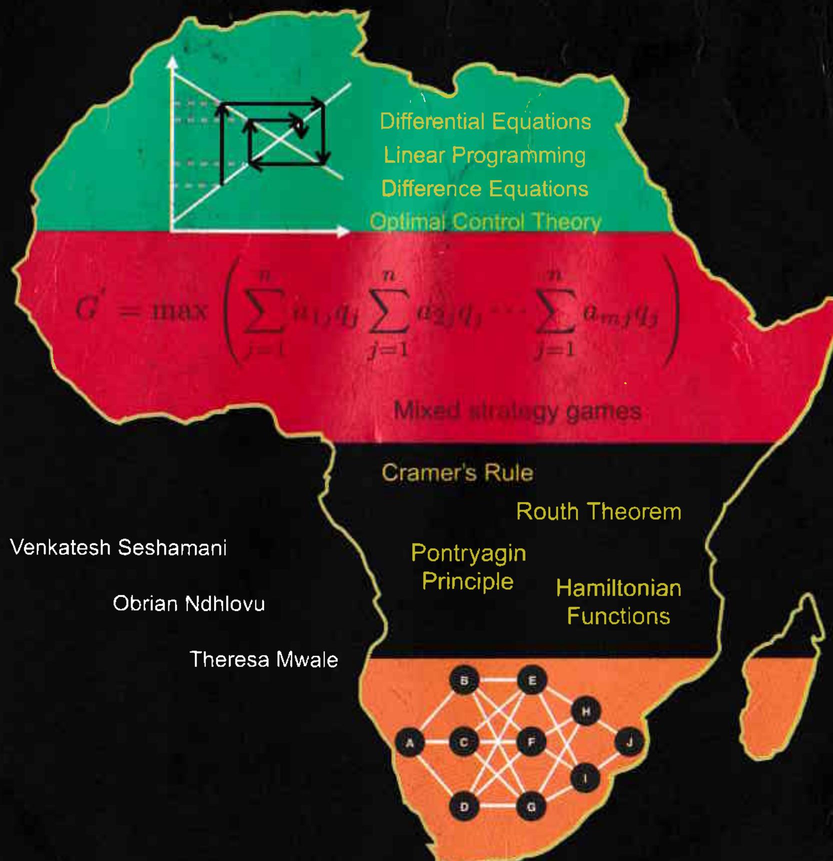
MATHEMATICAL OPTIMISATION AND PROGRAMMING TECHNIQUES FOR ECONOMIC ANALYSIS



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DEDICATION

To the memory of my parents-in-law,
To my mother
and
To my wife Lalitha, my fellow-pilgrim in the Journey

- Venkatesh Seshamani

To the memory of my Aunt Happy Nkosi
To my daughter Victoria

-Obrian Ndhlovu

To the memory of my father

-Theresa Mwale

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PREFACE

Although there are numerous textbooks on the application of mathematical methods to economic analysis, our motivation for writing this book is to provide a treatment of the subject from a perspective that would enable a better understanding and appreciation by potential users in Zambian universities and other universities in this region. The various techniques are illustrated with applications to situations with a local flavour. Many textbooks contain illustrations drawn from North American, European and other contexts that students in this part of the world may fail to adequately understand or appreciate. And understanding the contexts of problems is very critical for students to appreciate the value of the techniques used in solving the problems. The relevance of the techniques appears more meaningful and this in turn enhances the students' desire to grasp the techniques.

Mathematics for economics is a required course that is taught at various levels in universities. It is taught at undergraduate freshman, sophomore and higher levels in economics major programs; at masters level economics programs; in other masters programs such as programs in Economic Policy Management or Defence Studies that draw students with first degrees not only in economics but in a variety of social science disciplines; in economics courses that are taught in other schools such as Natural Resource Economics in the Schools of Natural Sciences, Engineering Economics in Engineering Schools, Health Economics in Schools of Medicine, Agricultural Economics in Schools of Agriculture; and so on. Our book is designed and sequenced in a way in which it can cater to the requirements of all these courses.

The three authors of this book have a variety of experiences. Venkatesh Seshamani is a Professor of Economics who has received specialised training in all areas of quantitative methods –mathematics, statistics, econometrics, programming and operations research – from the University of Mumbai and Stanford University. He has 45 years of teaching experience, 36 of which have been in African universities. Obrian Ndhlovu has an outstanding scholastic record from the University of Zambia and Oxford University and is currently a lecturer in the University of Zambia where he teaches economics courses, notably mathematics and other quantitative methods. Theresa Mwale is a very promising graduate student who is currently at an advanced stage of completion of her dissertation.

The combined insights of the authors garnered over the years have resulted in the production of this book. Feedback from various users of this book – students, teachers and practitioners – would be very welcome in improving later editions of the book.

Authors

PART I : PROLEGOMENON

Chapter 1

1 ECONOMICS AS A QUANTITATIVE SCIENCE

1.1 Background

In the course of its history since its birth as a formal discipline in 1776 with the publication of Adam Smith's magnum opus[1], economics has been subjected to a variety of concepts and definitions. Smith himself referred to economics as political economy and defined it as "an inquiry into the nature and causes of the wealth of nations". This definition in fact was the title of his work which earned him the epithet of "father of modern economics".

It was in the nineteenth century that the term economics came to be used to refer to what was nevertheless recognised as a science. John Stuart Mill (1844)[2] defined economics as "the science that traces the laws of such phenomena of society as arise from the combined operations of mankind in the production of wealth, in so far as those phenomena are not modified by the pursuit of any other object". Even when Thomas Carlyle (1849)[3] cynically described economics as a "dismal science", there was an implicit acknowledgement of the subject being a science.

Alfred Marshall (1890)[4] emphasised that economics "on the one side was a study of wealth and, on the other and more important side, a part of the study of man". Lionel Robbins (1932)[5] described economics as "the science that studies human behaviour as a relationship between ends and scarce means which have alternative uses". Thus both Marshall and Robbins underscored the nature of economics as a social behavioural science.

Economics has indeed travelled a long distance since its initial conception as political economy and its modern-day status as a science. But this has not sidelined the importance of political economy. In so far as it is recognised that institutional and legal frameworks, socio-cultural processes, environmental impacts of economic activity, etc. impinge on economic policy outcomes, political economy that subsumes all these factors within its canvas is a multidisciplinary subject in comparison to economics that is largely a more unified discipline. And in an age where economic and development policy analysts repeatedly discuss issues of

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[1] A. Smith (1776): An Inquiry into the Nature and Causes of the Wealth of Nations, Methuen and Company, London.
[2] J.S. Mill (1844): *On the Definition of Political Economy, and on the Method of Investigation Proper to It*, Essay V in Essays on Some Unsettled Questions of Political Economy, John W. Parker, London.
[3] T. Carlyle (1849): *Occasional Discourse on the Negro Question*, Fraser's Magazine, Vol. XL.
[4] A. Marshall (1890): *Principles of Political Economy*, Macmillan, London.
[5] L. Robbins (1932): An Essay on the Nature and Significance of Economic Science, Macmillan, London.

democracy, governance, human rights, transparency, accountability and other similar issues, such issues are more appropriately encompassed under political economy than under the more limited scope of economics. In other words, we regard political economy as a multidisciplinary field covering the interface between economics, politics, law, ethics, etc. and as a subject in its own right as distinct from economics as a social science. Indeed, today we even witness disagreements between economic scientists and political economists on a number of issues!

Economics is one of several social sciences that include, among others, anthropology, education, history, human geography, political science, psychology, sociology and social work. Social sciences differ from the natural sciences such as physics, chemistry, biology, astronomy and earth sciences in that while social sciences study human behaviour in a societal context, natural sciences study the natural world, natural behaviour and natural condition.

Subjects like logic, mathematics and statistics are also sciences but they are neither social nor natural. They belong to a class of their own.

1.2 Quantification in sciences

Every science contains a catalogue of concepts, laws and theories. Consider some typical sample illustrations of these. Economics has concepts such as production, distribution and consumption of goods and services; theories such as theory of consumer behaviour, theory of the firm and theory of distribution; and laws such as law of equi-marginal utility, law of demand, and law of supply. Political science has concepts such as democracy, freedom, citizenship, state, government; laws such as Greene's 48 laws of power and Duwerger's law; and theories such as Hobbes's theory of social contract, Rawls's theory of justice and Kropotkin's theory of anarcho-communism. Physics has concepts such as gravity, temperature and entropy; laws such as Newton's laws of motion, Coulomb's Inverse-square law, Kepler's laws of planetary motion and laws of thermodynamics; and theories such as Einstein's theory of relativity, big bang theory and string theory.

All theories (and laws) in every science postulate relationships between concepts (specified as variables) that can be tested empirically using quantitative and qualitative techniques. To the extent the concepts are *directly* measurable, they can be analyzed using quantitative techniques. And to the extent they can be more *accurately* measured, the more exact will be the results.

Natural sciences deal with concepts that are accurately measurable and hence are known as exact sciences. Social sciences can never be exact sciences since they deal with human behaviour. Nor can all key concepts be directly measured. The degree to which concepts can be quantified will vary from one social science to another.

1.3 Quantification in economics

It is not difficult to understand why economics is the most quantitative of the social sciences. Most of economic analysis deals with three variables that are directly measurable, namely, prices, outputs and values (which are products of prices and outputs).

But not all variables relate to prices, outputs and values. Some of the critical variables such as tastes and preferences and expectations cannot be exactly quantified. Also, many economic choices and decisions are of a qualitative nature. As Maddala and Nelson (1974)[6] state, decisions such as to buy or not to buy a car or a house, the mode of travel to use or the occupation one must choose are all of a qualitative nature. Economists do use a number of quantitative techniques using proxy and dummy variables[7] to measure such variables indirectly. For example, logit and multinomial logit models and random walk models are employed to analyze and predict outcomes. But the results will always be subject to errors and can never be exact as when one uses, for example, Einstein's famous equation $e = mc^2$ (energy equals mass times the speed of light squared), or when electrolysis of water invariably yields for every mole of water, a mole of hydrogen gas and a half-mole of oxygen gas in their diatomic form.

Notwithstanding the limitation described above, economics today is characterised by extensive use of quantitative methods. The application of quantitative methods in economics takes three major forms: mathematical economics, statistical economics/economic statistics and econometrics.

Mathematical economics is the application of mathematical methods to represent economic theories and analyze economic problems.

Statistical economics is the statistical analysis of economic relationships. It is more commonly referred to as economic statistics since it involves the collection, processing, compilation, dissemination and analysis of economic data.

Econometrics is a method that combines economic theory, mathematics and statistics in the application to the analysis of economic phenomena. It is concerned with the empirical determination of economic relationships.

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[6] G.S. Maddala & F.D. Nelson (1974): *Analysis of qualitative variables,* Working Paper No. 70, National Bureau of Economic Research.

[7] These are quantifiable substitutes for qualitative variables that cannot be quantified. For example, household income may be used as a proxy variable for household's standard of living. Again, the residential status of individuals may be of two categories: citizens and expatriates. A dummy variable may be used, with value 1 for citizens and value 0 for expatriates. Thus, the dummy variable is assigned numerical values to capture the categorical variable relating to an individual's legal status in a country.

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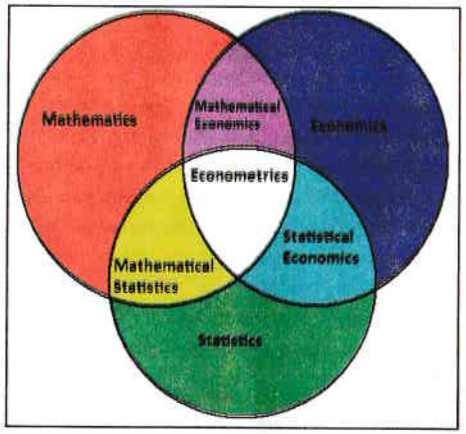
It may be noted that statistics itself can be studied using the tools of mathematics and such study of statistics from a mathematical standpoint is called mathematical statistics.

Figure 1.1: Illustration of interrelationships between mathematics, statistics and economics.



In this book, we deal with the application of mathematical methods.

1.4 The utility of mathematical methods

Mathematics today has virtually become a compulsory prerequisite language that anyone wanting to be a proficient economist has to learn.

Harvard economics professor Gregory Mankiw wrote in his blog[8] in 2006 the reasons why an aspiring economist today needs mathematics. Among the several reasons he cited were the following:

> Every economist needs to have a solid foundation in the basics of economic theory and econometrics. You cannot get this solid foundation without understanding the language of mathematics that these fields use.
>
> As a policy economist, you need to be able to read the academic literature to figure out what research ideas have policy relevance. That literature uses a lot of mathematics, so you need to be equipped with mathematical tools to understand it intelligently.
>
> Mathematics is good training for the mind. It makes you a more rigorous thinker.

[8] Greg Mankiw's Blog (2006): *Why Aspiring Economists Need Math,* September 15.

Turkish economist Dani Rodrik who also lectures at Harvard posed the above question in a dramatic way and also gave his answer.[9]

Question: Why do students of economics have to know about quasi-concavity and all that in order to improve the lives of the poor?

Answer: If you are smart enough to be a Nobel prize-winning economist maybe you can do without the math, but the rest of us mere mortals cannot. We need the math to make sure we think straight – to ensure that our conclusions follow from our premises and that we haven't left loose ends hanging in our argument. In other words, we use math not because we are smart but because we are *not smart enough*. We are just smart enough to recognise we are not smart enough. And this recognition, I tell my students, will set them apart from a lot of people out there with very strong opinions about what to do about poverty and development.

Similar arguments are also found in Calzi & Basille (2004)[10] who wrote: "Apparently, therefore, there is (by far!) more mathematics in economics than in any of the other social sciences and even than in more traditional scientific disciplines. It is therefore all the more important for economists to have a solid mathematical background, so as to avoid suffering from any inferiority complexes and to be able to distinguish between good and bad economics autonomously".

But what is the utility of mathematics that makes it seem such an indispensable tool for economists? The utility can be summed up in three words: clarity, brevity and efficiency. These three aspects are not independent but interlinked.

On the aspect of clarity, a very graphic statement was provided by Irving Fisher:[11] "The effort of the economist is to see, to picture the interplay of economic elements. The more clearly cut these elements appear in his vision, the better; the more elements he can grasp and hold in his mind, the better. The economic world is a misty region. The first explorers used unaided vision. Mathematics is a lantern by which what was dimly visible now looms up in firm, bold outlines. The old phantasmagorias disappear. We see better. We also see further".

Economics is one of those fields of knowledge where interactions and interrelationships among different elements are highly complicated and hence difficult to comprehend. According to Greek economist Stephan Valavanis, there are two disciplines – economics and astronomy –

[9] Dani Rodrik (2007): *Why We Use Math in Eeconomics*, Dani Rodrik's Weblog, September 4.

[10] M.L. Calzi & A. Basille (2004): *Economists and Mathematics* in M. Emmer (Ed): Mathematics and Culture I, Springer.

[11] I. Fisher (1926): *Mathematical Investigations in the theory of value and prices,* Section 10, Appendix 3, Yale University Press.

which deal with phenomena where "everything depends on everything else".[12] To give a simple illustration of economic interrelationships, the price of labour (wages) depends on the demand for labour and the supply of labour. The demand for labour will depend on how much of a commodity, say X, labour helps to produce. This will depend on the demand for commodity X. This demand in turn will depend on the incomes of the buyers. The incomes of the buyers who are mostly labourers depend on their wages. So we are back to wages again. Such relationships can be more clearly, more simply and more briefly expressed through mathematics than through tediously long verbal explanations.

The combination of clarity and brevity of expression makes for efficiency. Now, it is often said, and quite rightly, that there is very little in economics that mathematics can explain which cannot also be explained through verbal exposition. Then why bother to learn and use mathematics? The answer can be provided by means of an analogy. There is nothing that a tractor can do which cannot also be done by a plough or even strong bare hands. Then why are tractors used? It is because for the same amount of effort, a tractor can furrow the land much faster, deeper and wider than a plough or bare hands. In other words, the tractor is a more efficient tool of farming than a plough or bare hands. Likewise, mathematics is a more efficient method of economic analysis than verbal expositions.

Samuelson (1952)[13] brought out this idea using another analogy. He wrote: "To get to some destinations it matters a great deal whether you go afoot or ride by a train. No wise man studying the motion of a top would voluntarily confine himself to words, forswearing all symbols. Similarly, no sensible person who had at his command both the techniques of literary argumentation and mathematical manipulation would tackle by words alone a problem like the following: Given that you must confine all taxes to excises on goods or factors, what pattern of excises is optimal for a Robinson Crusoe or for a community subject to prescribed norms? I could go on and enumerate other problems. But that is not necessary. All you have to do is to pick up a copy of any economic journal and turn to the articles on literary economic theory, and you will prove the point a hundred times over".

Mathematics has also had a more fundamental role to play in economics. Not only has it contributed to a more efficient analysis of economic problems; the development of solutions to some problems in economics had to wait for development/application of mathematics. Many illustrations can be provided of such contributions of mathematics to economics. Here, we mention two instances.

[12] S. Valavanis (1959): Econometrics: An Introduction to Maximum Likelihood Methods, McGraw Hill and Company.
[13] P. Samuelson (1952): *Economic Theory and Mathematics – An Appraisal*, American Economic Review, 42.

One is the well-known "adding up problem". Total product produced at any time is distributed among the factors of production. What would constitute a fair distribution of the product among the factors? The answer was, in order to ensure a fair distribution, each factor should be paid a price equal to the value of its marginal product. But the question, as posed by Joan Robinson (1934)[14], was: "How do we know that if each factor is paid its marginal product, the total product is disposed of without residue, positive or negative?" The proof could be provided only by using the eighteenth century Swiss mathematician Euler's theorem relating to homogenous functions. Euler's theorem is explained in a later chapter of this book.

Our second illustration relates to the proof of the existence of a general equilibrium. French economist Leon Walras (1874)[15] developed the theory of general competitive equilibrium and it was proved that the equilibrium is efficient. But even though the equilibrium may be efficient, one needs to prove the existence of the equilibrium in the first place. This proof came through a series of developments. Nicola Giocoli (2003)[16] summarises the sequence as follows:

"The technique most frequently used in economics for establishing the existence of solutions to an equilibrium system of equations is that of searching for a fixed point of a suitably constructed function or correspondence...

...if we manage to transform our equilibrium problem into a fixed point problem we can easily derive existence results for economic equilibrium.

The application of fixed-point techniques in economics is one of the legacies of the game-theoretic revolution brought forth by mathematicians such as John von Neumann and John F. Nash. A well-defined sequence of contributions led to that outcome. The sequence started with von Neumann's first paper on game theory......., passed through his 1937 general equilibrium paper........and the 1944 Theory of Games and Economic Behaviour paper, written in collaboration with Oscar Morgenstern....., then featured Nash's new equilibrium concept......, and culminated in Arrow and Debreu's existence proof, which relied on game theory and Nash's equilibrium. This sequence is arguably one of the most important in the whole history of economics. Besides the fixed-point techniques, the above-mentioned works offered the economists' community a score of other tools and methods that became the backbone of modern mathematical economics: convex analysis, linear algebra, duality theory, etc."

[14] J. Robinson (1934): *Euler's Theorem and the Problem of Distribution*, The Economic Journal, Vol.44, No. 175, pp. 398 – 414.
[15] L. Walras (1874): *Elements of Pure Economics or the Theory of Social Wealth*, Routlege edition, Taylor and Francis, October 2010.
[16] N. Giocoli (2003): Fixing the Point: the Contribution of Early Game Theory to the Tool-box of Modern Economics, Journal of Economic Methodology, 10:1, pp. 1 – 39.

The developments delineated in the second illustration above involve fairly high-level mathematics and hence their detailed expose is beyond the scope of this book.

1.5 The inutility of mathematical methods

While the utility of mathematical methods in economics has been well appreciated, disillusionment with what has probably been an over-zealous application of the methods has also crept among economists (many of whom were themselves prominent appliers of mathematics in economics!) leading often to strident commentaries on the risks and limitations of doing so. As the saying goes, one can have too much of a good thing. And this seems to be the case with the use of mathematics in economics. Over the past decades, there has been a trend of increasing 'mathematicisation' of economic theory and, ironically, in tandem a rising trend in the criticism of this trend! The basic fear seems to be that there is a tendency for overuse, tantamount to misuse, of mathematics leading to a poorer instead of a heightened understanding of economic realities. In a scathing attack, Robert Kuttner (1985)[17] wrote: "Departments of economics are graduating a generation of idiot savants, brilliant at esoteric mathematics, yet innocent of actual economic life." [18]

In a candid critique[19] he wrote in the September 2, 2009 issue of the New York Times, noted Nobel prize-winning economist Paul Krugman said that just when economists had begun to believe that in the real world they had things under control and that the central problem of depression-prevention had been solved, the global crisis, unforeseen by anybody, occurred in 2008 and everything fell apart. Krugman attributed this to the economic profession going astray "because economists as a group mistook beauty, clad in impressive-looking mathematics for truth" and "the central cause of the profession's failure was the desire for an all-encompassing, intellectually elegant approach that also gave economists a chance to show off their mathematical prowess".

Krugman elaborated this point further. He wrote: "Unfortunately, this romanticized and sanitized vision of the economy led most economists to ignore all the things that can go wrong. They turned a blind eye to the limitations of rationality that often led to bubbles and busts; to the problems of institutions that run amok; to the imperfections of markets – especially financial markets – that can cause the economy's operating system to undergo sudden, unpredictable crashes; and to the dangers created when regulators don't believe in regulation".

[17] R. Kuttner (1985): *The Poverty of Economics*, The Atlanta Monthly, February, 1985, pp.74 – 84.

[18] We would like to make the point, however, that such criticism may be more applicable to top American universities. In many universities in sub-Saharan Africa, mathematics is still by and large a dreaded subject and many students of economics do math courses only because they are compulsory.

[19] P. Krugman (2009): *How did economists get it so wrong?*, New York Times.

Lest he be misunderstood that he was attacking the use of mathematics in economics, Krugman followed his write-up some ten days later with a clarifying note.[20] He said that mathematics in economics could be extremely useful but economists should have it as their servant and not as their master.

Krugman rightly pointed out that good mathematics does not imply good economics. Equally, you can have great work in economics with little or no mathematics. As an illustration of the former, he stated that the mathematics of real business cycle (RBC) models is much more elegant than New Keynesian models, but it does not make them less silly. As an example of the latter, he cited Ackerlof's market for lemons[21] that had virtually no explicit mathematics but was nevertheless transformative in its insight.[22]

Is it then possible to titrate the right dosage of mathematics in the context of any economic theorizing and analysis? Can one suggest a certain quantum of mathematics use as the most appropriate? In our view, it may not be easy to offer even rules of thumb here apart from providing a heuristic description of the dilemma that poses this challenge. And this dilemma is the age-old conflict between rigor and realism.

In a well-known article,[23] D.G. Champernowne wrote: "Unfortunately for the cautious economist, his economic models will be judged according to the degree in which they appear to be relevant to the real world; so that in avoiding the appearance of being wrong, he may yet appear to be silly in publishing a long article whose relevance to any practical issue seems to be superficial. This danger of manufacturing mere "toys" is especially great since the assumptions which are most convenient for model-building are seldom those which are appropriate to the real world". Champernowne further went on to emphasise that: "The ability to judge the relevance of an economic theory and its conclusions to the real world is but rarely associated with the ability to understand advanced mathematics".

The use of mathematics tends to warrant assumptions that are convenient for analysis but may not be appropriate to the real world. One can find 'learned' articles on say the theory of the firm which begin with the assumption of a continuum of firms or of a number of firms

[20] P. Krugman (2009a): *Mathematics and economics*, New York Times, September 11, 2009.

[21] George Ackerlof (1970):*The Market for lemons: Quality Uncertainty and the Market Mechanism*, Quarterly Journal of Economics, 84(3), pp.488 – 500. The paper discusses information asymmetry and is a seminal contribution to the economics of information. In 2001, Ackerlof shared the Nobel prize in economics for his contribution, with Michael Spence and Joseph Stiglitz.

[22] Debates and disagreements over several substantive contents of Krugman's first article continue. See, for instance, J.H. Cochrane (2011): *How did Paul Krugman get it so wrong?*, Economic Affairs, pp.36 – 40. However, in our view, the observations Krugman makes on the use of mathematics are largely valid.

[23] D. G. Champernowne (1954): *The Use and Misuse of Mathematics in Presenting Economic Theory*, Review of Economics and Statistics, Vol. 36(4), pp.369 – 372.

corresponding to points on a real line! These theories will undoubtedly score high on rigor and generality but what can one say about their practical relevance?

The conflict between rigour and realism was best brought out by Nobel laureate Tjalling Koopmans (1957)[24]: "As we strive for greater rigor and precision in the formulation of postulates and propositions, the inadequacies and lack of realism of these postulates are thereby made to stand out in stronger relief. As we succeed in incorporating one aspect of the reality in our models, our failure to incorporate other aspects becomes more apparent".

In sum, there is an inevitable trade-off between rigor and realism, between convenience of analysis and correspondence to reality and between the beauty of the mathematical models and the truth content of the models. The ingenuity of an economist lies in deciding on what constitutes the desired trade-off.

1.6 Conclusion

Whether one likes it or not, the use of mathematics is here to stay. It is a language in which every student of economics must gain at least a modicum of proficiency if he/she has to communicate with fellow economists and share ideas.

Some of the most advanced mathematical methods are used in economics today. However, such highly advanced mathematics may not be necessary for understanding the basic principles of economics. Most of basic economic theory can be well-grasped by a student who has also grasped well the basic principles of set theory, calculus and matrix algebra which are covered in this book.

Although economics is a science that is amenable to a high level of quantification, it is a subject in which a quote attributed to Einstein would well hold: "Not everything that can be counted counts; and not everything that counts can be counted". Never sideline the qualitative and eccentric aspects of human behaviour.

Finally, while learning mathematics, one can draw inspiration from another famous quote of Einstein: "Politics are only a matter of present concern. A mathematical equation stands for ever".

[24] T.C. Koopmans (1957): *Three Essays on the State of Economic Science*, Wiley and Sons, New York.

Chapter 2

2 ECONOMICS AS A SCIENCE OF OPTIMISATION

2.1 Introduction

In Chapter 1, we referred to Robbins's definition of economics as the science that studies human behaviour as a relationship between ends and scarce means that have alternative uses. As we would be already familiar from ECON 100 series, the two critical terms in this definition are 'scarce' and 'alternative uses'.

All relationships between ends and means need not in principle be characterised by scarcity. Conceptually, there could be four possible relationships:

1. Many ends, many resources;
2. Few ends, few resources;
3. Few ends, many resources;
4. Many ends, few resources.

It should be obvious that it is only in the last case that one is faced with the problem of scarcity. Scarcity then simply means that there are just not enough resources to meet all the ends, the ends here being goods and services that satisfy human wants. In such a case one has to make a choice regarding what ends to meet and to what extent, with the limited resources that are available.

The above kind of choice, however, assumes that such a choice is possible to make. Suppose the ends that we want to meet are various goods such as cooking oil, sugar, clothes, etc. Now imagine that the resources available are only in the form of ice boxes. Obviously, ice boxes can be used only for one purpose – to store ice. They cannot be used for anything else. In other words, they have only one use; they cannot be put to alternative uses. Hence we have to assume that resources have alternative uses. Money is one such resource. So are land, labour, and other factors of production.

The choice of putting the resources that one has to alternative uses is made with some objective in mind. The objective depends on the agent making the choice. If the agent is a consumer, the objective is to maximise satisfaction or utility from the consumption of goods and services. If the agent is a producer, the objective will be to maximise revenue or profit from or to minimise cost of the production of output of some good(s). If the agent is the government, the objective is to maximise the overall welfare of the society or citizens.

In brief, the objective of every choice-making agent is to maximise "profit" or minimise "cost" in a larger sense of the terms. Anything that is considered as benefit (utility, revenue, social welfare) is profit and anything that is considered as suffering (disutility, expenditure incurred in producing a good, labour expended to earn an income) is cost.

The maximisation of profit or minimisation of cost is called optimisation. In simple language, optimisation is making the most of given resources.

optimisation thus implies making the best choices possible in given situations. And making decisions is tantamount to making decisions on what actions to take. A set of actions that can be taken is called a programme. On could decide to choose among several programmes and the process of making the decision is called programming. The obvious aim is to choose the optimal programme, that is, one that optimises (maximises or minimises as required) the objective on hand. This is the reason that another expression used for optimisation is *mathematical programming*.

Economics in its quintessential nature involves optimisation. In other words, it can be regarded as a science of optimisation.

Now, decision-making can be done in one of three possible situations: certainty, risk and uncertainty. Situations of certainty are known as *deterministic* situations while situations of risk are also called *stochastic* situations.

Deterministic situations are those in which the outcomes of alternative courses of action are known with certainty. It is then possible for the decision-maker to evaluate these outcomes and choose that course of action to which the outcome with the highest valuation corresponds. This of course is the optimal outcome, and when a decision-maker selects such an outcome, he can be considered to have made an optimal decision.

Where the number of alternative actions and outcomes is pretty large, there are special techniques available for evaluating them. One such technique is linear programming which is explained in Chapter 13.

When the outcome in any situation is not certain, we can have risk or uncertainty. The technical distinction between risk and uncertainty was made by the American economist Frank Knight, best known for his book *Risk, Uncertainty and Profit*. A situation where the outcome is not known but the probabilities of alternative outcomes are known is a risk or stochastic situation. Where neither the outcomes nor the probabilities of alternative outcomes are known, the situation is described as one of uncertainty. Knight explained that the distinction between risk and uncertainty was a significant one. In the context of the theory of the firm, for instance, uncertainty could give rise to excess profits for a firm that could not be eliminated by perfect competition.

The gist of the above distinction is that while risk is measurable, uncertainty is not measurable. And yet, economic decision-makers are occasionally faced with situations of uncertainty. How can one make optimal decisions in such situations? The standard techniques provided by mathematics and statistics cannot readily apply to such decision-making. A number of alternative rules for decision-making have been proposed and the choice among these rules would depend on the decision-maker's attitude and psychological state of mind. Such rules include, among many, Maximin Rule, Maximax Rule, Hurwicz Rule, Minimax Regret Rule, and La Place Rule. The discussion of these rules is beyond the scope of this book which will deal with optimal decision making only in deterministic decisions that can be carried out with the help of mathematical techniques. Decision-making in situations of risk and uncertainty require knowledge of statistics, in particular the theory of probability.

2.2 Components of optimal decision-making

The following are the components of optimal decision-making in a situation of certainty.

A decision-maker with goals or objectives: a consumer who wants to maximise utility; a producer who wants to maximise output or minimise costs; a seller who wants to maximise sales; an investor who wants to maximise returns; a worker who wants to maximise wages or minimise disutility from labour, etc.

A known set of alternative actions: a consumer has to decide what goods to consume and in what quantities; an investor has to decide in what projects to invest and how much; etc.

A set of possible outcomes: different sets of quantities of goods and services consumed will yield different levels of utilities to the consumer; different portfolios of investment in various projects will yield different aggregate returns to the investor; etc.

An objective function or payoff function, $P(x, a)$ which defines the net payoff gain which accrues to the decision-maker: a utility function which is a function that shows how utilities are generated by the quantities of goods and services consumed; an earnings function which shows how earnings are generated by the levels of investment in various projects; etc.

A decision which is the action that is taken: that decision is the best which optimises the value of the objective function: a consumer will decide to maximise his utility function; a producer will decide to minimise his cost function; an investor will decide to maximise his earnings function; etc.

2.3 Types of Optimisation under Certainty

A decision-maker may be required to take decisions under various circumstances. In many situations, the results of the decision taken by a decision-maker are affected by the actions of others. This is the crux of *Game Theory* which deals with optimisation as a game involving two or more players. An individual player or decision-maker has to take into his reckoning the impact the actions of other players will have on his decisions. A game is *hostile* if the decision-maker is confronted by rivals who possess both the motive and the capacity to take such actions as will minimise the gains or payoffs to the decision-maker. But this need not be always the case. In many situations, the game may be non-hostile in the sense that the decision-maker or player is confronted by an indifferent opponent. In such a situation, the game effectively involves only one player, namely, the decision-maker in question.

Suppose there are two major producers of a certain product in the market. Each producer will try to grab the maximum space for himself/herself in the market. The rival producer will try to do the same. Thus the two players are rivals and the actions of one player will be hostile towards the other player.

Suppose a student has been given a certain monthly allowance to spend on various goods and services. He will decide to do so in a manner that will maximise his utility function. But this will not be of any concern to his fellow students who consequently will not do anything to undermine his actions aimed at his personal utility maximisation. This is an example of a non-hostile game which de facto involves only one player.

We deal with game Theory in Chapter 15. All the earlier chapters deal with optimisation by a sole decision-maker.

Now, the techniques of optimisation available to a single decision-maker themselves are of different kinds, depending on a number of factors. These are:

Choice variables: Suppose a firm produces a single product and wants to maximise its profits from the production and sale of that product. Here, the profit maximisation will be determined by the firm's choice of the level of output of that single product. If, however, the firm produces two or more products and wants to maximise its overall profit from the production and sale of those products, then the profit maximisation will be determined by the firm's choice of the levels of output of those products. The output of the product is the *choice* or *decision* variable and the technique of optimisation (maximisation in these examples) will depend on the number of choice or decision variables. The number of choice variables involved can be one, two, three and, in general, n.

Constraints: Suppose the firm in the above examples wants to maximise its profits but with the condition that it has a fixed amount of money to spend during the period of production. Then, the firm is not free to choose any level of output(s) but only such level as is feasible to produce under that condition. Such a condition is known as a *constraint* or *side relation*. It is also referred to as a *restraint* or *subsidiary condition*. An optimisation problem can have one, two, three, and, in general, m constraints. An essential requirement for solving a constrained optimisation problem is that the number of constraints must be fewer than the number of choice variables. In other words, $m < n$.

An optimisation problem which has no constraints is called *free* or *unconstrained* optimisation, while a problem involving constraints is called *constrained* optimisation. The distinction between the two types of optimisation has been humorously brought out by one writer in the following words: "The essence of an optimisation problem is 'catching a black cat in a dark room in minimal time'. A constrained optimisation problem corresponds to a room full of furniture." The complexity of the optimisation technique will depend on the number of choice variables and the number of constraints. The simplest case is unconstrained optimisation of an objective function in one choice variable. At the other end of the spectrum will be the case of optimisation of an objective function in n choice variables subject to m constraints ($m < n$)

Further, a constrained optimisation problem may involve constraints in the form of equations (=) or inequalities ($\geq$ or $\leq$). Solving constrained optimisation problems with inequality constraints requires techniques different from those required for solving constrained optimisation problems with equality constraints.

In order to understand the difference between equality and inequality constraints, suppose a firm has to produce output of its product subject to the constraint of a fixed amount of man hours of labour available during the production period. Then, such a constraint places an *upper bound* or ceiling on the quantity of labour that can be used to maximise profit from the production and sale of the product. In such a case, all of the available labour need not be used in maximising the profit function. If it is used, labour becomes a *binding* constraint; else it is *slack* at the optimum.

In other optimisation problems involving minimisation, the constraint imposed on the concerned variable may be of the ($\geq$) type. This imposes a *lower bound* on the value of the variable.

In all the above type of cases, the techniques used to solve optimisation problems with equality constraints may not be the most efficient ones to solve such problems involving inequality constraints.

Time: In a paper that he published in 1954 in *Oxford Economic Papers,* Nobel laureate Sir John Hicks distinguished between two categories of firms, *stickers* and *snatchers*. The snatchers were those interested in maximising immediate expected profits by snatching whatever was immediately available, irrespective of consequences. The stickers, on the other hand, would be wary of the longer-term possible adverse consequences of snatching now by losing consumer goodwill, and hence would opt for less than maximum profits now in order to maximise long-term expected profits. Reflecting on the number of banks that came to grief in Zambia in the 1990s in the post-liberalisation period, one may feel that they probably reaped the desserts of behaving like rapacious snatchers.

The distinction between stickers and snatchers is based on their respective perspective on time. The snatchers were interested in instantaneous profit maximisation; time did not enter into their model of optimisation. Such a model is called a *static* model and the corresponding optimisation static optimisation. It is a model without time. The stickers, on the other hand, were those who realised that their optimising decisions today would affect their decisions tomorrow. Time entered the model and the objective was to maximise profits over a period of time rather than at any given point in time. Such a model involving time is called a *dynamic* model and the corresponding optimisation, dynamic optimisation.

It is, however, important to understand that, while a static model is a timeless model, a model in which time enters need not be dynamic. As Silberberg and Suen (2001) in their book *The Structure of Economics: A Mathematical Analysis state:* "The fundamental property of dynamic models is that decisions made in the present affect decisions in the future". They further go on to explain that dynamic optimisation problems are those "where decisions are linked, that is, where a decision in one time period affects the level of some relevant variable in the future. In that case, simple replication of past decisions will not be optimal; each decision imposes an 'externality' on the future. It is only then that a problem becomes truly dynamic".

The above concepts will be elucidated later especially in chapter 12 on Dynamic optimisation.

Given a static optimal solution, one may analyze how a change in any choice variable may affect the solution. Such analysis once again is not dynamic but is referred to as *comparative statics.* It is comparative because we are comparing the initial static solution with a new solution arising from the change. But it is static because one does not analyze the time path of the change: how exactly the change took place. The well-known Keynesian 45° degree diagram showing shifts in equilibrium resulting from shifts in aggregate demand, covered in any intermediate macroeconomic course, is an example of comparative statics.

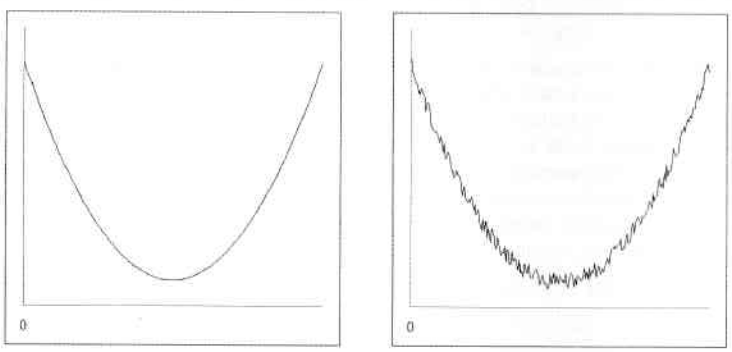
Dynamic models of optimisation are of two kinds, depending on how time is treated. For example, suppose the decision variable is price. Then price may be viewed as changing over time in two ways. The price may be adjusted continuously over time or it may be adjusted over successive periods of time, where each period may be a year, a quarter, etc. In the first case, time is a *continuous* variable and the optimisation will involve *differential* equations. In the second case, time is a *discrete* variable and the optimisation will involve *difference* equations.

Linearity: If, in a given optimisation problem, the objective function to be optimised and all the constraints are linear, then one can adopt the techniques of linear optimisation. Linear programming and some of its extensions dealt with in later chapters of this book are examples of linear optimisation.

On the other hand, if the objective function and/or some or all of the constraints are non-linear, then you have a problem of non-linear optimisation. Besides the standard calculus techniques, a number of *search* or *iterative* methods are available to solve non-linear optimisation problems. These include Conjugate Gradient, Double Dogleg, Neldor-Mead, Newton-Raphson, Quadratic and Trust Region methods among several others. Discussion of these methods is beyond the scope of this book.

Smoothness: Most of the optimisation techniques that are discussed as well as those that are not discussed in this book assume that the objective function to be optimised is *smooth*. In other words, it has no discontinuities or irregularities.[25] Consequently, these techniques cannot be applied if the objective function is non-smooth. In Figure 2.1, we provide a simple illustration of a cost function that is smooth and *non-smooth* or *noisy*.

Figure 2.1: smooth and Noisy Cost Functions



Smooth cost function

"Noisy" cost function

Techniques based on *Perturbation Theory* and *Pattern Search* methods are used to handle non-smooth functions. Their treatment again is beyond the coverage of this book.

[25] Technically, as will be explained in the chapter on Differentiation, the function must be twice differentiable.

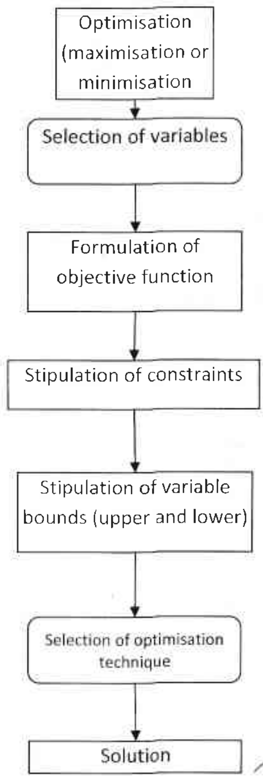
Objectives: Economists may at times wish to optimise more than one objective function. Simultaneously optimising multiple objectives is very difficult, if not impossible. The main reason for this is that the multiple objectives may be conflicting and it may not be possible to improve the value of one objective function without diminishing the value of another. Students of economics would immediately recall the concept of *Pareto* optimality which defines precisely such a situation. A solution would be deemed Pareto optimal if none of the individual objective functions can be improved without impairment of some other function.

Another prominent illustration is the achievement of *Nash* equilibrium (after the economics Nobel laureate John Nash who is basically a mathematician) through a process of bargaining among *n* decision-makers that makes all of them equally happy or unhappy.

The treatment of multiple objective optimisations is beyond the scope of this book.

2.4 Summary of the optimisation process

Figure 2.2: Steps in Optimisation on next page



PART II : PRELIMINARY CONCEPTS AND PRINCIPLES

Chapter 3

3 SETS, RELATIONS AND FUNCTIONS

3.1 Introduction

The principal objective of this chapter is to introduce the basic set-theoretical taxonomy that will be adopted throughout the text. We therefore start with an intuitive discussion of the notion of sets, covering its various types as well as representations, the basic algebra of sets, ordered pairs and Cartesian products, and relations. After the quick excursion to set theory, functions are introduced extensively. Since constraints in economic studies of optimisation are usually represented by sets and the critical issues of efficiency and economic growth are usually represented by functions which take on various graphical representations, the understanding of these aforementioned concepts is imperative to the full grasp of the idioms used in subsequent topics of this book. This chapter will therefore provide the necessary tools required for extensive understanding of the subsequent chapters. We assume here that the reader is familiar with the elementary properties of the real numbers.

3.2 What is a Set?

In microeconomic studies of production theory for instance, we use concepts such as the production *set* and input requirement *set*. The input requirement set is defined as that *set* of inputs required to produce a given level of outputs. The production *set* is therefore a *set* of all the possible outputs that can be produced using the input requirement *set*. In analyzing policies in the world today, we are concerned with alternative policies that will help us achieve a certain *set* of goals, for instance in monetary policy strategies such as inflation targeting, we are concerned with the *set* of instruments to use to achieve our objective. What then is a set?

Simply put, a set is a well defined collection of objects[26]. These objects are called the elements of the set. In the words of Georg Cantor, the great founder of abstract set theory in the 1870s, "a set is a Many which allows itself to be thought of as a One." Therefore, in many texts for easy comprehension, sets are usually denoted by capital letters such as $A, B, C \ldots . X, Y\ Z$, the elements are usually denoted by small letters such as $a, b, c \ldots \ldots x, y, z$. The symbol $\in$ is used to denote "belongs to" or "is a member of" while the symbol $\notin$ is used to denote "does not belong to".

[26] The notion of "object" is left undefined, that is, it can be given any meaning. However, the "objects" must be logically *distinguishable*. That is, if a *and* b are two objects, $a = b$ and $a \neq b$ cannot hold simultaneously, and the statement "either $a = b$ *or* $a \neq b$" is a tautology

Example 3.1

Suppose A is the input requirement set of a certain production technology with only two inputs, capital, denoted by k and labour denoted by l. Show using set notation that l and k are elements in A.

$l, k \in A$; read as l *and* k are elements that belong to the set A. If this statement is wrong, this can simply be written as $l, k \notin A$, read as l *and* k do not belong to A. If only l belongs to A and k does not belong to A or is not in A, this can be written as, $l \in A$, $k \notin A$.

The above example can be generalised to say that if A is an input requirement set with n elements, that is, $x_1, x_2, x_3 \ldots\ldots x_n$, then this is denoted in set notation as $x_1, x_2, x_3 \ldots\ldots x_n \in A$. It means the elements listed on the left are members of the set given on the right.

3.3 Representation of Sets

Sets can generally be represented by three methods: the tabular form or Roster method, rule method and the Venn-Euler diagrams method.

3.3.1 Tabular form or Roster method:

With this kind of representation, all the elements of the set are explicitly listed within the braces. This implies that if you describe a set, for instance, let M be a set of all the goals of monetary policy in Zambia, this means all the goals have to be listed within the braces regardless of their number. Assuming low and stable prices(p), high employment(e), economic growth(g), stability of financial markets(f), interest rate stability (i) and stability in foreign exchange markets(m) are the main monetary policy goals, this can be represented in tabular form as: $M = \{p, e, g, f, i, m\}$.

Similarly, if your set involves numerical values, such as a set X representing all single even digit values of inflation ever attained in Zambia since independence, this can be represented as: $X = \{2,4,6,8\}$

3.3.2 Rule method:

This is more of an implicit way of representing sets and usually more convenient compared to the tabular form. A set is indicated by stating the rule satisfied by its elements and written in braces. Consider the examples in section 3.3.1 above, the sets M *and* X using the rule method can be represented as follows:

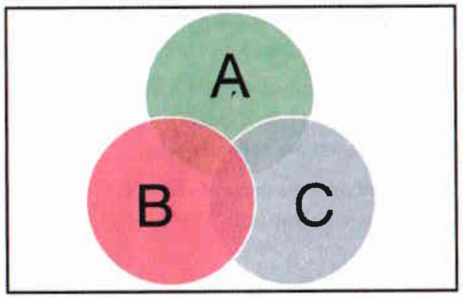
$M = \{m\!: m$ *is a monetary policy goal*$\}$, where the symbol m represents the element, that is, any element that was earlier listed in the braces.

$X = \{x\!: x \text{ is an even single digit inflation number ever attained in Zambia}\}$, where x represents the element, in this case the even inflation number. The colon symbol ":" is read as "such that".

3.3.3 Venn-Euler diagrams:

The elements are represented by points in a closed figure such as a circle, a square or an ellipse. A Venn diagram must contain all the possible zones of overlap between its curves, representing all combinations of inclusion/exclusion of its constituent sets, while in an Euler diagram, some zones might be missing. Thus, a Venn diagram is a restrictive form of an Euler diagram. Since the essence is to show the overlapping (or non-overlapping) of all the sets, involving both the Venn and Euler diagram concepts provide a more explicit representation of sets. An example of a Venn-Euler diagram consisting of 3 overlapping sets is shown in Figure 3.1 below:

Figure 3.1: Venn-Euler diagram for three sets



3.4 Types of sets

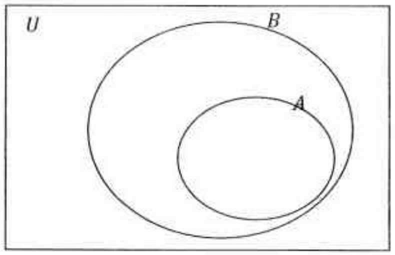
Finite set: This is a set with a finite number of elements; otherwise, the set is called an infinite set. For instance, $A = \{x\!: x \text{ is a natural number}\}$ is an infinite set while $B = \{Sunday, Monday, Tuesday\}$ is a finite set.

Empty set or null set: This is a set which has no elements. It is denoted by { } or ∅. An empty set does not mean it does not exist, it represents emptiness.

Subset: Set A is said to be a subset of B if every element of A is an element of B. We denote the relationship by the symbol ⊆ and write $A \subseteq B$. Note that a null set is a subset of every set.

Proper subset: Set A is a proper subset of another set B if every element of A is an element of B and the set B contains at least one element which is not an element of A. The symbol ⊂ is used to denote the relationship of proper subset. For instance, $A = \{2,4,6\}$ is a proper subset of $B = \{2,4,6,8,\}$. Subsets are represented using a Venn diagram as shown in Figure 3.1

Figure 3.1: Set and Subset



The empty set $\emptyset$ is a proper subset of every set.

Equal sets: Sets A and B are said to be equal if they contain the same elements, denoted as $A = B$.

Equivalent sets: The number of elements in one set must be equal to the number of elements in another set. For example, sets $A = \{1, 2, 3\}$ and $B = \{A, B, C\}$ are equivalent sets. Equal sets are equivalent sets but equivalent sets are not equal sets.

Ordered sets: A set is an ordered set if the order of elements is considered. This implies that if a set is $S = \{1,2,3,4,5\}$, then it is different from $S' = \{2,1,3,4,5\}$ though containing the same elements.

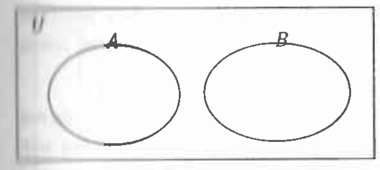
Similar sets: Sets are said to be similar if there is one-to-one correspondence established between ordered sets.

$$
\begin{array}{l}
S_1 = \{1,\ 2,\ 3,\ 4\} \\
\qquad\ \Downarrow\ \Downarrow\ \Downarrow\ \Downarrow \\
S_2 = \{4,\ 3,\ 2,\ 1\}
\end{array}
$$

A similar set is an equivalent set but a stronger version of an equivalent set. In a similar set, the elements are ordered, so the first element in S_1 must correspond to the first element in S_2 and so on, but in an equivalent set, there is just a one to one correspondence regardless of the order. That is to say, similarity is stronger than equivalence.

Disjoint sets: Two sets A and B are called disjoint sets if they have no common elements. For example, $A = \{1,2,3,4\}$ and $B = \{6,7,8,9\}$ are disjoint. They are depicted in Figure 3.2

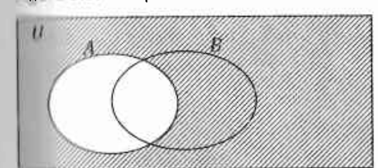
Figure 3.2. Disjoint Sets



Universal or parent set: this set consists of all the possible elements. In a Venn-Euler diagram, it consists of all the elements within and outside the sets (circles) but within the box. It is a super set consisting of all the sets under consideration. The universal set is normally denoted by the letter U or E (See Figure 3.2). For instance, when we discuss the set of letters in some words in English, the universal set is the set of 26 English alphabets.

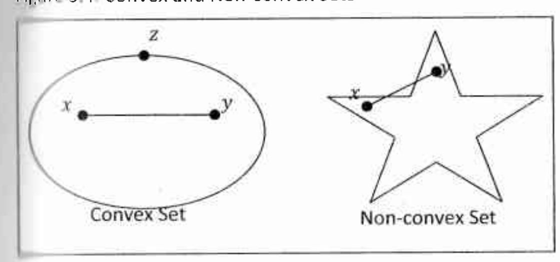
Complement of a set: If the set A is a subset of the universal set U, then the complement of A with respect to U is the set of all those elements of U which do not belong to A, denoted as A' or A^c. Alternatively, $A^C = \{x: x \in U \text{ and } x \notin A.\}$.Therefore the complement of any set is the set of all those elements that do not belong to the particular operation of the set but belong to the universal set. Figure 3.3 below shows A'.

Figure 3.3. Complement of a Set



Convex set: If we let $x \in S$ and $y \in S$ and $\lambda \in [0,1]$, and if $[\lambda x + (1 - \lambda)y] \in S$, then we say that set S is a convex set. This implies that for a set to be convex, any two points joined in the set must lie entirely within that set.

Figure 3.4. Convex and Non-convex Sets

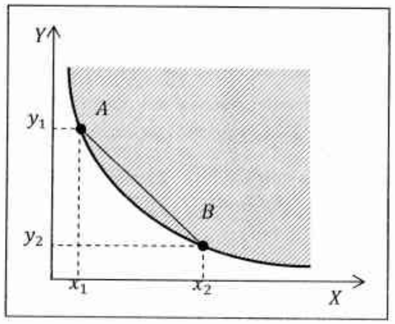


Closed and Open Sets: In Figure 3.4 above, point z is a *boundary point* where as x and y which lie strictly within the set are *interior points*. A set which includes the boundary points in addition to the interior points is called a *closed set*. A set which includes only the interior points and not the boundary points is an *open set*.

It is important to note that the concepts of being open and closed in the context of sets do not correspond to their being physically open and closed. As we shall see in Chapter 13 dealing with Linear Programming, the feasible region of a linear programme may appear physically open but is always a closed convex set.

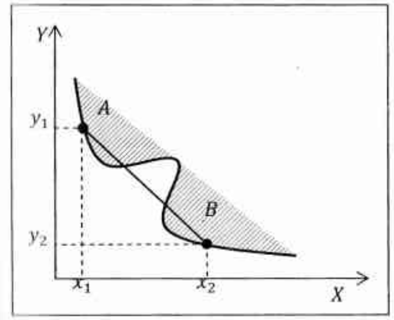
In economics, the concept of convexity is used in consumer theory when drawing the indifference curve, that is, it must be drawn in such a way that it conforms to the notion of the convex set. This is because the principle of diminishing marginal rate of substitution uses the mathematical notion of a convex set.

Figure 3.5. A convex Indifference Curve



The straight line lies completely in the shaded area. Point A contains more of Y and less of X. Point B contains more of X and less of Y. Therefore, more balanced bundles are preferred to extreme ones. As one moves down along the curve AB, the marginal rate of substitution of the good X for Y diminishes, this means you are willing to give up less and less of Y as we move down the curve. By contrast, consider Figure 3.6 below:

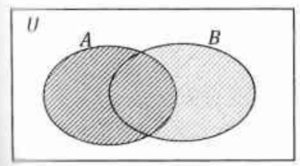
Figure 3.6. Non-convex indifference curve



This function does not obey the assumption of convexity and cannot be used to depict indifference curve. The counterpart of indifference curves is isoquants used in production theory which also must obey convexity for the various assumptions to hold.

3.5 Algebra of Sets

Union of sets: take two sets A and B, the union of the two sets denoted by $A \cup B$, consists of all elements in A and all elements in B and all elements common in A and B.This is represented by the shaded area in the diagram below.

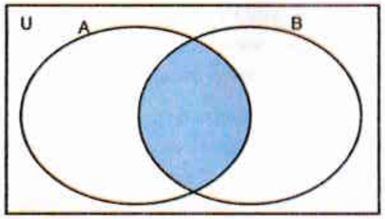


The following are some standard relations on the union of sets:

i) $A \cup A = A$, idempotent property
ii) $A \subseteq B$, then $A \cup B = B$ and if $B \subseteq A$, then $A \cup B = A$
iii) $A \cup A' = U$
iv) $A \cup \emptyset = A$, identity property
v) A and B are both subsets of $A \cup B$
vi) If $A \subset B$ and $B \subset C$, then $A \cup B \subset C$, Transivity.
vii) Commutative property, $A \cup B = B \cup A$.

Intersection of a set: This is denoted by $A \cap B$ and includes all the elements common to both A and B as shown by the shaded area in the diagram.

Figure 3.7. Set intersection



The following are some standard relations on intersection sets:

i) $A \cap A = A$, idempotent property
ii) If $A \subseteq B, A \cap B = A$
iii) $A \cap A' = \emptyset$
iv) $A \cap \emptyset = \emptyset$, identity law
v) $A \cap B$ is a subset of both A and B.
vi) If C is a subset of both A and B, then $C \subseteq A \cap B$.
vii) $A \cap B = B \cap A$, commutative property

Difference of two sets: $A - B$ is a set of all elements of A that do not belong to B.

$$A - B = \{x: x \in A, x \notin B\}$$

If $A - B = A$ and $B - A = B$, then A and B are disjoint.

$A - B = B - A$ Only when $A = B$

3.5.1 Summary of Set Theorems

The following theorems of sets will prove helpful in dealing with set problems.

1. **Equality of two sets:**
   Two sets A and B are equal if and only if A is a subset of B and B is a subset of A .Written as:
   $$A = B \text{ iff } A \subseteq B \text{ and } B \subseteq A$$
2. **Idempotent law:**
   $(a)\ A \cup A = A\ and\ (b)\ A \cap A = A$
3. **Identity law:**
   If A is any set, $\emptyset$ is the null set and U, the universal set,
   $$A \cup \emptyset = A$$
   $$A \cap U = A$$

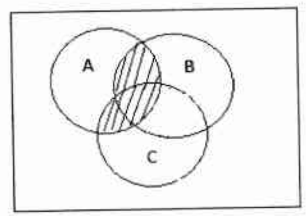
4. **Commutative law of union of two sets**

The order in which the sets in a Union are arranged is immaterial. The Union simply means elements belonging to some of the sets or all of them. It matters not which is written first.

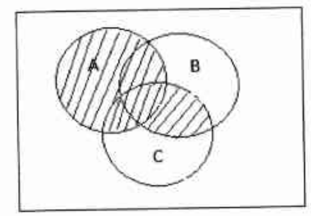
$$A \cup B = B \cup A$$

5. **Cumulative law of two sets:**
   The intersection of two sets given any order are equal, this is similar to the rule of numbers $ab = ba$. The law is expressed as:
   $$A \cap B = B \cap A$$
6. **Distributive property of union and intersection:**
   Similar to the rule $a(b + c) = ab + bc$, this can be illustrated as follows:
   The union of sets is distributive over the intersection:
   $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
   The intersection of sets is distributive over the union of sets:
   $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Figure 3.8. Illustration of Distributive law



i. $A \cap (B \cup C)$



ii. $AU(B \cap C)$

7. **Associative property of sets**
   $(A \cup B) \cup C = A \cup (B \cup C) = C \cup (A \cup B) = (C \cup A) \cup B$
   $(A \cap B) \cap C = A \cap (B \cap C) = C \cap (A \cap B) = (C \cap A) \cap B$
8. **De Morgan's law**
   The complement of the union of two sets A and B is the intersection of their complements. That is,
   $$(A \cup B)' = A' \cap B'$$
   The complement of an intersection of two sets A and B is the union of the complements.
   $$(A \cap B)' = A' \cup B'$$

9. Theorem:

a) $A \cap (B - C) = (A \cap B) - (A \cap C)$

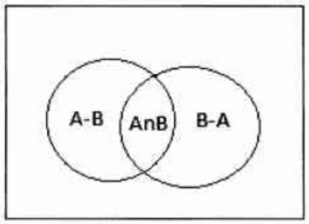
b) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

3.5.2 Number of elements in sets:

This is simply denoted by $n(A)$ and represents the total number of elements in a particular set. The total number in the union of two sets can be shown as follows:

$$\begin{aligned} n(A \cup B) &= n(A - B) + n(B - A) - n(A \cap B) \\ &= n(A) + n(B) - n(A \cap B) \\ &= n(A) + n(B) \text{ if A and B are disjoint} \end{aligned}$$

Figure 3.9.Illustration of number of elements in a Union set



In a case of 3 sets, A, B and C, the number of elements in the union set can be expressed as follows:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

Example 3.2

A marketing survey of a company wishes to find consumer preferences for three of its best selling products A, B and C. The following results were obtained:

The total number of consumers interviewed was 14,520. Of these, product A was preferred by 8150 consumers, product B was preferred by 5540 and product C by 4010. Products A and B were preferred by 3150 persons. Products B and C were preferred by 1820 persons and consumers who preferred A and C were 1030. The persons who did not prefer any of the three products were 2520.

Find:

a. The number of persons who liked all the three products.
b. The number of persons who liked A but not B.
c. The number of persons who liked A only.

The solution is as follows:

a. To find $n(A \cap B \cap C)$,since we know the number of persons who did not prefer any of the three products(2520) and also know the total number of persons interviewed (14520),we can find the number in the union set of the three products.

Therefore, the number of persons who liked at least one of the products is $14520 - 2520 = 12000$. Using the formula:

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$,

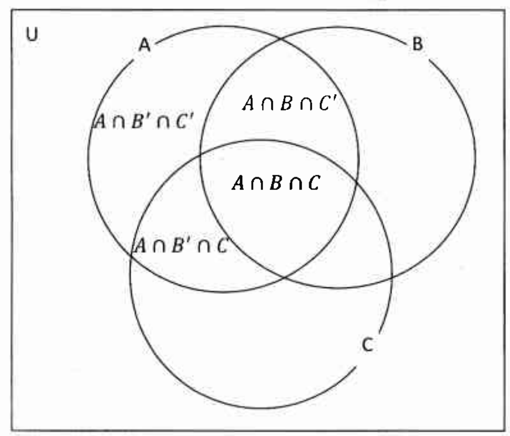
$12000 = 8150 + 5540 + 4010 - 3150 - 1820 - 1030 + n(A \cap B \cap C)$,

Therefore: $n(A \cap B \cap C) = 300$

b. To get the number that liked A but not B, we remind of the following set representations. $A \cap B' = A - B$, $B \cap A' = B - A$. Therefore, the number is derived using

$$\begin{aligned} n(A \cap B') &= n(A) - n(A \cap B) \\ &= 8150 - 3150 \\ &= 5000 \end{aligned}$$

c. **To find number of persons who liked A only, $n(A \cap B' \cap C')$, a glance at the graph below makes it clear:**



We proceed as follows: $n(A \cap B) = n(A \cap B \cap C) + n(A \cap B \cap C')$

$$3150 = 300 + n(A \cap B \cap C')$$

$$n(A \cap B \cap C') = 2850$$

And

$$n(A \cap C) = n(A \cap B \cap C) + n(A \cap B' \cap C)$$

$$1030 = 300 + n(A \cap B' \cap C)$$

$$n(A \cap B' \cap C) = 730$$

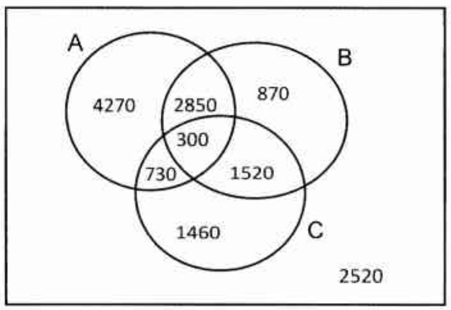
But

$$n(A) = n(A \cap B' \cap C) + n(A \cap B \cap C') + n(A \cap B' \cap C) + n(A \cap B \cap C)$$

$$8150 = n(A \cap B' \cap C') + 2850 + 730 + 300$$

$$n(A \cap B' \cap C') = 4270$$

Therefore, the number of persons who liked only A is 4270. The obtained solution can be represented as follows in a Venn diagram:



3.6 Ordered Pairs and Cartesian Products

Given the extensive introduction to sets, we are now in a position to introduce the concepts of ordered pairs and Cartesian products. It should be recalled that capital letters represents *sets* while small letters represent *elements*.

3.6.1 An Ordered pair

An ordered pair is a set of elements for instance (x, y). *where* x is the first element and y is the second element. However, $(x, y) \neq (y, x)$, that is, the two sets are not commutative if the set is an ordered pair. The two elements nevertheless need not be distinct: this simply means that (x, x) or (y, y) is possible where as if it is a set and not a pair, $\{x, x\} = \{x\}$. In general, if set A has n elements and set B has m elements, we can form $m \times n$ ordered pairs. This leads us to the concept of Cartesian products.

3.6.2 The Cartesian product

This is the product set of A and B, it consists of the set of all ordered pairs (x, y) where $x \in A$ *and* $y \in B$. For instance, if $A = \{a, b, c\}$, $B = \{1,2\}$, the Cartesian between the sets A *and* B denoted by $A \times B$ is shown as follows:

- $A \times B = \{(x, y): x \in A, y \in B\}$, this is a rule method of representation earlier discussed.
- $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$, this is a tabular way of representation. The product was obtained by "multiplying" every element in A with all the elements in B.This has formed "3 x 2" ordered pairs.

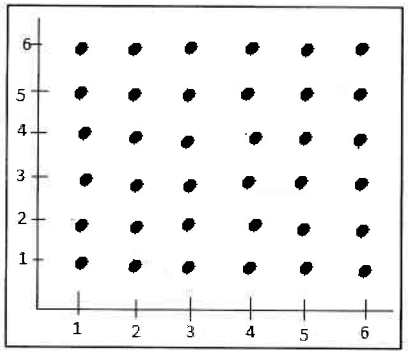
Note that the product $A \times B \neq B \times A = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}$ as earlier mentioned because the latter case is a "2 x 3". In the same way, the product $B \times B$ is given by $B \times B = \{(1,1)(1,2), (2,1), (2,2)\}$

3.7 What is a Relation?

Using the concept of ordered pairs and Cartesian products, you will notice that these terms are interrelated and that a function is a special type of a relation. Simply defined, a relation from x *to* y is a set of ordered pairs(x, y), such that to each $x \in X$, (each element that belongs to X) there corresponds at least one $y \in Y$. Note that x and y need not include all the elements in X *and* Y. To shed more light, let *X* and *Y* be two nonempty sets. A subset *R* of a Cartesian product $X \times Y$ is called a (*binary*) *relation from X to Y*. If *X* = *Y*, that is, if *R* is a relation from *X* to *X*, we simply say that it is a *relation on X*. Put differently, *R* is a relation on *X* iff (if and only if) $R \subseteq X^2(X \times X)$. If $(x, y) \in R$, then we think of *R* as associating the object *x* with *y*, and if $\{(x, y), (y, x)\} \cap R = \emptyset$, we understand that there is no connection between *x* and *y* as envisaged by *R*. In concert with this interpretation, we adopt the convention of writing *x R y* instead of (*x*, *y*) ∈ *R* throughout this text.

For illustration, consider all the possible outcomes when a die is tossed twice. Define set A to be outcomes from the first throw and set B to be outcomes from the second throw. Let the ordered pairs be denoted by(x, y), where $x \in A, y \in B$. The ordered pair $A \times B$ is a Cartesian product which contains 36 ordered pairs as shown below.

Figure 3.10. All outcomes of a throw of a pair of dice



Let R be the relation such that the sum of the 1st and 2nd throw is greater than 8. The solutions are the ordered pairs:

$$R = \{(3,6),(4,5),(4,6),(5,4),(5,5),(5,6),(6,3),(6,4),6,5),(6,6)\}$$

Alternatively, this can be represented in rule form as:

$$R = \{(x,y): x + y > 8, (x,y) \in A \times B\}$$

Therefore, given sets A and B, a relation R is a subset of the Cartesian product $A \times B$. The elements of the subset R are ordered which are ordered pairs (x,y) such that $x \in A, y \in B.x$ and y need not include all elements of A and B. In the above example for instance, x takes on values 3, 4,5 and 6, y too takes on the same values, therefore, the domain and range are identical in this case.

3.7.1 Domain, Range and inverse of a relation

Assuming two sets A and B as in the above example, where $x \in A,$ and $y \in B$. The *domain* and *range* of relation are defined as follows. The *domain* of the relation is defined as a set of elements x paired with y in (x,y) which belongs to R. This can be represented as:

$$X = \{x: for\ some\ y, (x,y)\ \in R\}$$

It is called the domain of the relation R. The *range* of the relation is defined as a subset of y in B over which y varies: Like the domain, it is represented by:

$$Y = \{y: for\ some\ y, (x,y) \in R\}$$

it is read as the range of R or range R. Therefore, the first element (x) in an ordered pair forms the domain while the second element (y) in an ordered pair forms the range.

Inverse of a relation: suppose R is a relation from $A\ to\ B$, the inverse of R denoted by R^{-1} is simply a relation from $B\ to\ A$. In other words, the domain and range are reversed.

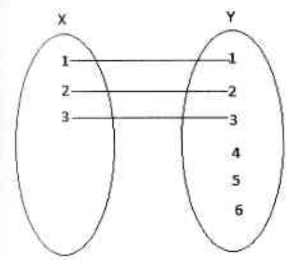
Basic notes of relations

i. A Relation R in a set A is called *reflexive* if $aRa, that\ is, (a,a) \in R$ for every $a \in R$
ii. A relation R in a set A is called *symmetric* if whenever $aRb, then\ bRa$, that is if $(a,b) \in R \Rightarrow (b,a) \in R$
iii. A relation R in a set A is called transitive if when aRb and bRa, then aRc.that is, $(a,b) \in R\ , (a,b) \in R \Rightarrow (a,c) \in R$

3.8 Mapping of Sets

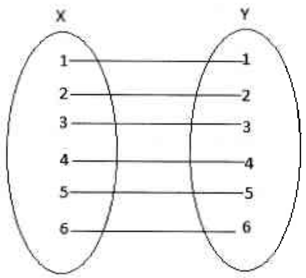
Ordered pairs obtained from functions (special type of relation) and relations can be represented by Mappings.

For instance, Consider two sets X and Y; $X = \{1,2,3\}$, $Y = \{1,2,3,4,5,6\}$



From the above mapping, it can clearly be seen that all elements of X have a corresponding element of Y, while only three elements of Y are paired with X. In such a correspondence where all elements of Y are not paired with the elements of X, we say that the set X has been ***mapped into the set*** Y. The Cartesian product gives a total of 18 ordered pairs, given $X = \{1,2,3\}$, $Y = \{1,2,3,4,5,6\}$.However; the above mapping shows only the ordered pair $(1,1),(2,2),(3,3)$. It is thus, a relation (also a function as will be seen later)

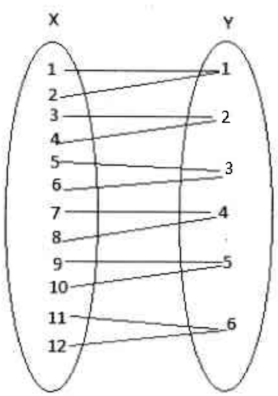
Consider another example, $X = \{1,2,3,4,5,6\}$, $Y = \{1,2,3,4,5,6\}$



If every element in X is matched with every corresponding element in Y and vice versa, then X has been ***mapped onto*** Y. For X and Y, there are 36 ordered pairs when their Cartesian product is obtained. The six ordered pairs constitute the above mapping which is a relation (this is also a function as will be seen later).

Consider another mapping:

X={1,2,3,4,5,6,7,8,9,10,11}, Y={1,2,3,4,5,6}



Since to every element in X there corresponds an element in Y and every element in Y is associated with at least one element in X, in this case, we also say that X has been ***mapped onto*** Y. The above mapping shows 12 ordered pairs out of the 12×6 ordered pairs of the Cartesian product $X \times Y$. From the examples shown above, what is common in each case is that regardless as to whether the set X was mapped *onto* or mapped *into* Y, for every element

of X, there is associated only one unique element of Y.This moves us to the introduction of functions.

3.9 What is a Function?

Functions are important in the economic study of the relationships between various cost curves such as average variable cost, total cost and marginal cost. Functions can be used to show explicitly the relationships between these costs and also shows us that if one of the cost curves has a particular shape, each of the others has another specific shape. The positions at which the average and marginal curves cross one another can also be exactly determined. In production theory, functions can be used to show the relationship between total, average and marginal products. This section thus gives insight as to what functions entail.

Simply defined, a function is a relation f (i.e. a subset of ordered pairs) such that to each element $x \in X$, there is a unique element $y \in Y$. It is a subset of ordered pairs characterised by certain given conditions, Denoted as $f: X \rightarrow Y$

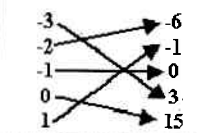
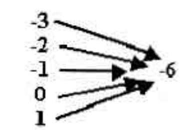
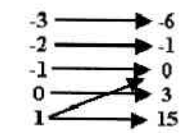
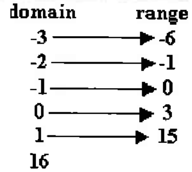
Read as: f is a function that maps X onto Y

i) The domain of f is equal to X
ii) To each X, there corresponds a unique $y \in Y$

$y = f(x)$ is an often-used expression to show the association between the element x and the corresponding unique element of the ordered pair (x, y) that is an element of the function f. x is sometimes called the *argument or independent variable* of the function and y is the entry into $y = f(x)$, called the *dependent* variable. The correspondences can be one of the following:

1. One-to-one: e.g., $y = x + 2$, for every x in X, there will correspond one and only one y in Y,and for every y in Y, there will correspond one and only one x in X.
2. Many-to-one: e.g. $y = x^2 + 3x - 2$, for one y in Y,there will correspond more than one x in X
3. One-to-many: e.g. $y^2 = x + 4$, for one x in X, there will correspond more than one y in Y
4. Many-to many: e.g. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, for more than one x in X, there will correspond more than one y in Y.

Therefore, the values taken by x form the *domain* while the values of y obtained from the *range*. Consider the following example of mappings to test your understanding of functions:

<table>
<tr><td></td><td>This is a function. You can tell by tracing from each x to each y. There is only one y for each x; there is only one arrow coming from each x. This is a one to one correspondence.</td></tr>
<tr><td></td><td>This is a function. There is only one arrow coming from each x; there is only one y for each x. It just so happens that it's always the same y for each x, but it is only that one y.</td></tr>
<tr><td></td><td>This one is not a function: there are two arrows coming from the number 1; the number 1 is associated with two different range elements. So this is a relation, but it is not a function.</td></tr>
<tr><td></td><td>Each element of the domain that has a pair in the range is nicely well-behaved. But what about that 16? It is in the domain, but it has no range element that corresponds to it! So then this is not a function. It is not even a relation.</td></tr>
</table>

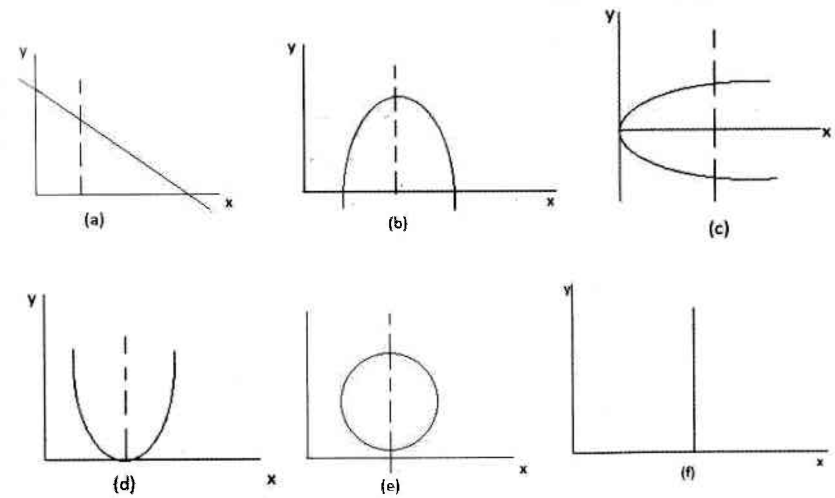
Example 3.3

This example will help exercise your graphing skill of equations. Thus, if you are given data, you should be able to produce total revenue curves, costs or any other curves in economics and various sciences.

Graph the following equations and determine which among them are functions.

a. $y = -2x + 6$
b. $y^2 = x$
c. $y = x^2$
d. $y = -x^2 + 6x + 14$
e. $x^2 + y^2 = 64$
f. $x = 5$

Solutions:



a) $y = -2x + 6$ is a function because for each value of the independent variable x there is one and only one value of the dependent variable y. For example, if $x = 1$, $y = -2(1) + 6 = 4$. The graph would be similar to (a).This is called a linear function used to represents some constraints in utility and profit maximisation.

b) $y^2 = x$, which is equivalent to $y = \pm\sqrt{x}$, is not a function because for each value of x, there are two values of y. For example, if $y^2 = 9$, $y = \pm 3$. The graph would be similar to that of (c) illustrating that a parabola whose axis is parallel to the x axis cannot be a function.

c) $y = x^2$ is a function. For each value of x there is only one value of y. For instance, if $x = -5, y = 25$. While it is also true that $y = 25$ when $x = 5$, it is irrelevant. The definition of a function simply demands that for each value of x, there be one value of y. The graph would be like (d).

d) $y = -x^2 + 6x + 14$ is a function. For each value of x there is a unique value of$y$. The graph would be like (b).

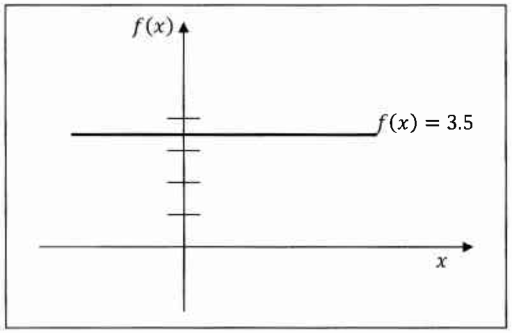
e) $x^2 + y^2 = 64$ is not a function. If $x = 0, y^2 = 64, and\ y = \pm 8$. The graph would be a circle, similar to (e).

f) $x = 5$ is not a function. The graph of $x = 5$ is a vertical line. This means that at $x = 5$, y has many values. The graph would be like (f).

3.10 Types of Functions

3.10.1 Constant Functions:

A function that maps every element of the domain to a single element of the co-domain is called a constant function .That is, the range of a constant function is always a *singleton* set. Generally, we write a constant function as $f(x) = k$, where k is a fixed number. For instance if $k = 3.5$, this can be shown graphically as:



As shown above, in the coordinate plane, the constant function will appear as a horizontal straight line. In economics, this graph is mainly used to depict the average revenue curve of the hypothetical competitive market which takes on a horizontal shape, implying that the prices, marginal revenue and average revenue are all equal, prices are plotted on the vertical axis are fixed.. Likewise, in national income models, when investment I is exogenuously determined, we may have an investment of the form $I = K10$ million, or $I = I_0$, which typifies the constant function.

3.10.2 Polynomial Functions:

A polynomial takes many different forms; it can be a constant function (as shown above), linear, cubic or even a quadratic function. Polynomials are widely used to model different maximisation problems of production as well as in consumer theory using the various cost, revenue, consumption, demand as well as supply functions. The word polynomial means "multi-term". A polynomial is an expression of one variable for instance y in the form

$y = a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_2x^2 + a_1x^1 + a_0x^0$,

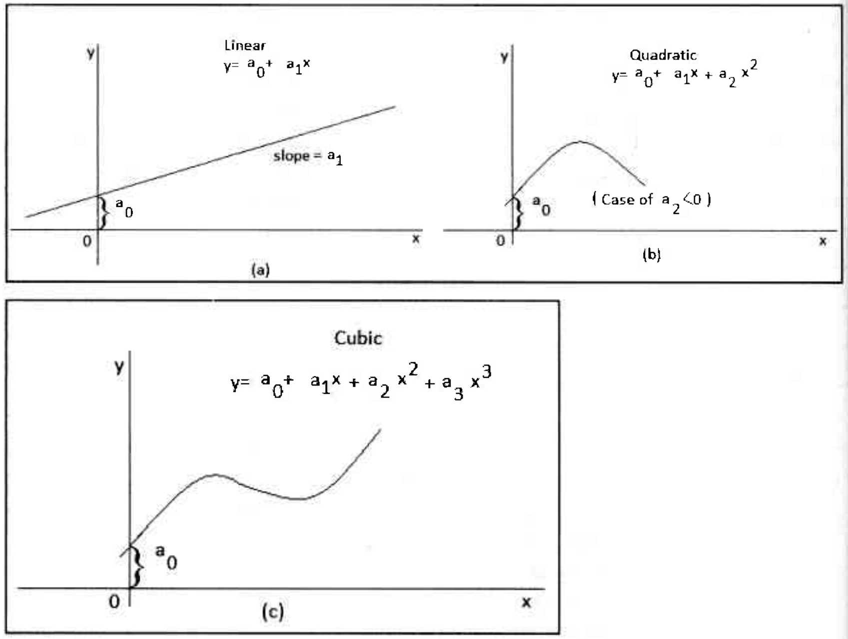
Where, $a_0, a_1, \ldots . a_n$ are real numbers with $a_n \neq 0$ and n is a positive integer. This implies that the last two expressions can simply drop to a_1x *and* a_0 respectively using the property of *exponents*[27]. Thus the constant function described above is actually a special case of polynomial functions, with $n = 0$. This can be derived as follows, substituting $n = 0$ in the above equation, all the terms drop except $y = a_0$. In the same line, various functions common to us can be derived by simply substituting n.Thus depending on the value of the integer n (which specifies the highest power of x), we have several subclasses of polynomial function:

Case of $n = 0$	$y = a_0$	**Constant function**
Case of $n = 1$	$y = a_1x + a_0$	Linear function
Case of $n = 2$	$y = a_2x^2 + a_1x + a_0$	Quadratic function
Case of $n = 3$	$y = a_3x^3 + a_2x^2 + a_1x + a_0$	Cubic function

When plotted in the coordinate plane, the aforementioned functions appear as follows. (a) illustrates the case of $a_1 > 0$, involving a positive slope and thus an upward-sloping line; if $a_1 < 0$, the line will be downward sloping. A quadratic function plots a parabola, a curve with a single built-in bump or wiggle. The particular illustration in figure (b) implies a negative a_2, a function with a maximum. In the case of $a_2 > 0$, the curve will open the other way, displaying a function with a minimum. The graph of a cubic function will, in general, manifest wiggles, as illustrated in figure (c), attaining both maximum and minimum values at its turning points.

