

# Computer algebra describes three dimensional turbulence flows over curved beds via the Smagorinski large eddy closure

A. J. Roberts\*      Meng Cao†

June 29, 2012

## Abstract

Consider the turbulent floods over curved beds. The Smagorinski closure for turbulence, with its linear dependence of eddy viscosity upon the shear-rate, models turbulent dissipation. A slow manifold model of the dynamics of the fluid layer allows for large changes in layer thickness provided the changes occur over a large enough lateral length scale. The slow manifold is based on two macroscopic modes by modifying the spectrum: here artificially modify the boundary conditions on the free surface so that, as well as a mode representing conservation of fluid, a lateral shear flow with slip is a neutral critical mode. Then remove the modification to recover a model for turbulent floods.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Uniform acceleration . . . . .	3
1.2	Overview . . . . .	5

---

\*School of Mathematical Sciences, University of Adelaide, South Australia 5005.  
<mailto:anthony.roberts@adelaide.edu.au>

†School of Mathematical Sciences, University of Adelaide, South Australia 5005.  
<mailto:meng.cao@adelaide.edu.au>

<b>2</b>	<b>Preamble</b>	<b>5</b>
2.1	Define order parameters . . . . .	5
2.2	Stretch the coordinates with the free surface . . . . .	6
2.3	Operators used elsewhere . . . . .	7
<b>3</b>	<b>Initialise with linear</b>	<b>9</b>
<b>4</b>	<b>Truncate the asymptotic expansion</b>	<b>10</b>
<b>5</b>	<b>Invoke the iterative loop</b>	<b>11</b>
<b>6</b>	<b>Update <math>w</math> with continuity and no flow penetrating the bed</b>	<b>11</b>
<b>7</b>	<b>Update the free surface evolution</b>	<b>12</b>
<b>8</b>	<b>Update pressure from vertical momentum and surface normal stress</b>	<b>12</b>
<b>9</b>	<b>Update stress, <math>u</math> and <math>v</math> from lateral momentum and surface tangential stress</b>	<b>13</b>
9.1	Smagorinski large eddy stress-shear closure . . . . .	13
9.2	Compute residuals of lateral momentum . . . . .	14
9.3	Solve for updates to lateral velocity and rate-of-strain . . . . .	17
<b>10</b>	<b>Postprocessing</b>	<b>18</b>
<b>A</b>	<b>Trace prints from sample execution</b>	<b>19</b>

# 1 Introduction

This approach to modelling turbulent floods develops from that of modelling non-Newtonian fluids which have a nonlinear dependence upon strain rate (Roberts 2007*a,b*). Bijvelds et al. (1999) required enhanced lateral mixing which is naturally predicted by this slow manifold approach. Roberts et al. (2008) reports preliminary results using the model derived with the computer algebra program documented herein.

Consider the three dimensional fluid flow over a curved bed, say  $\mathbf{b}(\mathbf{x}, \mathbf{y})$ . Let coordinates  $\mathbf{x}$  and  $\mathbf{y}$  measure the horizontal distances and coordinate  $z$

the vertical distance. Let the incompressible, irrotational turbulent fluid have thickness  $\eta(\mathbf{x}, \mathbf{y}, t)$ , constant density  $\rho$ , and the nonlinear constitutive relation of the Smagorinski closure (Kim 2002, Marstop 2006, Özgökmen, Iliescu, Fischer, Srinivasan & Duan 2007, e.g.). The fluid flows with some varying velocity field  $\mathbf{q}(\mathbf{x}, \mathbf{y}, z, t) = (\mathbf{u}, \mathbf{v}, w)$  and pressure field  $p(\mathbf{x}, \mathbf{y}, z, t)$ ; these fields are the turbulent mean fields, that is, the fields averaged over realisations.

