

# Computer algebra describes three dimensional turbulence flows over curved beds via the Smagorinski large eddy closure

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## Abstract

Consider the turbulent floods over curved beds. The Smagorinski closure for turbulence, with its linear dependence of eddy viscosity upon the shear-rate, models turbulent dissipation. A slow manifold model of the dynamics of the fluid layer allows for large changes in layer thickness provided the changes occur over a large enough lateral length scale. The slow manifold is based on two macroscopic modes by modifying the spectrum: here artificially modify the boundary conditions on the free surface so that, as well as a mode representing conservation of fluid, a lateral shear flow with slip is a neutral critical mode. Then remove the modification to recover a model for turbulent floods.

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# 1 Introduction

This approach to modelling turbulent floods develops from that of modelling non-Newtonian fluids which have a nonlinear dependence upon strain rate (Roberts 2007*a,b*). Bijvelds et al. (1999) required enhanced lateral mixing which is naturally predicted by this slow manifold approach. Roberts et al. (2008) reports preliminary results using the model derived with the computer algebra program documented herein.

Consider the three dimensional fluid flow over a curved bed, say  $\mathbf{b}(\mathbf{x}, \mathbf{y})$ . Let coordinates  $\mathbf{x}$  and  $\mathbf{y}$  measure the horizontal distances and coordinate  $z$

the vertical distance. Let the incompressible, irrotational turbulent fluid have thickness  $\eta(\mathbf{x}, \mathbf{y}, t)$ , constant density  $\rho$ , and the nonlinear constitutive relation of the Smagorinski closure (Kim 2002, Marstop 2006, Özgökmen, Iliescu, Fischer, Srinivasan & Duan 2007, e.g.). The fluid flows with some varying velocity field  $\bar{\mathbf{q}}(\mathbf{x}, \mathbf{y}, z, t) = (\mathbf{u}, \mathbf{v}, w)$  and pressure field  $\mathbf{p}(\mathbf{x}, \mathbf{y}, z, t)$ ; these fields are the turbulent mean fields, that is, the fields averaged over realisations.

## 1.1 Uniform acceleration

First, for the three dimensional fluid flowing a flat bed, do not allow any lateral variations,  $\partial_x = \partial_y = 0$ . Then a low order slow manifold, errors  $\mathcal{O}(\gamma^2 + g_x^2)$ , is that in terms of the scaled vertical coordinate  $\zeta = z/\eta$  the fluid fields are

$$w = 0, \quad (\text{shear flow}) \quad (1)$$

$$\mathbf{p} = g_z(1 - \zeta)\eta, \quad (\text{hydrostatic}) \quad (2)$$

$$\begin{aligned} \mathbf{u} = & \bar{\mathbf{u}} \frac{2(\zeta + c_u)}{1 + 2c_u} \\ & + \gamma \bar{\mathbf{u}} \frac{(1 + c_u)[(1 + 4c_u)(c_u + \zeta) - 2(1 + 2c_u)(3c_u\zeta^2 + \zeta^3)]}{4(1 + 2c_u)^2(1 + 3c_u + 3c_u^2)} \\ & + \frac{g_x \eta}{\bar{\mathbf{u}}} \frac{[(5 + 6c_u)(c_u + \zeta) - 6(2 + 7c_u + 6c_u^2)\zeta^2 + 6(1 + 2c_u)^2\zeta^3]}{48\sqrt{2}c_t(1 + 3c_u + 3c_u^2)}, \\ \dot{\epsilon} = & \frac{\bar{\mathbf{u}}}{\eta} \frac{\sqrt{2}}{1 + 2c_u} + \frac{\gamma \bar{\mathbf{u}}}{\eta} \frac{\sqrt{2}(1 + c_u)[(1 + 4c_u) - 6(1 + 2c_u)(2c_u\zeta + \zeta^2)]}{8(1 + 2c_u)^2(1 + 3c_u + 3c_u^2)} \\ & + \frac{g_x}{\bar{\mathbf{u}}} \frac{[(5 + 6c_u) - 12(2 + 7c_u)\zeta + 18(1 + 2c_u)^2\zeta^2]}{96c_t(1 + 3c_u + 3c_u^2)}. \end{aligned} \quad (3)$$

The parameter  $c_u\eta$  is a ‘slip’ length on the bed, see boundary condition (??), and  $c_t$  parametrises the strength of Smagorinski’s turbulent mixing, see the eddy viscosity (14); they are determined from observations. Figure 1 displays a sample of the vertical profile of the rate of strain  $\dot{\epsilon}$  and of the lateral velocity  $\mathbf{u}$ . This equilibrium flow resolves the shear in the lateral velocity and the increase in rate of strain near the bed. Our analysis does not attempt to resolve the turbulent log layer: we assume the details of dynamical interest are those determined by the relatively large scale of the fluid depth.

