22: MAP

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Maximum a Posteriori Estimator

Maximum Likelihood Estimator

Review

Consider a sample of n i.i.d. random variables $X_1, X_2, \dots, X_n$.

Maximum Likelihood Estimator (MLE)

What parameter θ **maximizes the likelihood** of our observed data $(X_1, X_2, \dots, X_n)$?

$$L(\theta) = f(X_1, X_2, \dots, X_n|\theta) = \prod_{i=1}^{n} f(X_i|\theta)$$

$$\theta_{MLE} = \arg\max_{\theta} \underbrace{f(X_1, X_2, \dots, X_n|\theta)}_{\text{likelihood of data}}$$

Observations:

- MLE determines θ value that maximizes the probability of observing the sample.
- If we're estimating θ, couldn't we just maximize the probability of θ?

Today: **Bayesian estimation** using the Bayesian definition of probability!

Maximum A Posteriori (MAP) Estimator

Not Review! New!

Consider a sample of n i.i.d. random variables $X_1, X_2, \dots, X_n$.

Maximum Likelihood Estimator (MLE)

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$$L(\theta) = f(X_1, X_2, \dots, X_n|\theta) = \prod_{i=1}^{n} f(X_i|\theta)$$

$$\theta_{MLE} = \arg\max_{\theta} f(X_1, X_2, \dots, X_n|\theta)$$

likelihood of data

Maximum a Posteriori (MAP) Estimator

Given the sample data $(X_1, X_2, \dots, X_n)$, what is the **most probable parameter** θ?

$$\theta_{MAP} = \arg\max_{\theta} f(\theta|X_1, X_2, \dots, X_n)$$

posterior distribution of θ

Maximum A Posteriori (MAP) Estimator

Consider a sample of n i.i.d. random variables $X_1, X_2, \ldots, X_n$.

def The **Maximum a Posteriori (MAP) Estimator** of θ is the value of θ that maximizes the **posterior** distribution of θ.

$$\theta_{MAP} = \arg\max_{\theta} f(\theta | X_1, X_2, \ldots, X_n)$$

Intuition with Bayes' Theorem:

$L(\theta)$, probability of data given parameter θ

After seeing data, posterior belief of θ

Before seeing data, prior belief of θ

$$\underbrace{P(\theta|\text{data})}_{\text{posterior}} = \frac{\overbrace{P(\text{data}|\theta)}^{\text{likelihood}}\overbrace{P(\theta)}^{\text{prior}}}{P(\text{data})}$$

Solving for θ_{MAP}

- Observe data: $X_1, X_2, \dots, X_n$, all i.i.d.
- Let likelihood be same as MLE: $f(X_1, X_2, \dots, X_n|\theta) = \prod_{i=1}^{n} f(X_i|\,\theta)$
- Let the prior distribution of θ be $g(\theta)$.

$$\theta_{MAP} = \arg\max_{\theta} f(\theta|X_1, X_2, \dots, X_n) = \arg\max_{\theta} \frac{f(X_1, X_2, \dots, X_n|\theta)g(\theta)}{h(X_1, X_2, \dots, X_n)}$$ (Bayes' Theorem)

$$= \arg\max_{\theta} \frac{g(\theta)\prod_{i=1}^{n} f(X_i|\,\theta)}{h(X_1, X_2, \dots, X_n)}$$ (independence)

$$= \arg\max_{\theta} g(\theta) \prod_{i=1}^{n} f(X_i|\,\theta)$$ ($1/h(X_1, X_2, \dots, X_n)$ is a positive constant w.r.t. θ)

$$= \arg\max_{\theta} \left(\log g(\theta) + \sum_{i=1}^{n} \log f(X_i|\,\theta) \right)$$

θ_{MAP}: Interpretation 1

- Observe data: $X_1, X_2, \ldots, X_n$, all i.i.d.
- Let likelihood be same as MLE: $f(X_1, X_2, \ldots, X_n|\theta) = \prod_{i=1}^{n} f(X_i|\,\theta)$
- Let the prior distribution of θ be $g(\theta)$.

$$\theta_{MAP} = \arg\max_{\theta} f(\theta|X_1, X_2, \ldots, X_n) = \arg\max_{\theta} \frac{f(X_1, X_2, \ldots, X_n|\theta)g(\theta)}{h(X_1, X_2, \ldots, X_n)}$$ (Bayes' Theorem)