## 1.1 Uniform acceleration

First, for the three dimensional fluid flowing a flat bed, do not allow any lateral variations,  $\partial_x = \partial_y = 0$ . Then a low order slow manifold, errors  $\mathcal{O}(\gamma^2 + g_x^2)$ , is that in terms of the scaled vertical coordinate  $\zeta = z/\eta$  the fluid fields are

$$w = 0, \quad (\text{shear flow}) \quad (1)$$

$$p = g_z(1 - \zeta)\eta, \quad (\text{hydrostatic}) \quad (2)$$

$$\begin{aligned} \mathbf{u} = & \bar{\mathbf{u}} \frac{2(\zeta + c_u)}{1 + 2c_u} \\ & + \gamma \bar{\mathbf{u}} \frac{(1 + c_u)[(1 + 4c_u)(c_u + \zeta) - 2(1 + 2c_u)(3c_u\zeta^2 + \zeta^3)]}{4(1 + 2c_u)^2(1 + 3c_u + 3c_u^2)} \\ & + \frac{g_x \eta}{\bar{\mathbf{u}}} \frac{[(5 + 6c_u)(c_u + \zeta) - 6(2 + 7c_u + 6c_u^2)\zeta^2 + 6(1 + 2c_u)^2\zeta^3]}{48\sqrt{2}c_t(1 + 3c_u + 3c_u^2)}, \\ \dot{\epsilon} = & \frac{\bar{\mathbf{u}}}{\eta} \frac{\sqrt{2}}{1 + 2c_u} + \frac{\gamma \bar{\mathbf{u}}}{\eta} \frac{\sqrt{2}(1 + c_u)[(1 + 4c_u) - 6(1 + 2c_u)(2c_u\zeta + \zeta^2)]}{8(1 + 2c_u)^2(1 + 3c_u + 3c_u^2)} \\ & + \frac{g_x}{\bar{\mathbf{u}}} \frac{[(5 + 6c_u) - 12(2 + 7c_u)\zeta + 18(1 + 2c_u)^2\zeta^2]}{96c_t(1 + 3c_u + 3c_u^2)}. \end{aligned} \quad (3)$$

The parameter  $c_u\eta$  is a ‘slip’ length on the bed, see boundary condition (??), and  $c_t$  parametrises the strength of Smagorinski’s turbulent mixing, see the eddy viscosity (14); they are determined from observations. Figure 1 displays a sample of the vertical profile of the rate of strain  $\dot{\epsilon}$  and of the lateral velocity  $\mathbf{u}$ . This equilibrium flow resolves the shear in the lateral velocity and the increase in rate of strain near the bed. Our analysis does not attempt to resolve the turbulent log layer: we assume the details of dynamical interest are those determined by the relatively large scale of the fluid depth.

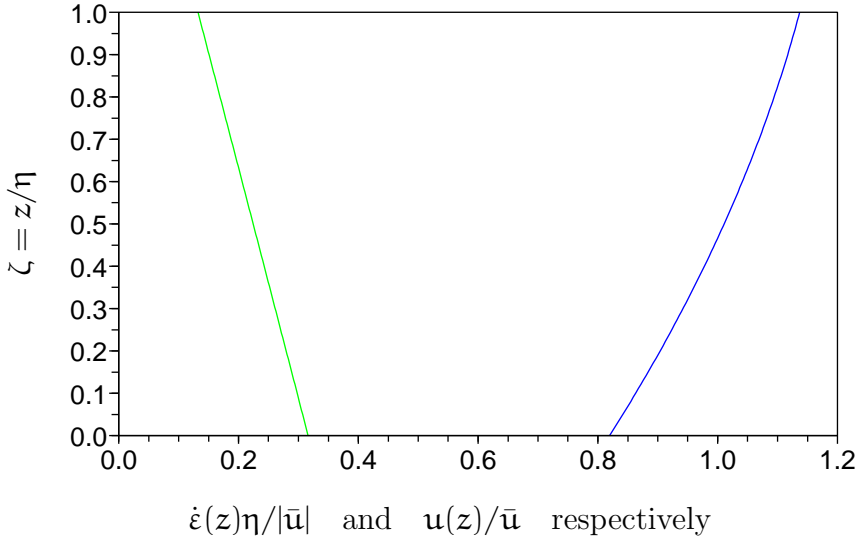


Figure 1: approximate vertical profiles at open channel flow equilibrium for which  $g_x \eta / \bar{u}^2 = 0.0031$ .