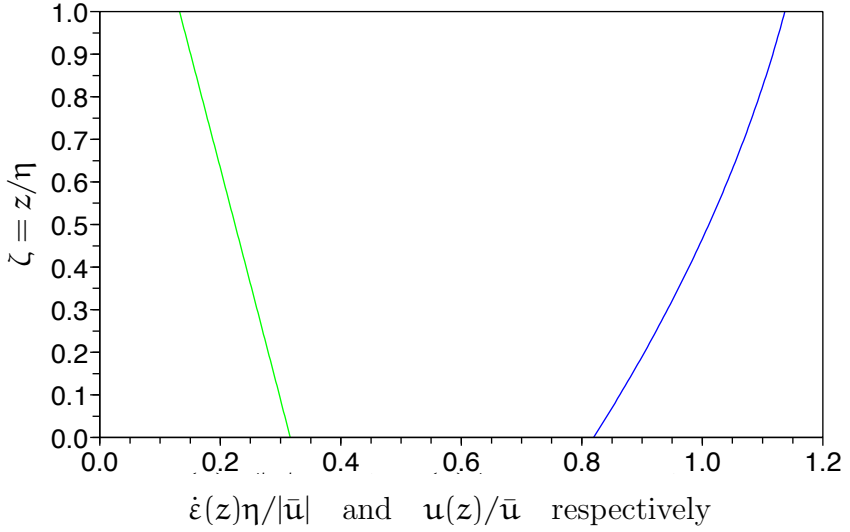


Figure 1: approximate vertical profiles at open channel flow equilibrium for which  $g_x\eta/\bar{u}^2 = 0.0031$ .

The set of such profiles in the vertical, and their nonlinear interactions, form a slow manifold. The evolution on this slow manifold of the mean lateral velocities  $\bar{u}$  and  $\bar{v}$  is dictated by turbulent bed drag limiting gravitational forcing:

$$\frac{d\bar{u}}{dt} = -\frac{\sqrt{2}3c_t(1+c_u)}{(1+2c_u)(1+3c_u+3c_u^2)}\frac{\gamma\bar{u}^2}{\eta} + \frac{\frac{3}{4}+3c_u+3c_u^2}{1+3c_u+3c_u^2}g_x + \mathcal{O}(\gamma^2 + g_x^2 + \partial_x) \quad (4)$$

$$\frac{d\bar{v}}{dt} = -\frac{\sqrt{2}3c_t(1+c_u)}{(1+2c_u)(1+3c_u+3c_u^2)}\frac{\gamma\bar{v}^2}{\eta} + \mathcal{O}(\gamma^2 + \partial_x) \quad (5)$$

Upon putting the artificial parameter  $\gamma = 1$  to recover the physical model, this evolution predicts an equilibrium channel flow at a mean velocity of

$$\bar{u} = \frac{1}{2} \left[ \frac{(1+2c_u)^3}{\sqrt{2}c_t(1+c_u)} \right]^{1/2} \sqrt{g_x\eta}. \quad (6)$$

For example, choosing  $c_t = 0.020$  and  $c_u = 1.848 \approx 13/7 \approx 11/6$  gives about the correct channel flow *and* gives about the correct eddy viscosity when compared with observations of open channel flow (Nezu 2005, e.g.).

## 1.2 Overview

Denote free surface thickness  $\eta(x, y, t)$  by  $h$ , curved bed  $b(x, y)$ , mean lateral velocities  $\bar{u}(x, y, t)$  and  $\bar{v}(x, y, t)$  by  $uu$  and  $vv$ , and their evolution  $\eta_t = gh$ ,  $\bar{u}_t = gu$  and  $\bar{v}_t = gv$ . The Reynolds number  $re$ , and the coefficients of lateral and normal gravitational forcing are Gravity numbers  $grx$  and  $grz$ . Construct an asymptotic solution of the Smagorinski equations in terms of  $\eta$ ,  $b$ ,  $\bar{u}$  and  $\bar{v}$  to some order of nonlinearity in  $\bar{u}$  and  $\bar{v}$  and some order of lateral derivatives  $\partial_x$  and  $\partial_y$ .

Decide upon how the asymptotic expansions of the solution are to be truncated. Then iteratively update the velocity and pressure fields to solve the Smagorinski equations and boundary conditions. The iteration continues until the governing equations are satisfied; that is, their residuals are zero to the order of truncation.

## 2 Preamble

Improve printing by factoring with respect to these variables. It is a matter of taste and may be different depending upon what one wishes to investigate in the algebraic expressions.

```
1 on div; off allfac; on revpri;
2 factor vv,uu,qq,rqq,h,ct,gx,gz,gam,r2,b;
```

### 2.1 Define order parameters

Use the operator  $h(m,n)$  to denote lateral derivatives of the fluid thickness,  $\partial_x^m \partial_y^n \eta$ ,  $b(m,n)$  present derivatives of the curved bed, and similarly  $uu(m,n)$  and  $vv(m,n)$  denotes lateral derivatives of the mean shear,  $\partial_x^m \partial_y^n \bar{u}$  and  $\partial_x^m \partial_y^n \bar{v}$ . Meanwhile define readable abbreviations for  $\eta$  and its first spatial derivatives. Consider the bed shape  $b(x,y)$  small and its first derivatives  $b_x$  and  $b_y$  smaller. Adopt  $d$  to count the number of lateral derivatives, so we can easily truncate the asymptotic expansion.

```
3 operator h; operator b; operator uu; operator vv;
4 eta:=h(0,0); etax:=h(1,0)*d$ etay:=h(0,1)*d$
```