[27] The subscript power (n) of the x is called the exponent. Any term raised to the power zero is one by property of exponents and any term raised to the power one is itself. The value of n, is often called the degree of the polynomial function; for instance, a quadratic function is a second-degree polynomial, and a cubic function is a third-degree polynomial. The order in which the several terms appear to the right of the equal sign is inconsequential; they may be arranged in descending order of power instead. Also, even though we put the symbol y on the left, it is also acceptable to write $f(x)$ in its place. (see Chiang & Wainwright, 2013.)

Figure 3.11. Illustration of functional forms



3.10.3 Rational functions

A *rational function* is any function which can be written as the ratio of two polynomial functions. Neither the coefficients of the polynomials nor the values taken by the function are necessarily rational numbers. According to this definition, any polynomial function must itself be a rational function because it can always be expressed as a ratio of 1, and 1 is a constant. Therefore, a constant function is a rational function.

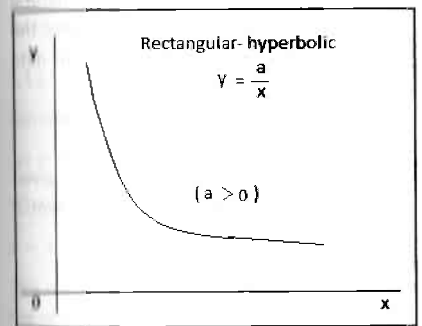
In the case of one variable x,a function is called a rational function if and only if it can be written in the form

$$f(x) = \frac{P(x)}{Q(x)}$$

Where P and Q are polynomial functions in x and Q is not the zero polynomial. The domain of f is the set of all points of x for which the denominator $Q(x)$ is not zero, where one assumes that the fraction is written in its lower degree terms, that is, P and Q have several

factors of the positive degree. A special rational function that has interesting applications in economics is the one depicted below. Since the product of two variables is a fixed constant, that is, $y = \frac{a}{x}$ implies $xy = a$, this can be used to depict the demand curve with prices (p) on the vertical axis and quantities (Q) on the horizontal axis. Similarly, the average fixed cost curve (AFC) is also rectangular-hyperbolic because AFC× Q (total fixed cost) is a constant.

Figure 3.12.Illustrations of a special form of rational function: Rectangular hyperbola



3.10.4 Exponential and logarithmic functions

Figure 3.13. Graphs of exponential and logarithmic functions

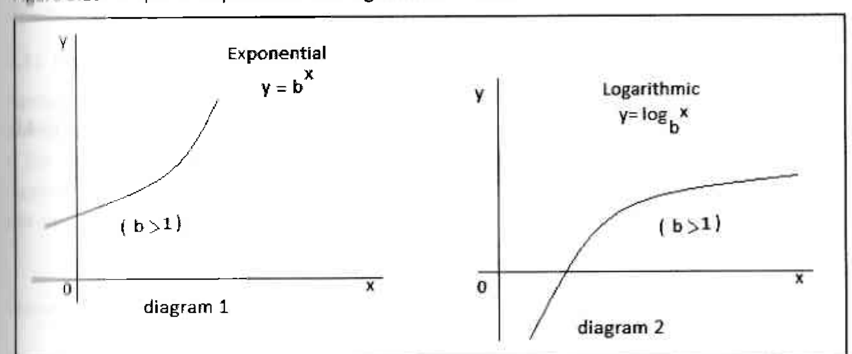


Diagram 1 depicts a function of the form $y = b^x$.This is called an exponential function where b is the base (positive real number) and x is the exponent. The prominent feature of this function is that the values of the function will always be positive for all real values of $x(-\infty, +\infty)$,. the domain is the set of real numbers while the range is $(0, \infty)$. Exponential functions with the base

e is greatly used in economics, where, e is a real number, whose value is approximately equals to 2.7182.For instance, the income y is an exponential function of time period t. That is, if y_0 is the initial income and r is the growth rate of income, then ,$y = y_0e^{rt}$.

Diagram 2 depicts a logarithmic function of the form $f(x) = log_b x$. Here b is a positive real number (b> 1) and called the base of the logarithmic function. Logarithmic functions are the inverse to the exponentional functions as clearly shown by the diagrams. Thus, its domain is $(0, \infty)$ and the range is $(-\infty, +\infty)$. When we use the number e as the base, we get natural logarithmic functions. When there is no ambiguity about the base, it is common to write the logarithmic function as: $f(x) = logx$. These are used in estimating growth rates from data points and also in compounding of interest rates in economics.

Example 3.4

Assume for a given principal P compounded annually at an interest rate i for a given number of years t will have a value S at the end of that time given by the exponential function,

$$S = P(1+i)^t$$

Example 3.5

A small firm with current sales of \$10,000 projects a 12 percent growth in sales annually. Its projected sales in 4 years are calculated in terms of an ordinary exponential function.

$$S = 10{,}000(1 + 0.12)^4$$

$$= 10{,}000(1.5735) = 15{,}735$$

Example 3.6

A 5-year development plan calls for boosting investment from 2.6 million a year to 4.2 million. What average annual increase in investment is needed each year?

$$4.2 = 2.6(1+i)^5$$

$$1.615 = (1+i)^5$$

$1 + i = \sqrt[5]{1.615}$ =1.10061

$$i = 0.10061 \approx 10\%$$

Example 3.7

A developing country wishes to increase savings from a present level of 5.6 million to 12 million. How long will it take if it can increase savings by 15 percent a year?

To solve an exponent such as t in $S = P(1+i)^t$, use the logarithmic transformation. Substituting the given values for the terms in the equations, we have

$$12 = 5.6(1 + 0.15)^t$$

Taking natural logs on both sides

$$\ln 12 = \ln 5.6 + t \ln 1.15$$

$$2.48491 = 1.72277 + 0.13976t$$

$$0.13976t = 0.76214, \quad \text{thus}, t \approx 5.45 \text{ Years}$$

1.11 Sequences

Consider the following expression,

Let $y = +\sqrt{4+x}$

When $x = 0, \ y = 2,$

$x = 1, \ \ y = +\sqrt{5}$, etc.

This is a sequence of values of y corresponding to integer values of x. A sequence is defined by the values of the function as x takes on positive values.

$y_n = \frac{1}{1+n}$, where $n = 1,2,3, \dots\dots,$ then $y_1, y_2, y_3, \dots\dots\dots\dots\dots\dots . . y_n$ will form a sequence. If $y = f(x)$, what then will be the value of y as x tends to some number? This brings us to the concepts of limits.

1.12 Limits of a Function

Fundamental question asked in limits is if you have any function, be it a consumption, production or demand function, what is the value of that function as it tends to a certain value? If the functional values $f(x)$ of a function f draws closer to one and only one finite real number l,for all values of x as x draws closer to a certain value "a" from both sides but does not equal "a" , l is called the limit of the function as x tends to a. This can be represented by,

$$\lim_{x \to a} f(x) = l$$

Therefore, for a limit to exist, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

Example 3.8

Find $f(x) = \frac{9x^2-4}{3x-2}$ as $x \to \frac{2}{3}$

Substituting the $x = \frac{2}{3}$ into the function gives an *indeterminate form* $f(x) = \frac{0}{0}$. The task is to change this to a determinate form by some algebraic operators.

$$\frac{9x^2-4}{3x-2}=\frac{(3x+2)(3x-2)}{3x-2}=3x+2$$

So,

$$\lim_{x\to\frac{2}{3}}\frac{9x^2-4}{3x-2}=4$$

Example 3.9

Given the function $f(x)=\frac{x^3-x^2-x-2}{x^3-3x^2+3x+2}$, find $\lim_{x\to 2} f(x)$

When 2 is substituted into the function, an indeterminate form results. The function gives zero both in the numerator and denominator. This means there is a common factor in the numerator and denominator which need to be factored out.

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2}\frac{x^2(x-2)+x(x-2)+(x-2)}{x^2(x-2)-x(x-2)+(x-2)}\\ &= \lim_{x\to 2}\frac{(x^2+x+1)(x-2)}{(x^2-x+1)(x-2)}\\ &= \lim_{x\to 2}\frac{(x^2+x+1)}{(x^2-x+1)}\\ &= {}^7/_3\end{aligned}$$

Example 3.10

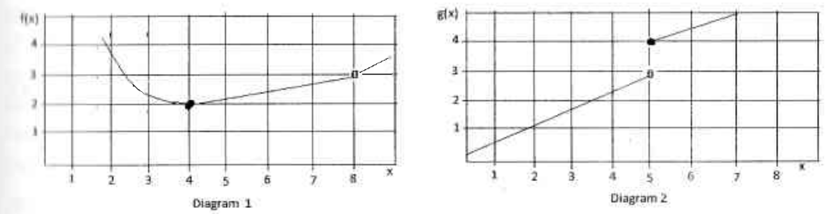
For the function $f(x)=\frac{(x+h)^3-x^3}{h}$, find $\lim_{h\to 0} f(x)$

Like in Example 3.9 above, the solution for this question cannot be found b merely substituting zero in place of h. This will result in an indeterminate form. To avoid this, a way has to be found that will negate the indeterminate form. We set the solution as follows.

$$\begin{aligned}\lim_{h\to 0} f(x) &= \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}\\ &= \frac{3x^2h+3xh^2+h^3}{h}\\ &= 3x^2+3xh+h^2\\ &= 3x^2\end{aligned}$$

Consider the following two functions, $f(x)$ and $g(x)$ graphically depicted in Figure 3.14:

Figure 3.14.Illustration of functions with and without limits



In diagram 1, it is clear that as the value of x approaches 4 from either side, the value of $f(x)$ approaches 2. This means the limit of $f(x)$ as x approaches 4 is the number 2, written as:

$$\lim_{x\to 4} f(x)=2$$

As x approaches 8 from either side in diagram 1, where the open circle in the graph of $f(x)$ signifies there is a gap in the function at that point, the value of $f(x)$ approaches 3 even though the function is not defined at that point.

In diagram 2, as x approaches 5 from the left (from values less than 5), written as $x\to 5^-$, $g(x)$ approaches 3, called a one-sided limit, as x approaches 5 from the right (from values greater than 5), written as $x\to 5^+$, $g(x)$ approaches 4, the limit does not exist, therefore, since $g(x)$ does not approach a single number as x approaches 5 from both sides.

Example 3.11

Suppose the cost function of a firm is $c(q)=Vq+K$, where q is output, V is variable cost per unit and K is fixed cost. What will be the average cost as output becomes very large?

Solution:

The average cost is simply the cost function divided by output; therefore, trying to find out the limit as q tends to infinity is as follows:

$$\lim_{q\to\infty}\left(V+\frac{K}{q}\right)=V$$

3.13 More on Sequences and Limits

Suppose you have the sequence

$$x_n=\frac{1}{n}$$

$$= 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots$$

As $n \to \infty, \frac{1}{n} \to 0$

If $x_n = \frac{1}{n} + 2$, as $n \to \infty$, the sequence tends to 2. In both cases, the sequence has a limit, it is a *convergent* sequence. A sequence that has no limit is called a *divergent* sequence.

e.g. $x_n = n + 3$; 4,5,6,7, … … … ..

3.13.1 Summary of properties for limits

Assuming that $lim_{x \to a} P(x)$ and $lim_{x \to a} Q(x)$ both exist, in a nutshell, the rules of limits are given below;

1. $\lim_{x \to a} h = h$ where h is a constant
2. $\lim_{x \to a} x^n = a^n$ where n is a positive integer
3. $\lim_{x \to a} hP(x) = h \lim_{x \to a} P(x)$ where h is a constant
4. $\lim_{x \to a}[P(x) \pm Q(x)] = \lim_{x \to a} P(x) \pm \lim_{x \to a} Q(x)$
5. $\lim_{x \to a}[P(x).Q(x)] = \lim_{x \to a} P(x). \lim_{x \to a} Q(x)$
6. $\lim_{x \to a}[P(x) \div Q(x)] = \lim_{x \to a} P(x) \div \lim_{x \to a} Q(x)$ where $[\lim_{x \to a} Q(x) \neq 0]$
7. $\lim_{x \to a}[P(x)]^n = \lim_{x \to a}[P(x)]^n$ Where $n > 0$

3.13.2 Key points about limits of functions

The following are the key points for limits of function:

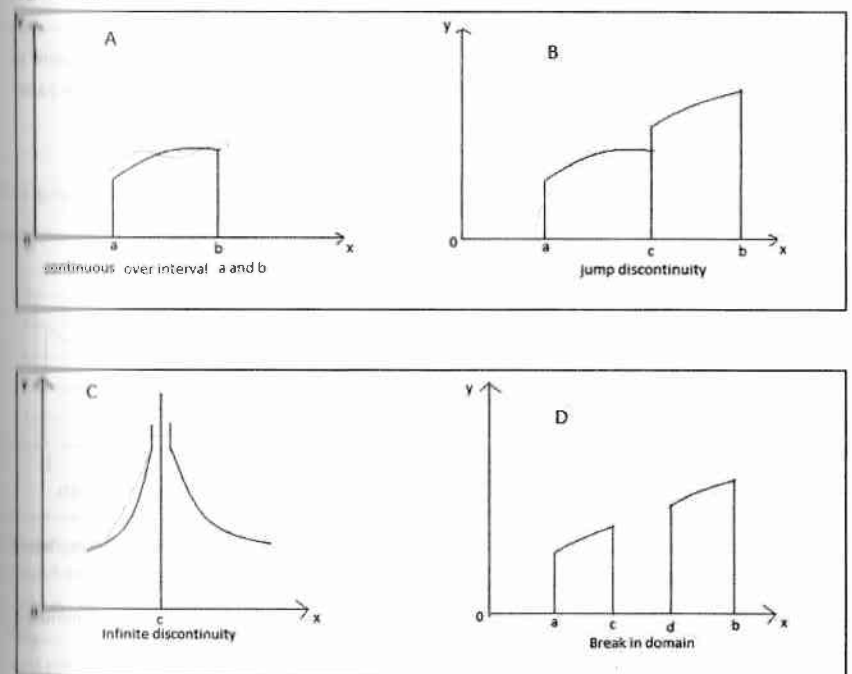
1. $\lim_{x \to x_{1+}}$ *means* the limit as x approaches x_1 from the right hand side and $\lim_{x \to x_{1-}}$ *is* the limit from the left hand side.
2. For a rational function (quotient of polynomials):
   a) If the degree of the numerator is less than the degree of the denominator, then the limits at ∞ and -∞ are both zero.
   b) If the degree of the numerator is the same as the degree of the denominator, then the limits at ∞ and -∞ are both the quotient of the coefficients of those of the highest degree.
   c) If the degree of the numerator is greater than the degree of the denominator, then as $x \to \infty$ or $x \to -\infty$, the function approaches either ∞ or -∞ according to the signs of the numerator and denominator.
3. A function $f(x)$ is continuous at x_1 in its domain if $\lim_{x \to x_1} f(x)$ exists and is $f(x_1)$. A continuous function is one that is continuous at every point in its domain.
4. Sums, products and quotients of continuous functions are again continuous functions. (Quotients are not defined where denominators become zero).
5. A differentiable function is continuous.

3.14 Continuity of a Function

In economic analysis, we are concerned with the continuity of functions as well as the existence of limits of those functions to forecast and predict certain outcomes in our economic. Suppose you are analyzing a discontinuous production function, what are the implications for the average and marginal product curves? Simply defined, a continuous function is defined as a function which has no breaks in its curve. Simply put, it can be drawn without lifting the pencil from the paper.

The following graphs give a clear picture of the concept of continuity and discontinuity.

Figure 3.15. Continuous and discontinuous functions



The above graphs are quite self explanatory. Graph (A) shows continuity over an interval a and b, this can clearly be drawn without lifting a pencil. Graph (B) shows a jump between a and b, thus it is discontinuous. Graph (C) shows infinite discontinuity as both curves will never meet despite showing convergence. Graph (D) shows a break in domain, the domain is from a to c, breaks off, then takes off from d to b. B and C are discontinuous in the range.

3.14.1 Point continuity

A function $f(x)$ is continuous at a point $x = a$ if for any positive value E, however small, there exists a positive number ∂ such that:

$| f(x) - f(a) | < E$, Provided $| x - a | < \partial$

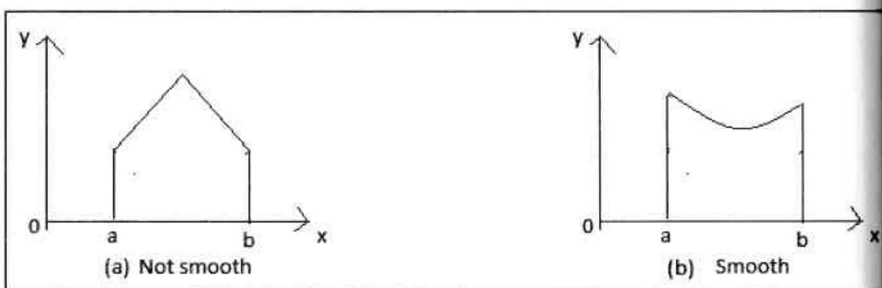
In other words, $f(x) \to f(a)$ as $x \to a$, where $x \to a$ either from the left or from the right. If the two one sided limits are unequal, the function is discontinuous at $x = a$. Therefore, a function f is continous at a point $x = a$ if the following hold:

1. $f(x)$ is defined, that is, it exits at $x = a$
2. $\lim_{x \to a} f(x)$ exists and
3. $\lim_{x \to a} f(x) = f(a)$

Therefore, a function is continuous over the interval (a, b) if it is continuous at every point in that interval. The graph between $x = a$ and $x = b$ can be drawn without lifting pen from paper as earlier stated.

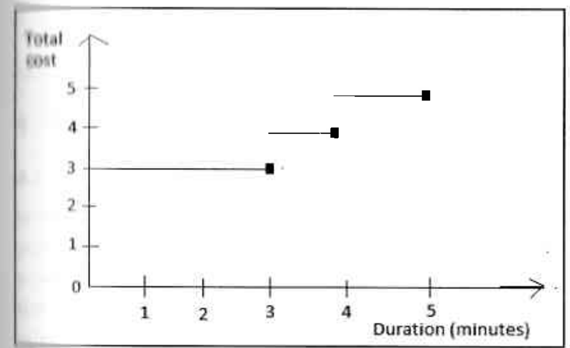
3.14.2 Types of continuous functions

Figure 3.16. Non-smooth and smooth functions



From the above graphs, (b) is a smooth continuous function whereas (a) is not completely smooth, it has a sharp points.

Consider the following discontinuous function: if international calls cost $K3$ up to 3 minutes, then a $K1$ every additional minute, the discontinuous function is obtained.



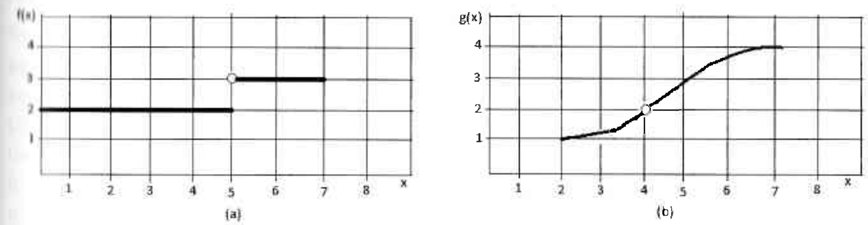
$$\lim_{t \to 3+} c(t) = 4, \lim_{t \to 3-} c(t) = 3$$

$$\lim_{t \to 4+} c(t) = 5, \lim_{t \to 4-} c(t) = 4,$$

$$\lim_{t \to 5+} c(t) = 6, \lim_{t \to 5-} c(t) = 5,$$

This goes on and on, showing discontinuity in the function.

Consider the following example to show that a limit may exist in a discontinuous function:



Since an open circle means a gap in the function and a continuous function can be sketched without ever removing pencil from paper, it is clear that $f(x)$ is discontinuous at $x = 5$ and $g(x)$ is discontinuous at $x = 4$. Diagram (a) shows that the limit does not exist ($\lim_{x \to 5-} f(x) = 2, \lim_{x \to 5+} f(x) = 3$). On the other hand, diagram (b) shows that the limit exists since $\lim_{x \to 4} g(x)$ is 2 approaching from both sides. The type of discontinuity depicted in (a) is a *jump discontinuity*. Discontinuity depicted in (b) is called a *removable discontinuity*, because if the function is redefined at x=4 such that $f(4) = \lim_{x \to a} f(4)$, it becomes continuous. Thus, a function may be discontinuous but a limit may exist. All polynomial functions are continuous, as are all rational functions, except where undefined, i.e., where that denominator equal zero.

Chapter 4

4 MATRIX ALGEBRA

4.1 Introduction

In the world of Economics, many characteristics or attributes are used to compare the performance or relative position of different countries, regions or even firms in a country. For a private firm for example, one might have to look at its market share, it size of sales revenue, the size of the work form and ultimately the level of profits either in absolute or in proportion to sales. To look at how two or more firms compare requires looking at this array of variables or information. For each firm, there must be a measure for market share, sales revenue, and the list goes on.

Given this information, the summation or subtraction for aggregation purpose must be permitted. If information is available for all the sectors in the economy, that is total output, the average wage rate, and so on, then it should be possible to compute, for the same variable, national statistics. If prices in different sectors change, *ceteris paribus,* should require getting new information on the *new* value of output? There should be no need to redo this because the change in prices can easily be factored in through multiplication.

This chapter is thus devoted to look at matrices (and their special forms, vectors) and their algebra. We explore with relevant examples the different operations of matrices and how matrices can be used to aid understanding of economic phenomena. As Chiang and Wainwright (2005)[28] put it, matrices provide a compact way of writing an equation system, a way of testing the existence of solution by evaluation of a determinant and ultimately providing a method of finding that solution (if it exists). Before endeavouring to look at the different operations of matrices, it is important to first consolidate the understanding of the matrix itself and the different forms it can manifest.

4.2 What is a vector?

Suppose that for each variable like GDP, there are observations for several countries, or for each country, observations are made on several variables. In the latter case, this gives an array of numbers representing the population size, level of GDP or per *capita* GDP and many other variables. The order in which these numbers are presented now matters as opposed to sets

[28] Chiang & K. Wainwright, 2013.

studied in the preceding chapter. The first number now stands for population size, the second for level of GDP and so on. This illustration gives rise to the vector.

Formally, a vector is defined as an ordered set of numbers and is denoted by X. Simply put, a vector is an ordered n-tuple. With n-elements, the vector is said to be of order n. The actual arrangement can take either a column way or a row way. In the case of the former, it is known as a column vector and the latter referred to as a row vector. Any given vector X with elements $x_1, x_2, \ldots,$ and x_n can be represented as

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

if it is a column vector and if a row vector, its appearance changes to

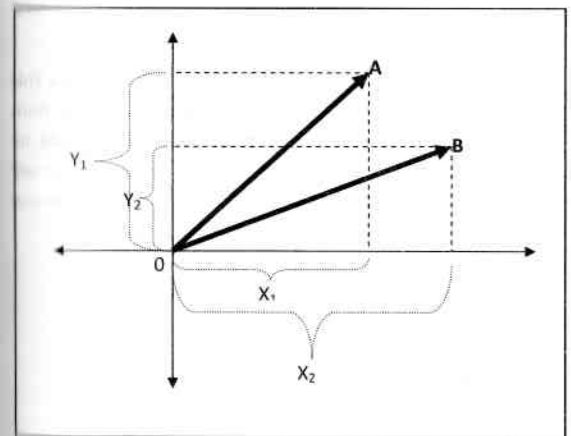
$$X_2 = (x_1 \ x_2 \ \cdots \ \cdots \ x_n)$$

The elements in the latter form are not separated by commas and this is not as a result of an error. It is deliberately so. In vectors, elements are not separated by commas but by spaces. Note that the natural order of a vector is taken to be a column. Changing from one form to another, from column to row or vice visa is called *transposing*. In the above two vectors, since the elements as well as the order remain the same, then one is a *transpose* of the other. This is denoted by a primed symbol or more unequivocally, by a superscript T. It is needless to say that a transpose of a transpose is a vector itself. Thus

$$X_2 = X_1' \quad \text{or} \quad X_2 = X_1^T$$

Since vectors are simply an ordered n-tuple, they can be sometimes be interpreted as a point on an n-dimensional space. Given two vectors **A** and **B** represented by $\binom{x_1}{y_1}$ and $\binom{x_2}{y_2}$ respectively. A and B can be represented as a points on the two dimensional Euclidian plane.

Figure 4.1: Diagramatical representation of Vectors



Having defined the vector and its forms, two special vectors deserve attention. These are *null and unit vectors*. If all the n-elements in a vector are identically zero, then the vector is referred to as a null vector. In the Euclidean plane, this is represented by a point at the origin. Since it is at the origin, it lacks both direction and magnitude, two critical requirements of a vector.

If however one and only one of the elements is unity and the rest remain identically zero, such a vector is call a unit vector. The term unit in this context emanates from its magnitude or length which, as will be discussed later in the chapter, is unit.

4.3 Vector Operations

Suppose there is a regional block of three countries, each with three production sectors or industries. These can be the Manufacturing, Agriculture and the Service sector and from each sector, output measures in physical units are obtained. In general, this is represented in a vector form $X = \begin{bmatrix} M \\ A \\ S \end{bmatrix}$ where the letters represent output from respective sectors. Subscripts can be added to represent countries. For a specific country, the output vector is written as $X_1 = \begin{bmatrix} 10 \\ 0 \\ 7 \end{bmatrix}$ which means Country one produced 10 units from the Manufacturing sector, nothing from the Agriculture and 7 units from the service sector. From the above vectors, many statistics can be generated. We can add country specific vectors to get output from the regional block as a whole. Alternatively, for each country, we may multiply each vector by a vector of prices for each sector's output. This gives a measure equivalent to the Gross Domestic Product. Thus vector operation is a look at how to add (subtract) vectors and multiplication of vectors. We begin with the former.

Since by definition vectors are ordered n-tuple, the summation then must also recognise this order. In the particular example under consideration, the first number represents output from the manufacturing sector. To add for different countries then, first-elements must add by themselves, the second-elements added and the same for the third. The resulting vector will show, for the regional block, how many units came from the Manufacturing, Agriculture and the Service sectors.

$$X = \begin{bmatrix} M_1 \\ A_1 \\ S_1 \end{bmatrix} + \begin{bmatrix} M_2 \\ A_2 \\ S_2 \end{bmatrix} + \begin{bmatrix} M_3 \\ A_3 \\ S_3 \end{bmatrix}$$

$$X = \begin{bmatrix} M_1 + M_2 + M_3 \\ A_1 + A_2 + A_3 \\ S_1 + S_2 + S_3 \end{bmatrix}$$

$$X = \begin{bmatrix} \sum_{i=1}^{3} M_i \\ \sum_{i=1}^{3} A_i \\ \sum_{i=1}^{3} S_i \end{bmatrix}$$

For each sector, the sum is made across the different countries so that the Manufacturing sector for example, its output in the three countries is added. The sum is the output from the sector in the regional block.

Example 4.1:

Given three vectors $A = \begin{bmatrix} 10 \\ 0 \\ 7 \end{bmatrix}$, $B = \begin{bmatrix} 8 \\ 5 \\ 11 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \\ 9 \\ 4 \end{bmatrix}$, find:

a) $A + B$,

$$A + B = \begin{bmatrix} 10 \\ 0 \\ 7 \end{bmatrix} + \begin{bmatrix} 8 \\ 5 \\ 11 \end{bmatrix}$$

Then add the individual elements in the two sets

$$A + B = \begin{bmatrix} 10 + 8 \\ 0 + 5 \\ 7 + 11 \end{bmatrix}$$

so that the final answer turns to be

$$A + B = \begin{bmatrix} 18 \\ 5 \\ 18 \end{bmatrix}$$

b) $B + A$

The procedure remains the same as in (a) above.

$$B + A = \begin{bmatrix} 8 \\ 5 \\ 11 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 7 \end{bmatrix}$$

Then add the individual elements in the two sets

$$B + A = \begin{bmatrix} 8 + 10 \\ 5 + 0 \\ 11 + 7 \end{bmatrix}$$

so that the final answer turns to be

$$A + B = \begin{bmatrix} 18 \\ 5 \\ 18 \end{bmatrix}$$

As would have been expected the answer is the same as in (a) above. Adding country A to B or country B to A should not change the results. This outcome leads us to perhaps one popular mathematical rule, that addition of vectors is *commutative*.

c) $A + B + C$,

$$A + B + C = \begin{bmatrix} 10 \\ 0 \\ 7 \end{bmatrix} + \begin{bmatrix} 8 \\ 5 \\ 11 \end{bmatrix} + \begin{bmatrix} 2 \\ 9 \\ 4 \end{bmatrix}$$

Then add the corresponding individual elements in the three sets

$$A + B + C = \begin{bmatrix} 10 + 8 + 2 \\ 0 + 5 + 9 \\ 7 + 11 + 4 \end{bmatrix}$$

so that the final answer turns to be

$A + B + C = \begin{bmatrix} 20 \\ 14 \\ 22 \end{bmatrix}$. This answer too enables us discover another rule related to commutative rule above. Notice that

$$(A+B)+C=\begin{bmatrix}18\\5\\18\end{bmatrix}+\begin{bmatrix}2\\9\\4\end{bmatrix}=\begin{bmatrix}20\\14\\22\end{bmatrix}$$

which is a necessary condition to show that addition of vectors is also *associative*

Two ingredients are needed for two or more vectors to be able to add. The first as noticed from the above example is that vectors should be of the same order, they should all have the same number of elements. In some cases however, one or more sectors are missing from one country such as the Aviation industry in Zambia. These can be shown with zeros against them as opposed to leaving the variable out. The second is that the vectors must be of the same type. There is need for all vectors in summation to be of one type, either column or row vectors.

The next and perhaps more intriguing operation of vectors is the multiplication of vectors. Vectors, like matrices, multiply in a rather special way. Elements of one row from the first vector correspond with elements of one column from the second vector. The multiplication is thus a scalar product of a row from the first vector and a column from the second. A necessary condition is that the two vectors be of the same order.

To illustrate this, take two vectors, $A=(a_1 \quad a_2 \quad \cdots \quad \cdots \quad a_n)$ and $B=\begin{bmatrix}b_1\\b_2\\\vdots\\\vdots\\b_n\end{bmatrix}$ The former is a row vector while the latter is column but of the same order since both have the same number (n) of elements. Since A is row and B is column, the requirement for multiplication is already met. This precludes the need to transpose any of the two. The multiplication thus proceeds as follows.

$$A*B=AB=(a_1 \quad a_2 \quad \cdots \quad \cdots \quad a_n)\begin{bmatrix}b_1\\b_2\\\vdots\\\vdots\\b_n\end{bmatrix}$$

$$AB=a_1b_1+a_2b_2+\cdots\cdots+a_2b_2$$

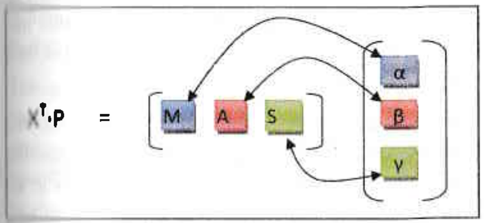
This is a scalar product that results from the multiplication of the two vectors.

In the above example of a three country regional block, multiplying any of the country-vectors is devoid of any economic meaning. Nonetheless, another vector can be defined so that its product with any of the above vectors carries economic meaning. Given two vectors **X** and **P** as follows:

$$X=\begin{bmatrix}M\\A\\S\end{bmatrix} \quad and\ P=\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}$$

The techniques of multiplication as shown in figure below.

Figure 4.2. Multiplication of Vectors



The first vector, transposed, is still the output vector developed earlier. The second is a vector of prices, giving prices for the respective sectors. The product of the two vectors is given by

$$A^T \cdot P=\alpha\cdot M+\beta\cdot A+\gamma\cdot S$$

This is invariably the value of total output, the GDP, because it is a summation of the value of output from all the three sectors.

Example 4.2:

A newly opened retail outlet in Woodlands market sells three brands of maize meal, National milling's Mama's pride, Simba milling's No1; and Superior milling's Mealile with the respective prices of $K46.50$; $K48.00$; and $K44.50$. If the volume of for the month of January is given by the vector $Q=\begin{bmatrix}q_{Na}\\q_{Si}\\q_{Su}\end{bmatrix}=\begin{bmatrix}45\\62\\51\end{bmatrix}$ find the Total Sales Revenue (TR) for the month of January.

$TR=P\cdot Q$ where P is the row vector of respective prices.

$$TR=(46.5 \quad 48 \quad 44.5)\begin{bmatrix}45\\62\\51\end{bmatrix}$$
$$TR=46.5*45+48*62+44500+51$$
$$TR=2{,}092.50+2{,}976.00+2{,}269.50$$
$$TR=K7{,}338.00$$

4.4 What is a matrix?

In the preceding subchapter of vectors, consideration was on a single economy with multiple sectors. Information however might need to be presented for a number of countries, each with respective sectors. This kind of information cannot be presented in a single vector but needs an array, with rows and columns. With rows representing countries, columns will be for sector. Each element in the array will be linked to a sector or a particular country.

In the case of a retail outlet, we considered only one outlet selling multiple maize meal brands. Suppose now that the outlet is just but part of a family of outlets. This must not be difficult to conceive. Imagine the same company operating the outlet at Woodlands market also has several others in different shopping malls around Lusaka. To present sales data, there must be a column for each brand of maize meal sold and a row for each outlet.

Since each element points to a sector of a particular country or a brand sold from a specific outlet, the arrangement of these elements is of critical importance. Each element has a specific position tied to a country and a sector and cannot be freely repositioned. This results in an ordered array or arrangement of numbers into columns and row. In mathematical language, such an array is called a matrix. Formally, a matrix is defined as an array of numbers, parameters or variables. Thus

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

is a matrix with three rows and two columns. It is consequently referred to as a three by two matrix commonly written in its short form $3 * 2$ and the matrix written as A_{3*2}. The subscripts on the elements represent the row and column. As a shorthand form, the above matrix can be written as $A = a_{ij}, i = 1,2,3 \; and \; j = 1,2$. The matrix a has elements a_{ij} where a_{ij} is an element in the i^{th} row and the j^{th} column. In general, we deal with matrices of dimensions $m * n$ where both any are any positive integers. If m or n but not both is unit, then the matrix is called a *vector*. Thus vector considered earlier are just a special type of matrices.

4.5 Types of Matrices

Matrices are of different dimensions or shape and can be written in other forms. Alternatively, they can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad A = \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{Vmatrix}$$

Matrices can take different forms and called by different names depending on the nature and arrangement of elements. If m and n remain unequal, the matrix is said to be rectangular because it takes a rectangular shape. When all the elements, regardless of the shape are identically zeros ($a_{ij} = 0, \forall \, i,j$), the matrix is called a null matrix denoted by O.

When m and n are equal, the number of row equal the number of columns, the matrix takes a square shape and is called a *square matrix*. The elements $a_{11}, a_{22}, \cdots, a_{mm}$ form the principal diagonal of the square matrix. For instance, the *Leontief Input-Output* analysis considered the linkages among many sectors of the economy. It shows how much each sector is producing and how that output is absorbed as input in other sectors as well as the sector itself. The rows and columns represent the input and output of the same sectors. As such, the input-output matrix is always a square matrix.

Taking the principal diagonal as a dividing line, the matrix is divided into two *triangles* with the same number of elements. These can be loosely called the upper and lower triangles. If all elements in one of these two triangles are zeros, then the matrix is called a *triangular matrix*. Specifically, the matrix is upper triangular if the non-zero elements are only in the upper triangle. When the non-zero elements are in the lower triangle, the matrix is called lower triangular matrix. The matrix A below is an example of a Lower triangular matrix while B represents the upper triangular case.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 7 & -2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & -3 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

But what if both triangles are zero? If all the elements off the principal diagonal are equal to zero, the matrix is known as a *diagonal matrix*. This is a matrix with both *triangles* equal to zero. The matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix of order 3 because non-zero elements are only along the principal diagonal and off the principal diagonal, all elements identically zero. In addition to being diagonal if the elements along the principal diagonal are not only non-zero, but are strictly unit, then the matrix is called a *unit* or *identity matrix* denoted by I. The identity matrix is of the form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ with zeros off the principal diagonal and units along. This is a special type of matrix which in matrix multiplication, behaves like a unit in scalar algebra.

The other pair of matrices deserving attention is the *symmetrical* and *skew-symmetrical* matrices. These relate to the mirror effect of the principal diagonal. If for a square matrix the mirror is placed along the principal diagonal, are elements in one triangle a reflection of elements in the other? If yes, then the matrix is symmetrical. The matrix M below is an example of a symmetrical matrix. The elements in the upper triangle are a reflection of the element in the lower triangle and vice-versa

$$M = \begin{bmatrix} 3 & 1 & 7 \\ 1 & 6 & -2 \\ 7 & -2 & 4 \end{bmatrix},$$

In some cases however, there could still be a reflection but the elements are changing signs, from positive to negative and vice-versa. Such a matrix is known as a *skew symmetrical matrix.* For the latter matrix, principal diagonal elements are equal to zero. Thus a square matrix $A = [a_{ij}]$ is *symmetrical* if $a_{ij} = a_{ji}$ for all i and j and is *skew symmetrical* if $a_{ij} = -a_{ji}$ for all i and j. Since the condition $a_{ii} = -a_{ii}$ cannot hold for non-zero elements, elements along the principal diagonal are equal to zero for a skew symmetrical matrix as stated a priori. An example of this type of matrix is given in N below.

$$N = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 2 \\ 3 & -2 & 0 \end{bmatrix}$$

The different operations and use of these matrices are considered in subsequent subsections.

4.6 Matrix Operations

Matrices have been defined as an array of data, representing a continuum of scenarios and contexts. There can be matrices representing output from various sectors of the economy, sales of different commodities, matrices of prices, and many other variables. Depending, on the type of data represented, there may be need to add (or subtract) or multiply matrices. Division of matrices is also an important part of matrix algebra as it enables equation solving. This section will deal with the addition and multiplication of matrices but postpone the discussion of the division aspect. Like under vectors, as special matrices, the subtraction of matrices can be imbedded in the addition and the discussion of the latter must be construed to include the former.

4.6.1 Addition of Matrices

In the addition of matrices, corresponding elements are added. This requires that for any two matrices to add, they must be of the same shape so that each element in one matrix has an element corresponding to its position in the other matrix. The position in matrices is defined by the row and column that the element is in. the sum will also be a matrix of the same order with elements equal to the sum of the corresponding elements of the two matrices adding. For two matrices A and B given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$$

The matrices are both of the order $4 * 3$ and each element in matrix A has an element in B corresponding to its position. This is also true about each element in B. To add the two matrices, we sum the corresponding elements as follows.

$$A + B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \\ a_{41}+b_{41} & a_{42}+b_{42} & a_{43}+b_{43} \end{bmatrix}$$

As a corollary, subtraction between any two matrices follows the same rules as addition. For the two matrices A and B defined above, then

$$A - B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} \\ a_{41}-b_{41} & a_{42}-b_{42} & a_{43}-b_{43} \end{bmatrix}$$

Both the sum and the difference are the sum and difference, respectively, of individual corresponding elements. The order of the resultant matrix also remains unchanged, in this particular case, $4 * 3$.

Example 4.3

Given two matrices defined by $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 5 & 4 \\ 4 & 7 & 3 \end{bmatrix}$ Find

a) A+B

$$A + B = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} -2 & 5 & 4 \\ 4 & 7 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 7 & 3 \\ 5 & 10 & 8 \end{bmatrix}$$

b) B-A

$$B - A = \begin{bmatrix} -2 & 5 & 4 \\ 4 & 7 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 3 & 5 \\ 3 & 4 & -2 \end{bmatrix}$$

is interested in the total number of each animal. This is given by the sum of $\begin{bmatrix} 2 \\ 7 \\ 15 \end{bmatrix} + \begin{bmatrix} 9 \\ 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \\ 25 \end{bmatrix}$ the two now have to decide whether to go to Livingstone, the tourist capital or Lusaka, the capital city. Both markets are of fairly the same distance hence the transportation cost. A rational farmer will prefer where one's total revenue is higher.

Prices in Lusaka for the three animal are given as follows: Goat: K 105, Pig: 320, Chicken: K 25, which can be expressed in a row vector $(105 \quad 320 \quad 25)$. For Livingstone, prices are slightly different and are represented by another row vector $(130 \quad 300 \quad 25)$. A price matrix can be defined with rows representing the two markets and columns for the different animals.

To get an idea of how farmers will decide, multiply the price matrix and a quantity matrix in which rows represent animals and columns for the two farmers.

$$\begin{bmatrix} 105 & 320 & 25 \\ 130 & 300 & 25 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 7 & 5 \\ 15 & 10 \end{bmatrix} = \begin{bmatrix} 2825 & 2795 \\ 2735 & 2920 \end{bmatrix}$$

In the product matrix, columns represent farmers while rows are markets. If they go to Lusaka, farmer 1 will make K 2, 825 while farmer 2 makes K 2, 795. Clearly, farmer 1 is better off with Lusaka while farmer 2 would prefer Livingstone.

Since they are using the same transport, they must choose one market. Which one they choose will depend on each farmer's bargaining power. In the extreme, farmer 2 may even consider compensating the other for the opportunity loss and still remain better-off.

4.6.3 Transpose

If columns of a matrix are written as rows, or *vice versa*, the rectangular shape of the matrix changes. If it was an $m \times n$, then after changing columns to rows and vice versa, the matrix becomes $n \times m$ in dimensions. The new matrix is known as a transpose of the former and vice versa. Like in the vectors discussed above, the transpose of a matrix is denoted by a primed symbol or a superscript T. Given a matrix A, then its transpose is denoted by A' or A^T. If $A = \begin{bmatrix} 1 & 4 \\ 3 & 7 \\ -2 & 5 \end{bmatrix}$, then A' or $A^T = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 7 & 5 \end{bmatrix}$

An element a_{ij} which is in the i^{th} row and j^{th} column in matrix A changes to a_{ji}, it is now in the j^{th} column and i^{th} row in the transposed matrix. In simple terms, to transpose is to mirror the matrix along the principal diagonal. Thus, transposing does not affect elements along the principal diagonal.

The above statement has implications on certain type of matrices. We defined above that symmetrical matrices have elements above the principal diagonal more of a reflection of elements below. Transposing such a matrix will leave it unchanged. Thus given a symmetrical matrix A, then $A' = A$. This result also extends to more special symmetrical matrices. The first is the diagonal matrices which has zero above and below the principal diagonal. For a diagonal matrix D, then $D' = D$. The second and perhaps more special is the identity matrix I which also conforms to the condition $I' = I$.

4.6.4 Three properties of transposes are worth stating:

Property I: When transposing a matrix, what are rows are written as columns and what are columns become rows. If this procedure is repeated, swapping again, then the result is the original matrix. The rows which had become columns are again rewritten as rows. Thus transposing a matrix twice leaves the matrix unchanged. This is the general result when the number of transposing is *even*. $(A')' = A$

Property II: Given two matrices A and B. If the product AB exists, then its transpose is equal to the product of transposed A and B taken in reverse order. That is $(AB)' = B'A'$.

Example 4.5

Given $A = \begin{pmatrix} 2 & -3 & 6 \\ 4 & 5 & 1 \\ 7 & -5 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 5 \\ -2 & 1 \\ 4 & 3 \end{pmatrix}$, then

$$(AB)' = \left[\begin{pmatrix} 2 & -3 & 6 \\ 4 & 5 & 1 \\ 7 & -5 & 4 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ -2 & 1 \\ 4 & 3 \end{pmatrix}\right]'$$
$$= \begin{bmatrix} 36 & 25 \\ 6 & 28 \\ 47 & 42 \end{bmatrix}' = \begin{bmatrix} 36 & 6 & 47 \\ 25 & 28 & 42 \end{bmatrix}$$

Notice that in this particular example, $A'B'$ does not even exist since the two matrices are not conformable. However, swapping the position in the multiplication evades the problem. The product now is the same as above, proving the law.

$$B'A' = \begin{bmatrix} 3 & 5 \\ -2 & 1 \\ 4 & 3 \end{bmatrix}' \begin{bmatrix} 2 & -3 & 6 \\ 4 & 5 & 1 \\ 7 & -5 & 4 \end{bmatrix}'$$
$$= \begin{bmatrix} 3 & -2 & 4 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 4 & 7 \\ -3 & 5 & -5 \\ 6 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 36 & 6 & 47 \\ 25 & 28 & 42 \end{bmatrix}$$

Property III: The transpose of a sum of matrices is a sum of the transposes. Algebraically, $(A+B)' = A' + B'$. Given two matrices A and B defined by $A = \begin{bmatrix} 3 & -1 \\ 5 & 4 \\ 2 & -4 \end{bmatrix}$ and

$B = \begin{bmatrix} 2 & 4 \\ -5 & 3 \\ 4 & 7 \end{bmatrix}$, then in the equation above we solve the two sides of the equation to show that the equation holds:

$$(A+B)' = A' + B'$$

$$\left[\begin{pmatrix} 3 & -1 \\ 5 & 4 \\ 2 & -4 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ -5 & 3 \\ 4 & 7 \end{pmatrix}\right]' = \begin{bmatrix} 3 & -1 \\ 5 & 4 \\ 2 & -4 \end{bmatrix}' + \begin{bmatrix} 2 & 4 \\ -5 & 3 \\ 4 & 7 \end{bmatrix}'$$

$$\begin{bmatrix} 5 & 3 \\ 0 & 7 \\ 6 & 3 \end{bmatrix}' = \begin{bmatrix} 3 & 5 & 2 \\ -1 & 4 & -4 \end{bmatrix} + \begin{bmatrix} 2 & -5 & 4 \\ 4 & 3 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 6 \\ 3 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 6 \\ 3 & 7 & 3 \end{bmatrix}$$

4.7 What is a determinant?

A determinant is a uniquely defined scalar or single number associated with a square matrix. For a matrix A, the determinant is denoted by $|A|$. This need not be confused with its use for absolute numbers. In scalar algebra, such type of brackets indicate that the algebraic sign of a number be ignored. In matrices however, this is just a way of showing a determinant which can unquestionably be negative. Care should also be exercised to distinguish this from the double bracket introduced earlier in the chapter. The determinant is an integral part of matrix algebra and will prove useful in latter concepts.

In a 2 × 2 matrix given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the determinant is of second order and is defined by

$$DetA = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$
$$= a_{11}a_{22} - a_{21}a_{12}$$

which is a scalar since it is a sum of products of scalars. For instance, the determinant of a matrix such as $\begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix}$ is obtained through cross multiplication as follows:

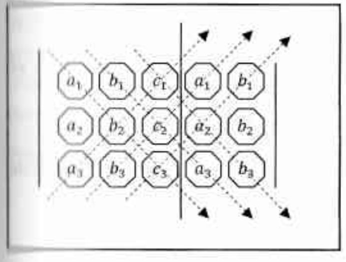
$$\begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} = 3 \times 1 - 5 \times 2$$
$$= 3 - 10 = -7$$

To obtain the determinant of higher order matrices, we have to resort to methods that will eventually reduce the calculation process to cross multiplication of second order determinants. Two methods stand out. The first is loosely referred to as *expansion by columns*. The second is the *Laplace Expansion*. We defer the latter and deal with the former now.

To get the determinant, for instance of a third order, write the matrix and repeat the first two columns after the third. This should result in five columns and the order should remain

unaltered. To clearly show this, the matrix is written with letters for columns and numerical subscripts for rows.

For the determinant $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, repeat the first two columns after the last one. The picture of the expanded matrix is of the form.



Each arrow represents a product of the three elements it joins. With the two pairs of arrows, sum the upward sloped and the downward sloped separately. The determinant is given by subtracting the upward arrow sum from the downward arrow sum. Thus

$$DetA = (a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1)$$

In the full equation, there are nine different elements, given a 3 × 3, and each element appear twice and only twice.

To understand the second method, we need to introduce some additional concepts which we do in the next subsection.

4.8 Minors and cofactors

The equation for the determinant given above can be factorised to make it look attractive. The selection of elements to factor has to be strategic however. The strategy is to factorise elements from one row or one column. For illustration purpose, we select the first column elements, the a_1 a_2 and a_3.

$$DetA = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$
$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$
$$= a_1\begin{vmatrix} * & \cdot & \cdot \\ \cdot & b_2 & c_2 \\ \cdot & b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} \cdot & b_1 & c_1 \\ * & \cdot & \cdot \\ \cdot & b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} \cdot & b_1 & c_1 \\ \cdot & b_2 & c_2 \\ * & \cdot & \cdot \end{vmatrix}$$

The parts in bracket or determinant bracket are determinants of 2 × 2 matrices. These are sub determinants of the matrix A which are multiplying with elements from the first column. Their composition is however not arbitrary. For the first sub determinant in the equation, it is as though the row and column of the element multiplying is deleted. That rule applies for the other two. The sub determinants are called *minors* denoted by M_{ij}. The minor M_{ij} is associated with the element in the i^{th} row and j^{th} column. It is gotten by 'deleting' the i^{th} row and j^{th} column and evaluating the determinant of the remaining sub matrix.

A concept more related with minors is that of *Cofactors* denoted by C_{ij}. A cofactor is a minor with an algebraic sign added. The rule for attaching a sign is as follows: if the sum of i and j for a specific minor is *even*, the minor takes a positive sign; otherwise it takes on a negative. This can be condensed in the equation $C_{ij} = (-1)^{i+j} M_{ij}$ where i and j are as defined *ex ante*.

Armed with this knowledge, we are now ready to evaluate a third-order determinant such as:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The method is as follows: the value of the determinant is obtained by the following equations where M_{ij} and C_{ij} are the minors and cofactors of the determinant.

$$\begin{aligned} DetA &= a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= \sum_{i=1}^{3} a_{ij}C_{ij} \end{aligned}$$

This method is what is referred to as the Laplace Expansion mention before. It is named after a 19^{th} century French mathematician and astronomer *Pierre-Simon Laplace*. We put it as the second and more formal method of evaluating a third order determinant. The above expression does not put any restriction on which column (or row) to use for expansion. The formulae works with any column or any row, provided the cofactor are adjusted accordingly. The expansion can still be used for higher order determinants but the procedure will be multistage[29]. In general, the determinant of a k^{th} order matrix A is given by

$$\begin{aligned} |A| &= \sum_{i=1}^{k} a_{ij}C_{ij} \\ &= \sum_{j=1}^{k} a_{ij}C_{ij} \end{aligned}$$

Example 4.6

Find the determinant of a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 6 & 2 & 5 \end{bmatrix}$

[29] At every use of Laplace Expansion, the cofactors are of order one less than the parent matrix. Thus given a fourth order determinant, the first application will produce third order Cofactors, which will in turn require the same procedure to reduce to second order.

Using the Laplace expansion, select the first row and expand using cofactor.

$$\begin{aligned} |A| &= 1\begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 5 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 6 & 2 \end{vmatrix} \\ &= 1(18) - 2(4) + 3(20) \\ &= -50 \end{aligned}$$

4.9 Properties of Determinants

With the determinant at hand, a couple of insights of the determinant are quite useful. These are called properties of determinants. It is essential to know how the determinants changes with the matrix changing form. This is simply an assessment of the behaviour of the determinant.

Property 1: If any two rows or columns are interchanged, the value of the determinant of the matrix only changes in sign. Swapping another pair of rows or columns (as the case may be) will again change the sign only. Since the change in sign is binary, negative or positive, an even number of row or column interchange will leave the determinant unchanged.

Consider a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ from the matrix A, if two matrices B and C are defined as

$$B = \begin{bmatrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \end{bmatrix} \text{ and } C = \begin{bmatrix} a_{12} & a_{13} & a_{11} \\ a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \end{bmatrix}$$

Matrix B is gotten by interchanging the second and third columns. The property states that the determinant of matrix B will be negative of the determinant of matrix A.

$$|B| = -|A|$$

The matrix C is derived by interchanging the first and third columns of matrix B. This means moving from A to C involves and even number of column swapping. Therefore, the determinant of C will equal the determinant of A.

$$|C| = |A|$$

Property 2: In a matrix, if any row or column can be expressed as a linear function of the remaining columns or row, then the determinant is zero. This embraces a more extreme case where two columns or rows are identical. Consider a matrix

$$A = \begin{bmatrix} 1 & 6 & 2 \\ 3 & 1 & 5 \\ 6 & 2 & 10 \end{bmatrix}$$

where the third row is twice the second one. We should then expect that the determinant is zero.

$$
\begin{aligned}
|A| &= \begin{vmatrix} 1 & 5 \\ 2 & 10 \end{vmatrix} - 3\begin{vmatrix} 6 & 2 \\ 2 & 10 \end{vmatrix} + 6\begin{vmatrix} 6 & 2 \\ 1 & 5 \end{vmatrix} \\
&= 1(0) - 3(56) + 6(28) \\
&= 0
\end{aligned}
$$

This confirms our expectation that if any two columns or rows are linearly related, the determinant is zero. Caution should however be exercised so as not to confuse a linear and non-linear relationship. This property only applies to linear relationship and does not include a case where one row or column is a square of another.

Property 3: If any row or column in a matrix is multiplied by a constant k, the value of the determinant also multiplies by the same constant k. If another row or column is multiplied by the same constant, the determinant will increase by a square of the constant. In a more general case, the determinant will increase by k^{μ} where μ is the number of columns or rows scaled up by k. Given a matrix $A = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$ its determinant can be calculated with ease $|A| = -7$. If the first row is multiplied by 3 for instance, the new matrix is $A_1 = \begin{bmatrix} 6 & 9 \\ 5 & 4 \end{bmatrix}$. The determinant of the new matrix is

$$
\begin{aligned}
\begin{vmatrix} 6 & 9 \\ 5 & 4 \end{vmatrix} &= 6(4) - 5(9) \\
&= -21 = 3\begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix}
\end{aligned}
$$

Property 4: if any row is changed by subtracting or adding a multiple of another row or column as the case may be, the value of the determinant is unaltered. Consider a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Its determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. From the first column, subtract the second column twice so that the new matrix is $\begin{bmatrix} a-2b & b \\ c-2d & d \end{bmatrix}$ and proceed to calculate the new determinant.

$$
\begin{aligned}
\begin{vmatrix} a-2b & b \\ c-2d & d \end{vmatrix} &= d(a-2b) - b(c-2d) \\
&= ad - \cancel{2bd} - bc + \cancel{2bd} \\
&= ad - bc \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix}
\end{aligned}
$$

4.10 Inverse of a matrix

In matrix algebra, the division of matrices is inconceivable. Matrices cannot be divided like scalars and thus dividing two matrices such as $\frac{A}{B}$ is not permissible. But then how can one solve an equation of the form $Ax = B$ other than by writing $x = {}^{B}/_{A}$ as in scalar algebra? The

concept of *inverse* of a matrix is the answer. Given a non-singular matrix A, a matrix that pre-multiplies or post-multiplies by A and gives an identity matrix is called an inverse of A. it is denoted by A^{-1} and may be simply referred to as a reciprocal of A. For a singular matrix, the inverse does not exist and this can be revealed at determinant stage. With the problem at hand, the solution can now be put simply as

$$
\begin{aligned}
A^{-1}Ax &= A^{-1}B \\
Ix &= A^{-1}B \\
x &= A^{-1}B
\end{aligned}
$$

For a matrix $A = [a_{ij}]$, define a matrix of cofactors where each element in A is replaced by its cofactor. The new matrix will be $C = [C_{ij}]$, where C_{ij} is a cofactor of a_{ij}. Then take the product AC' with C' commonly referred to as *adjoint* of A. It is symbolised by $adj\ A$. The two matrices are conformable since they are square matrices of the same order. Thus:

$$
AC' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \cdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}'
$$

$$
= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \cdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}
$$

$$
= \begin{pmatrix} \sum_{j=1}^{n} a_{1j}C_{1j} & \sum_{j=1}^{n} a_{1j}C_{2j} & \cdots & \sum_{j=1}^{n} a_{1j}C_{nj} \\ \sum_{j=1}^{n} a_{2j}C_{1j} & \sum_{j=1}^{n} a_{2j}C_{2j} & \cdots & \sum_{j=1}^{n} a_{2j}C_{nj} \\ \vdots & \vdots & & \cdots \\ \sum_{j=1}^{n} a_{nj}C_{1j} & \sum_{j=1}^{n} a_{nj}C_{2j} & \cdots & \sum_{j=1}^{n} a_{nj}C_{nj} \end{pmatrix}
$$

In this expression, each element is a Laplace expansion for the determinant of A. for elements along the principal diagonal, elements from a particular column are being multiplied by their respective cofactors and should equal the determinant of A denoted by $|A|$. In off diagonal elements however, column elements are multiplied by what was referred to as *alien* cofactors above (the row value i defers between the element and multiplying cofactor). We already know this should equal to zero.

This produces a diagonal matrix with $|A|$ along the principal diagonal and zeros elsewhere

$$AC' = \begin{pmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & \cdots & |A| \end{pmatrix} = |A| \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$AC' = |A|I$$

Pre-multiplying both sides of the equation by A^{-1}, the inverse of A and making it the subject of formulae yields $AC' = |A|$

$$A^{-1}AC' = |A|A^{-1}$$
$$C' = |A|A^{-1}$$
$$\therefore\ A^{-1} = \frac{1}{|A|}C'$$

Note that $|A|$ is not a matrix but a scalar and so can be divided like an ordinary scalar. The determinant plays a critical role in the inverse of a matrix. In the equation, if the determinant $|A| = 0$, then the inverse is undefined, it does not exist. Such a matrix, with no inverse, is called a *singular matrix*. The existence of an inverse depends on having a non-zero determinant which also depends on linear independence of rows and columns.

Since we defined $C' = Adj\ A$, the inverse of A can also be written as

$$A^{-1} = \frac{1}{|A|}Adj\ A$$

In summary, getting the inverse of A requires the following three steps

1. Find the determinant of the matrix, $|A|$
2. Replace all the elements in the matrix by their respective cofactors to get the matrix of cofactors, C
3. Transpose the new matrix of cofactors C to get the adjoint of A ($Adj\ A$)

Then, the inverse of A is defined

$$A^{-1} = \frac{1}{|A|}Adj\ A$$

Example 4.7

For the matrix $A = \begin{bmatrix} 2 & 0 & 5 \\ 1 & 2 & 2 \\ -2 & -1 & 6 \end{bmatrix}$ find its inverse A^{-1}.

The first step to get the inverse of a matrix is to find its determinant. Thus using the Laplace method, which the reader should now be conversant with, the determinant of matrix A is $|A| = 43$. The determinant is non zero, so the inverse is assured. Next is to find the Cofactor matrix and the adjoint, then A^{-1} can be derived.

$$M = \begin{bmatrix} \begin{vmatrix} 2 & 2 \\ -1 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -2 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} \\ \begin{vmatrix} 0 & 5 \\ -1 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ -2 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -2 & -1 \end{vmatrix} \\ \begin{vmatrix} 0 & 5 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 10 & 3 \\ 5 & 22 & -2 \\ -10 & -1 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 14 & -10 & 3 \\ -5 & 22 & 2 \\ -10 & 1 & 4 \end{bmatrix}$$

$$Adj\ A = \begin{bmatrix} 14 & -5 & -10 \\ -10 & 22 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{43}\begin{bmatrix} 14 & -5 & -10 \\ -10 & 22 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

4.11 Cramer's rule for solving simultaneous equations

Matrices are commonly used in linear algebra. With a linear algebra given in matrix form $Ax = d$, with A a coefficient matrix, x is a vector of variables and d is a vector of constants. As already known by now, vectors are just a special kind of matrices and multiplying them should not be unusual. The solution is found by pre multiplying by the inverse of A on both sides to obtain $x = A^{-1}d$. This is a tedious method as it requires finding the inverse of A.

A famous theorem called *Cramer's rule* proves a lot easier. It is named after an eighteenth century Swiss mathematician Gabriel Cramer, whose works brought it into the world. It is a lot easy and convenient to start with a system of two equations with two *unknowns*.

$$a_{11}x_1 + a_{12}x_2 = d_1$$
$$a_{21}x_1 + a_{22}x_2 = d_2$$

which can be expressed in matrix form $Ax = d$ as follows

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

using the elimination method of solving simultaneous equations, the solutions for x_1 is given by

$$(a_{11}a_{22} - a_{21}a_{12})x_1 = d_1a_{22} - d_2a_{12},$$

$$x_1^* = \frac{d_1 a_{22} - d_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}}$$

Before moving to x_2, a closer look at this one will be helpful. A closer examination of the numerator and the denominator will reveal that they are actually determinants of 2×2 matrices. A careful arrangement will give

$$x_1^* = \frac{\begin{vmatrix} d_1 & a_{12} \\ d_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Corollary, x_2 will also be given by

$$x_2^* = \frac{\begin{vmatrix} a_{11} & d_1 \\ a_{21} & d_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

In both solutions, the denominator is a determinant of A, the coefficient matrix. The numerator for x_1 is still a determinant though of a new matrix A_1 formed by replacing the first column of A with the vector of constants d. Similarly for x_2, the numerator is a determinant of a new matrix A_2 formed by replacing the second column with the vector d. With the two new determinants, the solutions can thus be written as

$$x_1^* = \frac{|A_1|}{|A|}$$
$$x_2^* = \frac{|A_2|}{|A|}$$

though this is illustrated using a smaller matrix, the method works for a coefficient matrix A of any order. The method does not require calculating the inverse of A. It only relies on determinants. This is the method referred to as Cramer's rule above. In a more general form for a coefficient of any order, the rule is written as

$$x_j^* = \frac{|A_j|}{|A|}$$

where A_j is formed by replacing the jth column in A with the vector of constants d.

Example 4.8

Given the following system of equations, use the Cramer's rule to find the equilibrium values x_1^*, x_2^* and x_3^*.

$$2x_1 - x_2 - x_3 = 4$$
$$3x_1 + 4x_2 - 2x_3 = 11$$
$$3x_1 - 2x_2 + 4x_3 = 11$$

the equations are arranged in a matrix form $Ax = d$ as

$$\begin{pmatrix} 2 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \\ 11 \end{pmatrix}$$

Using Cramer's rule (with the help of the Laplace expansion for determinant. First get the determinant of A and then A_1, A_2 and A_3.

$$|A| = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{vmatrix}$$
$$= 2\begin{vmatrix} 4 & -2 \\ -2 & 4 \end{vmatrix} - 3\begin{vmatrix} -1 & -1 \\ -2 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & -1 \\ 4 & -2 \end{vmatrix}$$
$$= 2(12) - 3(-6) + 3(6) = 60$$

$$|A_1| = \begin{vmatrix} 4 & -1 & -1 \\ 11 & 4 & -2 \\ 11 & -2 & 4 \end{vmatrix}$$
$$= 4\begin{vmatrix} 4 & -2 \\ -2 & 4 \end{vmatrix} - 11\begin{vmatrix} -1 & -1 \\ -2 & 4 \end{vmatrix} + 11\begin{vmatrix} -1 & -1 \\ 4 & -2 \end{vmatrix}$$
$$= 4(12) - 11(-6) + 11(6) = 180$$

Thus

$$x_1^* = \frac{|A_1|}{|A|}$$
$$= \frac{180}{60} = 3$$

Corollary, $x_2^* = 1$, $x_3^* = 1$

4.12 Rank of a matrix

An $m \times n$ matrix is a matrix with m rows and n columns. If all columns and rows are linearly independent, then whichever is lower between m and n, is the rank of such a matrix. The matrix is said to have a *full rank*. However, in some matrices, not all rows or columns will be linearly independent. When there is linear dependence among columns or rows, then the rank only considers the number of linearly independent columns and rows. Thus m and n need to be adjusted, by reducing each by the number of linearly dependent rows and columns respectively. Then whichever remains lowest is the rank. Formally, the rank of a matrix, denoted by $r(A)$, is defined as the number of independent rows, or columns, and where the two differ, the lower value counts. Linking this with the earlier discussion on the inverse of a matrix, we conclude that only matrices with full rank will be non-singular.

For a matrix given by $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 3 \\ 5 & 10 & 7 \end{bmatrix}$ is a square matrix with $m = n = 3$. There should not be a temptation to think that the rank $(A) = 3$. A closer look should reveal that the third row is a linear combination of the first two $r_3 = r_1 + 2r_2$. The number of linearly independent rows (or columns) is not three but two. The rank of matrix A is $r(A) = 3$.

4.13 Eigen Values and Eigen vectors

The whole idea behind matrices is the search for a better and convenient way of handling mathematical expressions. We continue to transform and explore different types of matrices and operations so as to equip the reader with mathematical prowess necessary to handle different economic problem. The next step is to consider what are known as *eigen values* and *eigen vectors*. The word *eigen* is a Germany word translating to *own*. So we are finding a matrix's own values and vectors. These are sometimes referred to as *characteristic roots*.

Consider a set of equation: $Ax = \lambda x$ where A is a square matrix of dimensions n; x is a non-null column vector with n elements and λ is a scalar called *eigen value* or *characteristic root*.

$$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$$

The *identity* matrix I is added for conformability, that is, to enable the two subtract. For a non-trivial solution to exist for this set of equations, the new coefficient matrix $A - \lambda I$ must be singular. This implies that its determinant $|A - \lambda I|$ must be zero.

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

An expansion of this determinant will produce a polynomial in λ often referred to as characteristic polynomial. The roots of the polynomial are called *eigen values* or *characteristic roots* will prove useful later. A vector x associated with each eigen value is called *eigen vector* or *characteristic vector*. With a singular coefficient matrix, the consequence is that a unique solution cannot be found. There will be linear dependence in the equations. One is a multiple of the other. The resulting vector will merely be a ratio of the two elements, which is only indicative of direction. Eigen vectors lack the second attribute of vectors, that is, magnitude. To impose this attribute, eigen vectors are normalised to a unit magnitude. Thus, the second equation

$${x_1}^2 + {x_2}^2 = 1$$

is used to get a unique eigen vector. The x_s are not vectors but elements of a single vector. To elucidate this distinction, the letter v is used to denote a vector with elements denoted by x_s.

Example 4.9

Given a square matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$, find the eigen values and the respective eigen vectors.

We require that

$$|A - \lambda I| = \begin{vmatrix} 4 - \lambda & 2 \\ 2 & 7 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (4 - \lambda)(7 - \lambda) - (2)(2) = 0$$

$$\Rightarrow \lambda^2 - 11\lambda + 24 = 0$$

$$\Rightarrow (\lambda - 3)(\lambda - 8) = 0$$

$$\lambda_1 = 3 \; and \; \lambda_2 = 8$$

which are eigen values of the matrix A.

For $\lambda_1 = 3$, we have $(A - 3I)x = 0$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

The coefficient matrix is singular as can be noticed from the rows or columns. $x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$ and the normalisation equation

$${x_1}^2 + {x_2}^2 = 1$$

the unique solution for $\lambda_1 = 3$ is

$$v_1 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

For $\lambda_2 = 8$, we have $(A - 8I)x = 0$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$2x_1 - x_2 = 0 \Rightarrow x_2 = 2x_1$ with the normalisation ${x_1}^2 + {x_2}^2 = 1$

$$v_2 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

4.14 Diagonalisation of a matrix

With eigen values and eigen vectors at finger tips, diagonalisation of a matrix is now within 'radius'. Diagonalisation, also known as *spectral decomposition*, is the transformation of an ordinary matrix into a *diagonal form*, with off-diagonal elements all equal to zero. The drive for diagonalisation is to find a matrix that is easy to manipulate. Diagonal matrices play a very important role in simplifying the mathematical operations of economics. For instance, diagonal matrices prove much easy to multiply than ordinary matrices. In addition, they provide a short route to determining the sign definiteness of quadratic forms that come later in the chapter.

When eigen values are distinct, the matrix A can be transformed into a diagonal matrix by using a transformation matrix T. This is a matrix of eigen vectors of A. That is

$$T = (v_1 \quad v_2 \quad \cdots \quad v_n)$$

for a square $n \times n$ matrix A. We have no intentions of developing the procedure, rather we just give the results. The procedure is well developed by Chiang and Wainwright (2005).[30] For the matrix A, its diagonal equivalent D is given by[31]

$$D = T^{-1}AT$$

The diagonal matrix D will have eigen values along the principal diagonal in the order which corresponds to the order in which the eigen vectors appear in the columns of T. That is, D will be given by

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

the λ_s are the respective eigen values or characteristic roots for the eigen vectors v_s in the matrix T.

Example 4.10

Given the matrix $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$, find the eigen values and vectors and the corresponding diagonal matrix D.

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$$
$$\Longrightarrow (4-\lambda)(2-\lambda) - (3)(1) = 0$$
$$\Longrightarrow \lambda^2 - 6\lambda + 5 = 0$$

Given this characteristic polynomial, the eigen characteristic roots are

$$\lambda_1 = 1 \text{ and } \lambda_2 = 5$$

For $\lambda_1 = 1$,

$$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$
$$3x_1 + x_2 = 0 \Longrightarrow x_2 = -3x_2$$

and the normalisation equation $x_1{}^2 + x_2{}^2 = 1$ the unique solution for $\lambda_1 = 1$ is

$$v_1 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$$

And for $\lambda_2 = 5$

[30] Chiang & Wainwright (2013)

[31] For orthogonal eigen vectors (v_i), it is permissible to use the transpose as opposed to inverse of the matrix T in the equation.

$$\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$
$$x_1 + x_2 = 0 \Longrightarrow x_2 = x_2$$

and the normalisation equation, the unique solution for $\lambda_1 = 5$ is $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

therefore, the matrix of eigen vectors T is given by $T = \begin{bmatrix} 1/\sqrt{10} & 1/\sqrt{2} \\ -3/\sqrt{10} & 1/\sqrt{2} \end{bmatrix}$ Matrix T is of order two, and its inverse must be pretty easy to find. The solution is left to the reader to prove[32]. For the equation

$$D = \quad T^{-1} \quad \times \quad A \quad \times \quad T$$

$$D = \begin{bmatrix} \sqrt{5}/\sqrt{8} & -\sqrt{5}/\sqrt{8} \\ 3/\sqrt{8} & 1/\sqrt{8} \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{10} & 1/\sqrt{2} \\ -3/\sqrt{10} & 1/\sqrt{2} \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{5}/\sqrt{8} & -\sqrt{5}/\sqrt{8} \\ 3/\sqrt{8} & 1/\sqrt{8} \end{bmatrix}\begin{bmatrix} 1/\sqrt{10} & 5/\sqrt{2} \\ -3/\sqrt{10} & 5/\sqrt{2} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

It should have been expected, that once eigen values are known, the diagonal matrix is implied. The result has just proved that the eigen values are the elements of the diagonal matrix. With the diagonal matrix known, there is no more need to go through that lengthy process. Once eigen values are known, so is the diagonal matrix. In the diagonal equation, we can still make A the subject of formulae and make use of the properties of a diagonal matrix. Simply pre multiply with T and post multiply by the inverse of T on both sides. Then

$$A = TDT^{-1}$$

4.15 Positive and non-negative square matrices

Positive and non-negative matrices must not be perceived as a new concept. Even for a 'novice' the names should give a clue. A *positive matrix* is one all of whose elements are positive. If all elements in a matrix are positive, then the matrix is said to be a positive matrix. On a number line, the zero divides it into two potions, the negative and the positive side. The zero itself does

[32] Swap the principal diagonal elements, change the signs of off-diagonal elements and divide the whole matrix by its determinant.

not fall on either side. As a consequence, a positive matrix does not include zero, because the number is not positive.

When a matrix has a zero in it but all other numbers are positive, this would imply that such a matrix has no negative number in it. It only has zero and positive numbers. It is thus called a *non-negative matrix*. It is defined as a matrix all of whose elements are non-negative. A more related matrix is a *semi-positive* matrix. This is a non-negative matrix with each row and column has at least one positive element.

4.16 Cayley-Hamilton Theorem

For any square matrix A, the characteristic equation is a polynomial in lambda (λ). The equation is denoted by $f(\lambda) = 0$. Cayley-Hamilton Theorem says any such matrix satisfies its own characteristic equation. If the matrix A is substituted in place of λ in the equation, the equation still holds. That is if $f(\lambda) = 0$, then $f(A) = 0$. For an illustration, take a matrix $A = \begin{bmatrix} 2 & 4 \\ -2 & 8 \end{bmatrix}$ Its characteristic polynomial or equation is given by

$$
\begin{aligned}
f(\lambda) &= |A - \lambda I| = 0 \\
&= \begin{vmatrix} 2-\lambda & 4 \\ -2 & 8-\lambda \end{vmatrix} = 0 \\
&= (2-\lambda)(8-\lambda) - 2(4) = 0 \\
&= \lambda^2 - 10\lambda + 24 = 0
\end{aligned}
$$

the equation should also hold if the matrix A is substituted for λ. The terms, in the equation, multiplying with the matrix will now be matrices themselves while the constant term remains a scalar. This will defy the rules of addition of matrices. To circumvent this, the constant is multiplied by the identity matrix I of the same order as the matrix A. Thus

$$
\begin{aligned}
f(A) &= A^2 - 10A + 24I = 0 \\
&= \begin{bmatrix} 2 & 4 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -2 & 8 \end{bmatrix} - 10\begin{bmatrix} 2 & 4 \\ -2 & 8 \end{bmatrix} + 24\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0 \\
&= \begin{bmatrix} -4 & 40 \\ -20 & 56 \end{bmatrix} - \begin{bmatrix} 20 & 40 \\ -20 & 80 \end{bmatrix} + \begin{bmatrix} 24 & 0 \\ 0 & 24 \end{bmatrix} = 0 \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0
\end{aligned}
$$

4.17 Input-output models

In an economy with many sectors, there is always some interdependence among all or some sectors. In addition to the primary input of labour (and land), output from various sectors is used as intermediate inputs in the other sectors. For instance, the agriculture sector depends on the industry for its implements; the industry on the other hand will rely on the agriculture for its raw material and so on. In some cases, a sector may use it own output as an intermediate

input. For instance, the agriculture sector uses output from the industry in form of farming implements as well as its own output as seed.

For each sector, there are two sources of demand. The first is from the same sectors as intermediate inputs in the continuation of production. The second is the final demand. When maize is used to brew beer, then it is said to be used as an intermediate input into the production of the final commodity beer. The total maize needed for domestic consumption is what is referred to as final demand. For equilibrium to occur, each sector must satisfy its total demand, that is, it must produce only enough to meet the input requirements of all the sectors as well as the total demand. This is also called the Leontief Input-Output model, after twentieth century Russian-American economist Wassily Leontief. Let us now develop the algebra.

Define a coefficient a_{ij} as the total amount of output i needed to produce a unit of j. But the production of j will not be restricted to a unit. As such the total of an intermediate input i that will be required to produce x_j of output j will be $a_{ij}x_j$. For instance, if it takes 20kgs of maize (in addition to other inputs) to produce a unit of beef, the it will require 100kg to produce five units of beef. Since each sector must satisfy its total demand,

$$x_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + d_1$$

On the left hand side is the total of commodity 1 produced. The first term on the right is how much of commodity 1 is needed to produce commodity 1. The second term is commodity 1 requirements into sector 2 and so on. The final demand for commodity 1 is depicted by d_1. It must suffice to state that nothing prevents any of the a_{ij}s or d_is from being zero. Sectors will not require inputs of every sector, making some a_{ij} zero. In addition, some commodities may be purely for intermediate use only. As such, their final demands are unequivocally zero. However, none of these elements can be negative.

For the other sectors, the equations are as follows:

$$x_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + d_2$$

$$\cdots$$

$$x_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + d_n$$

In matrix form, the model of the form.

$$
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}
$$

$$x = Ax + d$$

The matrix A is the coefficient matrix. It is square and nonnegative. It dimensions are determined by the number of sectors or industries and the prevailing production techniques determine the actual elements. Given the input output coefficient and the final demand vector, the task is to find output vector $\boldsymbol{x}$. The process is purely matrix algebra.

$$x = Ax + d$$

$$(I - A)x = d$$

Then pre-multiply by the inverse of $I - A$, also known as the Leontief inverse.

$$x = (I - A)^{-1}d$$

$$x = Bd, \qquad where\ B = (I - A)^{-1}$$

Example 4.11

A three sector economy has the input output matrix $A = \begin{bmatrix} .3 & .2 & .4 \\ .5 & .1 & .2 \\ .0 & .4 & .2 \end{bmatrix}$ and the final demand for the sectors' outputs is $d = \begin{bmatrix} 500 \\ 0 \\ 1000 \end{bmatrix}$. What level of output from all the three sectors will ensure that the final demand is satisfied?

Using the formula illustrated above, the first step is to find what was referred to as the Leontief inverse. This requires getting the inverse of the matrix $(\boldsymbol{I} - \boldsymbol{A})$. We assume the reader is already familiar with inverting a matrix discussed under section 4.10. $\boldsymbol{I} - \boldsymbol{A} = \begin{bmatrix} .7 & -.2 & -.4 \\ -.5 & .9 & -.2 \\ .0 & -.4 & .8 \end{bmatrix}$ and therefore $(\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{1}{0.288}\begin{bmatrix} .64 & .32 & .40 \\ .40 & .56 & .34 \\ .20 & .28 & .53 \end{bmatrix}$ therefore

$$x = (\boldsymbol{I} - \boldsymbol{A})^{-1}d$$

$$= \frac{1}{0.288}\begin{bmatrix} .64 & .32 & .40 \\ .40 & .56 & .34 \\ .20 & .28 & .53 \end{bmatrix}\begin{bmatrix} 500 \\ 0 \\ 1000 \end{bmatrix}$$

$$x^* = \begin{bmatrix} 2500 \\ 1875 \\ 2187.5 \end{bmatrix}$$

The above example is very interesting and worth a comment. In the final demand vector, the final demand for commodity 2 is zero. Nonetheless, the model still says 1875 units of the commodity must be produced. This should echo the point that some commodities may be produced purely to meet the input needs of all the sectors. The whole output of commodity 2 is used as an input into producing the same commodity 2 and in producing the other two commodities.