The set of such profiles in the vertical, and their nonlinear interactions, form a slow manifold. The evolution on this slow manifold of the mean lateral velocities  $\bar{u}$  and  $\bar{v}$  is dictated by turbulent bed drag limiting gravitational forcing:

$$\frac{d\bar{u}}{dt} = -\frac{\sqrt{2} 3c_t(1+c_u)}{(1+2c_u)(1+3c_u+3c_u^2)} \frac{\gamma \bar{u}^2}{\eta} + \frac{\frac{3}{4} + 3c_u + 3c_u^2}{1+3c_u+3c_u^2} g_x + \mathcal{O}(\gamma^2 + g_x^2 + \partial_x) \quad (4)$$

$$\frac{d\bar{v}}{dt} = -\frac{\sqrt{2} 3c_t(1+c_u)}{(1+2c_u)(1+3c_u+3c_u^2)} \frac{\gamma \bar{v}^2}{\eta} + \mathcal{O}(\gamma^2 + \partial_x) \quad (5)$$

Upon putting the artificial parameter  $\gamma = 1$  to recover the physical model, this evolution predicts an equilibrium channel flow at a mean velocity of

$$\bar{u} = \frac{1}{2} \left[ \frac{(1+2c_u)^3}{\sqrt{2}c_t(1+c_u)} \right]^{1/2} \sqrt{g_x \eta}. \quad (6)$$

For example, choosing  $c_t = 0.020$  and  $c_u = 1.848 \approx 13/7 \approx 11/6$  gives about the correct channel flow *and* gives about the correct eddy viscosity when compared with observations of open channel flow (Nezu 2005, e.g.).

## 1.2 Overview

Denote free surface thickness  $\eta(x, y, t)$  by  $h$ , curved bed  $b(x, y)$ , mean lateral velocities  $\bar{u}(x, y, t)$  and  $\bar{v}(x, y, t)$  by  $uu$  and  $vv$ , and their evolution  $\eta_t = gh$ ,  $\bar{u}_t = gu$  and  $\bar{v}_t = gv$ . The Reynolds number  $re$ , and the coefficients of lateral and normal gravitational forcing are Gravity numbers  $grx$  and  $grz$ . Construct an asymptotic solution of the Smagorinski equations in terms of  $\eta$ ,  $b$ ,  $\bar{u}$  and  $\bar{v}$  to some order of nonlinearity in  $\bar{u}$  and  $\bar{v}$  and some order of lateral derivatives  $\partial_x$  and  $\partial_y$ .

Decide upon how the asymptotic expansions of the solution are to be truncated. Then iteratively update the velocity and pressure fields to solve the Smagorinski equations and boundary conditions. The iteration continues until the governing equations are satisfied; that is, their residuals are zero to the order of truncation.

## 2 Preamble

Improve printing by factoring with respect to these variables. It is a matter of taste and may be different depending upon what one wishes to investigate in the algebraic expressions.

```
1 on div; off allfac; on revpri;  
2 factor vv,uu,qq,rqq,h,ct,gx,gz,gam,r2,b;
```

### 2.1 Define order parameters

Use the operator  $h(m,n)$  to denote lateral derivatives of the fluid thickness,  $\partial_x^m \partial_y^n \eta$ ,  $b(m,n)$  present derivatives of the curved bed, and similarly  $uu(m,n)$  and  $vv(m,n)$  denotes lateral derivatives of the mean shear,  $\partial_x^m \partial_y^n \bar{u}$  and  $\partial_x^m \partial_y^n \bar{v}$ . Meanwhile define readable abbreviations for  $\eta$  and its first spatial derivatives. Consider the bed shape  $b(x,y)$  small and its first derivatives  $b_x$  and  $b_y$  smaller. Adopt  $d$  to count the number of lateral derivatives, so we can easily truncate the asymptotic expansion.

```
3 operator h; operator b; operator uu; operator vv;  
4 eta:=h(0,0); etax:=h(1,0)*d$ etay:=h(0,1)*d$
```