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$$= \arg\max_{\theta} \left(\log g(\theta) + \sum_{i=1}^{n} \log f(X_i|\,\theta) \right)$$

θ_{MAP} maximizes
log prior + log-likelihood

θ_{MAP}: Interpretation 2

- Observe data: $X_1, X_2, \ldots, X_n$, all i.i.d.
- Let likelihood be same as MLE: $f(X_1, X_2, \ldots, X_n|\theta) = \prod_{i=1}^{n} f(X_i|\,\theta)$
- Let the prior distribution of θ be $g(\theta)$.

$$\theta_{MAP} = \arg\max_{\theta} f(\theta|X_1, X_2, \ldots, X_n) = \text{ar}\ldots$$

The mode of the posterior distribution of θ

(Bayes' Theorem)

$$= \arg\max_{\theta} \frac{g(\theta)\prod_{i=1}^{n} f(X_i|\,\theta)}{h(X_1, X_2, \ldots, X_n)}$$

(independence)

$$= \arg\max_{\theta} g(\theta) \prod_{i=1}^{n} f(X_i|\,\theta)$$

($1/h(X_1, X_2, \ldots, X_n)$ is a positive constant w.r.t. θ)

$$= \arg\max_{\theta} \left(\log g(\theta) + \sum_{i=1}^{n} \log f(X_i|\,\theta) \right)$$

θ_{MAP} maximizes log prior + log-likelihood

Mode: A statistic of a random variable

The mode of a random variable X is defined as:

(X discrete, PMF $p(x)$) $\arg\max_x p(x)$

$\arg\max_x f(x)$ (X continuous, PDF $f(x)$)

- Intuitively: The value of X that is "most likely".
- Note that some distributions may not have a unique mode (e.g., Uniform distribution, or Bernoulli(0.5))

$$\theta_{MAP} = \arg\max_\theta f(\theta | X_1, X_2, \ldots, X_n)$$

θ_{MAP} is the most likely θ given the data $X_1, X_2, \ldots, X_n$.

Bernoulli MAP: Choosing a prior

How does MAP work? (for Bernoulli)

Observe data n heads, m tails

Choose model Bernoulli(p)

Choose prior on θ (some $g(\theta)$)

Find $\theta_{MAP} =$
$\arg\max_{\theta} f(\theta | X_1, X_2, \dots, X_n)$

maximize
log prior + log-likelihood

$$\log g(\theta) + \sum_{i=1}^{n} \log f(X_i | \theta)$$

- Differentiate, set to 0
- Solve

MAP depends on what $g(\theta)$ we choose.

MAP for Bernoulli

- Flip a coin 8 times. Observe $n = 7$ heads and $m = 1$ tail.
- Choose a prior on θ. What is θ_{MAP}?

Suppose we pick a prior $\theta \sim \mathcal{N}(0.5, 1^2)$. $g(\theta) = \frac{1}{\sqrt{2\pi}} e^{-(p-0.5)^2/2}$

1. Determine log prior + log likelihood

$$\log g(\theta) + \log f(X_1, X_2, \dots, X_n | \theta)$$

$$= \log\left(\frac{1}{\sqrt{2\pi}} e^{-(p-0.5)^2/2}\right) + \log\left(\binom{n+m}{n} p^n (1-p)^m\right)$$

$$= -\log(\sqrt{2\pi}) - (p-0.5)^2/2 + \log\binom{n+m}{n} + n\log p + m\log(1-p)$$

2. Differentiate w.r.t. (each) θ, set to 0

$$-(p-0.5) + \frac{n}{p} - \frac{m}{1-p} = 0$$

We should choose a prior that's easier to deal with. This one is hard!

3. Solve resulting equations

cubic equations nope not doing it

A better approach: Use conjugate distributions

Observe data

n heads, m tails

Choose model

Bernoulli(p)

(choose conjugate distribution)

Choose prior on θ

(some $g(\theta)$)

Find $\theta_{MAP} = \arg\max_{\theta} f(\theta | X_1, X_2, \ldots, X_n)$

maximize
log prior + log-likelihood

$$\log g(\theta) + \sum_{i=1}^{n} \log f(X_i | \theta)$$

- Differentiate, set to 0
- Solve

⭐

Up next: Conjugate priors are great for MAP!

