Adaptive Fuzzy Double C-Means Clustering via Knowledge Transferring Latent Factor Analysis

I. Introduction

This is the supplementary file for the paper entitled "Adaptive Fuzzy Double C-Means Clustering via Knowledge Transfering Latent Factor Analysis" in IEEE Transactions on Industrial Informatics, which includes proofs of lemmas, proposed algorithms and experimental figures and tables cited by the paper. Specifically, Section 2 proves two important lemmas such as Lemma 1 and Lemma 2. Section 3 presents 3 algorithms and experimental figures and tables, as follows:

(a) Proofs of Lemma 1 and Lemma 2;
(b) Algorithm 1 NRNLF_KT;
(c) Algorithm 2 AFD_SS;
(d) Algorithm 3 NA;
(e) Figure of parameter optimization for datasets with different rates of missing values on NLF model. (i.e., 30% and 80%);
(f) Figure of recovery results for BSDS500 with 30% missing rate by using different image inpainting methods;
(g) Figure of recovery results for BSDS500 with 30% missing rate by using different clustering methods;
(h) Figure of NMIs for D1 with 30% and 80% missing values on AFD_SS model;
(i) Tables of performances of imputation algorithms for 8 datasets with 30% and 80% missing;
(j) Table of p-values for different datasets with different missing;
(k) Table of NMAE±STD for BSDS500;
(l) Table of parameter setting on 8 datasets;
(m) Table of performance of three clustering algorithms for 8 datasets;
(n) Table of performance of three clustering algorithms for 8 datasets with 30% and 80% missing;
(o) Table of segmentation accuracy for different estimation methods.

II. Supplementary Proofs of Lemma 1 and Lemma 2

A. Proof of Lemma 1

Proof: For $y_{m(a)}$ and $y_{m(b)} \in \mathcal{R}$, $\forall a, b \in \{1, 2, 3, \cdots, d\}$, we have

$$\begin{aligned}
&\nabla \varsigma_{m,n}(y_{m(a)}) - \nabla \varsigma_{m,n}(y_{m(b)}) \\
=&\frac{\partial \varsigma_{m,n}(y_{m(a)})}{\partial f(y_{m(a)})} f^{'}(y_{m(a)}) - \frac{\partial \varsigma_{m,n}(y_{m(b)})}{\partial f(y_{m(b)})} f^{'}(y_{m(b)}) \\
=&e_{m,n}(L_3 f^{'}(y_{m(b)}) - L_1 f'(y_{m(a)})) \\
&+ \lambda \tilde{e}_{m,n}(L_4 f'(y_{m(b)}) - L_2 f'(y_{m(a)}))
\end{aligned} \tag{1}$$

where $L_1 = \sum_{i=1}^{d}[(f(y_{a(i)}) + f(z_{a(i)}))f(y_{n(i)})]$, $L_2 = \sum_{i=1}^{d}[(f(y_{a(i)}) + f(\tilde{z}_{a(i)}))f(y_{n(i)})]$, $L_3 = \sum_{i=1}^{d}[(f(y_{b(i)}) + f(z_{b(i)}))f(y_{n(i)})]$, and $L_4 = \sum_{i=1}^{d}[(f(y_{b(i)}) + f(\tilde{z}_{b(i)}))f(y_{n(i)})]$.

Denoting $\lceil \cdot \rceil$ as an operation to round up "$\cdot$" to an integer, it can be stemmed from (1) that

$$\begin{aligned}
&||\nabla \varsigma_{m,n}(y_{m(a)}) - \nabla \varsigma_{m,n}(y_{m(b)})||_2 \\
\leq&||e_{m,n}((L_3 - L_1 + L_1) f^{'}(y_{m(b)}) - L_1 f'(y_{m(a)}))||_2 \\
&+ ||\lambda \tilde{e}_{m,n}((L_4 - L_2 + L_2) f'(y_{m(b)}) - L_2 f'(y_{m(a)}))||_2 \\
\leq&\Bigg(||e_{m,n} L_1||_2 + ||\lambda \tilde{e}_{m,n} L_2||_2 + \lceil \frac{||e_{m,n}(L_3 - L_1) f^{'}(y_{m(b)})||_2}{||f^{'}(y_{m(b)}) - f^{'}(y_{m(a)})||_2} \rceil \\
&+ \lceil \frac{||\lambda \tilde{e}_{m,n}(L_4 - L_2) f^{'}(y_{m(b)})||_2}{||f^{'}(y_{m(b)}) - f^{'}(y_{m(a)})||_2} \rceil \Bigg) ||f^{'}(y_{m(b)}) - f^{'}(y_{m(a)})||_2 \\
\leq&\Bigg(\varrho_{m,n}^{up} + \lceil \frac{||e_{m,n}^{up}(L_3 - L_1) f^{'}(y_{m(b)})||_2}{||f^{'}(y_{m(b)}) - f^{'}(y_{m(a)})||_2} \rceil \\
&+ \lceil \frac{||\lambda \tilde{e}_{m,n}^{up}(L_4 - L_2) f^{'}(y_{m(b)})||_2}{||f^{'}(y_{m(b)}) - f^{'}(y_{m(a)})||_2} \rceil \Bigg) ||f^{'}(y_{m(b)}) - f^{'}(y_{m(a)})||_2
\end{aligned} \tag{2}$$

where $\varrho_{m,n}^{up}=||e_{m,n}^{up}L_1||_2+||\lambda\tilde{e}_{m,n}^{up}L_2||_2$, $e_{m,n}^{up}=r_{m,n}-p_{m,n}^{up}$, $\tilde{e}_{m,n}^{up}=\tilde{r}_{m,n}-\tilde{p}_{m,n}^{up}$, $p_{m,n}^{up}=\sum_{k\neq a,b}^{d}\sum_{i=1}^{d}f(y_{m(k)})(f(w_{k(i)})+f(z_{k(i)}))f(y_{n(i)})$
$+2\sum_{i=1}^{d}N(f(w_{k(i)})+f(z_{k(i)}))f(y_{n(i)})$, $\tilde{p}_{m,n}^{up}=\sum_{k\neq a,b}^{d}\sum_{i=1}^{d}f(y_{m(k)})(f(w_{k(i)})+f(\tilde{z}_{k(i)}))f(y_{n(i)})+2\sum_{i=1}^{d}N(f(w_{k(i)})+f(\tilde{z}_{k(i)}))f(y_{n(i)})$, and $N=\max(f(y_{m(k)}))$, $(m,n)\in\Lambda, k\in 1,2,3,\cdots,d$.