These operators must depend upon time and lateral space. Then lateral derivatives transform as  $\partial_x h(m,n) = h(m+1,n)$ , for example. Also, a time derivative transforms into lateral derivatives of the corresponding evolution: for example,  $\partial_t h(m,n) = \partial_x^m \partial_y^n g h$ .

```

5 depend h,xx,yy,tt;
6 depend uu,xx,yy,tt;
7 depend vv,xx,yy,tt;
8 depend b,xx,yy;
9 let{ df(h(~m,~n),xx)=>h(m+1,n)
10      , df(h(~m,~n),yy)=>h(m,n+1)
11      , df(h(~m,~n),tt)=>df(gh,xx,m,yy,n)
12      , df(uu(~m,~n),xx)=>uu(m+1,n)
13      , df(uu(~m,~n),yy)=>uu(m,n+1)
14      , df(uu(~m,~n),tt)=>df(gu,xx,m,yy,n)
15      , df(vv(~m,~n),xx)=>vv(m+1,n)
16      , df(vv(~m,~n),yy)=>vv(m,n+1)
17      , df(vv(~m,~n),tt)=>df(gv,xx,m,yy,n)
18      , df(b,xx)=>b(1,0)
19      , df(b,yy)=>b(0,1)
20      , df(b(~m,~n),xx)=>b(m+1,n)
21      , df(b(~m,~n),yy)=>b(m,n+1)
22 };
```

## 2.2 Stretch the coordinates with the free surface

Use stretched coordinates  $zz$ ,  $xx$ ,  $yy$  and  $tt$  to denote  $Z = (z-b)/\eta$ ,  $X = x$ ,  $Y = y$  and  $T = t$ . The free surface is then simply  $Z = 1$ .

```

23 depend xx,x,y,z,t;
24 depend yy,x,y,z,t;
25 depend zz,x,y,z,t;
26 depend tt,x,y,z,t;
```

Then space-time derivatives transform according to

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X} - Z \frac{\eta_x}{\eta} \frac{\partial}{\partial Z} - \frac{b_x}{\eta} \frac{\partial}{\partial Z},$$

$$\begin{aligned}
\frac{\partial}{\partial y} &= \frac{\partial}{\partial Y} - Z \frac{\eta_Y}{\eta} \frac{\partial}{\partial Z} - \frac{b_Y}{\eta} \frac{\partial}{\partial Z}, \\
\frac{\partial}{\partial t} &= \frac{\partial}{\partial T} - Z \frac{\eta_T}{\eta} \frac{\partial}{\partial Z}, \\
\frac{\partial}{\partial z} &= \frac{1}{\eta} \frac{\partial}{\partial Z}.
\end{aligned} \tag{7}$$

We neatly insert an automatic count of lateral derivatives here, with  $\mathbf{d}$ , in between  $\partial_x$  and  $\partial_x$ , and in between  $\partial_y$  and  $\partial_y$ .

```

27 let{ df(~a,x)=>df(a,xx)*d-zz*etax/eta*df(a,zz)
28      -d*df(b,xx)/eta*df(a,zz)
29      , df(~a,y)=>df(a,yy)*d-zz*etay/eta*df(a,zz)
30      -d*df(b,yy)/eta*df(a,zz)
31      , df(~a,t)=>df(a,tt)-zz*gh/eta*df(a,zz)
32      , df(~a,z)=>df(a,zz)/eta
33      };