Consider for example a commodity like Lime. This is an industrial output. It is used in the agriculture sector and in the manufacture of cement. It is never used domestically implying that its final demand is zero. This should not mean that lime should not be produced since doing so will make impossible or inefficient the production of cement as well as cement which depend on lime.

4.11 Decomposable and indecomposable matrices

Consider a square matrix A with dimensions $n \times n$. The matrix is said to be decomposable if it is possible to interchange or rearrange its rows and columns is such a way as to obtain a matrix of the form

$$\hat{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Each element in the above matrix is a matrix. Thus decomposing is breaking the matrix into four sub matrices, with some pattern however. This pattern is what will distinguish a decomposable matrix from an ordinary. It is required that the sub matrices take to following form. Matrices A_{11} and A_{22} must be square matrices with dimensions $k \times k$ and $(n-k) \times (n-k)$ respectively. This will leave A_{12} and A_{21} with dimensions $k \times (n-k)$ and $(n-k) \times k$ respectively[33].

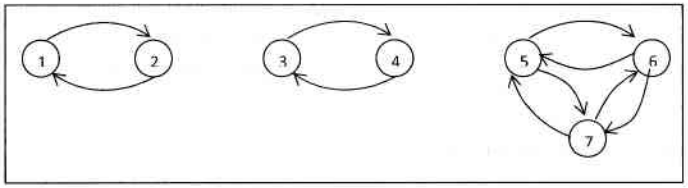
The beacon for decomposability lies in the matrix A_{21}. Is it possible to rearrange so that this matrix is null? A matrix is *decomposable* if after this rearrangement to form sub matrices, the A_{21} will be a null matrix, with zeros. If the above is not achievable, the matrix is said to be *indecomposable*. As a *rule of thumb*, a matrix will be indecomposable if there is a *zero* in every column or row.

In economics, a group of industries or sectors is said to be indecomposable if every industry in the economy is linked, directly or indirectly through others, to all other sectors. A sector is directly linked to another sector if one sector directly depends on the other for inputs or market for its output. If the linkage is only through the two sectors' linkage to a common third sector, it is called indirect linkage.

When some sectors directly or indirectly sell their output to others but do not buy from them, the economy is said to be decomposable. Assume a seven sector economy represented by Figure 4.4 below.

[33] The matrices would as well be vectors or scalars. These are just special types of matrices and form part of possible sub matrices

Figure 4.4. Seven sector completely decompasable model



Though the economy has seven-sector represented by the *numbered nodes*, the sectors are not all inter-linked. Sector 1 is only linked to sector 2 while sector 3 is linked only to sector 4. The same applies for the last three sectors.

The technological matrix representing the economy in Figure 4.4 is given by the matrix A below.

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & 0 & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} & a_{57} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} & a_{67} \\ 0 & 0 & 0 & 0 & a_{75} & a_{76} & a_{77} \end{bmatrix}$$

$$= \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$$

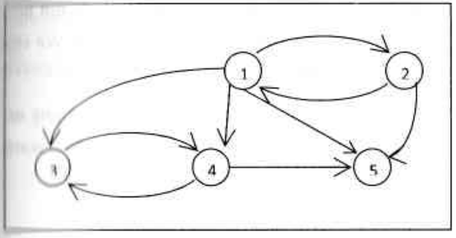
The above economy is divided into three independent major sectors. It can be viewed as having three separate economies. Its input-output model $(I - A)x = d$ can as well be broken into three separate input output models given by:

$$(I - A_1)x_1 = d_1$$
$$(I - A_2)x_2 = d_2$$
$$(I - A_3)x_3 = d_3$$

When, through appropriate interchange of rows and columns, the technological matrix of an economy can be represented by A above, the economy is said to be *completely decomposable*. This applies, as demonstrated above, when the economy comprises of sectors that can be categorised in completely independent groups. One group of sectors in an economy has nothing to do with sectors in other groups.

Consider another economy where the graph is as in Figure 4.5.

Figure 4.5. Five Sector Decomposable Model



In the above figure, all the sectors are linked, some directly in either a two-way fashion or one-way fashion and others indirectly. The technological matrix is given by matrix B below

$$B = \begin{bmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & \begin{pmatrix} b_{13} & b_{14} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} b_{15} \\ b_{25} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{pmatrix} & \begin{pmatrix} 0 \\ b_{45} \end{pmatrix} \\ (0 \quad 0) & (0 \quad 0) & (b_{55}) \end{bmatrix}$$

$$= \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & 0 & B_{33} \end{bmatrix}$$

The economy is divided into several subgroups of industries such that when the industries are properly numbered, the technological matrix is *block triangular*. The sub-matrices B_{ii} are square, each of them representing an indecomposable sub-matrix. Below the diagonals are only zeros and above the diagonals are blocks of non-negative elements with at least one positive element in each column. There may be some zeros above the main diagonal, but not all can be zero. If so, the system will be completely decomposable as in Figure 4.4.

The industries in the above graph fall into three categories:

- Group 1: Industries 1 and 2 which sell to each other and to Industries 3 and 4 in group 2 as well as sector 5 in group 3.
- Group 2: Industries 3 and 4 which sell to each other and to sector 5;
- Group 3: Only industry 5 is in this group and does not sell to any sector.

The matrices B_{ii} (forming the principle diagonal) are square, each of them representing an indecomposable matrix. Below the diagonal matrices are only zeros while above the diagonal are blocks of non-negative elements with at least one positive element in each column. There may be some zeros in these matrices but they cannot all be zero, else the system will be completely decomposable.

Decomposable economies with block triangular technology matrices are somewhat different from indecomposable economies. Suppose that final demand d_i is increased for some industry

i which falls in group k. Since every sector in group k sells directly of indirectly to all industries in the group, every output x_j in group k will increase. But since industries in group k do not buy from industries in groups indexed above k, x_j in such industries will remain unchanged. We say industries in group k do not have *backward linkages* with industries from such groups.

The x_j in some group r with an index smaller than k may or may not increase depending on whether one or more industries in group buy from one or more industries in group r. Backward linkages may exist with such groups.

Consider the matrix equation of the form $Ax = d$ given by.

$$\begin{bmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

To decompose coefficient matrix A, interchange row 1 and 2; then column 1 and 2. The resulting matrix is

$$\hat{A} = \begin{bmatrix} a_{22} & a_{21} & a_{23} \\ 0 & a_{11} & a_{13} \\ 0 & a_{31} & a_{33} \end{bmatrix}$$

From the above matrix, it is possible to define the four sub matrices required for decomposition. These are $A_{11} = [a_{22}]$, $A_{12} = [a_{21} \quad a_{23}]$, $A_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $A_{22} = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$

The coefficient matrix A is thus decomposable.

4.19 Perron-Frobenius theorem

If A is a semi-positive matrix and among its characteristic roots, there is one particular root λ^* called the *dominant root or Frobenius root* with an associated characteristic vector v^*, such that:

a) λ^* is a real number and non-negative
b) No other root exceeds λ^* in absolute value
c) λ^* is non-negative vector
d) For all $\mu > \lambda^*$, $\mu I - A$ is non singular and its inverse $(\mu I - A)^{-1}$ is a semi positive matrix.

If in addition to being semi positive, matrix A is also indecomposable, then (a), (c), and (d) can be strengthened to:

a) λ^* is positive and is not a repeated root
c) v^* is a strictly positive vector
d) $(\mu I - A)^{-1}$ is strictly positive matrix

This theorem is quite helpful in dealing with solutions involving linear system of equations and matrices. In the Leontief solution, $x^* = (I - A)^{-1}d$ where the final demand vector d is non negative, all that is needed to guarantee a strictly positive output vector $(x^* \gg 0)$ is that the

Frobenius root should be less than one $(\lambda^* < 1)$. Further, if the technological matrix A is indecomposable, then with $\lambda^* < 1$, then the matrix $(I - A)^{-1} \gg 0$ which implies that $x \gg 0$.

Example 4.12

Given the following input output matrix $A = \begin{bmatrix} 0 & 2 \\ 1/3 & 0 \end{bmatrix}$

Determine the Frobenius root

Find the matrix $(I - A)^{-1}$

Finding characteristic roots was introduced and discussed earlier in the chapter and the reader is now familiar with the steps. Therefore, we just state the roots and leave it to the reader to verify.

$$\lambda_1 = -\sqrt{\frac{2}{3}} \text{ and } \lambda_2 = \sqrt{\frac{2}{3}}$$

The root of interest, the Frobenius root, must be positive and no less than another root in absolute value. Therefore $\lambda^* = \sqrt{\frac{2}{3}}$. The associated characteristic vector (without normalisation) is $v^* = \begin{bmatrix} \sqrt{6} \\ 1 \end{bmatrix}$

$$I - A = \begin{bmatrix} 1 & -2 \\ -1/3 & 1 \end{bmatrix}$$

We now need to find its inverse $(I - A)^{-1}$. For a 2 × 2 matrix, the inverse is much simpler using the shortened cofactor method. The determinant is $\frac{1}{3}$. Then in the matrix, swap the principal diagonal elements and change the signs of the off-diagonal elements. Thus

$$(I - A)^{-1} = \frac{1}{1/3}\begin{bmatrix} 1 & 2 \\ 1/3 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix}$$

Note that the $(I - A)^{-1} \gg A$

We conclude that for every sector in an indecomposable system, the total requirement of every input exceeds its direct requirement.

Metzler's Theorem: if A is a non-negative matrix and λ^* is its Frobenius root, then a necessary and sufficient condition for λ^* is that all the principal minors of $(I - A)$ are positive.

4.20 Stochastic matrices

A good farmer ought to have a good focused of weather. S/he must know the chances of having a particular weather condition the following day given the current day's weather. In a market, a trader must make careful judgment of the price tomorrow, given today's, so as to decide on the optimal quantity to supply. Or one may ask, what is the probability that a person unemployed in one time period will have a job in the next period? Will still be unemployed even in the next period? The probability of getting into state i from state j is of critical importance, both for policy and decision making. Government only needs to intervene in the market if a distortion persists. But what is the probability that a distortion today will also be there tomorrow, *ceteris paribus*? This is best tackled using stochastic matrices.

A stochastic matrix is a non-negative square matrix whose column sums are unit. It is a matrix of probability or stochastic movements. The element a_{ij} gives the probability of having i given the previous period was in state j. Define a vector of employment status as

$$X = \begin{cases} 1, & employed \\ 2, & unemployed \end{cases}$$

Then the stochastic matrix is of the form

$$P = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The element a_{11} is the probability that someone who had a job in period 1 ($j = 1$) will have a job in period 2. This need not be the same job because the interest here is that someone employed, whether by the same company or a different one now. The element a_{21} is the probability that one who was employed ($j = 1$) will have lost employment by period 2. Since the state of being employed and being unemployed are mutually exclusive and exhaustive, their sum must be unit. Thus $a_{11} + a_{21} = 1$. This confirms the earlier statement that the column sums must be unit. The corollary applies to the second column, for someone unemployed in the initial period.

Example 4.13

Suppose there are 100,000 individuals participating in a particular labour market. If an individual is employed in the current period, there is a probability of 0.9 that the person will be employed in the next period. If the individual is unemployed in the current period, there is a probability of 0.4, the they will have found employment in the next period.

Write out the stochastic matrix.

What is the equilibrium rate of employment in the market?

To write the matrix, first compute the two remaining probabilities. For a person employed in the current period, the probability that they will be unemployed in the next period is

$1 - 0.9 = 0.1$ and for the unemployed person, the probability of being unemployed in the next period is $1 - 0.4 = 0.6$. Thus

$$P = \begin{bmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{bmatrix}$$

For the equilibrium rate of employment, we need to find the vector

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} employed\ workers \\ unemployed\ workers \end{bmatrix}$$

in $Px = x$

$$\Rightarrow (P - I)x = 0$$

$\Rightarrow |P - I| = 0$ since lambda the eigen value is unit ($\lambda = 1$) in this equation, then $(P - I)$ is singular.

$$\begin{bmatrix} -0.1 & 0.4 \\ 0.1 & -0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow x_1 = 4x_2$$

the equation give the employed unemployed ration of 4 to 1. The unemployment then must be one-fifth, equivalent to 20%. Alternatively, since the total number of employed and unemployed is always equal to 100,000, then using the equation $x_1 + x_2 = 100{,}000$, we have $x_1 = 800{,}000$ and $x_2 = 200{,}000$. Again unemployment rate is 20%.

A stochastic matrix can be column stochastic or row stochastic or doubly stochastic. For a *column* stochastic, columns add to unit and for *row* stochastic, it is rows that add to unit. If both rows and columns add to unit, then the stochastic matrix is said to be *doubly* stochastic. Consider a simple weather model.

The probability of a weather condition, given the weather on the preceding day can be represented by a *transition* matrix

$$P = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$$

A *rainy* year is 90% likely to be followed by a rainy year and a *drought* year is 50% likely to be followed by another drought year. Suppose the rainfall condition is known for the initial year to be 'rainy'. Then the vector $x^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives probabilities of rainy and drought for known initial year. For the following year, focused have to be made based on current condition and the transition matrix.

$$\begin{aligned} x^{(1)} &= Px^{(0)} \\ &= \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix} \end{aligned}$$

and for year 2

$$
\begin{aligned}
x^{(2)} &= Px^{(1)} \\
&= \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix} \\
&= \begin{bmatrix} 0.86 \\ 0.14 \end{bmatrix} \\
&\Rightarrow x^{(2)} = P^2 x^{(0)}
\end{aligned}
$$

The probability that year 2 will be rainy is 86% and chances of a drought is 14%. This focused is based on the initial year and will change as we become certain on year 1.Once year 1 is known, year 2 will no longer have to be based on the focused on year 1 but on actual outcome. But the adjustment is not effected in this manner. Once year 1 has passed and is known, then it becomes the present or initial period, and what was year two now becomes year 1 and so on. In general, with the latest period known with certainty taken as the initial period, the focus for the nth period made in the initial period is given by

$$x^{(n)} = Px^{(n-1)} = P^n x^{(0)}$$

The matrix P raised to power n need not be forbidding. Diagonal matrices discussed earlier in the chapter can be of service here. The stochastic matrix P can be transformed into a diagonal matrix and raising it to any power should not be a problem.

4.21 Quadratic forms

Consider a polynomial expression in several variables. If the sum of the exponents in each term of the polynomial is the same, the polynomial has a *form*. Alternatively, a form can be defined as a polynomial expression in which each term has a uniform degree. The uniform degree defines the order of the form. For example

$$f(x, y, z) = 4z - 9y + z$$

is a linear form because the degree is one. The expression

$$f(x, y, z) = 4x^2 - 9xy + 3y^2 + 5xz + 7z^2$$

is a form of the second degree. It is therefore known as a *quadratic form* in three variables.

Suppose two matrices X (a vector) and A are defined as $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then

$$
\begin{aligned}
X'AX &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} \\
&= a_{11}{x_1}^2 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}{x_2}^2 \\
&= a_{11}{x_1}^2 + (a_{12} + a_{21})x_1x_2 + a_{22}{x_2}^2
\end{aligned}
$$

which proves to be a quadratic form.

For every vector X other than the null vector, the function $X'AX$ is positive definite if $X'AX > 0$. This is an assurance that the function is throughout its domain above zero. It never touches the zero. For any values of $x_1, x_2 \cdots, x_k$ not all zero, the function is always positive. If however the function does touch zero but never cross into the negative, then the function is said to be positive semi-definite. It is defined by $X'AX \geq 0$. Alternatively, the function can be said to be *non-negative*.

A function is *negative definite* if it is negative for all values of $x_1, x_2 \cdots, x_k$, not all zero. Algebraically, a function is negative definite if $X'AX < 0$. If zero is permitted into the range, then the function becomes negative semi-definite. This is when $X'AX \leq 0$. For instance, if matrix A where $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, the form will be

$$
\begin{aligned}
X'AX &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 + x_2 & x_1 + x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= {x_1}^2 + x_1x_2 + x_2x_1 + {x_2}^2 \\
&= {x_1}^2 + 2x_1x_2 + {x_2}^2 \\
&= (x_1 + x_2)^2
\end{aligned}
$$

For whatever values of x_1 and x_2, not both zero, the function returns a positive number. This can be seen from the square which only produces a positive number. The quadratic form is positive definite. A slight modification of A will however alter the sign definiteness. Suppose now the matrix A is given by $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then the form will be

$$
\begin{aligned}
X'AX &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 - x_2 & -x_1 + x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= {x_1}^2 - x_1x_2 - x_2x_1 + {x_2}^2 \\
&= {x_1}^2 - 2x_1x_2 + {x_2}^2 \\
&= (x_1 - x_2)^2
\end{aligned}
$$

like the previous function, this one too has a square which will prevent a negative number occurring. The function will therefore be non negative. However, the difference $x_1 - x_2$ can be zero (and therefore its square) even if both variables are not zero. This kind of function can be described as positive semi definite.

4.22 Test for Sign Definiteness

The question yet to be answered is, 'under what conditions will the quadratic form be positive or negative definite?' What are the diagnostic tests for positive or negative definiteness? Two methods are used to determine the sign definiteness of a quadratic form. The first is known as the determinantal test, the name invariably coming from its use of the determinant in the test. The second is the Eigen value test, which uses Eigen values.

Assume a quadratic form described by the coefficient matrix $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ so that $Q = ax_1{}^2 + 2hx_1x_2 + bx_2{}^2$, then

$$\text{Q is } \begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix} \text{ definite if } \begin{Bmatrix} a > 0 \\ a < 0 \end{Bmatrix} \text{ and } ab - h^2 > 0$$

Now $ab - h^2$ is the value of the determinant of the coefficient matrix A. That is,

$$ab - h^2 = \begin{vmatrix} a & h \\ h & b \end{vmatrix}$$

The determinant is known as the *discriminant* of the quadratic form. A discriminant in other words is a determinant formed by the coefficient of the terms in the quadratic form. And since a quadratic form is a symmetric matrix, $a_{ij} = a_{ji}$. Hence the coefficient of the cross product term in the function is equally divided between a_{ij} and a_{ji}.

The condition for sign definiteness can be restated in a more general way. The function Q is positive definite if both principal minors, $|a|$ and $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$ are both positive. For a negative definite function Q, the first principal minor $|a|$ must be negative and the second $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$ positive. Notice that no specific condition is place on b. This however should not mean that the value of b is immaterial in the determination of sign definiteness. Instead the condition on b is hidden in $ab - h^2 > 0 \implies ab > h^2$. Since h^2, by virtue of the square, will always be positive, then ab can only exceed h^2 if and only if both a and b are of the same sign, positive or negative. Thus a condition on a is in fact a condition on b.

For instance, in the quadratic form $Q = 5X^2 + 3XY + 2Y^2$, the coefficient matrix is $A = \begin{bmatrix} 5 & 1.5 \\ 1.5 & 2 \end{bmatrix}$ and the discriminant $|D| = \begin{vmatrix} 5 & 1.5 \\ 1.5 & 2 \end{vmatrix}$. The principal minors are:

$$|a| = 5 > 0, \qquad |D| = 7.75 > 0$$

Therefore the quadratic form Q is positive definite.

In a more general case, with n-variables, a quadratic form is positive definite if all the principal minors are positive. That is $|D_1| > 0, |D_2| > 0, |D_3| > 0, \cdots, |D_n| > 0$. For a negative definite quadratic form, the principal minors must alternate in sign. Particularly, all odd-numbered principal minor must be negative and all even-numbered principal minors must be positive. Thus $|D_1| < 0, |D_2| > 0, |D_3| < 0, |D_4| > 0, \cdots$. If principal minors of any quadratic form fail to adhere to any of the two patterns, then such a form is indefinite. It is positive for some part of the domain and negative for some other part.

Example 4.14

Determine the sign definiteness of the following quadratic forms

a. $Q = x_1^2 + 6x_2^2 + 3x_3^2 - 2x_1x_2 - 4x_2x_3$
b. $Q = 2x_1^2 + 3x_2^2 - x_3^2 + 6x_1x_2 - 8x_1x_3 - 2x_2x_3$

The first step is to write out the coefficient matrix. For principal diagonal elements, these are coefficients of squared variables. For the off diagonal elements, halve each coefficient to create a symmetric matrix. For part (a)

$$Q = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Discriminant\ |D| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{vmatrix}$$

The principal minors will be given by

$$|D_1| = 1 > 0,$$

$$|D_2| = \begin{vmatrix} 1 & -1 \\ -1 & 6 \end{vmatrix} = 5 > 0,$$

$$|D_3| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{vmatrix} = 11 > 0$$

All the principal minors are positive. Therefore, the quadratic form is positive *definite*.

For part (b), $Q = 2x_1^2 + 3x_2^2 - x_3^2 + 6x_1x_2 - 8x_1x_3 - 2x_2x_3$

$$Q = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 2 & 3 & 4 \\ 3 & 3 & -1 \\ 4 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Discriminant\ |D| = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 3 & -1 \\ 4 & -1 & -1 \end{vmatrix}$$

The principal minors will be given by

$$|D_1| = 2 > 0,$$

$$|D_2| = \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = -3 < 0,$$

$$|D_3| = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 3 & -1 \\ 4 & -1 & -1 \end{vmatrix} = -23 < 0$$

Both positive and negative definite require that the second principal minor is positive, being even numbered. The case at hand is however opposite. This is enough evidence to suggest the quadratic form is *indefinite*. Nonetheless, even the third, by being different from the first (all odd numbered) confirms the results.

The second method involves the use of characteristic roots or eigen values. A concise discussion of eigen values is under *eigen values and eigen vectors*. Given a quadratic form $Q = X'AX$, the form has sign definiteness linked to the eigen values of the coefficient matrix A. We summarise the sign definiteness in the three statements below:

- Q is $\begin{Bmatrix} positive \\ negative \end{Bmatrix}$ definite if and only if every eigen value is $\begin{Bmatrix} positive \\ negative \end{Bmatrix}$
- Q is $\begin{Bmatrix} positive \\ negative \end{Bmatrix}$ semi definite if and only if every eigen value is $\begin{Bmatrix} non\ negative \\ non\ positive \end{Bmatrix}$
- Q is indefinite if some eigen values are positive and some are negative. This means the form Q is negative for some part of the domain and positive for another part.

Chapter 5

5 DIFFERENTIAL CALCULUS

5.1 Introduction

The subject of differential calculus is essentially concerned with the rate of change of a dependent variable with respect to an independent variable. In the study of many phenomena, we are concerned with changes in quantities or the rate of change. For instance, in comparative static analysis and the concept of margins in economics, the speed of a rocket and the study of voltage of an electrical signal in physics all involve the important underlying concept of "rate of change" of a variable. The concepts of limits and continuity discussed in chapter 3 are basic to the study of calculus. Furthermore, the discussion of variables and functions in the same chapter showed that as a variable changes, the values of all functions dependent on that variable also change. Thus, if a variable quantity x changes by an increment w, instead of x, we write $x + w$. Then functions of x such as x^2, x^3, $\frac{a+x}{x^2+a^2}$ take on new values. For instance, x^2 becomes, $x^2+2xw+w^2$.

5.2 The Concept of Derivative:

Consider a continuous function $y = f(x)$. If x changes by a certain quantity, denoted as Δx, then y will change by a certain quantity, Δy.

The ratio $\frac{\Delta y}{\Delta x}$ may be regarded as the *average rate of change* of the function with respect to x over the interval Δx. As Δx becomes smaller and smaller, Δy will also change and the limit of $\frac{\Delta y}{\Delta x}$, if this limit exists, as Δx tends to zero, may be called the *instantaneous rate of change* of the function with respect to x. Mathematically, it is called the derivative of the function y with respect to x and is denoted as $\frac{dy}{dx}$ or $f'(x)$ or y'. In other words,

$$\frac{dy}{dx} \text{ or } y' = lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Example 5.1

An apple grower agrees to supply crates of apples, with a dozen apples in each crate, according to the supply function $S(x) = 10x^2$ where x is the price per crate. As the price goes up, the supplier naturally supplies more apples.

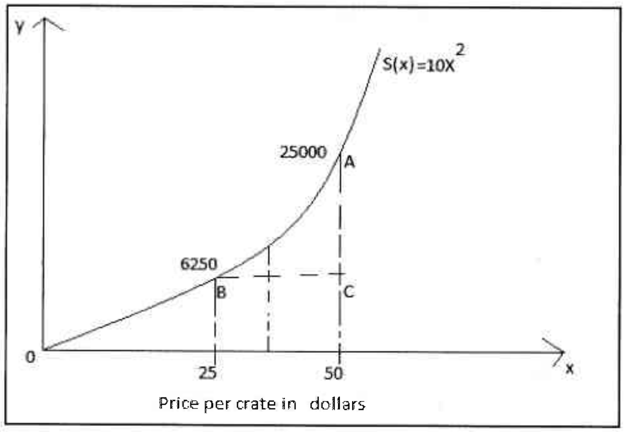
i. What is the average rate of change in supply when price changes from $K25$ to $K50$ per crate?

ii. What is the rate of change in supply from K25 per crate to $(25 + \Delta x)$ per crate?

iii. What value does $\frac{\Delta S}{\Delta x}$ in (ii) approach as Δx tends to zero?

Solution

Now at K25 per crate, the number of crates of apples supplied is $S(25) = 10(25)^2 = 6,250$ and at K50 per crate, the numbers of crates are $S(50) = 10(50)^2 = 25,000$.



The average rate in supply from K25 per crate to K50 per crate is given by $\frac{AC}{BC}$ in the above figure. That is,

$$\begin{aligned}\frac{\Delta S}{\Delta x} &= \frac{S(50) - S(25)}{50 - 25}\\ &= \frac{25,000 - 6,250}{25}\\ &= \frac{18,750}{25}\\ &= 750\end{aligned}$$

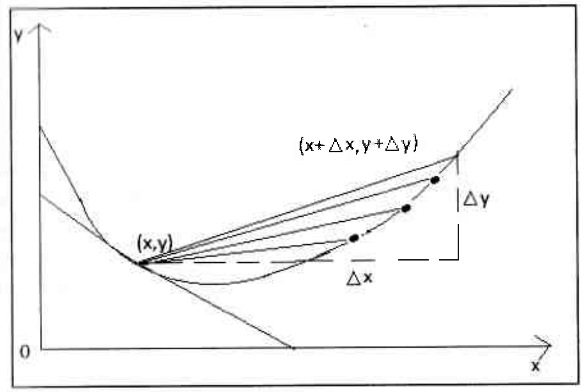
that is, 750 crates of apples for the increase of the price of a crate by every kwacha.

$$\begin{aligned}\frac{\Delta S}{\Delta x} &= \frac{S(25 + \Delta x) - S(25)}{\Delta x}\\ &= \frac{10(25 + \Delta x)^2 - 10(25)^2}{\Delta x}\\ &= \frac{1025^2 + 50\Delta x + (\Delta x)^2 - 25^2}{\Delta x}\end{aligned}$$

$$= 10(50 + \Delta x) = 500 + 10\Delta x$$

Taking the limit as $\Delta x \to 0$ in (ii), we get $\lim_{\Delta x \to 0} \frac{\Delta S}{\Delta x} = 500$ crates of apples per Kwacha increase in the price of the crate.

Let us now see the graphical meaning of the derivative. Refer to the diagram below.



We first have the point (x, y). Then, when x increases to $x + \Delta x$, y also changes to $y + \Delta y$. From elementary geometry, we know that the ratio $\frac{\Delta y}{\Delta x}$ is the slope of the chord joining the two points. As Δx becomes smaller and smaller, the point $(x + \Delta x, y + \Delta y)$ approaches more and more to the point (x, y) and the chord also shrinks. In the limit when Δx tends to zero, the chord becomes the tangent to the curve at the point (x, y) and the slope of the tangent is then given by the limit of the ratio $\frac{\Delta y}{\Delta x}$ when Δx tends to zero. Thus the derivative at the point (x, y) given by

$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ is nothing but the slope of the tangent to a curve at that point.

Let us now discuss the meaning of the derivative in economics and business. The concept of *margin* in economics corresponds exactly to the mathematical notion of the derivative. Margin is simply the derivative of the 'total' function. For instance, suppose you have the total cost function $C = f(Q)$, where C denotes cost and Q, output. Then the derivative $\frac{dC}{dQ}$ is the marginal cost or the rate of change in total cost at some value of the output Q. Let $c = 10 + 2Q + 0.5Q^2$ .Then

$$\frac{dC}{dQ} = \lim_{\Delta Q \to 0} \frac{f(Q+\Delta Q) - f(Q)}{\Delta Q}$$
$$= \lim_{\Delta Q \to 0} \frac{10 + 2(Q+\Delta Q) + 0.5(Q+\Delta Q)^2 - 10 - 2Q - 0.5Q^2}{\Delta Q}$$
$$= \lim_{\Delta Q \to 0} \frac{2\Delta Q + Q\Delta Q + 0.5\Delta Q^2}{\Delta Q}$$
$$= \lim_{\Delta Q \to 0} (2 + Q + 0.5\Delta Q)$$
$$= (2 + Q)$$

If $Q = 20$, then,

$$\left.\frac{dC}{dQ}\right|_{Q=20} = 2 + 20 = 22$$

The $\left.\frac{dC}{dQ}\right|_{Q=20}$ means the derivative $\frac{dC}{dQ}$ evaluated at $Q = 20$

A few other common marginal functions together with their mathematical counterparts in the form of derivatives are mentioned below.

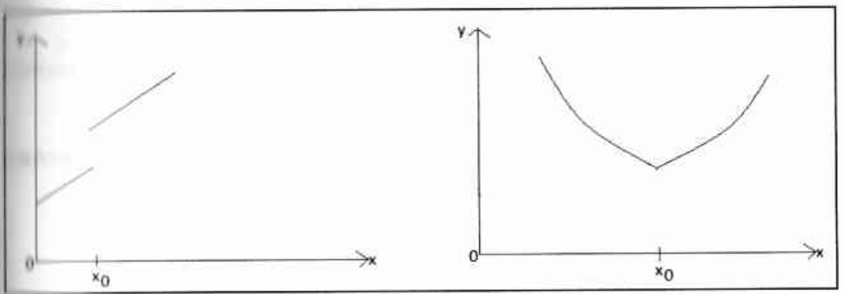
Function	Derivative
Utility function $U(x)$	$\frac{dU}{dx}$ = marginal utility
Revenue function $R = r(x)$	$\frac{dR}{dx}$ = marginal revenue
Production function $O = f(I)$	$\frac{dO}{dI}$ = marginal product
Consumption function $C = C(y)$	$\frac{dC}{dy}$ = marginal propensity to consume
Savings function $S = S(y)$	$\frac{dS}{dy}$ = marginal propensity to save

5.3 Non-differentiability

It is possible that a function $y = f(x)$ may possess derivatives at some values of x but not at some other values. When the derivative exists at every point (x, y), the function is said to be differentiable; else it is non-differentiable. It is obvious that a discontinuous function is non-differentiable. At the point of discontinuity the derivative does not exist. Graphically, it is not possible to draw a tangent to the curve at this point.

However, to be differentiable, it is not enough for a function to be continuous. It must also not contain any kink or sharp points. If a kink exists, then again no unique tangent can be drawn to the curve at this point. There will be an infinite number of lines that can be made at that point. The above two instances of non-differentiability are graphed below.

Figure 5.1 Non differentiable functions



In both the diagrams, derivative does not exist at the value x_0 of x. A function, in order to be differentiable must therefore be continuous and posses no kinks. In other words, it must be *smooth*.

5.4 Differentiation

Differentiation refers to the process of obtaining the derivative of a function. Though the derivative of any function can be obtained directly in a manner in which we obtained the marginal cost from the total cost in the previous section, there are rules which can often be applied mechanically to differentiate different types of functions. These rules are explained below.

(a) ***Constant functions:*** Suppose we have a function $y = a$ where a is a constant. then $\Delta y = 0$ so that $\frac{\Delta y}{\Delta x} = 0$ and hence the derivative $\frac{dy}{dx} = 0$.

(b) ***Power functions:*** The general form of a power function is $y = ax^b$. Where a and b are constants. The derivative of such a function is given by

$$\frac{dy}{dx} = bax^{b-1}$$

Example 5.2

Let $y = 5x^6$. Then $\frac{dy}{dx} = 30x^5$

Verify as a special case that if $y = x$ then $\frac{dy}{dx} = 1$

Note: Part (iii) of Example 5.1 could have just been solved by using the power function rule evaluated at K25 as follows:

$$S(x) = 10x^2$$
$$\frac{dS}{dx} = 20x$$

$$= 20(25)$$
$$= 500$$

Crates of apples per Kwacha increase in the price of the crate.

(c) ***Sum of equations:*** Suppose a function $y = f(x)$ is in fact a sum of two or more separate functions of x such as $f_1(x), f_2(x)$ etc. that is,

$$y = f_1(x) + f_2(x) + \cdots$$

Then the derivative of y with respect to x is the sum of the derivative of each of the separate functions with respect to x. That is,

$$\frac{dy}{dx} \text{ or } f'(x) = f_1'(x) + f_2'(x) + \cdots$$

Example 5.3

Let $y = 10 + 120x - 2x^3$. Then

$$\frac{dy}{dx} = \frac{d}{dx}(10) + \frac{d}{dx}(120x) + \frac{d}{dx}(-2x^3)$$
$$= 0 + 120 - 6x^2$$
$$= 120 - 6x^2$$

(d) ***Product of two functions:*** Suppose y is the product of two separate functions $f_1(x)$ and $f_2(x)$. That is ,

$$y = f_1(x).f_2(x)$$

Then

$$\frac{dy}{dx} = f_1(x)\frac{df_2(x)}{dx} + f_2(x)\frac{df_1(x)}{dx}$$

This is referred to as *product rule*

Example 5.4

Suppose $y = x^2(3x + 4)$. Then

$$\frac{dy}{dx} = x^2\frac{d(3x+4)}{dx} + (3x+4)\frac{d(x^2)}{x}$$
$$= x^2(3) + (3x+4)(2x)$$
$$= 3x^2 + 6x^2 + 8x$$
$$= 9x^2 + 8x$$
$$= x(9x + 8)$$

The rule can be extended but the extension becomes complicated. If

$y = f_1(x).f_2(x).f_3(x)$, then

$$\frac{dy}{dx} = f_1(x).f_2(x).f_3'(x) + f_1(x)f_3(x)f_2'(x) + f_2(x)f_3(x)f_1'(x)$$

In general, if y is a product of n functions of x, that is,

$$y = \prod_{i=1}^{n} f_1(x)$$

where each of n functions is differentiable, then

$$\frac{dy}{dx} = \sum_{i=1}^{n}\left\{f_i'(x)\prod_{\substack{j=1\\j\neq i}}^{n} f_j(x)\right\}$$

(e) ***Quotient of two functions:*** Let the variable y be equal to the quotient of two separate functions $f_1(x)$ and $f_2(x)$, that is,

$$y = \frac{f_1(x)}{f_2(x)}$$

Then

$$\frac{dy}{dx} = \frac{f_2(x).f_1'(x) - f_1(x).f_2'(x)}{[f_2(x)]^2}$$

This is called *Quotient rule*

Example 5.5

Let $= \frac{5}{x^5}$ then

$$\frac{dy}{dx} = \frac{(x^5)(0) - 5(5^4)}{(x^5)^2}$$
$$= \frac{-25x^4}{x^{10}}$$
$$= \frac{-25}{x^6}$$

(f) ***Function of a function:*** Suppose $y = f_1(u)$ and $u = f_2(x)$, then the derivative of y respect to x is given by

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$= \frac{d}{du}f_1(u).\frac{d}{dx}f_2(x).$$

This is called *chain rule*. This is because it involves differentiation of a chain of functions.

Example 5.6

Let $y = \sqrt{2x^2 - 1}$

Here, we may put $2x^2 - 1 = u$, so that we have $y = \sqrt{u} = u^{\frac{1}{2}}$, Then

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$= \frac{1}{2}u^{-\frac{1}{2}}.4x$$
$$= 2x(2x^2 - 1)^{-\frac{1}{2}}$$
$$= \frac{2x}{\sqrt{2x^2 - 1}}$$

(g) ***Inverse function:*** If $y = f(x)$, then the derivative of x with respect to y is given by

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}, \text{where the function } f \text{ is assumed one} - \text{to} - \text{one}$$

Example 5.7

let $y = 4x^2$, where the function is defined only for positive values of x.

$$\text{Then,} \quad \frac{dy}{dx} = 8x$$

What is $\frac{dx}{dy}$? we have $x = \sqrt{y/4}$, then,

$$\frac{dx}{dy} = \frac{1}{2}\left(\frac{y}{4}\right)^{-\frac{1}{2}}.\frac{1}{4}$$
$$= \frac{1}{8}\left(\frac{4x^2}{4}\right)^{-\frac{1}{2}}$$
$$= \frac{1}{8}(x^2)^{-\frac{1}{2}}$$
$$= \frac{1}{8}\frac{1}{\sqrt{x^2}} = \frac{1}{\frac{dy}{dx}}$$

(h) ***Exponential function:*** If $y = a^u$, where $u = f(x)$, then

$$\frac{dy}{dx} = a^u \log a \frac{du}{dx}$$

For example, let $y = 5^{x^2-1}$. Then,

$$\frac{dy}{dx}(5^{x^2-1})(\log 5)(2x)$$
$$= 2x.5^{x^2-1}\log 5$$

As a special case, if we put $a = e$ so that $y = e^u$ where $u = f(x)$, then

$$\frac{dy}{dx} = e^u\frac{du}{dx}$$

For example, let $y = e^x$, then $\frac{dy}{dx} = e^x$ and if $y = Ae^x$, then $\frac{dy}{dx} = Ae^x$

This is the *only* case of a function being equal to its own derivative.

(I) ***Logarithmic functions:***

i.	$y = \log x$	$\frac{dy}{dx} = \frac{1}{x}$
ii.	$y = \log u$	
	Where $u = f(x)$	$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}$
iii.	$y = \log_a x$	$\frac{dy}{dx} = \frac{1}{x}\log_a e$
iv.	$y = \log_a u$	
	Where $u = f(x)$	$\frac{dy}{dx} = \frac{1}{u}\log_a e\frac{du}{dx}$

Note that each succeeding function is a more generalised case of the preceding one.

Example 5.8

Let $y = \log\sqrt{x^2 - 1}$. Then, we can see that this corresponds to the function form in (ii) above. Hence,

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}\frac{1}{2}(x^2 - 1)^{-\frac{1}{2}}2x$$
$$= \frac{x}{x^2 - 1}$$

(j) ***Implicit function:*** When we have a functional relationship between x and y in the implicit form as $f(x,y) = 0$, we can differentiate each term on the *LHS* of the equation treating y as a function of x and then solve for $\frac{dy}{dx}$.

Example 5.9

Let $xy^2 - 2x^2 + y^3 = 0$

Differentiating with respect to x, we get

$$2xy\frac{dy}{dx} + y^2 - 4x + 3y^2\frac{dy}{dx} = 0$$
$$(2xy + 3y^2)\frac{dy}{dx} = 4x - y^2$$
$$\frac{dy}{dx} = \frac{4x - y^2}{2xy + 3y^2}$$

5.5 Economic Applications

The demand function for a commodity is given as $Q = 100 - P$, where P is the price per unit and Q is the number of units. Find the marginal revenue when $Q = 20$ units. We noted that marginal revenue is the derivative of the total revenue. The total revenue $R = PQ$. we have $P = 100 - Q$ hence $R = 100Q - Q^2$. Then, marginal revenue

$$\frac{dR}{dQ} = \frac{d}{dQ}(100Q - Q^2)$$
$$= 100 - 2Q$$

Note that the derivative of a function is itself a function and hence can be evaluated at particular values of the variable. Thus, when $Q = 20$, marginal revenue is equal to 60. Further, the marginal revenue reaches zero when $Q = 50$.

Let $y = f(x)$ be a differentiable function of x. The *elasticity* of the function is given by the ratio of the proportional change in y to proportional change in x. That is, elasticity of y with respect to x is given by:

$$e = \frac{\Delta y/y}{\Delta x/x}$$
$$= \frac{x}{y}\cdot\frac{\Delta y}{\Delta x}$$

This is in fact the *arc elasticity* of the function over the range Δx. We can get the point elasticity of the function at the point x as a limit of the arc elasticity as $\Delta x \to 0$. Point elasticity thus will be equal to $\frac{x}{y}\frac{dy}{dx}$.

Suppose we have a demand function $5x + 8y = 12$ where x the quantity is demanded and y is the price. We have

$$x = \frac{12}{5} - \frac{8}{5}y$$
$$\text{then, } \frac{dx}{dy} = -\frac{8}{5}$$

Therefore, point elasticity of demand is given by

$$\frac{y}{x}\frac{dx}{dy} = \frac{8y}{5x}$$
$$= \frac{-8y}{5\left(\frac{12-8y}{5}\right)}$$
$$= \frac{8y}{8y-12}$$
$$= \frac{2y}{2y-3}$$

5.6 Higher Order Derivatives

As has been pointed out, the derivative of a function $y = f(x)$ is also a function of x and hence can be differentiated with respect to x. The derivative of the original function $y = f(x)$ is hence often known as the *first derivative*. The derivative of the first derivative is the *second derivative;* the derivative of the second derivative is the *third derivative;* and so on.

As the first derivative is denoted as $\frac{dy}{dx}$ or $f'(x)$ or y', so the notation for the second derivative is $\frac{d^2y}{dx^2}$ or $f''(x)$ or y'', for the third derivative $\frac{d^3y}{dx^3}$ or $f'''(x)$ or y'''. In general, the nth derivative is denoted as $\frac{d^ny}{dx^n}$ or $f^n(x)$ or y^n.

For example, let $y = 5x^4$. Then

$$\frac{dy}{dx} = 20x^3$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 60x^2$$
$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = 120x$$
$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = 120$$
$$\frac{d^5y}{dx^5} = \frac{d}{dx}\left(\frac{d^4y}{dx^4}\right) = 0$$

And then, $\frac{d^6y}{dx^6}, \frac{d^7y}{dx^7}$ and all the subsequent derivatives are equal to zero.

We shall study in the subsequent chapters; second order derivatives have useful applications. If the first derivative tells us about the slope of a curve at a point or the rate of change in a function at a given value of its argument, the second derivative tells us about the slope of the slope of the curve at a point or the rate of change of the rate of change, in other words, the *acceleration* of the function. The first derivative tells us whether the function is increasing or decreasing. The additional knowledge about the second derivative value informs us whether the function is changing (increasing or decreasing) at an increasing rate or decreasing rate. These interpretations of the first and second derivatives will be discussed elaborately later.

5.7 Partial Derivatives and their Applications:

So far we have considered only functions of the form $y = f(x)$ involving one dependent variable and one independent variable. More generally, we may have a function in several variables of the form

$$y = f(x_1, x_2, \dots\dots\dots x_n)$$

With only one dependent variable, the derivative of the function $y = f(x)$ tells us the way y changes when x changes, x being assumed to be the sole influencing variable on y. For instance, if you take the production function $O = f(I)$. The derivative $\frac{dO}{dI}$ is the value of the marginal product of the input I. Now suppose the production function is $O = f(L, K)$ where the two inputs L and K are labour and capital respectively. We wish to study the separate influences of labour and capital on output. To do this, we may assume one of the inputs to be kept constant at a certain level (i.e. treat it as a constant) and differentiate output with respect to the other factor. This would tell us the rate of change in output when that factor alone varies. If say, capital is kept constant, then the derivative of output with respect to labour will give us the marginal product of labour; likewise, keeping labour constant, we can differentiate the function with respect to capital and obtain the marginal output of capital. But in either case, the function is differentiated *partially* with respect to one independent variable, keeping the other variable constant. In other words, the marginal products of labour and capital are obtained as *partial derivatives of output* with respect to labour and capital respectively. In general, if you have the variable y as a function of n independent variables, one can get the partial derivative for each of the n variables, treating the remaining $n - 1$ variables as constants.

Example 5.10

let $z = 2x^2 - 3xy + 5y^3 + 1$

We will have two first partial derivatives, one with respect to x and the other with respect to y. To distinguish a partial derivative from the ordinary derivative $\frac{dy}{dx}$, the former is denoted by a curled ∂. Thus,

$$\frac{\partial z}{\partial x} = 4x - 3y$$
$$\frac{\partial z}{\partial y} = -3x + 15y^2$$

$\frac{\partial z}{\partial x}$ can also be written as $\frac{\partial f}{\partial x}$ since $z = f(x, y), z_x$ or f_x and

$\frac{\partial z}{\partial y}$ can also be written as $\frac{\partial f}{\partial y}$, z_y or f_y

One can differentiate each of these first partial derivatives to obtain second order partial derivatives. But since the first derivatives are themselves functions of both x and y, each of them may therefore be differentiated with respect to x or y. We shall thus have four second order partial derivatives. Let us calculate them for Example 5.10 above.

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = z_{xx} = f_{xx}$$

$$= \frac{\partial}{\partial x}(4x - 3y)$$
$$= 4$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = z_{yy} = f_{yy}$$
$$= \frac{\partial z}{\partial y}(-3x + 15y^2)$$
$$= 30y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = z_{yx} = f_{yx}$$
$$= \frac{\partial z}{\partial x}(-3x + 15y^2)$$
$$= -3$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = z_{xy} = f_{xy}$$
$$= \frac{\partial}{\partial y}(4x - 3y)$$
$$= -3$$

The first two derivatives are similar to our simple derivatives of the form $\frac{d^2y}{dx^2}$. The latter two derivatives are called *cross* or *mixed partial derivatives.* In our example, we note that the values of the two cross partial derivatives are equal. This is generally the case. f_{yx} and f_{xy} will be equal for all values of x and y for which they are continuous. This has been proved by a theorem known as Young's Theorem.

5.7.1 *Economic Applications of Partial Derivatives*

Example 5.11

Suppose you have

$$O = f(L, K)$$

Where O is output, L is labour and K is capital. The production function is said to reflect constant returns to scale if when labour and capital are changed in a certain proportion, the output also changes in the same proportion. For example a doubling of the quantities of labour and capital doubles the output, and a 50% reduction in the amounts of labour and capital reduces output too by 50%. Mathematically, we have

$$f(\lambda L, \lambda K) = \lambda f(L, K) = \lambda O$$

Such a production function is said to be *homogenous of degree one* or *linear homogenous.* A theorem, known as *Euler's Theorem,* states that for a linear homogenous production function $O = f(L, K)$

$$L\frac{\partial O}{\partial L} + K\frac{\partial O}{\partial K} = O$$

This means that if the prices of labour and capital are set equal to their respective marginal products, then the total payments made to labour and capital will together be equal to the value of output produced. This in fact is a solution to the so called *adding up problem* or *product exhaustion problem*.

A popular example of a linear production function is the *Cobb-Douglas* production function:

$O = AL^aK^{1-a}$, where A and a are constants

$$\frac{\partial O}{\partial L} = aAL^{a-1}K^{1-a}$$
$$\frac{\partial O}{\partial L} = (1-a)AL^aK^{-a}$$

$$\therefore L.\frac{\partial O}{\partial L} + K.\frac{\partial O}{\partial K} = L(aAL^{a-1}K^{1-a}) + K[(1-a)AL^aK^{-a}]$$

Example 5.12

Let the demand functions for two goods be given as

$$Q_1 = f(p_1, q_1), \qquad Q_2 = g(p_2, q_2)$$

The four partial derivatives $\frac{\partial Q_1}{\partial p_1}, \frac{\partial Q_2}{\partial p_2}, \frac{\partial Q_2}{\partial p_1}$ and $\frac{\partial Q_1}{\partial p_2}$ are called marginal demand functions.

If the demand functions are well-behaved in the sense of adhering to the law of inverse relationship between the price of the good and its demand, then $\frac{\partial Q_1}{\partial p_1}$ and $\frac{\partial Q_2}{\partial p_2}$ will be negative. The signs of $\frac{\partial Q_1}{\partial p_2}$ and $\frac{\partial Q_2}{\partial p_1}$ will however depend on the nature of relationship between the two commodities. If the two goods are *complementary,* a fall in the price of either commodity will raise demand for both commodities so that both $\frac{\partial Q_1}{\partial p_2}$ and $\frac{\partial Q_2}{\partial p_1}$ will be negative. By similar reasoning, if both $\frac{\partial Q_1}{\partial p_1}$ and $\frac{\partial Q_2}{\partial p_2}$ are positive, the two goods are competitive. But if one of the partial derivatives is positive and the other negative, the goods are neither complementary nor competitive.

Once the marginal demand functions are obtained, one can also obtain the expression for the partial elasticities of demand by multiplying by the relevant price quantity ratio.

- $\frac{p_1}{Q_1}\frac{\partial Q_1}{\partial p_1}$ = partial elasticity of demand Q_1 with respect to p_1
- $\frac{p_2}{Q_2}\frac{\partial Q_2}{\partial p_2}$ = partial elasticity of demand Q_2 with respect to p_2
- $\frac{p_2}{Q_1}\frac{\partial Q_1}{\partial p_2}$ = partial elasticity of demand Q_1 with respect to p_2
- $\frac{p_1}{Q_2}\frac{\partial Q_2}{\partial p_1}$ = partial elasticity of demand Q_2 with respect to p_1

The last two elasticities are called *cross elasticities* of demand and they will have the same signs as those of their respective marginal and demand functions.

Example 5.13

If two commodities are jointly produced, their cost function will be

$$C = f(Q_1, Q_2)$$

The partial derivatives $\frac{\partial C}{\partial Q_1}$ and $\frac{\partial C}{\partial Q_2}$ are the marginal cost functions and are usually positive.

For example, let the joint cost function be given by

$$C = \log(10 + Q_1)Q_2 \text{. then,}$$
$$\frac{\partial C}{\partial Q_1} = \frac{Q_2}{10 + Q_1}$$
$$\frac{\partial C}{\partial Q_2} = \log(10 + Q_1)$$

5.8 Extensions to Two or More Variables: Total Differentials

Let $y = f(x_1, x_2)$. This means the variable y is a function of not only one variable but two. Assuming independence in the two explanatory variables, then x_1 and x_2 will vary independently. These variations will each have an effect on the dependent variable y. Since each x is causing a change in y, the total change in y will then be the sum of the two changes being caused by x_1 and x_2 respectively. This is given by:

$$dy = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2$$

It is called the *total differential* of y. It differs from the partial derivatives discussed in earlier sections in that the partial derivative assume variation in one variable at a time and look at the effect caused on y. In total derivatives however, all variables vary independently and our interest is the total change in y caused by independent changes in all the explanatory variables.

In a more general case, with many explanatory variables, the total derivative is the sum of the changes caused by each explanatory variable. With n explanatory variables, the total derivative is given below.

$$dy = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n$$

The total derivative above has two terms that show the change caused by the first variable and the second variable. As in the case of second order derivatives, one can also get the second order total derivative by differentiating the first order total derivative that we have discussed. To get to the second order total derivative, we must differentiate this with respect to the two variables in the first order total derivative again. It is simple to comprehend if we think of the first derivative as the original function which must be totally differentiated. Thus we must get the change (in the first total derivative) caused by the two variables and then sum them.

$$\begin{aligned} d^2y &= d(dy) \\ &= \frac{\partial(dy)}{\partial x_1}dx_1 + \frac{\partial(dy)}{\partial x_2}dx_2 \\ &= \frac{\partial(\frac{\partial y}{\partial x_1}dx_1 + \frac{\partial y}{\partial x_2}dx_2)}{\partial x_1}dx_1 + \frac{\partial(\frac{\partial y}{\partial x_1}dx_1 + \frac{\partial y}{\partial x_2}dx_2)}{\partial x_2}dx_2 \end{aligned}$$

At this stage, it is demonstrated that the second order total derivative is the derivative of the first. In the first total derivative, the two terms will each be differentiated with respect to each of the independent variables.

$$\begin{aligned} d^2y &= \left(\frac{\partial^2 y}{\partial x_1^2}dx_1 + \frac{\partial^2 y}{\partial x_1 \partial x_2}dx_2\right)dx_1 + \left(\frac{\partial^2 y}{\partial x_2 \partial x_1}dx_1 + \frac{\partial^2 y}{\partial x_2^2}dx_2\right)dx_2 \\ &= \frac{\partial^2 y}{\partial x_1^2}dx_1^2 + \frac{\partial^2 y}{\partial x_1 \partial x_2}dx_2 dx_1 + \frac{\partial^2 y}{\partial x_2 \partial x_1}dx_1 dx_2 + \frac{\partial^2 y}{\partial x_2^2}dx_2^2 \\ &= \frac{\partial^2 y}{\partial x_1^2}dx_1^2 + 2\frac{\partial^2 y}{\partial x_1 \partial x_2}dx_1 dx_2 + \frac{\partial^2 y}{\partial x_2^2}dx_2^2 \end{aligned}$$

In the last line, the expression for the second order derivative can be shortened by using the symbol f_{ij} in place of $\frac{\partial^2 y}{\partial x_i \partial x_j}$.

$$f_{ij} = \frac{\partial^2 y}{\partial x_i \partial x_j}$$

The left hand side is second order derivative of the function with respect to variable i and then j. By now we know using Young's theorem that the order in which the function is differentiated does not matter. This is the reason we stick to the order of starting with variable i in the

differentiation. Then the second order total derivative would simplify to

$$d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2$$

As we shall see later, the second order total derivative has an important role to play in optimisation.

Chapter 6

6 INTEGRAL CALCULUS

6.1 Introduction

In Chapter 5, we dealt with moving from a function to a derivative, and how this applies in Economics. In other economic problems however, this process has to be done in reverse. Given the derivative, what is the original function? This process of reverse differentiation is known as *integration*.

6.2 Inverse differentiation and the Indefinite Integral

Consider a function y whose derivative $\frac{dy}{dx} = 2x$. The derivative is known but the function itself is not known. Obviously one clue is that there must be a function which when differentiated, gives the derivative in question, $2x$. The easiest way to get round this is to differentiate a function and see how it transforms into a derivative. Take a simple quadratic polynomial

$$y = x^2 + 2x - 4$$

Using the techniques of differentiation covered in the preceding chapter, we find the derivative as $\frac{dy}{dx} = 2x + 2$. For every term of a polynomial ax^b where a is the coefficient, x is the variable and b is the exponent, the derivative bax^{b-1} is arrived at in two stages:

i. Multiply the term by its exponent
ii. Reduce the exponent by unit or subtract one from the exponent.

To carry out inverse differentiation, simply execute the above two processes in reverse. It is worth stating here that execution in reverse also means reversing the order of executing the two stages. Since differentiation starts with the first to end with the second, integration has to start with the second and end with the first step. It is also important to be mindful that the reverse of multiplication is division, while the reverse of addition is subtraction.

In reverse, the steps are therefore rearranged as

i. Increase the exponent by a unit or add one to the exponent
ii. Divide the term with the resulting exponent

Given the derivative bax^{b-1}, the new procedure should bring back the original function. The derivative above can then be reversed to become.

$$\int (2x + 2)\, dx = x^2 + 2x$$

Though this function looks similar to the original function above, one component is still missing. The constant is absent. In the original function the constant equals 4 and one may just opt to fix it in the above equation. However, it will not be known *a priori* that the value of the constant is 4. Hence in order to provide for the unknown constant, the right hand side of the above equation must include an arbitrary constant C. Thus we shall have

$$\int (2x + 2)\, dx = x^2 + 2x + C$$

Generally, an indefinite integral is of the form.

$$\int f(x)\, dx = F(x) + C$$

It is an indefinite integral because it lacks a definite value. It will be a function of x and hence vary with it. The opposite case, definite integral is discussed later in the chapter. In the integral, the symbol $\int$ which looks like an elongated s is called the integral sign. It is an instruction to *integrate*. The function $f(x)$ is what must be integrated. It is called the integrand. As a function in general, we expect that it will take on various form or types of functions. The last part dx specifies the variable of integration. It is similar to the dx under differentiation which specifies the variable of differentiation. The dx in integration means integration is to be executed with respect to variable x.

The C on the right side of the integral is an arbitrary constant. It is called the *constant of integration*. It represents the constant that disappears when a function is differentiated. Since integration reverses differentiation, the constant of integration is a place holder for the unknown constant, which can also be zero. This constant also serves to indicate the multiple parentage of the integrand. That is, a given derivative will result from multiple functions, differing in the vertical intercept or constant.

6.3 Fundamental Theorem of Calculus

In view of the discussion so far, applying differentiation and integration on a function, one after the other, should leave the function unchanged. This leads to the *fundamental theorem of calculus*. The theorem states that if $f(x)$ is a continuous function, then the derivative of the indefinite integral of $f(x)$ is the function $f(x)$ itself. Algebraically, this can be shown as follows

$$\frac{d}{dx}\int f(x) = f(x)$$

6.4 Rules of integration

The rules of integration show how integration of different functions is to be executed. As will be noticed, these rules are similar or related to the rules of differentiations studied earlier. It is assumed here that the reader is already familiar with the latter. In this section, we provide for each type of function its integral. In general, the rules are classified into four. These apply to the four classes of functions.

6.4.1 Power Rule:

As the name may suggest, this rule of integration applies to functions with powers. These are commonly referred to as polynomial functions and are discussed in chapter 3. Since polynomial simply means multi-term, polynomials are functions of several terms. In the integration, terms are integrated separately. This is exactly as put by Chiang and Wainwright (2005) when they state that 'the integral of the sum of finite number of functions is the sum of the integrals of those functions'. It must be cleared here that the term 'sum' as used in the statement also includes 'difference'. In this regard, each term in a polynomial can be taken as a function making a polynomial a sum of *power functions*. Thus the statement of the *power rule* refers to a single term of any function.

In general, for the function given by $f(x) = ax^n$ where a is the coefficient and n is the power, the power rules is expressed as follows

$$\int x^n\, dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$$

The only restriction on the rule is that the power must be non-negative-one. This restriction is very important for if ignored, the function may collapse. Other than that, n can take on any values. In particular, since $n = 0$ is admissible, the rule can be used to integrate functions of various power including constant functions without requiring any special treatment. This might be intriguing but it is easier to execute if a constant function $f(x) = a$ can be rewritten as $f(x) = ax^0$. Even if the variable x appears in the function, its power makes is logically absent.

Example 6.1:

Find $\int a\, dx$. This is a constant function explained above. It can be rewritten as $\int ax^0 dx$. With this the integral can be found with ease as follows.

$$\begin{aligned}\int a\, dx &= \int ax^0 dx \\ &= a\frac{1}{0+1}x^{0+1} + C \\ &= ax + C\end{aligned}$$

Example 6.2

Find $\int \sqrt{x}$. This type of function looks a little strange. It can however be transformed to a more familiar form, the power form. Notice that $\sqrt{x} = x^{\frac{1}{2}}$. Thus

$$\int \sqrt{x} = \int x^{\frac{1}{2}}$$

$$= \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} + C$$
$$= \frac{2}{3} x^{\frac{3}{2}} + C$$

6.4.2 The Exponential Rule

As the name implies, the rule applies to exponential functions. Though exponential is wrongly taken as synonymous to the constant $e \cong 2.71828$, the book takes a more general case. The general form of an exponential function is $f(x) = b^x$, where b is the base. For such a function, the integral is given by

$$\int b^x\, dx = \frac{b^x}{\ln b} + C$$

in the special case where $b = e \cong 2.71828$, the rule reduces to

$$\int e^x\, dx = e^x + C$$

since $\ln e = 1$. In some cases, the power in the function may not be a single variable x. The power may be a function of x. Such need not come as a strange concept. It is provided for by the variant of the exponential rule stated as

$$\int e^{f(x)}\, dx = \frac{e^{f(x)}}{f'(x)} + C$$

Example 6.3

Find $\int e^{x^2+2} dx$. This is an exponential function with base e. The power is specified and its integral should not be mystifying. Before dealing with the whole function, it is helpful for someone who acknowledges ineptitude for algebra to begin with the power alone. Take the power alone and differentiate it since the integral required the differential of the power. For a connoisseur, this may not be necessary.

The power is $f(x) = x^2 + 2$ and the derivative is $f'(x) = 2x$. The integral can then be worked out as

$$\int e^{x^2+2} dx = \int e^{f(x)} dx$$
$$= \frac{e^{f(x)}}{f'(x)} + C$$
$$= \frac{e^{x^2+2}}{2x} + C$$

6.4.3 The Logarithmic Rule

The logarithmic function is a special type of function with a special integral. As mentioned *a priori*, the rules of integration are derived from the rules of differentiation. Under the rules of differentiation, the derivative of a special function $f(x) = \frac{1}{x}$ is given by $\frac{d}{dx} \ln x = \left(\frac{1}{x}\right)$. The reverse of this differential will provide a clue for the integration of a logarithmic function. The logarithmic rule states as

$$\int \frac{1}{x} dx = \ln x + C$$

This rule comes as a relief because it provides a way out with a function that was not admissible under the power rule. The function $\frac{1}{x} = x^{-1}$ is a term with $n = -1$ and was not admissible under the power rule above. With the logarithmic rule on the menu, such a function will no longer be inadmissible.

The logarithmic rule can be magnified and used to solve more complex integrals. In its simple form, the rule refers to a quotient with unit in the numerator and $\boldsymbol{x}$ in the denominator. This is a very important statement which requires a closer look. Since the derivative of $\boldsymbol{x}$ is unit, then the function is a quotient of a derivative and the function in the numerator and denominator respectively. With this in mind, the rule can be stated in its variant, which is also more general as

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$$

where $\boldsymbol{f'(x)}$ in the numerator is the derivative of $\boldsymbol{f(x)}$ in the denominator. Given a quotient of functions for integration, it is a necessary condition that the numerator is a derivative of the denominator if the rule is to be applied. The method and procedure of determining the derivative relationship is left to the reader.

Example 6.4

Find $\int \frac{dx}{x \cdot \ln x}$. On face value, the function looks complex and seems not to fit into any of the integration rules looked at so far. Nonetheless, a non complex integral is easy to find. Split the integral into separate quotients as follows $\int \left(\frac{1}{x}\right)\left(\frac{1}{\ln x}\right) dx$ and let $g(x) = \frac{1}{x}$ and $h(x) = \ln x$. The integral can now be written as $\int \frac{g(x)}{h(x)} dx$ This exposes the relationship between the terms of the quotient. The derivative of $h(x) = \ln x$ is $g(x) = \frac{1}{x}$. With the new revelation, the integral takes the form $\int \frac{h'(x)}{h(x)} dx$ which now points to the simple logarithmic rule. Its execution has already been illustrated above.

$$\int \frac{dx}{x \cdot \ln x} = \frac{1/x}{\ln x} dx$$
$$= \int \frac{h'(x)}{h(x)} dx$$
$$= \ln h(x) + C$$
$$= \ln(\ln x) + C$$

These rules of integration need not be used in isolation. As will be discovered is course, some functions may require the use of more than one rule. There is no harm using one rule for a part of the function and another rule for other parts. This does not vitiate the answer.

Some functions however may be combined or expressed in more complex forms. For example, the methods covered so far have not provided for the integration of a function which is a product of independent functions. The term independent is used in a more restricted manner. It is used to mean the functions do not satisfy the relationship required for the logarithmic or exponential rules of integration to apply. The integrals of such complex functions can be found by way of substitution method or integration by parts. These are discussed below.

6.4.4 Substitution method

Since integration is the reverse of differentiation, we will often get back to differentiation to reload on the concepts applied in differentiation. This should make the work in integration a lot easier as will be demonstrated for more complex functions. One concept prominent in differentiation is the chain rule, which applies when one function is a function of another. That is, the function $G(h(x))$ has the differential

$$\frac{d}{dx} G\big(h(x)\big) = g(h(x)) \cdot h'(x)$$

The reverse of the above will be the integration of the right hand side of the equation to get the left hand side of it. Formally, this can be stated as

$$\int g(h(x)) \cdot h'(x) = G\big(h(x)\big) + C$$

This provides a way of integrating a product of functions. However, caution must be exercised before applying the formulae. The method requires that the derivative relationship between the two factors of a function is accurately identified. This is known as the substitution method of integration. It is used to integrate a function which is a product of functions which possess a special relationship. It is a requirement that this derivative relationship exist if the method is to be used.

The method allows the simplification of complex integration problem. As the name suggest, it involves substituting one variable for another to reduce a more complex function to a simple one. A new variable u replaces x time it is realised that the integrand is a chain rule derivative. The method of substitution is of critical importance as the outcome hinges on how this is done. For the above integral problem, the new variable is $u = h(x)$. Then $du = h'(x)dx$ and the new integrand is

$$\int g(u)\, du = G(u) + C = G\big(h(x)\big) + C$$

Example 6.5

Find $\int \frac{(2x+1)}{(x-2)(x+3)} dx$

Sometimes the way questions are presented, it may not be clear at first site which rule of integration must be used. In the above case, we can look at the two functions as the numerator and the denominator and check whether one is a derivative of the other. The rule of thumb for polynomial functions is that the derivative is always of a lower *order*. Combine this with the fact that the derivative, as a derived function cannot be in the denominator. This leaves only one possibility to be checked; whether the numerator is a derivative of the denominator, which we confirm.

Then introduce a new variable as follows:

$$u = (x-2)(x+3)$$
$$= x^2 + x - 6$$
$$du = (2x+1)dx$$

Then substitute into the integral

$$\int \frac{(2x+1)}{(x-2)(x+3)} dx = \int \frac{du}{u}$$
$$= \ln u + C$$
$$= \ln[(x-2)(x+3)] + C$$

6.4.5 Integration by parts

Some functions however may not possess the relationship required for the substitution method to apply. Suppose now we have to deal with the integral of a product of two completely independent functions. The substation method will not offer any clue, nor will any of the methods discussed earlier in the chapter. This however should not leave us with completely no alternative.

Though this may be over emphasising, it must always be in one's mind that the process of integration is the reverse of the process of differentiation. As such, there is no harm to ever trek back to consider differentiation concepts that are of use in integration. This time, the 'product rule'. Given a function $f(x) = uv$ with both u and v functions of x, its differential using the product rule is given by

$$f'(x) = \frac{d}{dx}(uv) = vu' + uv'$$

with this derivative at hand, integration can be done with ease. Integrate both sides of the above differential.

$$\int \frac{d}{dx}(uv)\, dx = \int (vu' + uv')\, dx$$

$$uv = \int vu'\, dx + \int uv'\, dx$$

The left hand side reduces because the integral of a differential of a function is a function itself[34]. On the right hand side, the integral of a sum of functions is the sum of the integral of the functions. The left hand side has no integration. The equation then needs to be rearranged to have the integration of a product of functions on the left hand side. Here it does not matter which of the two from the right transfers to the left. Nonetheless, the choice will have a bearing identification of the functions.

At this stage, the integration by parts can be stated formally and the authors expect that the reader will not have any problem. The function is

$$\int uv'\, dx = uv - \int vu'\, dx$$

read as the integral of a product of u and the derivative of v (uv'). This provides a method of integrating a product of two independent functions. However, this requires the two functions are correctly identified. By identification is meant identifying which function is u and which is v' which depends on the differentiability and integrability of the functions. Notice that the left side of the equation has u while the right has u'. This requires that u is differentiable. For the other function, the left side has v' while the right has its integral, v. This requires that for one function to be designated v', the integral must be known or it should be possible to integrate the function.

But the result side of the equation also has an integration of a product of functions. This must not be intimidating. Although it is true that the outcome of integration by parts will also require integration by parts, the latter will not be as complex as the former. This is made possible

[34] Refer to Fundamental Theorem of Calculus at 6.3

because with successive differentiation, the differentiable function must be getting to a constant. Thus the method of integration by parts will have to be performed as many times as it takes to differentiate the function u to a constant. Since both functions are likely to be differentiable, the strategy is differentiating one which ultimately becomes constant if repeatedly differentiated. For instance, the function $\frac{1}{x}$ as simple as may look cannot become constant with repeated differentiation. A function of the form ax^n can be reduced to a constant with repeated differentiation. This should guide the selection of which function is 'differentiable' and which one to 'integrate'.

6.5 Initial conditions and boundary values

As mentioned earlier in the chapter, the lost constant in differentiation cannot be retrieved from the differential alone. With the constant unknown, the best was to include an arbitrary C in its place. But the integral that results is a function which can be plotted on a Euclidean plane, but for the unknown vertical intercept as is sometimes called.

The shape and slope of the function is known but the exact location hinges on the unknown constant. But what is the constant? And why does it remain so critical? To the first question, the constant in the function is the values of the function when the explanatory variable assumes a null value. With a constant C, then the function passes through the point $(0, c)$. For the second question, the constant is not as critical as is normally put. It is only critical to the extent that it is simpler to locate or work with the point $(0, c)$ for any function.

In economics, the point $(0, c)$ is often referred to as the *initial condition*. Since most economic variables are restricted to the positive side of the plane, it is befitting to refer to $x = 0$ as the starting point. It gives the starting or initial value of the function. Any information about it essentially gives the precise point $(0, c)$ from which the constant of an indefinite integral can be definitised. From the foregoing, the arbitrary C in the indefinite integral is now definitised to a particular $C = c$, where c is the initial condition.

Alternatively, a point along the function may be given. This is equivalent to stating the value of the function for particular value(s) of the regressors. For instance, in a time series, information on the function may be available for a particular year. Thus in the function $Y(t) = \int f(t)dt + C$ a particular or definite value of C can be found by substituting the values of (t, Y) of a known point. Once the value of C is definitised, the integral ceases to be indefinite. It becomes a definite integral and gives a particular function with the *integrand* as its derivative

Example 6.6

The marginal cost of producing a given level of output is given by $MC = \frac{x}{10} + \frac{10}{x}$. Find the total cost function given that overhead costs total 15.

It is already clear from *microeconomics theory* that the marginal cost is the derivative of the total cost function. Therefore, the total cost function is an integral of the marginal cost function. So, proceed to integrate.

$$TC = \int \frac{x}{10} + \frac{10}{x} dx$$
$$= \frac{x^2}{10} + 10 \ln x + C$$

This is an indefinite integral. It has two parts, the variable part and the constant part. The variable part is the Total Variable Cost (TVC) and the constant is the Total Fixed Cost (TFC). Since the latter is given, then the value of the arbitrary constant is known, $C = TFC = 15$. To total cost function is therefore given as

$$TC = \frac{x^2}{10} + 10 \ln x + 15$$

6.6 Economic Applications

Integration is a useful concept in the field of economics in general. With a widespread use of the *marginal concept*, integration provides a critical link from the marginal function to the total function. For instance, if the marginal revenue curve is known to us, integration provides a way of getting the total revenue function which can further be used to get the average function. These are critical functions in *microeconomics*.

Consider the revenue example. Suppose the marginal revenue function is given by

$$MR = 25 - 6x$$

where x is the quantity of the commodity produced and we wish to find the total revenue function. Like the total cost function presented in Example 6.6Example 6.6 above, the total revenue function is the integral of the marginal revenue function.

$$TR = \int 25 - 6x\, dx$$

$$= 25x - 3x^2 + C$$

The marginal revenue function depicts the behaviour of total revenue as quantity changes. Its information is insufficient to determine the exact position (determined by the vertical intercept $C = c$) of the revenue function. So, an additional piece of information is required. When no such information is explicitly provided, then we must resort to theory.

In the above case, the initial condition is not explicitly provided. It can however be implied from theory. The guidance from theory is that revenue must be zero when no output is produced and sold.

$$TR_{x=0} = 0$$
$$\therefore\ C = 0$$
$$TR = 25x - 3x^2$$

6.7 Definite integration

All the integrals discussed so far are indefinite integrals. We explained already what makes them indefinite; it is the lack of a definite numerical value. Even when the constant of integration is know, there is still no specific value of the integral because it depends on the x-value.

$$\int f(x)\, dx = F(x) + C$$

If we identify two values of x, a and b such that $a < b$, we can substitute into the indefinite integral to get a definite numerical value.

$$[F(b) + C] - [F(a) + C] = F(b) - F(a)$$

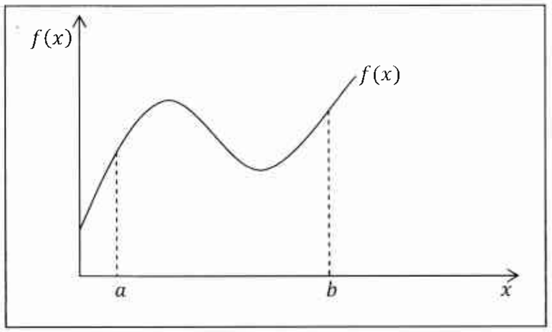
The constant will be common in both $F(a)$ and $F(b)$ and will disappear on differencing. The result is a definite value, free of the variable x and the arbitrary constant. This is what is referred to as *definite integral* from a to b. The first number a is known as the lower limit while the second b is the upper limit. In notation, the definite integral differs from the indefinite integrals by having limits below and above the integral sign. The lower limit will be below the integral sign while the upper limit takes the upper position.

This is provided by the *fundamental theorem of calculus*, which says the definite integral can be computed by subtracting the endpoints of an integral. For a function $f(x)$ continuous in the interval (a, b), the definite integral is

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

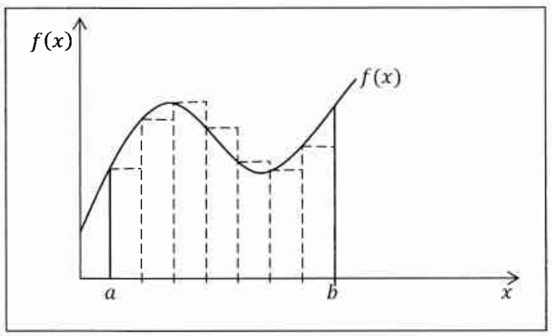
To put this in context, suppose we are interested in the area bound by the function $f(x)$, the horizontal axis and two limits, lower and upper. This is shown in the figure below.

Figure 6.1. Area under a Curve



For such a shape, geometry does not provide any method of precisely getting its area. Nonetheless, it should still be possible to use geometric methods to estimate the area. The starting point is the *Riemann sum*, named after a nineteenth Germany mathematician Bernhard Riemann. For the above area, partition it into rectangles whose area is a product of the width (w) and height (h).

Figure 6.2: Riemann Sum



In Figure 6.2 above, the area is partitioned into seven rectangles. The width of each rectangle is the difference between two successive values of x. That is $w = x_{i+1} - x_i = \Delta x$. The number of partitions we can form will depend on the width of each partition. The guiding formula is $\frac{b-a}{n} = \Delta x$. The partitions will be small the more they are or the number will increase with a reduction is the average width. For the rectangle bordered by x_i on the left and $x_i + \Delta x$ on the right, the height is the value of the function evaluated at x_i. We now state the area of each rectangle as:

$$A_i = w \times h = f(x_i) \times \Delta x = f(x_i)\Delta x$$

Since we are looking for the whole area partitioned into rectangles of various height (and as will be explained later, different widths) we sum all the rectangle areas. Thus, the estimate of the area between the curve and the x-axis bounded by $a = x_1$ on the left and $b = x_{n+1}$ on the right is given by

$$\tilde{A} = \sum_{i=1}^{n} f(x_i)\Delta x$$

Though the function assumes a fixed width for all the rectangles, they need not have equal width. The Riemann sum is actually defined for varying widths or changes in x. As such, a subscript is also attached as a demonstration that the width will also vary just like the height.

It is clear from Figure 6.2 that the area found by the Riemann sum is not the true area, and we denote it by $\tilde{A}$ with a tilde. It is an estimate. Nonetheless, the estimate will get closer to the true area if we make the partitions as small as possible. With finer divisions, we get a *near area*. The *Riemann integral* is defined as the limit of the Riemann sum as the partitions get smaller, $(\Delta x \to 0)$.

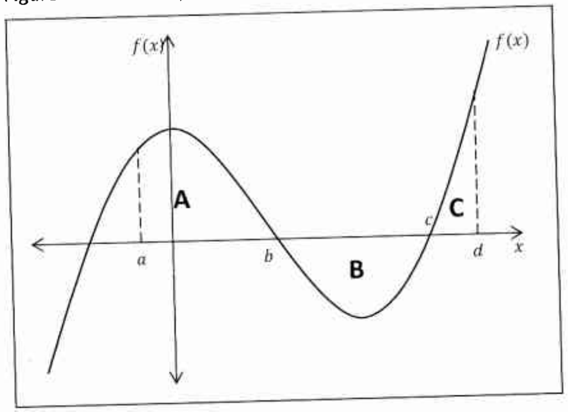
$$A = \int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

The integration takes care of infinitesimal division and sets the limit of summation as n becomes larger. The precise area can thus be given as

$$A = \int_a^b f(x)\,dx = F(b) - F(a)$$

The Riemann integral approaches integration from an area of a shape point of view. It calculates the area between the function and the x-axis. Since the function, as illustrated in Figure 6.2, both the height and width are positive and so will the area. When the function falls below the axis, the height of each partition becomes negative. With a positive width, the area will be negative. Consider Figure 6.3.

Figure 6.3: Possibility for a Negative Area



Suppose we are interested in the area between the function and the axis bound by a on the left and d on the right. Finding this area by simple *two limit Riemann integral*

$$\int_a^d f(x)\,dx$$

would be incorrect. It would be incorrect because within the two bounds, the function is above the horizontal axis for some part(s) and below it for some other(s). As a consequence, the area would be positive for some part(s) (those above the horizontal axis) and negative for other(s) (those below the horizontal axis). Thus in the above graph, the total area under the curve between a and d would be given by:

$$Total\ Area = A - B + C$$

$$= \int_a^b f(x)\,dx - \int_b^c f(x)\,dx + \int_c^d f(x)\,dx$$

Since area of a fixed shape is a definite number, definite integrals will always result in definite numbers. Thus, given, the marginal utility function, we are able to determine the total utility from consuming n units of a good. Given the marginal cost function, we are able to determine total cost of producing Q amount of output.

6.8 Properties of the definite integral

In the preceding discussion on integration, some properties emerge. Though it will be too much to highlight all of them, the book looks at a few key ones. The reader can infer from the few given to generate the rest.

Property I: Given a function with a constant coefficient, the constant coefficient can be factored out of the integral. Formally this is shown as follows

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

this property is also ideal for dealing with negative in an integral such as $\int_a^b -f(x)\,dx$. Perhaps the word negative mentioned may be a source of worry. The simple way to view the negative is that it is a constant coefficient and thus can be treated like k in the property above. In fact, most cases will present the negative as part of the constant, that is, the constant coefficient k is negative.

Property II: Since $\int_a^b f(x)\,dx = F(b) - F(a)$, then interchanging the limits of the integration only changes the sign of the definite integral.

$$\int_b^a f(x)\,dx = F(a) - F(b) = -\int_a^b f(x)\,dx$$

. The same property can also be used to prove that when the two limits are equal, the definite integral equal zero.