It can be easily inferred from (2) that the L-smooth of $\varepsilon_{m,n}$ is determined by the L-smooth of $f(\cdot)$. Fortunately, like Sigmoid function, Gaussian function, and so on, the LF-dependent mapping function $f(\cdot)$ easily satisfies the property of L-smooth. Thus, there exists a positive constant M for $f(\cdot)$ satisfying

$$||f^{'}(y_{m(a)})-f^{'}(y_{m(b)})||_2\leq M||y_{m(a)}-y_{m(b)}||_2. \tag{3}$$

Based on (2) and (3), we have

$$\begin{aligned}&||\nabla\varepsilon_{m,n}(y_{m(a)})-\nabla\varepsilon_{m,n}(y_{m(b)})||_2\\ \leq&M\Bigg(\varrho_{m,n}^{up}+\lceil\frac{||e_{m,n}^{up}(L_3-L_1)f^{'}(y_{m(b)})||_2}{||f^{'}(y_{m(b)})-f^{'}(y_{m(a)})||_2}\rceil\\ &+\lceil\frac{||\lambda\tilde{e}_{m,n}^{up}(L_4-L_2)f^{'}(y_{m(b)})||_2}{||f^{'}(y_{m(b)})-f^{'}(y_{m(a)})||_2}\rceil\Bigg)||y_{m(a)}-y_{m(b)}||_2\\ =&L||y_{m(a)}-y_{m(b)}||_2\end{aligned} \tag{4}$$

where $\varrho_{m,n}^{up}$ is the same as defined in (2), $L\triangleq M(||e_{m,n}^{up}L_1||_2+||\lambda\tilde{e}_{m,n}^{up}L_2||_2+\lceil\frac{||e_{m,n}^{up}(L_3-L_1)f^{'}(y_{m(b)})||_2}{||f^{'}(y_{m(b)})-f^{'}(y_{m(a)})||_2}\rceil+\lceil\frac{||\lambda\tilde{e}_{m,n}^{up}(L_4-L_2)f^{'}(y_{m(b)})||_2}{||f^{'}(y_{m(b)})-f^{'}(y_{m(a)})||_2}\rceil)$. According to Definition 2, $\varsigma_{m,n}$ is L-smooth. Thus, the proof is complete. ■

B. Proof of Lemma 2

Proof: For $\forall y_{m(a)}$ and $y_{m(b)}\in\mathcal{R}$, we consider the second-order Taylor expansion of $\varsigma_{m,n}(y_{m(a)})$ at $y_{m(b)}$, given by

$$\begin{aligned}\varsigma_{m,n}(y_{m(a)})\approx&\varsigma_{m,n}(y_{m(b)})+\nabla\varsigma_{m,n}(y_{m(b)})(y_{m(a)}-y_{m(b)})\\ &+\frac{1}{2}\nabla^2\varsigma_{m,n}(y_{m(b)})(y_{m(a)}-y_{m(b)})^2,\end{aligned} \tag{5}$$

which implies

$$\begin{aligned}\varsigma_{m,n}(y_{m(a)})-\varsigma_{m,n}(y_{m(b)})\approx&\nabla\varsigma_{m,n}(y_{m(b)})(y_{m(a)}-y_{m(b)})\\ &+\frac{1}{2}\nabla^2\varsigma_{m,n}(y_{m(b)})(y_{m(a)}-y_{m(b)})^2.\end{aligned} \tag{6}$$

Letting $\delta=\min(\nabla^2\varsigma_{m,n}(y_{m(k)}))$, we can obtain from (6) that

$$\begin{aligned}&\varsigma_{m,n}(y_{m(a)})-\varsigma_{m,n}(y_{m(b)})\\ \geq&\nabla\varsigma_{m,n}(y_{m(b)})(y_{m(a)}-y_{m(b)})+\frac{\delta}{2}(y_{m(a)}-y_{m(b)})^2.\end{aligned} \tag{7}$$

Based on Definition 3, it can be easily seen from (7) that $\varsigma_{m,n}$ has strong convexity, which completes the proof. ■

III. SUPPLEMENTARY ALGORITHMS, FIGURES AND TABLES

REFERENCES

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Algorithm 1 NRNLF_KT

Input: Λ, M, N, R, $\widetilde{R}$, d

Operation	Cost
Initialize $X^{(\|M\|+\|N\|)\times d}=0$, $(W^o)^{d\times d}=0$, $(Z^o)^{d\times d}=0$, $(\tilde{Z}^o)^{d\times d}=0$, $Y^{(\|M\|+\|N\|)\times d}$, $Z^{d\times d}$, $\tilde{Z}^{d\times d}$ and $W^{d\times d}$ at random	$\Omega\left((\|M\|+\|N\|+3d)\times d\right)$
Initialize $A^{\|M\|\times d}$, $\widetilde{A}^{\|M\|\times d}$	$\Omega(\|M\|\times d)$
Initialize $B^{\|N\|\times d}$, $\widetilde{B}^{\|N\|\times d}$	$\Omega(\|N\|\times d)$
Initialize $\eta, \lambda, t_1=1$, maxsteps=$\mathfrak{n}$	$\Omega(1)$
while not converge and $t_1 \leq \mathfrak{n}$	$\times\mathfrak{n}$
set A, $\widetilde{A}$with zeroes	$\Omega(\|M\|\times d)$
set B, $\widetilde{B}$with zeroes	$\Omega(\|N\|\times d)$
for each $r_{m,n}$ in Λ	$\times\|\Lambda\|$
$p_{m,n}=\sum_{k=1}^{d}\sum_{i=1}^{d} f\left(y_{m(k)}\right)\left(f\left(w_{k(i)}\right)+f\left(z_{k(i)}\right)\right) f\left(y_{n(i)}\right)$	$\Omega(d\times d)$
$\widetilde{p}_{m,n}=\sum_{k=1}^{d}\sum_{i=1}^{d} f\left(y_{m(k)}\right)\left(f\left(w_{k(i)}\right)+f\left(\widetilde{z}_{k(i)}\right)\right) f\left(y_{n(i)}\right)$	$\Omega(d\times d)$
for k= 1 to d	$\times d$
$b_{(n)i}=\sum_{i=1}^{d}\left(f\left(w_{k(i)}\right)+f\left(z_{k(i)}\right)\right) f\left(y_{n(i)}\right)$	$\Omega(d)$
$\widetilde{b}_{n(i)}=\sum_{i=1}^{d}\left(f\left(w_{k(i)}\right)+f\left(\widetilde{z}_{k(i)}\right)\right) f\left(y_{n(i)}\right)$	$\Omega(d)$
$y_{(n)i} \leftarrow y_{m(k)}+\eta f'\left(y_{m(k)}\right)\left(e_{m,n} b_{n(i)}+\lambda \widetilde{e}_{m,n} \widetilde{b}_{(n)i}\right)$	$\Omega(1)$
end for	
for i= 1 to d	$\times d$
$a_{m(k)}=\sum_{k=1}^{d} f\left(y_{(m)k}\right)\left(f\left(w_{k(i)}\right)+f\left(z_{k(i)}\right)\right)$	$\Omega(d)$
$\widetilde{a}_{m(k)}=\sum_{k=1}^{d} f\left(y_{m(k)}\right)\left(f\left(w_{k(i)}\right)+f\left(\widetilde{z}_{k(i)}\right)\right)$	$\Omega(d)$
$y_{m(k)} \leftarrow y_{(n)i}+\eta f'\left(y_{(n)i}\right)\left(e_{m,n} a_{m(k)}+\lambda \widetilde{e}_{m,n} \widetilde{a}_{m(k)}\right)$	$\Omega(1)$
for k= 1 to d	$\times d$
$w_{k(i)} \leftarrow w_{k(i)}+\eta\left(e_{m,n}+\lambda \widetilde{e}_{m,n}\right) f\left(y_{m(k)}\right) f\left(y_{n(i)}\right) f'\left(w_{k(i)}\right)$	$\Omega(1)$
$z_{(k)i} \leftarrow z_{(k)i}+\eta e_{m,n} f\left(y_{m(k)}\right) f\left(y_{n(i)}\right) f'\left(z_{k(i)}\right)$	$\Omega(1)$
$\widetilde{z}_{k(i)} \leftarrow \widetilde{z}_{k(i)}+\eta\lambda \widetilde{e}_{m,n} f\left(y_{m(k)}\right) f\left(y_{n(i)}\right) f'\left(\widetilde{z}_{k(i)}\right)$	$\Omega(1)$
end for; end for	
end for	
for k= 1 to d	$\times d$
for i= 1 to d	$\times d$
$w_{k(i)}^{o}=f\left(w_{k(i)}\right)$	$\Omega(1)$
$z_{k(i)}^{o}=f\left(z_{k(i)}\right)$	$\Omega(1)$
$\widetilde{z}_{k(i)}^{o}=f\left(\widetilde{z}_{k(i)}\right)$	$\Omega(1)$
for m= 1 to M	$\times\|M\|$
$x_{m(k)}=f\left(y_{m(k)}\right)$	$\Omega(1)$
for n= 1 to N	$\times\|N\|$
$x_{n(i)}=f\left(y_{n(i)}\right)$	$\Omega(1)$
end for; end for; end for	
end for	
$t_1=t_1+1$	$\Omega(1)$
end while	