Bernoulli MAP: Conjugate prior

Beta is a conjugate distribution for Bernoulli

Beta is a **conjugate distribution** for Bernoulli, meaning:

- Prior and posterior parametric forms are the same
- Practically, conjugate means easy update:
  Add numbers of "successes" and "failures" seen to Beta parameters.
- You can set the prior to reflect how fair/biased you think the experiment is a priori.

Prior	$\text{Beta}(a = n_{imag} + 1, b = m_{imag} + 1)$
Experiment	Observe n successes and m failures
Posterior	$\text{Beta}\left(a = n_{imag} + n + 1, b = m_{imag} + m + 1\right)$

Mode of $\text{Beta}(a, b)$: $\dfrac{a-1}{a+b-2}$

(we'll prove this in a few minutes)

Beta parameters a, b are called **hyperparameters.**
Interpret $\text{Beta}(a, b)$: $a + b - 2$ trials,
of which $a - 1$ are successes

How does MAP work? (for Bernoulli)

Observe data

n heads, m tails

Choose model

Bernoulli(p)

Choose prior on θ

(some $g(\theta)$)

(choose conjugate distribution)

Find $\theta_{MAP} =$

$\arg\max_{\theta} f(\theta|X_1, X_2, \dots, X_n)$

maximize
log prior + log-likelihood

$$\log g(\theta) + \sum_{i=1}^{n} \log f(X_i|\theta)$$

- Differentiate, set to 0
- Solve

Mode of posterior distribution of θ

(posterior is also conjugate)

Conjugate strategy: MAP for Bernoulli

- Flip a coin 8 times. Observe $n = 7$ heads and $m = 1$ tail.
- Choose a prior on θ. What is θ_{MAP}?

Define as data, D

1. Choose a prior

Suppose we pick a prior $\theta \sim \text{Beta}(a, b)$.

2. Determine posterior

Because Beta is a conjugate distribution for Bernoulli, the posterior distribution is $\theta | D \sim \text{Beta}(a + n, b + m)$

3. Compute MAP

$$\theta_{MAP} = \frac{a + n - 1}{a + n + b + m - 2}$$

(mode of Beta$(a + n, b + m)$)

MAP in practice

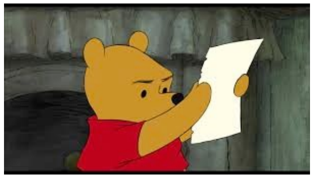
- Flip a coin 8 times. Observe $n = 7$ heads and $m = 1$ tail.
- What is the MAP estimator of the Bernoulli parameter p, if we assume a prior on p of Beta$(2, 2)$?

MAP in practice

- Flip a coin 8 times. Observe $n = 7$ heads and $m = 1$ tail.
- What is the MAP estimator of the Bernoulli parameter p, if we assume a prior on p of $\text{Beta}(2, 2)$?

1. Choose a prior

$\theta \sim \text{Beta}(2,2)$.



Before flipping the coin, we imagined 2 trials: 1 imaginary head, 1 imaginary tail.

2. Determine posterior

Posterior distribution of θ given observed data is $\text{Beta}(9, 3)$

3. Compute MAP

$$\theta_{MAP} = \frac{8}{10}$$

After the coin, we saw 10 trials: 8 heads (imaginary and real), 2 tails (imaginary and real).

Proving the mode of Beta

Observe data $\quad n$ heads, m tails

Choose model $\quad$ Bernoulli(p)

Choose prior on θ $\quad$ (some $g(\theta)$) $\quad$ (choose conjugate) Beta(a, b)

Find $\theta_{MAP} = \arg\max_{\theta} f(\theta | X_1, X_2, \dots, X_n)$

maximize
log prior + log-likelihood

$$\log g(\theta) + \sum_{i=1}^{n} \log f(X_i | \theta)$$

- Differentiate, set to 0
- Solve

Mode of posterior distribution of θ

(posterior is also conjugate)

These are equivalent interpretations of θ_{MAP}. We'll use this equivalence to prove the mode of Beta.

From first principles: MAP for Bernoulli, conjugate prior

- Flip a coin 8 times. Observe $n = 7$ heads and $m = 1$ tail.
- Choose a prior on θ. What is θ_{MAP}?

