1

$$\|G(x) - G(x^*)\| \leq L\|x - x^*\|$$

$$\tilde{s_k} = G_*^{1/2} s_k, \quad \tilde{y_k} = G_*^{-1/2} y_k, \quad \tilde{B_k} = G_*^{-1/2} B_k G_*^{1/2},$$

$$cos\tilde{\theta_k} = \frac{\tilde{s_k}^T \tilde{B_k} \tilde{s_k}}{\|\tilde{s_k}\| \|\tilde{B_k} \tilde{s_k}}, \quad \tilde{q_k} = \frac{\tilde{s_k}^T \tilde{B_k} \tilde{s_k}}{\|\tilde{s_k}\|^2}$$

$$\tilde{M_k} = \frac{\|\tilde{y_k}\|^2}{\tilde{y_k}^T \tilde{s_k}}, \quad \tilde{m_k} = \frac{\tilde{y_k}^T \tilde{s_k}}{\tilde{s_k}^T \tilde{s_k}}.$$

$$\tilde{B_{k+1}} = \tilde{B_k} - \frac{\tilde{B_k} \tilde{s_k} \tilde{s_k}^T \tilde{B_k}}{\tilde{s_k}^T \tilde{B_k} \tilde{s_k}} + \frac{\tilde{y_k} \tilde{y_k}^T}{\tilde{y_k}^T \tilde{s_k}}.$$

$$\begin{aligned} \psi(\tilde{B_{k+1}}) = \psi(\tilde{B_k} + (\tilde{M_{k+1}} - \ln(\tilde{m_k}) - 1) \\ + \left[1 - \frac{\tilde{q_k}}{cos^2\tilde{\theta_k}} + \ln\left(\frac{\tilde{q_k}}{cos^2\tilde{\theta_k}}\right)\right] \\ + \ln\left(cos^2\tilde{\theta_k}\right) \end{aligned} \tag{1}$$

$$y_k - G_* s_k = (\tilde{G_k} - \tilde{G_*}) s_k,$$

$$\tilde{y_k} - \tilde{s_k} = G_*^{-1/2} (\tilde{G_k} - \tilde{G_*}) G_*^{-1/2} \tilde{s_k}.$$

$$\|\tilde{y_k} - \tilde{s_k}\| \leq \|G_*^{-1/2}\|^2 \|\tilde{s_k}\| \|\tilde{G_k} - \tilde{G_*}\| \leq \|G_*^{-1/2}\|^2 \|\tilde{s_k}\| L \epsilon_k,$$

$$\epsilon_k = max\{\|x_{k+1} - x^*\|, \|x_k - x^*\|\}.$$

$$\frac{\|\tilde{y_k} - \tilde{s_k}\|}{\|\tilde{s_k}\|} \leq \bar{c}\epsilon_k, \tag{2}$$

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$$\|\tilde{y_k} - \tilde{s_k}\| \leq \bar{c}\epsilon_k \|\tilde{s_k}\|, \quad \|\tilde{s_k} - \tilde{y_k}\| \leq \bar{c}\epsilon_k \|\tilde{s_k}\|$$

$$(1 - \bar{c}\epsilon_k)\|\tilde{s_k}\| \leq \|\tilde{y_k}\| \leq (1 + \bar{c}\epsilon_k)\|\tilde{s_k}\| \tag{3}$$

$$(1-\bar{c}\epsilon_k)^2\|\tilde{s_k}\|^2 - 2\tilde{y_k}^T\tilde{s_k} + \|\tilde{s_k}\|^2 \le \|\tilde{y_k}\|^2 - 2\tilde{y_k}^T\tilde{s_k} + \|\tilde{s_k}\|^2 \le \bar{c}^2\epsilon_k^2\|\tilde{s_k}\|^2,$$

$$2\tilde{y_k}^T\tilde{s_k} \ge (1-2\bar{c}\epsilon_k + \bar{c}^2\epsilon_k^2 + 1 - \bar{c}^2\epsilon_k^2)\|\tilde{s_k}\|^2 = 2(1-\bar{c}\epsilon_k)\|\tilde{s_k}\|^2$$

$$\tilde{m_k} = \frac{\tilde{y_k}^T\tilde{s_k}}{\|\tilde{s_k}\|^2} \ge 1-\bar{c}\epsilon_k \tag{4}$$

$$\tilde{M_k} = \frac{\|\tilde{y_k}\|^2}{\tilde{y_k}^T\tilde{s_k}} \le \frac{1+\bar{c}\epsilon_k}{1-\bar{c}\epsilon_k} \tag{5}$$

$$\tilde{M_k} \le 1 + \frac{2\bar{c}}{1-\bar{c}\epsilon_k}\epsilon_k \le 1+\bar{c}\epsilon_k \tag{6}$$

3

$$\frac{-x}{1-x} - \ln(1-x) = h\left(\frac{1}{1-x}\right) \le 0$$

$$\ln(1-\bar{c}\epsilon_k) \ge \frac{-\bar{c}\epsilon_k}{1-\bar{c}\epsilon_k} \ge -2\bar{c}\epsilon_k$$

$$\ln(\tilde{m_k}) \ge \ln(1-\bar{c}\epsilon_k) \ge -2\bar{c}\epsilon_k \ge -2c\epsilon_k \tag{7}$$

$$0 < \psi(\tilde{B_{k+1}}) \le \psi(\tilde{B_k}) + 3c\epsilon_k + \ln(cos^2\tilde{\theta_k}) + \left[1 - \frac{\tilde{q_k}}{cos^2\tilde{\theta_k}} + \ln\left(\frac{\tilde{q_k}}{cos^2\tilde{\theta_k}}\right)\right] \tag{8}$$

$$\sum_{j=0}^{\infty}\left(\ln\left(\frac{1}{cos^2\tilde{\theta_j}}\right) - \left[1 - \frac{\tilde{q_j}}{cos^2\tilde{\theta_j}} + \ln\left(\frac{\tilde{q_j}}{cos^2\tilde{\theta_j}}\right)\right]\right) \le \psi(\tilde{B_0}) + 3c\sum_{j=0}^{\infty}\epsilon_j < +\infty$$

$$\lim_{j\to\infty}\ln\left(\frac{1}{cos^2\tilde{\theta_j}}\right) = 0, \qquad \lim_{j\to\infty}\left(1 - \frac{\tilde{q_j}}{cos^2\tilde{\theta_j}} + \ln\left(\frac{\tilde{q_j}}{cos^2\tilde{\theta_j}}\right)\right) = 0$$

$$\lim_{j\to\infty} cos\tilde{\theta_j} = 1, \qquad \lim_{j\to\infty}\tilde{q_j} = 1 \tag{9}$$

4

$$
\begin{aligned}
\frac{\|G_*^{-1/2}(B_k - G_*)s_k\|^2}{\|G_*^{1/2}s_k\|} &= \frac{\|(\tilde{B}_k - I)\tilde{s}_k\|^2}{\|\tilde{s}_k\|^2} \\
&= \frac{\|\tilde{B}_k\tilde{s}_k\|^2 - 2\tilde{s}_k^T\tilde{B}_k\tilde{s}_k + \tilde{s}_k^T\tilde{s}_k}{\tilde{s}_k^T\tilde{s}_k} \\
&= \frac{\tilde{q}_k^2}{cos\tilde{\theta}_k^2} - 2\tilde{q}_k + 1.
\end{aligned}
$$

$$
\therefore \quad \lim_{k\to\infty} \frac{\|(B_k - G_*)s_k\|}{\|s_k\|} = 0
$$