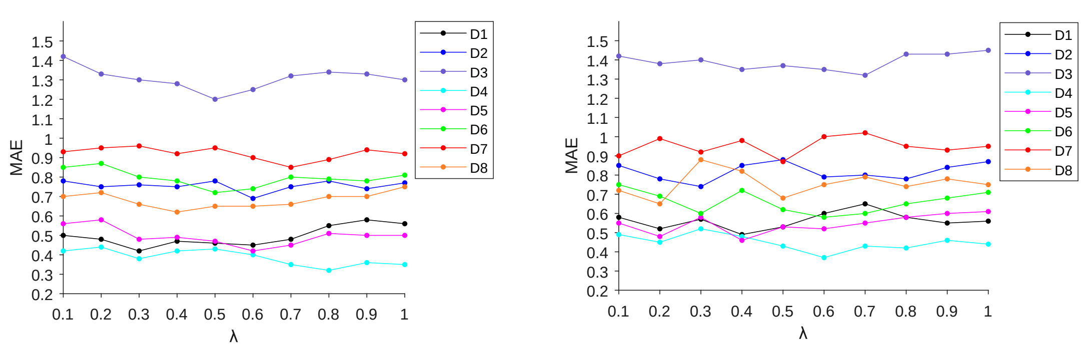
Output: non-negative latent factor matrices P, C, Q, H and $\widetilde{H}$

Algorithm 2 AFD_SS

Input: The set of samples $X = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)^T$,
the quantity of clusters C,
the quantity of features S,
fuzzy coefficient $\mathbf{m}$

Operation	Cost
Initialize the membership degree matrix U at random	$\Omega(N \times C)$
Initialize swarm size L, dimension D, $t_2 = 1$, maxsteps $\mathfrak{n}$	$\Omega(1)$
Initialize $r_1, r_2, c_1, c_2, w, gb$	$\Omega(1)$
Initialize pb^L	$\Omega(L)$
Initialize $O^{L \times D}$	$\Omega(L \times D)$
for each $l \in L$	$\times L$
if $f(\alpha_l) > pb_l$	$\Omega(1)$
$pb_l = f(\alpha_l)$	$\Omega(1)$
end if	
if $f(\alpha_l) > gb$	$\Omega(1)$
$gb = f(\alpha_l)$	$\Omega(1)$
end if	
end for	
for each $l \in L$	$\times L$
for each $d \in D$	$\times D$
$v_{l,d} = wv_{l,d} + c_1 r_1 (pb_{l,d} - \alpha_{l,d}) + c_2 r_2 (gb - \alpha_{l,d})$	$\Omega(1)$
$\alpha_{l,d} = \alpha_{l,d} + v_{l,d}$	$\Omega(1)$
end for	
end for	
while not converge and $t_2 \leq \mathfrak{n}$	$\times \mathfrak{n}$
obtain the discriminate feature set Z by solving SS model defined in (2)	$\Omega(N \times N)$
calculate the clustering centers $\mathbf{o}_i$ according to (27)	$\Omega(N \times S \times C)$
calculate the clustering centers $\hat{\mathbf{o}}_i$ according to (27)	$\Omega(N \times C)$
update the membership degree matrix U by using (27)	$\Omega(N \times C^2 \times S)$
$t_2 = t_2 + 1$	$\Omega(1)$
end while	

Output: the membership degree matrix U



(a) (b)

Fig. S1. Parameter optimization of datasets with different rates of missing values on NLF model. (a)30% and (b)80%.

Algorithm 3 NA

Input: The set of samples $X = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)^T$,
the quantity of clusters C,
the quantity of features S,
fuzzy coefficient $\mathbf{m}$

Operation	Cost
Initialize the membership degree matrix U at random	$\Omega(N \times C)$
Initialize swarm size L, dimension D, maxsteps $\mathfrak{n}$	$\Omega(1)$
Initialize $r_1, r_2, c_1, c_2, w, gb$	$\Omega(1)$
Initialize pb^L	$\Omega(L)$
Initialize $O^{L \times D}$	$\Omega(L \times D)$
calculate complete data set X by solving the NRNLF_KT model defined in (5)	
$t = t + 1$	
end while	
while not converge and $t \leq \mathfrak{n}$	$\times\mathfrak{n}$
obtain the discriminate feature set Z by solving SS model defined in (2)	$\Omega(N \times N)$
calculate the clustering centers $\mathbf{o}_i$ according to (27)	$\Omega(N \times S \times C)$
calculate the clustering centers $\hat{\mathbf{o}}_i$ according to (27)	$\Omega(N \times C)$
update the membership degree matrix U by using (27)	$\Omega(N \times C^2 \times S)$
for each $l \in L$	$\times L$
if $f(\alpha_l) > pb_l$	$\Omega(1)$
$pb_i = f(\alpha_l)$	$\Omega(1)$
end if	
if $f(\alpha_l) > gb$	$\Omega(1)$
$gb = f(\alpha_l)$	$\Omega(1)$
end if	
end for	
for each $l \in L$	$\times L$
for each $d \in D$	$\times D$
update the $v_{l,d}$ and $\alpha_{l,d}$ by using (25)	$\Omega(1)$
end for	
end for	
$t = t + 1$	$\Omega(1)$
end while	