These operators must depend upon time and lateral space. Then lateral derivatives transform as  $\partial_x h(m,n) = h(m+1,n)$ , for example. Also, a time derivative transforms into lateral derivatives of the corresponding evolution: for example,  $\partial_t h(m,n) = \partial_x^m \partial_y^n g h$ .

```

5 depend h,xx,yy,tt;
6 depend uu,xx,yy,tt;
7 depend vv,xx,yy,tt;
8 depend b,xx,yy;
9 let{ df(h(~m,~n),xx)=>h(m+1,n)
10      , df(h(~m,~n),yy)=>h(m,n+1)
11      , df(h(~m,~n),tt)=>df(gh,xx,m,yy,n)
12      , df(uu(~m,~n),xx)=>uu(m+1,n)
13      , df(uu(~m,~n),yy)=>uu(m,n+1)
14      , df(uu(~m,~n),tt)=>df(gu,xx,m,yy,n)
15      , df(vv(~m,~n),xx)=>vv(m+1,n)
16      , df(vv(~m,~n),yy)=>vv(m,n+1)
17      , df(vv(~m,~n),tt)=>df(gv,xx,m,yy,n)
18      , df(b,xx)=>b(1,0)*d
19      , df(b,yy)=>b(0,1)*d
20      , df(b(~m,~n),xx)=>b(m+1,n)
21      , df(b(~m,~n),yy)=>b(m,n+1)
22 };
```

## 2.2 Stretch the coordinates with the free surface

Use stretched coordinates  $zz$ ,  $xx$ ,  $yy$  and  $tt$  to denote  $Z = (z-b)/\eta$ ,  $X = x$ ,  $Y = y$  and  $T = t$ . The free surface is then simply  $Z = 1$ .

```

23 depend xx,x,y,z,t;
24 depend yy,x,y,z,t;
25 depend zz,x,y,z,t;
26 depend tt,x,y,z,t;
```

Then space-time derivatives transform according to

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X} - Z \frac{\eta_x}{\eta} \frac{\partial}{\partial Z} - \frac{b_x}{\eta} \frac{\partial}{\partial Z},$$

$$\begin{aligned}
\frac{\partial}{\partial y} &= \frac{\partial}{\partial Y} - Z \frac{\eta_Y}{\eta} \frac{\partial}{\partial Z} - \frac{b_Y}{\eta} \frac{\partial}{\partial Z}, \\
\frac{\partial}{\partial t} &= \frac{\partial}{\partial T} - Z \frac{\eta_T}{\eta} \frac{\partial}{\partial Z}, \\
\frac{\partial}{\partial z} &= \frac{1}{\eta} \frac{\partial}{\partial Z}.
\end{aligned} \tag{7}$$

We neatly insert an automatic count of lateral derivatives here, with  $\mathbf{d}$ , in between  $\partial_x$  and  $\partial_x$ , and in between  $\partial_y$  and  $\partial_y$ .

```

27 let{ df(~a,x)=>df(a,xx)*d-zz*etax/eta*df(a,zz)
28      -d*df(b,xx)/eta*df(a,zz)
29      , df(~a,y)=>df(a,yy)*d-zz*etay/eta*df(a,zz)
30      -d*df(b,yy)/eta*df(a,zz)
31      , df(~a,t)=>df(a,tt)-zz*gh/eta*df(a,zz)
32      , df(~a,z)=>df(a,zz)/eta
33      };

```

## 2.3 Operators used elsewhere

For integrating in the vertical, use the linear operator `wsolv` as it is quicker than native integration. This is the definite integral that is zero at the bed  $\zeta = 0$ .

```

34 operator wsolv; linear wsolv;
35 let {wsolv(zz^~~~n,zz) => zz^(n+1)/(n+1)
36      ,wsolv(1,zz) => zz };

```

Similarly, it is quicker to use operators than to use the native integration to find the pressure. This is the vertical integral (negative) that is zero at the surface  $\zeta = 1$ .

```

37 operator psolv; linear psolv;
38 let {psolv(zz^~~~n,zz) => (1-zz^(n+1))/(n+1)
39      ,psolv(1,zz) => (1-zz) };

```

Variable `qq` denotes the mean speed  $\bar{q} = \sqrt{U^2 + V^2}$ . Let `rqq` denotes its reciprocal. The following transformation rules should be correct. Either of the last two can be chosen.

```

40 depend qq,uu(0,0),vv(0,0);
41 let { qq^2=>uu(0,0)^2+vv(0,0)^2
42      , df(qq,~aa)=>(uu(0,0)*df(uu(0,0),aa)+vv(0,0)*df(vv(0,0),a
43      } ;
44 depend rqq,qq;
45 let { df(rqq,~aa)=>-rqq^2*df(qq,aa)
46      , rqq*qq=>1
47      , qq^2=>(uu(0,0)^2+vv(0,0)^2)
48      , vv(0,0)^2*rqq=>qq-uu(0,0)^2*rqq
49 %    , uu(0,0)^2*rqq=>qq-vv(0,0)^2*rqq
50      } ;

```

The linear operator `usolv` solves  $\partial_z^2 u' = \text{RHS}$  such that the bed boundary condition (??) is always satisfied and that the mean of the solution  $u'$  is always zero to ensure  $\bar{u}$  remains the mean later velocity.

```

51 operator usolv; linear usolv;
52 let { usolv(zz^~~~n,zz) => (zz^(n+2)
53      -(cu+zz)/(n+3)/(cu+1/2) )/(n+2)/(n+1)
54      , usolv(1,zz) => (zz^2 -(cu+zz)/3/(cu+1/2) )/2 } ;

```

Do not need it, but the linear operator `mean` quickly computes the average of some field over the fluid thickness.

```

55 operator mean; linear mean;
56 let { mean(zz^~~~n,zz) => 1/(n+1)
57      , mean(1,zz) => 1 } ;

```

To see how the iteration is proceeding, write out the number of terms in its residual throughout iteration for each equation. Could also write out time since last length written.

```

58 procedure mylength(res);
59 begin
60 %showtime;
61 %return res;
62 return if res=0 then 0 else length(res);
63 end;