Property III: The third property emanates from the fact that a given area can still be calculated as a sum of its partitions. Given an area under the function $f(x)$ bound by a and d on the left and right respectively, the area remains unaltered if partitioned into as many parts as possible. Define to other limits c and b such that $a < c < b < d$. The sum of the three partitions defined by the four limits equals the area bound by the lowest and highest limits. This is stated as follows.

$$\int_a^d f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \int_c^d f(x)\,dx$$

This property permits the evaluation of area defined by discontinuous function since the partitions can be made to follow the points of discontinuity. It is often referred to as *additivity property of integration*.

Property IV: The last of the four properties involves a sum of functions. This also includes the difference since a sum and difference apply the same principles, *mutatis* *mutandis*. Given a sum of functions $f(x) = g(x) \pm h(x)$, then the property states that

$$\int_a^b f(x)\,dx = \int_a^b [g(x) \pm h(x)]dx = \int_a^b g(x)dx \pm \int_a^b h(x)dx$$

This property allows separating a polynomial of function to apply the rule of integration separately. Often, polynomials in a function may not be of the same form and thus will require the use of different *rules of integration*.

6.9 Multiple integrals

Under differentiation, it is possible to differentiate a function with respect to more than one independent variables. This gives rise to the concept of partial derivatives in which the total derivative is found by the process of partial differentiation. In the same way, there is a concept of partial integration under integration. This is used to integrate a function with respect to more than one independent variables. This involves integrating with respect to one at a time while treating the others as constants.

Suppose one wants to integrate a function $z = f(x,y)$. The function may first be integrated with respect to x, and evaluated for the given limits. The resulting function is then integrated with respect to y and evaluated over given limits. The process is called partial integration because, while integrating with respect to one variable, one or more other variables are held constant. The same results can be obtain with the order of integrating with respect to x and y interchanged. This is shown using *double integrals* written as.

$$\int_a^b \int_{g(y)}^{h(y)} f(x,y)dx\,dy \text{ and}$$

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)dy\,dx \text{ respectively}$$

With more independent variables, the process is generalised by using multiple integrals in the place of double integrals used above.

Example 6.7

Evaluate $\int_0^1 \int_0^x (x^2 + y^2)dy\,dx$

This is an example of a multiple integral, it has more than one integrals. The functi[illegible] is integrated in two dimensions (two variables). The procedure is to start with the [illegible]er variable so that the resulting integral is of one variable only. Integrate the part in square brackets, treating the other variable as if a constant.

$$\int_0^1 \int_0^x (x^2 + y^2)dy\,dx = \int_0^1 \left[\int_0^x (x^2 + y^2)dy\right] dx$$

$$= \int_0^1 \left[x^3 + \frac{x^3}{3}\right] dx$$

$$= \int_0^1 \frac{4x^3}{3} dx$$

$$= \frac{1}{3}$$

6.10 Improper integrals

Sometimes in integration, one of the limits or both may be infinite. For example in the integration $\int_a^\infty f(x)\,dx$, the upper limit is infinite. Such a problem cannot be evaluated because the infinite on the right literary means its open on the right. Moreover, area can only be calculated for a closed space. This crumples the integral.

To deal with such a problem, it is necessary to assume a finite upper limit which can be assumed to approach infinite. This takes us back to the concept of limits studied in earlier chapters. The solution to the above example is found using limits as follows.

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

If the limit exists, the integral is said to be convergent. The term convergent is used with a very simple meaning. Since this refers to a space bounded on all the sides except the right, the space can still be considered a *closed space* if the function converges to zero as the explanatory variable approaches infinite. This is the case with asymptotic functions which virtually converge to zero as the explanatory variable approaches infinite. This permits calculating the area even though technically there is no upper limit. When the limit does not exist, then the function is divergent. This refers to functions that don't close-up with the axis.

Example 6.8

Evaluate

$$A = \int_{-\infty}^{0} e^x\,dx$$

This integral defines the area of a shape to the left of the y-axis bound by the function and the x-axis. Since the function in question never becomes zero, the shape is open and area cannot be found. Nonetheless, since the function becomes asymptotic as $x \to -\infty$, it can be assumed to close up with the x-axis so that the area is defined.

$$\begin{aligned} A &= \lim_{a\to-\infty} \int_a^0 e^x \, dx \\ &= \lim_{a\to-\infty} (e^x - e^a) \\ &= \lim_{a\to-\infty} (1 - e^a) \\ &= (1-0) \\ &= 1 \end{aligned}$$

6.11 L'Hôpital's Rule

In evaluating limits, it is possible to get outcomes of the form $\frac{0}{x} = 0$ and $\frac{x}{0} = \infty$. Answers to these quotients are non-trivial. The former is guided by the fact that dividing any number into zero gives zero. The latter is based on the fact that any number divided by zero is infinite. These forms, though involving zero, are determinate.

In some instances, it is possible to get expressions of the form

$$\frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

Such expressions have no determinate solutions. As such, any such expression is said to be indeterminate. The limit cannot be determined. Other indeterminate forms may involve a difference between two infinite values. As a remedy for such problems, we factorise and eliminate the common factor that drives both the numerator and denominator to zero. Though this sounds common and is used in many instances, it is not always possible to find this common factor. Some functions may lead to this scenario is though there is no common factor between the numerator and denominator.

The l'Hôpital's rule, named after a seventeenth century French mathematician Guillaume de l'Hôpital, provides a more general way of dealing with such functions. Given a function of the form.

$$f(x) = \frac{g(x)}{h(x)}$$

If

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = 0$$

then the limit of the function $f(x)$ as x approaches a particular value a is indeterminate. L'Hôpital's rule states that if the functions $g(x)$ and $h(x)$ are differentiable in the interval

containing the point a, then the limit can be evaluated more easily by replacing each function in the quotient by the respective derivatives. Formally,

$$\lim_{x\to a} f(x) = \frac{g'(x)}{h'(x)} \quad iff \ \lim_{x\to a} g(x) = \lim_{x\to a} h(x) = 0 \text{ or } \infty$$

If the first derivative still give the same output, l'Hôpital's rule allows differentiating the differentials. This leads to second order derivatives. The process of differentiating can continue until a determinate outcome is obtained. Caution must be exercised when to use the rule. It is important to bear in mind that the rule only applied in limited circumstances. It requires that the undifferentiated functions are simultaneously zero or infinite as approaches.

Example 6.9

Find

$$\lim_{x\to 0} \frac{e^x - 1}{x}$$

Substituting the zero into the function gives the indeterminate case of $\frac{0}{0}$. The quotient also does not have any common factor that may be factorised and eliminated. Thus l'Hôpital's rule remains the only option.

Differentiate the numerator and denominator separately to get

$$\begin{aligned} \lim_{x\to 0} \frac{e^x - 1}{x} &= \lim_{x\to 0} \frac{e^x}{1} \\ &= \lim_{x\to 0} e^x \\ &= e^0 \\ &= 1 \end{aligned}$$

6.12 Economic applications

As noted already, the concept of integration, both indefinite and definite can be applied to the study of economics in many ways. It provides a link between many econometrics problems. In another way, integration, like differentiation, provides an additional tool for working with functions. Given marginal utility, integration provides a method of finding the total utility or given the net investment rate, integration makes it possible to derive the level of capital as a function of time. This section gives practical economics examples in which integration is used.

The first scenario is where integration provides a link between marginal and total function. This is a broader area encompassing utility, revenue and cost functions. For illustration, we use the cost function but the reader must be able to apply the concept to other areas. Suppose

$$MC = 4 + 6x + 15x^2$$

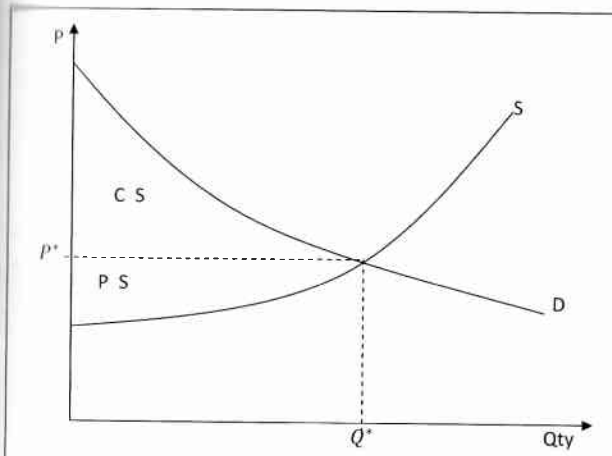
, what is the total cost function? It is known from theory that total cost function is the integral of marginal cost function. Therefore, proceed with integration.

$$TC = \int MC(x)\,dx$$
$$= \int (15x^2 + 6x + 4)\,dx$$
$$= 5x^3 + 3x^2 + 4x + C$$

This integration provides an opportunity to once again consider the arbitrary constant C. the constant C was not in the marginal cost function but only emerges in the total cost function. In addition, it does not depend on the level of output x. Microeconomic theory defines this type of cost as *fixed cost*, a type of cost related to overheads. Recall that the marginal cost measures the incremental costs, and must have nothing to do with overheads. It would however be misleading to ignore this type of cost in the total cost function because the firm incurs it.

The second application of integration as a measure of area under a curve is in the measurement of consumer and producer surpluses. The former is the area between the demand curve and the price line while the latter is measured by the area between the supply curve and the price line. For illustration again, there is no need to consider the two cases but the reader is left to *corollary* apply the concept to the other. The example used here is the consumer surplus. The figure below depicts the consumer surplus in the price-quantity plane.

Figure 6.4: Consumer and Producer Surplus



S is supply curve, D is demand curve, P^* and Q^* are equilibrium price and quantity, CS is consumer surplus and PS is producer surplus

The integration studied referred to an area bound by the axis on the lower side. On the contrary, consumer surplus in not bordered by the axis but by the price line, which is above the axis since price is eternally positive for a *good*. This should not prove problematic as some simple algebra is sufficient to get round the problem. The limits are not explicitly stated and it may be mistakenly taken as an *indefinite integration*. The area of interest is running from $Q = 0$ up to the last quantity. For the last quantity, microeconomics comes into play. It is the quantity where the demand intersects the given price line. The rationale is that a consumer will continue buying successive unit for as long as his/her marginal utility (demand) remains above the price. This point will define the upper limit of integration.

For the area below the price line, one must recognise that this is always a rectangle. It is defined by four straight lines meeting a *right angle*. The formula for its area is defined in *elementary mathematics* as a product of its two orthogonal sides. To sum it, consumer surplus is given by the area under the curve bound by the zero and equilibrium quantities less the rectangle.

$$CS = \int_0^{Q^*} P(Q)\,dQ - P^*Q^*$$

The rectangle being subtracted perhaps deserves attention and it may not suffice to just glossy over it. Recall that consumer surplus is defined as utility in excess of what the consumer pays for. This is the difference between total utility from consuming a good (area under the curve) and the value of money exchanged for the same goods (the rectangle).

The third area of application of integration is in the analysis of investment and the behaviour of capital stock, a function of time. In *Corporate Finance*, net investment $I(t)$ is the measure of net addition to capital $K(t)$. It is the rate of capital formation. Algebraically, this is shown as

$$\frac{dK}{dt} = I(t)$$

The stock of capital increases if net investment is positive and declines with a negative net investment. Thus capital stock is the sum of all net investments if time is discrete. For a continuous time case, capital stock is the integral of net investment. For a particular example, suppose net investment is given by $I(t) = 3t^{0.5}$ and that initial capital stock $K(0)$ is zero. What is the time path of capital stock?

The solution lies in integrating the investment function. Though no limits are given, the initial condition provided suffices to definitise the integral. Proceed as follows.

$$\begin{aligned} K(t) &= \int I(t)\,dt \\ &= \int 3t^{0.5}\,dt \\ &= 2t^{1.5} + C \end{aligned}$$

this is an indefinite integral because there is still an arbitrary constant in the function. Using the initial condition,

$$\begin{aligned} K(0) &= 2 \times (0)^{1.5} + C \\ &= C \end{aligned}$$

Since $K(0) = 0$, given in the condition, then $C = 0$. The time path for capital stock is thus

$$K(t) = 2t^{1.5}$$

The rate of capital formation is a function of time only. The accumulated capital also follows as a function of time only. This example must however not be taken to describe the general behaviour of capital. Its interpretation must be restricted to the above given scenario, that net investment is only a function of time.

PART III : CALCULUS TECHNIQUES

Chapter 7

7 STATIC OPTIMISATION: UNCONSTRAINED OPTIMISATION

7.1 Introduction

In a perfectly competitive market, each player is too small to influence the market prices. Take an example of a producer who hires L amount of labour and K worth of capital to produce output. The quantity produced is denoted by Q. The prices of output, labour and capital are given as P, w and r respectively. The sole objective of the producer is presumably profit maximisation. The producer optimises the profit given by

$$\pi = PQ - rK - wL$$

In this optimisation, the producer is not limited or restricted either by output or how much inputs can be utilised. We know that both capital and labour can be hired on assurance that they will be paid from the resulting sales. As such, the producer can hire as much capital and labour needed limit in order to maximise profits. He continues to engage more units of inputs as long as the Marginal Revenue Product (MRP) remains above the unit price of the particular input. No budget constraint exists since the producer can hire all the capital and labour that is needed on assurance that such will be paid from the resulting sales of output.

Because of the nature of the market, the producer is also not bound on how much to produce since the producer is too small to affect total supply. The same exists in the factor market where a single employer can employ as much as he is able without affecting the price or exhausting the supplies. Because there are no restrictions to the optimisation, the optimisation is referred to as *Unconstrained optimisation*.

7.2 What is optimisation?

As already explained in Chapter 2, optimisation is synonymous with the 'search for the best'. But searching can only take place where there are many alternatives. Optimisation involves choosing, from amongst many alternatives, a combination of factors that gives the best of what one is interested in. So in optimisation, there has to be choice.

The word *best* also needs to be understood in a dual way. If one is availed many bundles of goods from which to get only one, the interest from these goods is to generate benefit called *utility* in economic terminology. Utility or happiness is a desired characteristic or state. It is something one wants. In this context, best means getting the highest possible level of utility or satisfaction. Optimisation will therefore be synonymous to maximising. This entails searching for a bundle that gives the maximum utility.

If the commodity under consideration was pollution, the choice would be different because the objective would have changed. Pollution is something undesirable. No one wants to have it, only that nature imposes it on us. Given many bundles with different levels of pollution, *ceteris paribus,* the best option or bundle is one with the lowest level of pollution. Thus, while the terminologies *best* or *optimum* remain the same, the direction of optimisation changes depending on the nature of the variable. For a desirable variable, optimisation means maximisation. When the variable is undesirable such as pollution, payment, etc, optimisation implies minimisation.

Choice is not only restricted to bundles. In many cases, choice involves choosing the level of some given variable. It may be choosing how much to produce so that profits are maximised (optimised). It may also be choosing the quantity to produce so that the *per unit* cost is at its lowest. More precisely, optimisation refers to the choice of values of certain variables which could maximise or minimise the value of a function. For example, given a profit function $\pi = f(Q)$ where Q is the level of output, the objective is to choose that level of output which will maximise the profit. Alternatively, given an average cost function $C = f(x_1, x_2)$ where x_1 and x_2 are the outputs of two products, the objective would be to choose the values of outputs which minimise average cost. These problems are examples of unconstrained optimisation.

7.3 Meaning of the signs of first and second derivatives:

As noted in chapter 5, the first derivative measures the rate of change of the dependent variable as the independent variable changes by a unit. Consider a function in one variable, $y = f(x)$. The first derivative $\frac{dy}{dx}$ measures the rate at which y changes with respect to x. The function is changing positively or increasing when the first derivative is positive. It is changing negatively or decreasing when the first derivative is negative.

When the first derivative is zero, it means the function is not changing or, more commonly, constant. This rate of change may be constant as in the case of a linear function. Take for instance the function

$$y = ax + b$$

where a and b are constants. The derivative of this function is $\frac{dy}{dx} = a$. Since a was defined as a constant, the derivative is constant. It is the same for the entire domain of the function. Save for a constant function, linear functions have no finite maximum or minimum.

For a nonlinear function, the rate of change is not constant. It can only be specified for a specific point or value of the domain. Take an example of a quadratic function

$$y = ax^2 + bx + c$$

The derivative of the above (non linear) function is

$$\frac{dy}{dx} = 2ax + b$$

The derivative contains the variable x. This means it also varies. As the independent variable changes, so does the derivative which measures the slope. When $x < -\frac{b}{2a}$, the derivative is negative and the function is decreasing. When $x > -\frac{b}{2a}$, the derivative is positive and the function is increasing. Thus the function has two portions, one for which the function is decreasing and another for which it is increasing. These are separated by the point $x = -\frac{b}{2a}$.

The point $x = -\frac{b}{2a}$ is the demarcation. At this point, the derivative is neither positive nor negative. It is zero. It is a point at which the function changes direction, from decreasing to increasing and vice-versa. It is called a *turning point*. The turning point therefore is a point at which the function is changing direction or simply turning. At this point, the first derivative is zero, separating the negative and positive portions.

Thus when the first derivative is zero, the function is turning. Such a point is also the maximum or minimum of the function since the function is no longer going any further, either downwards or upwards. Both the maximum and minimum require that the first derivative is zero. The opposite will however not be definite. Getting a zero first derivative will not be specific on whether the point is maximum or minimum. For this reason, it is safer to simply call the point an optimum point since the word optimum encompasses both maximum and minimum. We will discover later however that, under rare circumstances, a zero first derivative may neither be minimum nor maximum.

Figure 7.1: A depiction of changing direction

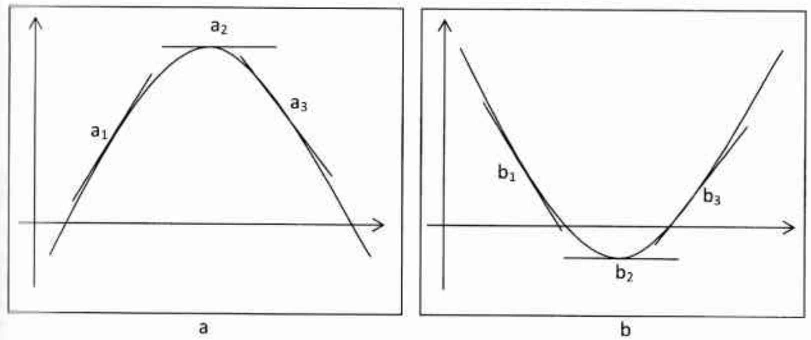


Figure 7.1 above shows two cases of a turning point. The first (a) is a maximum and the second (b) is a minimum. Points along the graph show points of different slopes. At point a_1, the slope is positive and the function is increasing; At point a_2, the slope is zero. The function is constant and on the graph, it is represented by a horizontal line. Point a_3 shows a negatively sloped part of the function. At this point, the function is decreasing. Similar comments can be made on part (b) of the figure.

The second derivative's sign sheds light on the *curvature* of the function. At the point $x = a$, if the second derivative, $\frac{d^2y}{dx^2}$ at a or $f''(a)$ is positive, the first derivative will be increasing and the function will be *concave upwards*. A function is concave downwards when its first derivative is decreasing so that the second derivative is negative, that is $f''(a) < 0$. If the function's acceleration is momentarily nil and the function is changing its curvature at that point (from concavity downward to concavity upward or vice versa), then there is a point of inflexion and $f''(a) = 0$ at that point.

Note however the following:

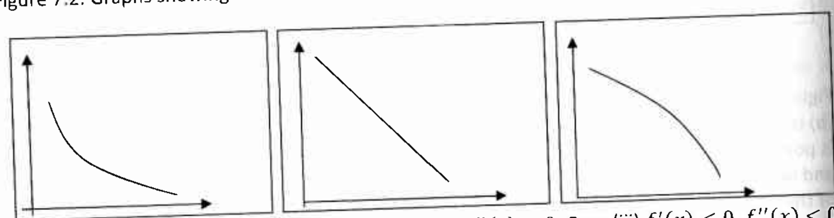
i. Concavity upward at $x = a$ implies $f''(a) > 0$ and $f''(a) > 0$ implies concavity upward;
ii. Concavity downward at $x = a$ implies $f''(a) < 0$ and $f''(a) < 0$ implies concavity downwards;
iii. Point of inflexion at $x = a$ implies $f''(a) = 0$ but $f''(a) = 0$ does not necessarily mean that $x = a$ is a point of inflexion unless we know that $f''(a)$ changes sign at $x = a$, i.e. $f'''(a) \neq 0$. in other words, zero value of the second derivative is a necessary but not a sufficient condition for a point of inflexion.

Since the first as well as the second derivative can take on positive, zero or negative values, we have a total of nine possible cases:

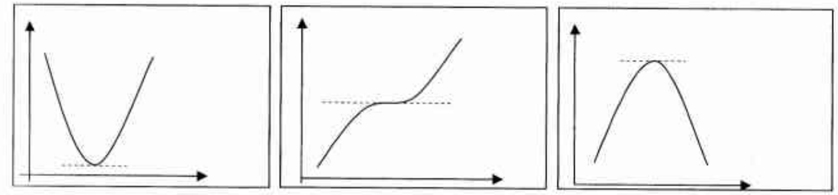
(i) $f'(x) < 0$; $f''(x) > 0$;
(ii) $f'(x) < 0$; $f''(x) = 0$; $f'''(x) \neq 0$;
(iii) $f'(x) < 0$; $f''(x) < 0$;
(iv) $f'(x) = 0$; $f''(x) > 0$;
(v) $f'(x) = 0$; $f''(x) = 0$; $f'''(x) \neq 0$
(vi) $f'(x) = 0$; $f''(x) < 0$;
(vii) $f'(x) > 0$; $f''(x) > 0$;
(viii) $f'(x) > 0$; $f''(x) = 0$; $f'''(x) \neq 0$
(ix) $f'(x) > 0$; $f''(x) < 0$;

The following graphs depict the above listed scenarios to help in understanding the interplay of the first and second derivatives in determining the shape of the graph.

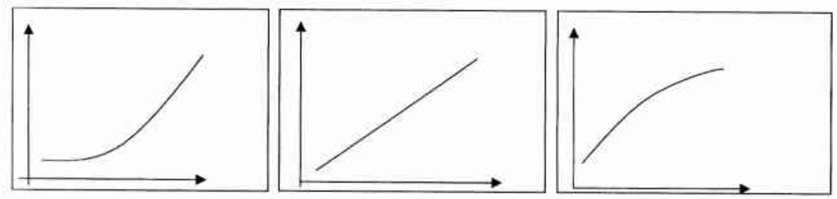
Figure 7.2. Graphs showing different combinations of first- and second-order derivatives



Case (i) $f'(x) < 0$, $f''(x) > 0$ Case (ii) $f'(x) < 0, f''(x) = 0$ Case (iii) $f'(x) < 0, f''(x) < 0$



Case (iv) $f'(x) = 0$, $f''(x) > 0$ Case (v) $f'(x) = 0, f''(x) = 0$ Case (vi) $f'(x) = 0, f''(x) < 0$



Case (vii) $f'(x) > 0$, $f''(x) > 0$ Case (iix) $f'(x) > 0, f''(x) = 0$Case (ix) $f'(x) > 0, f''(x) < 0$

It can be seen that in case (v), the first derivative was decreasing up to zero but instead of crossing into the negative, it suddenly starts to increase. Such a point, though having $f'(x) = 0$, is not a turning point since the function continues in the same direction. It is instead referred to as a point of inflexion. Cases (iv) and (vi) show minimum and maximum points respectively. Since, in optimisation, we are interested chiefly in such points we shall discuss them below.

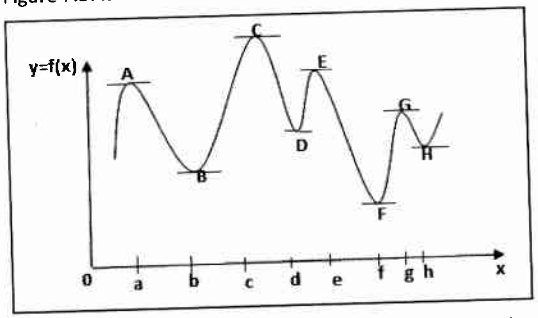
7.4 Unconstrained Optimisation in a Single Choice Variable:

If $y = f(x)$ is a twice differentiable function, that is, $f(x)$ and $f'(x)$ are both smooth and continuous at a particular point $x = a$, then the function has

(i) A minimum value at $x = a$ if
$f'(a) = 0$, and $f''(x) > 0$;
(ii) A maximum value at $x = a$ if
$f'(a) = 0$, and $f''(x) < 0$.
(iii) The condition in (i) and (ii) is a *sufficient* but *not* a *necessary* condition for a minimum or maximum. The *necessary* and *sufficient* condition for a minimum maximum) is that $f'(x) = 0$ and $f^{(n)}$ is positive (negative) where $f^{(n)}$ is the first non-zero higher-order derivative and n is an *even* integer.

The terms 'maximum' and 'minimum' in defining these points is not used in an absolute sense as they need not be unique for a function. A function may have more than one minimum or maximum in the sense of fulfilling conditions (i) and (ii) stated above. Consider the following graph:

Figure 7.3: Maximum and Minimum Points



The figure depicts four (4) points of maximum *A, C, E* and *G* and four points of minimum; *B, D, F* and *H*. Point *A* is not a maximum in the sense of it being the highest value of the function since there are several other values which are higher than *A*. it is a maximum only in a relative sense that the function has a highest value at $x = a$ than those at all neighbouring points $a + \Delta x$ where Δx is an arbitrarily small variation in x. In other words, $f(a) > f(a + \Delta x)$. Similarly, point *B* too is a minimum in a relative sense that $f(b) < f(b + \Delta x)$. Points such as *A* and *B* are therefore called points of *relative maximum* and *relative minimum* because they are maximum and minimum relative to the neighbouring points.

Point *C* is the highest of all maximum points. It should then be the highest of all values of the function. If this were true, point *C* would be called a *global maximum*. A global maximum is maximum globally, that is, for the entire domain. It is called global because it is higher than all other points on the function for the whole domain. Strictly speaking however, point *C* is not higher than all values of the function. The function has values higher than that of point *C* on the far right. As the x approaches infinite, the function also approaches infinite. It is unequivocal that infinite is greater than any finite value, particularly point C. Similarly, point *F* is the lowest of all minimums. It can be loosely call a global minimum.

There is no automatic way of locating the global maximum or global minimum from the first and second order conditions on the derivatives. It is necessary to compare all the relative maxima or relative minima to check which one/s is arc global.

Example 7.1

Find the extreme values of the function

$$y = 100 + 12x - x^3$$

The first step is to find the first derivative and locate a point where it is zero.

$$\frac{dy}{dx} = 12 - 3x^2$$

For an extreme value, we need $\frac{dy}{dx} = 0$.

$$12 - 3x^2 = 0$$

$$x = \sqrt{4}$$

$$x = \pm 2$$

The above solution shows that the function has extremes at two points. Therefore, obviously, one is a minimum and the other is a maximum. We therefore use the second order derivative to determine which one is maximum and minimum.

$$\frac{d^2y}{dx^2} = -6x$$

$$f''(2) = -12 < 0$$

Therefore, at $x = 2$, we have a maximum. The value of y is

$$f(2) = 100 + 12(2) - (2)^3$$
$$= 100 + 24 - 8 = 116$$

However, $f'(-2) = 12 > 0$, therefore,at $x = -2$, we have a minimum.The value of y is

$$f(-2) = 100 + 12(-2) - (-2)^3$$
$$= 100 - 24 + 8 = 84$$

Example 7.2

Find the extreme values of the function $y = f(x) = 8x^4$

The first step is always to take the first derivative and locate a point where it is zero

$$\frac{dy}{dx} = 32x^3$$

Clearly, the first derivative is zero at $x = 0$. Next, we get the second derivative to check for possible maximum or minimum. But the second derivative is also zero at this point. This leads to no decision as yet.

It must now be clear from section 7.3 that the *necessary* and *sufficient* condition for a minimum (maximum) is that $f'(x) = 0$ and $f^{(n)}$ is positive (negative) where $f^{(n)}$ is the first non-zero higher-order derivative and n is an *even* integer. Since the second order derivative is zero when evaluated at $x = 0$, we proceed to find a non-zero higher order derivative, therefore;

$$f'''(x) = 192x, \qquad f'''(0) = 0$$

$$f''''(x) = 192$$

The forth derivative turns out to be a non-zero constant. When evaluated at any point including the point of interest $x = 0$, it is positive. Thus the first non-zero value of the derivative at $x = 0$, occurs at $f^{(4)}$ which is an even order. Also, $f''''(0) = 192 > 0$. Hence the function has a minimum at $x = 0$. The minimum value of

$$\begin{aligned} y &= 8x^4 \\ &= 8(0)^4 = 0 \end{aligned}$$

7.5 Economic Application

Example 7.3

The total cost function for a commodity is given by $C = 4x - 6x^2 + 3x^3$. Find the value of the output x for which the average cost is lowest.

The average cost is a cost per unit of output. It is gotten by dividing the total cost by the level of output. Let $\bar{C}$ denote average cost. Then

$$\bar{C} = \frac{C}{x} = 4 - 6x + 3x^2$$

One must always be careful not to go for the optimisation of the cost function. Moreover, such a point seldom exists since the total cost function is normally a monotonically increasing function. As a custom now, the first order derivative is.

$$\frac{d\bar{C}}{dx} = -6 + 6x$$

The first order derivative is zero at $x = 1$. From economic theory, we know this point represents a minimum. However, this should never preclude the need for a test using the second order condition.

$$\frac{d^2\bar{C}}{dx^2} = 6$$

$$\left.\frac{d^2\bar{C}}{dx^2}\right|_{at\ x=1} = 6 > 0$$

Thus average cost is minimum at output $x = 1$. The minimum average cost is

$$\begin{aligned} \bar{C} &= 4 - 6(1) + 3(1)^2 \\ &= 4 - 6 + 3 = 1. \end{aligned}$$

In the above example, the marginal cost function would have been given by taking the first derivative of the cost function, that is,

$$MC = \frac{dC}{dx} = 4 - 12x + 9x^2$$

The relationship between the average cost and marginal cost curves (for U-shaped cost curves) is as follows; when average cost is minimum, marginal cost is equal to average cost; when marginal cost is below average cost, the average cost is falling; and when marginal cost is higher than average cost, average cost is rising.

Since the average cost function, $\bar{C}$, given above has a single minimum at $x = 1$, it is U-shaped. The minimum average cost is 1. At $x = 1$, the marginal cost

$$MC = 4 - 12(1) + 9(1)^2 = 13 - 12 = 1$$

This proves the equality of the average cost and marginal cost when the former is at its minimum. The other two prepositions about the relation between average and margin can be similarly verified. The reader can also check that the marginal cost reaches its minimum even earlier at output level $x = \frac{2}{3}$.

Example 7.4

Let a monopolists' demand function be given as $P = 15 - 2x$ and his cost function as $C = x^2 + 4x$. Let a tax of t kwacha per unit of out-put be imposed on the monopolist. We wish to find the maximum profit obtainable by the monopolist and the tax revenue obtainable by the government.

The total profit for the producer is the total revenue less the production cost. Since the producer must also pay some tax, the total tax payment or obligation is treated as a cost since it eats on the profits.

$$\begin{aligned} \pi &= R - C - tx \\ &= (15 - 2x)x - (x^2 + 4x) - tx \\ &= 11x - 3x^2 - tx \end{aligned}$$

To maximise profits, we need

$$\frac{d\pi}{dx} = 0$$

$$\frac{d\pi}{dx} = 11 - 6x - t$$

$$\therefore\ 11 - 6x - t = 0$$

$$x^* = \frac{11 - t}{6}$$

The second order derivative gives $\frac{d^2\pi}{dx^2} = -6 < 0$. This is negative for all values of x. Therefore, profit is maximised at $x^* = \frac{11-t}{6}$. To get the level of profit at the optimal level of output, substitute the optimal x^* into the profit function.

$$\begin{aligned} \pi_{max} &= (11 - t)x - 3x^2 \\ &= (11 - t)\left(\frac{11 - t}{6}\right) - 3\left(\frac{11 - t}{6}\right)^2 \\ &= \frac{(11 - t)^2}{6} - \frac{(11 - t)^2}{12} \\ &= \frac{(11 - t)^2}{12} \end{aligned}$$

The government's total tax revenue $= tx$, substituting x, we get the following;

$$\begin{aligned} T &= \frac{t(11-t)}{6} \\ &= \frac{11}{6}t - \frac{1}{6}t^2 \end{aligned}$$

The profit level in Example 7.4 above is the maximum for the producer. For as long as the tax rate, price and the production technology remain the same; profits cannot be increased any further. For the government however, the tax revenue is for a given tax rate. It does not represent the optimum tax level for the government. There still remains room for the government to increase tax revenue, either by increasing or reducing the rate. In either case, the tax level cannot be increased indefinitely. Though an increase in tax rate should increase the tax collection, at a certain point, the fall in output resulting from increased taxation will outweigh the increase in the rate.

But how can such a point be identified? This scenario is similar to the duopoly setup where each market player takes into account the reaction of the rival. In this case, government sets a tax rate that when the producer will maximise his profits, tax collection will also be maximised. The question is: Given the tax function, find the tax rate for which tax is optimised. We proceed as follows:

To maximise tax revenue, we set

$$\frac{dT}{dt} = \frac{d}{dt}\left(\frac{11}{6}t - \frac{1}{6}t^2\right) = 0$$

$$\frac{11}{6} - \frac{1}{3}t = 0$$

$$t = \frac{11}{2} = 5.5$$

The second order condition $\frac{d^2}{dt^2}(tx) = -\frac{1}{3} < 0$, for all t. This means the set tax rate gives the maximum tax level, given the producer maximises profits. Thus the maximum level of tax revenue given all the above conditions is.

$$\begin{aligned} T &= 5.5\left(\frac{11-5.5}{12}\right) \\ &= \frac{121}{48}. \end{aligned}$$

7.6 A note on points of inflexion

As already noted, a point of inflexion occurs when the concavity of the curve changes. While a point of maximum or minimum is necessarily a stationary point, a point of inflexion may or may not be stationary.

The geometric feature of an inflexional point is that the tangent at the point crosses the curve.

Example 7.5

Let

$$y = \frac{1}{6}x^3 - x^2 + \frac{3}{2}x + 10$$

$$\frac{dy}{dx} = \frac{1}{2}x^2 - 2x + \frac{3}{2}$$

$$\frac{d^2y}{dx^2} = x - 2$$

$$\frac{d^2y}{dx^2} = 0 \; at \; x = 2$$

$$\frac{d^3y}{dx^3} = 1 \neq 0$$

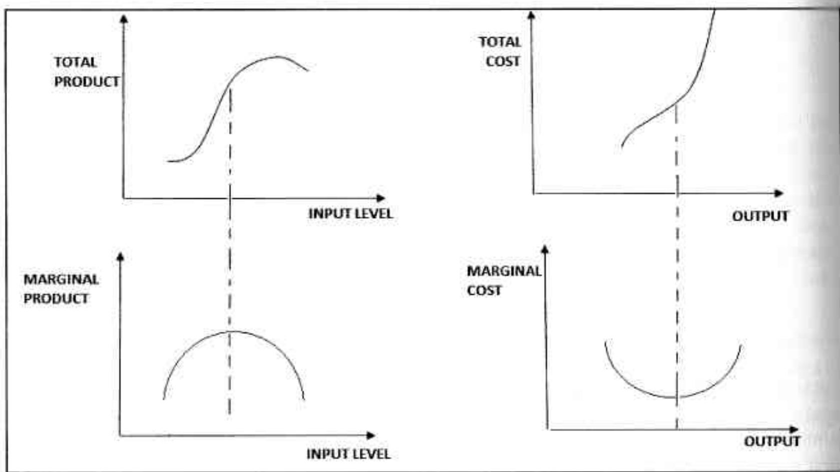
We have a point of inflexion at $x = 2$.

Total cost curves and total productivity curves often display points of inflexion. If the marginal cost curve is U-shaped, it means that over initial ranges of output it is falling, and then reaches a minimum and thereafter it starts rising. Correspondingly it means that the total cost curve first rises at a decreasing rate when the firm is enjoying economies of scale. Subsequently, it rises at an increasing rate as diseconomies of scale set in. At the output level, when the marginal cost is minimum, we have a point of inflexion on the total cost curve. The curve at that point ceases to be concave downward and becomes concave upward.

Owing to the laws of returns, the marginal product curve is generally inverted U-shaped. This means that the total product curve first increases at an increasing rate and thereafter increases at a decreasing rate. At the input level where the marginal product is at a maximum, we have a point of inflexion on the total product curve. The curve at that, point ceases to be concave upward and becomes concave downward.

The typical total product and total cost curves are graphed below.

Figure 7.4. Total vs marginal functions



7.7 Hessian and Jacobian determinants

Hessian and Jacobian determinants are also widely used in optimisation. The Hessian matrix was developed in the 19th century by the German mathematician Ludwig Otto Hesse and later named after him, while the Jacobian matrix is named after another 19th century German mathematician Carl Gustav Jacob Jacobi.

A Jacobian is a matrix of first order partial derivatives while a hessian is a matrix of second order partial derivatives. Jacobian determinants $|J|$ are used to test for functional dependence for both linear and non linear functions while a Hessian determinant is used to check for the second order conditions in optimisation.

Consider a column vector X $= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and let t be a scalar parameter on which each of the x_i's depends. The full derivative is then given by $\dfrac{dX}{dt} = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ \vdots \\ dx_n/dt \end{bmatrix}$ because it is a derivative with respect to only t. Now let $y = f(x) = f(x_1, x_2, x_3 \dots \dots x_n)$, then the gradient vector is given as follows:

$$\frac{\partial f(x)}{\partial x} = \left[\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \frac{\partial f(x)}{\partial x_3} \dots \dots \frac{\partial f(x)}{\partial x_n}\right]$$

Consider now the derivative of the gradient vector with respect to (w.r.t) an $n \times 1$ column vector X.

$$\frac{\partial^2 f(x)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f(x)}{\partial x}\right)$$

$$= \begin{bmatrix} \frac{\partial^2 f(x)}{\partial {x_1}^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial {x_2}^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial {x_1}^2} \end{bmatrix}$$

$$= \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

This is an $n \times n$ matrix called a *Hessian matrix*. Notice that the hessian matrix is made up of all the second-order partial derivatives, with the second-order direct partial derivatives on the principal diagonal and the second-order cross partials off the principal diagonal.

Next, consider a column vector of m functions.

$$g(x) = \begin{bmatrix} g_1(x) \\ g_1(x) \\ \vdots \\ \vdots \\ g_m(x) \end{bmatrix}$$

Where x is an n-dimensional column vector. Then the derivative of g w.r.t x involves differentiating each function in g w.r.t to each variable in x. There are m functions in g and n variables in x. This gives an $m \times n$ matrix with rows representing functions and columns representing variables, given by:

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix}$$

This $m \times n$ matrix is called a *Jacobian Matrix*. Notice that the elements of each row are the partial derivatives of one function g_i with respect to each of the independent variables $x_1, x_2, \dots \dots x_n$, and the elements of each column are the partial derivatives of each of the functions $g_1(x) g_1(x) \dots \dots g_m(x)$ with respect to one of the independent variables x_j.

It can be noted from the above definitions of the Hessian and the Jacobian that there are two points of distinction between them:

- A Hessian is a matrix of second-order partial derivatives while a Jacobian is a matrix of first-order partial derivatives.
- A Hessian is always square while a Jacobian need not always be square.

7.8 Unconstrained Optimisation in two choice variables

So far we have looked at unconstrained optimisation in one choice variable, denoted by x in our previous section. This is where a dependent variable such as utility or profit depends on entirely one variable. If a person consumes only one commodity, then utility will only depend on how much of that commodity is consumed. For a single-product firm, its profit will only depend on how much of that product it produces. All these were cases of single choice variable. Decision is only made on one variable.

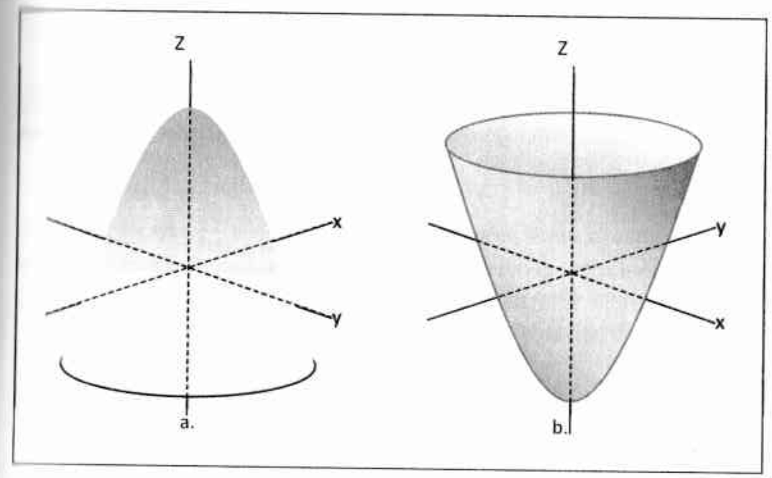
An extension of the single-choice variable is a two-choice variable. We recognise that in reality, utility depends on quantities of many goods that an individual consumes. But we are conservative enough not to rush to multiple-choice variable. It is important to tackle this in a gradual manner. Therefore, as a step towards reality, we first look at two-choice variable case. In this case, utility does not depend on a singly quantity because it is now naive to imagine this. On the other hand, it does not depend on many quantities, because it would be too much to analyse. Instead, two quantities determine utility.

The general form of such functions is given by

$$Z = f(x, y)$$

where Z is a function of x and y. In total, the equation has three variables and therefore drawn in three dimensions, one for each variable. This will form a shape with volume. The actual shape will be determined by the specific function. The two basic forms are shown in Figure 7.5 below.

Figure 7.5: Three dimensional graphs for optimisation



In the figure above, part a. shows something like a dome or a hill which has a summit. It has a maximum point. Part b. can be described as a trough, something one can use to fetch water. It has a minimum point. But the minimum or maximum points may not always exist. They may both be absent in various functions.

For a two variable function such as $Z = f(x, y)$, the necessary conditions for the function to have an extremum at the value $x = a$ and $y = b$ is that at this pair of values, we must have the following three conditions met:

(i) $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$, the first order partial derivatives must equal zero simultaneously. This means at this given point (a, b), the function is neither increasing nor decreasing.

(ii) The second–order direct partial derivatives, when evaluated at the critical point (a, b), must be both negative for a relative maximum and positive for a relative minimum. This ensures that from a relative plateau at (a, b), the function is concave and moving downward in relation to the principal axes in the case of a maximum and convex and moving upward in relation to the principal axes in the case of a minimum. This condition is summarised as $\frac{\partial^2 z}{\partial x^2} < 0$ and $\frac{\partial^2 z}{\partial y^2} < 0$; for maximum and $\frac{\partial^2 z}{\partial x^2} > 0$ and $\frac{\partial^2 z}{\partial y^2} > 0$; for a minimum. Note that these second-order direct partial derivatives are found along the principle diagonal of the Hessian matrix.

(iii) The product of the second-order direct partial derivatives evaluated at the critical point must exceed the product of the cross partial derivatives also evaluated at the critical point. This added condition is needed to preclude an inflection point or saddle point condition (iii) is summarised by; $\left(\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$.

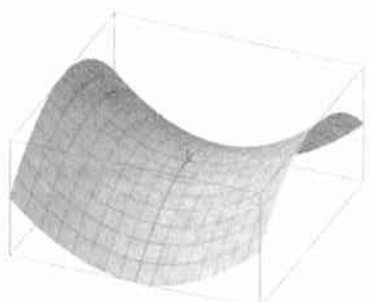
NOTE

1) According to Young's theorem, the cross partial derivatives are equal, that is: $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$
2) If we have condition (i), (ii) met and instead of condition (iii), we have

$$\left(\frac{\partial^2 Z}{\partial x^2}.\frac{\partial^2 Z}{\partial y^2}\right) - \left(\frac{\partial^2 Z}{\partial x \partial y}\right)^2 < 0$$

this means we have a *saddle point*. This occurs when $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ have different signs. This point is called saddle because the function is at a maximum when viewed from one axis but at a minimum when viewed from the other axis. See Figure 7.6 below.

Figure 7.6: A diagram showing a saddle point



When this diagram is viewed from the x-axis, it appears to be a maximum, but when viewed from the y-axis, it appears to be a minimum.

3) However, If we have condition (i), (ii) met and instead of condition (iii), we also have $\left(\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 < 0$, but $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ have the same sign, then we have an *inflexion point*.

4) If $\left(\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$, then further investigation is needed before arriving at any conclusion. The test is therefore inconclusive.

Example 7.6

Given $Z = x^2 + xy + y^2 - 6x + 5$, find its extremum point.

To locate an extreme value, we need

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0.$

So we have

$$\frac{\partial Z}{\partial x} = 2x + y - 6$$

$$\frac{\partial Z}{\partial y} = x + 2y$$

This results into two simultaneous equations.

$$2x + y - 6 = 0$$

$$x + 2y = 0$$

Since this chapter is not devoted to solving this kind of equation, we simply state the answer without solving. We are assuming the reader is already conversant with this. The two values are $x = 4$ and $y = -2$. This must not be confused with a two point scenario encountered in the preceding section. The two values here are two values of two respective variables representing a single point.

Given this point, the step that follow is to use the second order condition to determine the nature of the point, whether it is maximum or minimum or no of the two. Recall that the second order condition for two choice variables requires getting the second order direct partial derivatives and also get the cross partial derivatives, we get the following;

$$\frac{\partial^2 Z}{\partial x^2} = 2$$

$$\frac{\partial^2 Z}{\partial y^2} = 2$$

$$\frac{\partial^2 Z}{\partial x \partial y} = 1$$

At the point $x = 4$ and $y = -2$,

$$\frac{\partial^2 Z}{\partial x^2} > 0, \quad \frac{\partial^2 Z}{\partial y^2} > 0$$

and

$$\left(\frac{\partial^2 Z}{\partial x^2}.\frac{\partial^2 Z}{\partial y^2}\right) - \left(\frac{\partial^2 Z}{\partial x \partial y}\right)^2 = (2)(2) - (1)^2 = 3 > 0$$

The point $x = 4$ and $y = -2$ is a minimum for the function. The whole shape must look like an inverted dome. The lowest value can be found by evaluating the function at the turning point.

$$
\begin{aligned}
Z(4,-2) &= (4)^2 + (4)(-2) + (-2)^2 - 6(4) + 5 \\
&= 6 - 8 + 4 - 24 + 5 \\
&= -7
\end{aligned}
$$

Example 7.7

Two producers (duopolists) face the same market. Though they decide on output independently, each is well informed that the market price will depend on total quantity supplied on the market or more precisely, on the sum of their individual outputs. Let

x_1 = supply of the first duopolist

x_2 = supply of the second duopolist

The market price P, is given by $P = f(x_1 + x_2)$ with $\frac{\partial P}{\partial x_1} < 0$ and $\frac{\partial P}{\partial x_2} < 0$. The cost functions for the two duopolists are respectively given as

$$
\begin{aligned}
C_1 &= g(x_1) \\
C_2 &= h(x_2)
\end{aligned}
$$

The duopolists' profits π_1 and π_2 are

$$\pi_1 = Px_1 - C_1$$

$$\pi_2 = Px_2 - C_2$$

If each duopolist determines the output which maximises individual profit, the first order conditions are:

$$\frac{\partial \pi_1}{\partial x_1} = \frac{\partial \pi_2}{\partial x_2} = 0$$

For the first duopolist,

$$
\begin{aligned}
\frac{\partial \pi_1}{\partial x_1} &= \frac{\partial}{\partial x_1}(Px_1 - C_1) \\
&= \frac{\partial}{\partial x_1}(x_1 \cdot f(x_1 + x_2) - g(x_1)) \\
&= f(x_1 + x_2) + x_1.f'(x_1 + x_2) - g'(x_1)
\end{aligned}
$$

When firm one is deciding on output, it does not know what the other firm will produce. So it takes the output of the other as a constant. To get the optimum profit, the first derivative is equated to zero.

$$f(x_1 + x_2) + x_1.f'(x_1 + x_2) - g'(x_1) = 0$$

$$f(x_1 + x_2) + x_1.f'(x_1 + x_2) = g'(x_1)$$

$$f + x_1.f' = g'$$

In summary,

$$f + x_1.f' = \frac{dC_1}{dx_1} \qquad (1)$$

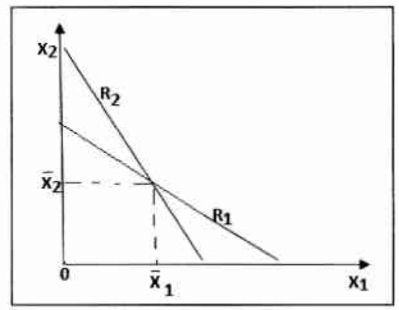
Corollary, for the second duopolist, we will get

$$f + x_2.f' = \frac{dC_2}{dx_2} \qquad (2)$$

The assumption of fixed x_2 for the first duopolist and fixed x_1 for the second duopolist means that each duopolist assumes that there will be no reaction from his rival to his own action. Equation (1) can be used to solve for x_1 in terms of x_2 and equation (2) can be used to solve for x_2 in terms of x_1. These solutions can be written as

$$x_1 = R_1(x_2), \qquad x_2 = R_2(x_1)$$

These two functions are called reaction functions and give the output of each duopolist as a function of the other. If the equations corresponding to the two reaction functions are solved simultaneously, they will give the equilibrium outputs of both the duopolists, which may be denoted as $\bar{x}_1$ and $\bar{x}_2$. This is shown in the figure below.



7.9 Hessian and Jacobian Determinants in Optimisation

Jacobian determinants $|J|$ are used to test for functional dependence for both linear and non linear functions. If $|J| = 0$, the equations are not all functionally independent. If $|J| \neq 0$, the equations are functionally independent. By functional dependence, we mean that a function can be obtained by the linear combination or any other non linear operation of another function. Consider the following example.

Example 7.8

Suppose :

$$g_1 = 5x_1 + 3x_2$$

$$g_2 = 25{x_1}^2 + 30x_1x_2 + 9{x_2}^2$$

First, we take the first- order partials,

$$\frac{\partial g_1}{\partial x_1} = 5$$

$$\frac{\partial g_1}{\partial x_2} = 3$$

$$\frac{\partial g_2}{\partial x_1} = 50x_1 + 30x_2$$

$$\frac{\partial g_2}{\partial x_2} = 30x_1 + 18x_2$$

Setting up the Jacobian, we have $|J| = \begin{vmatrix} 5 & 3 \\ 50x_1 + 30x_2 & 30x_1 + 18x_2 \end{vmatrix}$

Evaluating the determinant,

$$\begin{aligned} |J| &= 5(30x_1 + 18x_2) - 3(50x_1 + 30x_2) \\ &= 150x_1 + 90x_2 - 150x_1 - 90x_2 \\ &= (150 - 150)x_1 + (90 - 90)x_2 \\ &= 0 \end{aligned}$$

Since $|J| = 0$, there is functional dependence between the equations. Though the Jacobian does not specify the specific relationship, it is easy to tell that,

$$(5x_1 + 3x_2)^2 = 25x_1^2 + 30x_1x_2 + 9x_2^2$$

That is, $g_2 = (g_1)^2$

Recall the 2nd order conditions for optimisation, that is;

(i) $\frac{\partial^2 z}{\partial x^2} < 0$ and $\frac{\partial^2 z}{\partial y^2} < 0$; for a maximum and $\frac{\partial^2 z}{\partial x^2} > 0$ and $\frac{\partial^2 z}{\partial y^2} > 0$; for a minimum

(ii) Where $\left(\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$ must hold for both cases in (i).

Now consider a second order Hessian matrix given as follows:

$$H = \begin{bmatrix} \frac{\partial^2 Z}{\partial x^2} & \frac{\partial^2 Z}{\partial x \partial y} \\ \frac{\partial^2 Z}{\partial y \partial x} & \frac{\partial^2 Z}{\partial y^2} \end{bmatrix}$$

Example 7.9

Refer to Example 7.6 and let us use the Hessian to check for the second order conditions. Given to find the extremum of $Z = x^2 + xy + y^2 - 6x + 5$, The above stated conditions imply that for a maximum, $|H_1| = \partial^2 Z < 0$, $|H_1| = \partial^2 Z > 0$ for a minimum, where $|H_1|$ is the determinant of the first principal minor. This is consistent with (i) above. In our example, we have

$$\frac{\partial^2 Z}{\partial x^2} = 2 > 0$$

$$\frac{\partial^2 Z}{\partial y^2} = 2 > 0$$

$$\frac{\partial^2 Z}{\partial x \partial y} = 1$$

Setting up the Hessian, we have

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

As shown above, both values of the principle minor are clearly greater than zero. Note that they need not be equal in value.

On the other hand, (ii) above is the same as $|H_2| > 0$, where $|H_2|$ is the determinant of the second principal minor, in this case, the determinant of the Hessian above. Solving using our example,

$$|H_2| = (2)(2) - (1)(1) = 4 - 1 = 3 > 0$$

Therefore, just as in Example 7.6, using the Hessian, we arrive at the same conclusion that the function is a minimum at the optimal points.

In general, a Hessian matrix must be negative definite for a maximum. In a similar fashion, for a minimum, the Hessian must be positive definite. In the above example, this means $|H_1| = \partial^2 Z > 0$ and $|H_2| > 0$.This is also consistent with the second order conditions for optimisation in (i) and (ii).

7.10 Unconstrained Optimisation in n-choice variables

Let

$$y = f(x_1, x_2 \dots \dots \dots, x_n)$$
$$\partial y = f_1 dx_1 + f_2 dx_2 + \cdots f_n dx_n$$

For a critical value, we require all first order partial derivatives to be equal to zero. The second order differential $\partial^2 y$ will be the quadratic form. Its coefficients properly arranged will be the symmetric Hessian matrix.

$$|\mathrm{H}| = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

The principal minors for a maximum must all alternate in sign starting with negative. That is $-, +, -, +$........, where $f_{11} = |\mathrm{H}_1|$, $\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = |\mathrm{H}_2|$, and

$$|H_n| = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

For a minimum in a n-variable case, just like the two variable case, all the principal minors must be positive.

7.11 Economic Applications

Optimisation is useful is the study of economics in many ways. The principal aim of economics is to optimise. As such, tools of optimisation serve a vital role in economics or decision making. Though many areas of application can be cited, this section will deal with profit optimisation. See the examples below.

Example 7.10

A firm producing two goods; x and y has the profit function given by

$$\pi = 64x - 2x^2 + 4xy - 4y^2 + 32y - 14$$

What quantities of the two outputs should the firm produce in order to maximise the profit?

Partially differentiate the function with respect to the two choice variables.

$$\pi_x = 64 - 4x + 4y$$

$$\pi_y = 4x - 8y + 32$$

Equate the partial derivatives to zero and solve them simultaneously for x^* and y^*.

$$64 - 4x + 4y = 0$$

$$4x - 8y + 32 = 0$$

After solving the two equations above simultaneously, the optimal levels of output emerge to be,

$$x^* = 40$$
$$y^* = 24$$

Next, take the second order direct partial derivatives and check whether both are negative, as required for a relative maximum. The second order partial derivatives are

$$\pi_{xx} = -4$$
$$\pi_{yy} = -8$$

Both the direct partials are negative but this is just the first condition. In the second, we must check whether $\pi_{xx}\pi_{yy} > (\pi_{xy})^2$. The cross partial derivative is required in addition to the two partial already at hand.

$$\pi_{xy} = \pi_{yx} = 4$$

$$(\pi_{xy})^2 = 16$$

For the left hand side

$$\pi_{xx}\,\pi_{yy} = (-4)(-8)$$
$$= 32$$

Therefore, $\pi_{xx}\,\pi_{yy} > (\pi_{xy})^2$ and profits are indeed maximised at $x = 40$ and $y = 24$. At that point, profit $\pi = 1650$

Example 7.11

In monopolistic competition, producers must determine the price that will maximise their profit. Assume that a producer offers two different brands of a product, for which the demand functions are:

$$Q_1 = 14 - 0.25P_1$$

$$Q_2 = 24 - 0.5P_2$$

The producer has a joint cost function

$$TC = Q_1^2 + 5Q_1Q_2 + Q_2^3$$

The profit maximisation level of output, the price that should be charged for each brand, and the profits are determined as follows:

First, establish the profit function $\boldsymbol{\pi}$ in terms of Q_1 and Q_2. Total revenue will be the sum of revenues from the two brands. Since profit is total revenue less total costs, then

$$\pi = P_1Q_1 + P_2Q_2 - TC$$
$$= P_1Q_1 + P_2Q_2 - (Q_1^2 + 5Q_1Q_2 + Q_2^3)$$

The two demand functions given above can be converted to inverse demand functions so that they are substituted into the profit function.

$$P_1 = 56 - 4Q_1$$

$$P_2 = 48 - 2Q_2$$

The profit equation continues as

$$\pi = (56 - 4Q_1)Q_1 + (48 - 2Q_2)Q_2 - Q_1^2 - 5Q_1Q_2 - Q_2^3$$
$$= 56Q_1 - 5Q_1^2 + 48Q_2 - 3Q_2^3 - 5Q_1Q_2$$

The profit function above is the final equation. This is the function that must be maximised. The process of maximisation must be more familiar now. Equate the partial derivatives to zero and solve the resulting equations simultaneously.

$$\frac{\partial \pi}{\partial Q_1} = 56 - 10Q_1 - 5Q_2$$
$$\frac{\partial \pi}{\partial Q_2} = 48 - 6Q_2 - 5Q_1$$

Therefore

$$56 - 10Q_1 - 5Q_2 = 0$$

$$48 - 6Q_2 - 5Q_1 = 0$$

Which, when solved simultaneously, give $Q_1 = 2.75$ and $Q_2 = 5.7$. Take the second derivative to be sure profit is maximised:

$$\frac{\partial^2 \pi}{\partial Q_1^2} = -10$$
$$\frac{\partial^2 \pi}{\partial Q_1^2} = -6$$
$$\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = \frac{\partial^2 \pi}{\partial Q_2 \partial Q_1} = -5$$

With both second order partial derivatives negative and

$$\frac{\partial^2 \pi}{\partial Q_1^2} \cdot \frac{\partial^2 \pi}{\partial Q_1^2} > \left(\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2}\right)^2$$

the function is maximised at the critical values. Finally, substitute $Q_1 = 2.75$ and $Q_2 = 5.7$ into the inverse demand functions to find the profit-maximising prices.

$$P_1 = 56 - 4(2.75)$$
$$= 45$$
$$P_2 = 48 - 2(5.7)$$
$$= 36.6$$

Prices should be set at $K45.00$ for brand 1 and $K36.60$ for brand 2, leading to sales of 2.75 of brand 1 and 5.7 of brand 2. The maximum profit is:

$$\pi = P_1Q_1 + P_2Q_2 - (Q_1^2 + 5Q_1Q_2 + Q_2^2)$$
$$= 45(2.75) + 36.6(5.7) - (2.75)^2 + 5(2.75)(5.7) + (5.7)^2)$$
$$= 213.94$$

Chapter 8

8 STATIC OPTIMISATION: CONSTRAINED OPTIMISATION WITH EQUALITY CONSTRAINTS

8.1 Introduction

Constrained optimisation is a mathematical concept mostly essential to the subject of economics. Economics is essentially a framework for understanding the world in which individuals, firms and governments make their seemingly best decisions given the intrinsic limitations confronting their behaviour. The most common examples of constraints are limitations on time, money and other resources available to the use of man. For the most part, these constraints are likely to be binding, meaning that man tends to use up all the available resources in making his best decisions.

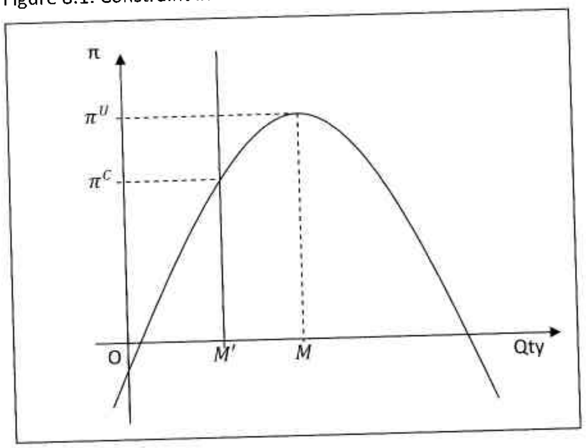
We often have problems of *constrained* optimisation, where a function has to be maximised or minimised subject to certain constraints or *side relations*. For instance, if we have a joint cost function $C = f(x, y)$, we may have to choose the values of x and y which will minimise the cost subject to the condition that $x + y = Z$, i.e the output of both goods combined must be equal to a certain value, this equal sign in the constraint gives it the name, equality constraint. Or, if we have a sales function $S = f(x, y)$ where x and y are the kwachas spent on two advertising media, we may want to choose the values of x and y which will maximise sales subject to the constraint of a given advertising budget Z which must be used fully, i.e $x + y = Z$. Constrained optimisation thus implies optimising the value of the objective function subject to certain side conditions. When these side conditions are absent, you have an unconstrained optimum discussed in the previous chapter.

Sometimes, the constraints may be in the form of inequalities. In the constrained cost minimisation problem above, the constraint may be $x + y \geq Z$, i.e, the quantities produced of the two goods must be at least equal to the value of Z. However, in this chapter, we focus on equality constraints. In the context of optimisation, the function to be optimised is called the *objective function*. The variables whose values have to be chosen are called the instruments and (if the optimisation is constrained) the set of values of the instruments which satisfy all the constraints is called the *opportunity set*.

Obviously, a constrained optimum can never give a better value for the objective function than the unconstrained optimum. It will generally give a less favourable value for the objective function and at best may leave the value of the objective function given by an unconstrained optimum unchanged. To illustrate the idea, consider the profit curve in the figure below. In the absence of constraints, the output level OM will be produced and the maximum profit will be π^U.

Suppose now a limit is imposed on resources which does not permit output beyond a certain level to be produced. This is a constraint since it constrains the use of resources or production. If the limit is set above the optimal level, the latter must remain unchanged. This is because it is within the permitted region. In this case, the limit will be ineffective. However, due to the nature of equality constraints, which imposes production at the limit, output will increase with the limit but this will cause profits to dwindle.

Figure 8.1. Constraint in one choice case



If however the limit is below the optimal level, then it precludes production at the optimal level. Output must adjust downwards so that it is in conformity with the new limit. For instance, if the limit is set at OM', then no output beyond this point is permissible. The constrained output level will now fall to the new level OM'.

Since the initial level, unconstrained optimum, was the best among all possible levels including the new level, deviating from the former cannot in any way make profits better. In fact, profits will decline in line with a reduction in output coming as a result of a constraint. In general, a constrained optimum is never better that an unconstrained optimum. In the case of maximisation, this can be shown as

$$\pi^C \leq \pi^U$$

As stated earlier, this chapter will mostly be devoted to the main techniques that deal with optimisation with equality constraints. These are constraints that hold exactly. This is equivalent to using up all the available resources or capacity. In reality however, this may not be the case. The case of *second best* is one such example. If the optimum is not permissible, the second best or constrained optimum may require significantly deviating from the first best or unconstrained optimum. Secondly, the indivisibility of commodities will make it impossible to consume certain amounts. In particular, it restricts quantities to integers. It rules out the consumption of fractions of commodities as may be dictated by the available resources. Dealing with such a situation involves the technique of *Integer programming* which is not covered in this book.

8.2 Optimisation of a function in two variables subject to a single constraint

Two methods have been developed for solving constrained optimisation problems. The first, and perhaps the simplest, is the substitution method. In this method, an equality constraint is used to substitute one variable for the other in the objective function. This reduces the problem to a single variable case. The second is the Lagrangean method. This method introduces a new variable called the Lagrangean multiplier and combines the objective function and constraint(s) into a single function. This method is more robust and can be used to deal with multiple constraint cases. The method is discussed in section 8.3.

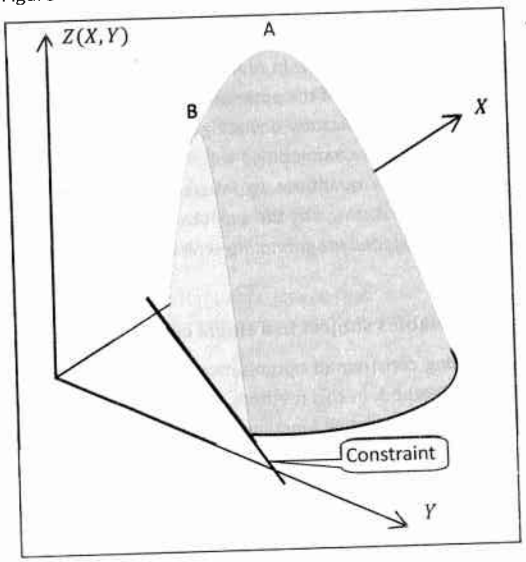
The standard constrained optimisation problem can be stated as below:

$$\underset{\{x_1,x_2\}}{optimise} f(x_1,x_2)$$

$$s.t.\ g(x_1,x_2) = m$$

We can see that we have three variables in the above problem, namely x_1, x_2 and $f(x_1,x_2)$. Our task is to choose the independent variables (x_1,x_2) so that the objective function $f(x_1,x_2)$ can be made as large as possible, as long as when we plug these values into the constraint $g(x_1,x_2)$, we obtain exactly m. m is a parameter which is taken as a constant in the optimisation problem. In essence, the constraint imposes restrictions on the domain of the function. Hence, the solution to a constrained optimisation problem is the optimum value that the function takes on, over the restricted part of the domain that is consistent with the constraint. The graph form is presented in Figure 8.2 below.

Figure 8.2: Constrained Optimisation



In the above figure, point A is the maximum, it is the optimal point. But because of the constraint, which exclude some values of the independent variables, this point is no longer available. Another point, which satisfies the constraint, must be found. This corresponds to point B in the figure.

The first thing to do when using the substitution method is to solve the constraint for one of the independent variables as a function of the other variable and the parameter m. In the particular case given above, express x_2 as a function of x_1 and m. That is,

$$x_2 = h(x_1, m)$$

The next is to substitute x_2 out of our given problem in order to obtain the following optimisation problem;

$$\underset{\{x_1\}}{optimize}\, f\big(x_1, h(x_1, m)\big)$$

The parameter m is a constant. As such, the above optimisation is now in one choice variable x_1. The substitution method makes it possible to translate a problem into a single variable problem. The method is quite simple as it essentially reduces the problem to an unconstrained

problem. The problem is evaluated with respect to the single choice variable x_1. Once the optimal value of x_1 is known, then the other variable can be evaluated using the equality constraint.

The first order condition of this problem is given by the following expression:

$$\frac{df}{dx_1} = f_{x_1} + f_{x_2}\frac{dh}{dx_1}$$

where f_{x_1} is a partial derivative with respect to the independent variable x_1, f_{x_2} is a partial derivative with respect to x_2 and $\frac{dh}{dx_1}$ is the effect of x_1 on h. Therefore, the solution to the problem is given by the value of x_1 that makes the first order condition exactly equal to 0. If the value of the parameter m is unknown, the maximising value of x_1 will be a function of m. To indicate that the maximising value of x is a number, we write it as $x_1{}^*$. Therefore, the maximising value of x_1 is written as $x_1{}^*(m)$ and it solves the following equation:

$$f_{x_1} + f_{x_2}\frac{dh}{dx_1} = 0$$

To obtain the maximising value of x_2, use $x_1{}^*$ in the implicit function obtained above:

$$x_2{}^*(m) = h(\ x_1{}^*(m), m)$$

The two equations above determine the optimum values of x_1 and x_2.

Example 8.1

Optimise the following function

$$Z = xy$$

Subject to the constraint

$$x + y = m$$

We want to find the maximum value for $f(x,y)$ over the domain of x, y that satisfy $x + y = m$, If we solve for $h(x,m)$, we have the following:

$$y = m - x = h(x,m)$$

Substituting for y, we obtain the following equivalent maximisation problem:

$$\underset{\{x\}}{max}\, x(m - x)$$

$$\Rightarrow \underset{\{x\}}{max}(xm - x^2)$$

Taking the first order conditions of this problem, we have the following:

$$m - 2x^* = 0$$
$$m = 2x^*$$
$$x^* = \frac{m}{2}$$

The Second order condition is -2, signifying that it is at a maximum.

To obtain the optimal value of y, we use the equation $y = m - x = h(x, m)$

$$y^* = m - x^*$$
$$y^* = \frac{m}{2}$$

Example 8.2

Optimise the following:

$$\min_{\{x,y\}} x^2 + 3y^2 + xy - 20$$
$$s.t. \quad x + y = 1$$

Hence,

$$f(x,y) = x^2 + 3y^2 + xy - 20$$
$$g(x,y) = x + y$$

In this case, we want to find the minimum value of $f(x,y)$over the domain of x, y that satisfies $x + y = 1$.

If we solve for $h(x,m)$, we have the following:

$$y = 1 - x = h(x, 1)$$

Substituting for y, we obtain the following equivalent maximisation problem:

$$\max_{\{x\}} (x^2 + 3(x^2 - 2x + 1) + x(1 - x) - 20)$$
$$\Rightarrow \max_{\{x\}} (3x^2 - 5x - 17)$$

This reduces to unconstrained optimisation in a single variable x rather than constrained optimisation in two variables x and y. Hence we can repeat the same procedure as in Example 8.1.

To obtain the first order condition, equate the partial derivatives to zero whereby we have

$$6x^* - 5 = 0$$
$$x^* = \frac{5}{6}$$

The second order derivative is 6, signifying that the extremum point is a minimum.

Solving for y, we have,

$$y^* = 1 - x^*$$
$$= 1 - \frac{5}{6}$$
$$y^* = \frac{1}{6}$$

The substitution method discussed above seems quite easy, but it gets rather complicated with the increase of variables and also their powers, therefore, replacing the constraint into the objective function by use of the substitution method becomes difficult and complicated. An alternative and easier way to go about such problems is the use of the Lagrange multiplier method discussed in the next section.

8.3 The Lagrangean Multiplier Method

The *Lagrangean multiplier method* introduces one more variable, λ, into the problem. This variable is known as the Lagrangean multiplier and has an important economic interpretation which will be explained later. The method of Lagrange relies on maximising an associated function, called the Lagrangean function. We form the Lagrangean function by adding λ times the constraint to the objective function and maximising over the independent variables which now include the Lagrange multiplier.

Suppose we wish to optimise (maximise or minimise) a function $y = f(x_1, x_2)$ subject to a constraint $g(x_1, x_2) = 0$. We form a *Lagrangean* expression:

$$L = f(x_1, x_2) - \lambda g(x_1, x_2), \qquad \lambda \neq 0$$

where λ is called the *Lagrangean Multiplier*.

It can be seen that L is a function of three variables x_1, x_2 and λ. We differentiate L partially with respect to x_1, x_2 and λ separately and set each of the partial derivatives to zero to solve for x_1, x_2 and λ. We will get

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} = 0$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} = 0$$

$$\frac{\partial L}{\partial \lambda} = g(x_1, x_2) = 0$$

Solving the above equations simultaneously, we get the critical values of x_1 and x_2. The critical values have to be examined to check whether the function has a maximum or a minimum at those values. This is done by applying the same second-order conditions for unconstrained optimisation in two variables.

Take an instance of a producer using two inputs, labour and capital. Both factors must ordinarily exhibit diminishing marginal productivity. For this kind of production, the production will have two choice variables, labour and capital. The optimising firm has to determine the optimal levels of capital and labour. The objective of the firm is to maximise output but this must be accomplished within the available resources. The amount of labour and capital used cannot be increased indefinitely because these have to be bought using finite resources.

Take the following illustration.

Optimise $y = f(x_1, x_2)$ subject to $g(x_1, x_2) = 0$

The Lagrangean expression is

$$L = f(x_1, x_2) - \lambda g(x_1, x_2), \qquad \lambda \neq 0$$

The first step in optimisation is to equate the first order partial derivatives to zero which is equivalent to equating the first order total derivative to zero. Thus,

$$\frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial x_2} = \frac{\partial L}{\partial \lambda} = 0$$

$$\Rightarrow \mathrm{d}L = 0$$

Once the optimum point(s)s has (have) been established using the first order condition, the step that follows is to determine whether such a point is maximum or minimum. In the unconstrained optimisation presented in Chapter 7, it was easy to check the second order condition and verify whether the point is maximum or minimum. The Hessian determinant was used to simplify this. The Hessian determinant is denoted by $|\mathrm{H}|$.

$$|\mathrm{H}| = \begin{vmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{vmatrix}$$

However, since there is a constraint, we have now to form a *bordered Hessian* determinant by bordering the Hessian with the partial derivatives of the constraint function as follows:

$$|\overline{\mathrm{H}}| = \begin{vmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{vmatrix}$$

The second order conditions then is:

For a maximum $|\overline{\mathrm{H}}| > 0$

For a minimum $|\overline{\mathrm{H}}| < 0$

Example 8.3

Find the optimal values of

$$z = xy \text{ subject to } x + y = 6$$

$$L = xy - \lambda(x + y - 6)$$

$$\mathrm{d}L = 0 \begin{Bmatrix} \frac{\partial L}{\partial x} = y - \lambda = o \\ \frac{\partial L}{\partial y} = x - \lambda = 0 \\ \frac{\partial L}{\partial \lambda} = x + y = 0 \end{Bmatrix}$$

Solving, $x = 3, y = 3, z = 9$

Second Order Conditions

$$\frac{\partial^2 L}{\partial x^2} = 0, \quad \frac{\partial^2 L}{\partial x \partial y} = 1, \quad \frac{\partial^2 L}{\partial y \partial x} = 1, \quad \frac{\partial^2 L}{\partial y^2} = 0, \quad \frac{\partial g}{\partial x} = 1, \quad \frac{\partial g}{\partial x} = 1$$

$$|\overline{\mathrm{H}}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2 > 0$$

The function has a maximum

Example 8.4

Evaluate the critical points for the following problem:

optimise $3x_1^2 - x_1x_2 + 4x_2^2$

subject to $2x_1 + x_2 = 21$

We form the Lagrangean function

$L = 3x_1^2 - x_1x_2 + 4x_2^2 - \lambda(2x_1 + x_2 - 21)$,

Next, we take partial derivatives with respect to (w.r.t.) x_1, x_2 and λ

$$\frac{\partial L}{\partial x_1} = 6x_1 - x_2 - 2\lambda = 0 \ (1)$$

$$\frac{\partial L}{\partial x_2} = -x_1 + 8x_2 - \lambda = 0 \ (2)$$

$$\frac{\partial L}{\partial \lambda} = -2x_1 - x_2 + 21 = 0 \ (3)$$

Next, solve equations (1) and (2) simultaneously, you could do it easily by the elimination method, that is, eliminate one variable. We choose to eliminate the variable λ so that we can remain with two variables, x_2 and x_1 which are also found in equation (3). To eliminate λ, multiply equation (2) by two, this does not change the equation in any way. Next, subtract (2) from (1). This process eliminates λ to give the following equation:

$$8x_1 - 17x_2 = 0 \ (4)$$

Next solve (3) and (4) simultaneously

$$2x_1 + x_2 = 21$$
$$8x_1 - 17x_2 = 0$$

we obtain

$$x_1 = 8.5$$

$$x_2 = 4$$

Thus, the critical point is (8.5, 4)

To check whether the critical point is a maximum or minimum, we use the bordered Hessian determinant introduced earlier. It is given by

$$|\bar{H}| = \begin{vmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{vmatrix}$$

From the objective function, the second order partial derivatives are

$$\frac{\partial^2 L}{\partial x_1^2} = 6, \quad \frac{\partial^2 L}{\partial x_2^2} = 8, \quad \frac{\partial^2 L}{\partial x_1 \partial x_2} = -1$$

And the constraint will give the following first order partial derivatives needed for the border:

$$\frac{\partial g}{\partial x_1} = 2, \quad \frac{\partial g}{\partial x_2} = 1$$

The border Hessian matrix $|\bar{H}|$ will be:

$$|\bar{H}| = \begin{vmatrix} 0 & 2 & 1 \\ 2 & 6 & -1 \\ 1 & -1 & 8 \end{vmatrix}$$
$$= -42$$

The bordered Hessian Determinant ($|\bar{H}| = -42$) is negative. This implies that the second order total differential is positive. Therefore, the function has a minimum at the critical point (8.5, 4).

Example 8.5

Optimise $Z = x + y$ subject to $x^2 + y^2 = 1$

Setting up the Lagrangean, we get;

$$L = x + y - \lambda(x^2 + y^2 - 1)$$

Following the steps in the previous example, we leave it up to the reader to solve the steps,

$$\frac{\partial L}{\partial x} = 1 - 2\lambda x = 0$$
$$\frac{\partial L}{\partial y} = 1 - 2\lambda y = 0$$

We get $x = y$

$$\therefore 2x^2 = 1$$

$$\text{or} \quad x = \pm\sqrt{\frac{1}{2}}$$

Critical points are $\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$ and $\left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$. Economics variables generally do not admit negative values. As such, we only take the point falling in the nonnegative quadrant. The second order partial derivatives are as follows:

$$\frac{\partial^2 L}{\partial x^2} = -2\lambda, \qquad \frac{\partial^2 L}{\partial y^2} = -2\lambda, \qquad \frac{\partial^2 L}{\partial x \partial y} = 0$$

And the first order partial derivatives from the constraints given by:

$$\frac{\partial g}{\partial x_1} = 2x = \sqrt{2}, \qquad \frac{\partial g}{\partial x_2} = 2y = \sqrt{2}$$

The resultant bordered Hessian at the critical point is

$$|\bar{H}| = \begin{vmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -2\lambda & 0 \\ \sqrt{2} & 0 & -2\lambda \end{vmatrix} = 8\lambda$$

Given the positive restriction on the lambda λ, we know even without getting the actual value that the product is positive and that is all that matters. The function is maximum at the critical point.

Note that the Lagrangean multiplier method can be generalised to the case of optimising a function in n variables subject to m constraints, where $m \leq n$.This is the major weakness of the substitution method which cannot easily solve such problems.

8.4 Economic Interpretation of the Lagrangean multiplier

Another reason for using the Lagrangean method is that is provides additional insights into the nature of economic problem. Also, it enables us to carry out sensitivity analysis; that is, it shows how the solution varies when we change the parameters of the problem. One can work out the comparative statics properties.

The insights that the Lagrangean multiplier λ provides is in terms of the interpretation of the Lagrangean multiplier λ. In general, it represents the shadow values of the constraints. For example, in consumer theory, it is called the marginal utility of income. It shows the amount by which the maximum value of the objective function changes when the constraint changes by one unit:

$$\frac{df^*}{dm} = \lambda^*$$

That is, if m were to increase by one unit, how much higher would f^* go? In utility maximisation, m represents the available resources and f^* is the highest possible level of satisfaction. This question is therefore equivalent to knowing the effect of additional income on one's optimal utility. If an additional unit of resource (to spend on buying consumables) is made available, what would be the resulting change in optimal utility?