Suppose we pick a prior $\theta \sim \text{Beta}(a, b)$. $g(\theta = p) = \frac{1}{\beta} p^{a-1}(1-p)^{b-1}$

normalizing constant, β

1. Determine log prior + log likelihood

$$\log g(\theta) + \log f(X_1, X_2, \dots, X_n | \theta) = \log\left(\frac{1}{\beta} p^{a-1}(1-p)^{b-1}\right) + \log\left(\binom{n+m}{n} p^n (1-p)^m\right)$$

$$= \log\frac{1}{\beta} + (a-1)\log(p) + (b-1)\log(1-p) + \log\binom{n+m}{n} + n\log p + m\log(1-p)$$

2. Differentiate w.r.t. (each) θ, set to 0

$$\frac{a-1}{p} + \frac{n}{p} - \frac{b-1}{1-p} - \frac{m}{1-p} = 0$$

3. Solve

(next slide)

From first principles: MAP for Bernoulli, conjugate prior

- Flip a coin 8 times. Observe $n = 7$ heads and $m = 1$ tail.
- Choose a prior θ. What is θ_{MAP}?

Suppose we pick a prior $\theta \sim \text{Beta}(a, b)$. $g(\theta) = \frac{1}{\beta} p^{a-1}(1-p)^{b-1}$

normalizing constant, β

3. Solve for p

$$\frac{a-1}{p} + \frac{n}{p} - \frac{b-1}{1-p} - \frac{m}{1-p} = 0$$

(from previous slide)

$$\Longrightarrow \frac{a+n-1}{p} - \frac{b+m-1}{1-p} = 0$$

$$\Longrightarrow (a+n-1) - (a+n-1)p = (b+m-1)p$$

$$\Longrightarrow p(a+n+b+m-2) = a+n-1$$

$$\theta_{MAP} = \frac{a+n-1}{a+n+b+m-2}$$ ✅

The mode of the posterior, Beta$(a+n, b+m)$!

If we choose a conjugate prior, we avoid calculus with MAP: just report mode of posterior.

Conjugate distributions

Conjugate distributions

MAP estimator:

$$\theta_{MAP} = \arg\max_{\theta} f(\theta | X_1, X_2, \ldots, X_n)$$

The mode of the posterior distribution of θ

Distribution parameter	Conjugate distribution
Bernoulli p	Beta
Binomial p	Beta
Multinomial p_i	Dirichlet
Poisson λ	Gamma
Exponential λ	Gamma
Normal μ	Normal
Normal σ^2	Inverse Gamma

Don't need to know Inverse Gamma…
but it will know you ☺

CS109: We'll only focus on MAP for Bernoulli/Binomial p, Multinomial p_i, and Poisson λ.

Multinomial is Multiple times the fun

Dirichlet$(a_1, a_2, \dots, a_m)$ is a conjugate for Multinomial.

- Generalizes Beta in the same way Multinomial generalizes Binomial:

$$f(x_1, x_2, \dots, x_m) = \frac{1}{B(a_1, a_2, \dots, a_m)} \prod_{i=1}^{m} x_i^{a_i - 1}$$

Prior Dirichlet$(a_1, a_2, \dots, a_m)$
Saw $\left(\sum_{i=1}^{m} a_i\right) - m$ imaginary trials, with $a_i - 1$ of outcome i

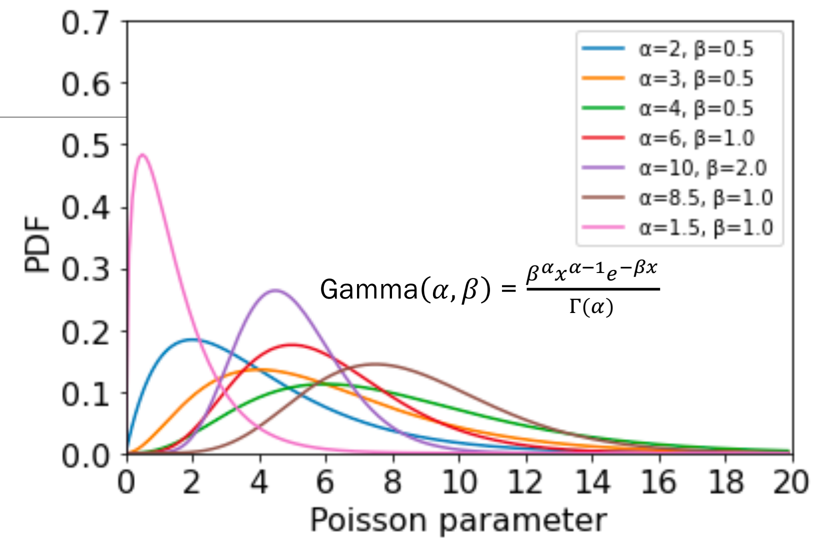
Experiment Observe $n_1 + n_2 + \cdots + n_m$ new trials, with n_i of outcome i

