Our objective is to devise a control strategy for a peculiar class of spatially extended systems. The setup of the problem is rather simple. Consider a one-dimensional "large" domain of length L (the "big box" (BB)) inside which a smaller domain (line) of length l is embedded (the "small box" (SB), see Figure 1). For convenience, we place the origin at the center of the system, so that the boundaries of the "big box" are localized at –L/2 and +L/2 while the boundaries of the "small box" are $-l/2$ and $+l/2$, respectively.

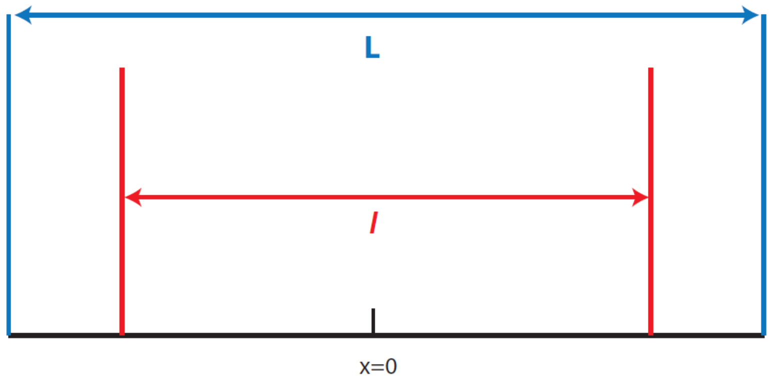

Figure 1: A schematic representation of the one-dimensional type of support considered here.

This setup forms the support for a partial differential equation for a variable u=u(x,t), which at this stage should be seen as a grey box model, i.e. we know that it is a PDE of a given order but do not know its exact form – however, (a) we can perform simulations with this grey box; (b) we can observe what occurs anywhere in it; and (c) we can only "talk" to this box by varying the boundary conditions at the edges of the "big box", L/2 and –L/2.

We want to control some of the properties of the solution – in particular, we want to control features of what happens at the boundaries of the "small box", SB. The basic idea is to understand how to act on the boundary conditions of the PDE at the borders of the BB, in order to obtain prescribed boundary conditions for the PDE at $-l/2$ and $+l/2$ "as soon as possible". Since the exact dynamics is supposed to be unknown, our idea was to use some appropriate feedback loop that would change the external boundary conditions following measurements done on the boundaries of the SB.

Let us take a simple example to illustrate this. Consider the following reaction-diffusion equation

$$ut=uxx-u$$

that would take place on the entire aforementioned domain. For reasons that appear arbitrary at the moment, but can be justified in a more general setting, we consider the case where one could actually only impose (time-dependent) *Dirichlet* boundary conditions on the edges of the grey box, while our objective would be to have a prescribed *flux* at $-l/2$ and $+l/2$. The "open loop" dynamics is rather simple here, and leads to a well-known stationary profile as depicted in

Figure 2. With the imposed BB boundary conditions (see figure legend), the stationary fluxes at $-l/2$ and $+l/2$ are respectively -0.0721 and – 0.000181 (here, we chose l =0.6 L).

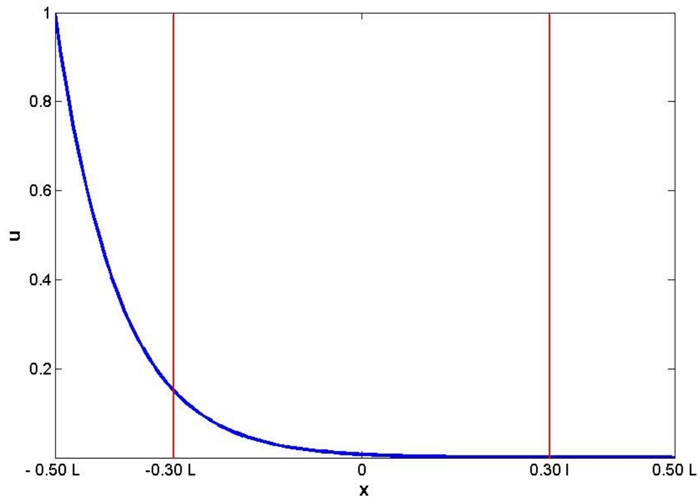

Figure 2: Stationary profile for $u(x,t)$, with L=100, $u(-0.5L, t)=1.0$, $u(0.5L, t)=0.0$.
The initial condition is a linear profile connecting the two boundary values. The boundaries of the SB are depicted as vertical red lines.

Imagine that we want the fluxes at the boundaries of the SB to be instead -0.06 and + 0.0002, respectively. One could here implement different types of feedback based on a measurement performed at these points. For example, we here chose in a totally arbitrary way to implement a purely integral feedback on the Dirichlet boundary conditions of the large box:

$$u(-0.5L,t)=u(-0.5L,0)+\int_0^t K_{I}^{left}\left(j_{-l/2}(\tau)+0.06\right)d\tau$$
$$u(0.5L,t)=u(0.5L,0)+\int_0^t K_{I}^{right}\left(j_{+l/2}(\tau)-0.0002\right)d\tau,$$

where $j_{-l/2}$ and $j_{+l/2}$ are the fluxes at the left and right boundaries of the SB, respectively. The numerical integrations reveal that, in this way, one can indeed get the desired fluxes at the boundaries of the SB, using an appropriate set of values for the gains (see Figure 3 for results and parameter values). One can see however that the “control actions” imposed in the form of “big box” boundary conditions need some time to propagate to the interior of the system as demonstrated in Figure 4. We see there that there is a delay between the time when the Dirichlet boundary condition is modified and the time when this action becomes visually apparent at the “small box” inner boundary. As expected, in view of our choice of feedback, one also encounters appreciable overshoots.

