^2x^2 \pm b^2}\right|$ is useful here.

$$\therefore I = \frac{1}{\sqrt{2}}\ln\left|\sqrt{2}\sec x + \sqrt{2\sec^2 x - 1}\right| + C.$$

This answer is not fully correct. We assume that $\sec x\sqrt{1+\sin^2 x} = \sqrt{\sec^2 x + \tan^2 x}$, where we drop the sign of $\sec x$.

Advanced trig sub

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The sub $u = \sin x \pm \cos x$ is another good friend, because $u^2 = 1 \pm \sin 2x$. E.g, with

$$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}}$$

Advanced trig sub

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$$\int \frac{\sin x - \cos x}{(\sin x + \cos x)\sqrt{\sin x \cos x + \sin^2 x \cos^2 x}}\, dx$$

Reciprocal sub

Earlier in this section, we ran into the integral

$$I = \int \frac{dx}{x^2\sqrt{x^2-1}},$$

which I claimed we needed trig sub to do. That was me spreading misinformation. Notice

$$\begin{aligned} I &= \int \frac{(1/x)}{\sqrt{1-(1/x)^2}}\, d(1/x) \\ &= \sqrt{1-(1/x)^2} + C \\ &= \frac{\sqrt{1-x^2}}{x} + C. \end{aligned}$$

The substitution $u = \dfrac{1}{x}$ is the smallest child of a family of powerful subs.

Reciprocal sub

Consider
$$I = \int \frac{x^2+1}{x^4+1}\,dx.$$
Some of you may recognise $x^4+1 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$, and be inclined to bash with partial fractions. Instead, consider the following:
$$I = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \int \frac{d(x-1/x)}{(x-1/x)^2+2}.$$
The sub $u = ax + b/x$ is quite strong. Often, it is just $x \pm 1/x$, which you see here.

Reciprocal sub

$$\int \frac{x^2 - 1}{x^4 + 3x^2 + 1}\, dx$$

Reciprocal sub

$$\int \frac{(x+1)^4(x-1)^2\sqrt{x^4+x^2+1}}{(x^2+1)^2(x^6-1)}\,dx$$

I and J

How would you approach $\displaystyle\int \frac{\sin x}{\sin x + \cos x}\,dx$?

I and J

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$$\int \frac{\cos^3 x}{\sin x - \cos x}$$

I and J

CPMSOC

$$\int \sqrt{\cot x}\, dx$$

Cancelling integrals through by parts

We have all seen how in finding $\int e^x \sin x \, dx$ you have to integrate by parts twice to recover the original integral. This is somewhat similar.

$$\int \frac{x^2 - x - 1}{x^2 - 2x + 1} e^x \, dx$$

Hint: Consider splitting into

$$\int \frac{x^2 - x}{x^2 - 2x + 1} e^x \, dx - \int \frac{e^x}{x^2 - 2x + 1}$$

Random asspull manipulation

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$$\int \frac{x^{2023}}{x + x^{2025}}\,dx$$

Random asspull manipulation

$$\int \frac{dx}{x\sqrt{x^n - 1}}$$

Reverse product rule

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$$\int \frac{1+\sin x}{1+\cos x} e^x \, dx$$

Reverse quotient rule

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$$\int \frac{x\cos x - \sin x}{x^2 + \sin^2 x}\,dx$$

Kings rule

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$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Very commonly used. Especially for trig, where there is a special rule for it.

$$\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx$$

Kings rule

CPMSOC

$$\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\, dx$$

Kings rule

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$$\int_1^3 \frac{\ln x}{\ln(4x - x^2)}\,dx$$

Reciprocal sub

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$$\int_{\frac{1}{a}}^{a} f(x)\,dx = \int_{\frac{1}{a}}^{a} \frac{f(1/x)}{x^2}\,dx$$

Very useful when there are $\ln$s anywhere in the integral, or $1 + x^2$s in the denominator.

Reciprocal sub

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$$\int_0^\infty \frac{dx}{(1+x)(\pi^2+(\ln x)^2)}$$

Together now!

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If $\varphi = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio, evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\tan x}{\varphi^2 + 4\ln(\tan x)^2}\,dx$$

Involution

Notice that both of these substitutions took the bounds and changed them into the other bound. Specifically, $f(b) = a$ and $f(a) = b$, where $f(x) = a + b - x$ or $f(x) = 1/x$. These functions are called “involutions” because they satisfy $f(f(x)) = x$. Another involution is

$$f(x) = \frac{1-x}{1+x}.$$

Use $u = f(x)$ to evaluate

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx$$

Interval splitting

A trick which is usually only helpful for trig integrals, where cases would be required due to the invertible requirement for u sub:

$$\int_0^{2a} f(x)\,dx = \int_0^{a} (f(x) + f(2a - x))\,dx$$

In the case when $a = \pi$ or $a = \frac{\pi}{2}$, this has a couple applications.

Go practice.

Pizza should arrive soon

Attendance form :D

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