Output: the membership degree matrix U

TABLE S1
PERFORMANCES OF IMPUTATION ALGORITHMS FOR DIFFERENT DATASETS WITH 30% MISSING

	Matric	Mean	KNN [1]	UNLFA [2]	TF-SNLF [6]	LSCF [3]	NRNLF_KT
D1	M1	0.8603±0.0012	0.7591±0.0016	0.6248±0.0042	0.6168±0.0011	0.5622±0.0011	**0.4692±0.0016**
	M2	1.1603±0.0018	1.1030±0.0013	0.9811±0.0063	0.9665±0.0008	0.8602±0.0016	**0.7718±0.0022**
D2	M1	0.9855±0.0011	0.9943±0.0008	0.8583±0.0011	0.8664±0.0016	0.7816±0.0009	**0.6986±0.0005**
	M2	1.2516±0.0020	1.2990±0.0006	1.1566±0.0051	1.1627±0.0017	1.0679±0.0005	**0.9842±0.0016**
D3	M1	1.7346±0.0014	1.6714±0.0017	1.5046±0.0068	1.4170±0.0021	1.3283±0.0008	**1.2051±0.0014**
	M2	1.9822±0.0019	1.9522±0.0022	1.7700±0.0036	1.6762±0.0019	1.5721±0.0019	**1.4508±0.0022**
D4	M1	0.7448±0.0008	0.6932±0.0016	0.6474±0.0045	0.6029±0.0066	0.4392±0.0013	**0.3268±0.0010**
	M2	0.9105±0.0013	0.8912±0.0023	0.8646±0.0022	0.8365±0.0025	0.5829±0.0017	**0.5108±0.0008**
D5	M1	0.6324±0.0017	0.5744±0.0019	0.5349±0.0016	0.5474±0.0048	0.4624±0.0013	**0.4215±0.0002**
	M2	0.9983±0.0012	0.8577±0.0013	0.7815±0.0015	0.7930±0.0006	0.7595±0.0006	**0.7157±0.0006**
D6	M1	0.9014±0.0005	0.9219±0.0022	0.7830±0.0036	**0.7133±0.0010**	0.7485±0.0022	0.7224±0.0011
	M2	1.1354±0.0015	1.1501±0.0025	0.9269±0.0019	**0.8211±0.0013**	0.8744±0.0018	0.8315±0.0013
D7	M1	1.3915±0.0011	1.2771±0.0017	1.1344±0.0056	1.1100±0.0008	0.9810±0.0011	**0.8577±0.0005**
	M2	1.7025±0.0019	1.5713±0.0013	1.4315±0.0023	1.3912±0.0019	1.1860±0.0026	**1.0977±0.0004**
D8	M1	0.8937±0.0006	0.8570±0.0024	0.7620±0.0016	0.7752±0.0026	0.6878±0.0020	**0.6254±0.0005**
	M2	1.1286±0.0018	1.0663±0.0013	0.9472±0.0013	0.9695±0.0013	0.9136±0.0023	**0.8797±0.0011**

TABLE S2
PERFORMANCES OF IMPUTATION ALGORITHMS FOR DIFFERENT DATASETS WITH 80% MISSING

	Matric	Mean	KNN	UNLFA	TF-SNLF	LSCF	NRNLF_KT
D1	M1	0.8746±0.0003	0.7910±0.0011	0.7248±0.0012	0.7040±0.0012	0.6280±0.0005	**0.4997±0.0005**
	M2	1.1824±0.0005	1.1259±0.0013	1.0556±0.0008	1.0328±0.0003	0.9329±0.0011	**0.7929±0.0002**
D2	M1	0.9902±0.0010	1.0483±0.0008	0.9318±0.0016	0.9264±0.0011	0.8639±0.0013	**0.7483±0.0009**
	M2	1.2753±0.0009	1.3452±0.0020	1.2157±0.0025	1.2372±0.0026	1.1593±0.0005	**1.0300±0.0012**
D3	M1	1.7308±0.0006	1.5377±0.0016	1.4505±0.0013	1.3707±0.0023	1.3561±0.0009	**1.3263±0.0010**
	M2	1.9805±0.0008	1.7436±0.0003	1.6150±0.0033	1.6331±0.0015	1.6099±0.0022	**1.5603±0.0011**
D4	M1	0.7239±0.0007	0.7599±0.0006	0.6918±0.0026	0.6588±0.0016	0.4816±0.0022	**0.3704±0.0015**
	M2	0.9045±0.0009	0.9320±0.0005	0.9284±0.0028	0.8819±0.0022	0.6243±0.0019	**0.5111±0.0007**
D5	M1	0.7157±0.0008	0.6175±0.0010	0.5530±0.0046	0.5642±0.0019	0.4881±0.0023	**0.4620±0.0013**
	M2	1.0235±0.0006	0.9289±0.0013	0.8168±0.0030	0.8280±0.0015	0.7956±0.0026	**0.7503±0.0022**
D6	M1	0.9147±0.0011	0.8638±0.0015	0.7925±0.0015	0.7635±0.0024	**0.7504±0.0015**	0.7672±0.0015
	M2	1.1549±0.0010	1.0938±0.0006	0.9565±0.0009	0.9671±0.0030	**0.9256±0.0026**	0.9493±0.0026
D7	M1	1.3838±0.0013	1.2385±0.0009	1.1337±0.0050	1.1451±0.0028	1.0721±0.0023	**0.8744±0.0019**
	M2	1.6895±0.0005	1.5286±0.0007	1.4414±0.0016	1.4556±0.0019	1.2648±0.0030	**1.1284±0.0017**
D8	M1	0.9231±0.0008	0.8299±0.0012	0.7644±0.0059	0.7320±0.0011	0.7526±0.0016	**0.6570±0.0013**
	M2	1.1715±0.0009	1.1248±0.0016	1.0653±0.0022	1.0283±0.0006	0.9916±0.0022	**0.9144±0.0024**

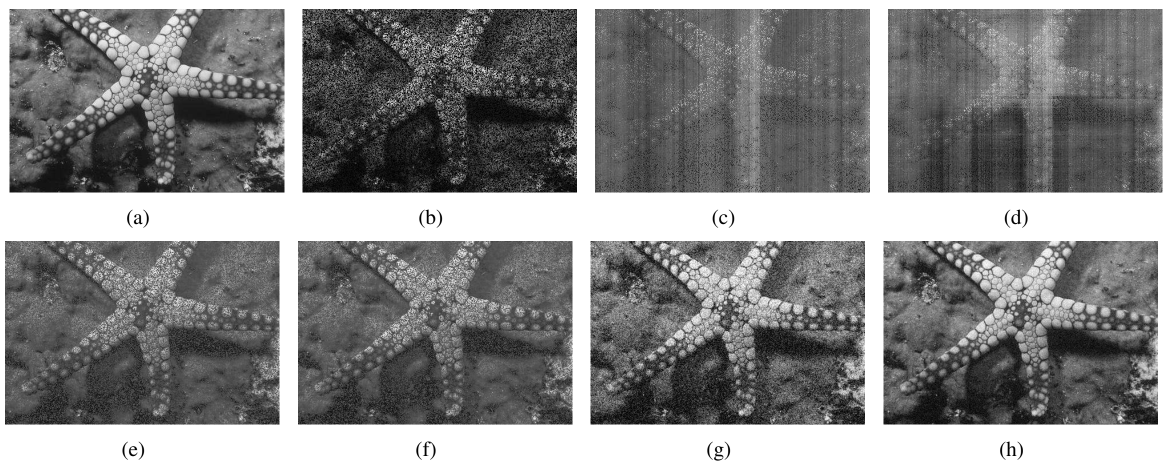


Fig. S2. Recovery results for BSDS500 with 30% missing rate by using different methods. (a)original image, (b)original image with 30% missing, (c)Mean, (d)KNN, (e)UNLFA, (f)TF-SNLF, (g)LSCF, (h)NRNLF_KT.