The answer to all these questions lies in knowing the value of the multiplier λ^* in the optimal solution. More formally, m is a parameter fixed over the optimisation. All the optimal values of maximisers x^*, y^* and λ^* are dependent on m. They can be written as $x^*(m), y^*(m)$ and $\lambda^*(m)$ to emphasise their dependence on the parameter m. Therefore, the maximum value of the objective function is obtained by plugging the maximisers into it:

$$f^*(m) = f(x^*(m), y^*(m))$$

Taking the derivative of the objective function with respect to m, we obtain the following:

$$\frac{df^*}{dm} = f_x^* \frac{dx^*}{dm} + f_y^* \frac{dy^*}{dm}$$

Recall that the first order conditions of the Lagrangean function are

$$f_x^* - \lambda^* g_x^* = 0$$

$$f_y^* - \lambda^* g_y^* = 0$$

Substituting into the former equation, the derivative with respect to the constraint, turns to:

$$\frac{df^*}{dm} = \lambda^* \left(g_x^* \frac{dx^*}{dm} + g_y^* \frac{dy^*}{dm}\right)$$

Since the constraint must hold with equality, that is

$$g(x^*(m), y^*(m)) = m.$$

We can take the derivative of the constraint with respect to m and obtain the following:

$$g_x^* \frac{dx^*}{dm} + g_y^* \frac{dy^*}{dm} = 1$$

In the equation $\frac{df^*}{dm}$, the term in brackets must sum to a unit. This is proved by the preceding equation which differentiates the constraint. This leads to a proof that:

$$\frac{df^*}{dm} = \lambda^*$$

Thus λ shows how the objective function changes with the constraint parameters. Given a utility maximising individual, the multiplier shows how much additional utility the consumer will gain if the income was increased by a unit. Though this multiplier comes out as a constant, it actually varies. If all other factors are held constant and only income is allowed to vary, the marginal utility of income, measured by the multiplier λ, will vary. Usually it will increase at first but start to decline so that the consumer reaches a point of satiation. At the point of satiation, λ is zero since any additional income does not result in increased satisfaction or enjoyment.

In production, the Lagrange multiplier λ shows how output changes in response to changes in the amount of resources spent on production. Since the input ratio remains unchanged, increasing the available resources will scale up all factors of production by the same factor. Labour, land and capital will all increase by the same proportion. This is referred to as scaling up/down of production. The resulting increase in output is called *return to scale*. In this context, the Lagrange multiplier measures the return to scale. When it is equal to one, then there is constant return to scale. When λ is greater than one, the production function exhibits increasing return to scale and decreasing returns when it is less than one.

We shall have more discussion on λ in later chapters

8.5 Optimisation of a function in n-variables subject to a single constraint

With n-choice variables and a single constraint, the process of finding the optimal solution remains the same.

$$y = f(x_1, x_2, \dots, x_n)$$
$$s.t.\ g(x_1, x_2, \dots, x_n) = 0$$

The Lagrange equation is formed as

$$L = f(x_1, x_2, \dots, x_n) - \lambda g(x_1, x_2, \dots, x_n)$$

First order conditions

$$\mathrm{dL} = 0$$
$$\Rightarrow \frac{\partial L}{\partial x_i} = 0,$$
$$\Rightarrow \frac{\partial L}{\partial \lambda} = 0$$

Second order conditions

$$|\bar{H}| = \begin{vmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} & \cdots & \frac{\partial g}{\partial x_n} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \frac{\partial g}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} & \cdots & \frac{\partial^2 L}{\partial x_3 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial g}{\partial x_N} & \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \frac{\partial^2 L}{\partial x_n \partial x_3} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{vmatrix}$$

For a maximum, we require the last $n-1$ principal minors to alternate in sign starting with a positive. That is, of the last $n-1$ principal minors, the first must be positive, followed by a negative and so on.

For a minimum, we require the last $n-1$ principal minors to be all negative.

The last $n-1$ principal minors are $|\bar{H}_2|$, $|\bar{H}_3|, \dots, |\bar{H}_n| = |\bar{H}|$. Thus for instance,

$$|\bar{H}_2| = \begin{vmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{vmatrix}$$

$$|\bar{H}_3| = \begin{vmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial g}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{vmatrix}$$

And so on until $|\bar{H}_n| = |\bar{H}|$

8.6 Optimisation of a function in n variables subject to m constraints ($m < n$)

The most general case is of optimisation of a function in n variables subject to m constraints with $m < n$. The formulation of this general case is as follows:

Optimise $y = f(x_1, x_2, \cdots, x_n)$

Subject to

$$\begin{aligned} g_1(x_1, x_2, \cdots, x_n) &= 0 \\ g_2(x_1, x_2, \cdots, x_n) &= 0 \\ &\vdots \\ g_m(x_1, x_2, \cdots, x_n) &= 0 \end{aligned}$$

The solution procedure is as follows:

Form the Lagrangean function

$$L = f(x_1, x_2, \cdots, x_n) - \sum_{i=1}^{m} \lambda_i g_i(x_1, x_2, \cdots, x_n)$$

Note that this is a problem in n variables and m lambdas (λ). There are thus $m + n$ variables to be solved for.

The first order conditions are:

$$\mathrm{dL} = 0$$

$$\frac{\partial L}{\partial x_i} = 0, \qquad i = 1, \dots, n$$

$$\frac{\partial L}{\partial \lambda_j} = 0, \qquad j = 1, \dots, m$$

For the second order conditions, we consider the bordered Hessian determinant corresponding to the matrix:

$$\bar{\mathrm{H}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ 0 & 0 & \cdots & 0 & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & & & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ \frac{\partial g_1}{\partial x_n} & \frac{\partial g_2}{\partial x_n} & \cdots & \frac{\partial g_m}{\partial x_n} & \frac{\partial^2 L}{\partial x_1 \partial x_n} & \frac{\partial^2 L}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix}$$

Note that the above matrix is obtained by bordering the Hessian matrix by the Jacobian matrix of the constraint functions.

For a maximum, the last n-m principal minors of $|\bar{\mathrm{H}}|$ must alternate in sign, the sign of $|H_{m+1}|$ being $(-1)^{m+1}$. For the minimum, the sufficient condition requires that the bordered principal minors all take the same sign, namely, that of $(-1)^m$. Note that it makes an important difference whether we have an odd or even number of constraints. This is so because raising (-1) to an even power will yield the opposite sign as when raised to an odd power.

8.7 Economic Applications

8.7.1 Utility Maximisation and Consumer Demand

Suppose the utility function is given as $U(x, y)$. The available resources to buy the two consumables is M so that the resulting budget constraint is $P_x x + P_y y = M$. What is the optimum level of utility or satisfaction?

The standard consumer problem is that he must maximise utility such that he spends all his income M on purchasing two goods $x, y,$ and the prices of both goods are market determined and hence exogenous. It is also assumed that the marginal utility functions are continuous and positive, that is, $U_x, U_y > 0$.

We form a Lagrangean function and get the first order condition, thus;

$$L = U(x, y) + \lambda(M - P_x x - P_y y)$$

The first order conditions are:

$$L_x = U_x - \lambda P_x = 0$$
$$L_y = U_y - \lambda P_y = 0$$
$$L_\lambda = M - P_x - P_y = 0$$

Solving for λ, we obtain

$$\frac{U_x}{P_x} = \frac{U_y}{P_y} = \lambda$$

In this case, we can interpret the Lagrangean multiplier as the marginal utility of income when utility is maximised, thus:

$$\frac{dU^*}{dM} = \lambda^*$$

To verify whether the utility function is maximised, we check the second order conditions by forming the bordered hessian. The bordered Hessian is set here below.

$$|\overline{H}| = \begin{vmatrix} 0 & P_x & P_y \\ P_x & U_{xx} & U_{xy} \\ P_y & U_{yx} & U_{yy} \end{vmatrix}$$

The largest principal minor is 2×2. Therefore, $|\overline{H}| = |\overline{H}_2|$. We only have to solve for $|\overline{H}_2|$ and arrive at a conclusion.

$$|\overline{H}_2| = 2P_yP_xU_{xy} - P_x^2U_{yy} - P_y^2U_{xx}$$

It is not clear yet whether $|\overline{H}_2| \gtreqless 0$. However, assuming the law of diminishing marginal utility, one can infer that U_{yy}, $U_{xx} < 0$. All the prices are positive numbers but the sign of U_{xy} remains unclear. If positive, then $|\overline{H}_2|$ is unambiguously positive. If negative, the sign of $|\overline{H}_2|$ will depend on the sign of which term outweighs the other, the negative $2P_yP_xU_{xy}$ or the positive $\left(-P_x^2U_{yy} - P_y^2U_{xx}\right)$.

Chapter 9

9 STATIC OPTIMISATION: CONSTRAINED OPTIMISATION WITH INEQUALITY CONSTRAINTS

9.1 Inequality Constraints

In Chapter 8, we considered optimisation with equality constraints. That is to say. The constraints were of the (=) type. In this chapter, we consider optimisation with inequality constraints. Inequalities can be *strong* or *weak*. Strong inequalities are of the (>) or (<) type. Weak inequalities are of the (≥) or (≤) type. In other words, a weak inequality is one that permits the equality case.

In economics, we mostly deal with optimisation problems that involve weak inequalities. In this chapter, therefore, we look at solutions to problems of this type. Also, the objective function and the constraints are assumed to be concave and convex respectively and hence the problems are known as *concave-programming* problems. In Chapter 13, we discuss linear programming which is a special case of concave programming.

9.2 Binding and Non-binding (Slack) Constraints

In chapter 2, we briefly explained the concept of a constraint as binding or slack. Here, we amplify on this concept a little more. Formally, we make the following propositions:

I. If an inequality constraint holds with *equality* at the optimal point, the constraint is *binding*;
II. If an inequality constraint holds as a *strict* inequality at the optimal point (that is, does not hold with equality), the constraint is *non-binding*.
III. If a constraint is non-binding, the solution for the optimisation problem would be the same as in the absence of that constraint. To put it in another way, if a binding constraint is changed, the optimal solution will also change. But if a non-binding constraint is changed, the optimal solution is unaffected.

Take a simple illustration. Suppose a manufacturing firm produces steel products that use two inputs, labour and steel. There is a limited supply of labour and steel available during a given production period. The firm's objective is to maximise profits. Suppose now that when the optimal (profit maximising) outputs are produced, all the labour is used up, but there is some unused quantity of steel, say two (2) tonnes. This unused quantity of the resource is called *slack value*.

In the above example, labour is a binding constraint. If an additional unit of labour was made available, the optimum solution would change and the profit would increase. On the other hand, steel is a non-binding constraint. The availability of an additional unit of steel will not affect the optimal solution or raise profits. In economic terminology, labour has a positive *shadow cost*, where as steel has zero shadow cost. In general, binding constraints have positive shadow cost and non-binding constraints have zero shadow cost.

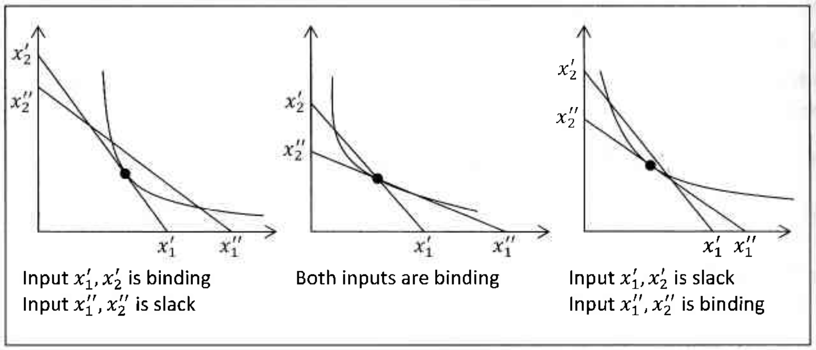
Let us use diagrams to illustrate the possibilities for a firm that wishes to produce optimal output subject to two input constraints. The problem is:

$$Maximise\ O = f(x_1, x_2)$$

$$\begin{aligned} subject\ to\ x_1 &\leq a \\ x_2 &\leq b \\ x_1, x_2 &\geq 0 \end{aligned}$$

The nature of the constraints is shown in the diagrams below.

Figure 9.1. Binding and Slack constraints



In the solution method that we shall explain in the next section, we assume we are dealing with concave-programming problems; that is, we assume the objective function is concave and the constraint functions are convex. The reason for this is that the necessary conditions for an optimal solution also become sufficient conditions.

Most problems in economics involving constrained optimisation with inequality constraints are maximisation problems, and hence we shall consider maximisation problems. However, it is not as though the solution method for a minimisation is different. Maximising an objective function

is the same as minimising the negative of the objective function. That is,

$$Max\ f(x)\ subject\ to\ g(x) \leq a$$

is the same as

$$Min\ -f(x)\ subject\ to\ g(x) \leq a$$

9.3 Karush-Kuhn-Tucker Conditions

Consider the following problem:

$$\begin{aligned} &Max\ f(x_1, x_2) \\ &\text{subject to } g(x_1, x_2) \leq b \\ &x_1, x_2 \geq 0 \end{aligned}$$

The first-order necessary conditions for solving the above problems are known as the *Karush-Kuhn-Tucker Conditions*, more frequently referred to as the Kuhn-Tucker conditions. This is because they were initially named after American and Canadian mathematicians Harold W. Kuhn and Albert W. Tucker respectively, who published the conditions in 1951. It was later discovered that another American mathematician William Karush had stated the conditions in his master's thesis in 1939. Here we shall refer to them as KKT conditions. The KKT conditions in fact constitute one of the most important results in non-linear programming.

For the above stated problem, the KKT conditions are given after forming the Lagrangean function:

$$L = f(x_1, x_2) - \lambda[g(x_1, x_2) - b]$$

The KKT conditions are :

1. $\frac{\partial L}{\partial x_1} \leq 0,\ \frac{\partial L}{\partial x_2} \leq 0,\ \frac{\partial L}{\partial \lambda} \geq 0$
2. $x_1 \geq 0,\ x_2 \geq 0,\ \lambda \geq 0$
3. $x_1 \frac{\partial L}{\partial x_1} = x_2 \frac{\partial L}{\partial x_2} = \lambda \frac{\partial L}{\partial \lambda} = 0$

Conditions 3 are called *complementary slackness conditions*. What do they imply? Now $\frac{\partial L}{\partial \lambda} = g(x_1, x_2) - b$. If this constraint is binding, then its value is zero in which case $\lambda \geq 0$. But if the constraint is not binding, then $\lambda = 0$. The parameter λ is thus the shadow cost we spoke about. A non-binding constraint entails zero shadow cost, since there is some slack. A binding constraint implies no slack and hence can have positive shadow cost.

Again, $\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} = 0$ where f and g and objective and constraint functions respectively. If we assume $\lambda = 0$, the above first order condition reduces to $\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} = 0$. This implies that the objective function is optimised freely, without a constraint. A similar line of reason holds for x_2 as well.

The overall upshot of the KKT conditions is the following:

I. An inequality constraint $g_i(x) \leq b_i$ is binding at some feasible point x_0 if it holds with equality $[g_i(x) = b_i]$ and not binding if it holds with strict inequality $[g_i(x) < b_i]$.
II. Only binding constraints matter since only they impact on the optimal value.
III. If it is known from the beginning which of the constraints are binding, it is permissible to ignore the non-binding constraints from the problems. The KKT problem will then reduce to a Lagrangean multiplier problem with only equality constraints.
IV. One important distinction between the KKT conditions and the Lagrangean condition for optimisation with only equality constraints is that $\lambda \geq 0$ for the KKT conditions while $\lambda \neq 0$ for the Lagrangean conditions.

The distinction between binding constraints and non-binding constraints is of relevance only in cases of optimisation with inequality constraints. In optimisation problems with equality constraints, all constraints are binding.

Example 9.1

Consider the following problem

$$\max Z = 4x_1 + 3x_2$$
$$\textit{subject to } 2x_1 + x_2 \leq 10$$
$$x_1, x_2 \geq 0$$

The Lagrangean for the problem is $L = 4x_1 + 3x_2 - \lambda(2x_1 + x_2 - 10)$. The KKT conditions for the maximum are:

$$\frac{\partial L}{\partial x_1} = 4 - 2\lambda \leq 0, \qquad x_1 \geq 0, \qquad x_1(4 - 2\lambda) = 0 \tag{1}$$

$$\frac{\partial L}{\partial x_2} = 3 - \lambda \leq 0, \qquad x_2 \geq 0, \qquad x_2(3 - \lambda) = 0 \tag{2}$$

$$\frac{\partial L}{\partial \lambda} = 2x_1 + x_2 - 10 \geq 0, \qquad \lambda \geq 0, \qquad \lambda(2x_1 + x_2 - 10) = 0 \tag{3}$$

Now $x_1(4 - 2\lambda) = 0$ implies that either of the two factors is zero. That is $x_1 = 0$ or $\lambda = 2$. We then test this value of λ from the second equation, that is $3 - \lambda \leq 0$. Clearly, this is false. Hence, we must have $x_1 = 0$. Then $x_2(3 - \lambda) = 0$ implies that either $x_2 = 0$ or $3 = \lambda$. We already concluded from conditions (1) that $x_1 = 0$ and to assume that $x_2 = 0$ would imply that $10\lambda = 0$. But this is not consistent with $\lambda \geq 2$ from equations (1). Hence, $x_2 > 0$ and $\lambda = 3$.

Using equation (3), we get

$$3(x_2 - 10) = 0$$
$$\Rightarrow x_2 = 10$$

Thus the solution is $x_1 = 0,\ x_2 = 10,\ \lambda = 3$

The value of the objective function is $Z = 4(0) + 3(10) = 30$

9.4 Sufficient Conditions

The KKT conditions constitute only the necessary conditions. They will also be sufficient conditions if:

I. The objective function is differentiable and concave in the non-negative orthant (quadrant for 2-dimentional), that is, the region where each $x_i \geq 0$;
II. Each constraint function is differentiable and convex in the non-negative orthant;
III. A point x_0 satisfies the KKT conditions.

In Example 9.1 above, both the objective function and the constraint are linear. Since all linear functions are both concave and convex, we can assume that conditions (I) and (II) are fulfilled. In addition, the point $(0, 10)$ satisfies the KKT conditions. Hence we can be sure this is a point of maximum.

9.5 Optimisation with Mixed Constraints

Not all optimisation problems involve only equality constraints or only inequality constraints. There can be problems where some constraints are in the form of equations and some in form of inequalities. These are problems with *mixed constraints*.

The general formulation of a constrained optimisation problem with mixed constraints would be:

$$\max f(x) = f(x_1, x_2, \ldots x_k) \text{ subject to}$$
$$g_1(x) \leq b_1, \qquad g_2(x) \leq b_2, \qquad \cdots, \qquad g_m(x) \leq b_m$$
$$h_1(x) = c_1, \qquad h_2(x) = c_2, \qquad \cdots, \qquad h_n(x) = c_n$$

This is a problem where an objective function in k-variables has to be maximised subject to m-inequality constraints and n-equality constraints. In the optimal solution for such a problem, namely x^*, one could have the first m_0 of the inequality constraints binding and the remaining $(m - m_0)$ not binding.

To solve this kind of a problem, the Lagrangean function to formulate would be:

$$\mathcal{L}(x_1,\dots,x_k,\lambda_1,\dots,\lambda_m,\mu_1,\dots,\mu_n)=f(x)-\sum_{i=1}^{m}\lambda_i(g_i(x)-b_i)-\sum_{j=1}^{n}\mu_j\big(h_j(x)-c_j\big)$$

The optimal solution (x^*,λ^*,μ^*) would be one that satisfies the conditions.

$$\frac{\partial y}{\partial x}(x^*,\lambda^*,\mu^*)=0, \qquad \forall\, x_s$$

$$\lambda_i^*[g_i(x^*)-b_i]=0, \qquad i=1,2,\dots,m$$

$$h_j(x^*)-c_j=0, \qquad j=1,2,\dots,n$$

$$\lambda_i^*\geq 0, \qquad \forall\, i=1,2,\dots,m$$

$$g_i(x^*)\leq b_i, \qquad i=1,2,\dots,m$$

Example 9.2

Consider the problem:

$$\max Z = x_1 - x_2^2$$
$$\text{subject to } x_1^2+x_2^2=4,$$
$$x_1\geq 0,$$
$$x_2\geq 0$$

The first step is to form the Lagrangean equation.

$$\mathcal{L}=x_1-x_2^2-\mu(x_1^2+x_2^2-4)+\lambda_1x_1+\lambda_2x_2$$

The Lagrange equation has five variables. The five corresponding first order conditions are:

$$\frac{\partial\mathcal{L}}{\partial x_1}=1-2\mu x_1+\lambda_1=0 \tag{1}$$

$$\frac{\partial\mathcal{L}}{\partial x_2}=-2x_2-2\mu x_2+\lambda_2=0 \tag{2}$$

$$\frac{\partial\mathcal{L}}{\partial \mu}=-(x_1^2+x_2^2-4)=0 \tag{3}$$

$$\lambda_1\frac{\partial\mathcal{L}}{\partial \lambda_1}=\lambda_1x_1=0 \tag{4}$$

$$\lambda_2\frac{\partial\mathcal{L}}{\partial \lambda_2}=\lambda_2x_2=0 \tag{5}$$

$$\lambda_1\geq 0 \tag{6}$$

$$\lambda_2\geq 0 \tag{7}$$

$$x_1\geq 0 \tag{8}$$

$$x_2\geq 0 \tag{9}$$

From (1), we have

$$1+\lambda_1=2\mu x_1$$

Based on (4), either λ_1 or x_1 must be zero. But given the condition from (6) that $\lambda_1\geq 0$, it means x_1 cannot be zero. Equation (1) would never balance. Therefore, we have $\lambda_1=0,\ \mu\geq 0$ and $x_1\geq 0$

From (2), we have

$$2x_2(1+\mu)=\lambda_2$$

Since $1+\mu>0$, there are two possibilities. The two, x_2 and λ_2, are either both zero or both positive. But from (5) they cannot be concurrently positive. Hence $x_2=\lambda_2=0$.

Looking at (3), we now obtain $x_1=2$. Then from (1), we get $\mu=\frac{1}{4}$. In conclusion, the solution thus is

$$x_1=2, \qquad x_2=0, \qquad \mu=\frac{1}{4}, \qquad \lambda_1=0, \qquad \lambda_2=0$$

Chapter 10

10 ECONOMIC DYNAMICS IN CONTINUOUS TIME: DIFFERENTIAL EQUATIONS

10.1 Introduction

The growth rate of GDP has become a common statistic used in many spheres. This also applies to the growth in population and many other variables. Though these growth statistics are commonly mentioned, the actual values of GDP, population and many other economic variables are seldom mentioned. This chapter (and its successor) is devoted to use techniques of calculus to deduce, from the known growth rate, the behaviour of the variables with time. The key questions are: Given a differential equation, what is the time path of the variable? When the instantaneous rate of change is known, can we know how the actual variable behaves over time? The answer to the questions lies in finding the time path of the variable of interest. This time path is therefore a function expressing the variable as a function of time and time only.

The chapter proceeds by first looking at the differential equation. An attempt is made to answer the question, 'What is a differential equation?' in a manner that can be easily understood by an ordinary reader. The chapter will then proceed to consider differential equations of different orders and their application in economics, particularly in mathematical economics.

10.2 What is a differential equation?

To attempt to answer this question, it is always helpful to reconsider what has been learnt so far or in the earlier chapters. Chapter 5 of this book dealt with Differential Calculus and the concept of differentiation must be clear at this stage. With this assumption at hand, the simple definition of a differential equation is that; 'this is an equation in which a derivative appears'. A point of emphasis though is that since the main concern is the behaviour of a variable with time, the derivative ought to be with respect to time. It must be the change in a particular variable for a one unit change in time. The unit of measurement of time must not arise, for it matters not. A differential equation is therefore an equation in which a derivative, $\frac{dy}{dt}$, appears.

The power on the derivative determines the order of the differential equation. If the derivative appears only in the first degree, the equation is said to be of first order. When the derivative appears with a second degree, the differential equation is of second order. With a higher degree on the derivative, higher order differential equations set in with increasing complexity.

The chapter will however concentrate on the first order case from which inference will be made for the latter cases.

10.3 First-order linear differential equations

The title of this section brings out two points. The first is that this is a *first order* differential equation. This comes out quite clearly in the heading and implies that the derivative appears in the first degree. Second, there is mention of linearity. The word linearity must be understood in its ordinary meaning, which refers to *line* function. This is a function represented by a straight line. To bring this to context, there is need to look at what is behind a linear function. For a general discussion on functions, refer to Chapter 3.

The function $f(x) = x^2$ is called quadratic function while a function of the form $f(x) = x$ is said to be linear. So where does the difference lie? One may argue that the unit degree on the independent variable defines a linear function. We don't challenge such an answer but our worry is that it does not offer any help to solving the problem at hand, linear differential equation. The second and perhaps more candid definition of a linear function is that it is a function with a constant gradient, or precisely whose gradient is independent of the explanatory variable. The gradients of the two function given above are $\frac{dy}{dx} = 2x$ and $\frac{dy}{dx} = 1$ respectively. Clearly, the first derivative of the former is dependent on the explanatory variable. In the latter case however, the first derivative is constant and hence independent of the explanatory variable.

In short, this is referring to a differential equation where $\frac{dy}{dt}$ occurs only in the first degree. The equation has no product of the form $y\frac{dy}{dt}$. Broadly, a first order linear differential equation takes the form

$$\frac{dy}{dt} + u(t)y = \omega(t)$$

where $\frac{dy}{dt}$ occurs only once. The equation shows that the rate of growth of the variable y is not only a function of time (t) but also its own level. The functions $u(t)$ and $\omega(t)$ are both functions of time and determine the role of time on the derivative. However, they need not be time dependent always since even a constant can be a function of some variable. For instance, a function may be given as $f(x) = a$ even though the right side of the equation has no x. In the same way, $u(t)$ and $\omega(t)$ may actually be constants or in extremes zero but still be referred to as functions of time.

The actual forms of the two functions are of critical importance here as they determine the form of the differential equations. To make the exposition simple, the possible values of the two functions are categorised into three categories; zero, constant and varying with time. This lessens the complexity of working with differential equation and allows them to be tackled in a gradual manner. It enables us to start with a much simpler form of the differential equation, from which the solution can be derived with ease. As the function assumes more complicated forms, we often have to rely on its simpler forms to understand the solution.

10.3.1 Constant Coefficient and Constant Term

We start the exposition with an assumption that the two functions $u(t)$ and $\omega(t)$ are constants. The differential equation is written as.

$$\frac{dy}{dt} + ay = b$$

This is a first order linear differential equation. The derivative is of first degree and the parameters are time invariant. The particular solution to such an equation will depend on the actual values of a and b. As usual, we set out with a simpler case.

10.3.1.1 The Homogeneous Case

A homogeneous equation is defined in various ways. One is the Euler's theorem which states that for homogeneous functions of degree n, the sum of the products of each independent variable and its partial derivative is equal to n times the value of the functions. It however suffices to just note that a homogeneous function is one which remains valid even after multiplying all the variables by a constant factor. In the case at hand, the variables are $\frac{dy}{dt}$ and y. The differential equation can only be homogeneous if the constant b is zero so that the resulting equation is

$$\frac{dy}{dt} + ay = 0$$

The particular solution for a homogeneous case is pretty simple. The rearrangement of the equation, by dividing by y and multiplying by dt throughout, gives

$$\frac{1}{y}dy = (-a)dt$$

Then applying integration on both sides of the equation will give

$$\int \frac{1}{y}dy = \int (-a)dt$$

the left side of the equation is integrated with respect to y while the right side is integrated with respect to t. Techniques of integration are covered in chapter 6. This chapter assumes familiarity with the technique. Therefore, it must proceed as follows.

$$\int \frac{1}{y}dy = \int (-a)dt$$

$$\ln y = -at + C^{35}$$
$$y = e^{-at+C}$$
$$y = e^{C}e^{-at}$$
$$y(t) = Ae^{-at}$$

where $A = e^C$. Since C is an arbitrary constant, so is A, its product. This is the general solution to the homogeneous differential equation. The generality is based on the arbitrary constant contained in the solution. If any particular value is substituted for the arbitrary A, then the solution is called a particular. Where A is not known, it can be definitised using the initial condition. The initial condition requires that $y_{t=0} = y(0)$ where $y(0)$ denotes the initial value of y. This yields $y(0) = A$ which gives the definite solution as

$$y(t) = y(0)e^{-at}$$

The solution is now definite because it has ridden the arbitrary constant. The initial condition which replaces the arbitrary constant is itself definite. Once a specific time point t is known, a definite value of y is calculated.

Example 10.1

Given $\frac{dy}{dt} + 4y = 0$; $y(0) = 1$, find the time path of y. The time path can be found using two routes. The first is to derive like the general solution was derived. The second, if permissible, is to use the general solution by substituting the specific values. We use the latter here. For this kind of differential equation, the general solution is given by

$$y(t) = y(0)e^{-at}$$

By substituting the respective values, we obtain the time path as

$$y(t) = e^{-4t}$$

10.3.1.2 The Non Homogeneous Case

Homogeneity in the differential equation was defined based on the value of the constant. When the constant is zero, the equation is said to be homogeneous. Non homogeneous case is the opposite and an equation is said to be non homogenous if the constant, $b \neq 0$

$$\frac{dy}{dt} + ay = b$$

The equation is non homogeneous as explained above. In particular, the equation is referred to as a *complete equation* with the former referred to as the *reduced equation*. It contains all the components of a differential equation. Thought the interest lies in getting the solution for the latter, the former will prove useful.

[35] With integration on both sides, the two emerging arbitrary constants will sum to a single arbitrary constant.

A simpler route to the solution is to consider the complete equation as consisting of two parts; the complementary function and the particular function. With y as the variable of interest, the two parts are denoted by y_c and y_p respectively. The complementary function is the solution of the homogeneous version, referred to as the reduced equation. Its general solution is only unaltered to depict its new status, as a component of the complete equation by writing:

$$y_c = Ae^{-at}$$

The particular on the other hand is defined as any particular solution of the complete equation. This definition is loose ended as it allows us to make assumption on the form of the complete equation. Of course it is *unwise* to assume a complicated and *indigestible* complete equation. The choice of the form will not alter the solution since any details are taken care by the complementary part of the equation. To proceed, we assume the simplest form of the equation, that y is constant. This selection has a bearing on the derivative and enables us to find a non trivial solution. For a constant function y, its derivative with time $\frac{dy}{dt} = 0$.

Using this condition, we find the solution to the particular equation by substituting $\frac{dy}{dt} = 0$ into the complete equation and solving the equation for y which we denote with a subscript p for a particular equation. Thus

$$y_p = \frac{b}{a}$$

This is valid for $a \neq 0$. The solution to the complete equation involves summing the two components or parts. The sum is

$$y = y_c + y_p = Ae^{-at} + \frac{b}{a}$$

This is the solution to the complete equation. It is however not definite but general because of the presence of an arbitrary constant A. The last step will hence involve definitising the solution using the initial condition. It must be stressed that the presence of an arbitrary is not coincidental but a matter of procedure. Definitising must be the last step and no attempts should be made to eliminate the arbitrary before the full solution is derived. Once we arrive at the final general solution, eliminating the arbitrary is pretty simple. Simply use the condition $y_{t=0} = y(0)$. This will yield a particular solution $A = y(0) - \frac{b}{a}$ which when substituted in the general solution produces a definite solution given by

$$y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a}$$

this solution, like the particular equation solution above, requires that $a \neq 0$. The solution has three parameters; $y(0)$, a and b and varies with time t. Once a specific time point t is known, a definite value of the variable $y(t)$ can be calculated.

Example 10.2

For the differential equation given by $\frac{dy}{dx} + 2y = 6; y(0) = 10$ find the time path of y. Again, there is no need for now to go for derivation since a derived solution is at our disposal. All that remains critical is to identify the two parameters as $a = 2$ and $b = 6$. The initial condition is also given. Using the general solution given by

$$y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a}$$

the specific solution or the time path for the problem at hand is given by

$$y(t) = \left[10 - \frac{6}{2}\right]e^{-2t} + \frac{6}{2}$$
$$= 7e^{-2t} + 3$$

We have considered two scenarios based on the value of the constant in the complete model. The homogeneous is based on zero constant while non homogeneous is based on the converse. In addition, both assumed a non zero coefficient a. In the third scenario is a *zero coefficient* scenario. With a zero coefficient, the differential equation reduces to

$$\frac{dy}{dt} = b$$

This can be solved without recourse to the complicated methods discussed earlier. It is an ordinary derivative requiring an ordinary method.

10.3.2 Variable Coefficient and Variable Term

This section relaxes the assumption imposed in the preceding section that the coefficient and term in the differential equation are constants. Of course that made the differential equation simple, both to look at and solve. With that assumption gone, the differential equation reverts to the general form

$$\frac{dy}{dt} + u(t)y = \omega(t)$$

where the coefficient and term vary with time t. Like the preceding section, this section also provides for a homogeneous and non-homogeneous differential equation. The only difference here is that the coefficient and term are function of time t. This however does not preclude them from being zero. We first deal with the homogeneous case before moving to a more complex non-homogeneous case.

10.3.2.1 Homogeneous Case

A first order linear differential equation is homogeneous when the term $\omega(t)$ equals zero. With a zero term, the differential equation simplifies to

$$\frac{dy}{dt} + u(t)y = 0$$

Since the equation is linear, the technique of separation of variables discussed in section 10.5.1 is applicable. Divide through by y and multiply by dt. This results in an equation with two terms, both on one side. Take one term across the equal sign to form an equation

$$\frac{1}{y}dy = -\,u(t)dt$$

This procedure is the same as that used in section 10.3.1 under a homogeneous equation case. The next step is to integrate both sides of the equation with respect to respective variables.

$$\int \frac{1}{y}dy = \int -\,u(t)\,dt$$

Again refer to Chapter 6 on exactly how to integrate the above equation. The specific form of $u(t)$ is not given, hence its integral cannot be determined. For the left hand side, it is the natural log technique.

$$\ln y = -C - \int u(t)\,dt$$

taking antilog

$$y_t = Ae^{-\int u(t)dt}, \qquad where\ A = e^{-C}$$

This gives an indefinite time path of y. Once the initial condition $y(0)$ is known, the arbitrary coefficient A is determined hence a definite time path.

Example 10.3

Given $\frac{dy}{dt} + 3t^2y = 0$, find the time path of y.

Multiply the equation by dt and then divide by y. Then take on term across the equal sign to get the equation of the form

$$\frac{1}{y}dy = -3t^2dt$$

integrate the left and right side of the equation with respect to y and t respectively.

$$\int \frac{1}{y}dy = -\int 3t^2\,dt$$
$$\ln y = -t^3 + C$$

Then apply antilog

$$y_t = Ae^{-t^3}, \qquad where\ A = e^C$$

10.3.2.2 Non-homogeneous Case

The linear differential equation is non-homogeneous when the term $\omega(t)$ is non-zero, that is, when $\omega(t) \neq 0$. This is a more general form of a first order linear differential equation. It put no restrictions on both the coefficient and term in the equation. The derivation of the solution however is complex. A complete expose is given by Chiang and Wainwright (2005). They develop the solution using the method of *exact differential equations*. In this book, the solution will be provided in retrospective. We state the solution, without derivation, and prove its correctness by proving its reverse process of differentiation. For a general first order linear differential equation given by $\frac{dy}{dt} + u(t)y = \omega(t)$, the indefinite solution is given by

$$y_t = e^{-\int u(t)dt}\left(A + \int \omega(t)e^{\int u(t)dt}\,dt\right)$$

the arbitrary constant A makes the solution indefinite. With information on the initial condition available, the constant can be definitised so that the solution is definite. Of course even differentiating the above equation is not easy. To keep the exposition *within reach*, we show the proof by way of an example.

Example 10.4

Find the time path for the differential equation $\frac{dy}{dt} + 2ty = t$

To solve such a question, deriving the solution must be out of question now. The first step is to identify the two functions $u(t)$ and $w(t)$ which after necessary integration are substituted into the general solution formula. This is by far the easiest route. In the problem at hand

$$\omega(t) = t$$
$$u(t) = 2t$$
$$\int u(t)\,dt = \int 2t\,dt = t^2 + K$$

the solution then is given by

$$y_t = e^{-t^2-K}\left(A + \int te^{t^2+K}\,dt\right)$$
$$y_t = e^{-t^2-K}\left(A + e^K\int te^{t^2}\,dt\right)$$
$$= e^{-t^2}\left(Ae^{-K} + \frac{1}{2}e^{t^2} + C\right)$$
$$= Ae^{-K}e^{-t^2} + \frac{1}{2} + Ce^{-t^2}$$
$$= (Ae^{-K} + C)e^{-t^2} + \frac{1}{2}$$

$$= Be^{-t^2} + \frac{1}{2}$$

The constants A, K and C are arbitrary and so will $B = (Ae^{-K} + C)$. with this time path, we leave it to the reader to prove that

$$\frac{dy}{dt} + 2ty = t$$

10.4 Economic applications

Differential equations play an important role in the understanding of how some economic variables behave over time. A look at this kind of behaviour has up to now been overshadowed by assumptions of equilibrium in the market. This assumption means the market is always in equilibrium and so does not put any stress on the price to change. In reality however, equilibrium is more synonymous with the long run. In the short run, some disequilibrium is inevitable. There might be a period of low price followed by a period of high price, normally triggered by some shocks on the market. A shock is simply an unanticipated change (huge) in either the demand or supply (or both) of a commodity. But how exactly does the price behave in the short run? This question can be best answered by resorting to differential equations.

Assume a simple market model with demand and supply given by

$$Q^d = \alpha - \beta P, \qquad (\alpha, \beta > 0)$$
$$Q^s = -\gamma + \delta P, \qquad (\gamma, \delta > 0)$$

given this system of equation, the equilibrium price P can be solved which will lead to equilibrium quantities traded. We leave it to the reader to show that equilibrium price is given by

$$\bar{P} = \frac{\alpha + \gamma}{\beta + \delta}$$

For as long as $P(t) = \bar{P}$ the market clears and no pressure is exerted on the price. If price deviates from its long run equilibrium however, it would be naive to think that it will simply jump back to equilibrium. There should be some adjustment that must take place over time until equilibrium is reached, if at all it can be reached.

We know from theory that changes in prices are caused by the inequality of quantity demanded and supplied at the initial price. In particular, excess demand drives the price up. This allows expressing the change in price (with respect to time) as a function of excess demand. This should not limit the analysis to cases of excess demand only since an excess supply can be regarded as a negative excess demand. The specific function can be expressed as follows.

$$\frac{dP}{dt} = j(Q^d - Q^s), \qquad j > 0$$

where j is the adjustment coefficient. It measures the responsiveness of price to a deficit. It is clear from the above specification that price will only remain constant if and only if the market clears, $Q^d = Q^s$.

Suppose now the price is disturbed by a shock such that $P(0) \neq \bar{P}$, how will it behave over time? This requires solving the differential equation above.

$$\begin{aligned}\frac{dP}{dt} &= j(Q^d - Q^s)\\ &= j[(\alpha - \beta P) - \gamma + \delta P)\\ &= j[(\alpha + \gamma) - (\beta + \delta)P]\\ &= j(\alpha + \gamma) - j(\beta + \delta)P\end{aligned}$$

when rearranged will yield a differential equation of the form

$$\frac{dP}{dt} + j(\beta + \delta)P = j(\alpha + \gamma)$$

with P as the variable of interest, the differential equation corresponds to the general form $\frac{dy}{dt} + ay = b$ whose definite solution is given by $y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a}$. For the specific problem at hand, identify the two parameters and substitute accordingly. The time path is given by

$$\begin{aligned}P(t) &= \left[P(0) - \frac{\alpha + \gamma}{\beta + \delta}\right]e^{-(\beta+\delta)t} + \frac{\alpha + \gamma}{\beta + \delta}\\ &= [P(0) - \bar{P}]e^{-(\beta+\delta)t} + \bar{P}\end{aligned}$$

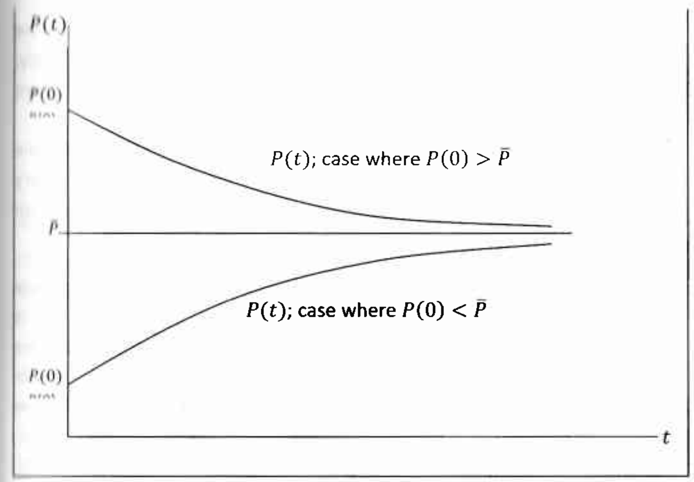
The time path enables the examination of one important characteristic of time paths. Does the time path converge to its long run equilibrium $\bar{P}$ after a disturbance? Since both $P(0)$ and $\bar{P}$ are constants, the convergence (or divergence) will depend only on $e^{-(\beta+\delta)t}$. Using the law of indices and limit theorem, it must be clear that

$$\lim_{t\to\infty} e^{-(\beta+\delta)t} = 0, \qquad \beta, \delta > 0$$

which means the price always converges to the long run equilibrium. In economics, such an equilibrium is said to be *dynamically stable* since the variable has a tendency to revert back to the equilibrium after any disturbance. It requires the asymptotic vanishing of the complementary function over time ($t \to 0$) leaving only the particular integral.

The particular integral in this context is $\bar{P}$, which is constant[36]. With a constant remnant $\bar{P}$ in the time path, the price is said to have a *stationary equilibrium*, otherwise it would have been a *moving equilibrium*.

[36] With the same demand and supply schedules defined, there can only be one equilibrium price.



10.5 Non-linear differential equations of the first order and first degree

The previous section concentrated on linear differential equations. This subsection introduces non-linear differential equation of the first order and first degree. These are equations for which the variable y appears with an exponent other than unit. They are non-linear equations, though still of first order because the differential is of first order, the variable y appears with a power. It may be a quadratic power like y^2. Alternatively, the variable may manifest in a hyperbolic form, written in a rather special form $y\frac{dy}{dt}$. All these forms are non-linear and this section provides the way out. Generally, non-linear differential equations of first order and first degree are of the form

$$f(y,t)\,dy + g(y,t)\,dt = 0 \text{ or}$$

$$\frac{dy}{dx} = h(y,t)$$

Basically two methods are available for solving this kind of differential equations. Depending on the exact form of the equation at hand, they may be separable variables or if not, conversion to linear form is used to convert the non-linear equation to a linear differential equation and use methods discussed under linear differential equations. We start with the former method.

10.5.1 Separable Variables

The differential equation given above is of general form. For some equations however, the function $f(y,t)$ may only be a function of y. Similarly, $g(y,t)$ may also be a function of t only. If this holds simultaneously, the differential equation appears in a much simpler form

$$f(y)\,dy + g(t)\,dt = 0$$

The two variables are separable since each term only has one variable. It is possible to take one on the other side of the equation so that the left hand side only has one variable and corollary for the right hand side. Separating the above equation transforms it to the form

$$f(y)\,dy = -g(t)\,dt$$

The variables have been separated and a simple integration techniques is sufficient to produce the desired results. Proceed by integrating on both sides of the equation, the left hand side is integrated with respect to y while the right is integrated with respect to t. The actual outcome of integration and the necessary steps to get the time path will depend on the exact forms on the functions $f(y)$ and $g(t)$. Obviously, simple $f(y)$ and $g(t)$ will require simple steps to arrive at the time path while complex one will demand more complex procedures.

Example 10.5

Given the differential equation $\frac{dy}{dt} + 3y^2t^2 = 0$, find the time path of y.

With this kind of non-linear differential equation, the first step is to separate the variable so that the equation is expressed in the form $f(y)\,dy = -g(t)\,dt$. This involves some simple algebra. Multiply the equation by dt and then divide by y^2. After taking one term across the equal sign, the equation is transformed to

$$\frac{1}{y^2}dy = -3t^2dt$$

Integrate both sides of the equation

$$\int \frac{1}{y^2}dy = \int -3t^2\,dt$$

The techniques of integration are discussed in Chapter 6 of this book. Here we just state the results without showing the requisite method, which the reader can verify.

$$-\frac{1}{y} = -t^3 - C$$

both integrations will produce an arbitrary constant but the two can be summed to give only one arbitrary constant C.[37] The final step is to make y the subject of formula. The time path is

$$y_t = \frac{1}{t^3 + C}$$

If the initial condition was known, it could be used to make definite the arbitrary constant C. Without the initial condition or any information on known $y_{t=a}$, the time path can only be solved up to this far.

10.5.2 Equations Reducible to Linear Form: Bernoulli Equations

For some non-linear differential equation, it may not be possible or practical to separate the variables. This is common with Bernoulli Equations. Consider a non-linear differential equation

$$\frac{dy}{dt} + Ry = Ty^m$$

where R and T are both functions of t and m is between zero and one, that is, $0 < m < 1$. Such a non-linear equation is called a Bernoulli Equation, named after its discoverer, the Swiss Mathematician, Astronomer and Theologian Jacob Bernoulli. He was one of the prominent Mathematicians of the Bernoulli family.

For this family of equation, separation of variables is not tenable. Opportunely, they can be reduced to linear differential equations. The procedure is as follows. Divide both throughout by y^m to get the equation

$$\frac{1}{y^m}\frac{dy}{dt} + Ry^{1-m} = T$$

then define a new variable Z as $Z = y^{1-m}$ Using the product rule of differentiation, the new variable can be differentiated with respect to t as

$$\frac{dZ}{dt}\left[= \frac{dZ}{dy}\frac{dy}{dt}\right] = (1-m)y^{-m}\frac{dy}{dt}$$

our interest however is a transformed derivative equation

$$\frac{1}{1-m}\frac{dZ}{dt} = \frac{1}{y^m}\frac{dy}{dt}$$

which can be substituted into the differential equation. Replace y with Z in the differential equation to get. $\frac{1}{1-m}\frac{dZ}{dt} = RZ = T$ Multiply through by $(1-m)$ to get

$$\frac{dZ}{dt} + (1-m)RZ = (1-m)T$$

this is a linear first order differential equation in which Z has replaced y. It has a coefficient $a = (1-m)R$ and a constant $b = (1-m)T$. Techniques applicable to linear differential

[37] Note that it does not matter the sign that an arbitrary C takes. Here it is assigned a negative sign strategically because all the other terms in the equation have negative signs.

equation, discussed in section 10.3, can be used to derive the time path Z. After solving for the time path of Z, a reverse substitution replaces Z with y to give the time path of y.

Example 10.6

Find the time path given by the differential equation $\frac{dy}{dt} + ty = 3ty^2$

For this kind of differential equation, separation of variables is not feasible. Its form however provides for reduction to a linear difference equation using the method just discussed. We note that for the sake of conforming to the formula, $m = 2$. Divide through by y^2. The new equation will be

$$\frac{1}{y^2}\frac{dy}{dt} + t\frac{1}{y} = 3t$$

then define a new variable $Z = \frac{1}{y}$ and $\frac{dZ}{dy} = -\frac{1}{y^2}$ Since $\frac{dZ}{dt} = \frac{dZ}{dy}\frac{dy}{dt}$ which implies that

$$\frac{dZ}{dt} = -\frac{1}{y^2}\frac{dy}{dt} \Leftrightarrow -\frac{dZ}{dt} = \frac{1}{y^2}\frac{dy}{dt}$$

then substituting into the differential equation to eliminate y

$$\frac{dZ}{dt} - tZ = -3t$$

The new differential equation is linear, though with variable coefficient and term. We must turn to section 10.3.2 for the solution. Again, there is no need to derive it, simply substitute into the formula. We identify the variable term and coefficient as

$$\omega(t) = -3t$$
$$u(t) = -t$$
$$\int u(t)\,dt = \int -t\,dt = -\frac{1}{2}t^2 + K$$

where K is an arbitrary constant. Substitute these component into the general formula

$$Z_t = e^{-\int u(t)dt}\left(A + \int \omega(t)e^{\int u(t)dt}\,dt\right)$$
$$Z_t = e^{\frac{1}{2}t^2-K}\left(A + \int -3te^{-\frac{1}{2}t^2+K}\,dt\right)$$

Let $v = -\frac{1}{2}t^2 + K$, then $dv = -tdt$

$$Z_t = e^{\frac{1}{2}t^2-K}\left(A + \int 3e^{v}\,dv\right)$$
$$Z_t = e^{\frac{1}{2}t^2-K}\left(A + 3e^{-\frac{1}{2}t^2+K} + C\right)$$
$$Z_t = (A + C)e^{-K}e^{\frac{1}{2}t^2} + 3$$
$$Z_t = Be^{\frac{1}{2}t^2} + 3, \qquad where\ B = (A + C)e^{-K}$$

This is the time path of Z and is not the final solution since the question is asking for the time path of y. To get to y, the equation that defined the variable Z is used, in a reverse substitution. The reversed equation is $y = \frac{1}{Z}$ and the time path of y is.

$$y_t = \frac{1}{Z_t}$$
$$= \frac{1}{Be^{\frac{1}{2}t^2} + 3}$$
$$= \left(Be^{\frac{1}{2}t^2} + 3\right)^{-1}$$

10.6 Economic applications

Consider the Solow growth model and its implications. The model is described below.

The Solow growth model is an advancement of the Domar growth model. The Domar model assumes that capital and labour are used in the same proportion. That is to say the capital-labour ratio is constant. As such, output can be determined exclusively by one factor say capital. Labour may be said to be complementing capital in the same proportion. Solow however relaxes this assumption. The latter recognises that capital and labour may grow at different rates which alter the capital labour ratio. Then output cannot be exclusively determined by capital alone.

Assume a linearly homogeneous production function given by

$$Q = f(K, L)$$

Quantity Q is a function of capital K and labour L where Q is net output after adjusting for depreciation. While labour is assumed to grow at a fixed rate equivalent to population growth rate $\frac{\dot{L}}{L} = n$, the change in capital is dependent on the presumably fixed savings rate. If a fixed proportion of output is saved (and invested) then $\dot{K} = sQ$. Since the production function is linearly homogeneous, by scaling all factors by $1/_L$, this changes the above equation to $Q = Lf\left(\frac{K}{L}, 1\right) = Lf(k)$

Output is now expressed as a function of the capital labour ratio k. Given this expression, the interest is to know how k, the capital labour ratio grows, since this directly determines how output grows. Since k is a quotient of capital and labour, it can be differentiated, with respect to time, using the quotient rule[38]. $k = \frac{K}{L}$

[38] Refer to Chapter 5

$$\dot{k}\left(\equiv \frac{dk}{dt}\right) = \frac{\dot{K}L - K\dot{L}}{L^2}$$
$$= \frac{\dot{K}}{L} - k\frac{\dot{L}}{L}$$
$$= \frac{sLf(k)}{L} - nk, \qquad since\ Q = Lf(k)$$
$$\frac{dk}{dt} = sf(k) - nk$$

At this stage, the specific form of the function $f(k)$ is unknown since the parent production function $f(K, L)$ was now specified. If it is now assumed that the parent function takes the Cobb-Douglas production function of the form

$$Q = K^{\alpha}L^{1-\alpha}$$

Since the function is linearly homogeneous, it can be expressed as follows $Q = Lk^{\alpha}$. A comparison with the earlier function leads to the conclusion that the specific function $f(k) = k^{\alpha}$. Then the differential equation above is rewritten, taking the negative term to left hand side of the equation, as

$$\frac{dk}{dt} + nk = sk^{\alpha}$$

This is a Bernoulli equation discussed in the preceding section with $R = n$ and $T = s$. It is non linear but reducible to a linear form. This enables us determine the time path of k and hence the time path of *per capita* income $y\left(\equiv \frac{Q}{L}\right) = f(k)$.

To reduce the differential equation to a linear form, we follow the Bernoulli exposition discussed in the preceding section. divide the equation by k^{α} to get.

$$\frac{1}{k^{\alpha}}\frac{dk}{dt} + nk^{1-\alpha} = s$$

Define a new variable Z as $Z = k^{1-\alpha}$. Differentiating with time t, we get

$$\frac{dZ}{dt} = (1-\alpha)\frac{1}{k^{\alpha}}\frac{dk}{dt}$$
$$\frac{1}{(1-\alpha)}\frac{dZ}{dt} = \frac{1}{k^{\alpha}}\frac{dk}{dt}$$

Then substitute into the non-linear differential equation in k. The new equation in Z is

$$\frac{dZ}{dt} + (1-\alpha)nZ = (1-\alpha)s$$

The new differential equation is linear. Since both the population growth rate and savings rates are constant, then we have a constant coefficient $a = (1-\alpha)n$ and a constant term $b = (1-\alpha)s$. We proceed to get the time path of Z using techniques applicable to linear differential equations. We will state the solution without deriving it. For the full procedure of deriving the general solution, we refer the reader to section 10.3.1.

Given the two parameters $a = (1-\alpha)n$ and $b = (1-\alpha)s$, the general solution of Z is given by

$$Z(t) = \left[Z(0) - \frac{s}{n}\right]e^{-(1-\alpha)nt} + \frac{s}{n}$$

with the time path of Z known and how it relates to the variable of interest k, the time path of k is found by $Z = k^{1-\alpha} \Leftrightarrow k = Z^{\frac{1}{1-\alpha}}$

$$k = \left(\left[k(0)^{1-\alpha} - \frac{s}{n}\right]e^{-(1-\alpha)nt} + \frac{s}{n}\right)^{\frac{1}{1-\alpha}}$$

Since both $(1-\alpha)$ and the population growth n are positive, the exponential expression will tend to zero as $t \to \infty$. This means the capital labour ratio will ultimately converge to a stead state

$$k = \left(\frac{s}{n}\right)^{\frac{1}{1-\alpha}}$$

It will increase with an increase in the savings rate and fall with an increase in the growth of population. These conclusions follow quite naturally. With an increased savings rate, there would be an increase in the build up of capital which, *ceteris paribus*, leads to an increase in the capital labour ratio. Population increase causes capital spread which reduces the concentration of capital per labour. So an increase in the population growth rate will cause a fall in the stead state capital labour ratio.

10.7 Higher-order differential equations

The first order differential equations may not always be robust enough to explain the behaviour of many economic variables. Such variables require higher order differential equation, which become broader the higher the order. By definition, higher order differential equations are differential equations with higher differentials. For instance, a second order differential equation has a second order differential. The third order will have a third order differential and this goes on to the n^{th} order differential equation.

$$\frac{d^2y}{dt^2} = ky$$

Specifically, the order of a differential equation is based on the highest order of differential in the equation. This needs to be clear because often reader will come across differential equations with two or more differentials of different order. Therefore, the highest order rules.

It is also permissible to have a differential equation constituted by a polynomial of differentials. For instance, an n^{th} order differential equation will have a chain of lower order differentials. A simple variety of a linear differential equation of order n is:

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots\cdots + a_{n-2} \frac{d^2 y}{dt^2} + a_{n-1} \frac{dy}{dt} = b$$

this is a general form of an n^{th} order linear differential equation. The exact form need not have the entire decreasingly ordered differentials. Their absence may alternatively be simply indicated by a zero coefficient.

Consider a differential equation given by

$$y_t'' + a_1 y_t' + a_2 y = b, where\ y_t' = \frac{dy}{dt}$$

This is a second order differential equation. For its solution, there might be need to review how the first order case was dealt with. Since this is a non homogeneous case, the solution must be divided into two; the particular and the complementary solution denoted by y_c and y_p respectively. The time path is the summation of the two. The former is of less controversy so we start with the latter.

The particular solution becomes easy when dealt with in phases or stages. The stages are defined by the assumption made on the nature of the variable y. The first and simplest is to assume that the variable is constant, that is, $y = a$. With this assumption, both the first and second order derivatives will be zero. The differential of a constant is zero and any subsequent differentiation will yield zeros. This simplifies the equation (for the particular solution) to

$$a_2 y = b$$

so that by making y the subject of formula

$$y_p = \frac{b}{a_2}$$

The subscript p indicates that this is a solution for the particular part of the equation. This solution is strictly based on assuming that the variable is constant. This assumption however comes with its own restrictions. It restricts a_2 from attaining zero, otherwise the solution collapses.

Example 10.7

For the differential equation $y_t'' + y_t' - 2y = -10$, find the particular solution y_p.

The coefficient on y is non-zero. This allows assuming y as a simple constant function. All its derivatives will be zero. The equation then reduces to

$$-2y = -10$$

A simple division yields the particular solution

$$y_p = \frac{-10}{-2}$$
$$= 5$$

But what if $a_2 = 0$? Well, the answer is simple. The first assumption of a constant time path must be revisited.

When $a_2 = 0$, there is no option but to assume a non-constant value for y. Again it would not do any good to assume a more complicated function of y. The closest look is to think of y as a linear function of time t. This will meet the condition of non-constant since it changes with time. If we take the simple form of

$$y = kt$$

where k is an arbitrary constant, then $y' = k$ and $y'' = 0$ bearing in mind that this form is coming because$a_2 = 0$, the new equation is $a_1 k = b$, $\therefore k = \frac{b}{a_1}$ given this expression for k, then $y_p = \frac{b}{a_1} t$ represents the second possible particular solution for y_p. Recall that the particular solution represents the long run equilibrium of a time path. Since this specific particular solution is non-constant, but changes with time, it represents a moving equilibrium scenario. This is typical of most prices in the economy. Prices fluctuate around some equilibrium but the equilibrium price itself may have a tendency to increase (or in rare circumstances, reduce).

Such a solution however is dependent on a_1 being non-zero. When the converse is true, that is $a_1 = 0$, the solution collapses. The zero left hand side of the equation is equated to a non-zero right hand side. The latter has $b \neq 0$ while the former will have zero because of zero coefficients and a zero second order differential. This leads to the third and perhaps the last scenario, when both parameters a_1 and a_2 are zero. The simple linear function will not suffice to give a particular solution under these circumstances. A more complex function, specifically with a non-zero second derivative, is needed.

It is easy to generate the function, in indefinite form, from the differential equation. When a_1 and a_2 are zero, the differential equation reduces to a second order derivative

$$y'' = b$$

the process of integration, reverse differentiation, will enable finding y from its derivative. Though each function has an infinite number of integrals, we retain the right to assume the arbitrary constant is zero. This will produce a unique definite integral, giving the particular solution. Using the simple power rule of integration, the particular solution is found as

$$y_p = \frac{b}{2}t^2$$

This is a quadratic function. It is easy to prove that this is the correct integral by differentiating the function twice and see if the outcome gets to the initial derivative. Like in the earlier case, this again is a moving average case. In particular, the long run equilibrium explodes with time, given the nature of a quadratic function.

The discussion of the particular solution might be lengthy but still falls short of all possible scenarios. The three scenarios given are not exhaustive but meant to be a guide to the insurmountable scenarios not covered. They give a broader idea of dealing with various forms of the functions.

The complimentary solution, as earlier stated, does not need many suppositions. Its solution is more straightforward. It relies on assuming a homogeneous differential equation, one with $b = 0$. It is the general solution of the reduced equation $y_t'' + a_1 y_t' + a_2 y = 0$.

Recall that in the first order differential equation, the complementary solution was of the form $y = Ae^{rt}$. If this is adopted as a trial solution, then the first and second order derivatives are $y' = rAe^{rt}$ and $y'' = r^2Ae^{rt}$ respectively. When substituted into the reduced form, the differential equation becomes

$$r^2Ae^{rt} + a_1rAe^{rt} + a_2Ae^{rt} = 0$$

$$Ae^{rt}(r^2 + a_1r + a_2) = 0$$

In the equation, there are two possible explanations. Either $A = 0$ or $r^2 + a_1r + a_2 = 0$. The first case is not plausible since $A = y_{t=0}$. The arbitrary constant A is the initial condition which cannot be assumed to be zero. This leads us to be certain that the second case holds. That is:

$$r^2 + a_1r + a_2 = 0$$

This is a characteristic equation of the reduced equation. It is a quadratic equation in r and its solutions are called characteristic roots. As usual, such an equation will produce two roots, r_1 and r_2 which will give rise to two complementary solutions $y_1 = A_1e^{r_1t}$ and $y_2 = A_2e^{r_2t}$ respectively. To get to a unique complementary solution, simply combine the two by averaging.

In the actual algebra however, only summation is prominent because the dividing is swallowed in the arbitrary constant(s). $y_c = y_1 + y_2$

This general result may need to be modified depending on the actual characteristics realised. Recall that there are three possible outcomes from a characteristic equation, depending on the value of the discriminant $D = {a_1}^2 - 4a_2$. There may be *two distinct roots, two repeated roots* and *two complex conjugates*. We explore the three in succession.

10.7.1 Case 1: Two distinct roots

This case occurs when the discriminant $D = {a_1}^2 - 4a_2$ is strictly positive. The characteristic equation produces two distinct roots, r_1 and r_2. The two roots are used separately, to derive the two possible complementary solutions, y_1 and y_2 respectively which are summed to give the complementary solution. It takes the form

$$\begin{aligned} y_c &= y_1 + y_2 \\ &= A_1e^{r_1t} + A_2e^{r_2t} \end{aligned}$$

The solution still contain to two arbitrary constant A_1 and A_2. No need to be bothered with finding the definite values of the two at this stage. The two are only part of the complementary solution. Getting the definite values of the arbitrary constants must always be the last step, after putting together the complementary and particular solution, dealt with earlier, together.

Example 10.8

Given the differential equation $y_t'' + y_t' - 2y = -10$, find the time path of y.

This equation is one and the same as Example 10.7 which solved for the particular solution. It still remains the same and we simply adopt it here as

$$y_p = 5$$

Even though the example asks for the general solution, the emphasis is on the complementary solution. The way to get the particular solution is explained in the said example. The first step is to write the reduced form of the equation, one with a zero term.

$$y_t'' + y_t' - 2y = 0$$

From the first order differential equation, the general solution is of the form $y = Ae^{rt}$ which we adopt here. Then the first and second derivatives are

$$\begin{aligned} y' &= rAe^{rt} \\ y'' &= r^2Ae^{rt} \end{aligned}$$

substitute into the reduced equation and factorise to generate

$$\begin{aligned} r^2Ae^{rt} + rAe^{rt} - 2Ae^{rt} &= 0 \\ Ae^{rt}(r^2 + r - 2) &= 0 \end{aligned}$$

The first factor on the left hand side cannot be zero. Since the arbitrary constant is associated with the initial condition and the exponential has no zero in the *range*. So the second must be. This gives the anticipated characteristic equation

$$r^2 + r - 2 = 0$$

We do not have to labour to explain how to get the two roots. We can safely state them and refer the reader to other *texts* on quadratic equations. The roots are $r_1 = 1,\ r_2 = -2$ and the two solutions are

$$y_1 = A_1 e^t$$
$$y_2 = A_2 e^{-2t}$$

The complementary solution is a sum of the two.

$$\begin{aligned} y_c &= y_1 + y_2 \\ &= A_1 e^t + A_2 e^{-2t} \end{aligned}$$

This is the complementary solution of the differential equation. Since the particular solution is available, the general solution of the complete differential equation is found by summing the two.

$$\begin{aligned} y_t &= y_c + y_p \\ &= A_1 e^t + A_2 e^{-2t} + 5 \end{aligned}$$

only after putting the two components together should definitising come. Emphasis must be put that definitising is the last step in solving differential equations. In this particular case, no initial information is provided. Moreover, the solution has two unknowns which would require that two time points are known.

In the interest of having a complete solution, assume now that two initial conditions are known. These are $y_0 = 12$; $y_0' = -2$. The initial level of the variable is known as well as its first derivative. Using this information, create two equation and then solve them simultaneously. The first equation is substituting $t = 0$ into the indefinite solution and equating to 12. The second equation is to substitute $t = 0$ into the first derivative of the indefinite solution and equating to -2.

$$\begin{aligned} y_t &= A_1 e^t + A_2 e^{-2t} + 5 \\ y_{t=0} &= A_1 + A_2 + 5 \\ 12 &= A_1 + A_2 + 5 \\ A_1 + A_2 &= 7 \end{aligned}$$

$$\begin{aligned} y_t' &= A_1 e^t - 2A_2 e^{-2t} \\ y_{t=0}' &= A_1 - 2A_2 \\ -2 &= A_1 - 2A_2 \\ A_1 - 2A_2 &= -2 \end{aligned}$$

We do not need to go into how to solve this pair of equations. We instead assume the author is already conversant with simultaneous equation. So we just state the answers as

$$A_1 = 4$$
$$A_2 = 3$$

so that the definite solution to the differential equation is

$$y_t = 4e^t + 3e^{-2t} + 5$$

10.7.2 Case 2: Repeated Real Roots

When the discriminant in the characteristic equation equals zero, the equation produces two repeated real roots. In other words, the two roots r_1 and $r_2$are equal. This means there is only one root r without a numerical subscript since this is meant to identify two different numbers. Given the two roots are equal, there is no need to identify them.

With a single root, the general form of the solution is slightly different from one presented in the *two distinct root* scenario. If we were to continue with this form, the solution would collapse to a single constant. That is, in the expression

$$y_c = A_1 e^{rt} + A_2 e^{rt} = A_3 e^{rt}, \qquad A_3 = A_1 + A_2$$

with only one arbitrary constant. This is not sufficient for the second order differential equation. As will be notice even in the next chapter on difference equation, the general solution in the case of two repeated roots is given by

$$y_c = A_1 e^{rt} + A_2 t e^{rt}$$

this does not collapse to a single constant. The rest of the procedure remains the same as in the previous case. When the root r is known and two initial conditions, then a definite time path is calculated by solving the resulting simultaneous equations.

Example 10.9

Solve $y_t'' + 6y_t' + 9y_t = 27$ for the time path.

For the particular solution, since the coefficient on y_t is non zero, it is permissible to assume a simple constant for the function. this will have all the derivatives equal to zero. The equation will reduce to

$$9y_t = 27$$

This gives a particular solution

$$y_p = \frac{27}{9} = 3$$

The complementary solution is found by taking a reduced form of the differential equation, the homogeneous one. This is given by $y_t'' + 6y_t' + 9y_t = 0$. Then assume $y = Ae^{rt}$. The first and second order derivates are

$$y' = rAe^{rt}$$
$$y'' = r^2Ae^{rt}$$

Substitute the three into the reduced equation to get an equation in r and then factorise the common Ae^{rt}.

$$r^2Ae^{rt} + 6rAe^{rt} + 9Ae^{rt} = 0$$
$$Ae^{rt}(r^2 + 6r + 9) = 0$$

Again, we must determine which factor(s) makes the equation zero. Of course it cannot be the arbitrary constant A because this must be allowed to be arbitrary. The exponent also never becomes zero. The range of an exponential function is strictly positive. This *elimination* method leaves one factor, called the characteristic root.

$$r^2 + 6r + 9 = 0$$

Since quadratic equations are no longer an issue at this stage, it must suffice to state the root(s) without having to show all the necessary steps. The reason we dodge the working is not that it is irrelevant but is on assuming that the reader is already well vested on that. This characteristic equation has the root

$$r = -3$$

With the root now at hand, substitute into the complementary solution form to get complementary solution.

$$y_c = A_1e^{rt} + A_2te^{rt}$$
$$y_c = A_1e^{-3t} + A_2te^{-3t}$$

Since we already found the particular solution, the next step is to put the two together by summation. Recall that the general solution is a sum of the two separate solutions.

$$y_t = y_c + y_p$$
$$= A_1e^{-3t} + A_2te^{-3t} + 3$$

This is an indefinite general solution. It is indefinite because it contains indefinite constants. Suppose it is known that $y_{t=0} = 5$ and $y'_{t=0} = -5$? This information allows getting a definite solution since the two indefinite constants can now be made definite. The procedure involves generating two equation simultaneous equations using the two initial conditions.

$$y_{t=0} = A_1 + 3$$
$$5 = A_1 + 3$$
$$A_1 = 2$$

$$y_t' = -3A_1e^{-3t} - 3A_2te^{-3t} + A_2e^{-3t}$$
$$y'_{t=0} = -3A_1 + A_2$$
$$-5 = -3(2) + A_2$$
$$A_2 = 1$$

Now the constants are definite and the definite general solution is

$$y_t = 2e^{-3t} + te^{-3t} + 3$$

10.7.3 Case 3: Two complex conjugates

When the discriminant in the characteristic equation $f(r) = 0$ is negative, the characteristic equation has no real valued roots. The solution will require getting the square root of a negative number, which is known to have no solution in the real number system. This case is discussed after a discussion of *Complex numbers*.

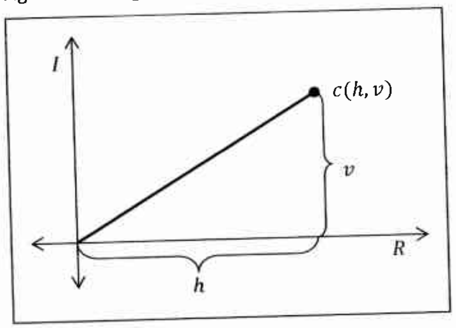
10.8 Complex Numbers

We have stated that when $a_1{}^2 - 4a_2 < 0$ in the characteristic equation, the solution will lie outside the real number system. The problem is that the solution requires us to evaluate the square root of a negative. To do this, a special number is defined. Let $i = \sqrt{-1}$. This is called an *imaginary number*. This number allows the evaluation of a square root of any negative number since the negative can be separated using the law of indices. For example, $\sqrt{-2} = \sqrt{2}\sqrt{-1} = \sqrt{2}i$, $\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$. A *complex number* is a number that can be written as a sum of a real part and an imaginary part.

In general, a complex number will take the form $(h + vi)$ where h and v are two real numbers. They can be negative or positive or even zero. This means any real number is also a complex number with $(v = 0)$. In the same way, all imaginary numbers can be written as complex numbers with $(h = 0)$. Thus sets of all real numbers and all imaginary numbers are both subsets of the set of all complex numbers. Examples of complex numbers include $5 + 2i$, $8 - i$ and so on.

Complex numbers can also be represented by an *Argand diagram* or a *complex plane*. It is called Argand after a nineteenth century French mathematician Jean-Robert Argand. This is similar to the *Euclidean plane* used for functions. The Argand diagram is a plane measuring the real part of a complex number in the horizontal axis and the imaginary part in the vertical axis.

Figure 10.2. Argand diagram



Using Pythagoras theorem, we know that $R^2 = h^2 + v^2$, where R is the modulus of the complex number $(h + vi)$.

Recall that the characteristic equation was given by

$$r^2 + a_1 r + a_2 = 0$$

Its solution is

$$r = \frac{-a_1 \pm \sqrt{a_1{}^2 - 4a_2}}{2}$$

When $a_1{}^2 - 4a_2 < 0$, the equation has no real valued roots. This part can be rewritten as $-1(4a_2 - a_1{}^2) < 0$. Though the left hand side still remains negative, the part in parentheses is now positive and we can proceed to find its square root.

$$r = \frac{-a_1 \pm \sqrt{-1(4a_2 - a_1{}^2)}}{2}$$

$$= \frac{-a_1 \pm \sqrt{-1}\sqrt{(4a_2 - a_1{}^2)}}{2}$$

$$= \frac{-a_1 \pm i\sqrt{(4a_2 - a_1{}^2)}}{2}$$

$$= \frac{-a_1}{2} \pm i\frac{\sqrt{(4a_2 - a_1{}^2)}}{2}$$

Now let $h = \frac{-a_1}{2}$ and $v = \frac{\sqrt{(4a_2 - a_1{}^2)}}{2}$, then

$$r = h \pm vi$$

The solution to the complementary function remains $y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$ substituting the two distinct roots

$$y_c = A_1 e^{(h+vi)t} + A_2 e^{(h-vi)t}$$

$$= e^{ht}\left(A_1 e^{vit} + A_2 e^{-vit}\right)$$

The solution has imaginary exponential functions. But how do we interpret imaginary exponential functions? To aid understanding, we take a look at two key concepts in Mathematics. The first is the Maclaurin series and the second is the Trigonometric functions. The Maclaurin series is similar to the Taylor series. The two differ only on the point around which the function is expanded. While the Taylor series expands the function around any point $x_0 = a$ in the domain, the Maclaurin series locks the point of expansion to the origin. That is $x_0 = 0$. According to the Maclaurin series, a differentiable function can be expanded around zero as follows:

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f''''(0)x^4}{4!} + \cdots$$

$$f(x) = \frac{f(0)}{0!} + \frac{f^1(0)x}{1!} + \frac{f^2(0)x^2}{2!} + \frac{f^3(0)x^3}{3!} + \frac{f^4(0)x^4}{4!} + \cdots$$

If then $f(v) = e^{vi}$

Then $f(0) = 1$,

$$f'(v) = ie^{vi}, \qquad f'(0) = i$$

$$f''(v) = i^2 e^{vi}, \qquad f'(0) = i^2$$

$$f'''(v) = i^3 e^{vi}, \qquad f'(0) = i^3$$

$$f''''(v) = i^4 e^{vi}, \qquad f'(0) = i^4$$

$$f'''''(v) = i^5 e^{vi}, \qquad f'(0) = i^5$$

Since $i = \sqrt{-1}$, then $i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1$

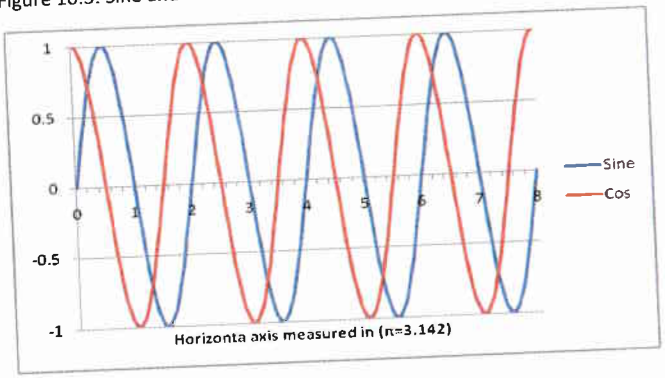
So that the series for the function can now be written as

$$f(v) = e^{vi} = \frac{1}{0!} + \frac{iv}{1!} + \frac{-v^2}{2!} + \frac{-iv^3}{3!} + \frac{v^4}{4!} + \frac{iv^5}{5!} + \frac{-v^6}{6!} + \cdots$$

$$e^{vi} = \frac{1}{0!} + \frac{iv}{1!} - \frac{v^2}{2!} - \frac{iv^3}{3!} + \frac{v^4}{4!} + \frac{iv^5}{5!} - \frac{v^6}{6!} + \cdots$$

Next we expand in the same way the two trigonometric functions; the sine and cosine. The graphs of the two functions are plotted in Figure 10.3 below.

Figure 10.3: Sine and Cosine functions



The graphs show two important characteristics of the two functions. First, the functions are *periodical*. Their values repeat themselves for every 2π radians. That is, each function has a period of 2π. Second, the functions have constant *amplitude* of fluctuation.

The two will differ in phases. If the $\cos\theta$ is shifted rightward by $\frac{\pi}{2}$, the two will coincide. That is:

$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$$

Since the two are smooth functions, they are differentiable. Given $f(\theta) = \sin\theta$ the higher order derivatives[39] are as follows.

$$f'(\theta) = \cos\theta, \quad f'' = -\sin\theta, \quad f''' = -\cos\theta, \quad f'''' = \sin\theta, \quad f''''' = \cos\theta$$

Similarly, given the function $g(\theta) = \cos\theta$, the derivatives go as follows:

$$g'(\theta) = -\sin\theta, \quad g''(\theta) = -\cos\theta, \quad g'''(\theta) = \sin\theta, \quad g''''(\theta) = \cos\theta,$$
$$g'''''(\theta) = -\sin\theta$$

In both functions, the forth derivative is equal to the function itself and the process repeats every after four derivatives. We can then write the expansion of the cosine function around $v = 0$. From Figure 10.3 above, $\cos(0) = 1$ and $\sin(0) = 0$

[39] The derivative results are valid only if θ is measured in radian

$$f(v) = \cos v = \frac{(\cos 0)}{0!} - \frac{(\sin 0)v}{1!} - \frac{(\cos 0)v^2}{2!} + \frac{(\sin 0)v^3}{3!} + \frac{(\cos 0)v^4}{4!} + \cdots$$

$$= \frac{1}{0!} - \frac{0}{1!} - \frac{v^2}{2!} + \frac{0}{3!} + \frac{v^4}{4!} - \frac{0}{5!} - \frac{v^6}{6!} + \cdots$$

$$\cos v = \frac{1}{0!} - 0 - \frac{v^2}{2!} + 0 + \frac{v^4}{4!} - 0 - \frac{v^6}{6!} + \cdots$$

Similarly, the *sine* v function can be expanded using the same procedure.

$$f(v) = \sin v = \frac{(\text{sine } 0)}{0!} + \frac{(\cos 0)v}{1!} - \frac{(\text{sine } 0)v^2}{2!} - \frac{(\cos 0)v^3}{3!} + \frac{(\text{sine } 0)v^4}{4!} + \cdots$$

$$= \frac{0}{0!} + \frac{v}{1!} + \frac{0}{2!} - \frac{v^3}{3!} + \frac{0}{4!} + \frac{v^5}{5!} + \frac{0}{6!} - \cdots$$

$$\sin v = 0 + \frac{v}{1!} - 0 + \frac{-v^3}{3!} + 0 + \frac{v^5}{5!} + \cdots$$

As we delve into this algebra, it is important to restrain ourselves from being carried away. The objective should not be forgotten. It was to find a way of dealing with the imaginary number. So far, this hasn't made any appearance in the trigonometric functions. To introduce it, multiply on both sides of the expanded *sine* v function by $i = \sqrt{-1}$.

$$i\sin v = \frac{iv}{1!} - \frac{iv^3}{3!} + \frac{iv^5}{5!} + \cdots$$

Now adding the new equation to the expanded $\cos v$. Note that the *cos* v is zero for all odd number powers of v while *sine* v is zero only for even number power of v. We can therefore rearrange the sum so that there is ascending power of v.

$$\cos v + i\sin v = \frac{1}{0!} + \frac{iv}{1!} - \frac{v^2}{2!} - \frac{iv^3}{3!} + \frac{v^4}{4!} + \frac{iv^5}{5!} - \frac{v^6}{6!} + \cdots$$

The emerging expression is identical to the expression obtained by expanding e^{vi}. We obtain the following *Euler Relation* also called Euler's formula.

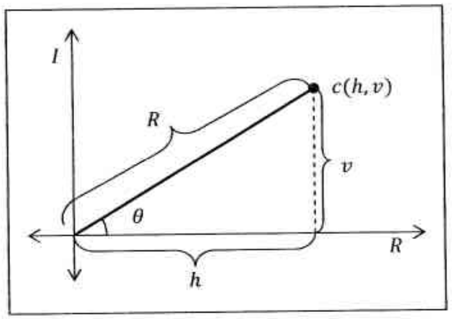
$$e^{vi} = \cos v + i\sin v$$

$$e^{-vi} = \cos v - i\sin v$$

10.8.1 Alternative Representation of Complex Numbers

Complex numbers can also be represented using polar coordinates. Let us use the Argand diagram presented in Figure 10.2. The figure is repeated with some details added.