```

## 2.3 Operators used elsewhere

For integrating in the vertical, use the linear operator `wsolv` as it is quicker than native integration. This is the definite integral that is zero at the bed  $\zeta = 0$ .

```

34 operator wsolv; linear wsolv;
35 let {wsolv(zz^~~~n,zz) => zz^(n+1)/(n+1)
36      ,wsolv(1,zz) => zz };

```

Similarly, it is quicker to use operators than to use the native integration to find the pressure. This is the vertical integral (negative) that is zero at the surface  $\zeta = 1$ .

```

37 operator psolv; linear psolv;
38 let {psolv(zz^~~~n,zz) => (1-zz^(n+1))/(n+1)
39      ,psolv(1,zz) => (1-zz) };

```

Variable `qq` denotes the mean speed  $\bar{q} = \sqrt{U^2 + V^2}$ . Let `rqq` denotes its reciprocal. The following transformation rules should be correct. Either of the last two can be chosen.

```

40 depend qq,uu(0,0),vv(0,0);
41 let { qq^2=>uu(0,0)^2+vv(0,0)^2
42      , df(qq,~aa)=>(uu(0,0)*df(uu(0,0),aa)+vv(0,0)*df(vv(0,0),a
43      } ;
44 depend rqq,qq;
45 let { df(rqq,~aa)=>-rqq^2*df(qq,aa)
46      , rqq*qq=>1
47      , qq^2=>(uu(0,0)^2+vv(0,0)^2)
48      , vv(0,0)^2*rqq=>qq-uu(0,0)^2*rqq
49 %    , uu(0,0)^2*rqq=>qq-vv(0,0)^2*rqq
50      } ;

```

The linear operator `usolv` solves  $\partial_z^2 u' = \text{RHS}$  such that the bed boundary condition (??) is always satisfied and that the mean of the solution  $u'$  is always zero to ensure  $\bar{u}$  remains the mean later velocity.

```

51 operator usolv; linear usolv;
52 let { usolv(zz^~~~n,zz) => (zz^(n+2)
53      -(cu+zz)/(n+3)/(cu+1/2) )/(n+2)/(n+1)
54      , usolv(1,zz) => (zz^2 -(cu+zz)/3/(cu+1/2) )/2 } ;

```

Do not need it, but the linear operator `mean` quickly computes the average of some field over the fluid thickness.

```

55 operator mean; linear mean;
56 let { mean(zz^~~~n,zz) => 1/(n+1)
57      , mean(1,zz) => 1 } ;

```

To see how the iteration is proceeding, write out the number of terms in its residual throughout iteration for each equation. Could also write out time since last length written.

```

58 procedure mylength(res);
59 begin
60 %showtime;
61 %return res;
62 return if res=0 then 0 else length(res);
63 end;

```



### 3 Initialise with linear

Start the iteration from the linear solution that the lateral velocity  $\mathbf{u} = \bar{\mathbf{u}}(\mathbf{c}_u + \zeta)/(\mathbf{c}_u + \frac{1}{2})$ ,  $\mathbf{v} = \bar{\mathbf{v}}(\mathbf{c}_u + \zeta)/(\mathbf{c}_u + \frac{1}{2})$ ,  $\mathbf{p} = \mathbf{g}_z(1 - \zeta)\eta$  and  $\mathbf{w} = 0$ . The parameter  $\mathbf{c}_u$  determines the turbulent slip on the bed and is to be determined to best fit experiment and/or observations. The evolution of the ‘order parameters’ is also zero:  $\bar{\mathbf{u}}_t = \mathbf{g}\mathbf{u} = 0$ ,  $\bar{\mathbf{v}}_t = \mathbf{g}\mathbf{v} = 0$  and  $\eta_t = \mathbf{g}\mathbf{h} = 0$ .

```
64 let r2^2=>2; % r2=sqrt2
65 u:=uu(0,0)*(cu+zz)/(cu+1/2);
66 v:=vv(0,0)*(cu+zz)/(cu+1/2);
67 p:=grz*(1-zz)*eta;
68 w:=gh:=gu:=gv:=0;
```