Posterior Dirichlet$(a_1 + n_1, a_2 + n_2, \dots, a_m + n_m)$

MAP:

$$p_i = \frac{a_i + n_i - 1}{\left(\sum_{i=1}^{m} a_i\right) + \left(\sum_{i=1}^{m} n_i\right) - m}$$

Good times with Gamma



Gamma(α, β) is a conjugate for Poisson.

- Also conjugate for Exponential, but we won't delve into that
- Mode of gamma: $(\alpha - 1)/\beta$

Prior $\theta \sim \text{Gamma}(\alpha, \beta) = \dfrac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$

Saw $\alpha - 1$ total imaginary events during β prior time periods

Experiment Observe n events during next k time periods

Posterior $(\theta | n \text{ events in } k \text{ periods}) \sim \text{Gamma}(\alpha + n, \beta + k)$

MAP: $$\theta_{MAP} = \frac{a + n - 1}{\beta + k}$$

MAP for Poisson

Gamma(α, β) is conjugate for Poisson

Mode: $\frac{\alpha-1}{\beta}$

Let λ be the average # of successes in a time period.

1. What does it mean to have a prior of $\theta \sim \text{Gamma}(11{,}5)$?

Observe 10 imaginary events in 5 time periods, i.e., observe at Poisson rate = 2

Now perform the experiment and see 11 events in next 2 time periods.

2. Given your prior, what is the posterior distribution?

3. What is θ_{MAP}?

MAP for Poisson

Gamma(α, β) is conjugate for Poisson   Mode: $\frac{\alpha-1}{\beta}$

Let λ be the average # of successes in a time period.

1. What does it mean to have a prior of $\theta \sim \text{Gamma}(11,5)$?

Observe 10 imaginary events in 5 time periods, i.e., observe at Poisson rate = 2

Now perform the experiment and see 11 events in next 2 time periods.

2. Given your prior, what is the posterior distribution?

$(\theta | n \text{ events in } k \text{ periods}) \sim \text{Gamma}(22, 7)$

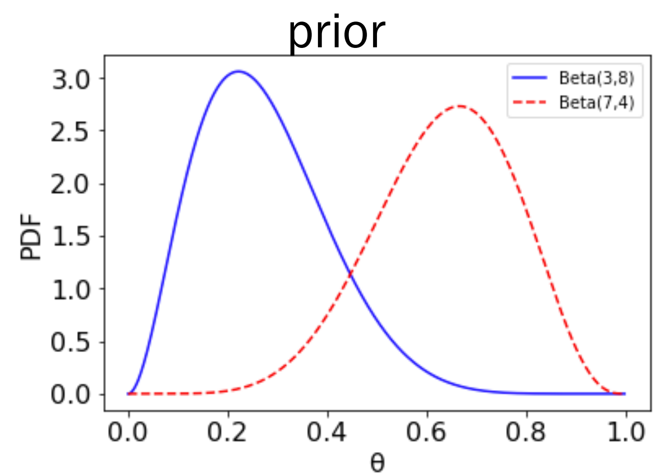
3. What is θ_{MAP}?

$\theta_{MAP} = 3$, the updated Poisson rate

Choosing hyperparameters for conjugate prior

Where'd you get them priors?

- Let θ be the probability a coin turns up heads.
- Model θ with 2 different priors:
  - Prior 1: Beta(3,8): 2 imaginary heads, 7 imaginary tails — mode: $\frac{2}{9}$
  - Prior 2: Beta(7,4): 6 imaginary heads, 3 imaginary tails — mode: $\frac{6}{9}$



Now flip 100 coins and get 58 heads and 42 tails.

1. What are the two posterior distributions?
2. What are the modes of the two posterior distributions?

Where'd you get them priors?