Figure 10.4. A detailed Argand diagram



The Cartesian coordinates of the complex number are still given by (h, v) which defines point c. The modulus or absolute length is given by $R = \sqrt{h^2 + v^2}$. This is derived using the now famous *Pythagoras theorem* dealing with a right angled triangle. Given the angle between the complex number representation and the horizontal axis, it is possible to form some equations from trigonometry. In particular, h and v can be expressed as functions of the angle and the modulus.

In the triangle, R manifests as the *hypotenuse*, h is the *adjacent* while v is the *opposite*. generate the following equations.

$$\sin\theta = \frac{v}{R}, \qquad v = R\sin\theta$$
$$\cos\theta = \frac{h}{R}, \qquad h = R\cos\theta$$

Thus, the complex number will be given by

$$h \pm vi = R\cos\theta \pm i\,R\sin\theta$$
$$= R(\cos\theta \pm i\sin\theta)$$

Where R and θ are called polar coordinates of the complex number as opposed to h and v which are Cartesian coordinates. Through Euler relations, the complex number can be further translated into the exponential from. The Euler relations state that

$$e^{\pm vi} = \cos v \pm i\sin v$$

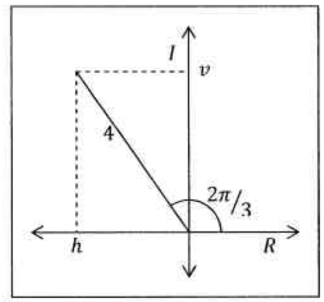
Since

$$h \pm vi = R(\cos\theta \pm i\sin\theta)$$
$$= R(e^{\pm\theta i}), \qquad 0 \le \theta < 2\pi$$

Example 10.10

Find the Cartesian form of the complex number $4e^{\frac{2\pi}{3}i}$

The modulus $R = 4$ and the angle $\theta = \frac{2\pi}{3}$. The Argand diagram will take the following shape.



Though the Argand diagram may not be needed, it is important to sketch it in order to have insights on the expected values. In the above case, the given angle means the point fall in the *second quadrant* ($\frac{\pi}{2} < \theta < \pi$). As such, we already know that h must be negative while v will be positive. We proceed as follows:

$$h = R\cos\theta = 4\cos\frac{2\pi}{3} = 4(-0.5) = -2$$

$$v = R\sin\theta = 4\sin\frac{2\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$

The Cartesian form is $h + vi = -2 + 2\sqrt{3}i$

10.8.2 de Moivre's Theorem

The theorem is named after a French mathematician Abraham de Moivre. It states that:

$$(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$$

By extension, the complex number $h + vi$ raised to power n will give:

$$(h + vi)^n = [R(\cos\theta \pm i\sin\theta)]^n$$
$$= R^n(\cos\theta \pm i\sin\theta)^n$$
$$= R^n(\cos n\theta \pm i\sin n\theta)$$

That is, to raise a complex number to the n^{th} power, raise R to the n^{th} power and multiply the angle θ by n.

Now we return to the analysis of the complex root case that was suspended earlier. The question posed earlier regarding how to interpret the imaginary exponential function can be loosely answered. The way out is to replace it with trigonometric functions in which the imaginary number is stand-alone. We now proceed solving for the complementary function

$$y_c = e^{ht}\left(A_1 e^{vit} + A_2 e^{-vit}\right)$$

$$\begin{aligned} y_c &= e^{ht}(A_1(\cos v + i\sin v)^t + A_2(\cos v - i\sin v)^t) \\ &= e^{ht}\left(A_1(\cos vt + i\sin vt) + A_2(\cos vt - i\sin vt)\right)^{40} \\ &= e^{ht}\left((A_1 + A_2)\cos vt + i(A_1 - A_2)\sin vt\right) \\ &= e^{ht}(A_3\cos vt + A_4\sin vt) \end{aligned}$$

This is the complementary function. As shown in Figure 10.3, the Sine and Cosine functions have a constant amplitude and period. We expect that the time path will have a smooth oscillation around the around (constant or moving depending on the particular solution). The number of complete cycles in a unit of time (also known as the frequency) is dependent on the v. In particular, the frequency f is given by $f = \frac{v}{2\pi}$.

Since the term in brackets results in an oscillation of fixed amplitude (non converging and non diverging) with a constant period, the question of convergence is only dependent on the exponential e^{ht}. In particular, h determines whether the time path converges or not. But h derives from $h = \frac{-a_1}{2}$. Thus the coefficient of the first derivative in the differential equation determines the convergence of the time path. That is, when $a_1 > 0$ so that $h < 0$, the time path converges. It diverges when the opposite is true.

Example 10.11

Find the time path of the differential equation $y_t'' + y_t' + 4y = 2$

[40] Given $e^{vit} = (\cos v + i\sin v)^t$. If we let $\theta = vt$, then $e^{\theta i} = \cos\theta + i\sin\theta = \cos vt + i\sin vt$ which implies that $e^{vit} = \cos vt + i\sin vt$. In the same way, $e^{-vit} = \cos vt - i\sin vt$

For the particular solution, simply assume the function is constant. Given this assumption, all the derivatives will be zero. The remnant of the differential equation is:

$$4y = 2$$

This leads to a particular solution

$$y_p = \frac{1}{2}$$

For the complementary solution, we must assume a homogeneous differential equation. Further, assume the time path is of the form $y = Ae^{rt}$. Given this form, the first and second order derivatives are $y' = rAe^{rt}$ and $y'' = r^2Ae^{rt}$ respectively. Then substitute into the assumed homogeneous equation.

$$r^2Ae^{rt} + rAe^{rt} + 4Ae^{rt} = 0$$

This translates to the characteristic equation

$$r^2 + r + 4 = 0$$

This equation has the roots

$$\begin{aligned} r &= \frac{-1 \pm \sqrt{1^2 - 4(4)}}{2} \\ &= \frac{-1 \pm \sqrt{-15}}{2} \\ &= \frac{-1}{2} \pm \frac{i\sqrt{15}}{2} \end{aligned}$$

$$h = \frac{-1}{2}, \qquad v = \frac{\sqrt{15}}{2}$$

The complementary solution is

$$\begin{aligned} y_c &= e^{ht}(A_3\cos vt + A_4\sin vt) \\ &= e^{\frac{-1}{2}t}\left(A_3\cos\frac{\sqrt{15}}{2}t + A_4\sin\frac{\sqrt{15}}{2}t\right) \end{aligned}$$

To get the general solution, add the two components.

$$y_t = y_c + y_p$$

$$y_t = e^{\frac{-1}{2}t}\left(A_3\cos\frac{\sqrt{15}}{2}t + A_4\sin\frac{\sqrt{15}}{2}t\right) + \frac{1}{2}$$

This is the general solution. It is not definite because of the two arbitrary constants. Suppose it is now known that $y(0) = 5\frac{1}{2}$ and $y'(0) = 5$, then what the definite time path? Using the two known initial conditions, we create two equations that are simultaneously solved to get definite values of A_3 and A_4.

$$y(0) = e^0(A_3\cos(0) + A_4\sin(0)) + \frac{1}{2}$$

$$= A_3 + \frac{1}{2} = 5.5$$

$$\therefore\ A_3 = 5$$

$$y_t' = \frac{-1}{2}e^{\frac{-1}{2}t}(A_3\cos\phi t + A_4\sin\phi t) + e^{\frac{-1}{2}t}(-A_3\phi\sin\phi t + A_4\phi\cos\phi t), \qquad \varphi = \frac{\sqrt{15}}{2}$$

$$y'(0) = \frac{-1}{2}(A_3\cos(0) + A_4\sin(0)) + \left(-A_3\frac{\sqrt{15}}{2}\sin(0) + A_4\frac{\sqrt{15}}{2}\cos(0)\right)$$

$$= \frac{-1}{2}(A_3) + \left(A_4\frac{\sqrt{15}}{2}\right) = 5$$

$$-(5) + \left(A_4\sqrt{15}\right) = 10$$

$$\therefore\ A_4 = \frac{15}{\sqrt{15}}$$

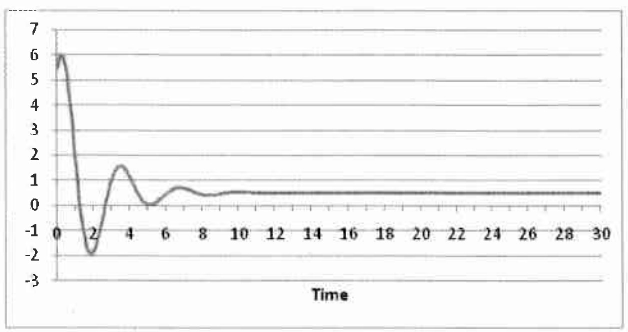
$$= \sqrt{15}$$

Therefore, the definite time path is

$$y_t = e^{\frac{-1}{2}t}\left(5\cos\left(\frac{\sqrt{15}}{2}t\right) + \sqrt{15}\sin\left(\frac{\sqrt{15}}{2}t\right)\right) + \frac{1}{2}$$

The graph is shown in Figure 10.5 below.

Figure 10.5: An oscillatory time path plot



The time path converges since the power on the exponential is negative. This does not come at any surprise. The coefficient $a_1 = 1 > 0$ implying that $h < 0$. Since convergence hinges on $h < 0$, we could conclude right away that the time path will be convergent.

Sometimes, given an n^{th} order differential equation, it may not be easy to find the time path. The characteristic equation will be an n^{th} degree polynomial. This gives many possibilities on the outcome. There may be a mix of repeated roots, distinct roots as well as complex conjugates. The problem of the higher degree characteristic equation is not an easy one. Nonetheless, it is still possible to ascertain the convergence or divergence of a time path without necessarily finding the roots. This is done on the basis of the Routh theorem. It was named after an English mathematician Edward John Routh. It states that for a normalised ($a_0 = 0$) characteristic polynomial

$$a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n = 0$$

the real parts of all the roots are negative if and only if the first n of the following sequence of determinant are all positive.

$$|a_1|, \quad \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

The theorem makes it possible to determine the convergence of a time path without actually finding the roots. Once the above condition is satisfied, we conclude that all the real parts (h) of the roots are negative. This is a sufficient test for convergence.

Example 10.12

Determine whether the time path of the following differential equation will diverge or converge to long run equilibrium. $y_t'''' + 6y_t''' + 14y_t'' + 16y_t' + 8y = 24$

The characteristic root for a forth order differential equation is of the form

$$a_0r^4 + a_1r^3 + a_2r^2 + a_3r + a_4 = 0$$

for the particular example, the characteristic equation is

$$r^4 + 6r^3 + 14r^2 + 16r + 8 = 0$$

Without actually finding the roots, we straight away formulate the sequence of determinants based on Routh theorem.

$$|a_1| = |6| = 6,$$

$$\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} 6 & 16 \\ 1 & 14 \end{vmatrix} = 68,$$

$$\begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 6 & 16 & 0 \\ 1 & 14 & 8 \\ 0 & 6 & 16 \end{vmatrix} = 800,$$

$$\begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix} = \begin{vmatrix} 6 & 16 & 0 & 0 \\ 1 & 14 & 8 & 0 \\ 0 & 6 & 16 & 0 \\ 0 & 1 & 14 & 8 \end{vmatrix} = 6400$$

All the principal determinants are positive. The time path is convergent.

10.9 Economic applications

Higher order differential equations are very useful in understanding the behaviour of certain markets. Take for instance a market governed by expectations. Either the demand or the supply of a commodity will not only depend on the current market price or what price is expected to be at a particular time. The long term behaviour of the price becomes critical. Players in the market are interested in knowing whether the price is on an increase or now ($\frac{dP}{dt} = P'$). Secondly, players also take into consideration whether price is changing (increasing or decreasing) at an increasing or decreasing rate ($\frac{d^2P}{dt^2}$).

Take a specific example of the agriculture sector. The interplay of the demand and supply will determine the ruling price. The quantity demanded will then be determined by the market price since each consumer will buy based on the price obtaining at that particular moment. Thus

$$Q^D = \alpha - \beta P, \qquad \alpha, \beta > 0$$

The supply side is however characterised by a long gestation period. There is a time lag between the time one decides to produce and the time the produce is actually offered for sale on the market. At the time of making the decision, the price that will obtain on the market in unknown to the producer. Moreover, the producer will also be interested in future prices because the decisions to invest in certain fixed assets take into account the life time value of such assets. For instance, growing tobacco requires investing in *tobacco kilns* and the farmer may not profit from such in one period only. The asset is available for a much longer period so the farmer must consider the price of tobacco for the life-time of the kilns.

To make a rational decision, the farmer decides on the quantity to supply (based on how much to invest) based on the knowledge of the behaviour of the price. For a simple model, assume the farmer only cares to know how the price changes and the nature of that change. Hence

$$Q^S = -\gamma + \delta P + \theta P' + \lambda P'', \qquad \gamma, \delta > 0$$

While there are restrictions on γ and δ based on traditional supply theory, no restrictions are placed on the new parameters θ and λ.

With the given demand and supply schedules, the equilibrium equation is expressed as

$$\alpha - \beta P = -\gamma + \delta P + \theta P' + \lambda P''$$

$$\alpha + \gamma = (\delta + \beta)P + \theta P' + \lambda P''$$

$$P'' + \frac{\theta}{\lambda}P' + \frac{(\delta + \beta)}{\lambda}P = \frac{\alpha + \gamma}{\lambda}$$

This is a second order differential equation. The aim here is to just show how higher (second) order differential equations can be used in economics. The solution, which we do not attempt to solve, will depend on the actual values of the parameters in the equation.

Notice that the order was restricted by the assumption that the producer only cares to know up to the second differential. With complicated price functions however, third, fourth up to the n^{th} order differential equations emerge. The Taylor's formula is useful in understanding the link between the order of the differential equation and the nature of the price function.

10.10 Simultaneous differential equations

The word simultaneous is no longer new. Its prominence is in finding the point of intersection of two or more functions. It is synonymous to the word concurrence which implies togetherness. In this context, it is used when two or more differential equations have a relationship or are interlinked. These may be rightly referred to as *interacting patterns of change*. This is common with variable that are related so that a change in one variable is not Independent of another. Their changes are interrelated.

For this kind of variables, their differential equations form a set of equations. Since each differential equation has more than one variable, it is then not possible to find its solution, independent of other equations. The number of unknowns exceeds the number of equations. Recall that in order for a system of equations to be solved simultaneously, the number of equations must at least be as many as the number of unknowns or variables. This is what is referred to as *simultaneous differential equations*.

The main area of focus is the analysis of a system of simultaneous dynamic equations or interacting patterns of change. Take for instance a multi-sector model where each sector is described by a differential (dynamic) equation which impinges on at least one of the other sectors. The specific example may be sectors like the industry whose dynamic equation cannot be independent of the education sector, let alone the health. The health sector also relies on the education sector for the personnel. Growth in agriculture cannot be independent of the performance of the industrial sector and *vice-versa*. The former depends on the latter for equipment and the output from the Agriculture is raw material for the industry. With this kind of model, a single dynamic equation will contain more than one variable, in dynamic form because the different sectors are interlinked or interacting. To find the time path of each sector's output, all the interacting equations must be solved simultaneously.

Consider a model with two sectors. The sectors are represented by x and y and the respective differential equations are given as

$$2x_t' + y_t' + 2x_t + y_t = 14$$
$$5y_t' + x_t + 3y_t = 12$$

This set of equation can be cast in matrix notation of the form

$$Ju + Mv = g$$

where J and M are coefficient matrices, u and v are vectors of variables and g is a vector of constants. Taking this form, the various matrices are defined as

$$J = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}; u = \begin{bmatrix} x_t' \\ y_t' \end{bmatrix}; M = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}; v = \begin{bmatrix} x_t \\ y_t \end{bmatrix}; g = \begin{bmatrix} 14 \\ 12 \end{bmatrix}$$

Just as in independent differential equation, the first step to dealing with interacting differential equations is to find the particular solutions. Since there are two equations for presumably two sectors, there must be found two particular solutions x_p and y_p. To find these, we try with the solution that the two variables are constant, that is $x_t = \bar{x}$ and $y_t = \bar{y}$. In the single variable case, this would equate to the term b. With more than one variable however, the term in each equation is a combination of the two variables. With an assumption that the variables are constant, the first derivatives equal to zero. That is, $x_t' = y_t' = 0$

The equation is now reduced to

$$Mv = g$$
$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 14 \\ 12 \end{bmatrix}$$

This is a set of simultaneous equation. Various methods are available on how to solve the equation. The actual details are presented in Chapter 4 of this book. Here we simply state the results as

$$\begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

For the complementary solution, assume a reduced equation. This is achieved by simply getting rid of non-zero constants or terms in the complete equations. Then take trial solutions

$$x_t = me^{rt}, \qquad y_t = ne^{rt}$$

The two variables are of the same pattern but only differ in the coefficient. One rises or falls faster than another depending on how the two arbitrary coefficients compare. The respective derivatives are

$$x_t' = rme^{rt}, \qquad y_t' = rne^{rt}$$

In matrix form, the two variables and the respective derivatives can be summarised as $v = \begin{bmatrix} m \\ n \end{bmatrix} e^{rt}$, $u = \begin{bmatrix} m \\ n \end{bmatrix} re^{rt}$ Substituting these into the reduced equation $Ju + Mv = 0$ will give

$$J\begin{bmatrix} m \\ n \end{bmatrix} re^{rt} + M\begin{bmatrix} m \\ n \end{bmatrix} e^{rt} = 0$$
$$(Jr + M)\begin{bmatrix} m \\ n \end{bmatrix} e^{rt} = 0$$
$$(Jr + M)\begin{bmatrix} m \\ n \end{bmatrix} = 0$$

Since the exponential is unambiguously non-zero, its multiplicative inverse can be multiplied on both sides to eliminate it from the equation. The remnant is similar to the problem encountered under independent differential equations. There are two factors whose product is zero. In matrix however, a zero product does not imply that one of the factors is zero. Here it implies that the coefficient matrix, now $(Jr + M)$, is singular. In algebra, singularity of a matrix is indicated by a zero determinant. Therefore, the resulting equation is

$$|Jr + M| = 0$$

Since the two matrices J and M are known, we can proceed to find the value of r. This becomes

$$\left| r\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \right| = 0$$
$$\left| \begin{pmatrix} 2r & r \\ 0 & 5r \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \right| = 0$$
$$\begin{vmatrix} 2r + 2 & r + 1 \\ 1 & 5r + 3 \end{vmatrix} = 0$$

Using the formula for the determinant; we generate a non matrix equation

$$(2r + 2)(5r + 3) - 1(r + 1) = 0$$
$$10r^2 + 15r + 5 = 0$$

$$2r^2 + 3r + 1 = 0$$

This is a characteristic equation, similar to what was encountered earlier. Again we state the roots without showing the working as $r_1 = -1$, and $r_2 = -0.5$

With these characteristic roots, we solve the equation $(Jr + M)\begin{bmatrix} m \\ n \end{bmatrix} = 0$ for $\begin{bmatrix} m \\ n \end{bmatrix}$ corresponding to both roots. The solutions will not give specific values of the vector components. This because the coefficient matrix, with the given value of r, is singular. The solution will merely be a relationship between the two variables. For $r_1 = -1$,

$$\begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} m_1 \\ n_1 \end{bmatrix} = 0$$

$$m_1 = 2n_1$$

$$If \;\; n_1 = A_1, \qquad then \; m_1 = 2A_1$$

For $r_2 = -0.5$,

$$\begin{bmatrix} 1 & 0.5 \\ 1 & 0.5 \end{bmatrix} \begin{bmatrix} m_2 \\ n_2 \end{bmatrix} = 0$$

$$m_2 = -0.5n_2$$

$$If \;\; n_2 = 2A_2, \qquad then \; m_2 = -A_2$$

In general, given r_i, m_i and n_i, the complementary solutions are

$$x_c = \sum_i m_i e^{r_i t}$$

$$y_c = \sum_i n_i e^{r_i t}$$

in the particular case with $r = -0.5, -1$ the respective complementary solutions for the two variables are

$$\begin{aligned} x_c &= m_1 e^{r_1 t} + m_2 e^{r_2 t} \\ &= 2A_1 e^{-t} - A_2 e^{-0.5t} \end{aligned}$$

$$\begin{aligned} y_c &= n_1 e^{r_1 t} + n_2 e^{r_2 t} \\ &= A_1 e^{-t} + 2A_2 e^{-0.5t} \end{aligned}$$

Now both the particular solution and the complementary solution are known, with arbitrary constants for the latter. The general solution is the summation of the two. To keep with the matrix format, the summation is done as

$$\begin{aligned} \begin{bmatrix} x_t \\ y_t \end{bmatrix} &= \begin{bmatrix} x_c \\ y_c \end{bmatrix} + \begin{bmatrix} x_p \\ y_p \end{bmatrix} \\ &= \begin{bmatrix} 2A_1 e^{-t} - A_2 e^{-0.5t} \\ A_1 e^{-t} + 2A_2 e^{-0.5t} \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2A_1 e^{-t} - A_2 e^{-0.5t} + 6 \\ A_1 e^{-t} + 2A_2 e^{-0.5t} + 2 \end{bmatrix} \end{aligned}$$

This is the general solution. It is not definite since it still has arbitrary constants. The constants can be made definite if the initial conditions are known. For instance, assume that information is now available that $x_{t=0} = 2$, $y_{t=0} = 5$. The resulting linear equations are

$$2A_1 - A_2 + 6 = 2$$

$$A_1 + 2A_2 + 2 = 5$$

We leave it up to the reader to show or prove that $A_1 = -1$ and $A_2 = 2$. The definite solution then is

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} -2e^{-t} - 2e^{-0.5t} + 6 \\ -e^{-t} + 4e^{-0.5t} + 2 \end{bmatrix}$$

When time paths have expressions e^{rt}, the convergence or divergence will depend on the value of r. The time path converges when r is negative and diverges when positive. In the particular example at hand, r is throughout negative in both x_t and y_t. Therefore, both time paths converge to their respective equilibriums.

10.11 Economic applications

Simultaneous differential equations are applied in many aspects of mathematical economics. They provide useful tools for working with interrelated variables in a dynamic way. For instance, the inflation-unemployment model in which changes in either inflation or the level of unemployment cannot be isolated from the other variable. This is dealt with using the framework presented above, the simultaneous differential equations.

In general, the framework can also be used with more than two interlinked variables. This is ideal for an economy wide dynamic analysis where the fundamentals are assumed to change continuously. For the purpose of an example, take Leontief input-output model discussed in Chapter 4. The model is static in nature and assumes that there is clearance in all the sectors. If we relax the two assumptions and assume production is done continuously so that the model conforms to continuous time. Since markets do not clear, there must be a continuous adjustment of output. This adjustment will be a function of the deficit, that is, if there was a shortage, then there must be some increase in output and *vice-versa*.

Take a two sector economy. The output for sector one is used as an input in all the sectors as well as to meet the final demand for that particular output.

$$x_1 = a_{11}x_1 + a_{12}x_2 + d_1$$

where the left hand denotes the supply and the right hand denotes the demand. If output changes by exactly the difference, then the differential is written as.

$$x_1' = a_{11}x_1 + a_{12}x_2 + d_1 - x_1, \;\; where \; x_1' = \frac{dx_1}{dt}$$

the equation is already a differential equation. When the change in output is not exactly the difference, a coefficient can be attached to the difference to take care of the *less than* or *more than* deficit change. Rearrange by taking all variable terms to the left side of the equation. This gives the equation of a more familiar form.

$$x_1' + (1 - a_{11})x_1 - a_{12}x_2 = d_1$$

Similarly, the differential equation involving sector 2, that is, x_2 can be gotten. The two will form simultaneous differential equations whose time paths can be gotten using the framework illustrated earlier.

Chapter 11

11 ECONOMIC DYNAMICS IN DISCRETE TIME: DIFFERENCE EQUATIONS

11.1 Introduction

In the preceding chapter, we dealt with the behaviour of a variable over time taking time as a continuous variable. The change in the variable was assumed to occur continuously. In reality however, not all economic variables can be assumed to change continuously. For instance, while the exchange rate, as a price of one currency in terms of another, can be changing continuously when disturbed, the price of commodities like labour, seasonal crops only change in discrete time. Instead of taking time as a continuous variable, there is need to modify the model so that it takes into account cases where changes in the variable are not continuous. Such dynamics are in discrete time and make use of *difference equations* as opposed to *differential equations* studied earlier.

While in continuous time each time value represented a specific time point, in discrete time, time is taken as an interval or a period. If the price changes annually, then there would be year 1, 2, 3 and so on which represent periods of time. Instead of looking at the instantaneous change in the variable, this type of dynamics is more concerned with how a variable changes from one period to another. If the price of say tobacco was higher than the long run equilibrium, how will it adjust to equilibrium, if at all it reverts to equilibrium? This type of questions is best handled by difference equation.

11.2 What is a difference equation?

It is much easier to define the difference equation by first looking at their use. Difference equations are used for dynamic analysis where time is treated as a discrete (categorical) variable rather than a continuous variable. It is a tool for deriving the solution to equations where the current value of some variable depends upon the value of the same variable in earlier time periods. Thus a *difference equation* is one where a variable is a function of its past values. More precisely, it is an equation of the form

$$y_t = a + by_{t-1}$$

where the current value of the variable depends on the previous value denoted by a lagged time indicator.

The highest time lag between the explained and explanatory variable defines the order of the equation. For instance, in the above equation, the current value depends on the immediate past value giving a unit time lag. It is therefore of first order. Second, the equation is also linear in the variable. Combining the two traits gives it the name of first order linear difference equation.

The second order linear difference equation will have an added *two-time-lag* variable y_{t-2} in the equation. Recall that the order is determined by the *highest* time lag and will thus remain definite even with multiple lagged variables on the right hand side of the equation. In general, an n^{th} order linear difference equation is of the form

$$y_t = a + b_1 y_{t-1} + b_2 y_{t-2} + \cdots \cdots + b_n y_{t-n}$$

the highest time lag between the regressed and the regressands is n, giving the order of the equation.

11.3 First-order linear difference equations

Given the above definition of linear difference equations, the first order linear difference equation is then one given by

$$y_t = a + by_{t-1}$$

It portrays a case where a current value of a variable depends only on the immediate past values of the variable. The knowledge of last year's level of output tells us how much output will be produced this year. Or given this year's maize price, we can anticipate the price next year to be at a certain level. These expressions hinge on the first order linear difference equations, which provide a link between the current level and the immediate past level of a variable.

The solution of this order of linear difference equation can be found using a simple lagging process. It is known that the current value is a function of the immediate past. In the same way, moving one-time back, the immediate past value is a function of value preceding it. Using this link, the difference equation can be expanded up to the initial value $y(0)$.

The difference equation for this period is given by $y_t = a + by_{t-1}$. Using the same equation, we can derive a chain of first order linear difference equations corresponding to all the past periods. The equations are

$$\begin{aligned} y_t &= a + by_{t-1} \\ y_{t-1} &= a + by_{t-2} \\ y_{t-2} &= a + by_{t-3} \\ \cdots &= a + b\cdots \\ y_1 &= a + by_0 \end{aligned}$$

This chain of equations is such that every equation can be substituted into the equation immediately on top. Substituting in this pattern will give rise to an equation of the form

$$y_t = a + ab + ab^2 + ab^3 + \cdots\cdots + ab^{t-1} + b^t y_{t-t}$$

obviously $y_{t-t} = y_0$. Like in the differential equation case, the above equation can also be divided into two parts. The first, composing of constants only, is the particular sum. The term *sum* is used here because in discrete time, summation is used in place of integration. Let's denote this by y_p which is given by $y_p = a + ab + ab^2 + ab^3 + \cdots\cdots + ab^{t-1}$. This is a *geometric progression* with the initial term a and the common ratio b. Its finite sum is given by

$$y_p = S_t = \frac{a(1-b^t)}{1-b}, \qquad b \neq 1$$

The second part is one with the lagged variable in it. In the continuous time case, it was referred to as a complementing sum, *mutatis mutandis*. We can denote it with $y_c = b^t y_0$ and requires no further modifications.

To derive the general solution of the first order difference equation, we sum the two components of the general sum. This process can be summarised as below

$$\begin{aligned} y_t &= y_p + y_c \\ &= \frac{a(1-b^t)}{1-b} + b^t y_0 \\ &= \frac{a}{1-b} - \frac{ab^t}{1-b} + b^t y_0 \\ y_t &= \left(y_0 - \frac{a}{1-b}\right) b^t + \frac{a}{1-b}, \qquad b \neq 1 \end{aligned}$$

The equation has three components which deserve attention. These are $\left(y_0 - \frac{a}{1-b}\right)$, b^t and $\frac{a}{1-b}$. It is easier to understand the three if we start with the third. The third component gives the long run equilibrium of the variable y. When the long run equilibrium is attained, the variable stabilised so that it does not change. Given the primary difference equation $y_t = a + by_{t-1}$, at long run equilibrium will reduce to $y = a + by$. The time subscript disappears because the variable no longer changes with time. The solution of the equation is $y = \frac{a}{1-b}$ which gives the long run equilibrium level.

The first component can now be defined using the definition of the third. It is the deviation of the initial level from the long run equilibrium level. In other words, it shows how far the starting point is from the long run equilibrium. Since this is a one-observable point, it is not expected to change, the reason it is not connected to time. It can be looked at as a measure of the magnitude of the disturbance on the variable. If for instance price is disturbed from its long run level, the displacement of the price from its long run level gives an idea of how big the disturbance was. Depending on whether the value falls or rises, this component which appears in the equation as a coefficient of the second component, can take on both negative and positive value. When it is zero, the variable is at its long run equilibrium and the difference equation collapses to the long run equilibrium level.

The second component shows how the variable is linked to time. The example behaviour of the variable with time can be told depending on the value of the base b since time t is in the exponent. When the absolute value of the base is less than unit, the component approaches zero as time approaches infinite. In this way, the variable is said to be dynamically stable since it always reverts back to equilibrium after a disturbance. This behaviour can be summarised as follows.

Table 11.1. The Significance of b

Base b	**Convergence**	**Oscillatory**	**Behaviour**
$b < -1$	Divergent	Oscillatory	Oscillating divergence
$-1 < b < 0$	Convergent		Dampened oscillatory path
$0 < b < 1$	Convergent	Non oscillatory	Smooth convergence
$b > 1$	Divergent		Smooth divergence

Example 11.1

Solve $y_t = 3 - 0.5y_{t-1}, \ y_0 = 4$

This is a first order linear difference equation. Its solution comprises of two parts: the particular solution and the complementary solution. The general solution is given by

$$y_t = \left(y_0 - \frac{a}{1-b}\right)b^t + \frac{a}{1-b}, \qquad b \neq 1$$

In the current example, $a = 3$ and $b = -0.5$. Substituting these parameters into the general solution yields

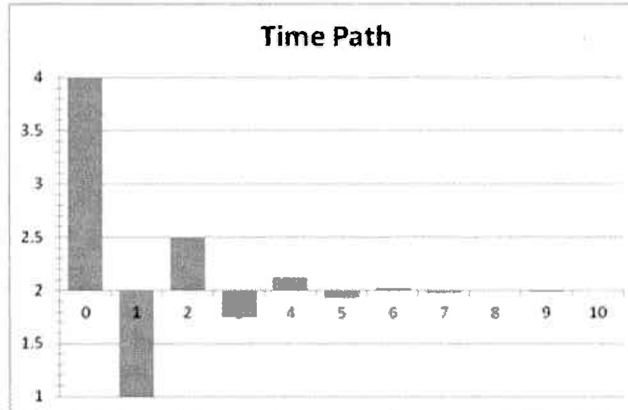
$$\begin{aligned} y_t &= \left(y_0 - \frac{3}{1-(-0.5)}\right)(-0.5)^t + \frac{3}{1-(-0.5)} \\ &= (4-2)(-0.5)^t + 2 \\ &= 2(-0.5)^t + 2 \end{aligned}$$

All the parameters of the solution are definite. Therefore, the solution is also definite.

We now comment on the above solution in line with Table 11.1 above. The solution is a sum of two components. The first varies with time and the second is constant. Our interest lies in the first part which determines the behaviour of the time path. Firstly the base is negative. The implication is that this part of the solution will be positive for *even numbered* time periods and negative for *odd numbered* time periods. That means there will be oscillations around the constant.

Second, the base is also less than unit. As time t becomes larger (with the passage of time) $(-0.5)^t$ will tend to zero. Ultimately, the time path will just equal the constant. Thus there is convergence to constant long run equilibrium. See Figure 11.1 below.

Figure 11.1: An oscillatory time path[41]



Example 11.2

Solve $y_t = 16 + 3y_{t-1}, \ y_0 = 5$ and determine the nature of the time path.

The previous example did not derive the solution but merely used the provided formula. In this example however, we provide the derivation. The process is as follows.

$$\begin{aligned} y_t &= 16 + 3y_{t-1} = 16 + 3(16 + 3y_{t-2}) \\ &= 16 + 3(16) + 3^2 y_{t-2} = 16 + 3(16) + 3^2(16 + 3y_{t-2}) \\ &= 16 + 16(3) + 16(3^2) + 3^3 y_{t-3} \\ &= 16 + 16(3) + 16(3^2) + \cdots + 16(3^{t-1}) + 3^t y_0 \end{aligned}$$

Using the geometric series, the first t terms can be summarised as the geometric series with $a = 16$ and the common ratio $r = 3$.

$$\begin{aligned} y_t &= 16\frac{(3^t - 1)}{3 - 1} + 3^t y_0 \\ &= 8(3^t - 1) + 3^t y_0 \end{aligned}$$

[41] The bars show deviations from the long run equilibrium as opposed to actual level

$$= (y_0 + 8)3^t - 8$$
$$= (5 + 8)3^t - 8$$
$$= 13(3)^t - 8$$

The base in the time path is positive and greater than unit. Therefore, the time path is *nonoscillatory* and explosive in nature. It never reverts to any particular value as time grows.

11.4 Economic applications

The application of difference equations in economic problems is enormous. They can be applied to any area of economics where a variable has an autoregressive process. This is where the variable is a function of its lagged values. Though these may not manifest to a form that is easily identifiable as difference equation, some modification and transformation may make them look so. We put the question, 'can the given equation be expressed in a form that shows a difference equation?'

Take an example of the agriculture sector where a particular crop like cotton or tobacco can be taken. Their production is seasonal with a complete cycle corresponding to a year. This makes time discrete. At any particular time (year), there is equilibrium because the price can adjust to equate demand and supply. The demand for the product is not different from the ordinary market where it adjusts with the price. The supply on the other hand does not conform to traditional supply behaviour. Because of gestation lags between decisions to plant and subsequent harvest, the supply decision are not altered by current price but depend on price expected when the crop is harvested.

This brings in the role of expectations. Depending on how expectations of the harvest price are formed, the decision to supply will be made. Given the two types of expectations; *rational* and *adaptive*, and the complications of the former, it is acceptable to assume that farmers use the latter. This is based on assuming the low level of technology and capacity among the farming communities in most developing countries to make use of the former. Besides, this assumption is outside the realm of this book and it must suffice to take it as an assumption.

With adaptive expectations at play, the price is expected to be the same as in the previous period. The shorter form is to say the supply of a commodity will be a function of the previous price. The long run equilibrium is reached only when expected price at time t is equal to actual price at time t.

Assume demand and supply of say tobacco is given by

$$Q^d{}_t = \alpha - \beta P_t, \qquad (\alpha, \beta > 0)$$

$$Q^s{}_t = -\gamma + \delta EP_t, \qquad (\gamma, \delta > 0)$$
$$EP_t = P_{t-1}$$

At equilibrium, $Q^d{}_t = Q^s{}_t$, a market clearing condition. The constant long run equilibrium price is given by.

$$\bar{P} = \frac{\alpha + \gamma}{\beta + \delta}$$

This example is similar to the one given in continuous time case. The difference though is that now time has become important since there is time variation in the model. This was not the case in the previous case. Solving for price in the above system of equation will produce

$$\alpha - \beta P_t = -\gamma + \delta P_{t-1}$$
$$P_t = \frac{\alpha + \gamma}{\beta} - \frac{\delta}{\beta} P_{t-1}$$

this equation corresponds to the general form given above. Using the solution of the general form, we substitute $\frac{\alpha+\gamma}{\beta} = a$ and $-\frac{\delta}{\beta} = b$ to arrive at the particular solution

$$P_t = \left(P_0 - \frac{\alpha + \gamma}{\beta + \delta}\right)\left(-\frac{\delta}{\beta}\right)^t + \frac{\alpha + \gamma}{\beta + \delta}$$

This is a particular solution for the linear difference equation looking at the price of an agricultural product. Given this particular solution, there is need to determine the behaviour of price based on the dictates on Table 11.1 above. It must be clear from the solution that the question of convergence or non convergence and whether there is oscillation or not are dependent on the slopes of the demand and supply functions. The slopes are denoted by β and δ respectively. These parameters are defined positive for downward sloped demand and upward sloped supply. It means therefore that for normal demand and supply function, the ultimate b in the solution is negative. This means the price, when disturbed, will oscillate around its long run equilibrium level.

The issue of convergence remains trivial. The information at hand is not sufficient to ascertain whether there will be convergence or not. In the above solution, convergence will require the absolute value of the base to be less than unit. This will hold if and only if $\frac{\delta}{\beta} < 1 \Rightarrow \delta < \beta$ this simplifies to having the demand schedule being steeper than that of the supply schedule. When the opposite hold, there will be divergence and the market will be dynamically unstable.

The following graphs that show variations of the *cobweb model* illustrate the above situations of convergence and non-convergence.

Figure 11.2. Cobweb model: Case of convergence

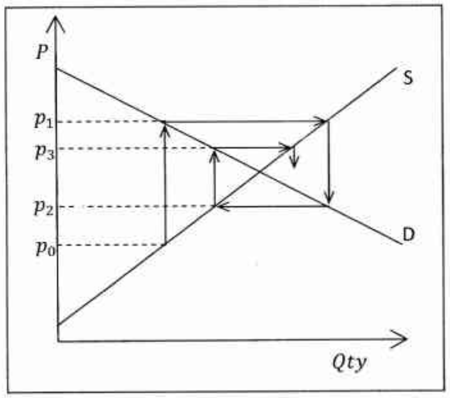


Figure 11.3. Cobweb model: Non convergence and non divergence

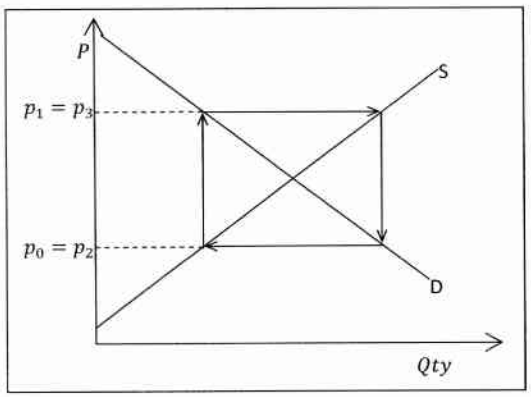
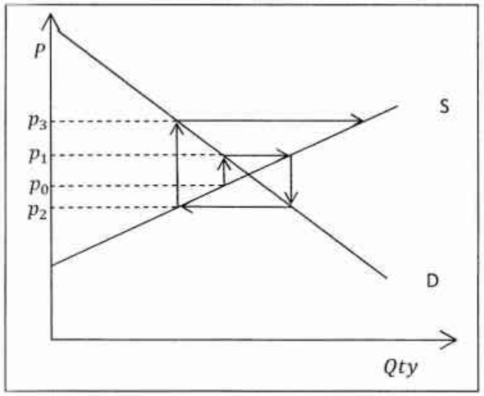


Figure 11.4. Cobweb model: Case of divergence



11.5 Second-order difference equations

The second order linear difference equations are difference equations where the current level is a function of two preceding values; the immediate past and the one preceding the immediate past. It is a situation where the current price for example is a function of the price two years ago. Such equations are common in situations where the effects of a disturbance take time to disappear. In particular, second order linear difference equations apply where the recovery takes two periods. The time lag on the regressands extends to two. In notation, they are written in the form

$$y_t = \alpha + \beta_1 y_{t-1} + \beta_2 y_{t-2}$$

the parameters in the model are arbitrary. Though this may sound ordinary, it has a strong implication on how the parameters can be treated in algebra. This is precisely what is about to take place. As arbitrary numbers, changing their signs does not affect the model! We assume the reader is familiar with this and we proceed to change the signs on the coefficients of the lagged variables. The motivation is to change the form of the model. After changing the signs, bring all the variables on one side so that their sum equates to a constant.

This yields the model of the form.

$$y_t + \beta_1 y_{t-1} + \beta_2 y_{t-2} = \alpha$$

The model remains unchanged except for the presentation. It must be mentioned here that this presentation does not invalidate the former; rather it facilitates a quicker and easy way to derive the general solution. The equation is quite complex and deriving the solution needs a well thought strategy. By strategy, it implies that what we apply is not the sole method but rather what we consider candid.

Like in the continuous time case, this equation can be looked at in two forms; the homogeneous and non-homogeneous. The homogeneous part is found by letting the constant, on the right hand side of the equation, equal to zero. This results in a homogeneous equation of the form

$$y_t + \beta_1 y_{t-1} + \beta_2 y_{t-2} = 0$$

The equation does not have a unique route to the solution but can be solved by trying different possible forms of the solution. This is often referred to as a *trial and error* method. We now try $y_t = Am^t$ as the solution to the equation. Then substitute this assumed solution into the equation, lagging appropriately for the lagged variables. The equation that emerges is

$$Am^t + \beta_1 Am^{t-1} + \beta_2 Am^{t-2} = 0$$

This equation can be simplified since Am^{t-2} is common in all the terms. After this simplification, the equation turns to

$$m^2 + \beta_1 m + \beta_2 = 0$$

This is an ordinary quadratic equation. Since the problem is ordinary, the solution need not be special. The solution of a quadratic equation is given by the formulae

$$m = \frac{-\beta_1 \pm \sqrt{\beta_1{}^2 - 4\beta_2}}{2}$$

and the general solution of the homogeneous part of the difference equation is given by

$$y_t = A_1 m_1{}^t + A_2 m_2{}^t$$

The two bases m_1 and m_2 are the two roots of the quadratic equation above. A_1 and A_2 are two arbitrary constants. Once definitised, the solution will be definite. The homogeneous equation does not have the particular solution so that the solution of the complementary function is the general solution. When we move to non homogeneous functions however, the solution above will not be final, it will be a solution of the complementary function. At that stage, it will become necessary to add a superscript identifier to the solution. With a homogeneous equation, this is not critical.

But the solution to the quadratic equation come in three forms depending on the value of the discriminant $D = \beta_1{}^2 - 4\beta_2$. These are: *distinct real roots* $m_1 \neq m_2$; the *repeated real roots* $m_1 = m_2$ and the *complex conjugates* where the roots are a pair of *nonreal* numbers. We use examples to illustrate the three cases in succession.

11.5.1 Distinct real roots case

Example 11.3

Find the time path of the following difference equation: $y_t - y_{t-1} - 2y_{t-2} = 0$

We start the solution by trying the solution of the form $y_t = Am^t$ which reduces the difference equation above to a quadratic equation of the form $m^2 - m - 2 = 0$ The quadratic equation has two distinct real roots $m = -1$ and 2. Given the two values of m, substitute them into the general solution equation $y_t = A_1 m_1{}^t + A_2 m_2{}^t$. This gives the time path;

$$y_t = A_1(-1)^t + A_2(2)^t$$

Since there are now two arbitrary constants to be made definite, an initial condition will not be sufficient. This follows the general rule of equations that there must be at least as many equations as there are unknowns. To make definite the two constants, two conditions must be known, which can be used to generate two simultaneous equations. These conditions need not include the initial condition but can be at any time period.

For instance, suppose it is known that $y_1 = 11$ and $y_2 = 13$. The arbitrary constants can be made definite using the derived equations

$$y_1 = 11 = -A_1 + 2A_2$$
$$y_2 = 13 = A_1 + 4A_2$$

Solving the two simultaneously gives $A_1 = -3$ and $A_2 = 4$. The definite solution or time path is

$$y_t = -3(-1)^t + 4(2)^t$$

this is a definite solution because all the parameters are definite. Once time is known, the level of the variable at hand can be known as well.

11.5.2 Repeated real roots case

Example 11.4

Given the difference equation $y_t + 1.2y_{t-1} + 0.36y_{t-2} = 0$ and that $y_0 = 8$, $y_1 = 7.2$, find the time path y_t.

The solution follows the same path as in Example 11.3 above. First use the solution form $y_t = Am^t$, simplify the equation to remain with a quadratic equation which is solved for the two values m_1 and m_2. The first three steps will proceed as follows.

$$Am^t + 1.2Am^{t-1} + 0.36Am^{t-2} = 0$$

which after dividing by the highest common multiple of all the terms $36Am^{t-2}$, the equation reduces to a quadratic form.

$$m^2 + 1.2m + 0.36 = 0$$

Using the formula for a quadratic equation, the roots turn out to be $m_1 = m_2 = -0.6$. With repeated roots at hand, does $y_t = A_1 {m_1}^t + A_2 {m_2}^t$ still apply?

Well, let's see. When modified for $m_1 = m_2 = m$, it will still apply. The equation is factorised for the common root as $y_t = (A_1 + A_2)m^t$. Since the two parameters are arbitrary, their sum will equally be arbitrary and can be denoted by A so that the solution equation to use now is

$$y_t = Am^t$$

Before going any further, this solution implies that $y_t = my_{t-1}$, a first order homogeneous difference equation. The solution differs from the original equation, which is of second order. Thus the form $y_t = A_1 {m_1}^t + A_2 {m_2}^t$ does not generally apply to repeated root case.

When there are repeated roots, the solution of the function takes the form

$$y_t = A_1 m^t + tA_2 m^t$$

And substituting the root,

$$y_t = A_1(-0.6)^t + tA_2(-0.6)^t$$

Using two known condition y_0 and y_1, the arbitrary constants are made definite using

$$y_0 = A_1$$
$$y_1 = -0.6A_1 - 0.6A_2$$

The first equation is a complete definitising equation for A_1. The reader should be able to show that the remaining constant is

$$A_2 = \frac{-y_1 - 0.6y_0}{0.6}$$

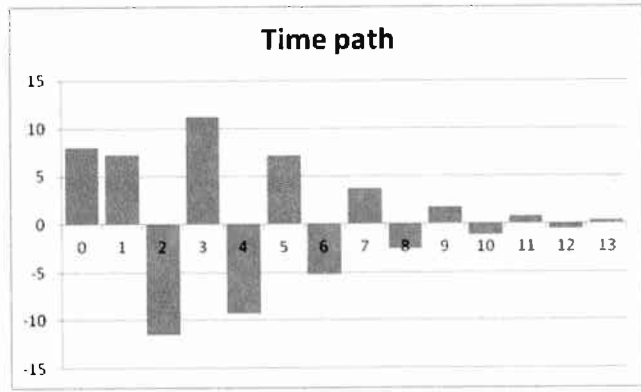
so that the definite solution is expressed as

$$y_t = y_0 m^t + t\left(\frac{-y_1 - 0.6y_0}{0.6}\right)m^t$$

with $y_0 = 8$, $y_1 = 7.2$ given, the particular solution for the problem at hand is

$$y_t = 8(0.6)^t + t\left(\frac{-7.2 - 0.6(8)}{0.6}\right)(0.6)^t$$
$$y_t = 8(-0.6)^t - 20t(-0.6)^t$$
$$= 4(2 - 5t)(-0.6)^t$$

Figure 11.5: Oscillatory time path



11.5.3 The complex conjugate case

When the discriminant of the characteristic equation is negative, the characteristic equation does not have real roots. Roots can nonetheless still be found in another domain. This is the complex number domain. The characteristic equation will produce complex conjugates. This is a pair of complex numbers with an equal real number part and an imaginary part of equal magnitude but of opposite signs.

We are looking for the two roots given by

$$m = \frac{-\beta_1 \pm \sqrt{{\beta_1}^2 - 4\beta_2}}{2}$$

This time around however, ${\beta_1}^2 - 4\beta_2 < 0$ so that it is not possible to get a square roots. We rewrite this inequality as $(-1)(4\beta_2 - {\beta_1}^2) < 0$ the second factor is now positive. Placing this into the quadratic formula, we get

$$m = \frac{-\beta_1 \pm \sqrt{(-1)(4\beta_2 - {\beta_1}^2)}}{2}$$
$$= \frac{-\beta_1}{2} \pm \sqrt{-1}\frac{\sqrt{4\beta_2 - {\beta_1}^2}}{2}$$

$$= \frac{-\beta_1}{2} \pm i\frac{\sqrt{4\beta_2 - {\beta_1}^2}}{2}$$
$$\equiv h \pm vi$$

The two roots can then be substituted into the general solution formula

$$y_c = A_1 {m_1}^t + A_2 {m_2}^t$$

Homogeneous difference equations do not have the particular solution. As such, the complementary solution is in fact the general solution.

$$y_t = A_1(h + vi)^t + A_2(h - vi)^t$$

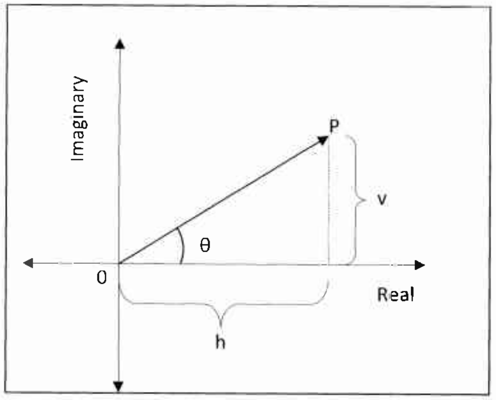
Define R, the magnitude of the complex number as $R = \sqrt{h^2 + v^2}$. The above equation can then be written, in view of the defined magnitude of the roots, as

$$y_t = A_1 R^t \left(\frac{h}{R} + \frac{v}{R} i\right)^t + A_2 R^t \left(\frac{h}{R} - \frac{v}{R} i\right)^t$$

The effect of the R is neutral. It has the same power in the numerator as well as in the denominator. This would ultimately cancel out but we have an interest in this format.

The solution requires raising a complex number to some power t. A similar scenario was encountered in Chapter 10 when second order differential equations produce complex conjugates as roots to the characteristic equation. An available option is to use trigonometric functions. Since a complex number is a number in two dimensions, it can be presented in Euclidean plane as shown below.

Figure 11.6: Argand diagram (Difference equations)



Given the angle θ and the SOH-CAH-TOA[42] mnemonic, it is possible to link the trigonometric function with the parameters of the complex number. The length of the vector OP was defined as $R = \sqrt{h^2 + v^2}$ using the Pythagoras theorem. Trigonometry states that

$$\cos\theta = \frac{h}{R}$$
$$\sin\theta = \frac{v}{R}$$

Substitute the two equations into the solution

$$y_t = A_1 R^t \left(\frac{h}{R} + \frac{v}{R} i\right)^t + A_2 R^t \left(\frac{h}{R} - \frac{v}{R} i\right)^t$$
$$y_t = A_1 R^t (\cos\theta + i\sin\theta)^t + A_2 R^t (\cos\theta - i\sin\theta)^t$$
$$y_t = A_1 R^t (\cos\theta t + i\sin\theta t) + A_2 R^t (\cos\theta t - i\sin\theta t)$$
$$y_t = R^t [(A_1 + A_2)\cos\theta t + i(A_1 - A_2)\sin\theta t]$$
$$y_t = R^t (A_3 \cos\theta t + A_4 \sin\theta t)$$

Example 11.5

Find the time path for the difference equation $y_t - y_{t-1} + \frac{5}{2} y_{t-2} = 0$, $y_0 = 2$ and $y_1 = 7$

[42] Sine equals Opposite over Hypotenuse, Cosine equals Adjacent over Hypotenuse, Tangent equals Opposite over Adjacent (SOH-CAH-TOA).

We make use of the general solution form $y_t = Am^t$. The difference equation transforms to

$$Am^t - Am^{t-1} + \frac{5}{2}Am^{t-2} = 0$$

the resulting characteristic equation is $m^2 - m + \frac{5}{2} = 0$ since the discriminant is negative, the roots will be complex numbers

$$m = -\frac{b}{2} \pm i\frac{\sqrt{4ac - b^2}}{2}$$

$$= \frac{1}{2} \pm i\frac{\sqrt{4(2.5) - (-1)^2}}{2}$$

$$= \frac{1}{2} \pm \frac{3}{2}i$$

To get the solution, two parameters are needed. The first is the magnitude of the complex number R and the second is angle θ. By definition,

$$R = \sqrt{h^2 + v^2}$$

$$= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$$

$$= \sqrt{2.5}$$

Given the two dimensions of the complex number, we can proceed to get the parameter θ using the *Tan* function. This function, unlike the Cosine and Sine, does not require the magnitude of the number. It only relies on the imaginary and real dimensions.

$$\operatorname{Tan}\theta = \frac{v}{h} = \frac{\frac{3}{2}}{\frac{1}{2}}$$

$$\Rightarrow \theta = \tan^{-1} 3 = 1.249$$

The time path then is

$$y_t = \sqrt{2.5}^t\,[A_3 \cos(1.249)t + A_4 \sin(1.249)t]$$

Using the two initial conditions, we can find the particular solution as follows

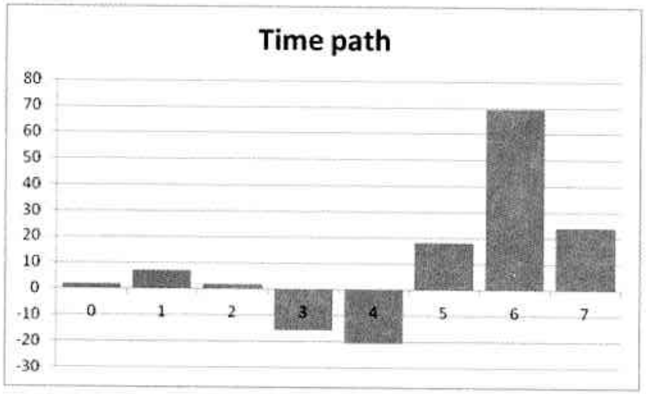
$$y_0 = [A_3] = 2$$

$$y_1 = \sqrt{2.5}[2\cos(1.249) + A_4 \sin(1.249)] = 7$$

$$A_4 \sin(1.249) = \frac{7}{\sqrt{2.5}} - 2\cos(1.249)$$

$$A_4 = \frac{\frac{7}{\sqrt{2.5}} - 2\cos(1.249)}{\sin(1.249)} = 4$$

$$y_t = \sqrt{2.5}^t\,[2\cos(1.249)t + 4\sin(1.249)t]$$



The above figure shows the time path for the particular solution $y_t = \sqrt{2.5}^t\,[2\cos(1.249)t + 4\sin(1.249)t]$. The time path oscillates around some long run averages and is divergent in nature. Though oscillatory in nature, the deviations from the long run average increase with time.

Suppose the assumption of homogeneity is relaxed, what happens? The assumption is relaxed when the constant in the difference equation is allowed to assume values other than zero. This is called the *non-homogeneous* second order linear difference equation. Having found solution for the former, the solution for the latter is pretty simple. Recall that the solution to the former is but a solution to the complementary part of the latter. When the non-homogeneous equation is divided into two parts, the complementary and particular sum, the solution of the complementary is already known through the homogeneous case. The particular component of the equation is found by assuming the variable is constant.

For the difference equation

$$y_t + \beta_1 y_{t-1} + \beta_2 y_{t-2} = \alpha$$

the particular component requires equating all the variables, the present and lagged variables. If we set $y_t = y_{t-1} = y_{t-2}$, the parent equation can be solved for the solution to the particular equation. The solution will be

$$y_t{}^p = \frac{\alpha}{1+\beta_1+\beta_2}$$

which holds for as long as $1+\beta_1+\beta_2 \neq 0$

The solution to the homogeneous case, which is also the solution to the complementary equation, will still depend on the kind of roots that come out of the characteristic equation. It is now denoted as $y_t{}^c$ where the *superscript* as earlier mentioned, indicate that it is a solution to the complementary equation. The procedure however remains the same as in the homogeneous case alone. The general solution is found by summing the two components as in the continuous time case studied in the earlier chapter.

$$y_t = y_t^c + y_t^p$$

We modify the above two homogeneous case examples to illustrate the non-homogeneous case. The examples are modified by adding a non-zero constant to the difference equation. The solutions to the complementary parts of the equations will not change and so will be taken as already found. For the particular part of the equation, the solution only varies with the selected constant α.

Example 11.6

Find the time path of the following difference equation: $y_t - y_{t-1} + 2y_{y-2} = 4$

This example is coming from Example 11.3, where the complementary solution was found as

$$y_t{}^c = A_1(-1)^t + A_2(2)^t$$

Recall that the solutions of the two components must never be in definite form. Definitising is the last procedure, after combining the two components. With $\alpha = 4$, the solution to the particular part is

$$y_t{}^p - y_t{}^p + 2y_t{}^p = 4$$

$$y_t{}^p(1-1+2) = 4$$

$$y_t{}^p = \frac{4}{2} = 2$$

The complete solution is

$$y_t = y_t{}^c + y_t{}^p$$

$$= A_1(-1)^t + A_2(2)^t + 2$$

This is an indefinite solution because it still has arbitrary constants. If there was information for some two periods, the constants could be made definite so that the solution is also definite. To use the information available to Example 11.3, two options are available. The first one is to maintain $y_1 = 11$ and $y_2 = 13$ which will result in a new pair of constants. The second option is to realise that changing the constant from zero to four is equivalent to having all observations go up by four. New available information will then be $y_1 = 15$ and $y_2 = 17$. This will leave the constants unchanged.

The second option is used, as it gives an opportunity to prove the results. The two simultaneous equation, when simplified become

$$11 = -A_1 + 2A_2$$
$$13 = A_1 + 4A_2$$

This is the same set of equations used in Example 11.3 and so the results should not be different, $A_1 = -3$ and $A_2 = 4$. The definite time path is

$$y_t = -3(-1)^t + 4(2)^t + 2$$

Example 11.7

Given the difference equation $y_t + 1.2y_{t-1} + 0.36y_{t-2} = 12.8$ and that $y_0 = 8$, $y_1 = 7.2$, find the time path y_t.

This is coming from Example 11.4, except the constant has been changed to make it non-homogeneous. Again, the solution to its complementary part is already found in Example 11.4 and there is absolutely no need to start the calculations again. Simply take it as given in its indefinite form.

$$y_t{}^c = A_1(-0.6)^t + tA_2(-0.6)^t$$

The particular component is not provided. It has to be found but the procedure must be familiar by now.

$$y_t{}^p + 1.2y_t{}^p + 0.36y_t{}^p = 12.8$$

$$y_t{}^p(1+1.2+0.36) = 12.8$$

$$y_t{}^p = \frac{12.8}{2.56} = 5$$

Then combining the two gives the indefinite general solution

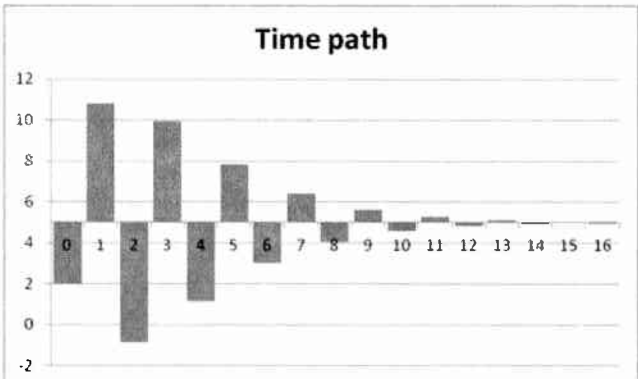
$$y_t = y_t{}^c + y_t{}^p$$
$$= A_1(-0.6)^t + tA_2(-0.6)^t + 5$$

In addition, the information $y_0 = 8$, $y_1 = 7.2$ is available. In the previous example, the given conditions were modified to keep the constants unchanged. In this example, we keep the initial condition unchanged so as to observe what happens to the constants. Using the two pieces of available information, the two resulting equations are

$$3 = A_1$$
$$2.2 = -0.6A_1 - 0.6A_2$$

which give $A_1 = 3$ and $A_2 = 6\frac{2}{3}$. The definite general solution is then given by

$$y_t = 3(-0.6)^t - \frac{40t}{6}(-0.6)^t + 5$$



Compare the solutions in Example 11.4 and Example 11.7. Maintaining the initial condition after adding non-zero constants forces the constants to adjust.

11.6 Economic applications

The application of linear difference equation to economic problems is enormous. Some economic variables may be too complex for the first order linear difference equations. Second order equations become the best-fit expressions which can be used to work with such variables.

In macroeconomics, the static Keynesian model is used to describe an economy. For an economy closed to international trade, the model is

$$Y = C + I + G$$
$$C = a + c(Y - T)$$
$$I = \bar{I}$$

$$G = \bar{G}$$

now relax a few assumptions. The first is consumption. If time is measured in months, then it is inconceivable that this month's consumption depended on income this month. One would be more comfortable to assume it as a function of last month's income. We get a salary at the end of the month for consumption in the coming month. If this lag is sustained on an annual basis, then the consumption function can be best described by the function

$$C_t = a + c(Y_{t-1} - \bar{T})$$

Investment has always been assumed to be exogenous. But this may not be close to reality. Much as it is independent of current level of income, it is undeniable that it is a function of past incomes. Particularly, it can be arguably assumed that it is a function of the increase in the previous period's income over its predecessor. The level of investment (saving) in December depends on how the income for November month end exceed that of October month end, since the December income is not yet received. This is described by the function

$$I_t = j(y_{t-1} - y_{t-2})$$

government expenditure, as well as taxation can still be assumed to be lump-sum and exogenous to the model. With this information, the economy is now best described by the following set of equations

$$Y_t = C_t + I_t + G_t$$
$$C_t = a + c(Y_{t-1} - \bar{T})$$
$$I_t = j(y_{t-1} - y_{t-2})$$

The problem is to find the time path of income, and using this, the time path of consumption and investment.

Putting the last two equations into the first, with some simplification leads to a second order linear difference equation given by

$$Y_t - (c + j)Y_{t-1} + jY_{t-2} = a + (1 - c)\bar{G}$$

with $\beta_1 = -(c + j)$ and $\beta_2 = j$. Once the autonomous part of consumption a, the marginal propensity to consume c and the responsiveness of investment to change in income are known, a particular time path for income (as well as consumption and investment) can be found using the second order linear difference equation method. Such time path is crucial for planning because it enables a more accurate forecasting into fur future periods.

11.7 A note on higher-order difference equations

The principles discussed in this chapter on second order difference equation do apply, *mutatis mutandis*, to difference equations of higher orders. For the particular solution, very little changes. The solution proceeds in the same way as though one was dealing with a second order difference equation. The idea is to assume that the variable is constant. The implication is that the variable as well as its lagged values will be equal. Of course with higher order difference equations, there will be more lagged variables.

With the complementary solution, the strategy of solving a characteristic equation still applies. It is worth stating that the degree of the characteristic equation is the same as the order of the difference equation. As such, higher order difference equation will require solving polynomial equations of higher degrees. Just as the second order difference equation results into a second degree polynomial equation commonly called quadratic equation, an n^{th} order difference equation will result into an n^{th} degree polynomial equation.

An n^{th} degree polynomial equation must essentially produce n-roots. However, the actual number of roots may be less than the degree of the equation. This is because some roots may be reaping (repeated roots). There is also a possibility of having complex conjugates among the n-roots. Once roots are established, their treatment follows that used in the second order difference equation.

Since a higher degree polynomial can produce roots in the three possible categories (distinct roots; repeated roots and complex conjugates) at once, the general solution will also combine the three methods for the three different kind of roots. For instance, a fifth degree characteristic equation emanating from a fifth order difference equation can have the following possibility.

Distinct Roots	Repeated roots (pair)	Complex conjugates(pair)
5	0	0
3	1	0
3	0	1
1	1	1

Chapter 12

12 DYNAMIC OPTIMISATION: AN INTRODUCTION TO OPTIMAL CONTROL THEORY

12.1 Introduction

Mining is a very significant activity in many countries in Africa. In Zambia, copper mining has been the main stay of the economy even before the country attained political independence in 1964.

Since the liberalisation of the economy began in 1991, a number of foreign companies have swarmed into the mining sector. Harking back to our discussion in Chapter 2, these companies are likely typically to be stickers, rather than snatchers. They come with their huge capital investment not to make some quick profits and leave but to maximise returns from their operations over a longer period of time. In other words, their production decisions would be made so as to choose a time path of investment that would maximise their profit over time. In short, the objective is not one of static optimisation but *dynamic optimisation*. *Optimal control Theory* is a technique that is used to solve such dynamic optimisation problems.

12.2 An Illustrative Example

Let us start with a simple static production function

$$Q = f(K)$$

where Q is mineral output and K is capital that includes a slew of mining equipment. Let p be the price per unit of output and c the unit cost of capital. Then the profit (π) will be given by:

$$\pi(K) = p \cdot f(K) - cK$$

We already know that the first order condition for maximum profit is

$$\pi'(K) = p \cdot f'(K) - c = 0$$

But now, the firm may not be interested in simply maximising its current profit but the sum of discounted profits over a period of time between now ($t = 0$) and a stipulated time horizon, T. It would then want to maximise the function

$$S[K(t)] = \int_0^T e^{-rt}\pi[K(t)]\,dt$$

where r is the rate of discount and e^{-rt} is the discounting factor.

$S[K(t)]$ is in fact not a function as we understand it. It is more appropriately called a *functional*. The distinction between a function and a functional is as follows.

A function maps a single value for a variable like capital K into a single value such as current profit π. A functional maps a function like $K(t)$ into a single number like the discounted sum of profits. To put it in other works, one has to choose a function of time, $K(t)$ (a time path of K values) to maximise S and not just choose a single value K to maximise profit π.

But at this stage, the important thing to understand is that the maximisation of the sum of discounted profits does not necessarily mean dynamic optimisation! True, one has to choose not just a single output but a stream of outputs that would maximise profits over time. But if current output affects only current profits, then in choosing current output, one is concerned only with its effect on current profit. The solution to the optimisation problem in such a case would be nothing more than a sequence of solutions to a sequence of static optimisation problems.

If however, current output affects not only current profits but future profits, then in choosing current output, one has to be concerned with its effect on current and future profits. The problem now becomes a dynamic one. Recall once again our quote in Chapter 2 from Silberberg and Suen (2001) on what makes the problem dynamic.

Now, why and how would current output impact on future profit? The answer is this: current profit depends on current output. Current output depends on the amount of current capital used. Although capital is durable, it has a limited lifetime. Capital stock depreciates over time. The more it is used to produce current output, the faster will be its physical wear and tear and shorter its effective lifetime. It will have to be replaced through new purchase of capital (investment) which will push up cost and cut into profits.

Let I(t) be investment or amount of capital bought in time t and δ be the rate of depreciation in capital stock. $\dot{k}$ is the change in the capital stock K(t) available is time t. Then,

$$\dot{k} = I(t) - \delta K(t)$$

The above equation means that at any given point in time, the firm's capital stock increases by the amount of investment and decreases by the amount of depreciation.

Suppose $c[I(t)]$ is the cost of investment $I(t)$, then the profit at time t is given by the functional

$$S[I(t)] = \int_0^T e^{-rt}\pi[K(t), I(t)]\,dt$$

Subject to $\dot{k} = I(t) - \delta K(t)$ and $K(0) = K_0$ where K_0 is the initial capital stock.

Thus, it will be clear that in order to maximise S given K_0, the problem is to choose a time path of investment, $I(t)$. For once the path of $I(t)$ is determined, the path of $K(t)$ will also be determined given $K(0) = K_0$ and the optimal value of S will be solved. The technique of choosing the path of $I(t)$ is optimal control theory.

12.3 Concepts Relating to Optimal Control Theory

Let us begin with a general formulation of a dynamic optimisation problem. The problem is:

$$\max S = \int_0^T f[x(t), y(t), t]\,dt$$

$$\text{subject to } \dot{x} = g[x(t), y(t), t], \qquad x(0) = x_0 > 0$$

The solution to the above problem is guided by a set of necessary conditions emanating from a principal known as *Pontryagin's Maximum Principle*, after the twentieth century Soviet mathematician Lev Semenovich Pontryagin. These necessary conditions are stated in terms of a *Hamiltonian function*. But before we come to this function, it is necessary to define the terms in the above optimisation problem.

- S is the value of the function to be maximised;
- $x(t)$ is called the *state variable*;
- $y(t)$ is called the *control variable*;
- $x(T)$, the final value of the state variable is called the *endpoint*. If this final value is fixed, it is known as a *fixed endpoint*. If the value is unrestricted and free to be chosen optimally, it is known as a *free endpoint*.

In the example we have been dealing with, the amount of mining equipment (capital) the company has, is the state variable. The amount of investment the company makes is the control variable. The problem has a free endpoint since no limit is placed on the amount of capital stock. If such a limit had been placed, it would have been a fixed endpoint problem.

The solution to the problem of maximising S is tantamount to finding the optimal solution path for the control variable, $y(t)$.

15.1 The Hamiltonian function and Necessary Condition for Optimisation

The Hamiltonian function H, for the problem specified in the preceding subsection is

$$H[x(t), y(t), \lambda(t), t] = f[x(t), y(t), t] + \lambda(t) g[x(t), y(t), t]$$

In this equation, $\lambda(t)$ is called the *co-state variable* or the *shadow pricing function* for x.

The optimal solution path for $y(t)$ is one that satisfies the following necessary conditions:

i. $\frac{\partial H}{\partial y} = 0$
ii. $\frac{\partial \lambda}{\partial t} = \dot{\lambda} = -\frac{\partial H}{\partial x}$
iii. $\frac{\partial x}{\partial t} = \dot{x} = \frac{\partial H}{\partial \lambda} = g[x(t), y(t), t]$
iv. $x(0) = x_0$
v. $x(T) = 0$

The first three equations together constitute the maximum principle. The last two are called the boundary conditions. Note that one can readily discern a similarity between the necessary condition of the maximum principle and the Lagrangean function that was introduced in Chapter 8 to solve constrained optimisation problems. The boundary condition in equation (v) assumes that $x(T)$ is a fixed endpoint. If it is a free endpoint, the equation would be substituted by $\lambda(T) = 0$. This is called the *transversality condition*.

12.4 Sufficient Conditions

The maximum principle provides only the necessary conditions for optimisation. It can be shown that these conditions will also be sufficient if the following are satisfied:

i. $f(x, y, t)$ is differentiable and jointly concave in x and y
ii. Any one of the following holds;
  - $g(x, y, t)$ is linear in (x, y);
  - $g(x, y, t)$ is concave in (x, y) and $\lambda(t) \geq 0,\ t \in (0, T)$;
  - $g(x, y, t)$ is convex in (x, y) and $\lambda(t) \leq 0,\ t \in (0, T)$

Example 12.1

Solve $\max \int_0^1 (x - y^2)\, dt$ subject to $\dot{x} = 2y \text{ and } x(0) = 2$

We form the Hamiltonian function which takes the form

$$H[x(t), y(t), \lambda(t), t] = f[x(t), y(t), t] + \lambda(t) g[x(t), y(t), t]$$

Thus $H = x - y^2 + \lambda \cdot 2y$

The necessary conditions are

$$\frac{\partial H}{\partial y} = 0 = -2y + 2\lambda$$

This gives $y = \lambda(t)$

Now $\dot{\lambda} = -\frac{\partial H}{\partial x} = -1$

We now have the two linear differential equations

$$\dot{\lambda} = -1$$
$$\dot{x} = 2\lambda$$

The boundary conditions are $x(0) = 2,\ \lambda(3) = 0$

The solution to the first differential equation is $\lambda(t) = C_1 - t$, where C_1 is an arbitrary constant of integration. Since the boundary condition $\lambda(1) = 0$, has to be satisfied, we will have $C_1 = 1$. Hence

$$\lambda(t) = 1 - t$$

Substituting this value in the second differential equation $\dot{x} = 2\lambda$, we get

$$\dot{x} = 2(1 - t)$$
$$= 2 - 2t$$

this gives the solution $x(t) = C_2 + 2t - t^2$ which using the boundary condition on x will give $C_2 = 2$. Thus, the time path for x will be

$$x(t) = 2 + 2t - t^2$$

Given the time path of λ, we also substitute to get the time path for y. We get

$$y(t) = \lambda(t)$$
$$= 1 - t$$

as the solution path to the control variable. At $t = o$, $y(t) = 1$. It then declines over time and finishes at $y(1) = 0$.

12.5 Economic Application: A Mining Problem

An investor has bought a mine and has been given rights to extract the ore between date 0 and date T. At time 0, there is x_0 ore in the ground. The stock of ore remaining at any time t, $x(t)$ will depend on the rate of extraction of the ore. The cost of extraction is $C = \frac{y(t)^2}{x(t)}$. The market price of the ore is p.

The decision to be made by the owner is the optimal rate of extraction in time t, $y(t)$ that will maximise profits over the period of ownership. For simplicity, let us assume there is no time-discounting. The problem now is

$$\max \pi = \int_0^T \left[p \cdot y(t) - \frac{y(t)^2}{x(t)} \right] dt$$

subject to $x(t) = -y(t)$

We form the Hamiltonian function $H = p \cdot y(t) - \frac{y(t)^2}{x(t)} - \lambda(t)y(t)$

The necessary conditions (rearranged) are:

$$\frac{\partial H}{\partial y} = p - \lambda(t) - 2\frac{y(t)}{x(t)} = 0$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = -\left(\frac{y(t)}{x(t)}\right)^2$$

$$x(0) = x_0, \qquad \lambda(T) = 0$$

The two solutions to the differential equations are

$$\dot{\lambda}(t) = \frac{[p - \lambda(t)]^2}{4}$$
$$y(t) = x(t)\frac{p - \lambda(t)}{2}$$

PART IV : PROGRAMMING TECHNIQUES

Chapter 13

13 LINEAR PROGRAMMING

13.1 Introduction

The constrained optimisation problems studied in Chapter 8, involved an equality constraint. In addition, the objective and constraint functions were not simultaneously linear. That is, if the objective function is linear, then the constraint needed to be nonlinear. This condition was very cardinal, for a simple reason. The method of finding the optimal value relied on classical methods of optimisation, based principally on the calculus. It involved finding a point of tangency between the two functions.

In practice however, a firm is not obliged to use up all the available resources. A consumer is at liberty to consume less than is actually at their disposal. These statements imply that the constraint should not always be equality. Moreover, the indivisibility of certain commodities make is impossible to spend all the money. For instance, consider a consumer with $K5$ income to buy apples costing $K2$ each. An apple is indivisible, that is, it cannot be cut into half or anything less than unit. The consumer then can only buy utmost two which leaves some resources unutilised.

Further, the methods of Chapter 8 fail to deal with cases where the objective and constraint functions are simultaneously linear. With linear objective and constraint function, the respective slopes are constant. Therefore, they are either equal throughout or never equal at all. This inhibits the use of tangency rule of optimisation. A more appropriate approach is necessary to deal with such practical cases.

This chapter therefore brings out a non-classical method of optimisation known as linear programming. As the name may suggest, this method deals the cases where both the objective and constraint (inequality) functions are linear. The constraints are of the form $g(x,y) \leq c$ rather than $g(x,y) = c$.

13.2 What Is Linear Programming?

Define the process by which an input is transformed into an output as an activity. Thus an activity is a process. It links inputs to output. For instance, production is an activity whereby inputs such as land, labour, raw materials and other factors of production are converted into outputs of final goods and services. Consumption is an activity where final goods and services are transformed into utility or satisfaction to the consumer.

Suppose an activity A transforms inputs I into output O. The activity remains the same but it is possible to alter the level of inputs. In particular, consider

$$I_1 \overset{A}{\rightarrow} O_1$$
$$I_2 \overset{A}{\rightarrow} O_2$$

$K2.00$. The monthly demand for the next six months is forecast at 800, 1000, 1400, 1500, 1300 and 1550 units respectively. What is the production schedule for the firm?

13.4.1.1 Formulation of Problem 1

The optimum production schedule would be one that meets the monthly demands at minimum cost. The total cost would include cost of regular production, cost of overtime production and storage cost. But since regular production costs have to be incurred anyway, one needs to consider only storage costs and additional cost of overtime production to minimise total cost.

Let $X_1, X_2, X_3, X_4, X_5, X_6$ be regular production in the six months.

$Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$ be overtime production in the six months.

$Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$ be number of units held over from each month to the immediately following month.

p= unit penalty cost for unsold items.

Then the problem is:

Minimise $C = 7\sum_{i=1}^{6} Y_i + 2\sum_{i=1}^{6} Z_i + pZ_6$

Subject to:

$$\left.\begin{array}{l} X_i \leq 1000, i = 1, \ldots .6 \\ Y_i \leq 500, i = 1, \ldots \ldots 6 \end{array}\right\} \textit{production limitations}$$

$$\left.\begin{array}{l} X_1 + Y_1 - Z_1 = 800 \\ X_2 + Y_2 - Z_2 = 1000 \\ X_3 + Y_3 - Z_3 = 1400 \\ X_4 + Y_4 - Z_4 = 1500 \\ X_5 + Y_5 - Z_5 = 1300 \\ X_6 + Y_6 - Z_6 = 1550 \end{array}\right\} \textit{Monthly sales storage balance equation.}$$

$$\left.\begin{array}{l} X_i \geq 0, \ \forall i \\ Y_i \geq 0, \ \forall i \\ Z_i \geq 0, \ \forall i \end{array}\right\} \textit{Non} - \textit{negativity restrictions.}$$

13.4.2 Problem 2

A big firm has five factories which are manufacturing a product which is to be sold at four places. The cost of transporting one unit of the product from any factory F_i to any place L_j is shown below.

	F_1	F_2	F_3	F_4	F_5
L_1	12	14	15	10	4
L_2	10	5	11	10	6
L_3	9	8	10	5	9
L_4	7	6	12	3	7

The supply at each of the five factories is 1500, 4000, 3000, 2500 and 3500 respectively. The demand at each of the places is 6000, 3000, 1500 and 2500 respectively. The firm must meet all these demands at minimum transport cost. What transportation strategy will minimise the cost?

13.4.2.1 Formulation of Problem 2

Let X_{ij}= amount transported from factory F_i to the place of demand L_j. Then,

Minimise $C = 12X_{11} + 10X_{12} + 9X_{13} + 7X_{14}$

$+14X_{21} + 5X_{22} + 8X_{23} + 6X_{24}$

$+15X_{31} + 11X_{32} + 10X_{33} + 12X_{34}$

$+10X_{41} + 10X_{42} + 5X_{43} + 3X_{44}$

$+4X_{51} + 6X_{52} + 9X_{53} + 7X_{54}$

Subject to:

$$\left.\begin{array}{l} X_{11} + X_{12} + X_{13} + X_{14} \leq 1500 \\ X_{21} + X_{22} + X_{23} + X_{24} \leq 4000 \\ X_{31} + X_{32} + X_{33} + X_{34} \leq 3000 \\ X_{41} + X_{42} + X_{43} + X_{44} \leq 2500 \\ X_{51} + X_{52} + X_{53} + X_{54} \leq 3500 \end{array}\right\} \textit{Supply contraints}$$

$$\left.\begin{array}{l} X_{11} + X_{21} + X_{31} + X_{41} + X_{51} \geq 6000 \\ X_{12} + X_{22} + X_{32} + X_{42} + X_{52} \geq 3000 \\ X_{13} + X_{23} + X_{33} + X_{43} + X_{53} \geq 1500 \\ X_{14} + X_{24} + X_{34} + X_{44} + X_{54} \geq 2500 \end{array}\right\} \textit{Demand requirements}$$

$$\left.\begin{array}{l} X_{ij} \geq 0, \ i = 1, \ldots 5; \\ \quad j = 1, \ldots 4 \end{array}\right\} \textit{non} - \textit{negativity constraints}$$

This is developed into a transportation model. The model is discussed in Chapter 14.