The situation worsens as the distance separating the outer and inner boundaries becomes larger. For identical gain and initial offset and initial profile, one can observe that the settling time needed to reach the setpoint rapidly increases with L-l.

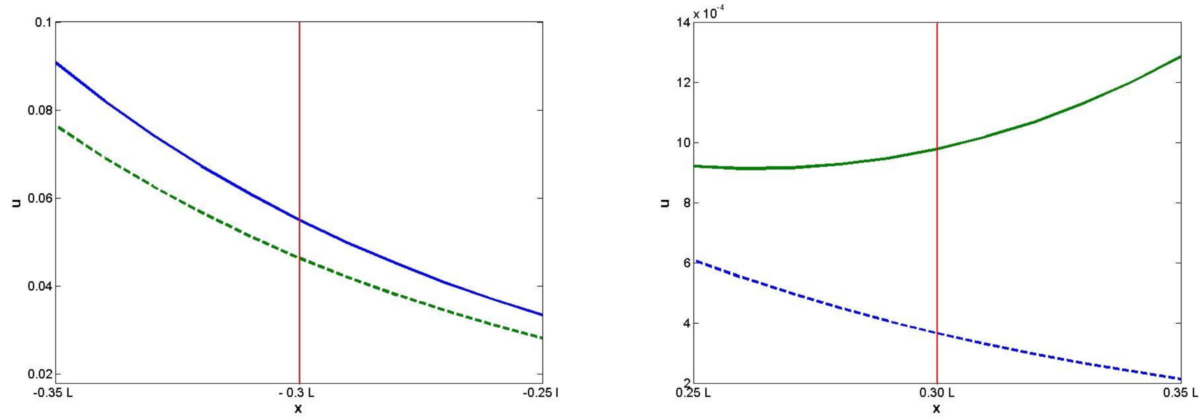

Figure 3: Stationary profile for u without (blue curves) and with (green curves) the SB flux feedback. The conditions are the same as in Figure 2. The first subfigure is a zoom on the left SB boundary, and the second one on the right SB boundary. The values for Kl have been taken to be 10 for the left boundary and -10 for the right one.
The setpoint values for the fluxes are given in the text.

For example, with the parameters used here, the settling time of the closed-loop dynamics becomes 4-5 times longer than the time it takes for the open-loop system to reach steady state, whenever l <0.4L. This problem can be partially taken care of by using slightly more sophisticated feedback loops, such as PI or PID where the gains are estimated using the Ziegler-Nichols rules. Even that approach however gives mediocre results as the distance between the boundaries increases (specifically, one can observe important overshoots). This is more than probably related to the inefficiency of traditional PID gains tuning in systems with large delays.

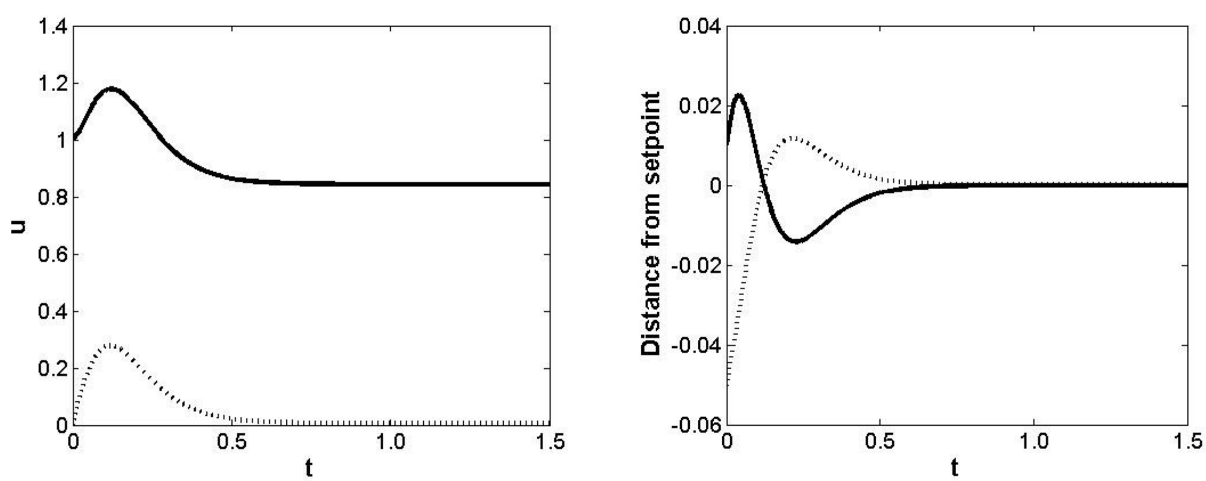

Figure 4: Time evolution of the variable u at the external boundaries (first subfigure) and distance from the setpoint (second subfigure). The plain curves are representative of the left part of the system; the dotted ones stand for the evolution in the right one. All conditions are the same as before.

Obviously, such a simple controller is not the optimal choice. On top of the aforementioned issues, using PID controllers also means tuning 1 to 3 different gains at each boundary – which is all right for a single box, but could become very heavy and time consuming for a collection of boxes (which we envision to do in a near future). We were wondering what would be the best choice of (anticipative) controller that would minimize the time to reach the setup value in an efficient way.