```



### 3 Initialise with linear

Start the iteration from the linear solution that the lateral velocity  $\mathbf{u} = \bar{\mathbf{u}}(\mathbf{c}_u + \zeta)/(\mathbf{c}_u + \frac{1}{2})$ ,  $\mathbf{v} = \bar{\mathbf{v}}(\mathbf{c}_u + \zeta)/(\mathbf{c}_u + \frac{1}{2})$ ,  $\mathbf{p} = \mathbf{g}_z(1 - \zeta)\eta$  and  $\mathbf{w} = 0$ . The parameter  $\mathbf{c}_u$  determines the turbulent slip on the bed and is to be determined to best fit experiment and/or observations. The evolution of the ‘order parameters’ is also zero:  $\bar{\mathbf{u}}_t = \mathbf{g}\mathbf{u} = 0$ ,  $\bar{\mathbf{v}}_t = \mathbf{g}\mathbf{v} = 0$  and  $\eta_t = \mathbf{g}\mathbf{h} = 0$ .

```
64 let r2^2=>2; % r2=sqrt2
65 u:=uu(0,0)*(cu+zz)/(cu+1/2);
66 v:=vv(0,0)*(cu+zz)/(cu+1/2);
67 p:=grz*(1-zz)*eta;
68 w:=gh:=gu:=gv:=0;
```

Also set initial strains from the linear solution.

```
69 exx:=df(u,x);
70 eyy:=df(v,y);
71 ezz:=df(w,z);
72 exz:=(df(u,z)+df(w,x))/2;
73 exy:=(df(u,y)+df(v,x))/2;
74 eyz:=(df(v,z)+df(w,y))/2;
```

Initially approximate the magnitude  $\dot{\epsilon}$  of the strain-rate tensor: the above iteration step assumes the strain rate is  $(\bar{\mathbf{u}}, \bar{\mathbf{v}})\sqrt{2}/\eta/(1 + 2\mathbf{c}_u)$  to leading approximation.

```
75 ros:=qq*r2/eta/(1+2*cu);
```

In the Smagorinski model (Özgökmen, Iliescu, Fischer, Srinivasan & Duan 2007, e.g.)  $\mathbf{c}_t = (\mathbf{c}_s\Delta/\eta)^2$  where arguments indicate  $\mathbf{c}_s \approx 0.2$ . To match observations (Nezu 2005, e.g.) of open channel flow equilibria we set  $\mathbf{c}_t = 0.020$  from which the appropriate filter scale  $\Delta \approx 0.7\eta$  consistent with significant mixing across the fluid layer as seen in Figures 14–15 by Janosi et al. (2004).

```
76 ct:=1/50;
```

Set initial values for the stress.

```

77 txx:=2*ct*eta^2*ros*exx;
78 tyy:=2*ct*eta^2*ros*eyy;
79 tzz:=2*ct*eta^2*ros*ezz;
80 txz:=2*ct*eta^2*ros*exz;
81 txy:=2*ct*eta^2*ros*exy;
82 tyz:=2*ct*eta^2*ros*eyz;

```

The bed slip-drag law requires coefficient.

```

83 cu:=11/6;

```

## 4 Truncate the asymptotic expansion

There are lots of ways to truncate the asymptotic model. The small parameters available are:

- **d** counting the number of **x** derivatives of the slowly varying lateral spatial structure in any term;
- the homotopy parameter **gam** varying between  $\gamma = 0$  for the artificial base problem and  $\gamma = 1$  for the physical fluid equations; and
- **grx**, **gry** and **grz** being the lateral and normal components of gravity.

Because of the velocity of the flow not small, consider the parameter  $\bar{u}$  and  $\bar{v}$  finite. Usually we will make lateral gravity fairly small by scaling with the magnitude of  $\partial_x$  and  $\partial_y$ .

Initially omit all **x** derivatives by setting  $\mathbf{d} = 0$ , later we scale **d** with **eps** to get the relatively simple but interesting model with errors  $\mathcal{O}(\gamma^{3/2} + g_x^{3/2} + g_z^3 + \partial_x^3)$ .<sup>1</sup> We need not make normal gravity  $g_z$  small as here, but doing so removes some messy terms.

```

84 d:=eps;
85 grz:=gz;
86 grx:=eps*gx;
87 gry:=0;
88 gamm:=eps*gam;
89 factor eps;

```

---

<sup>1</sup>We need to check the algorithm at  $\mathcal{O}(g_x^3)$ .

For now truncate to relatively low order,  $\mathcal{O}(\gamma^{3/2} + g_x^{3/2} + g_z^3 + \partial_x^3)$ , in spatial derivatives and boundary condition artifice (can do  $\mathcal{O}(\epsilon^4)$  if no spatial variations):

```
90 let { eps^3=>0 };
```

## 5 Invoke the iterative loop

```
91 for iter:=1:6 do begin ok:=1;
92 write "ITERATION ",iter;
```

## 6 Update $w$ with continuity and no flow penetrating the bed

The nondimensional continuity equation for the incompressible fluid flow is

$$\vec{\nabla} \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (8)$$

Let the vector  $\vec{n} = \frac{1}{\sqrt{1+b_x^2+b_y^2}}(-b_x, -b_y, 1)$  present the unit vector normal to the curved bed. No flow penetrating the curved bed indicates  $\vec{q} \cdot \vec{n} = 0$ , that is,

$$w = ub_x + vb_y \text{ on } z = b. \quad (9)$$

As with all field variables in this model, the quantities  $u$ ,  $v$ ,  $w$  and  $p$  are averaged over the ensemble of turbulent flows. Compute the residual of the continuity equation and the boundary condition (9), then update the vertical velocity  $w$  by integrating from the bed,  $\zeta = 0$ . The variable `ok` stores whether all residuals are so far zero in this iteration.

```
93 resc:=df(u,x)+df(v,y)+df(w,z);
94 resa:=sub(zz=0,w-u*df(b,x)-v*df(b,y));
95 write length_resc:=mylength(resc);
96 write length_resa:=mylength(resa);
97 ok:=if {resc,resa}={0,0} then ok else 0;
98 w:=w+(dw:=-eta*wsolv(resc,zz))-resa;
```