13.4.3 Problem 3

A company owns a small paint factory that produces both interior and exterior house paints for wholesale distribution. Two basic raw materials A and B are used to manufacture the paints.

The maximum availability of A is 6 tons a day; that of B is 8 tons a day. The daily requirements of the raw materials per ton of interior and exterior paints are shown below.

	Exterior	Interior
Raw material A	1	2
Raw material B	2	1

A market survey has established that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. The survey also shows that the maximum demand for interior paint is limited to 2 tons daily. The wholesale price per ton is K12 million for exterior paint and K8 million for interior paint. How much interior and exterior paints should the company produce to maximise gross income?

13.4.3.1 Formulation of Problem 3

Let X_E= tons of exterior paint produced daily and X_I= tons of interior paint produced daily.

The objective is to maximise total revenue $R = 12X_E + 8X_I$ subject to:

$$\left.\begin{array}{l} X_E + 2X_I \leq 6 \\ 2X_E + X_I \leq 8 \end{array}\right\} \textit{raw material availabilty constraints}$$

$$\left.\begin{array}{l} -X_E + X_I \leq 1 \\ X_I \leq 2 \end{array}\right\} \textit{demand restictions}$$

$$\left.\begin{array}{l} X_E \geq 0 \\ X_I \geq 0 \end{array}\right\} \textit{non} - \textit{negativity constraints}$$

15.1.1.1 Problem 4 (Diet Problem)

Suppose a meal consists of two goods; a cereal and meat, to get the requirements of proteins, fats, calcium, iron, vitamin A and B. The table below shows the nutrition value (quantity per unit of good) of the two ingredients of a meal.

	Cereal	Meat
b_1 Protein	a_{11}	a_{12}
b_2 Fat	a_{21}	a_{22}
b_3 Calcium	a_{31}	a_{32}
b_4 Iron	a_{41}	a_{42}
b_5 Vit A	a_{51}	a_{52}
b_6 Vit B	a_{61}	a_{62}

Let X be the amount of cereals consumed and Y represent the accompanying Meat consumed. X and Y are food activities which makes a program. The two commodities must be bought on the market. Assume the price of cereal and meat is given by P_X and P_Y respectively.

The objective is to minimise the cost $C = P_X X + P_Y Y$. With no lower limit on the quantities, one would decide to starve, by not buying anything at all. The cost will be at its lowest, at zero. This, however, is not tenable because the continuation of life requires that certain amounts of nutrition are consumed. Suppose now to maintain a healthy life, b_i of each food requirement must be consumed. If any nutrient is not necessary, a value of zero is assigned to it.

Therefore, the cost $C = P_X X + P_Y Y$ must be minimised subject to minimum nutrition requirements, that is;

$$\begin{array}{c} a_{11}X + a_{12}Y \geq b_1 \\ a_{21}X + a_{22}Y \geq b_2 \\ a_{31}X + a_{32}Y \geq b_3 \\ a_{41}X + a_{42}Y \geq b_4 \\ a_{51}X + a_{52}Y \geq b_5 \\ a_{61}X + a_{62}Y \geq b_6 \end{array}$$

The quantities of the two commodities consumed cannot fall below zero. That is, $X \geq 0,\ Y \geq 0$

Optimal program: This is one that minimises the cost function. A vector which fulfils all the constraints is called a feasible vector. There can be many feasible vectors but only the one with a minimum cost is optimal. Thus the equation is solved using two things; feasible program of feasible vectors and select the optimal feasible program.

Example 13.1

A firm produces three types of furniture. These are Bookshelves, TV cabinets and Dining tables. Let X_1 be the number of bookshelves, X_2 the number of TV cabinets and X_3 the number of Dining tables. The market prices for the three products are P_1, P_2 and P_3 respectively. the objective of the firm is to maximise revenue

$$R = P_1X_1 + P_2X_2 + P_3X_3$$

Two inputs are required, in varying quantities to produce the three furniture, wood and labour. Though the firm can get different units of inputs are a constant prices, the supply of the two inputs is limited. Suppose there is only b_1 of wood and b_2 of labour available to the firm. The input-output matrix is given in the table below.

	Wood	Labour
Bookshelf	a_{11}	a_{21}
TV cabinet	a_{12}	a_{22}
Dining table	a_{13}	a_{23}

The objective is to maximise revenue $R = P_1X_1 + P_2X_2 + P_3X_3$ subject to availability of inputs. The constraints are based on input availability.

$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 \leq b_1$

$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 \leq b_2$

The feasible production must fulfil all constraints.

13.5 Duality

Consider the general form of the Linear programming problem:

Minimise $C'X$

Subject to $AX \geq B,\ X \geq 0$

The dual of the above primal linear program is written as follows:

Maximise $B'Y$

Subject to $A'Y \leq C,\ r \geq 0$

Specifically consider the following linear program;

Minimise $50X_1 + 45X_2 + 25X_3 + 30X_4$

Subject to $2X_1 + 6X_2 + X_3 \geq 5$

$8X_1 - 3X_2 + 4X_3 + X_4 \geq 7$

$X_1, X_2, X_3, X_4 \geq 0$

The dual of this problem will be written as:

Maximise $5Y_1 + 7Y_2$

Subject to

$$\begin{aligned} 2Y_1 + 8Y_2 &\leq 50 \\ 6Y_1 - 3Y_2 &\leq 45 \\ Y_1 + 4Y_2 &\leq 25 \\ Y_2 &\leq 25 \\ Y_1, Y_2 &\geq 0 \end{aligned}$$

It can be seen that both the primal and dual problems have the same data inputs, that is, the matrix and the vectors A, B and C. What differs are the objectives and the program vectors X and Y. The following points are worth attention:

a) If the objective of the primal problem is to be minimised, the objective function of the dual problem is to be maximised.
b) The right hand side constants of the primal problem become the coefficients of the objective function of the dual problem and likewise the coefficients of the objective function functions of the primal problem become the right hand side constants of the dual problem.

c) The matrix of coefficients of the constraints in the problem is transposed to obtain the matrix of coefficients of the constraints in the dual problem.
d) The sign of the inequalities in the constraints of the primal are reversed in the dual.

From (b) and (c) it can be understood that if there are m program variables and n constraints in the primal, there will be n program variables and m constraints in the dual.

13.5.1 Theorems on Duality

The following theorems state the relationships between the primal and dual programs, especially in respect of their solutions.

i) The dual of the dual is the primal;
ii) If either the primal or the dual has a finite optimal solution, so does the other;
iii) If feasible solutions to both the primal and dual systems exist, then both have optimal solutions;
iv) If either problem has an unbounded optimal solutions then the other has no feasible solution;
v) Both the primal and dual may be feasible;
vi) If (ii) holds, then the objective functions of both problems have the same optimum value.

13.6 Graphical Solution of Linear Programmes

Linear programs can be solved using two methods. The first is the graphical method which employs graphs to find the optimal solution. The second, and more robust, is the Simplex or algebraic method. This section deals with the former and the latter is discussed in the succeeding section. The graphical method involves plotting all the constraints on one graph which defines a feasible solution. Owing to the difficulties of drawing graphs of more than two dimensions, the discussion of this method is restricted to problems having two choice variables only.

Example 13.2

Consider the following linear programming problem in mathematical form:

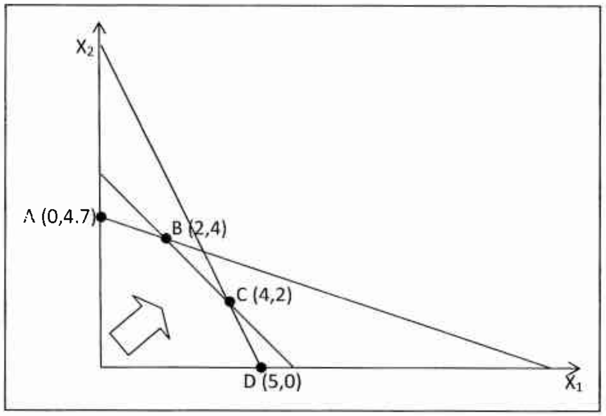
Maximise $6X_1 + 8X_2$ subject to

$$\begin{aligned} 2X_1 + X_2 &\leq 10 \\ X_1 + X_2 &\leq 6 \\ X_1 + 3X_2 &\leq 14 \end{aligned}$$

$$X_1 \geq 0, \qquad X_2 \geq 0$$

Draw all the three constraints or feasibility conditions on the same graph. If for each constraint the unfeasible side is shaded, then the feasible region which satisfies all the constraints would emerge. It is the five sided polygon lying on the left of all the constraints. It is the region bound by the origin and the points A, B, C and D.

Figure 13.1. Illustration of feasible solution



Any point falling in the region, satisfies all the three constraints. Since the two variables cannot be negative, the vertical and horizontal axes are in fact boundaries for the respective sides. A feasible vector for this linear program is any point (X_1, X_2) which satisfies all the three constraints simultaneously. In other words, such a point should lie in the region, which may be called the *feasible region*, which at once satisfies all the three inequalities.

In the figure above, the feasible region is a *closed, convex set*. It contains infinity of points or feasible solutions or program vectors$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, from which the optimal solution is to be found. Since the objective function is one of maximisation, we select the point (X_1, X_2) as much away from the origin and upwards and to the right as possible. That is, our *preference direction* is towards the north-east. It is clear that the optimal point would be somewhere on the boundary of the convex set. To find this, consider the iso-profit line.

The objective is to maximise $6X_1 + 8X_2$. The unit value of X_1 is 6 and that of X_2 is 8. That is, their unit values are in the ratio of 3:4. This means that 4 units of X_1 would give the same profit as 3 units of X_2. Hence the straight line joining points (4,0) and (0,3) give all the other points or combinations of X_1 and X_2 which give the same profit. To maximise profit (objective function) subject to the constraints, push the iso-profit line parallel outwards and see which point is in feasible region that falls on the highest iso-profit line. In the diagram above, the optimal solution is at point B. The coordinates of B, (2, 4) are the optimal values of X_1 and X_2.

We have already stated that the feasible region is a closed, convex set. It can be proved (though we shall not prove it) that an optimal feasible solution is one of the extreme points of this convex set. Thus, although there is an infinity number of feasible solutions, to locate an optimal solution only the points A, B, C and D, in the diagram (the origin (0,), though an extreme point is obviously trivial and need never be considered) need to be examined.

Extreme point	Value = $6X_1 + 8X_2$
$A: (0,4\frac{2}{3})$	$6(0) + 8\left(4\frac{2}{3}\right) = 37\frac{1}{3}$
$B: (2,4)$	$6(2) + 8(4) = 44$
$C: (4,2)$	$6(4) + 8(2) = 40$
$D: (5,0)$	$6(5) + 8(0) = 30$

Thus the objective function is maximised with $X_1 = 2$ *and* $X_2 = 4$. The maximum value of the objective function is 44.

If the Linear programming program had been a minimisation problem, our search for an optimal feasible solution would be tantamount to finding the point on the feasible region which lies on the lowest iso-cost line. The preference direction would have been towards the origin, which corresponds with minimisation. This can be seen in the graph corresponding to the following linear program:

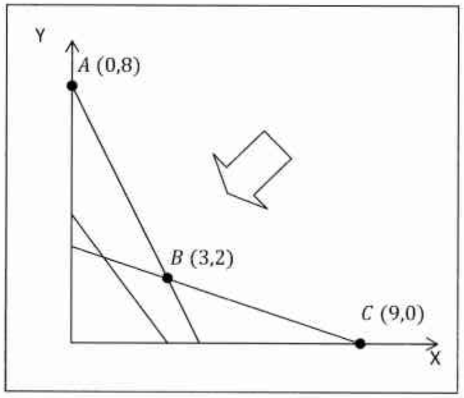
Example 13.3

Minimise $C = 20X + 40Y$ subject to

$$2X + Y \geq 8$$
$$4X + 3Y \geq 12$$
$$X + 3Y \geq 9$$
$$X \geq 0, \qquad Y \geq 0$$

Again, draw the three constraints on one graph. From the graph, identify the feasible solution. This is the region which satisfies all the constraints.

Figure 13.2. Illustration of irrelevant constraint



One feature to be noted in the above graph is that the second constraint $4X + 3Y$ does not play any part in determining the feasible region. It entirely lies outside the feasible region. Hence, it is irrelevant for solutions of the linear program.

We can examine the three possible points A, B and C and look for one giving the lowest cost because the problem is that of cost minimisation. Point A gives 320, point B gives **140 and point C gives 180. Thus point B is the optimal point. It has the lowest cost.**

Also recollect what we had said about closed and open sets in Chapter 3. In the above diagram, the feasible region is physically open but is nonetheless a closed convex set.

13.7 Nature of Linear Programming Solutions

It was already stated in an earlier section that the optimal solution to a linear program need not be finite and unique. There are other possibilities. These are illustrated graphically.

13.7.1 Case 1: No feasible solution

Suppose you have the following problem:

Minimise $Z = 10X + 15Y$ subject to

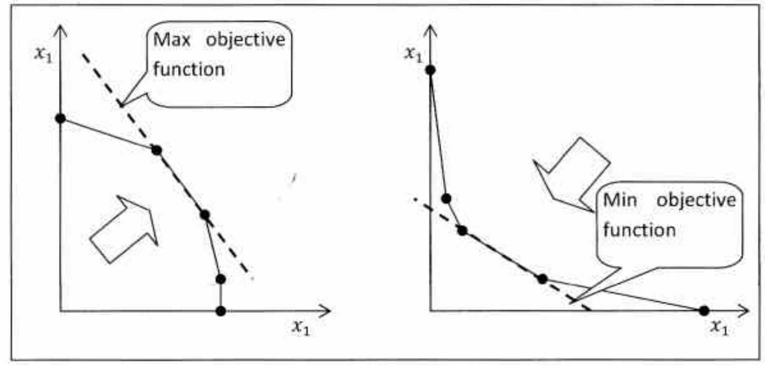
$$X + 2Y \geq 8$$
$$2X + Y \geq 10$$
$$4X + 3Y \leq 12$$
$$X \geq 0, \qquad Y \geq 0$$

There is no single point that will simultaneously satisfy the second and third constraint. There is therefore, no point which can at once satisfy all the three constraints. In other words, the feasible region is an empty set.

13.7.2 Case 2: Multiple optimal solution

This situation occurs if the slope of the iso-profit line (or iso-cost line) equals the slope of the boundary edges of the convex feasible region which is in fact the slope of the line corresponding to one of the constraints. This is depicted in the following two diagrams.

Figure 13.3. Case of multiple optimal solutions



In each of the above diagrams, points a and b and all other points on the boundary edge or line segment connecting a and b represent the optimal feasible solutions.

13.7.3 Case 3: Unbounded optimal solutions

These are the kind of solutions that are not bound, in other words not exact as in the above scenarios demonstrated. The necessary but not sufficient condition for this situation to occur is that the feasible region be unbounded.

Example 13.4

Minimise $c_1X_1 + c_2X_2 + c_3X_3$

Subject to $a_{11}X_1 + a_{12}X_2 + a_{13}X_3 \geq b_1$

$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 \geq b_2$

$X_1,\ X_2,\ X_3 \geq 0$

Maximise B^TY

Subject to $A^TY \leq C$

$Y \geq 0$

Example 13.5

Maximise $b_1Y_1 + b_2Y_2$

Subject to $a_{11}Y_1 + a_{21}Y_2 \leq C_1$

$a_{12}Y_1 + a_{22}Y_2 \leq C_2$

$a_{13}Y_1 + a_{23}Y_2 \leq C_3$

$Y_1,\ Y_2 \geq 0$

The difference is that the program variables are different between the primal and dual problems. The coefficients of the objective function and right hand side constraints have been interchanged. The signs are different. There is also interchange of rows and column. In general, if you have n program variables and m constraints if $n < m$, it is easier to solve than when $n >$ m. Fewer program variables make a problem easier to solve. The program variables of the primal and dual are different.

Maximise $P_1X_1 + P_2X_2 + P_3X_3$ ....Primal

Subject to

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 \leq b_1$$
$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 \leq b_2$$
$$X_1 \geq 0, \quad X_2 \geq 0, \quad X_3 \geq 0$$

This is a resource allocation problem among the products X_1, X_2, X_3.

Minimise $b_1Y_1 + b_2Y_2$ ....dual

Subject to

$$a_{11}Y_1 + a_{21}Y_2 \geq P_1$$
$$a_{12}Y_1 + a_{22}Y_2 \geq P_2$$
$$a_{13}Y_1 + a_{23}Y_3 \geq P_3$$
$$Y_1 \geq 0, \quad Y_2 \geq 0$$

b_1 and b_2= are units of resources

Y_1= price of one ton of e.g. labour

Y_1 and Y_2 are the input prices.

This is an input pricing problem. $a_{11}Y_1$ is the price of producing one unit of Y_1. In actual sense, the constraints may encompass the equal sign due to the fact that if firms make super normal profits, assuming perfect competition, the profits will be normal again.

13.8 The Simplex Method (Algebraic)

The algebraic analogue of the graphical method of solving a linear programme is known as the *simplex method.* It was formulated by George Dantzig, the late American mathematician. In the former method, locating an optimal point required examining all the extreme points. These are vertexes of the feasible solution shape. The simplex procedure is even more efficient in that it avoids examining all the extreme points. It reduces on the points to be examined before

reaching the optimal solution. It arrives at the optimal solution through an iterative process or algorithm. The idea is to start with an initial feasible solution and go on improving the solution at each iteration until no further improvement is possible and the optimal solution is arrived at. Before illustrating this method, there is need to introduce a few prerequisite ideas and concepts.

13.8.1 Prerequisite ideas and concepts for the Simplex Method.

i) Basic variables are variables in a linear programming problem which are solved from the constraints of the problem in terms of other variables which are called *non-basic* variables. Since we require as many equations as there are unknowns to be solved for, we can obviously have, at any time, only as many basic variables as there are constraints. The collection of basic variables is termed the *basis* and the remaining group of non-basic variables, *the non-basis*.

ii) *Basic solutions* are solutions of the basic variables that result from equating the values of the non-basic variables to Zero. Basic solutions which satisfy all the constraints are known as *basic feasible solutions* (b.f.s.). These solutions correspond to the extreme points of the convex feasible region.
To find the optimal solution, only basic feasible solutions are considered. This is based on the *Basic Theorem* in linear programming (whose proof is not provided here) which states that if a feasible solution exists, then a basic feasible solution exists and that if finite optimal solutions exist, then at least one of them is a basic feasible solution (b.f.s.). That is to say, if a unique optimum solution exists, then it is a *basic feasible solution*. If multiple optimum solutions exist, at least one of them must be a b.f.s.

There could be non-basic optimal solutions. Reverting to our diagram of case 2 examined earlier, points 'a' and 'b' represent b.f.s. because they are extreme points. Points lying on the line segment connecting 'a' and 'b' though optimal solutions are actually non-basic. They are non-basic because they are not extreme points. They lie on a straight line and not a corner.

iii) *Slack and surplus variables*:
Consider the inequality

$$a_1X_1 + a_2X_2 + \cdots\cdots + a_nX_n \leq b$$

The left hand side (LHS) is less than the right hand side (RHS). The actual difference is not specific and can be treated as an arbitrary number. If a non-negative arbitrary number assumed to be equal to the difference is added, then the inequality turns into equality. The two sides of sides will now be equal since the lower side has been raised by an arbitrary number equal to the initial difference. This will make up for the difference, that is, how less the LHS was to the RHS

$$a_1X_1 + a_2X_2 + \cdots\cdots + a_nX_n + S = b, \qquad S > 0$$

Similarly, the inequality below has the RHS less than the LHS

$$a_1X_1 + a_2X_2 + \cdots\cdots + a_nX_n \geq b$$

It can be turned into equality by adding a positive arbitrary constant on the right side of the inequality. Alternatively, the equality is achieved by subtracting the same constant from the left side of the inequality. This will become:

$$a_1X_1 + a_2X_2 + \cdots\cdots + a_nX_n - t = b, \qquad t \geq 0$$

In the resulting equations, S is called a *slack* variable and t, a *surplus* variable. A surplus variable may be regarded as a negative slack variable.

13.8.2 Illustration of the Simplex procedure:

To illustrate this method, it is much easier to make use of examples. Since the graphical and the simplex methods must ideally arrive at the same conclusion, this subsection will as a strategy use examples solved with the former method. This will provide a check on the solutions as well as provide a detailed comparison of the two methods.

Example 13.6

Revisit Example 13.2. Solve this example using the Simplex method.

Maximise $Z = 6x_1 + 8x_2$

$$\begin{aligned} 2x_1 + x_2 &\leq 10 \\ x_1 + x_2 &\leq 6 \\ x_1 + 3x_2 &\leq 14 \\ x_1, \quad x_2 &\geq 0 \end{aligned}$$

The first step is to restate the problem by converting the constraint inequalities into equations by adding slacks. The problem now is

Maximise $Z = 6x_1 + 8x_2 + 0x_3 + 0x_4 + 0x_5$ Subject to

$$\begin{aligned} 2x_1 + x_2 + S_1 \qquad\qquad &= 10 \\ x_1 + x_2 \qquad + S_2 \qquad &= 6 \\ x_1 + 3x_2 + \qquad\qquad S_3 &= 14 \end{aligned}$$

$$x_1, x_2, S_1, S_2, S_3 \geq 0$$

We must start with an initial basic feasible solution. Usually, as a convenient starting point, the slack variables are taken as the initial basis. This basis is, of course, trivial in the sense that the value of the objective function would be zero. In our problem then, we begin with S_1, S_2 and S_3 as the basis. Equating non-basic variables x_1 and x_2 to zero, we get

$$1)\ S_1 = 10 - 2x_1 - x_2 = 10$$

$$2)\ S_2 = 6 - x_1 - \quad x_2 = 6$$

$$3)\ S_3 = 14 - x_1 - 3x_2 = 14$$

$$Z = (0)(10) + (0)(6) + (0)(14) = 0$$

The question now is: by including any non-basic variable into the basis can the value of Z improve? The answer is of course in the affirmative. Both x_1 and x_2 have positive

coefficients in the objective function, thus increasing them will improve the objective function. However, the simplex procedure allows only one variable to enter the basis at a time. Further, to make room for the variable entering the basis, one of the existing basic variables must leave the basis and become non-basic. The number of variables in the solution is limited by the number of equations will directly depends on the number of constraints.

But the model does not state which variable will enter and which one leaves. The guidance nonetheless exists. Since the objective function is one of maximisation, then the variable that add most is considered first. By adding most, we mean the value of the coefficient since it measures how the objective function changes with a change in the concerned variable. Between the two variables, x_2 has a coefficient larger than that of x_1. A unit of x_2 will add 8 units to the value of Z while a unit of x_1 will add only 6 units.

But which variable must leave the model to create room for x_2? The strategy is as follows. The existing variables should be such that the solution corresponding to the new basis should be feasible. Suppose x_3 is made to leave the basis. Then our current basis would consist of x_2, S_2 and S_3 and non-basis of x_1, S_1.

From equation (1) we will get $x_2 = 10$. But if this value is substituted in equations (2) and (3) it will give $S_2 = -4$ and $S_3 = -1$ respectively. These values violate the non-negativity restrictions on x_2 *and* x_4 and hence is not feasible. Thus S_1 cannot be the exiting variable.

If S_2 is taken out of the basis, from equation (2), we will get the value of x_2 as 6. If this value is inserted in equation (5), we get $S_3 = -4$. Thus S_2 too cannot be made to leave the basis. If S_3 leaves the basis, $x_2 = \frac{14}{3} = 4\frac{2}{3}$. Using this value of x_2 we get $S_1 = 5\frac{1}{3}$ from equation (1) and $S_2 = 1\frac{1}{3}$. This is a feasible solution. our new basis is, therefore, (x_2, S_1, S_2) and non basis (x_1, S_3) .the value of $Z = (0)\left(5\frac{1}{3}\right) + (0)\left(1\frac{1}{3}\right) + (8)\left(4\frac{2}{3}\right) = 37\frac{1}{3}$

We can improve the value of Z further by including x_1 now into the basis. Which variable will it replace? The choice is between S_1 and S_2. Suppose S_1 is chosen to leave the basis. (x_1, x_2, S_2) will constitute the new basis. We solve for x_1, x_2, S_2 in terms of S_1 and S_3. From equations (1) and (3) we get,

$$2x_1 + x_2 = 10$$

$$x_1 + 3x_2 = 14$$

Solving these two equations we get $x_1 = 3\frac{1}{5}$, $x_2 = 3\frac{3}{5}$, if these values are substituted in equation(2), we will get $S_2 = 6 - 3\frac{1}{5} - 3\frac{3}{5} = -\frac{4}{5} < 0$. this is not a feasible solution. Hence S_1 cannot leave the basis. If S_2 becomes part of non-basis, we shall now solve equations (2) and (3) simultaneously. We get

$$x_1 + x_2 = 6$$

$$x_1 + 3x_2 = 14$$

These equations yield $x_1 = 2$ and $x_2 = 4$. And with these values, from equation (1), we get $S_1 = 2$.The value of $Z = (6)(2) + (8)(4) + (0)(2) = 44$. Since the current non basis consists of the variables S_2 and S_3 and since both these have zero coefficients in the objective function, the value Z cannot be improved further and hence is optimal.

The simplex method can also be worked in a more summarised way using the simplex tableau. Assume the following linear programming problems with slack variables already added to turn inequality constraints into equality constraints.

$$\max Z = c_1X_1 + c_2X_2 + c_3X_3$$

subject to

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + S_1 = b_1$$

$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + S_2 = b_2$$

$$a_{31}X_{31} + a_{32}X_2 + a_{33}X_3 + S_3 = b_3$$

$$a_{41}X_1 + a_{42}X_2 + a_{43}X_3 + S_4 = b_4$$

$$X_1, X_2, X_3 \geq 0$$

Since we have to start from some feasible solution, we start from the origin, where all the X's are zero and form a special matrix called *simplex tableau* as follows.

Table 13.1. The Simplex Tableau

c_1	c_2	c_3	0	0	0	0	Obj fn Coef		
X_1	X_2	X_3	S_1	S_2	S_3	S_4	RHS	CBV	Ratio
a_{11}	a_{12}	a_{13}	1	0	0	0	b_1	0	
a_{21}	a_{22}	a_{23}	0	1	0	0	b_2	0	
a_{31}	a_{32}	a_{33}	0	0	1	0	b_3	0	
a_{41}	a_{42}	a_{43}	0	0	0	1	b_4	0	
c_1	c_2	c_3					Marginal Net Gain		

In Table 13.1, the first seven columns pertain to the seven variables (choice variables plus the slack variables) indicated in the second row for each column. The first row of the table is a representation of the objective function. It gives, for each variable, the coefficient in the objective function. Since slack variables are not in the objective function, the coefficients are zero.

After the first two rows, there are as many rows as are constraints, each row representing a constraint. In the particular case used above, there are four constraints, therefore four rows. The element a_{ij} is the coefficient of the j^{th} variable in the i^{th} constraint. The first line for instance pertains to the first constraint

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + S_1 = b_1$$

Only one slack variable appears in this constraint with a coefficient of one. The other three therefore appear with coefficients of zero in the simplex tableau. The column headed RHS is the

constant in each constraint, or the Right Hand Side (RHS) of the constraint. In the case of the first constraint, this is b_1. In the CBV column is the Coefficient of the Basis Variable in the objective function. In the initial basic solution, it is the slack variables that form the basis solution. Since these are not part of the objective function, the CBV will have zeros only. These will change as some variables enter and others leave the basis.

The last row of the tableau is the Marginal Net Gain (G) for each variable including slack variables. It is calculated as follows

$$G_j = c_j - \left[a_{1j}CBV_1 + a_{2j}CBV_2 + a_{3j}CBV_3 + a_{4j}CBV_4\right]$$

For each constraint coefficient in the column, multiply by the CBV in that row. Then subtract a sum of these from the Objective function coefficient. The marginal net gain shows how much the objective function would gain if the particular variable enters the basis.

The next step involves identifying the column with the highest Net Gain. Amongst variables not in the basis, determine one with the highest positive value. The chosen column becomes the pivot column. The variable in that column will enter the basis. If all the Net gains are non-positive, it means no change of variables will improve the objective function. This entails the optimal solution has been reached. The searching ends.

Once the pivot column has been identified, we turn to the Ratio column (last column in the simplex tableau). This column is blank in Table 13.1. For each row, divide the RHS by the coefficient in the pivot column. This will give the ratio for each row. Then select the row with the lowest (positive) ratio. This is the pivot row, which identifies that variable that will leave the basis.

Once the pivot column and row are identified, a few operations are necessary. In the pivot row, divide (multiply) all the elements in the row by an appropriate number so that the coefficient in the pivot column becomes one (1). Then make all the other elements in the pivot column zero by subtracting from that row multiples of the pivot row. Remember that whatever multiplication or division in a row must be done for all the elements in the row. Once this is done, one of the slack variables leaves the basis as one choice variable enters. Virtually all the elements will change in the tableau.

This process must be repeated until all the Net gains are either zero or negative. When all the net gains are non-positive, the optimal solution is found and no further searching is needed. The optimal values of the variable will be read using the coefficient one (1) in each column. That is, to read the optimal value of the variable in the j^{th} column, check the row in which the coefficient one (1) in that column is. The RHS element in that row is the optimal value.

Let us now take an example. Since the graphical and the simplex methods must ideally arrive at the same conclusion, this subsection will as a strategy use an example solved with the former method. This will provide a check on the solutions as well as provide a detailed comparison of the two methods.

Example 13.7

Revisit Example 13.2. Solve this example using the Simplex tableau.

Maximise $Z = 6x_1 + 8x_2$

$$2x_1 + x_2 \leq 10$$
$$x_1 + x_2 \leq 6$$
$$x_1 + 3x_2 \leq 14$$
$$x_1, \quad x_2 \geq 0$$

The first step is to restate the problem by converting the constraint inequalities into equations by adding slacks. The problem now is

Maximise $Z = 6x_1 + 8x_2 + 0S_1 + 0S_2 + 0S_3$ Subject to

$$2x_1 + x_2 + S_1 = 10$$
$$x_1 + x_2 + S_2 = 6$$
$$x_1 + 3x_2 + S_3 = 14$$
$$x_1, x_2, S_1, S_2 S_3 \geq 0$$

As explained already, we start with the origin as the basic solution. The simplex tableau will be

6	8	0	0	0	Obj fn Coef		
x_1	x_2	S_1	S_2	S_3	RHS	CBV	Ratio
2	1	1	0	0	10	0	10
1	1	0	1	0	6	0	6
1	3	0	0	1	14	0	4.7
6	8				Marginal Net Gain		

Since all the CBVs are zero, the Net gains will equal the objective function coefficient for each variable. The highest Net Gain is in column 2, that is, variable x_2. Now divide the RHS by the corresponding element in the pivot column, column 2 to identify the lowest ratio. The last row is identified. The row and column are identified and we now look to changing the 3 in the pivot row and column to 1.

Divide every element in the pivot row by 3. Since the pivot column elements in row 1 and 2 are already 1, we subtract the pivot row without necessarily getting its multiples. Simply row 1 less the new row 3 and the same for row 2. The new tableau will be.

6	8	0	0	0	Obj fn Coef		
x_1	x_2	S_1	S_2	S_3	RHS	CBV	Ratio
$^5/_3$	0	1	0	$-^1/_3$	$^{16}/_3$	0	3.2
$^2/_3$	0	0	1	$-^1/_3$	$^4/_3$	0	2
$^1/_3$	1	0	0	$^1/_3$	$^{14}/_3$	8	14
$^{10}/_3$	0				Marginal Net Gain		

In the new tableau, the first column has the highest gain and therefore the pivot column. We then generate the ratio to determine the slack variable that leaves the basis.

The second row is the pivot row. Therefore, we have to turn $^2/_3$ to one (1) in the pivot column and all the other elements to zero by subtraction. Multiply every element in the pivot row by $^3/_2$. Then subtract $^5/_3$ times row 2 from row 1 and a third of row 2 from row 3. This operation will ensure that the rest of elements in the pivot column are zero. The new tableau is:

6	8	0	0	0	Obj fn Coef		
x_1	x_2	S_1	S_2	S_3	RHS	CBV	Ratio
0	0	1	$-^5/_2$	$-^7/_3$	2	0	
1	0	0	$^3/_2$	$-^1/_2$	2	6	
0	1	0	$-^1/_2$	$^1/_2$	4	8	
0	0	0	-5	-1	Marginal Net Gain		

The Net gains are all zero or below. No improvements are possible. Therefore, the optimal values are $x_1 = 2$, $x_2 = 4$ and $S_1 = 2$. The first slack being non-zero means the first constraint is not binding. It has a slack at the optimal solution. The value of $Z = 6(2) + 8(4) = 44$.

13.9 Comparison between the Simplex and Graphical Methods

In the preceding section, we solved using the simplex method the same problem that we had solved by the graphical method in section 13.6. By dint of the linear programming theorem which tells us that if an optimal feasible solution exists, it will be located in one of the extreme points of the closed convex set constituted by the feasible region. We had to evaluate only the points A, B, C and D in the graph in Example 13.2. But all the four points had to be evaluated in order to identify the optimal feasible solution. The Simplex method however has two advantages over the graphical method.

- Starting from an arbitrary feasible solution which is the origin (which will obviously yield a zero value for the objective function), it leads us step by step (that is, by changing one variable at a time) to improved solutions until the optimal solution is reached. The procedure gives us a clear indication when the optimal solution is reached so that we do not have to evaluate any remaining extreme points.
- There is one further advantage of the simplex method. In the example provided, one could have moved to the optimal point B by moving from O to A and then to B or by moving from O to D, D to C and then C to B. The direction of movement in the first case would have involved two steps whereas the direction of movement in the second case would have involved three steps. The first case therefore provides a shorter route from the origin to the optimal feasible solution in comparison to the second case. In here, the simplex procedure tells us to move in the direction suggested by the first case and not the second case.

The above two advantages show that the simplex (algebraic) method is more efficient than the graphical method in solving a linear programme involving only two programme variables.

Needless to say in linear programmes involving more than two programme variables, one has no option but to resort to the algebraic method.

13.10 Recent developments in Solving Linear Programming Problems

13.10.1 Algorithmic developments

The simplex method explained in the preceding section basically involves proceeding from one vertex to another vertex of the feasible region until the optimal solution is attained. In large scale problems this could be very time-consuming (sometimes weeks in optimisation problems involving communication networks) and hence inefficient. Attempts have been made to develop more efficient algorithms. One such algorithm was developed in 1984 by an Indian mathematician Narendra Karmarkar.

15.1.2 Karmarkar's algorithm

Karmarkar's algorithm solves linear programming problems by using an *interior point method* first invented by John von Neumann. It cuts through the requirement of proceeding from one vertex to another by traversing the interior of the feasible region. Consequently, the solution time gets drastically reduced. Weeks could be reduced to days. This in turn enables faster policy decisions.

Karmarkar's algorithm has stimulated the development of several other efficient methods of solving linear programming problems.

Discussion of all these algorithms is beyond the scope of this book.

Chapter 14

14 SOME EXTENSIONS OF LINEAR PROGRAMMING

14.1 Introduction

The linear programming model discussed in chapter 13 can be extended to analyze more complex problems. It is itself a tool for an easy way to reduce complex problems to something much easy to grasp and understand. In this chapter, we apply the notion of linear programming to understand three models. The Transportation model is presented in section 14.2 while its special case, the Assignment model is presented in section 14.4. The Transhipment model, which is an extension of the Transportation model is presented in section 14.5.

14.2 Transportation model

One area of application of the linear programming techniques is in the decision making process by firms. Suppose we are given one hypothetical firm with multi production plants and several markets scattered across the country or across the world for a multinational corporation. The firm has m production sites and n market points. The production sites and markets are looked at as points because the main interest is the transportation problem. It is pretty easy to define a transportation problem with specific points of origin and destinations. From what is provided, there are m sources of the product to satisfy the demand in n markets. There is need at this stage to assume the product from all the production points are identical so that there is no restriction on sources and destinations. Output from any source can be transported to any destination.

The different plants have varying production capacities. We denote output from the i^{th} plant as s_i. Total output for the firm will be a summation of various outputs from all the plants. This constitutes the supply of the commodity and can be denoted as

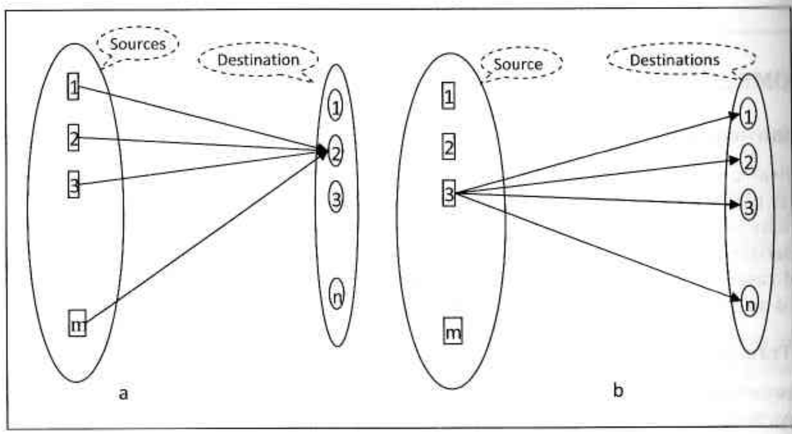
$$S = \sum_{i=1}^{m} s_i$$

The demand side is also made up of individual demands from various markets. Let us denote the demand from the j^{th} market as d_j. The total demand by the n markets is given by

$$D = \sum_{j=1}^{n} d_j$$

this is summarised in Figure 14.1 below

Figure 14.1. Source and Destination nodes



Part (a) of the figure shows all the sources supplying to destination (2). The sum of these shipments will equal the demand from destination (2). Part (b) shows it the other way round. It has one source (3) supplying to all the destinations. This means the total output at source node (3) must equal the sum of shipments from this node to all the destination nodes.

We know that the firm cannot overproduce nor can it under produce, at least in the long run. This means the quantity supplied equals the quantity demanded and the model is said to be balanced.

The model is not balanced when total supply does not equal total demand. There is either a deficit (when the demand exceeds the supply) or a surplus (when supply exceeds the demand). The linear programming model cannot be used in an unbalanced model. This is because the linear programming assumes a balanced model.

As an option for the above problem of unbalanced model, a dummy source or destination is assumed in the model. When there is a deficit, a dummy supply is brought into the model to make up for the difference. The same applies in the case of a surplus where a dummy source is assumed.

The next variable we define is the cost of transporting the output from the production plants to the destination markets. If each and every plant supplied to all the markets, there would be $m \times n$ transportation routes. In reality however, production plants will not necessarily have to supply all the markets and markets do not have to receive from all the plants. The output from all the plants is identical so the specific source is of no interest. The cost of transporting one unit of output from source i to destination j is c_{ij}. The total cost will depend on how much output is transported.

The objective of the model is to find the combinations of output to be transported that will results in the lowest transportation cost. Formally, the model is stated as

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_i$$

That is, output from all sources is transported and all markets are supplied needed quantities respectively.

When there is a dummy source or destination in the model, there is possibility that the dummy distorts the optimal outcome. In the optimum case, the dummy source or destination may be assigned a quantity different from the quantity it meant to account for. For instance, suppose there is a deficit of 100 units so that a dummy source is added which is supposedly producing the 100 units. To ensure the dummy source is assigned 100 units in the optimum outcome, the unit transportation cost from the dummy source to any destination note is assigned at zero.

Different transportation routes have different per unit costs. Finding the optimal transportation strategy involves two stages. The first is to find what is referred to as *feasible solution*. Since these methods do not always result in the optimal strategy, there is need to test whether the feasible solution found in the first step is the optimal solution. This becomes the second step.

To find the feasible solution, three alternative methods are used. The first is the *Northwest corner rule*. The second is the *lowest cost entry method*. As the name suggests, this method involves searching for lowest individual transportation routes. The third is known as *Vogel's Approximation Method (VAM)*. This method usually produces an optimal or near optimal starting solution.

14.2.1 Northwest corner rule

The objective of this method is to find the feasible solution rather than an optimal solution. It is concerned with ensuring that all supply is transported without paying attention to the resulting cost. In the transportation tableau with sources indicated in rows and destinations in columns, this method requires starting with the northwest corner, that is, the first column first row. Exhaust that cell by transporting everything from the first source to the first destination. If the source is exhausted before the destination, move to the next row in the same column. If it is the destination satisfied before exhausting the source, then move to the next column in the same row. Once a row or column is satisfied, move to the next until ending in the southeast corner.

This method is equivalent to the *first-come first-served* rule. It deals with sources and destinations as they come without paying attention to the unit costs. When any two rows or columns are interchanged, the method will identify a different feasible solution. Let us consider the following example.

Example 14.1

The Ministry of Education has received funding to buy lockers for distribution to four boarding schools in the following districts: Monze (50 lockers), Kabwe (65 lockers), Kaoma (25 lockers) and Chongwe (40 lockers). The tenders have been awarded to three bidders for each to make 60. One bidder is in Lusaka, another in Livingstone and another in Ndola. The unit cost of transporting each locker from each source and destination is given in the table below.

		Destination			
		Monze (50)	Kabwe (65)	Kaoma (25)	Chongwe (40)
Sources	Lusaka (60)	2	1.5	4	0.5
	Livingstone (60)	4	6	7	5
	Ndola (60)	5.5	2	8	4

The method requires starting with the first row first column. Transport the maximum possible from Lusaka to Monze. Lusaka has 60 lockers but Monze only needs 50. So transport the 50 to Monze and the whole Monze column is exhausted. Then move to the next destination (column). Transport the remaining 10 from Lusaka to Kabwe. Kabwe now remains with a deficit of 55 lockers and everything from Lusaka is taken.

Next transport 55 to Kabwe from Livingstone. Kabwe is satisfied but Livingstone still has 5 remaining. Take these to Kaoma. Next take 20 from Ndola to Kaoma and the remaining 40 from Ndola go to Chongwe. The feasible solution will be given by.

		Destination				
		Monze (50)	Kabwe (65)	Kaoma (25)	Chongwe (40)	**Total**
Sources	Lusaka (60)	50	10			**60**
	Livingstone (60)		55	5		**60**
	Ndola (60)			20	40	**60**
	Total	**50**	**65**	**25**	**40**	**180**

We leave it to the reader to verify that the total cost will be $K800$. The feasible solution will change when the order in which sources and destinations are presented and any solution from this method will only be optimal by chance.

14.2.2 Lowest Cost Entry method

Since the objective is to minimise the total transportation cost, then we must use least cost routes whenever possible. Given different available routes, with different unit costs, the decision must be using one that is least cost. Only when such a route is no longer available, because the source or destination in that particular route has been exhausted should the next least cost be used. In summary, given the cost matrix, exhaust the routes in the order of ascending cost. This is easier to illustrate with specific examples as given below.

Example 14.2

Zambeef is a leading producer of beef in the country with many abattoirs and retail outlets across the country. To make the illustration simple, assume there are only three abattoirs located in Sinazongwe, Chisamba and Chipata with respective outputs of 350, 400, 270 units of beef. The three market destinations are Lusaka, Livingstone and the Copperbelt. The demands from the market are 380, 200 and 440 respectively. The unit cost of transporting from each abattoir to a market is as given in the table below.

	Lusaka 380	Livingstone 200	Copperbelt 440
Sinazongwe 350	7.5	5.0	10.5
Chisamba 400	3.0	11.0	6.0
Chipata 270	7.5	14.5	13.5

Determine the optimal transporting strategy.

There is a total of 9 transportation routes available with different unit costs. Of the 9, the Chisamba-Lusaka is the cheapest. Chisamba has a total of 400 units of the commodity but Lusaka only takes 380. So, 380 units is transported from Chisamba to Lusaka. Lusaka now has all it needs but Chisamba still has the remainder of 20 units. Since the Lusaka destination is exhausted, then the whole column referring to Lusaka must be removed as it refers to an exhausted destination.

Now 6 possible routes remain. The least cost route is now Sinazongwe-Livingstone. Using the same procedure as above, Sinazongwe has 350 and Livingstone only needs 200 units, so 200 units is transported which exhausts the Livingstone destination. Sinazongwe still remains with 150 units.

With two destinations exhausted, there is only one destination remaining. All the remaining output should then be destined to the Copperbelt. Extra care must be taken with numbers here especially the remainders. Just to remind ourselves, Chisamba remained with 20 units, Sinazongwe with 150 units and Chipata still has all the 270 units. If there was need for prioritisation, we would start with the 20 units from Chisamba since it has the least cost to the Copperbelt. But there is no need to set any order since everything must be transported to the same destination. The table below is a summary of the optimal transportation strategy.

	Lusaka	Livingstone	Copperbelt	**Total**
Sinazongwe	0	200	150	**350**
Chisamba	380	0	20	**400**
Chipata	0	0	270	**270**
Total	**380**	**200**	**440**	**1020**

To get the total cost, simply multiply the quantity transported with the unit cost of transportation for each used route. This gives the cost incurred on each route. The total cost will be a sum of all the individual route costs. The calculation is provided in the table below.

Route	Quantity	Unit Cost	Total Cost
Sinazongwe-Livingstone	200	5.0	1,000
Sinazongwe-Copperbelt	150	10.5	1,575
Chisamba-Lusaka	380	3.0	1,140
Chisamba-Copperbelt	20	6.0	120
Chipata-Copperbelt	270	13.5	3,645
Total	**1020**		**7,480**

Example 14.3

Suppose there is a critical food shortage in three districts of Zambia, that is, Zambezi, Mpika and Katete. The actual deficits are estimated to be 4500, 2700, and 3600 respectively. Fortunately, government has stocks of maize available in four of the six grain storage silos across the country. The available quantities are Kabwe 2900; Kitwe 2800; Lusaka 2500; and Monze with 2600. How should these quantities be transported to the needy districts so that minimum transportation cost is incurred? Assume the following cost matrix based on the distance chart.

	Zambezi 4500	Mpika 2700	Katete 3600
Kabwe 2900	87	50	63
Kitwe 2800	66	58	85
Lusaka 2500	77	64	49
Monze 2600	96	83	68

Using the lowest cost entry method, search for the smallest number in the matrix. The cheapest route is the Lusaka-Katete so allocate the maximum possible of 2500 which exhausts the supply at Lusaka. So delete the Lusaka source (row) and reduce the quantity for Katete by 2500 since it has received this from Lusaka.

With the Lusaka source gone, the next cheapest is the Kabwe-Mpika route. A maximum of 2700 is transported which satisfies the Mpika needs. This destination is eliminated as well and Kabwe now has 200 units $(2900 - 2700)$ remaining.

Using the same procedure, the next is Kabwe-Katete where all the remaining units in Kabwe of 200 is transported. Katete still remains with unmet demand of the remainder. Then Kitwe supplied 2800 to Zambezi which exhausts the source.

Three sources are exhausted and there is only one remaining to satisfy the two destinations. Simply apportion according to destination needs. Zambezi will take 1700 while Katete takes 900. The summary table is given below

	Zambezi	Mpika	Katete	**Total**
Kabwe		2700	200	**2900**
Kitwe	2800			**2800**
Lusaka			2500	**2500**
Monze	1700		900	**2600**
Total	**4500**	**2700**	**3600**	

This allocation satisfies all the sources and destinations. It is therefore a feasible solution. An optimality test is needed to ascertain the optimality of this allocation. The total cost will be given by

Route	Quantity	Unit Cost	Total Cost
Kabwe-Mpika	2700	50	135,000
Kabwe-Katete	200	63	12,600
Kitwe-Zambezi	2800	66	184,800
Lusaka-Katete	2500	49	122,500
Monze-Zambezi	1700	96	163,200
Monze-Katete	900	68	61,200
Total	**10800**		**679,300**

14.2.3 Vogel's Approximation Method (VAM)

Vogel's method is unique and a little complex compared to the former. It is a realisation that the lowest cost may not always be optimal. Instead of looking at one route at a time, Vogel insists that some routes, though cheapest, may prevent us from using other cheap routes. The absolute cost is not enough to determine the optimality of a route but when considered in relation to others. Take a simple case of two source two destination model with the cost matrix given in Table 14.1. Assume that the sum of output at the two sources equal total demand from the two destinations so that the model is balanced.

Table 14.1.Simple Transportation problem

	D_1	D_2
S_1	3	4
S_2	4	7

With the above cost matrix, the lowest cost method will begin with S_1D_1 because it has the lowest cost of 3. The next potential number is 4, because it is less than 7. However, the two routes which cost 4 are no longer available because all the output at S_1 has been exhausted to satisfy the demand at D_1. It now becomes mandatory to use the most expensive route in the model because what looked 'more attractive' has precluded the neighbourhoods. The total cost will be higher than if the absolutely low cost was avoided.

To find the feasible solution, the VAM applies five steps. They are as follows:

Step 1: Determine for each row and column the difference between the two lowest unit costs or cell elements in the cost matrix, including dummies.

Step 2: identify the row or column with the highest difference. When there is a tie, take one from the highest arbitrarily.

Step 3: Allocate as much as possible of the good to the lowest cell(s) in the row/column identified in step 2 above.

Step 4: Eliminate the exhausted row or column and repeat the first three steps. Stop when all the rows and columns are satisfied.

Step 5: Test whether the result is an optimal solution.

Example 14.4

Redo Example 14.2 using the VAM and compare the answers.

When working with the VAM, the cost matrix is very important and so is the quantity table. We take the matrix table and replicate it below. A row and a column are added in which the difference between the two lowest cells is presented.

	Lusaka 380	Livingstone 200	Copperbelt 440	
Sinazongwe 350	7.5	5.0	10.5	2.5
Chisamba 400	3.0	11.0	6.0	3
Chipata 270	7.5	14.5	13.5	6
	4.5	6	4.5	

There are two highest values, the second column and third row. We arbitrarily pick on the former which has the lowest cost on the Sinazongwe-Livingstone route. The maximum quantity on this route is 200 units corresponding to the total demand in Livingstone. So the Livingstone market is satisfied and hence the second column is eliminated.

After adjusting for the remaining quantity from Sinazongwe, because it has already supplied Livingstone, and calculating the new differences, we have the following table.

	Lusaka 380	Copperbelt 440	
Sinazongwe 150	7.5	10.5	3
Chisamba 400	3.0	6.0	3
Chipata 270	7.5	13.5	6
	4.5	4.5	

The third row has the highest difference. Allocate the maximum possible cargo for the Chipata-Lusaka route. The route takes all the supply from Chipata but the Lusaka market still has a deficit of 110. We delete the Chipata source and make the necessary adjustments to the difference row and column.

	Lusaka 110	Copperbelt 440	
Sinazongwe 150	7.5	10.5	3
Chisamba 400	3.0	6.0	3
	4.5	4.5	

There is a tie again between Lusaka and Copperbelt destinations. Arbitrarily, take all the output from Chisamba to the Copperbelt. This only leaves one source. So the remaining output in Sinazongwe is apportioned to Lusaka and Copperbelt. This completes the transportation strategy which we present below.

	Lusaka	Livingstone	Copperbelt	**Total**
Sinazongwe	110	200	40	**350**
Chisamba			400	**400**
Chipata	270			**270**
Total	**380**	**200**	**440**	**1020**

This strategy has the following cost.

Route	Quantity	Unit Cost	Total Cost
Sinazongwe-Livingstone	200	5.0	1,000
Sinazongwe-Copperbelt	40	10.5	420
Sinazongwe-Lusaka	110	7.5	825
Chisamba-Copperbelt	400	6.0	2,400
Chipata-Lusaka	270	7.5	2,025
Total	**1020**		**6,670**

The strategy has not utilised the least cost Chisamba-Lusaka route but the total cost falls far below the total arrived using the previous method. This is a 'litmus-test' which proves that the earlier method did not yield an optimal outcome. However, it must not imply that the VAM has itself produced an optimal strategy.

Example 14.5

Redo **Example 14.3** using the VAM method. Which method produces the lower cost?

The example in question has the following cost matrix

	Zambezi 4500	Mpika 2700	Katete 3600
Kabwe 2900	87	50	63
Kitwe 2800	66	58	85
Lusaka 2500	77	64	49
Monze 2600	96	83	68

For each row and column, calculate the difference between the lowest and second lowest cell element. These are inserted as follows

	Zambezi 4500	Mpika 2700	Katete 3600	
Kabwe 2900	87	50	63	13
Kitwe 2800	66	58	85	8
Lusaka 2500	77	64	49	15
Monze 2600	96	83	68	15
	11	8	14	

There is a tie between the last two rows or sources. So we pick the Lusaka source which is all transported to Katete. The Lusaka source is no more so we eliminate the corresponding row. The resulting table with new differences calculated is

	Zambezi 4500	Mpika 2700	Katete 1100	
Kabwe 2900	87	50	63	13
Kitwe 2800	66	58	85	8
Monze 2600	96	83	68	15
	21	8	5	

Now we have the first column. The lowest cost is from Kitwe. Transport all output from Kitwe to Zambezi and the Kitwe source is closed.

	Zambezi 1700	Mpika 2700	Katete 1100	
Kabwe 2900	87	50	63	13
Monze 2600	96	83	68	15
	9	33	5	

Then transport 2700 units from Kabwe to Mpika. This closes the Mpika destination but the source still has some remaining output. The resulting table is:

	Zambezi 1700	Katete 1100	
Kabwe 200	87	63	24
Monze 2600	96	68	28
	9	5	

From Monze transport 1100 to Katete. The Katete demand is satisfied so eliminate the corresponding column.

	Zambezi	
Kabwe 200	87	24
Monze 1500	96	28
	9	

There is only one destination remaining so no need to calculate the cost difference. The output is transported accordingly. From Kabwe to Zambezi 200 units and Monze to Zambezi 1500. The final table will appear as follows.

	Zambezi	Mpika	Katete	**Total**
Kabwe	200	2700		**2900**
Kitwe	2800			**2800**
Lusaka			2500	**2500**
Monze	1500		1100	**2600**
Total	**4500**	**2700**	**3600**	**10800**

The total cost is given in the table below.

Route	Quantity	Unit Cost	Total Cost
Kabwe-Mpika	2700	50	135000
Kabwe-Zambezi	200	87	17400
Kitwe-Zambezi	2800	66	184800
Lusaka-Katete	2500	49	122500
Monze-Zambezi	1500	96	144000
Monze-Katete	1100	68	74800
Total	**10800**		**678,500**

14.3 Test for Optimality

As stated earlier, the solutions that come from the above three methods are not always optimal. The solution may be optimal or may not at all be optimal. This calls for the need to test for the optimality of the solution. Because the optimality of the solution is not certain yet, the solution cannot be called optimal. An alternative name, *basic feasible solution* is used to distinguish it from a tested optimal solution.

To test for optimality, we must start with a *non-degenerate* basic feasible solution. The solution is non degenerate if it possesses the following two properties

a. The solution must contain exactly $m + n - 1$ number of individual allocations. The variables m and n denote the number of sources and destinations respectively.

b. The allocations must be such that it is impossible to form any closed loop by drawing vertical and horizontal lines through these allocations.

Two methods are used to test for the optimality of the solution. These are:

- The *Stepping stone method*.
- the *u-v method* (also called the *Modified distribution method* or *MODI*).

14.3.1 The stepping stone method

In the stepping stone method, the objective is to evaluate the effect (on cost) of using one or more of the unused routes. The method determines whether there is a cell with no allocations that would reduce the cost if used. We ask ourselves whether it is possible to reduce cost by allocating one unit to an unused route. This means trying all the empty or unused transportation routes. To illustrate this method, let us use the basic solution obtained by the lowest cost entry method in Example 14.3. The basic solution is

	Zambezi	Mpika	Katete	**Total**
Kabwe		2700	200	**2900**
Kitwe	2800			**2800**
Lusaka			2500	**2500**
Monze	1700		900	**2600**
Total	**4500**	**2700**	**3600**	

The solution has $6 = 4 \times 3 - (4 + 3 - 1)$ as required by the non degenerate condition. These are Kabwe-Zambezi route, Kitwe-Mpika, Kitwe-Katete, Lusaka-Zambezi, Lusaka-Mpika and Monze-Mpika routes. Let us start with the first route. Allocate one unit to the Kabwe-Zambezi route. To ensure the basic solution is not changed, subtract one unit from the Kabwe-Katete route, add a unity to the Monze-Katete rout and subtract one from the Monze-Zambezi routes. This will ensure that the row and column total which represent the demands and supplies are not altered. This must be done for all the empty or unused cells. If all reallocations show positive change in total cost, the basic solution is optimal since no reallocation would successfully reduce cost. If however one cell shows a negative change in cost, then the basic solution is not optimal yet. The total cost can be reduced by utilising the 'discovered' route. So allocate as much as possible to that route and recheck the optimality.

We show the changes in the table below.

	Zambezi	Mpika	Katete	**Total**
Kabwe	+1	2700	200-1	**2900**
Kitwe	2800			**2800**
Lusaka			2500	**2500**
Monze	1700-1		900+1	**2600**
Total	**4500**	**2700**	**3600**	

The change in total cost is $87 - 63 - 96 + 68 = -4$

The change in total cost is negative. That means this reallocation will lead to a reduction in cost implying that the basic solution is not optimal yet.

Reallocate the whole 200 units from the Kabwe-Katete to Kabwe-Zambezi and then take another 200 from Monze-Zambezi to Monze-Katete. We obtain the following matrix

	Zambezi	Mpika	Katete	**Total**
Kabwe	200	2700		**2900**
Kitwe	2800			**2800**
Lusaka			2500	**2500**
Monze	1500		1100	**2600**
Total	**4500**	**2700**	**3600**	

There is no guarantee yet that this allocation is optimal. We still need to test it for optimality by repeating the method on the new allocation.

There are 6 empty slots in the matrix. Nonetheless, the Kabwe-Katete route was already ruled out. So we start with the Kitwe-Mpika route. We form the following loop.

	Zambezi	Mpika	Katete	**Total**
Kabwe	200+1	2700-1		**2900**
Kitwe	2800-1	+1		**2800**
Lusaka			2500	**2500**
Monze	1500		1100	**2600**
Total	**4500**	**2700**	**3600**	

The change in cost resulting from allocating one unit to this route is $87 - 66 - 50 + 58 = 29$. This reallocation would result in increase in cost and therefore inferior to the current allocation.

We try the Kitwe-Katete route. We form the following loop.

	Zambezi	Mpika	Katete	**Total**
Kabwe	200	2700		**2900**
Kitwe	2800-1		+1	**2800**
Lusaka			2500	**2500**
Monze	1500+1		1100-1	**2600**
Total	**4500**	**2700**	**3600**	

The change in cost resulting from allocating one unit to this route is $85 - 66 - 68 + 96 = 47$. Again, this reallocation will lead to increased cost and therefore not ideal.

Now we try allocating one unit to the Lusaka-Zambezi route. The following loop is used.

	Zambezi	Mpika	Katete	Total
Kabwe	200	2700		**2900**
Kitwe	2800			**2800**
Lusaka	+1		2500-1	**2500**
Monze	1500-1		1100+1	**2600**
Total	**4500**	**2700**	**3600**	

The change to cost resulting from this reallocation is $77 - 49 - 96 + 68 = 0$. This reallocation has a zero additional cost. Whether it is used or not, the cost would stay the same.

We now move on to consider the Lusaka-Mpika route. The loop will be slightly different from the rest because it affects many routes.

	Zambezi	Mpika	Katete	Total
Kabwe	200+1	2700-1		**2900**
Kitwe	2800			**2800**
Lusaka		+1	2500-1	**2500**
Monze	1500-1		1100+1	**2600**
Total	**4500**	**2700**	**3600**	

The resulting change in cost is $87 - 50 + 64 - 49 + 68 - 96 = 24$. The reallocation will cause total cost to go up.

Lastly, we try the Monze-Mpika route. Reallocate one unit to this route and adjust the others accordingly to ensure the model remains feasible. We form the following loop.

	Zambezi	Mpika	Katete	Total
Kabwe	200+1	2700-1		**2900**
Kitwe	2800			**2800**
Lusaka			2500	**2500**
Monze	1500-1	+1	1100	**2600**
Total	**4500**	**2700**	**3600**	

The change in total cost is $87 - 50 - 96 + 83 = 24$. This reallocation will lead to increased cost. Therefore, it is not favoured.

We have shown that all the reallocations to the empty cells or unused routes have positive changes in costs except in one case, that is, the Lusaka-Zambezi route. This means there is no feasible reallocation that will succeed in reducing cost. Therefore, the obtained allocation is optimal. The existence of a zero additional cost in utilising the Lusaka-Zambezi route only

entails the existence of another optimal allocation. This is a case of multiples optimal allocations.

14.3.2 The *u-v method*

The method has this name because it uses u and v in the procedure. The following steps summarise this procedure:

Step 1: Determine a set of $m + n$ numbers $u_i(i = 1,2, \dots m)$ and $v_j(j = 1,2, \dots n)$ in such as way that for each occupied cell, the unit cost

$$c_{ij} = u_i + v_j$$

The method uses the occupied cells or utilised routes to determine a new set of variable u_i and v_j. However, there is only a total of $m + n - 1$ occupied cells equivalent to $m + n - 1$ independent equations to determine $m + n$ *unknowns*. This defies a mathematical condition on simultaneous equations, that the number of equations must at least be as many as are unknowns. To counteract this problem, one unknown must be arbitrarily determined.

Step 2: using the $u_i(i = 1,2, \dots m)$ and $v_j(j = 1,2, \dots n)$ from step 1, calculate for each empty cell (i, j) the *cell evaluation*. This is a unit cost difference d_{ij} given by $d_{ij} = c_{ij} - (u_i + v_j)$

Step 3: examine the matrix of cell evaluation and conclude as follows:

I. If all $d_{ij} > 0$, that is, all cell evaluations are strictly positive, basic feasible solution is not only optimal but also unique. Its total cost is the lowest. Any alternative solution will have higher total transportation cost.

II. If all $d_{ij} \geq 0$, that is, all cell evaluations are non negative, the basic feasible solution is optimal. However, an alternate solution also exists with the same level of total cost.

III. If at least one $d_{ij} < 0$, the solution is not optimal. This outcome means repeating the above process.

Example 14.6

Test the optimality of the basic feasible solution from Example 14.2 on page 289.

The basic feasible solution in the stated example is given in the table below. The table shows the unit cost of transportation for the utilised routes.

	Lusaka	Livingstone	Copperbelt	**v**
Sinazongwe		5	10.5	4
Chisamba	3		6	-0.5
Chipata			13.5	7
u	3.5	1	6.5	

It has $3 + 3 - 1 = 5$ utilised routes which are in independent positions. Therefore the basic feasible solution is non-degenerate. We can proceed with the u-v method.

With the u_s and v_s now determined, the cell evaluations table is given below.

	Lusaka	Livingstone	Copperbelt	v
Sinazongwe	0			4
Chisamba		10.5		-0.5
Chipata	-3	6.5		7
u	3.5	1	6.5	

The cell evaluation matrix has one negative element. There is no need here to talk about the other zero since the negative element overrides the zero elements. The conclusion is that this solution is not optimal. It is possible to change the transportation arrangement and reduce the costs.

Let us then test the allocation obtained by the VAM method in Example 14.4 which uses the same data. The following feasible allocation was obtained where the number in parenthesis is the unit cost.

	Lusaka	Livingstone	Copperbelt	**Total**
Sinazongwe	110 (7.5)	200 (5.0)	40 (10.5)	**350**
Chisamba	(3.0)	(11.0)	400 (6.0)	**400**
Chipata	270 (7.5)	(14.5)	(13.5)	**270**
Total	**380**	**200**	**440**	**1020**

There are three steps to follow.

Step 1: Determining the u_s and v_s. We use the following equations.

$$u_1 + v_1 = 7.5$$
$$u_2 + v_1 = 5$$
$$u_3 + v_1 = 10.5$$
$$u_3 + v_2 = 6$$
$$u_1 + v_3 = 7.5$$

Let $v_1 = 3$, we have the following:

	Lusaka 380	Livingstone 200	Copperbelt 440	v
Sinazongwe 350	7.5	5.0	10.5	3
Chisamba 400			6.0	-1.5
Chipata 270	7.5			3
u	4.5	2	7.5	

Step 2: Cell evaluation of empty cells, we have

	Lusaka 380	Livingstone 200	Copperbelt 440	v
Sinazongwe 350				3
Chisamba 400	0	10.5		-1.5
Chipata 270		9.5	3	3
u	4.5	2	7.5	

Step 3: Conclusion. On examination, all the cost differences $d_{ij} = c_{ij} - (u_i + v_j)$ of the unused routes are non-negative. We conclude that the basic feasible solution is optimal. There nonetheless exists another optimal solution with the same level of cost based on the zero element in the cell evaluation.

14.4 Assignment Model

In the Transportation model, we dealt with many sources and many destinations. Each source and destination is producing or demanding more than a unit of the commodity. The problem was to minimise the total cost of transporting all output from all the sources to satisfy all the demand by all the destinations. The question we pose here is what happens when each source is only producing one unit of output and each destination takes only one unit of output?

The question still under consideration is how much output from which source goes to which destination. This is a value-laden question but is precisely what the transportation model deals with. When each node (source and destination) only takes one unit of output, the question now is which source supplies which destination. The problem reduces from considering the quantities to only looking at the pairing of source and destination nodes. Each source must have a corresponding destination. The pairing could be done haphazardly since all nodes either produce one unit or demand one unit of output. However, source nodes will not supply to destination nodes at the same cost. With differing costs for different pairs, pairs must be strategically made so that the cost (transportation) is minimised. This is called the *assignment model*.

It is named assignment model because instead of deciding on quantities, its central problem involves assigning each source to a particular destination in a way that minimises transportation cost. As a special case of the transportation model, it is best described by the assignment of workers to jobs or chores. In the model, there are n workers and n jobs so that the model is balanced. Each worker can take any job but because of differing skills and job requirements, the performance on the job will vary depending on the job assigned to.

This can be an output maximisation problem. With the duality theorem at hand, the problem can be turned into cost minimisation. The cost is measured as the cost of producing a unit of output. Alternatively, it means a more productive worker will be less costly. We denote c_{ij} as the cost of assigning the i^{th} worker to the j^{th} job. This will vary depending on how productive a worker is at that particular job, which depends on skill-match. That is, how one's skills match the job requirements. The objective of the model is to determine the optimal (least cost) assignment of workers to jobs.

Formally, the assignment model is defined by the optimisation problem of

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} x_{ij} = 1$$

where x_{ij} defines whether a particular worker has been assigned to a particular job. It is defined as

$$x_{ij} = \begin{cases} 1, & if\ worker\ i\ assigned\ job\ j \\ 0, & otherwise \end{cases}$$

When there is a mismatch in the number of workers and available jobs, the model is said to be unbalanced. It is unbalanced in the sense that the number of workers and jobs do not equal. There may be more workers than jobs in which case some workers will not be assigned any job. The opposite is where the number of jobs exceeds the number of workers so that some jobs will not be assigned any workers. Since linear programming only workers with a balanced model, we must make-up for the lesser jobs or workers so that the model is balanced. This is achieved by adding *fictitious* workers or jobs as the case may be. When there are fewer jobs than workers, add as many fictitious jobs as is the short-fall. The same applies when the number of workers exceeds the number of available jobs.

14.4.1 The Hungarian method

The optimal solution is found using the Hungarian method. The method was developed by an American mathematician Harold William Kuhn based on the work of two *Hungarian* mathematicians; Denes Konig and Jeno Egervary from which the method derived its name. The method has five steps involving the cost matrix. Given the cost matrix

Step 1: Subtract the smallest entry in each row from all elements of its row.

Step 2: Subtract the smallest entry in each column from all the entries of its column.

Step 3: Draw lines through appropriate rows and columns so that all the zero entries of the cost matrix are covered and the minimum number of such lines used.

Step 4: Test for optimality

- If the minimum number of covering lines is equal to n, (assuming the cost matrix is $n \times n$), an optimal assignment of zeros is possible and we have reached the end of the process.
- If the minimum number of covering lines is less than n, optimality is not yet achieved and we go for step 5.

Step 5: Determine the smallest entry not covered by any line. Subtract this entry from each uncovered row and then add it to each covered column. Then return to step 3.

This has the lowest cost of assigning the n tasks to n workers. Any re-assignment will result in higher total cost of performing the tasks.

Example 14.7

A lodge supervisor has three workers to perform three tasks. He must assign one to work in the Bar, one in the laundry and another in the restaurant. The cost, measured in man-hours per unit of output, is given form each worker in the table below.

	Bar	Laundry	Restaurant
Banda	12	15	11
Mwiya	9	7	8
Phiri	12	13	17

How should he assign the three workers?

Let us follow the five step Hungarian method outlined earlier.

Step 1. Subtract 11 from row 1, 7 from row 2 and 12 from row 3 to obtain.

	Bar	Laundry	Restaurant
Banda	1	4	0
Mwiya	2	0	1
Phiri	0	1	5

Step 2: Subtract 0 from column 1, 0 from column 2 and 0 from column 3 and we get (the same matrix in this particular example).

	Bar	Laundry	Restaurant
Banda	1	4	0
Mwiya	2	0	1
Phiri	0	1	5

Step 3: Cover all zeros with the minimum number of horizontal or vertical lines. We get

	Bar	Laundry	Restaurant
Banda	~~1~~	~~4~~	~~0~~
Mwiya	~~2~~	~~0~~	~~1~~
Phiri	~~0~~	~~1~~	~~5~~

Step 4: Since the minimum number of lines is 3 (the number of assignments to be made), we have an optimal assignment as follows:

	Bar	Laundry	Restaurant
Banda	1	4	$\boxed{0}$
Mwiya	2	$\boxed{0}$	1
Phiri	$\boxed{0}$	1	5

The optimal assignment is shown by the zeros in independent positions. The corresponding cost will be:

	Bar	Laundry	Restaurant
Banda	12	14	$\boxed{11}$
Mwiya	9	$\boxed{7}$	1
Phiri	$\boxed{12}$	13	17

So Banda should be assigned to the restaurant, Mwiya to the laundry while Phiri to the Bar. The total cost will be $11 + 7 + 12 = 30\ manhours$

Example 14.8

Keren Motors Ltd provides a wide range of equipment for hire to mining companies in Zambia. These include bulldozers, excavators and other earth moving machines.

Suppose the company has four large caterpillar machines at four different sites. These caterpillars have to be moved to four different mining areas. The distances in kilometres between the caterpillar locations and the mining areas are given below.

	Mining Area			
Caterpillar	**A**	**B**	**C**	**D**
1	90	75	75	80
2	35	85	55	65
3	125	95	90	105
4	45	110	95	115

The problem is: Which caterpillar should be assigned to which mining area in order to minimise the total 'mobilisation' cost measured by distance travelled?

To find the optimal solution, we have to follow the five steps of the Hungarian method.

Step 1: subtract 75 from row 1, 35 from row 2, 90 from row 3 and 45 from row 4. We obtain the following matrix.

	Mining Area			
Caterpillar	**A**	**B**	**C**	**D**
1	15	0	0	5
2	0	50	20	30
3	35	5	0	15
4	0	65	50	70

Step 2: subtract 0 from column 1, 0 from column 2, 0 from column 3 and 5 from column 4 to obtain.

	Mining Area			
Caterpillar	**A**	**B**	**C**	**D**
1	15	0	0	0
2	0	50	20	25
3	35	5	0	10
4	0	65	50	65

Step 3: Cover all the zeros of the matrix with the minimum number of horizontal or vertical lines. We get:

	Mining Area			
Caterpillar	**A**	**B**	**C**	**D**
1	~~15~~	~~0~~	~~0~~	~~0~~
2	0	50	20	25
3	35	5	0	10
4	0	65	50	65

Step 4: Since the minimum number of lines is less than 4, we have to proceed to Step 5.

Step 5: The smallest entry not covered by any line is 5. So subtract it from each uncovered row. We get:

	Mining Area			
Caterpillar	**A**	**B**	**C**	**D**
1	15	0	0	0
2	-5	45	15	20
3	30	0	-5	5
4	-5	60	45	60

Now add 5 to each covered column. We get:

	Mining Area			
Caterpillar	**A**	**B**	**C**	**D**
1	20	0	5	0
2	0	45	20	20
3	35	0	0	5
4	0	60	50	60

Step 3: Cover all the zeros with the minimum number of horizontal or vertical lines.

	Mining Area			
Caterpillar	**A**	**B**	**C**	**D**
1	20	0	5	0
2	0	45	20	20
3	35	0	0	5
4	0	60	50	60

Step 4: An optimal assignment is still not possible since the minimal number of covering lines is less than 4, the number of assignments needed. So we get to Step 5 again.

Step 5: The smallest element not covered by any line is 20. Subtract 20 from each uncovered row. We get.

	Mining Area			
Caterpillar	**A**	**B**	**C**	**D**
1	20	0	5	0
2	-20	25	0	0
3	35	0	0	5
4	-20	40	30	40

Then add 20 to each covered column and obtain:

	Mining Area			
Caterpillar	**A**	**B**	**C**	**D**
1	40	0	5	0
2	0	25	0	0
3	55	0	0	5
4	0	40	30	40

Then return to Step 3.

Step 3: Cover all the zeros of the matrix with the minimum number of horizontal or vertical lines

	Mining Area			
Caterpillar	**A**	**B**	**C**	**D**
1	40	0	5	0
2	0	25	0	0
3	55	0	0	5
4	0	40	30	40

Step 4: Now the minimal number of covering lines is 4. So an optimal assignment is possible.

	Mining Area			
Caterpillar	**A**	**B**	**C**	**D**
1	40	0	5	$\boxed{0}$
2	0	25	$\boxed{0}$	0
3	55	$\boxed{0}$	0	5
4	$\boxed{0}$	40	30	40

The optimal assignment is shown by the zeros in independent positions. The corresponding cost will be:

	Mining Area			
Caterpillar	**A**	**B**	**C**	**D**
1	90	75	75	$\boxed{80}$
2	35	85	$\boxed{55}$	65
3	125	$\boxed{95}$	90	105
4	$\boxed{45}$	110	95	115

So caterpillar 1 should be sent to mining area D. Caterpillar 2 sent to mining area C, caterpillar 3 to mining area B and caterpillar 4 to mining area A. The total distance (a measure of mobilisation cost) is $80 + 45 + 95 + 45 = 265km$

14.4.2 König's Theorem

Underlying the assignment algorithm are the following theorems:

1. If a number is added to or subtracted from all of the entries of any one row or column of a cost matrix, then an optimal assignment for the resulting cost matrix is also an optimal assignment for the original cost matrix.
2. The maximum number of independent zero positions in a matrix is equal to the minimum number of lines (known as *covering index*) required to cover all the zeros in the matrix

14.5 Transhipment model

Transhipment, as the name suggests is the shipment of goods to an intermediate destination and later on to the final destination. There are many reasons for transhipment, legal and illegal. Here we just dwell on the legal and economical reasons. One reason is to change the means of transport during the journey, for instance, from road to rail or vice versa. This is known as transloading.

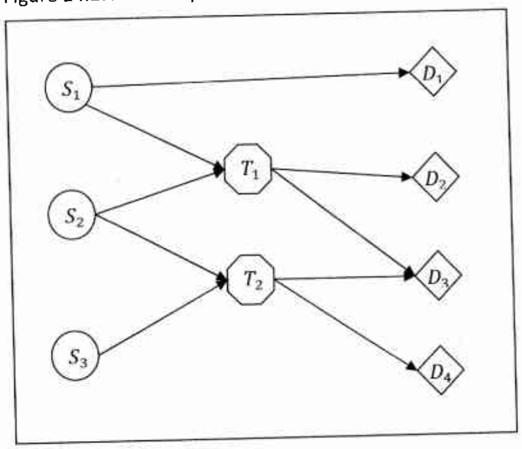
The other is to combine small shipments into large shipments and vice versa. Quantities of commodities to be transported from satellite depots may not be large enough to warrant the use of large trucks which have low per unit costs. So smaller vehicles, with high per unit cost are used to ferry goods to centres. These centres are not the final destinations so the large trucks then transport to final destinations. This is known as consolidation and the opposite is deconsolidation.

Transhipment brings in the concept of intermediate destination, a destination which is not final and itself having no demand for the goods transported. The goods are meant for another destination called the final destination. It is nonetheless still possible for some goods to be transported directly from the source to the final destinations without having to pass through some transient destinations.

In other cases, some final destinations or source may actually be used as transient destinations. For instance, the distribution of educational or health supplies is normally through provincial centres for final disbursement to districts. But the provincial centre is also a district with a portion of the final demand. This is a case of a final destination functioning as a transient point. When the transportation direction is reversed, some sources become transient destinations.

The diagram showing the different transportation routes is portrayed in Figure 14.2. It has three primary sources, two intermediate destinations and four final destinations. The model must be treated as an extension of the transportation model by allowing for transloading and/or consolidation. Therefore, the direct routes that were available under the transportation model are still available, save with a possibility of combining cargo from or to different points.

Figure 14.2: A transhipment model



The model provides for many shipments routes. These do not need to be used all together. Economic rationale will dictate which routes are used. Moreover, usage of certain routes rules out the possibility of using certain other routes.

The rationale again is to determine the routes that minimise total transportation costs. We continue to take c_{ij} as the cost of transporting a unit of output from source i to destination j. This is a direct route. Then define C_{ik} as the cost of transporting a unit of output from source i

to an intermediate destination k. Then it will cost C_{kj} to transport a unit of output from an intermediate destination k to the final destination j. This allows categorising costs at three levels. The first is the cost of direct routes, from a source to a final destination. The second will be costs incurred to transport from sources to intermediate destinations and the third category is the shipment cost from these intermediate destinations to final destinations.

Formally, the transhipment model is stated as follows.

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{i=1}^{m}\sum_{k=1}^{l} c_{ik}x_{ik} + \sum_{k=1}^{l}\sum_{j=1}^{n} c_{kj}x_{kj}$$

This is the total cost that will be incurred in the shipment. There are conditions however that must be met regarding the total quantities from the sources and destinations. There are basically three conditions, in addition to having all quantities x_s as non negative.

The first is that the output from every source must be transported

$$\sum_{j=1}^{n} x_{ij} + \sum_{k=1}^{l} x_{ik} = a_i, \qquad \forall\, i$$

Second, all final destinations must be satisfied

$$\sum_{i=1}^{m} x_{ij} + \sum_{k=1}^{l} x_{kj} = b_j, \qquad \forall\, j$$

Third, everything arriving at a transient point must depart to final destinations. Every such point must dispatch as much as it receives.

$$\sum_{i=1}^{m} x_{ik} = \sum_{j=1}^{n} x_{kj}, \qquad \forall\, k$$

For illustration purpose, consider the following example

Example 14.9

Consider the Zambeef case used in Example 14.2. In that example, we looked at a case where beef emanates from abattoirs and has to be transported to various markets. In this example, we now take into account the source of animals for the abattoirs. The full model will now have three points: The farms will be the primary sources which will supply animals to abattoirs. In this case, abattoirs will be destinations, but only intermediate destinations. After the slaughtering and all the required processes, the beef now has to be transported from abattoirs to final destinations, the markets. In the second stage, abattoirs are acting as sources (intermediate sources) while the markets are final destinations.