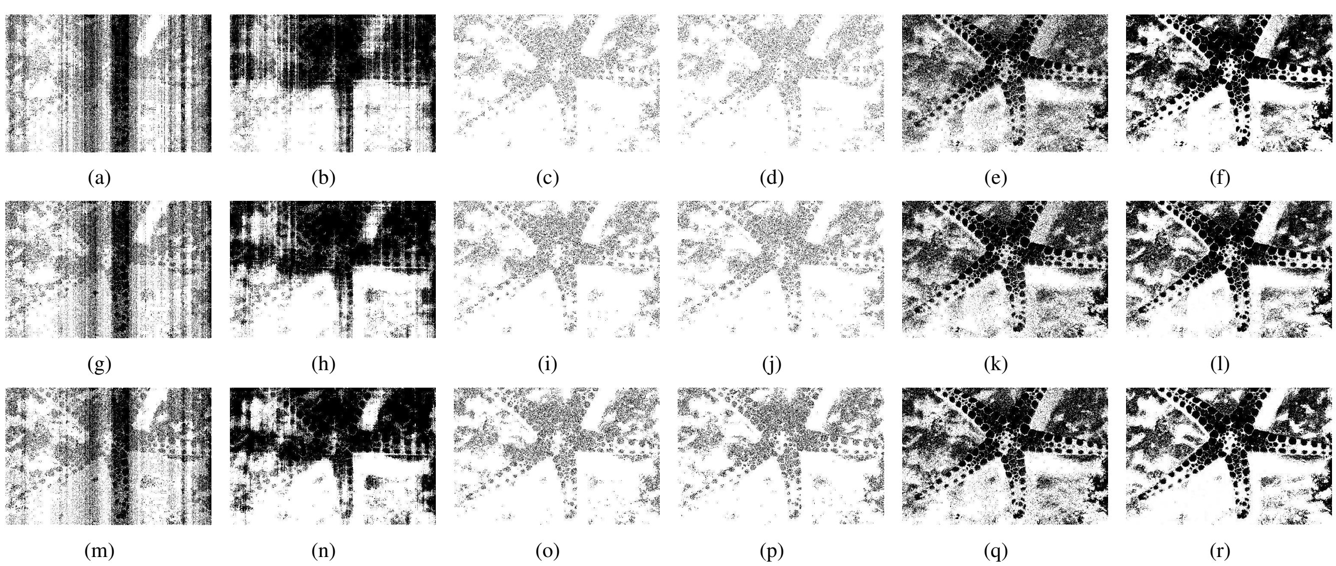
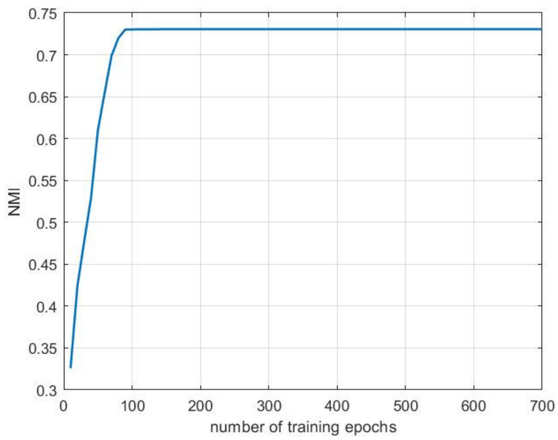
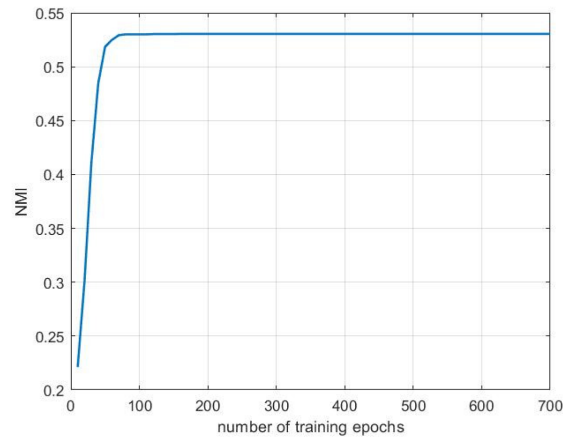


Fig. S3. Recovery results for BSDS500 with 30% missing by using different methods.



(a)



(b)

Fig. S4. NMIs for D1 with different rates of missing values on AFD_SS model. a)30% and b)80%.

TABLE S3
P-VALUES OF DIFFERENT DATASETS WITH DIFFERENT MISSING

	Mean		KNN		UNLFA		TF-SNLF		LSCF		NRNLF_KT	
	30%	80%	30%	80%	30%	80%	30%	80%	30%	80%	30%	80%
D1	0.4252	<u>0.0049</u>	0.5771	0.1108	0.6139	0.0899	0.6302	0.1596	0.6899	0.1395	0.7139	0.2414
D2	0.6271	0.0634	0.6386	0.0968	0.6238	<u>0.0116</u>	0.6559	0.1201	0.6849	0.0833	0.6901	0.1558
D3	0.4157	<u>0.0022</u>	0.4768	<u>0.018</u>	0.4951	0.0789	0.5159	<u>0.0167</u>	0.5302	<u>0.0477</u>	0.5897	0.1078
D4	0.5209	0.0573	0.6068	0.0965	0.6216	0.1026	0.6587	0.1558	0.6657	0.1324	0.6705	0.1658
D5	0.5433	<u>0.0006</u>	0.5522	<u>0.0109</u>	0.5801	0.0632	0.6509	<u>0.0154</u>	0.6816	0.0258	0.6999	<u>0.0445</u>
D6	0.4572	<u>0.0039</u>	0.4823	0.0601	0.4934	<u>0.0131</u>	0.5135	0.0506	0.5323	<u>0.0124</u>	0.5713	0.0746
D7	0.4970	0.0582	0.5305	<u>0.0421</u>	0.5672	0.1049	0.6231	0.1281	0.6304	0.1794	0.6728	0.1883
D8	0.5513	<u>0.0085</u>	0.5834	0.1018	0.6081	0.1931	0.6161	0.1511	0.6577	0.2204	0.7364	0.2537