The  $\zeta$  component of the stress and rate-of-strain tensor should be, and might need to be, updated from this correction to the normal velocity; the magnitude of the rate-of-strain tensor is unaffected (to leading order).

```

99 ezz:=ezz+df(dw,zz)/eta;
100 tzz:=tzz+2*r2*ct/(1+2*cu)*qq*df(dw,zz);

```

## 7 Update the free surface evolution

The kinematic condition at the free surface,

$$\frac{\partial \eta}{\partial t} + u \frac{\partial (\eta + b)}{\partial x} + v \frac{\partial (\eta + b)}{\partial y} = w \quad \text{on} \quad z = \eta + b, \quad (10)$$

gives the evolution of the fluid thickness  $h$ . It dominantly arises from the normal velocity.

```

101 gh:=sub(zz=1, w-u*etax-v*etay-u*df(b,x)-v*df(b,y));

```

## 8 Update pressure from vertical momentum and surface normal stress

The nondimensional momentum equation is

$$\frac{\partial \vec{q}}{\partial t} + \vec{q} \cdot \vec{\nabla} \vec{q} = -\vec{\nabla} p + \vec{\nabla} \cdot \vec{\tau} + \vec{g}, \quad (11)$$

where  $\vec{\tau}$  is the nondimensional deviatoric eddy stress tensor, and  $\vec{g}$  is the direction of gravity; when the substrate slopes,  $\vec{g}$  is not normal to the substrate. The vertical momentum equation is solved with the surface condition that the turbulent mean, normal stress to the free surface is zero, that is,

$$-p + \frac{\tau_{zz} - 2\eta_x \tau_{xz} - 2\eta_y \tau_{yz} + \eta_x^2 \tau_{xx} + 2\eta_x \eta_y \tau_{xy} + \eta_y^2 \tau_{yy}}{1 + \eta_x^2 + \eta_y^2} = 0 \quad \text{on} \quad z = \eta + b. \quad (12)$$

Compute the residuals of the vertical momentum equation and the zero normal stress on the free surface. The recent change to the normal velocity affects the pressure update.

```

102 resw:=df(w,t)+u*df(w,x)+v*df(w,y)+w*df(w,z)+df(p,z)
103     +grz-df(txz,x)-df(txy,y)-df(tzz,z);
104 restn:=sub(zz=1,-p*(1+(etax+df(b,x))^2+(etay+df(b,y))^2)
105     +tzz-2*(etax+df(b,x))*txz-2*(etay+df(b,y))*tyz
106     +(etax+df(b,y))^2*txx+2*(etax+df(b,x))
107     *(etay+df(b,y))*txy+(etay+df(b,y))^2*tyy);
108 write length_resw:=mylength(resw);
109 write length_restn:=mylength(restn);
110 ok:=if {resw,restn}={0,0} then ok else 0;

```

Update the pressure field  $p$  by integrating down from the free surface  $\zeta = 1$ ; we use the linear operator `psolv` to solve  $\partial_\zeta p' = -\text{RHS}$  such that  $p' = 0$  at  $\zeta = 1$ .

```

111 p:=p+eta*psolv(resw,zz)+restn;

```

## 9 Update stress, $u$ and $v$ from lateral momentum and surface tangential stress

### 9.1 Smagorinski large eddy stress-shear closure

Now the strain-rate tensor

$$\dot{\epsilon}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

```

112 exx:=df(u,x);
113 eyy:=df(v,y);
114 ezz:=df(w,z);
115 exz:=(df(u,z)+df(w,x))/2;
116 exy:=(df(u,y)+df(v,x))/2;
117 eyz:=(df(v,z)+df(w,y))/2;

```

Then the stress tensor for the fluid is  $\sigma_{ij} = -p\delta_{ij} + 2\rho\nu\dot{\epsilon}_{ij}$ : when the viscosity  $\nu$  is constant this models a Newtonian fluid; but here the Smagorinski closure for turbulent flow is that this eddy viscosity varies

linearly with strain-rate magnitude (analogous to a shear thickening non-Newtonian fluid). Define the magnitude  $\text{ros} = |\dot{\epsilon}|$ , the second invariant of the strain-rate tensor, where

$$|\dot{\epsilon}|^2 = \sum_{i,j} \dot{\epsilon}_{ij}^2. \quad (13)$$

```

118 rese:=exx^2+ezz^2+eyy^2+2*exz^2+2*exy^2+2*eyz^2-ros^2;
119 write length_rese:=mylength(rese);
120 ok:=if rese=0 then ok else 0;
121 ros:=ros+rese*eta*(cu+1/2)/r2*rqq;
```