Assume that Zambeef has m farms producing beef cattle at different scales, l abattoirs and n destination markets (of different sizes).These could be town or district centres. The reader should realise two notions from this example. The first is that transportation from the primary source to final destination is not permissible since the product has to undergo some processing at what we call transhipment points or abattoirs.

Second, because of this processing, unit cost will also include the cost of processing. But this cost is more likely to be equal for all abattoirs since the mother company will more likely build equal capacity in them. Nonetheless, the cost is simply factored into the unit cost. The abattoirs neither have a fixed supply nor fixed demand. In reality however, there may be issues of capacity where an upper limit exists.

The problem for Zambeef is to devise a transhipment mechanism that will result in lowest cost of turning cattle from different farms into beef in different outlets. This cost will not be minimised simply by minimising the distance to the abattoir or using an abattoir nearest to the market. That is

$$\min \sum_{i=1}^{m} \sum_{k=1}^{l} c_{ik} x_{ik} + \sum_{k=1}^{l} \sum_{j=1}^{n} c_{kj} x_{kj}$$

This minimisation is subject to the requirement that the total quantities dispatched from the primary sources equal the total arriving at final destinations and the non-negativity constraints on the quantities.

Chapter 15

15 INTRODUCTION TO GAME THEORY

15.1 Introduction

In the optimisation problems studied so far, a decision by the firm on price is what solely determines the quantity and consequently, its profits. For instance, a farmer's profit will be solely determined by his own decision regarding quantity and price which will depend on the market. If the firm is operating in a perfectly competitive market, its profit will solely depend on the quantity it decides to produce. If operating in a monopolistic market, it can either decide on price or quantity (never on both) and the profit will be determined accordingly.

In this chapter however, we introduce a different scenario. A scenario where a firm's profit or gain does not only depend on its own decisions. They will depend on what other players in the market decide. Think of a tender process. A tender will specify the size and/or quality of expected output in what is known as *bill of quantities*. The bidding firm will then maximise profits by working on the mark-up. With a higher mark-up, the firm will make a higher profit. But there is a limit to this. Many firms will participate in the tender and the tender is offered to the lowest bidder. That is, increasing the mark up will increase the profits but only so long as other bidders have put a much higher mark up. When the mark-up exceeds that of the next lowest bidder, the firm looses the bid.

The bidders each makes a decision on the mark-up with a view to maximise own profits. Each bidder does not know what the others have decided until the opening of bids. When bids have been opened, only one firm gets the job. The acquisition of the job here is not only dependent on the firm's bid price but also on what all the other firms have put. Once the job is secured, the actual profit will then be a function of its own price.

Thus each firm will have to play it '*right*' if it has to be given the job. It must strategise on the price to set. In so doing, it has to take into consideration what the others are likely to decide. This is not different from how a game like poker is played. In the same manner, firms play games when they decide on strategies to maximise profits. They must decide on whether to advertise or not, to lower prices or not and many other decisions which will only impact on profits depending on the steps taken by other players in the market.

This chapter is devoted to introducing the theory behind such decision making, the *game theory*. As an introduction, the chapter will not provide a 'mouth-full' of a discussion. It will merely introduce the theory by simply mentioning the various aspects of game theory.

15.2 What is a game?

In answering this question, there is no need to restrict oneself to economics. Games are played in virtually all spheres of life and take different forms. Think of a game like soccer, poker and teller. In each of these games, there is more than one player. Each player plays in such a way as to optimise own desires which normally are in conflict with those of the other players.

In economics, the understanding of a game is in relation to markets behaviour. Each market has producers (suppliers) and consumers (buyers). Each producer has an objective of maximising profits by having as many buyers as possible. But since a market has a finite number of buyers, an increase in one producer's buyers logically means a reduction in another's. This turns into some competition in which each producers takes actions to increase one's sales.

This above scenario can be described as a game. The many producers are the players in the game. Their range of actions is strategies. The profits that result are the outcomes of the game. More precisely, a game is a situation involving more than one decision maker where the value of the objective function of one decision maker depends not only on his choices but also on the choices of others. In Downing and Clark (1988), a game is defined as 'an activity wherein each of a group of players is faced with a fixed set of choices and each possible combination of choices among players has a fixed outcome'.

A game can also be illustrated using the famous *Prisoner's dilemma*. It is as follows. Two people, Chanda and Mweemba, are arrested for a crime which carries say 15 years imprisonment if convicted. The police however lack evidence for the charge. Without the critical evidence, the two cannot be convicted of the main crime. Nonetheless, they can still be convicted on a lesser charge. Assume the minor charge carries one year imprisonment. If they both confess, they get convicted of the principal charge but the Judge will exercise lenience and reward their honest by sentencing them to 6 years each. The police therefore offer each suspect, in solitary confinement, a bargain.

Each suspect is told that if he/she confesses and testifies against the other, he gets discharged. Each suspect is therefore faced with the following *possibility matrix*. The matrix shows the number of years that each is sentenced to for each combination of pleas. The first number refers to Chanda and the second to Mweemba.

Table 15.1.Prisoners' Dilemma Illustration

GAME		Mweemba	
		Confess	Don't Confess
Chanda	Confess	6, 6	0, 15
	Don't Confess	15, 0	1, 1

Clearly, the suspects are better off not confessing. They only get one year each compared to 6 years if they confess. If they were interrogated together, they would keep their mouths shut. But since interrogation is in confinement, each one knows the danger of not confessing if the other confesses. One actually knows that the other may get attracted to the possibility of discharge and so will be more likely to confess. Once one thinks the other may confess, the best decision is to confess also.

But the other one does not confess because of the possibility of discharge. He confesses because he/she too is worried of what the other is worried of. The two are in a dilemma. At the end, they both confess. This is a decision not in their best interest. They were going to be better off not confessing.

The above illustration is a traditional narrative of the prisoner's dilemma. In economics, we can explain it with an economic illustration from international trade involving two countries. Suppose that two countries Zambia and Zimbabwe are contemplating on liberalising trade. From a theoretical view, trade liberalisation would benefit the two countries on condition that both liberalise. If one liberalises and the other doesn't, the one liberalising will lose out. Let us assume the measured losses and gains are as given in the payoff matrix below.

Table 15.2. Game of Trade liberalisation

GAME		Zimbabwe	
		Liberalise	Control
Zambia	Liberalise	10,10	-15, 20
	Control	20, -15	0, 0

The two countries are making decision in the 'dark'. That is, each does not know what the other will decide upon. Collectively, the two countries make the highest gains if they both liberalise. With only one liberalising, the total net gain is only 5 to the non-liberalising country.

Though each country knows the benefit from liberalisation, they are both worried of the losses if the other does not follow suit. As such, the optimal decision for each will be to retain controls on trade. Each country will after liberalising because it believes the other may not liberalise. Thus both will remain in the 'no trade liberalisation' where they forgo the positive benefits from trade liberalisation.

The subject of Game Theory specifically deals with behaviours in a game. It seeks to explain how players will behave depending on the nature of games. By nature of games, we mean the way a particular game relates the players. Does the game provide the scope for conflict or cooperation? Two extreme levels of relationship are *pure conflict* and *whole hearted cooperation*. The latter subsist in situations where the objective functions of two or more players are positively related. The move by one to increase one's profits automatically makes the other player better-off.

The latter on the other hand exists where objective functions are negatively related. When firms are fighting for buyers, they can never cooperate since the behaviour of one will always be to the detriment of the other. As a consequence, each player will take actions to avoid being made worse-off.

15.3 The terminology of game theory

Two key terminologies in game theory deserve attention. The first is 'players' and the second is 'strategies'. In game theory, the term "players" still retains its natural connotation. It refers to the entities that are taking part in the game. These are the people or firms or institutions depending on the nature of the game that are in the competition. In the case of a bid, the firms participating are the players.

A strategy on the other hand is the steps or moves that players make. These are choices or decisions made by players. These choices are not just made randomly but are carefully calculated to improve the welfare of the player. They are strategically made. When deciding on whether to reduce the price or not, the producer has to consider the likely reaction from the rivals as the effect on profitability will depend not only on the choices of the player but also on the rival's reaction.

For instance, advertising would definitely lead to increased sales as long as rival firms don't decide to advertise also. A firm sends unchallenged messages that its product has improved qualities and therefore worth trying. This leads to increased sales as more consumers buy the commodity following the advertising. If however, other competing firms decide to advertise also, the customers will learn of the good attributes for all the goods and are unlikely to be swayed by a single firm. This is the reason why they are called 'strategies'.

Some strategies may however dominate others. In a multi strategy game, one strategy is said to dominate another if it is preferred over the other regardless of the rival's strategy. Given a payoff matrix, if the payoff from one strategy is always better than another for all the possible choices of the rival, then the strategy is said to dominate another. It is better regardless of what the rival will play. If in the extreme, is single strategy pays better than all other regardless of what the rival plays, then such is strategy is referred to as a *dominant strategy*. A rational player will always go for the dominant strategy (if it exists) because doing otherwise reduces the payoff.

Payoff Function: in every game, there is an objective function. This is precisely what each player wants to achieve. In the case of profit maximising firms, the profit is the objective function. That is the payoff a firm gets by playing the game. We have stated already that the payoff to a particular player does not only depend on his/her choices. It also depends on the choices of all the other players. That is to say, the payoff is dependent on the strategies played by all the players. This completes the meaning of the name 'payoff function'. It is a payoff because it shows what players are paid or benefit after playing the game. It is a function because it is determined by the different strategies played by all the players.

Let's take a look at Table 15.3 below. It shows a game with two players, A and B. Each player is making a decision on whether to advertise or not. The strategy 'Yes' means 'Advertise' while 'No' means 'Not Advertising'. In the payoff matrix, each cell shows two numbers. The first is the payoff or profit for 'Player A' and the second pertains to 'Player B'

Table 15.3. Decision to Advertise or Not

GAME		Player B	
		Yes	No
Player A	Yes	10, 12	22, 2
	No	2, 23	16, 17

It is clear from the table that the two firms collectively make huge profits if they avoid the cost of advertising. That is, without advertising, A gets 16 and B gets 17. The sum for the two firms is 33. They would make this profit if they collude, that is, if they agree not to advertise. But collusion is illegal in most if not all jurisdictions. Rival firms are not allowed to engage each other in the market as this amount to anti-competition behaviour.

With no collusion, each firm has its own profit to safeguard or optimise. It knows too well that the other firm is also doing the same. Each player is comfortable with 'No Advertising' but fears that the other firm might just decide to advertise. This is because advertising, *ceteris paribus*, would increase profits. Player A's profit would go up to 22 if B does not advertise while B's would increase to 23 if A does not advertise. Each player's fears are genuine because there is an incentive for the other to play as feared.

With these fears, each player will not want to be left behind because failure to advertise when the other player has advertised has grave consequence on the profit. By not advertising when the other advertises, the player that does not advertise would have its profit fall to only 2. As such, the two players will simultaneously decide to advertise. When they all advertise, their profits however fall. This fall in profits is nonetheless better than a fall in one player that does not advertise when the other does.

Example 15.1

> Suppose there are two companies in Lusaka, Aquarite and Aquamina, selling mineral water. Each company has a fixed cost of $K50{,}000$ per month, regardless of whether they sell anything or not. The two companies are selling in the same market and each must choose between a higher price ($K2.00$ per bottles) or a low price ($K1.00$ per bottles).
>
> At the price of $K2.00$, quantity demanded equal $50{,}000$ while at $K1.00$, the quantity demanded is $100{,}000$. If both companies charge the same price, they split the sales evenly between them. Otherwise the company with the lower price sells the whole amount. Taking the payoff as the profit, formulate the problem as a two person game and obtain the solution.

To answer the question, we must start by calculating the profit each will make for every price combination. Note that the companies have no variable cost, only a fixed cost. The profit or payoff table is given below.

GAME		Aquamina (B)	
		K1	K2
Aquarite (A)	K1	0, 0	K50,000, -K50,000
	K2	-K50,000, K50,000	0, 0

In the table, each cell has two numbers which really are profits. The first related to player A, in this case Aquarite. The second is for Aquamina. The table shows that the firm that charges a higher price will not sell anything but will still incur the fixed cost. This is the explanation for the negative value which is indicative of a loss.

The strategy for each player is to consider the outcome for each choice. The player will get the worst case scenario for each choice and will choose the strategy with the highest worst case scenario. This is synonymous to maximising the minimum. It is often called the maxmin strategy. Clearly, both companies will charge the lower price.

15.4 Types of Games

Not all games are the same. They will differ on a number of aspects. This subsection therefore discusses the different differentiating aspect of a game. Particularly, it discusses the factors that define the type of a game. Basically, three factors define the type of a game. These are:

15.4.1 Number of Players

You will know by reasoning that a game cannot have one player only. It needs a minimum of two players. Since two is only a minimum, it is possible to have three or even more players. Thus, games would be differentiated based on the number of players involved. The example given in Example 15.1 is an example of a two-player game.

15.4.2 Number of Strategies

The number of strategies refers to the options that each player can choose from. In the case of deciding to do something, there are only two strategies; *to do* or *not to do*. In the case of changing price, every firm has three options; to increase, to maintain or to reduce. This is a three strategy game. In general, the number of strategies can be increased indefinitely.

15.4.3 Type of Payoffs

The payoff functions come in different forms. We basically distinguish three types of pay-off functions. The first is known as a *zero sum game*. This is a game in which the sum of payoffs for the players is always zero. The profits one player makes exactly equals the loss of the other. For instance, take a two player game with players A and B. When player A plays strategy i and B plays strategy j, the two players A and B will each get a_{ij} and b_{ij} respectively. As a zero sum game, then the following equation must hold.

$$a_{ij} + b_{ij} = 0$$

$$a_{ij} = -b_{ij}$$

Because the benefits for the two players are equal, save for the negation, the payoff function can be sufficiently shown by only showing the payoff for one player. This is typical of unproductive games such as gambles. This kind of game has no room for cooperation. It is a pure conflict game.

The second type is called *non-zero sum game*. The sum of payoff for the players is not zero. It depends on the strategies played. One player can lose while the other gains or they can all lose or gain together. This kind of game provides room for both conflict and cooperation.

The third type is called *constant difference games*. As the name may suggest, the difference between the payoff for one player and the other is constant whatever the strategy. This entails that the players lose or gain together. As a consequence, the players will always cooperate. This is because the gain made by one player always equal the gain made by the other. Each player benefits from the optimal strategy of the other. That is, if Player 1 chooses a strategy which maximises his welfare, Player 2's welfare is also maximised by the same strategy.

15.4.4 A Two Person Zero-sum game

Assume a two person zero sum game. Player A has m strategies while player B has n strategies. The payoff matrix is given in the table below.

Table 15.4.A general payoff matrix

		Player B			
		1	2	.....	n
Player A	1	a_{11}, b_{11}	a_{12}, b_{12}	.....	a_{1n}, b_{1n}
	2	a_{21}, b_{21}	a_{22}, b_{22}	.....	a_{2n}, b_{2n}
	...	.....	.....		.....
	m	a_{m1}, b_{m1}	a_{m2}, b_{m2}	.....	a_{mn}, b_{mn}

Since it is a zero sum game, the table above can be simplified by looking at the payoff for player A only. The reader must know by now that the payoff for player B is simply negative of the payoff for player A. The new table is shown below.

		Player B			
		1	2	.....	n
Player A	1	a_{11}	a_{12}	.....	a_{1n}
	2	a_{21}	a_{22}	.....	a_{2n}
	...	.....	.....		.....
	m	a_{m1}	a_{m2}	.....	a_{mn}

To optimise the objective function, Player A (whose gain is shown in the table) will choose a strategy yielding the highest possible payoff. But A does not know yet the strategy that B will play. As such, A will have to prepare for whatever step the rival plays. To do this, player A will have to resort to the *maxmin strategy*. That is, for each strategy, mark the worst case scenario or the minimum gain. Given these minimums, player A chooses a strategy with the highest minimum. This is a strategy with the maximum of the minimums.

For player B, the gain is the opposite or negative of A's. As such, B will optimise the objective function in pretty the same manner as A. The only difference is that B is maximising own gains indirectly by minimising A's gain. The steps will be to get the maximum (for A) for each strategy and then find the lowest among these maximums. This will be a *minmax strategy*. It differs from A's only in the order in which maximisation and minimisation are applied.

Formally, player A will maxmin strategy i with

$$\max_{i}\min_{j} a_{ij}$$

This represents the lowest that player A is expecting from the game. Player B on the other hand will minmax strategy j with

$$\min_{j}\max_{i} a_{ij}$$

This will represent the highest that player B is willing to concede. The two players are strategising on the same variable, except in different directions. One maximises and the other minimises. If they both arrive at the same answer, that is,

$$\max_{i}\min_{j} a_{ij} = \min_{j}\max_{i} a_{ij} = a_{ij}^{*}$$

then the point a_{ij}^{*} is called a *saddle point* of the game. It is a saddle point in the ordinary sense of optimisation. Recall that a saddle point is a maximum when viewed from one angle and a minimum when viewed from another. In game theory, it is maximum for one player and a minimum for another. It is the lowest that the maximising player is willing to accept from the game and at the same time, the highest that the minimising player is willing to concede from the game.

The saddle point is also referred to as a *Nash Equilibrium*, named after an American mathematician John F. Nash, Jr. In simpler definition, a Nash Equilibrium is where each player in a game feels they have done the best. There is no incentive for any play to play otherwise. Since there is no incentive to change, a Nash Equilibrium is generally stable.

But not all games will have a saddle point! If a two person zero sum game has a saddle point, then it is called a *strictly determined* game. It has only one point where the two players agree. Since there is only one acceptable outcome, the game is deterministic. If such a point does not exist however, the game is known as *non-strictly determined* game. The game fails to precisely point on the strategies that the player will take.

Example 15.2

Consider the following two-person zero-sum game and determine whether it has a saddle point or not. The payoff shown pertain to player A.

	Player B		
Player A	3	1	5
	-2	0	6

The two players have opposite objectives on the same payoff matrix. Player A wants the best while B is looking for the waste. To get to the optimal strategies, we have to create a column of minimums for each of A's strategies and a row of maximums for each of B's strategies. This is shown in the table below.

	Player B			Row min
Player A	3	1	5	**1**
	-2	0	6	-2
Col max	3	**1**	6	

Clearly, player A will go for the first strategy. Of the minimums, it gives the maximum of 1. This implies that this is the lowest A will accept as a gain from the game. Player B on the other hand has (among the maximums) the lowest of 1. This is what B is willing to accept as loss. Of course, a loss to B is a gain to A. Since the two are equal, the game has a saddle point.

Example 15.3

Consider another game similar to one in Example 15.2 but with a different payoff matrix. Does the game have a saddle point?

	Player B		
Player A	8	-3	3
	-5	5	4

To get to the answer, we follow the same strategy. Player A will have to look for the maximum of all minimums. For each strategy, a minimum outcome must be identified and then go for the highest among minimums. Player B will look at the payoff matrix in the opposite. The player knows too well that the payoff pertain to Player A and since this is a *zero-sum* game, he must minimise the function. To achieve this, Player B will look for the maximum for each of the strategy and then pick the minimum of all. Notice that indirectly, Player B is also getting the maxmin of own payoff.

Given the above strategies, we construct the payoff matrix showing the column minimums and row maximums for the two players. The new table is presented below.

	Player B			min
Player A	8	-3	3	**-3**
	-5	5	4	-5
max	8	5	**4**	

It is clear from the above table that Player A will go for the first strategy. Given this strategy, the lowest A is expecting to gain is -3. This is actually a loss of 3. But player B, the rival, will play strategy 3 because it gives the lowest of all minimums. In this strategy, B is willing to concede or lose at the most 4.

In this game, Player B is willing to lose or concede an amount which is seven (7) more than what Player A is expecting to gain. It is clear the two players are not maximising on the game. Player A, by adopting the maxmin strategy hopes to gain no less than -3 (or loss of not more than 3) but the opponent is prepared to concede 4. This means Player A is not maximising his gains. There is still room to increase gains. Player B on the other hand by adopting the minmax strategy hopes not to lose more than 4 but the opponent does not expect him to lose, let alone that amount. This mean Player B is also not maximising his position.

In such games, the solution for each player is to behave unpredictably in the selection of strategies in order to keep the rival guessing. The probability of selecting each strategy should provide minimum knowledge if any regarding the strategy he will choose. In this way, each player can increase own payoff over time. This leads to the discussion of *Mixed Strategies*.

15.5 Mixed strategies

In the succeeding subsection, we looked at pure games. These are games where there is only one optimal strategy. Each player will know, based on the payoffs, what the rival will go for. This was the case because the games have what was referred to as a *saddle point*. For games that have no saddle point however, there is scope for each player to increase payoff by selecting strategy other than the maxmin or minmax. The opponent must not know which strategy a player will go for. This is known as *Mixed Strategy* game.

More precisely, Downing and Clark (1988) have defined a mixed strategy as a decision to play various choices or strategies at various probabilities, and is described by a probability vector. In short, a player does not just go for a single choice or strategy as in pure strategy. Under mixed strategy, a player can play any of more than one choices. Each strategy is played by a fixed probability.

For a two player game, a mixed strategy for Player A is a probability vector

$$P = (p_1 \quad p_2 \quad \cdots \quad p_m)$$

where p_i is the probability of playing strategy i. As a probability vector, each element in the vector must be non-negative and the summation of all elements must equal unit. That is

$$p_i \geq 0, \forall i \text{ and } \sum_{i=1}^{m} p_i = 1$$

The probability vector for player A is in row format because A is a row player. His/her strategies are listed in rows.

Corollary, for player B (column player) the mixed strategy is given by the probability vector

$$Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

which must also meet the same conditions.

$$q_j \geq 0, \forall j \text{ and } \sum_{j=1}^{n} q_j = 1$$

Given the mixed strategies for the two players, the problem now is to find the optimal values of the vectors P and Q. This is done by calculating for P (and Q) which will give the highest possible return to player A regardless of what the opponent plays.

Example 15.4

A two player zero sum game between players A and B has the following payoff matrix M. The payoffs relate to player A.

$$M = \begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}$$

Find the optimal values of the probability vectors P and Q.

Let us start with player A whose probability vector is $P = (p_1 \quad p_2) = (p_1 \quad 1 - p_1)$. If player B goes for the first strategy, then the value G for player A will be

$$\begin{aligned} G_1 &= 5p_1 + 2(1 - p_1) \\ &= 3p_1 + 2 \end{aligned}$$

This results from multiplying the first column elements by the probability vector which shows precisely how they will be played. If player B will go for the second strategy, then the value of the game for Player A will change. The new value denoted by G_2 will be based on the second column values.

$$\begin{aligned} G_2 &= 3p_1 + 4(1 - p_1) \\ &= 4 - p_1 \end{aligned}$$

Player A can hope to gain $G = \min(G_1, G_2)$. Since A is looking for the lower value of the two, he is better off if the two are equalised. He knows too well that increasing one over the other will make him worse off. This piece of information will allow the creation of two equations in two *unknowns*. That is

$$\begin{aligned} G &= 3p_1 + 2 \\ G &= 4 - p_1 \end{aligned}$$

Solving the two simultaneously, we arrive at

$$P = \begin{pmatrix} .5 \\ .5 \end{pmatrix}, \qquad G = 3.5$$

At the optimal, player A will play the two strategies half the time each. In this way, A will be assured of gaining on average $G = 3.5$.

Similarly, Player B will have to consider his take given the different strategies that A can play. The mixed strategy for B is given by $Q = \binom{q_1}{q_2} = \binom{q_1}{1-q_1}$. If player A goes for the first strategy, the value for B is

$$\begin{aligned} G_1 &= 5q_1 + 3(1-q_1) \\ &= 2q_1 + 3 \end{aligned}$$

If A plays the second strategy, the value for B will be

$$\begin{aligned} G_2 &= 2q_1 + 4(1-q_1) \\ &= 4 - 2q_1 \end{aligned}$$

the objective of player B is to minimise the gains for player A so as to indirectly optimise his. He will thus choose the minimum of the two game values. At the optimum, he chooses a mixed strategy that will give equal value regardless of what the opponent plays. This will result in a system of simultaneous equations

$$\begin{aligned} G &= 2q_1 + 3 \\ G &= 4 - 2q_1 \end{aligned}$$

By solving the two equations simultaneously, we conclude that Player two will play the first strategy a quarter of the times. This means the second strategy will be played three-quarters of times. The value of the game for player B is $G = 3.5$.

$$Q = \binom{.25}{.75}, \qquad G = 3.5$$

With mixed strategies, both players are maximising gains from the game. In the above example, Player A is hoping to gain 3.5 from the game. This is exactly equal to what player B (the rival player) is willing to concede. Player B is also only willing to concede an amount equal to what A is willing to accept a gain from the game.

The solution for a two player zero sum game can also be evaluated using the theory of Linear Programming. This is discussed in the section that follows.

15.6 Game theory as a linear programme

A two player zero sum game can be formulated as a Linear Programming problem. Consider a game involving two players; player 1 and player 2. The corresponding payoff matrix (for player) is given by the matrix

$$A = (a_{ij}), \qquad i = 1, \ldots, m \; and \; j = 1, \ldots, n$$

The payoff matrix is of dimensions $m \times n$. This means player 1 has m strategies while player 2 has n. Let us now assume that the game has a value of $G = \bar{G}$. The mixed strategies for the two players are as given in the previous section. They are $P' = (p_1 \quad p_2 \quad \cdots \quad p_m)$ and $Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$.

In the given payoff matrix, the objective of player 1 is to get the highest possible value. The

lowest player 1 is expecting from the game is the value of the game itself. For player 2, since the payoff is the converse of what player1 gets, the objective is to make the gain as low as possible. Similarly to player 1, the maximum player 2 is expecting to concede is the value of the game. This illustration gives two linear programmes which are duals of each other.

In the first, player 1 wants to maximise the objective function. Using a mixed strategy, player 1 has the following expectations which depend on what player 2 plays.

$$G = min\left(\sum_{i=1}^{m} a_{i1}p_i \quad \sum_{i=1}^{m} a_{i2}p_i \quad \cdots \quad \sum_{i=1}^{m} a_{in}p_i\right)$$

There are n possible outcomes for the mixed strategy of player 1. Each related to a particular strategy that player 2 can go for. These represent different gains for player 1. In the worst case, player 2 is not expecting anything less than the minimum of the n possible outcomes. Thus the minimum in the vector provides the lower limit on the gains for player 1. Anything above is desirable for the player.

The linear programming formulation of the above problem is

$$\max_{0 \le p_i \le 1} G, \qquad i = 1, 2, \ldots, m$$

subject to

$$\begin{aligned} &\sum_{i=1}^{m} a_{ij}p_i \ge G, \qquad j = 1, 2, \ldots, n \\ &\sum_{i=1}^{m} p_i = 1 \\ &p_i \ge 0, \qquad i = 1, 2, \ldots, m \end{aligned}$$

The problem above has many unknowns than are equations. It would therefore be difficult to arrive at a desirable point. Moreover, it does not yet look like the linear programming encountered in the earlier chapter. One more step is needed to transform the problem into the familiar linear programming problem. Let us introduce a new system of variables which will serve a very critical purpose, that is, to swallow one unknown.

Let $x_i = \frac{p_i}{G}, i = 1, 2, \ldots, m$. We use this new system of variables to transform the problem. Though the order is to start with the objective function and then the constraints, it is inevitable that the order is altered here. First consider how the constraints will change. The new constraints are now

$$\begin{aligned} &\sum_{i=1}^{m} a_{ij}x_i \ge 1, \qquad j = 1, 2, \ldots, n \\ &\sum_{i=1}^{m} x_i = \frac{1}{G} \\ &x_i \ge 0, \qquad i = 1, 2, \ldots, m \end{aligned}$$

In the first constraint, the value of the game, G, was common on both sides of the inequality and so was factored out. The requirement now is that the left hand side does not fall below a known value of 1. In matrix notation, this is expressed as $X'A \geq 1$ which replaces the earlier one with P. In the second constraint, the action of bringing the new variable is equivalent to dividing by a common value G on both sides of the equation. Since by supposition $x_i = \frac{p_i}{G}$, the left hand side has the new variable.

The third constraint needs some closer attention. Each probability value is divided by a constant G. The resulting x_i's would only remain non-negative if the divisor is positive. If this is not met, then the third constraint would not hold. This has the potential to put the solution in jeopardy. To prevent this from occurring, it is made a condition that the value of the game is always positive. But the value of the game is part of what we are looking for. It is not known yet. So, how can we ensure it is positive? Well, this is achieved by ensuring that every entry in the payoff matrix is positive. If this is not the case, an arbitrary number large enough to make all entries positive is added to every element in the matrix. This is equivalent to player 2 transferring a given sum to player 1 before the start of the game. This can be reversed once the actual value is known.

Let us now turn to the objective function. It remains to maximise the value of the game G. But G no longer appears in any of the constraints. Its equivalent must be used instead. To do this, we examine the second constraint. It is

$$\sum_{i=1}^{m} x_i = \frac{1}{G}$$

from the binding equation above, the objective of maximising G is equivalent to minimising the left side of the equation. This is because the G being maximised is in the denominator. Maximising it actually minimises the quotient. Thus, the new objective is now that of minimising $\sum_{i=1}^{m} x_i$ subject to the constraints listed above.

Formally, the resulting linear programming formulation is summarised as

$$\min \sum_{i=1}^{m} x_i$$

subject to

$$\sum_{i=1}^{m} a_{ij} x_i \geq 1, \qquad j = 1, 2, \ldots, n$$
$$x_i \geq 0, \qquad i = 1, 2, \ldots, m$$

Once the optimal values of x_i's are know, the value of the game G can be calculated. Then the individual p_i's can be calculated based on the x_i's and the value of the game G.

Example 15.5

A two person zero sum game for player 1 and player 2 has the following payoff matrix which pertain to player 1. is $A = \begin{pmatrix} 5 & 9 \\ 9 & 6 \\ 0 & 3 \end{pmatrix}$

a. Formulate the linear programming problem.
b. Find the optimal strategies for the two players.

To get to the linear programming, there is need to look at the nature of player 1's mixed strategies. The player has three choices and the mixed strategy must reflect this. But let us look at the gain to player 1 for each of the three strategies. It should suffice to state that the benefits of the first strategy are greater that the third, irrespective of what the opponent plays. The same hold between the second and third strategy. In game theory, we say that the first two strategies dominate the third. As such, the player will never consider the third strategy. His choice will now be between the two. This information helps is simplifying the payoff matrix. The effective matrix is thus $A = \begin{pmatrix} 5 & 9 \\ 9 & 6 \end{pmatrix}$

Therefore, $P' = (p_1 \quad p_2)$. The P vector is transformed to the X vector by dividing by the yet to be known value of the game G. Therefore $X = (x_1 \quad x_2)$. The linear programming problem is

$$\min \sum_{i=1}^{m} x_i$$

subject to

$$(x_1 \quad x_2)\begin{pmatrix} 5 & 9 \\ 9 & 6 \end{pmatrix} \geq 1$$
$$x_i \geq 0, \qquad i = 1, 2, 3.$$

Or in a more familiar way, the linear programme is presented as

$$\min x_1 + x_2$$

subject to

$$5x_1 + 9x_2 \geq 1$$
$$9x_1 + 6x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

Two methods can be used to find the optimal values of x_1 and x_2. Once these are known, the value of the game G and the mixed strategy P can be gotten. We leave is to the reader to verify the following solution.

$$x = \begin{pmatrix} 1/17 \\ 4/51 \end{pmatrix}$$

$$G = {}^{51}/_{7}, \qquad P = \begin{pmatrix} {}^{3}/_{7} \\ {}^{4}/_{7} \\ 0 \end{pmatrix}$$

The mixed strategy P must show that there is one option that player 1 will dare not consider. A zero in the third element shows that this option is available but because of it being dominated by other options, the player will never consider it.

In the second problem, we look at the game from player 2's perspective. Given the payoff matrix, player 2's objective is to minimise the value of the outcome. Given player 2's mixed strategy, the following are the possible outcome depending on Player 1's unknown mixed strategy.

$$G' = max\left(\sum_{j=1}^{n} a_{1j}q_j \quad \sum_{j=1}^{n} a_{2j}q_j \quad \cdots \quad \sum_{j=1}^{n} a_{mj}q_j\right)$$

The outcome vector has a total of m elements. Like in the earlier case, each element corresponds to the possible strategy played by the rival. In the worst case for of the game, player 2 is only expecting to lose the maximum of the m possible outcomes.

The linear programming formulation for player 2 is to

$$\min_{0\le q_j \le 1} G, \qquad j = 1, 2, \dots, n$$

subject to

$$\sum_{j=1}^{n} a_{ij}q_j \le G, \qquad i = 1, 2, \dots, m$$
$$\sum_{j=1}^{n} q_j = 1$$
$$q_j \ge 0, \qquad j = 1, 2, \dots, n$$

Like in the previous case where we looked at the mixed strategy for player 1, the above formulation is not yet the familiar linear programming. It must be transformed further, to a stage where the G will not appear in the inequality. To attain this, a new variable must be introduced. In the dual case, the variable that was introduced was x. This time around, y is used.

Let $y_j = \frac{q_j}{G}$. With the new variable, we can straight away go into changing the constraints. The method that was applied to player 1 also applies, *mutatis mutandis*. The new constraints are

$$\sum_{j=1}^{n} a_{ij}y_j \le 1, \qquad i = 1, 2, \dots, m$$
$$\sum_{j=1}^{n} y_j = \frac{1}{G}$$
$$y_j \ge 0, \qquad j = 1, 2, \dots, n$$

Recall that for player 2, the objective is not to maximise the value of the game. Player 2 seeks to minimise it. In so doing, player 2 will be indirectly maximising own gain. But as stated in the dual case, minimising G is equivalent to maximising $\sum_{j=1}^{n} y_j$. In summary, player 2 will be faced with the following problem

$$\max \sum_{j=1}^{n} y_j$$

subject to

$$\sum_{j=1}^{n} a_{ij}y_j \le 1, \qquad i = 1, 2, \dots, m$$
$$\sum_{j=1}^{n} y_j = \frac{1}{G}$$
$$y_j \ge 0, \qquad j = 1, 2, \dots, n$$

Once the values of y have been established, the value of the game and consequently the mixed strategy would be known.

Example 15.6

Redo Example 15.5 from Player 2's perspective.

The summary of the problem in Example 15.4 is as follows (from player 2's perspective)

$$\min y_1 + y_2$$

subject to

$$5y_1 + 9y_2 \le 1$$
$$9y_1 + 6y_2 \le 1$$
$$y_1, y_2 \le 1$$

Using either the graphical or simplex method, the optimal values of y are

$$y = \begin{pmatrix} {}^{1}/_{17} \\ {}^{4}/_{51} \end{pmatrix}$$

From this, the value of the game and the mixed strategy are given as

$$G = {}^{51}/_{7}, \qquad q = \begin{pmatrix} {}^{3}/_{7} \\ {}^{4}/_{7} \end{pmatrix}$$

Player 2 will play the first strategy thrice every seven times and play the second strategy four times every seven times.

16.2 Steps in dynamic programming problem

The following are the steps involved in any dynamic programming problem:

1. Divide the problem into *stages* with a policy decision required at each stage;
2. Each stage has a number of *states* associated with it;
3. The effect of a policy decision is to transform the current state into a state associated with the next stage;
4. Given the current state, an optimal policy for the remaining stages is independent of the policy adopted in the previous stages; (technically this is known as the *Markovian property);*
5. The solution procedure begins by finding the optimal policy for each state in the last stage;
6. A *recursive relationship* is available which identifies the optimal policy for each state with *n* stages remaining, given the optimal policy for each state, with *(n-1)* stages remaining;
7. Using the recursive relationship, the solution procedure moves backward stage by stage, each time finding the optimal policy for each state at that stage, until it finds the optimal policy when starting at the initial stage.

16.3 The classic stagecoach problem

The above steps have been frequently illustrated in many books on dynamic programming with what is considered as a classic problem, namely, the stagecoach problem. A mythical fortune seeker in Missouri decided to go west to join the gold rush in California in the mid-nineteenth century. The main transport available was the stagecoach which would pass through unsettled country with the danger of being attacked by bandits. Although the starting point and destination were fixed, the traveller had a choice of the possible routes he could take. The possible routes are shown in Figure 15.1 where each state is shown by a circled letter. The direction of the travel was from left to right starting from Missouri (A) to California (J), and required four stages to be passed through by using stagecoaches.

In view of the danger present in the travel, the stagecoaches offered life insurance policies to the traveller and the cost of the policy would vary directly with the extent of danger perceived. Hence, in order to maximise his security, the traveller decided to choose the route with the minimum total cost of the insurance policy. The cost of the policy for going from state i to state j, C_{ij} is shown below:

Figure 16.1: Cost of moving from one state to another

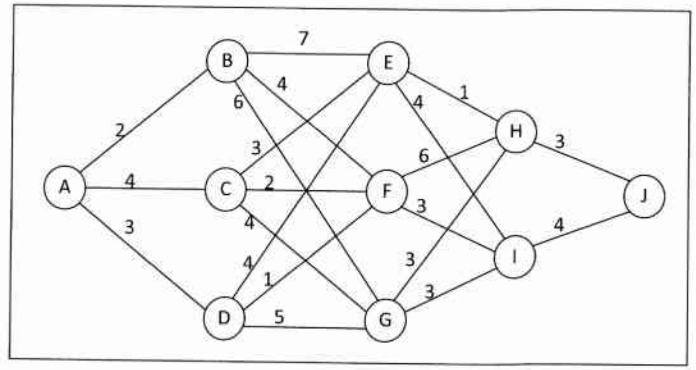
	B	C	D
A	2	4	3

	E	F	G
B	7	4	6
C	3	2	4
D	4	1	5

	H	I
E	1	4
F	6	3
G	3	3

	J
H	3
I	4

Figure 16.2: Route Network with associated costs available to the traveler



To give the problem a more contemporary and local flavour, let us assume that a man working in Mfuwe in Zambia's Eastern Province has just heard that one of his long-lost uncles is actually living in Dundumwezi in Southern Province and so he wants to visit him. The roads are not easy to pass and the distance is long (over 1000 km). There could be many mishaps along the way. But there are many alternative routes to choose from with varying costs. He wants to choose the route with minimum cost.

The data on the route network between Mfuwe and Dundumwezi could still be similar to what is shown in Figure 15.1 for the stagecoach problem. So let us see how this can be solved as a problem in dynamic programming.

The insightful point that we shall gain at the end of this exercise is that *the overall optimal decision is <u>not necessarily</u> the sum of optimal decisions at each stage.* For example, the cheapest route by each successive stage is:

$$A \rightarrow B \rightarrow F \rightarrow I \rightarrow J = 13$$

But the following route yields a lower cost:

$$A \to D \to F \to I \to J = 11$$

With the above point in mind, we shall now see the analytics of the solution.

16.4 Solving a dynamic programming problem

Let X_n be the immediate destination when there are n more stages to go. Then the route to be chosen is:

$$A \to X_4 \to X_3 \to X_2 \to X_1, \qquad where\ X_1 = J.$$

Let $F_n(S, X_n)$ be the cost of transport for the last n stages, given that the traveller is in state S and selects X_n as the immediate destination.

Given S and n, let X_n^* denote the value of X_n which minimises $F_n(S, X_n)$. That is to say,

$$F_n^*(S) = F_n(S, X_n^*)$$

We have to solve for $F_4^*(1)$. We do so by successively finding $F_1^*(S)$, $F_2^*(S)$, $F_3^*(S)$ and $F_4^*(1)$. The sequence of backward computations would then be as follows:

With only one more stage to go

S	$F_1^*(S)$	X_1^*
H	3	10
I	4	10

With two more stages to go

S	H	I	$F_2^*(S)$	X_2^*
E	4	8	4	H
F	9	7	7	I
G	6	7	6	H

With three more stages to go

S	E	F	G	$F_3^*(S)$	X_3^*
B	11	11	12	11	E or F
C	7	9	10	7	E
D	8	8	11	8	E or F

With four more stags to go

S	B	C	D	$F_4^*(S)$	X_4^*
A	13	11	11	11	C or D

Thus the optimal routes are:

$A \to C \to E \to H \to J$;

$A \to D \to E \to H \to J$; and

$A \to D \to F \to I \to J$;

All the above routes yield a minimal total cost of 11.

The recursive relation is given by:

$$F_n^*(S) = min\{C_{SX_n} + F_{n-1}^*(X_n)\}$$

16.5 Bellman's Principle of Optimality

The above recursive relation is known as the *Bellman equation* after the mathematician Richard Bellman who formulated the *Principle of Optimality*. The Bellman Principle of Optimality states that whatever the initial state and initial decision(s), the remaining decisions must constitute an optimal policy with regard to the state resulting from the initial decision.

Another writer Aris (1964) summarised this principle as: If you don't do the best with what you happen to have got, you'll never do the best you might have done with what you should have had.

This is the essence of the Markovian property stated in section 16.2. The optimal decision in the current state depends only on the current state and not on how you got there. Given the current state, an optimal policy for the remaining stages is independent of the policy decisions in the previous stages. This is because the knowledge of the current state of the system conveys all the information about its previous behaviour necessary for determining the optimal policy henceforth. Any problem lacking this Markovian property cannot be formulated as a dynamic programming problem. (see Hillier et al, 2012).

PART V : POSTSCRIPT

SOME KEY MESSAGES

The Main Takeaway

If there is one takeaway that we hope readers of this book will carry, it is this: the relation between problems and techniques for their solution in not a one-to-one relation. A given problem can often be solved using more than one technique. And a given technique can often be used to solve more than one type of problem. It is this *flexibility* of the problems that allow plural techniques for their solution and the *versatility* of the techniques that have the capability to solve a variety of problems that one must learn to appreciate.

Flexibility in Problem Solving

As an illustration of problem flexibility in terms of its solution, consider the problem below $\max Z = 2x_1 + 5x_2$ subject to

$$\begin{aligned} 2x_1 + x_1 &\leq 430 \\ 2x_2 &\leq 460 \\ x_1, x_2 &\geq 0 \end{aligned}$$

this can be solved as a linear programming problem. We transform the inequality constraints into equality constraints by adding slack variables. We then have

$$\begin{aligned} 2x_1 + x_1 + S_1 &= 430 \\ 2x_2 + S_2 &= 460 \end{aligned}$$

We start with an initial arbitrary basis consisting of S_1 and S_2 and using the simplex procedure, we finally attain the optimal basic feasible solution.

But the above problem can also be solved as a constrained optimisation problem involving inequality constraints. And since the objective function and the constraints are all linear, it can be viewed as an exercise in concave (convex) programming problem for which the fulfilment of the necessary conditions stipulated by the Karush-Kuhn-Tucker conditions will also ensure fulfilment of the sufficient conditions.

The Karush-Kuhn-Tucker conditions for the above problem will be attained as follows: from the Lagrangean function

$$L = 2x_1 + 5x_2 - \lambda_1(2x_1 + x_1 - 430) - \lambda_1(2x_2 - 460)$$

the necessary conditions for a maximum are:

$$\begin{aligned} \frac{\partial L}{\partial x_1} \leq 0, \qquad x_1 \geq 0, \qquad x_1\frac{\partial L}{\partial x_1} = 0 \\ \frac{\partial L}{\partial x_2} \leq 0, \qquad x_2 \geq 0, \qquad x_2\frac{\partial L}{\partial x_2} = 0 \end{aligned}$$

$$\frac{\partial L}{\partial \lambda_1} \geq 0, \qquad \lambda_1 \geq 0, \qquad \lambda_1 \frac{\partial L}{\partial \lambda_1} = 0$$
$$\frac{\partial L}{\partial \lambda_2} \geq 0, \qquad \lambda_2 \geq 0, \qquad \lambda_2 \frac{\partial L}{\partial \lambda_2} = 0$$

Using the above equations, one can obtain the optimal solution.

The problem can also be solved as a problem in dynamic programming! There are two resources and hence there are two states. Let (m_j, n_j) be the states are stage $j, (j = 1, 2)$. Then the recursive equation is given by:

At stage 2

$$f_2(m_2, n_2) = \max_{\substack{0 \leq x_2 \leq m_2 \\ 0 \leq 2x_1 \leq n_2}} [5x_2]$$

At stage 1

$$f_1(m_1, n_1) = \max_{0 \leq 2x_1 \leq n_2} [2x_1 + f_2(m_1 - 2x_1, n_1)]$$

Solving the recursive equations will yield the optimal solution. We leave it to the reader to verify that whatever technique is applied, the optimal solution will be $x_1^* = 100, x_2^* = 230$.

As another example of problem flexibility, we had shown earlier in Chapter 15 how a game theoretic problem can also be expressed as a linear programming problem.

Versatility of Techniques

And now we give an example to illustrate the versatility of techniques.

Most reader would know that there are four blood groups: A, B, AB and O. And individuals belonging to these different blood groups have different capabilities in terms of donating and receiving blood:

Blood group A individuals can receive blood only from individuals of groups A or O and can donate to individuals with type A or AB;

Blood group B individuals can receive blood only from individuals of groups B or O and can donate blood to individuals with type B or AB;

Blood group AB individuals can receive blood from any group (they are universal recipients) but can donate blood only to individuals with type AB.

Blood group O individuals can receive blood only from individuals of group O but can donate blood to individuals with type A, B, AB or O (they are universal donors).

The above characteristics are summarised in the following table

Recipient	Donor			
	A	B	AB	O
A	✓			✓
B		✓		✓
AB	✓	✓	✓	✓
O				✓

Now consider a hospital which has patients belonging to different blood groups who are in need of blood. Let us indicate them as P_A, P_B, P_{AB} and P_O, respectively. the hospital has has a blood bank which stores supplies of different types of blood. Let us indicate them as S_A, S_B, S_{AB} and S_O, respectively.

There is a cost associated with administering each unit of blood to each patient. Let us denote this as C_{ij} where i and j represent the blood group and the patent group respectively. that is, C_{ij} is the cost of administering a unit of blood in the i^{th} group to a patient in the j^{th} group. The problem is to use the quantities of blood available in the blood bank to meet the patients' need for blood at minimum total cost.

Here, the different types of blood stored in the blood bank are the "origins" and the patients belonging to different blood groups are the "destinations". The quantities of blood available in the blood bank are the supplies of blood and the patients' requirements are the demands.

The problem then is: How should the supplies of blood be "transported" to meet the demand at the destinations so as to minimise "total transportation cost"?

The problem can thus be viewed as a "transportation" problem and be solved by the techniques described in Chapter 14, even though there is no transport involved. No doubt, in a very elementary sense, one can think of blood being physically transported from the blood bank to the different wards where the patients are admitted. But then, the cost we a trying to minimise is not this physical transport cost.

Efficiency

While problem flexibility and technique versatility may be advantageous, it is important to bear in mind another critical factor – *efficiency* in problem solving. Efficiency is determined both by the amount of time taken and by the complexity involved in problem solving. More often than not, time and complexity are directly related. The more complex the solution process, the more time it will take to complete. We had some simple illustrations of this in the chapter on Linear Programming.

We noted that the complexity of solution of a linear programming depends more on the number of programme variables than on the number of constraints. We also noted that the number of programme variables and constraints are interchanged between a primal and a dual problem. Hence if a primal problem has more programme variables than constraints, then it is more efficient to solve the dual problem since it will yield the same optimal solution.

Again, we noted that large scale programmes can take an enormous amount of time for their solution if they are sought to be solved by the standard algorithmic procedure of moving from one extreme point of a feasible region to another extreme point. Hence, efforts are being made to develop new algorithms (such as Karmarkar's algorithm) that will significantly cut down on the solution time.

With the advent of computer technology, one disciple that is fast developing in economics is *computational economics*. Computational economics uses computer based economic modelling for solution of analytically and statistically formulated economic problems.

The quest for time-efficient solutions for computationally hard optimisation problems will continue, unbounded in time.

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Appendix I

CONCEPTS AND THEOREMS NAMED AFTER MATHEMATICIANS

Argand Diagram: Named after the French mathematician Jean-Robert Argand (1768 – 1822)

Bellman's Principle: Named after the American mathematician Richard E. Bellman (1920 – 1984) celebrated as the inventor of Dynamic Programming in 1953.

Bernoulli equation: named after the Swiss mathematician Daniel Bernoulli (1700 – 1782).

Georg Cantor: German mathematician (1845 – 1918), known as the inventor of set theory.

Cayley-Hamilton Theorem: Named after the British mathematician Arthur Cayley (1821 – 1895) who founded the modern British school of pure mathematics; and the Irish mathematician William Rowan Hamilton (1805 - 1865) noted for his contributions to classical mechanics, optics and algebra.

Cartesian product: Named after the French mathematician René Descartes (1596 – 1650) credited as the father of analytical geometry.

Cobb-Douglas function: Named after the American mathematician and economist Charles Cobb (1875 – 1949) and the American economist Paul Howard Douglas (1892 – 1976) who tested the functional form of production functions.

Cramer's Rule: Named after the Swiss mathematician Gabriel Cramer (1704 – 1752).

De Moivre's Theorem: Named after the French mathematician Abraham de Moivre (1667 – 1754) well-known for his formula that links complex numbers and trigonometry and his work on normal distribution and probability theory.

Euler relation or Euler's formula: Named after the Swiss mathematician and physicist Leonhard Euler (1707 – 1783) who, among his many contributions, introduced the concept of a mathematical function.

Euler's Theorem: also named after Leonhard Euler

Hamiltonian function: Named after the Irish mathematician and astronomer William Rowan Hamilton (1805 – 1865).

Hessian matrix, determinant: Named after the German mathematician Ludwig Otto Hesse (1811 – 1874) who developed the matrix and the determinant.

Hungarian method: Named after two Hungarian mathematicians Dénes König (1884 – 1944) and Jenő Egerváry (1891 - 1958)

Jacobian matrix, determinant: Named after the German mathematician Carl Gustav Jacob Jacobi (1804 – 1851) who made fundamental contributions to areas such as differential equations and number theory.

Karmarkar's algorithm: Named after Indian mathematician Narendra Karmarkar (b 1957).

König's Theorem: Named after the Hungarian mathematician Dénes König (1884 – 1944) who wrote the first textbook in the field of Graph Theory.

Karush-Kuhn-Tucker conditions: Named after American mathematicians William Karush (1917 – 1997), Harold W. Kuhn (1925 -) and the Canadian mathematician Albert W. Tucker (1905 – 1995).

Lagrangean multiplier: Named after the Italian mathematician and astronomer Joseph-Louis Lagrange (1736 – 1813).

Laplace expansion: Named after a French mathematician and astronomer Pierr-Simon Laplace (1749 – 1827)

Leontief Input Output Matrix: Named after a Russian-American economist Wassily Leontief (1906-1999). Though Leontief was not a mathematician, his matrix, the Leontief input Output matrix is actually a mathematical concept.

L'Hôpital's rule: Named after the French mathematician Guillaume de l'Hôpital (1661 – 1704).

Markovian property: Named after the Russian mathematician Andrey Markov (1856 – 1922).

Nash Equilibrium: Named after the American mathematician John Forbes Nash, Jr. (b. 1928)

Perron-Frobenius Theorem: Named after the German mathematicians Oscar Perron (1880 – 1975) and Ferdinand Georg Frobenius (1849 – 1917).

Pontryagin's Maximum Principle: Named after the Soviet mathematician Lev Pontryagin (1908 – 1988).

Riemann Integral: Named after the German mathematician Bernhard Riemann (1826 – 1866)

Routh theorem: Named after the British mathematician Edward John Routh (1831 – 1907).

Schurr's Theorem: Named after the mathematician Issai Schur (1875 – 1941). He was born in Belarus (then Russian Empire) and died in Tel Aviv, (then Palestine, now Israel) but worked in Germany most of his life.

Venn Diagram: Named after the British logician and philosopher John Venn (1834 – 1923).

Vogel's Approximation Method: Named after William R. Vogel.

Young's Theorem: Named after the British mathematician William Henry Young (1863 – 1942). He made significant contributions to the study of functions of several complex variables. He was the husband of an equally renowned mathematician Grace Chisholm Young who initially published her papers under her husband's name! Their son Lawrence Chisholm Young was also a brilliant mathematician who contributed significantly to measure theory, calculus of variations and optimal control theory.

Appendix II

EXAMINATION TYPE QUESTIONS

PAPER ONE QUESTIONS

Question One

a. Given the following matrix

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{bmatrix}$$

i. Construct the matrix of Eigen vectors for the following matrix:

ii. Using the Eigen values, determine the sign definiteness of the above matrix?

b. The coordinator of a research project wants to recruit researchers and research assistants. A researcher is paid K500 per day and a research assistant is paid K300 per day. The project is estimated to cost:

$$Z = 6000 + 6x^3 - 36xy + 3y^2$$

where x is the number of researchers, and y is the number of research assistants.

i. How many researchers and research assistants should be recruited for the project to minimise cost?

ii. What will be the actual staffing cost of the project.

iii. Use the second order condition to show that the answer above is indeed minimum.

Question Two

a. Diagonalise the following matrix:

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

b. Let P_1, P_2 and P_3 denote the profits which a multi-product corporation earns from the production and sale of tea, coffee and cocoa respectively. The corporation's economics department believes that he profits are linked as follows:

$$2P_1 - P_2 + P_3 = 5$$
$$P_1 - 3P_2 + 2P_3 = 2$$
$$2P_1 + P_2 + 4P_3 = -3$$

Using a 3×3 marix, work out the profits of each product

Question Three

An individual is HIV positive. Doctors have prescribed two drugs for him and his health status depends on the consumption levels of the two drugs. If the health status of the individual is measured on a scale of 0 to 10 (0= extremely poor health, 10=perfectly normal health) and if his health function is given as:

$$Y = -(X_1 - 1)^2 - (X_2 - 2)^2 + 10$$

Where Y = health status, X_1 = dosage of Drug 1, X_2 = dosage of Drug 2.

a) What is the optimal dosage of the two drugs

b) What is the best health status he can attain with the optimal consumption of the two drugs?

c) Suppose we are also told that the individual can tolerate only one drug dose per day. What will be the best health status he can now attain?

Question Four

a. Check the sign definiteness of the following quadratic form:

$$Q = 2x_1^2 + 5x_2^2 + 13x_3^2 + 6x_1x_2 + 10x_1x_3 + 14x_2x_3$$

b. Given the following market model:

$$Q_t^D = 180 - 0.75P_t$$
$$Q_t^S = -30 + 0.3P_{t-1}$$
$$P_0 = 220$$

i. Find the price at all times t in the market

ii. Comment on the dynamics of the price path.

Question Five

a. The demand function for a good X is given as:

$$Q = 3000 - 4P_X + 5\ln(P_Y)$$

Where Q is quantity demanded of good X, P_X is the price of the good X and P_Y is the price of another good, good Y.

i. What is the cross price elasticity of demand when $P = 5$ and $P' = 10$?

ii. On the basis of your answer, what can you say about the relationship between the two goods?

b. For the utility function $U = U(x, y)$ to be maximised subject to the budget constraint $xP_x + yP_y = B$,

Set out the first and second order conditions that must be fulfilled.

PAPER TWO QUESTIONS

Question One

a. Given the following differential equation:

$$\frac{dy}{dt} + 4y = 12$$

i. Obtain the general solution showing clearly the complementary function and the particular solution.

ii. Test for the dynamic stability of the equilibrium.

Question Two

a. A manufacturing company estimates that the marginal costs for its business activities follows the following function:

$$MC = 4Q^2 + \frac{300}{Q} + 2Q + 12$$

the fixed cost is 150,000

i. Find the total cost function and average cost functions

ii. Calculate the total cost if 50 units are made

b. Due to large-scale mining activity on the Copperbelt, there are opportunities for small companies to do business by way of providing a variety of ancillary services to large-scale mining companies. At any time, there are 6000 small business companies that are actually engaged in business or are seeking to do business. From past data, we know that a company that is doing business this year has a 30 percent chance of losing its business next year, and for a company without business this year, there is 60 percent chance that it will have business next year. Find the equilibrium number of small companies that will be engaged in business with the large companies.

Question Three

a. Suppose there are two firms in the market selling a product at some fixed price. Advertising does not affect the total market demand but each firm's share of the market will depend on the relative advertising levels chosen by it. Each firm chooses between two advertising levels: 'high' H and 'low', L. The total gross profit for both firms is $K1000$. The cost of H for each firm is $K400$ and of L it is $K200$. When both firms advertise at the same level, they split the market and hence the net profit, fifty-fifty else the highest advertising firm gets K800 while the other gets $K200$ For example, if both firms advertise at a high level, each firm has a gross profit of $K500$ and a net profit (gross profit minus advertising cost) of 100.

i. What is the optimal decision for firm 1 and firm 2?

ii. Does the optimal decision look inferior? If yes, why is it still optimal?

b. Consider the following equation:

$$Y_t = 16 + 3Y_{t-1}$$

Analyze the time path of Y with the first value of Y being given as 5.

Question Four

a. State the order and degree of the following differential equation:

$$\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + X^2 = 0$$

b. A firms profit function is given by:

$$\pi = 12X_1 - X_1^2 + 24X_2 - 1.5X_2^2$$

The firm is faced with a resource constraint given by: $2X_1 + X_2 = 27$

i. Calculate the profit-maximising outputs and the maximum profit

ii. Compare the results in (a) with the results you would have obtained in the absence of the resource constraint.

PAPER THREE QUESTIONS

Question One

a. Given the demand and supply functions below

$$Q^D = \frac{40}{3} - \frac{1}{3}P$$
$$Q^S = 2p - 3$$

i. Find the equilibrium price and quantity

ii. Calculate the consumer and producer surpluses at the equilibrium.

b. Determine the decomposability of the following matrices.

i. $A = \begin{bmatrix} 7 & 9 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}$

ii. $B = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 8 \\ 2 & 0 & 0 \end{bmatrix}$

Question Two

a. Find the time path for the following differential equation.

$$\frac{d^2y}{dt^2} + 0.25y = 4, \qquad y(0) = 16, y'(0) = 2$$

b. Given the demand and supply function for a market below:

$$Q^d = 42 - 4P - 4\frac{dP}{dt} + \frac{d^2P}{dt^2}$$
$$Q^S = -6 + 8P, \qquad P(0) = 6, \qquad P'(0) = 4$$

i. Find the time path P_t

ii. Comment on the stability of the market.

Question Three

a. Given the production function $Q = K^3 + 3L^2$, what is the marginal rate of technical substitution between capital and labour?

b. An enterprise invested K20,000 in the development of a new product. They can manufacture it for K2, per unit. They then hire a marketing consultant, Conda Marketing Agency Limited in Lusaka, whose conclusions were: If the enterprise spends X kwacha on advertising and sell the product at price P per unit, the quantity sold will be:

$$20{,}000, +4\sqrt{X} - 20P$$

Their profit function will then be:

$$F(p, X) = \left(20{,}000 + 4\sqrt{X} - 20P\right)(P - 2) - X - 20{,}000$$

i. Obtain the optimal level of advertising X^* and price P^*

ii. Using the Hessian determinant, obtain the second-order conditions for profit maximisation.

Question Four

Given the consumption function

$$C = 0.01y^2 + 0.4y + 100$$

calculate the marginal propensity to save when

a. y = 25

b. y = 10

Question Five

The following set of equations describes the behaviour in the market of a particular commodity:

$$Q_t^D = 120 - 0.5P_t$$
$$Q_t^S = -30 + 0.3P_t$$
$$P_t = P_{t-1} + \alpha(Q_{t-1}^D - Q_{t-1}^S)$$

where Q^D is quantity demanded, Q^S is quantity supplied, P is the price and α is a positive parameter. The market does not always clear but the price adjusts depending of the deficit from the previous period.

a. Solve for the long run equilibrium price P^*

b. Solve the first order difference equation and find the particular solution if $P_{t=0} = 200$.

c. For what values of α will there be damped oscillations in the price?

PAPER FOUR QUESTIONS

Question One

a. Suppose Zambia had an initial stock of 32,000 bags of maize in the year 2000. Each year half of the existing stock was consumed and another 8000 bags of maize were produced.
   i. What is the equilibrium quantity of maize?
   ii. What will happen to the actual quantity of maize in Zambia over time?

b. A consumer is known to have a Cobb-Douglas utility function of the form

$$u(x,y) = x^{a}y^{1-a}$$

where the parameter a is unknown. However, it is known that when faced with the following utility maximisation problem:

$$\max x^{a}y^{1-a} \text{ subject to } x + y = 3$$

The consumer chooses $x = 1, y = 2$,
   i. find the value of a
   ii. How much utility can be obtained given the choices of x and y.

Question two

a. The amount of money deposited in a bank is proportional to the interest rate the bank pays on this money. Furthermore, the bank can reinvest the money at 7%.
   i. Find the interest rate the bank should pay to maximise its profit.

b. An economy's output at time $t = 0$ was 100 and the rate of change of output is given by:

$$\frac{dy}{dt} = 0.1y$$

   i. Find the time-path of output of the economy.
   ii. Comment on the time path pattern.

Question three

a. Given the following differential equation:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2, \qquad y(0) = 4, \qquad y'(0) = -1$$

   i. Find the time path
   ii. Test for convergence

b. Find the second-order partial derivatives of the following function:

$$Z = e^{x^2+3xy+y^2}$$

c. Solve the following problem using the Lagrange multiplier method:

$$\max Z = log(x) + log(y) \text{ subject to } x + y = m$$

Question four

a. One major problem in the developed countries is the mental disorder schizophrenia. Three views on the cause of the development of schizophrenia are: a) environmental conditions b) heredity interaction between environment and c) heredity.

At a convention of 80 psychologists, 50 psychologists felt that schizophrenia was due to the interaction between environmental conditions and heredity. Another 10 psychologists felt that schizophrenia was due to heredity alone. Determine:

   i. How many psychologists believed that schizophrenia was due to environmental conditions alone?
   ii. How many felt that heredity had something to do with the development of the disease?

b. Determine if the following equations will give rise to a convergent time path:

$$y'''(t) + 11y''(t) + 34y'(t) + 24y = 5$$

PAPER FIVE QUESTIONS

Question one

a. Diagonalise if you can the following matrices.

i. $A = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$

ii. $B = \begin{pmatrix} 5 & -3 \\ 3 & -1 \end{pmatrix}$

b. A consumer's utility function is given as: $U(X_1, X_2, X_3)$ and he wants to maximise this function subject to the constraint that $P_1X_1 + P_2X_2 + P_3X_3 = 1$.

i. Write out the first-order condition for utility maximisation

ii. Use the Hessian to show the second order conditions.

c. Evaluate:

$$\int_0^{\pi/2} \int_0^1 y \, sinx \, dxdy$$

Question two

a. Given the following market model

$$Q_{dt} = 180 - 0.75P_t$$
$$Q_{st} = -30 + 0.3P_{t-1}$$
$$P_0 = 220$$

i. Find the time path

ii. comment on the dynamics of the time path of price P_t.

b. consider the differential equation:

$$12ytdy + 4y^2dt = 0$$

i. Is it exact?

ii. If not, make it exact and obtain its solution.

Question three

a. Suppose two oil marketing companies Petroda and Puma have to decide where to locate their service stations along Great East Road between Kafue roundabout and Manda hill. Their market share will depend on their choice of six possible locations on this route. It is assumed that if both choose the same location, the market will be divided equally between them. The table below shows the market shares for Petroda. Puma's share is Petroda's share subtracted from 100%

		Petroda					
		1	**2**	**3**	**4**	**5**	**6**
Puma	**1**	50	10	20	30	40	50
	2	90	50	30	40	50	60
	3	80	70	50	50	60	70
	4	70	60	50	50	70	80
	5	60	50	40	30	50	90
	6	50	40	30	20	10	50

i. Using concepts of dominating and dominated strategies, reduce the above matrix to an effective payoff matrix

ii. Does the resulting matrix in (i) have a saddle point?

Question Four

Although South Sudan seceded from Sudan in July 2011, the relationship between the two countries remains tense and is often on the brink of war. In May 2012, despite the United Nations Security Council Resolution 2046 calling for immediate cessation of hostilities, President Bashir of Sudan stated: "If they [South Sudan] want to change the regime in Khartoum, we will work to change the regime in Juba. If they want to attrite us, we will attrite them. And if they want to support our rebels, we will support theirs".

In the context of the above situation, explain the following concepts in Game theory and illustrate them with examples of defence strategies that may be adopted by the two countries and their consequences:

a. Nash equilibrium

b. Prisoner's dilemma

Question Five

Find the extreme point of:

$$Y = 5\ln X_1 + 10\ln X_2 + 15\ln X_3$$

Subject to

$$X_1 + X_2 + X_3 = 6$$

Is the extreme point a maximum or a minimum?

PAPER SIX QUESTIONS

Question one

a. Solve the problem:

$$\max Z = X_1^{\ 2} + X_2^{\ 2} \ \text{ subject to } X_1 + X_2 = 1$$

b. In each of the following cases, comment on the continuity/discontinuity of the function.

i. A welfare program offers unemployed individuals K600 per month. Once an individual earns income, the payment is stopped. Suppose an individual can earn K10 per hour. Then the income, y, of the person is a function of the hours worked, h. That is:

$$y(h) = \begin{cases} 600, & h = 0 \\ 10h, & h > 0 \end{cases}$$

ii. The salary of a salesperson in a company has 3 components: (i) a basic salary of $K800$; (ii) a commission of 2 percent of one's sales, and (iii) a bonus of $K500$ if the sales person's sales for the month reach or exceed $K20,000$ per month.

Question two

a. For the matrix A given below.

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

Find:

i. The Eigen values and their respective eigenvectors

ii. Diagonalise the matrix

b. Consider the following specific Cobb-Douglas production function:

$$Y = 50(LK)^{0.5}$$

i. Find the second-order partial derivatives and determine the signs;

ii. What is the economic interpretation of the signs of these derivatives?

Question three

a. Find the solution $y(t)$ of the following differential equation, given that $y(0) = -2$

$$\frac{dy}{dt} = \frac{ty^2}{\sqrt{1+t^2}}$$

b. A publishing company employs typist on an hourly basis. There are five typists for service and their charges and speeds are different. According to an earlier understanding, only one job is given to one typist and the typist is paid for full hour even if he/she works for a fraction of it. Given the data in the following tables, find the optimal assignment of typist to jobs.

Typist	Rate per hour	Typing speed
A	5	12
B	6	14
C	3	8
D	4	10
E	4	11

Job	Size (pages)
P	199
Q	175
R	145
S	198
T	178

Question four

a. A firm uses one input, Labour (L) to produce output (Q). The marginal production function is $MP(L) = 10 - 10L^{2/3}$. Assume that $Q = 0$ if $L = 0$. Find the production function $Q(L)$.

b. A firm begins at time $t = 0$ with a capital stock $K(0) = K500,000$, and in addition to replacing any depreciated capital, is planning to invest in new capital at the rate $I(t) = 600t^2$ over the next 10 years. Compute the planned level of capital stock 10 years from now.

Question five

An international organisation intends to initiate a large research project. It is willing to pay principal researchers K250 a day and research assistants K50 a day. The man power cost of the project is estimated to be:

$$C = 6000 + 6X^3 - 36XY + 3Y^2,$$

Where X is the number of principal researchers and Y is the number of research assistants.

a. How many principal researchers and research assistants should be assigned to the project to minimise cost?

b. Calculate the total cost given these values.

PAPER SEVEN QUESTIONS

Question one

a. **Check the sign definiteness of the following matrix using eigen value and determinantal tests:**

$$A = \begin{bmatrix} -3 & -1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

b. A Publisher agrees to pay the author of a book a royalty of 15%. The demand for the book is

$$x = 200 - 5p$$

and the production cost is

$$c = 10 + 2x + x^2$$

i. Find the optimal sales from both the author's and the publisher's perspective

ii. Comment on your result

Question two

Given the demand function:

$$Q_1 = 60 - 3P_1 + 2P_2 + 0.25Y$$

Where $P_1 = 5$, $P_2 = 10$ and $Y = 80$, Calculate:

i. Own price elasticity of demand

ii. Cross price elasticity of demand

iii. Income elasticity of demand

iv. What is the relationship between good 1 and good 2?

v. Is good 1 an inferior or a superior good?

Question three

a. **Given the production function $Q = K^3 + 3L^2$, what is the marginal rate of technical substitution between capital and labour?**

b. Solve $\frac{dy}{dx} + e^{x+y} = 0$

Question four

A market for some agricultural product, because of gestation lags between the decision to plant and the subsequent harvest, supply decisions are conditioned by the price which is expected to rule at the time the crop is harvested; so the supply at time t depends upon the expected price for the crop at time t which is assumed to equal to the actual price at time$t - 1$. The demand for the crop at time t depends upon the actual price at time t. the actual price moves to clear the market in each period, so the market is always in short-run equilibrium. But long-run

equilibrium is reached only when expectations are realised and the expected price at time t equal to the price which is actually on served at time t.

i. Formulate the model corresponding to the above situation

ii. **Obtain the particular solution for the model**

iii. Under what conditions will convergence on the long-run equilibrium price occur?

Question Five

A taxi company has to assign each taxi to each passenger as fast as possible. The following matrix shows the time to reach the passenger in minutes.

	Taxi 1	Taxi 2	Taxi 3	Taxi 4	Taxi 5
Passenger 1	12	8	11	18	11
Passenger 2	14	22	8	12	14
Passenger 3	14	14	16	14	15
Passenger 4	19	11	14	17	15
Passenger 5	13	9	17	20	11

Find the optimal assignment of the taxies.

PAPER EIGHT QUESTIONS

Question one

a. Find the second-order partial derivatives of the following function:

$$Z = X^2e^y + y^2 + X^2 + \ln(X^2)$$

b. Using integration techniques find:

i. $\int 2x\sqrt{x^2+1}\ dx$

ii. $\int_{-1}^{4} 3(x+2)^2\, dx$

Question two

a. Consider the first order linear difference equation:

$$Y_t = b^t(Y_0 - Y^*) + Y^*$$

Draw pictures showing the stability or instability of the system when

i. $0 < b < 1$,

ii. $-1 < b < 0$;

iii. $b > 1$,

iv. $b = -1$;

v. $b = 1$

b. Two companies A and B are promoting two competing products. Each product currently controls 50% of the market. Because of recent modifications in the two products, the two companies are now preparing to launch a new advertisement campaign. If no advertisement is made by either of the two companies, the present status of the market shares will remain unchanged. However, if either company launches a stronger campaign, the other company will certainly lose a proportional percentage of its customers. A survey of the market indicated that 50% of the potential customers can be reached through television, 30% through newspapers, and the remaining 20% through radio. The objective of each company is to select the appropriate advertising media.

i. Formulate the problem as a two-person zero-sum game

ii. Does the problem have a saddle point?

Question three

A refinery must transport a finished good to some storage tanks. There are two pipelines A and B to do the transporting. The cost of transporting x units on A is ax^2; and the cost of transporting y units on B is by^2, where $a > 0$ and $b > 0$ are given.

a. What will be the minimum cost of transporting Q units?

b. What happens to the cost if Q increases by $r\%$?

Question four

a. Solve the following system of equations using Cramer's rule:

$$\begin{aligned} 3X_1 + 5X_3 &= 69 \\ 2X_2 + 4X_3 &= 46 \\ X_1 + X_2 + 4X_3 &= 30 \end{aligned}$$

b. The following is the input coefficient matrix A for a two-sector input-output model:

$$A = \begin{bmatrix} 0.10 & 0.03 \\ 0.05 & 0.20 \end{bmatrix}$$

i. Calculate the total input requirement matrix $(I - A)^{-1}$ for producing a unit output of each sector.

ii. Comment on the sign definiteness of the above matrix

Question five

The following table provides data on unit transportation cost from each of the three production factories A, B and C to each of the three distributors of the output X, Y and Z. for each destination, a second indicates an initial basic feasible solution obtained by the Matrix Minimum Method.

Factory	**X**		**Y**		**Z**		**Supply**
A	2		1	10	5		**10**
B	7	**13**	3	12	4		**25**
C	6	**2**	5		3	18	**20**
Requirement		**15**		**22**		**18**	**55**

i. Check if the cost of the initial feasible solution is optimal

ii. If not, what is the optimal cost?

PAPER NINE QUESTIONS

Question one

a. Given the profit function $\pi = 160x - 3x^2 - 2xy - 2y^2 + 120y - 18$ for a firm producing two goods x and y
   i. Find the profit maximising level of outputs
   ii. What is the profit in (i) above
   iii. Test the second order conditions

b. A monopolistic firm has the following demand functions for each of its products x and y.

$$x = 72 - 0.5P_x$$
$$y = 120 - P_y$$

The combined cost function is

$$c = x^2 + xy + y^2 + 35$$

and the maximum joint production is 40. Thus, the constraint is $x + y = 40$. Find the profit maximising level of
   i. Output
   ii. Price
   iii. Profit

Question two

Find the following integrals

i. $\int (4x^3 + 9x^2)(x^4 + 3x^3 + 6)^3 dx$

ii. $\int_2^5 \frac{3x}{(x+1)^2}$

iii. $\int_{-\infty}^{0} \frac{1}{(3-x)^2} dx$

Question three

a. A profit maximising monopolist has the following demand function: $p = 100 - Q^2$ The marginal cost facing the monopolist is given by: $MC = 2 + 3Q$
   i. Find the profit maximising level of price and quantity
   ii. Find the consumers' surplus at the profit maximising price and quantity.

b. Mrs. Patricia Banda is the owner-manager of a beauty salon specialising in contemporary hairstyles. The annual demand and supply for a standard treatment are given by:

$$Q^D = P^2 - 175P + 7500$$

$$Q^S = P^2 + 25P - 1250$$

Over the relevant ranges of P and Q. Price adjusts according to excess demand as follows: $\frac{dP}{dt} = 0.01(Q^D - Q^S)$
   i. Find the equilibrium price, P^*

   ii. If price is initially 20, deduce an equation for price, P, at time t.how does P approach P^*.

Question four

a. A firm's production technology can be specified by the following Cobb-Douglas production function:

$$Q = F(K, N) = 10K^{0.5}N^{0.5}$$

What are the cost–minimising quantities of its two inputs, capital services, K, and labour services, N if the firm wishes to produce an output, Q, of 500 units; given that the wage rate is 8 and the price of a unit of capital services is 2?

b. Given the following differential equation,

$$y'''(t) + 6y''(t) + y'(t) + 8y = 8$$

   i. State the Routh Theorem;
   ii. Using the Routh theorem, check whether the differential equation has a convergent time path

Question five

a. The payoff table for a two-person zero-sum for a mixed strategy game is presented in the table below. The numbers relate to player X's gain:

	Y_1	Y_2
X_1	2	10
X_2	6	1

Find the best combination of strategies for each player and the value of the game.

b. Test the dynamic stability of the following system:

$$\frac{dx}{dt} = -5x - 0.5y + 12$$
$$\frac{dy}{dt} = 6x - y - 8$$

$$x(0) = 3, \quad y(0) = 6$$

PAPER TEN QUESTIONS

Question One

Suppose a firm producing chemical products can buy a chemical for $K10$ per ounce. There are only 17.25 ounces available. The firm can transform this chemical into two products: A and B. Transforming to A costs $K3$ per ounce, while transforming to B costs $K5$ per ounce. If x_1 ounces of A are produced, the price the firm will get for A is $P_1 = 30 - x_1$; if x_2 ounces of B are produced, the price the firm will get for B is $P_2 = 50 - x_2$. The question is: how much chemical should the firm buy and what should it transform it to? Let x_3 be the amount of chemical the firm purchases. The model to be maximised is:

Maximise

$$x_1(30 - x_1) + x_2(50 - x_2) - 3x_1 - 5x_2 - 10x_3$$

Subject to

$$x_1 + x_2 - x_3 \leq 0$$
$$x_3 \leq 17.25$$

a. State the Karush-Kuhn-Tucker conditions for the above problem;
b. Obtain the solution for the problem.

Question Two

The demand for some agricultural product produced at time t is given by:

$$Q_t^D = 120 - 4P_t$$

Where Q^D is quantity demanded and P is the price.

The supply of the produce at time t is given by:

$$Q_t^S = -20 + 3EP_t$$

Where Q^S is quantity supplied and EP is expected price. Assume that $EP_t = P_{t-1}$ and that the market clears period after period.

a. If $P_0 = 25$, obtain the time path of price and comment on its nature;
b. Assume the coefficient on the expected price in the supply function rises from 3 to 5. Does the market converge to the long-run equilibrium price?

Question Three

Consider the following problem:

Maximise

$$\int_0^t (K - \alpha K^2 - I^2)dt$$

Subject to

$$\dot{K} = I - \delta K$$

$$K(0) = K_0 \qquad \text{(given)}$$

Where K is the capital stock and I is investment.

a. Form the Hamiltonian for the above problem;
b. State the necessary conditions based on Pontryagin's maximum Principle;
c. State the boundary conditions;
d. Form the system of differential equations and obtain the solution.

Question Four

Believe it or not, the following poem whose author is not known is a linear programming problem! Read the poem and

a. Formulate the problem mathematically;
b. Solve the problem using the Simplex algorithmic procedure.

SERENDIPITY
The three princes of Serendip
Went on a little trip.
They could not carry too much weight.
More than 300 pounds made them hesitate.
They planned to the ounce. When they returned to Ceylon
They found their supplies were just about gone
When, what to their joy Prince William found
A pile of coconuts on the ground.
"Each will bring 60 rupees", said Prince Richard with a grin
When he almost tripped over a lion skin.
"Look out!" cried Prince Robert with glee
As he spied more lion skins under a tree.
"These are worth even more – 300 rupees each
If we can carry them just down to the beach".
Each skin weighed fifteen pounds and each coconut five.
But they carried them all and made it alive.
The boat back to the island was very small,
Fifteen cubic feet capacity – that was all.
Each lion skin took up one cubic foot
While each coconut the same space took.
When everything was stowed, they headed to the sea

And on the way calculated what their new wealth might be.
"Eureka" cried Prince Robert, "our wealth is so great
That there is no other way we could return in this state.
And any other skins or nut which we might have brought
Might have left us poorer. And now I know what –
I will write my friend Horace in England, for surely,
Only he can appreciate our serendipity".

Question Five

A person wants to go from City 1 to City 10 by the shortest distance. The journey involves four legs and the distances (in kms) for each leg are shown below.

Leg 1			
	C_2	C_3	C_4
C_1	50	50	60

Leg 2			
	C_5	C_6	C_7
C_2	40	70	80
C_3	80	100	50
C_4	40	50	70

Leg 3		
	C_8	C_9
C_5	60	80
C_6	90	70
C_7	50	70

Leg 4	
	C_{10}
C_8	80
C_9	90

a. In the above problem, identify the stage, state and policy decision variable.
b. Write out the recursive relationship.
c. Assuming Markovian property and using Bellman's Principle of Optimality, solve the problem through backward iterations.

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