TABLE S4
NMAE%±STD% FOR BSDS500

Algorithm	Mean	KNN	UNLFA	TF-SNLF	LSCF	NRNLF_KT
NMAE%±STD%	32.11±2.35	27.59±2.71	22.82±1.66	21.42±1.02	19.07±1.67	**16.25±1.28**

TABLE S5
PARAMETER SETTING ON DIFFERENT DATASETS

Parameter	D1	D2	D3	D4	D5	D6	D7	D8
d	6	4	2	6	5	19	8	9
λ_1	0.3	0.6	0.5	0.8	0.6	0.5	0.7	0.4
λ_2	0.4	0.3	0.7	0.6	0.4	0.3	0.5	0.4

TABLE S6
PERFORMANCE OF THREE CLUSTERING ALGORITHMS FOR DIFFERENT DATASETS

Dataset	Matric	FCM [4]	FD_SS [5]	FWFDCM [2]	AFD_SS
D1	M3	54.72±5.16	57.04±4.08	59.53±2.24	**60.82±3.28**
	M4	77.86±2.28	78.03±2.70	78.65±3.13	**79.70±3.11**
D2	M3	58.67±3.45	65.06±3.12	67.55±4.35	**69.11±2.16**
	M4	82.77±6.12	84.58±4.86	86.53±1.45	**87.89±3.53**
D3	M3	58.67±8.03	65.06±5.74	68.35±5.75	**69.11±4.66**
	M4	57.38±0.78	58.54±1.06	57.98±3.20	**60.28±0.95**
D4	M3	15.65±1.89	17.23±2.22	18.06±4.10	**18.54±3.26**
	M4	58.10±4.56	60.21±2.09	60.84±1.75	**61.25±1.42**
D5	M3	57.57±3.35	59.07±3.97	59.34±2.44	**60.91±2.13**
	M4	76.06±1.61	76.86±1.26	76.99±5.12	**77.21±0.97**
D6	M3	18.56±5.73	22.56±4.49	**24.17±3.27**	23.78±5.08
	M4	60.32±4.33	62.21±3.56	**62.94±4.72**	62.82±3.88
D7	M3	19.67±7.26	23.47±5.14	25.64±2.27	**26.47±2.33**
	M4	61.47±5.42	64.16±3.84	65.56±3.56	**67.00±4.56**
D8	M3	57.33±2.46	60.67±3.06	61.75±4.36	**63.36±1.91**
	M4	78.95±3.15	81.44±4.82	82.22±5.75	**83.31±2.48**