Also set initial strains from the linear solution.

```
69 exx:=df(u,x);
70 eyy:=df(v,y);
71 ezz:=df(w,z);
72 exz:=(df(u,z)+df(w,x))/2;
73 exy:=(df(u,y)+df(v,x))/2;
74 eyz:=(df(v,z)+df(w,y))/2;
```

Initially approximate the magnitude  $\dot{\epsilon}$  of the strain-rate tensor: the above iteration step assumes the strain rate is  $(\bar{\mathbf{u}}, \bar{\mathbf{v}})\sqrt{2}/\eta/(1 + 2\mathbf{c}_u)$  to leading approximation.

```
75 ros:=qq*r2/eta/(1+2*cu);
```

In the Smagorinski model (Özgökmen, Iliescu, Fischer, Srinivasan & Duan 2007, e.g.)  $\mathbf{c}_t = (\mathbf{c}_s\Delta/\eta)^2$  where arguments indicate  $\mathbf{c}_s \approx 0.2$ . To match observations (Nezu 2005, e.g.) of open channel flow equilibria we set  $\mathbf{c}_t = 0.020$  from which the appropriate filter scale  $\Delta \approx 0.7\eta$  consistent with significant mixing across the fluid layer as seen in Figures 14–15 by Janosi et al. (2004).

```
76 ct:=1/50;
```

Set initial values for the stress.

```

77 txx:=2*ct*eta^2*ros*exx;
78 tyy:=2*ct*eta^2*ros*eyy;
79 tzz:=2*ct*eta^2*ros*ezz;
80 txz:=2*ct*eta^2*ros*exz;
81 txy:=2*ct*eta^2*ros*exy;
82 tyz:=2*ct*eta^2*ros*eyz;

```

The bed slip-drag law requires coefficient.

```

83 cu:=11/6;

```

## 4 Truncate the asymptotic expansion

There are lots of ways to truncate the asymptotic model. The small parameters available are:

- **d** counting the number of **x** derivatives of the slowly varying lateral spatial structure in any term;
- the homotopy parameter **gam** varying between  $\gamma = 0$  for the artificial base problem and  $\gamma = 1$  for the physical fluid equations; and
- **grx**, **gry** and **grz** being the lateral and normal components of gravity.

Because of the velocity of the flow not small, consider the parameter  $\bar{u}$  and  $\bar{v}$  finite. Usually we will make lateral gravity fairly small by scaling with the magnitude of  $\partial_x$  and  $\partial_y$ .

Initially omit all **x** derivatives by setting  $\mathbf{d} = 0$ , later we scale  $\mathbf{d}$  with **eps** to get the relatively simple but interesting model with errors  $\mathcal{O}(\gamma^{3/2} + g_x^{3/2} + g_z^3 + \partial_x^3)$ .<sup>1</sup> We need not make normal gravity  $g_z$  small as here, but doing so removes some messy terms.

```

84 d:=eps;
85 grz:=0*gz;
86 grx:=0*eps*gx;
87 gry:=0;
88 gamm:=0*eps*gam;
89 factor eps;

```

---

<sup>1</sup>We need to check the algorithm at  $\mathcal{O}(g_x^3)$ .

For now truncate to relatively low order,  $\mathcal{O}(\gamma^{3/2} + g_x^{3/2} + g_z^3 + \partial_x^3)$ , in spatial derivatives and boundary condition artifice (can do  $\mathcal{O}(\epsilon^4)$  if no spatial variations):

```
90 let { eps^3=>0 };
```