- Let θ be the probability a coin turns up heads.
- Model θ with 2 different priors:
  - Prior 1: Beta(3,8): 2 imaginary heads, 7 imaginary tails — mode: $\frac{2}{9}$
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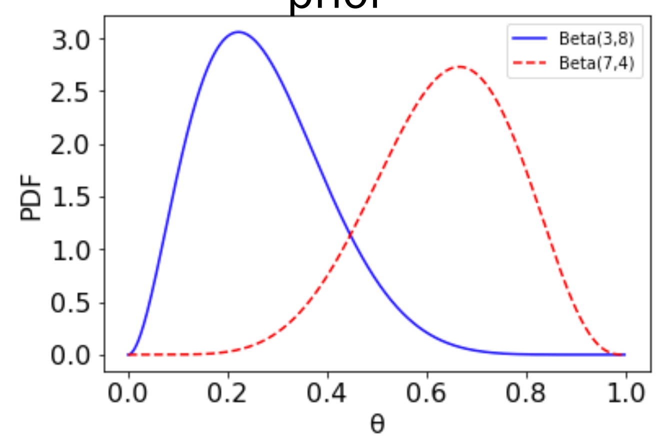
prior

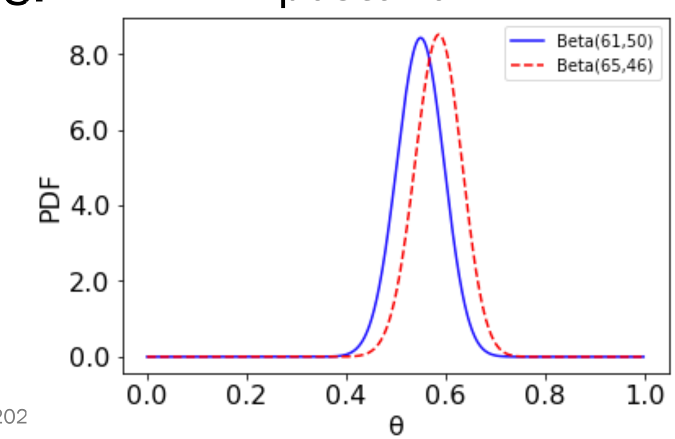


Now flip 100 coins and get 58 heads and 42 tails.

Posterior 1: Beta(61,50) mode: $\frac{60}{109}$

Posterior 2: Beta(65,46) mode: $\frac{64}{109}$

posterior



Provided we collect enough data, posteriors will converge to the true value and choice of priors will matter less and less.

Laplace smoothing

MAP with Laplace smoothing: a prior which represents k imagined observations of each outcome.

- Categorical data (i.e., Multinomial, Bernoulli/Binomial)
- Also known as additive smoothing

Laplace estimate Imagine $k = 1$ of each outcome
(follows from Laplace's "law of succession")

Example: Laplace estimate for coin probabilities from aforementioned experiment (100 coins: 58 heads, 42 tails)

heads $\frac{59}{102}$ tails $\frac{43}{102}$

Laplace smoothing:
- Easy to implement/remember

Back to our happy Laplace

Consider our previous 6-sided die.

- Roll the dice $n = 12$ times.
- Observe: 3 ones, 2 twos, 0 threes, 3 fours, 1 fives, 3 sixes

Recall θ_{MLE}: $p_1 = 3/12, p_2 = 2/12, p_3 = 0/12,$
$p_4 = 3/12, p_5 = 1/12, p_6 = 3/12$

What are your Laplace estimates for each roll outcome?

Back to our happy Laplace

Consider our previous 6-sided die.

- Roll the dice $n = 12$ times.
- Observe: 3 ones, 2 twos, 0 threes, 3 fours, 1 fives, 3 sixes

Recall θ_{MLE}: $p_1 = 3/12, p_2 = 2/12, p_3 = 0/12,$ ⚠️
$p_4 = 3/12, p_5 = 1/12, p_6 = 3/12$

What are your Laplace estimates for each roll outcome?

$$p_i = \frac{X_i + 1}{n + m}$$

$p_1 = 4/18, p_2 = 3/18, p_3 = 1/18,$ ✅
$p_4 = 4/18, p_5 = 2/18, p_6 = 4/18$

Laplace smoothing:
- Easy to implement/remember
- **Avoids estimating a parameter of 0**