TABLE S7
PERFORMANCE OF THREE CLUSTERING ALGORITHMS FOR DIFFERENT DATASETS WITH 30% MISSING

		Mean		KNN		UNLFA		TF-SNLF		LSCF		NRNLF_KT	
		M3	M4	M3	M4	M3	M4	M3	M4	M3	M4	M3	M4
D1	F1	34.68±5.13	62.82±6.59	33.39±3.58	62.55±1.47	36.98±1.13	64.80±6.88	35.91±4.77	63.96±5.28	37.99±4.31	67.52±1.81	38.88±7.92	67.99±4.35
	F2	34.91±4.62	63.00±5.26	34.02±4.65	62.38±7.54	37.64±7.69	65.25±4.33	37.33±7.04	66.49±5.43	38.43±1.74	68.04±5.29	42.91±6.45	68.55±2.67
	F3	**36.43±2.24**	**63.43±3.42**	35.43±2.33	62.38±5.43	37.98±6.53	65.87±3.65	38.24±6.43	66.49±5.45	39.80±2.23	68.84±4.32	43.45±5.35	69.04±3.45
	F4	35.91±4.20	63.09±3.13	**36.33±3.16**	**63.82±2.78**	**38.28±4.41**	**66.96±1.47**	**39.67±4.36**	**68.44±7.47**	**40.27±4.53**	**69.70±4.24**	**44.78±1.13**	**70.90±4.84**
D2	F1	45.41±3.33	77.12±5.06	50.90±5.74	78.94±3.08	55.01±6.14	80.48±5.45	59.62±4.63	82.02±6.56	58.90±6.34	80.86±5.11	61.57±7.07	83.94±3.31
	F2	48.76±3.65	78.07±5.89	52.93±4.95	78.52±6.86	56.83±4.17	80.03±7.92	58.90±2.46	80.85±6.58	59.01±3.14	81.45±5.78	61.54±4.75	84.12±4.02
	F3	49.56±2.57	79.10±3.25	53.11±5.05	79.01±5.78	57.12±3.45	81.50±6.54	58.66±2.98	81.94±3.80	**60.57±2.11**	**82.05±6.62**	62.78±2.83	84.20±2.15
	F4	**50.54±3.01**	**80.92±4.16**	**54.55±3.26**	**80.19±3.41**	**58.28±2.11**	**82.03±7.54**	**60.71±2.13**	**82.93±5.33**	59.96±4.68	81.98±1.41	**63.78±7.26**	**85.02±6.96**
D3	F1	3.83±6.12	47.69±7.26	4.90±7.02	47.18±3.57	5.37±2.65	48.97±7.25	5.89±5.01	52.41±1.73	5.77±4.13	50.30±2.13	6.33±2.57	52.52±4.47
	F2	5.09±5.87	50.08±6.98	5.45±6.59	49.33±7.16	6.09±1.07	50.31±1.07	7.16±3.75	53.56±4.67	7.44±5.38	54.39±4.66	8.95±6.02	53.56±7.28
	F3	6.13±4.32	52.08±3.45	5.87±5.27	49.96±6.78	7.25±2.47	50.79±2.57	7.16±3.75	53.12±5.07	8.63±3.07	54.88±5.12	9.42±3.45	53.56±7.28
	F4	**7.22±4.69**	**53.22±5.02**	**6.55±6.32**	**51.15±4.78**	**8.61±5.45**	**52.15±7.4**	**9.88±2.92**	**54.18±6.76**	**9.31±4.98**	**55.70±5.61**	**10.61±1.47**	**54.79±5.88**
D4	F1	10.38±1.64	58.82±4.26	12.46±5.94	56.79±2.24	12.81±4.95	57.12±6.83	14.50±7.26	58.93±5.46	15.26±5.03	60.93±4.29	15.29±6.39	60.91±4.55
	F2	13.55±0.96	59.51±2.16	15.50±4.26	58.22±3.04	15.61±4.76	59.29±3.67	16.22±2.27	59.23±2.24	16.11±3.12	60.88±5.17	18.45±6.89	61.70±6.41
	F3	13.98±1.26	60.01±1.26	16.10±3.49	58.89±2.65	17.34±4.66	59.87±3.78	18.02±1.65	61.07±3.24	**18.53±5.85**	**61.88±4.61**	20.24±6.35	63.54±5.24
	F4	**14.27±0.45**	**60.95±2.59**	**16.30±2.19**	**59.77±5.47**	**18.31±3.70**	**60.22±2.86**	**19.62±7.72**	**62.52±6.44**	17.12±4.57	61.59±6.98	**23.78±3.87**	**65.48±6.70**
D5	F1	38.84±2.59	73.40±2.11	42.49±3.49	50.47±1.51	44.69±2.89	76.27±5.85	43.10±7.19	76.17±1.59	47.49±2.12	78.60±6.42	48.71±4.95	77.18±6.51
	F2	39.90±2.07	74.22±2.56	45.65±3.86	52.52±1.85	46.17±6.07	77.92±3.35	44.26±7.81	77.84±1.91	48.05±4.93	78.37±4.15	50.14±1.95	78.20±5.37
	F3	40.11±3.16	75.02±5.12	**46.21±5.83**	**52.78±2.45**	47.22±5.13	77.54±2.13	44.87±4.23	77.84±1.91	49.46±3.24	78.64±3.15	50.88±2.45	79.02±2.42
	F4	**40.95±3.15**	**75.63±3.19**	45.25±2.94	52.20±2.12	**48.56±1.85**	**78.21±2.62**	**45.67±1.12**	**78.10±6.55**	**50.66±5.28**	**79.65±1.78**	**51.39±6.98**	**79.74±1.45**
D6	F1	23.09±2.55	62.87±7.16	22.68±1.65	62.35±4.53	23.33±1.88	65.04±6.47	24.84±4.78	67.83±5.02	24.30±5.98	67.50±2.74	27.13±3.01	68.01±7.93
	F2	25.13±3.12	65.30±5.26	25.70±1.78	65.18±7.88	24.56±6.45	66.59±1.72	27.33±5.42	68.79±6.61	24.35±2.42	68.21±4.02	25.51±4.99	69.75±3.42
	F3	26.62±2.16	66.78±5.42	25.70±1.78	65.95±6.35	25.07±3.51	67.00±2.45	28.35±2.45	68.67±2.45	25.80±2.45	68.21±4.02	26.13±3.44	69.32±2.44
	F4	**27.61±5.46**	**67.99±6.08**	**26.43±4.65**	**67.26±3.75**	**26.22±2.28**	**67.24±2.63**	**29.12±2.46**	**69.07±3.51**	**26.45±7.89**	**69.31±5.12**	**26.86±6.67**	**70.22±1.31**
D7	F1	20.04±3.19	52.54±5.16	20.70±5.16	53.93±6.4	21.80±1.4	54.76±2.46	21.06±1.76	53.69±7.62	24.12±2	56.81±5.72	23.09±1.34	55.89±1.71
	F2	23.26±2.99	54.02±4.59	23.72±6.54	55.71±3.86	24.58±4.82	57.02±5.88	24.43±7.31	57.42±6.47	26.56±5.85	58.49±6.93	27.19±7.56	60.17±5.56
	F3	23.57±2.45	55.24±3.89	23.45±4.35	56.75±1.22	24.89±2.14	57.63±3.45	25.30±5.32	57.88±5.36	27.21±3.25	59.47±6.66	28.11±5.33	60.50±4.43
	F4	**24.01±2.16**	**56.33±4.97**	**24.94±5.97**	**57.20±6.72**	**25.60±2.22**	**58.05±6.81**	**25.82±3.55**	**58.05±5.42**	**27.91±5.09**	**60.39±1.46**	**28.74±4.76**	**61.40±4.72**
D8	F1	23.27±6.48	49.13±5.16	22.45±3.66	50.40±7.75	27.27±7.56	56.22±5.65	30.43±6.57	58.81±2.92	37.25±3.84	64.00±4.24	40.63±4.12	66.77±5.25
	F2	27.41±5.23	51.26±6.49	28.78±2.57	53.09±2.92	32.22±7.65	58.35±5.96	35.22±2.41	61.55±1.44	39.16±4.12	64.57±3.35	45.77±6.77	68.73±6.62
	F3	28.33±2.45	51.76±4.37	29.12±2.45	53.87±3.45	33.45±6.21	59.54±2.47	35.78±1.36	59.38±2.45	39.75±1.22	67.13±3.25	46.32±5.60	69.21±3.25
	F4	**29.22±5.16**	**52.18±4.21**	**29.71±3.29**	**54.94±3.31**	**35.88±1.72**	**60.48±4.69**	**36.36±4.46**	**61.50±5.43**	**40.40±5.62**	**69.41±6.72**	**47.40±3.85**	**70.55±5.52**