Approximate the eddy viscosity at any point in the fluid as proportional to the local strain-rate magnitude,

$$\nu = c_t \eta^2 \dot{\epsilon}, \quad (14)$$

where  $c_t$  is a dimensionless constant to be chosen to fit experiments and/or observations. For whatever  $c_t$  is chosen, the deviatoric stress tensor is  $\tau_{ij} = 2\nu \dot{\epsilon}_{ij} = 2c_t \eta^2 \dot{\epsilon}_{ij}$ .

```

122 txx:=2*ct*eta^2*ros*exx;
123 tyy:=2*ct*eta^2*ros*eyy;
124 tzz:=2*ct*eta^2*ros*ezz;
125 txz:=2*ct*eta^2*ros*exz;
126 txy:=2*ct*eta^2*ros*exy;
127 tyz:=2*ct*eta^2*ros*eyz;
```

## 9.2 Compute residuals of lateral momentum

There must be no turbulent mean, tangential stress at the free surface,

$$(1 - \eta_x^2) \tau_{xz} + \eta_x (\tau_{zz} - \tau_{xx}) - \eta_y (\tau_{xy} + \eta_x \tau_{yz}) = 0 \quad \text{on } z = \eta + b \quad (15)$$

$$(1 - \eta_y^2) \tau_{yz} + \eta_y (\tau_{zz} - \tau_{yy}) - \eta_x (\tau_{xy} + \eta_y \tau_{xz}) = 0 \quad \text{on } z = \eta + b \quad (16)$$

Also, put a slip law on the mean bed to provide bed drag:

$$\vec{q}_{\text{tan}} = c_u \eta \frac{\partial \vec{q}_{\text{tan}}}{\partial \vec{n}} \quad \text{on } z = b. \quad (17)$$

for some constant  $\mathbf{c}_u \approx 11/6$  to match open channel flow observations, where the term  $\vec{q}_{\text{tan}}$  represents the velocity tangential to the curved bed and the vector  $\vec{n}$  is the unit vector normal to the bed. To solve equation (17), assume  $\vec{t}_x = \frac{1}{\sqrt{1+b_x^2}}(1, 0, b_x)$  and  $\vec{t}_y = \frac{1}{\sqrt{1+b_y^2}}(0, 1, b_y)$  are unit vectors tangential to the curved bed in the  $x$ - and  $y$ - directions. For simplicity, consider  $\vec{q}_{\text{tan}} = (\vec{t}_x \cdot \vec{q})\vec{t}_x + (\vec{t}_y \cdot \vec{q})\vec{t}_y$ . Thus, the boundary condition (17) becomes

$$\begin{aligned}\vec{t}_x \cdot \vec{q}_{\text{tan}} &= c_u \eta \frac{\partial}{\partial \vec{n}} (\vec{t}_x \cdot \vec{q}_{\text{tan}}) \quad \text{on } z = b, \\ \vec{t}_y \cdot \vec{q}_{\text{tan}} &= c_u \eta \frac{\partial}{\partial \vec{n}} (\vec{t}_y \cdot \vec{q}_{\text{tan}}) \quad \text{on } z = b,\end{aligned}$$

that is,

$$\frac{1}{\sqrt{1+b_x^2}}(u + wb_x) = \frac{c_u h}{\sqrt{1+b_x^2+b_y^2}} \frac{\partial}{\partial \vec{n}} (u + wb_x), \quad (18)$$

$$\frac{1}{\sqrt{1+b_y^2}}(v + wb_y) = \frac{c_u h}{\sqrt{1+b_x^2+b_y^2}} \frac{\partial}{\partial \vec{n}} (v + wb_y). \quad (19)$$

To get centre manifold theory support for the slow manifold model of shallow water flow, modify the surface condition (15) and (16) on the tangential stresses to have an artificial forcing proportional to the square of the local, free surface, velocity:

$$\begin{aligned}& [(1 - \eta_x^2)\tau_{xz} + \eta_x(\tau_{zz} - \tau_{xx}) - \eta_y(\tau_{xy} + \eta_x\tau_{yz})] \\ &= \frac{(1 - \gamma)\sqrt{2}c_t}{(1 + c_u)(1 + 2c_u)} u \sqrt{u^2 + v^2} \quad \text{on } z = \eta + b.\end{aligned} \quad (20)$$

$$\begin{aligned}& [(1 - \eta_y^2)\tau_{yz} + \eta_y(\tau_{zz} - \tau_{yy}) - \eta_x(\tau_{xy} + \eta_y\tau_{xz})] \\ &= \frac{(1 - \gamma)\sqrt{2}c_t}{(1 + c_u)(1 + 2c_u)} v \sqrt{u^2 + v^2} \quad \text{on } z = \eta + b.\end{aligned} \quad (21)$$

When we evaluate at  $\gamma = 1$  this artificial right-hand side becomes zero so the artificial surface conditions (20) and (21) reduces to the physical surface condition (15) and (16). However, when both the parameter  $\gamma = 0$  and the lateral derivatives are negligible,  $\partial_x = \partial_y = 0$ , then the lateral shear  $u, v \propto c_u + \zeta$  becomes a neutral mode of the dynamics. The Euler parameter

of a toy problem suggests introducing a factor  $(1 - \frac{1}{6}\gamma)$  into the left-hand side of the tangential stress boundary conditions (20) and (21) in order to improve convergence in the parameter  $\gamma$  when evaluated at the physically relevant  $\gamma = 1$ . This needs further exploration. For the moment omit such a factor.

Compute the residuals of the lateral momentum equation, an artificial tangential stress on the free surface, and the bed boundary. See that when  $\gamma = 0$  the free surface condition is effectively  $\eta \partial_z \mathbf{u} = \mathbf{u}/(1 + \mathbf{c}_u)$ , leading to our neutral mode  $\mathbf{u} \propto \mathbf{c}_u + \zeta$ , namely  $\mathbf{u} = \bar{\mathbf{u}}(\mathbf{x}, t)(\mathbf{c}_u + \zeta)/(\mathbf{c}_u + \frac{1}{2})$ ; whereas when  $\gamma = 1$  the free surface condition reduces to zero tangential stress. In the computer algebra, we expand the square root terms in conditions (18) and (19) in Taylor series.

```

128 resu:=df(u,t)+u*df(u,x)+v*df(u,y)+w*df(u,z)+df(p,x)
129      -grx-df(txx,x)-df(txy,y)-df(txz,z);
130 resv:=df(v,t)+u*df(v,x)+v*df(v,y)+w*df(v,z)+df(p,y)
131      -gry-df(tyy,y)-df(txy,x)-df(tyz,z);
132 resbu:=sub(zz=0,(-u-w*df(b,x))*(1-df(b,x)^2/2)
133      +cu*eta*(1-df(b,x)^2/2-df(b,y)^2/2)*(-
134      -df(u+w*df(b,x),x)*df(b,x)
135      -df(u+w*df(b,x),y)*df(b,y)
136      +df(u+w*df(b,x),z)));
137 resbv:=sub(zz=0,(-v-w*df(b,y))*(1-df(b,y)^2/2)
138      +cu*eta*(1-df(b,x)^2/2-df(b,y)^2/2)*(-
139      -df(v+w*df(b,y),x)*df(b,x)
140      -df(v+w*df(b,y),y)*df(b,y)
141      +df(v+w*df(b,y),z)));
142 write length_resu:=mylength(resu);
143 write length_resv:=mylength(resv);
144 write length_resbu:=mylength(resbu);
145 write length_resbv:=mylength(resbv);
146 ok:=if {resu,resv,resbu,resbv}={0,0,0,0} then ok else 0;

```



### 9.3 Solve for updates to lateral velocity and rate-of-strain

Update the lateral fields using an as yet unknown change in the evolution for the lateral mean velocities. The lateral fields are coupled by the nonlinear stress-strain relation of the Smagorinski turbulent flow.

```
147 u:=u+(du:=eta*(1+2*cu)*r2/(4*ct)*rqq^3*usolv(
148     +(qq^2+vv(0,0)^2)*resu
149     -uu(0,0)*vv(0,0)*resv ,zz));
150 v:=v+(dv:=eta*(1+2*cu)*r2/(4*ct)*rqq^3*usolv(
151     +(qq^2+uu(0,0)^2)*resv
152     -uu(0,0)*vv(0,0)*resu ,zz));
153 ros:=ros+(uu(0,0)*df(du,zz)+vv(0,0)*df(dv,zz))/(r2*eta)*rqq;
```

Now use the tangential stress on the free surface to determine the evolution corrections gud and gvd. But first need to update part of the stress from changes in lateral velocity and rate-of-strain: these updates to stress and strain are indeed needed to ensure that the residual of the lateral momentum equations are satisfied.

```
154 exz:=(df(u,z)+df(w,x))/2;
155 eyz:=(df(v,z)+df(w,y))/2;
156 txz:=2*ct*eta^2*ros*exz;
157 tyz:=2*ct*eta^2*ros*eyz;
```

Then compute residuals of tangential stress equations.

```
158 resttu:=(-sub(zz=1,
159     (1-0*gamm)*((1-(etax+df(b,x))^2)*txz
160     +(etax+df(b,x))*(tzz-txx)
161     -(etay+df(b,y))*(txy+(etax+df(b,x))*tyz))
162     -(1-gamm)*r2*ct/(cu+1)/(2*cu+1)*u*qq ));
163 write length_rettu:=mylength(resttu);
164 resttv:=(-sub(zz=1,
165     (1-0*gamm)*((1-(etay+df(b,y))^2)*tyz
166     +(etay+df(b,y))*(tzz-tyy)
167     -(etax+df(b,x))*(txy+(etay+df(b,y))*txz))
```

```

168      -(1-gamm)*r2*ct/(cu+1)/(2*cu+1)*v*qq ) );
169 write length_resttv:=mylength(resttv);
170 ok:=if {resttu,resttv}={0,0} then ok else 0;

```

Update the lateral evolution based upon these residuals.

```

171 gu:=gu-3*(1+2*cu)*(1+cu)
172      /2/eta/(3+11*cu+12*cu^2)/(1+3*cu+3*cu^2)/(3+4*cu)
173      *(((1+5*cu+8*cu^2)*uu(0,0)^2*rqq^2
174      -(9+45*cu+80*cu^2+48*cu^3))*resttu
175      +(1+5*cu+8*cu^2)*uu(0,0)*vv(0,0)*rqq^2*resttv);
176 gv:=gv-3*(1+2*cu)*(1+cu)
177      /2/eta/(3+11*cu+12*cu^2)/(1+3*cu+3*cu^2)/(3+4*cu)
178      *(((1+5*cu+8*cu^2)*vv(0,0)^2*rqq^2
179      -(9+45*cu+80*cu^2+48*cu^3))*resttv
180      +(1+5*cu+8*cu^2)*uu(0,0)*vv(0,0)*rqq^2*resttu);

```

## 10 Postprocessing

End the iterative loop.

```

181 showtime;
182 if ok then write iter:=100000+iter;
183 end;

```

We may use these transformations to check on the dimensionality of various expressions. But not at the moment.

```

184 dims:={ h(~m)=>nh*ll
185         , uu(~m)=>mu*ll/tt
186         , gx=>ngx*ll/tt^2 }$

```

Write out the final evolution on the slow manifold.

```

187 r2:=sqrt(2)$ %eps:=1$
188 on rounded; print_precision 4;
189 write dhdt:=length(gh);
190 write dudt:=length(gu);
191 write dvdt:=length(gv);

```

Finish.