F1, F2, F3 and F4 represent FCM, FD_SS, FWFDCM and AFD_SS

TABLE S8
PERFORMANCE OF THREE CLUSTERING ALGORITHMS FOR DIFFERENT DATASETS WITH 80% MISSING

		Mean		KNN [1]		UNLFA		TF-SNLF		LSCF		NRNLF_KT	
		M3	M4	M3	M4	M3	M4	M3	M4	M3	M4	M3	M4
D1	F1	25.92±4.78	42.53±6.07	28.25±4.32	52.46±1.26	29.12±5.48	55.08±1.27	33.00±1.05	54.40±2.11	34.37±2.79	53.78±1.30	32.65±6.31	53.36±1.71
	F2	27.25±2.34	947.24±4.73	31.32±1.86	54.62±4.1	32.75±1.79	57.06±1.29	37.33±7.04	66.49±5.43	33.41±3.24	56.36±5.19	34.60±2.33	56.36±3.82
	F3	27.88±2.35	49.45±3.54	31.66±0.45	54.90±3.45	33.45±2.41	57.06±1.29	38.02±5.23	67.12±3.45	34.50±2.42	57.02±3.45	35.00±4.21	57.22±1.17
	F4	**28.41±4.18**	**50.64±5.32**	**32.13±4.91**	**55.25±2.81**	**34.74±1.36**	**58.95±3.47**	**39.67±4.36**	**68.44±7.47**	**34.72±4.46**	**58.17±5.17**	**35.29±5.07**	**58.36±1.49**
D2	F1	38.05±2.29	52.15±1.64	40.23±3.64	58.13±4.45	42.69±3.51	59.53±1.04	59.62±4.63	82.02±6.56	43.93±5.12	64.51±1.71	49.85±4.57	64.36±5.14
	F2	39.63±4.15	55.15±1.76	42.11±6.24	59.61±5.39	44.56±4.47	62.85±4.11	58.90±2.46	81.94±3.8	46.04±4.99	67.43±5.11	50.77±5.67	67.36±3.85
	F3	39.23±2.13	56.87±2.54	41.05±3.26	60.23±6.25	45.11±3.26	63.46±3.20	59.16±3.26	82.56±2.58	46.98±3.65	68.23±5.77	**52.25±345**	68.38±2.68
	F4	**40.58±6.55**	**57.80±2.33**	**42.51±3.57**	**60.67±5.94**	**45.75±3.28**	**64.02±1.77**	**60.71±2.13**	**82.93±5.33**	**48.13±4.14**	**69.04±1.44**	51.43±5.52	**69.36±5.85**
D3	F1	2.16±4.25	30.66±2.29	5.01±3.36	54.76±4.51	5.71±1.25	38.77±2.59	5.89±5.01	52.41±1.73	4.49±5.57	35.09±4.01	5.43±5.9	35.36±6.65
	F2	2.50±5.06	32.06±1.51	5.26±2.37	56.18±2.19	5.79±4.86	39.35±2.41	7.16±3.75	53.56±4.67	4.93±2.26	36.47±4.94	6.02±5.63	36.36±1.13
	F3	2.87±3.25	33.57±2.33	5.22±1.02	57.34±3.21	5.80±3.22	39.87±2.41	8.25±4.11	53.87±2.56	5.02±1.12	37.55±3.45	6.12±2.42	38.23±2.11
	F4	**3.29±3.87**	**35.37±4.99**	**5.35±5.18**	**58.35±6.27**	**5.98±2.25**	**40.18±4.13**	**9.88±2.92**	**54.18±6.76**	**5.85±4.48**	**40.29±2.28**	**6.30±4.63**	**40.36±6.94**
D4	F1	9.38±4.77	40.01±2.17	12.72±3.77	42.76±1.65	12.75±5.97	43.57±3.59	14.50±7.26	58.93±5.46	12.78±2.81	40.42±2.84	15.03±5.08	40.36±5.19
	F2	11.84±4.47	44.82±4.57	14.52±4.62	45.19±3.76	14.70±5.01	45.36±4.22	16.22±2.27	59.23±2.24	15.29±5.92	45.31±1.33	16.84±1.50	45.36±4.66
	F3	11.72±2.12	44.99±3.25	14.87±2.36	46.22±4.23	15.02±3.21	45.88±2.30	17.52±3.06	59.23±2.24	15.58±3.22	46.25±0.26	16.95±2.00	46.12±2.36
	F4	**12.00±4.03**	**45.12±5.04**	**15.01±2.50**	**47.23±4.04**	**15.87±1.57**	**47.40±5.57**	**19.62±7.72**	**62.52±6.44**	**16.71±1.48**	**48.17±1.84**	**17.07±3.88**	**48.36±4.94**
D5	F1	27.52±5.28	38.65±5.7	26.16±3.25	37.78±4.11	34.60±6.9	44.50±3.89	43.10±7.19	76.17±1.59	35.88±2.11	42.22±2.72	37.21±3.25	42.36±5.93
	F2	30.68±6.06	41.23±3.09	30.54±2.99	40.85±6.82	**36.75±1.74**	**46.58±4.45**	44.26±7.81	77.84±1.91	38.09±1.68	47.32±4.91	39.61±4.21	47.36±4.87
	F3	31.32±5.11	43.21±2.11	30.84±1.36	41.30±2.26	34.43±2.36	44.90±2.42	44.33±2.45	77.63±2.13	38.49±0.36	47.98±5.32	40.23±5.32	47.88±5.36
	F4	**34.35±1.86**	**45.82±6.59**	**32.84±4.85**	**43.21±1.68**	35.73±6.24	45.42±5.03	**45.67±1.12**	**78.10±6.55**	**40.00±5.19**	**49.10±4.18**	**41.52±2.86**	**49.36±2.73**
D6	F1	11.76±1.53	42.88±5.86	13.56±3.05	46.57±2.19	10.75±6.81	48.14±1.88	24.84±4.78	67.83±5.02	15.31±1.47	50.38±1.01	18.52±3.43	50.36±1.49
	F2	13.05±6.09	44.03±4.62	15.27±3.08	49.42±2.06	14.78±2.02	50.47±5.07	27.33±5.42	68.79±6.6	17.50±5.04	52.92±4.65	19.22±1.92	52.36±2.81
	F3	13.90±5.32	45.55±3.35	14.87±2.45	**50.42±3.22**	14.98±3.25	50.01±2.43	28.98±4.35	68.43±2.65	17.88±6.53	53.01±2.35	19.87±2.35	52.55±3.45
	F4	**15.00±1.33**	**47.24±5.25**	**15.59±1.12**	49.23±2.75	**15.89±3.92**	**50.72±5.74**	**29.12±2.46**	**69.07±3.51**	**18.05±5.09**	**53.55±1.56**	**20.77±2.73**	**53.36±3.91**
D7	F1	11.74±4.68	42.12±5.54	15.44±6.01	46.98±4.29	15.86±1.25	48.79±2.27	21.06±1.76	53.69±7.62	16.51±4.85	48.43±2.97	19.97±5.75	48.36±2.75
	F2	15.04±2.08	46.46±1.92	18.31±5.25	50.34±6.66	20.75±1.76	51.08±4.01	24.43±7.31	57.42±6.47	19.79±4.73	51.32±1.14	21.88±1.09	51.36±4.08
	F3	15.88±3.24	47.21±2.01	18.87±4.32	50.75±2.35	20.99±0.35	52.23±3.45	24.85±3.56	57.88±3.52	19.97±2.37	52.01±2.36	22.01±5.34	51.86±3.24
	F4	**16.73±5.61**	**48.56±5.31**	**19.75±6.95**	**52.35±3.1**	**21.75±1.48**	**53.33±2.64**	**25.82±3.55**	**58.05±5.42**	**21.02±5.74**	**53.09±5.99**	**22.24±6.39**	**53.36±4.05**
D8	F1	31.30±1.87	47.63±1.86	29.50±5.73	44.90±4.24	31.75±2.2	47.99±3.71	30.43±6.57	58.81±2.92	31.49±1.19	55.92±4.71	30.20±4.99	55.36±4.21
	F2	33.06±1.27	48.16±2.94	30.85±1.61	47.72±4.23	33.75±4.75	49.58±2.09	35.22±2.41	60.55±1.44	33.99±5.23	58.28±3.67	34.84±3.30	58.36±6.04
	F3	33.25±2.44	48.75±1.35	31.02±2.75	47.12±2.35	32.45±1.52	49.75±2.55	35.01±3.77	61.10±0.24	34.09±2.35	59.54±5.22	35.02±2.22	60.22±2.78
	F4	**33.91±1.20**	**49.35±6.89**	**32.80±3.36**	**48.90±1.93**	**34.75±3.86**	**50.21±4.98**	**36.36±4.46**	**61.50±5.43**	**35.09±4.23**	**61.88±1.07**	**36.52±4.63**	**61.36±5.13**

TABLE S9
SEGMENTATION ACCURACY OF DIFFERENT ESTIMATION METHODS

Algorithm	Mean	KNN	UNLFA	TF-SNLF	LSCF	NRNLF_KT
FCM	75.25±3.02	76.12±5.84	79.96±1.54	79.35±2.71	80.86±5.93	81.27±4.13
FD_SSR	77.68±4.65	78.29±3.81	81.19±1.66	81.26±4.12	82.54±3.56	83.89±2.46
AFD_SSR	78.52±3.92	79.75±4.59	83.01±4.29	82.84±2.37	83.64±2.04	84.41±1.38