192 end;

## A Trace prints from sample execution

193 d := eps

194

195 grz := gz

196

197 grx := gx\*eps

198

199 gry := 0

200

201 gamm := gam\*eps

202

203 ITERATION 1

204

205 length\_resc := 9

206

207 length\_resa := 3

208

209 length\_resw := 74

210

211 length\_restn := 3

212

213 length\_rese := 51

214

215 length\_resu := 93

216

217 length\_resv := 88

218

219 length\_resbu := 0

220

221 length\_resbv := 0

222

223 length\_rettu := 153

```
224
225 length_resttv := 134
226
227 Time: 250 ms
228
229 ITERATION 2
230
231 length_resc := 377
232
233 length_resa := 0
234
235 length_resw := 337
236
237 length_restn := 94
238
239 length_rese := 249
240
241 length_resu := 780
242
243 length_resv := 818
244
245 length_resbu := 0
246
247 length_resbv := 0
248
249 length_resttu := 143
250
251 length_resttv := 152
252
253 Time: 5039 ms plus GC time: 150 ms
254
255 ITERATION 3
256
257 length_resc := 380
258
259 length_resa := 0
260
```

```
261 length_resw := 317
262
263 length_restn := 99
264
265 length_rese := 268
266
267 length_resu := 682
268
269 length_resv := 728
270
271 length_resbu := 0
272
273 length_resbv := 0
274
275 length_rettu := 143
276
277 length_rettv := 152
278
279 Time: 11660 ms   plus GC time: 410 ms
280
281 ITERATION 4
282
283 length_resc := 0
284
285 length_resa := 0
286
287 length_resw := 125
288
289 length_restn := 43
290
291 length_rese := 0
292
293 length_resu := 285
294
295 length_resv := 303
296
297 length_resbu := 0
```

```
298
299 length_resbv := 0
300
301 length_restvu := 0
302
303 length_rettv := 0
304
305 Time: 14591 ms   plus GC time: 550 ms
306
307 ITERATION 5
308
309 length_resc := 0
310
311 length_resa := 0
312
313 length_resw := 0
314
315 length_retn := 0
316
317 length_rese := 0
318
319 length_resu := 0
320
321 length_resv := 0
322
323 length_resbu := 0
324
325 length_resbv := 0
326
327 length_restvu := 0
328
329 length_rettv := 0
330
331 Time: 14489 ms   plus GC time: 550 ms
332
333 iter := 100005
334
```

335 dhdt := 4  
 336  
 337 dudt := 175  
 338  
 339 dvdt := 175

## References

- Bijvelds, M. D. J. P., Kranenburg, C. & Stelling, G. S. (1999), ‘3d numerical simulation of turbulent shallow-water in square harbor’, *Journal of Hydraulic Engineering* **125**(1), 26–31. [http://dx.doi.org/10.1061/\(ASCE\)0733-9429\(1999\)125:1\(26\)](http://dx.doi.org/10.1061/(ASCE)0733-9429(1999)125:1(26)).
- Janosi, I. M., Jan, D., Szabo, K. G. & Tel, T. (2004), ‘Turbulent drag reduction in dam-break flows’, *Experiments in Fluids* **37**, 219–229. <http://dx.doi.org/10.1007/s00348-004-0804-4>.
- Kim, T. K. (2002), A Modified Smagorinsky Subgrid Scale Model for the Large Eddy Simulation of Turbulent Flow, PhD thesis, Univeristy of California, Davis. [<http://arXiv.org/abs/math.sc/0209377>].
- Marstop, L. (2006), Subgrid-scale modelling for large-eddy simulation including scalar mixing in rotating turbulent shear flows, PhD thesis, Royal Institute of Technology KTH Mechanics SE-100 44 Stockholm, Sweden.
- Nezu, I. (2005), ‘Open-channel flow turbulence and its research prospect in the 21st century’, *J. Hydraulic Engineering* p. 229. [http://dx.doi.org/10.1061/\(ASCE\)0733-9429\(2005\)131:4\(229\)](http://dx.doi.org/10.1061/(ASCE)0733-9429(2005)131:4(229)).
- Özögökmen et al.
- Özögökmen, T. M., Iliescu, T., Fischer, P. F., Srinivasan, A. & Duan, J. (2007), ‘Large eddy simulation of stratified mixing and two-dimensional dam-break problem in a rectangular enclosed domain’, *Ocean Modelling* **16**, 106–140. <http://dx.doi.org/10.1016/j.ocemod.2006.08.006>.
- Roberts, A. J. (2007a), Computer algebra models the inertial dynamics of a thin film flow of power law fluids and other generalised newtonian fluids, Technical report, University of Southern Queensland. <http://eprints.>

[usq.edu.au/archive/00002010/](http://usq.edu.au/archive/00002010/).

**URL:** <http://eprints.usq.edu.au/archive/00002010/>

Roberts, A. J. (2007*b*), ‘The inertial dynamics of thin film flow of non-newtonian fluids’, *Physics Letters A* **372**, 1607–1611. <http://dx.doi.org/10.1016/j.physleta.2007.10.014>.

Roberts, A. J., Georgiev, D. J. & Strunin, D. V. (2008), Model turbulent floods with the Smagorinski large eddy closure, Technical report, <http://arxiv.org/abs/0805.3192>.