## 5 Invoke the iterative loop

```
91 for iter:=1:6 do begin ok:=1;
92 write "ITERATION ",iter;
```

## 6 Update $w$ with continuity and no flow penetrating the bed

The nondimensional continuity equation for the incompressible fluid flow is

$$\vec{\nabla} \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (8)$$

Let the vector  $\vec{n} = \frac{1}{\sqrt{1+b_x^2+b_y^2}}(-b_x, -b_y, 1)$  present the unit vector normal to the curved bed. No flow penetrating the curved bed indicates  $\vec{q} \cdot \vec{n} = 0$ , that is,

$$w = ub_x + vb_y \text{ on } z = b. \quad (9)$$

As with all field variables in this model, the quantities  $u$ ,  $v$ ,  $w$  and  $p$  are averaged over the ensemble of turbulent flows. Compute the residual of the continuity equation and the boundary condition (9), then update the vertical velocity  $w$  by integrating from the bed,  $\zeta = 0$ . The variable `ok` stores whether all residuals are so far zero in this iteration.

```
93 resc:=df(u,x)+df(v,y)+df(w,z);
94 resa:=sub(zz=0,w-u*df(b,x)-v*df(b,y));
95 write length_resc:=mylength(resc);
96 write length_resa:=mylength(resa);
97 ok:=if {resc,resa}={0,0} then ok else 0;
98 w:=w+(dw:=-eta*wsolv(resc,zz))-resa;
```

The  $\zeta$  component of the stress and rate-of-strain tensor should be, and might need to be, updated from this correction to the normal velocity; the magnitude of the rate-of-strain tensor is unaffected (to leading order).

```

99 ezz:=ezz+df(dw,zz)/eta;
100 tzz:=tzz+2*r2*ct/(1+2*cu)*qq*df(dw,zz);

```

## 7 Update the free surface evolution

The kinematic condition at the free surface,

$$\frac{\partial \eta}{\partial t} + u \frac{\partial (\eta + b)}{\partial x} + v \frac{\partial (\eta + b)}{\partial y} = w \quad \text{on} \quad z = \eta + b, \quad (10)$$

gives the evolution of the fluid thickness  $h$ . It dominantly arises from the normal velocity.

```

101 gh:=sub(zz=1, w-u*etax-v*etay-u*df(b,x)-v*df(b,y));

```

## 8 Update pressure from vertical momentum and surface normal stress

The nondimensional momentum equation is

$$\frac{\partial \vec{q}}{\partial t} + \vec{q} \cdot \vec{\nabla} \vec{q} = -\vec{\nabla} p + \vec{\nabla} \cdot \vec{\tau} + \vec{g}, \quad (11)$$

where  $\vec{\tau}$  is the nondimensional deviatoric eddy stress tensor, and  $\vec{g}$  is the direction of gravity; when the substrate slopes,  $\vec{g}$  is not normal to the substrate. The vertical momentum equation is solved with the surface condition that the turbulent mean, normal stress to the free surface is zero, that is,

$$-p + \frac{\tau_{zz} - 2\eta_x \tau_{xz} - 2\eta_y \tau_{yz} + \eta_x^2 \tau_{xx} + 2\eta_x \eta_y \tau_{xy} + \eta_y^2 \tau_{yy}}{1 + \eta_x^2 + \eta_y^2} = 0 \quad \text{on} \quad z = \eta + b. \quad (12)$$

Compute the residuals of the vertical momentum equation and the zero normal stress on the free surface. The recent change to the normal velocity affects the pressure update.

```

102 resw:=df(w,t)+u*df(w,x)+v*df(w,y)+w*df(w,z)+df(p,z)
103     +grz-df(txz,x)-df(txy,y)-df(tzz,z);
104 restn:=sub(zz=1,-p*(1+(etax+df(b,x))^2+(etay+df(b,y))^2)
105     +tzz-2*(etax+df(b,x))*txz-2*(etay+df(b,y))*tyz
106     +(etax+df(b,y))^2*txx+2*(etax+df(b,x))
107     *(etay+df(b,y))*txy+(etay+df(b,y))^2*tyy);
108 write length_resw:=mylength(resw);
109 write length_restn:=mylength(restn);
110 ok:=if {resw,restn}={0,0} then ok else 0;

```

Update the pressure field  $p$  by integrating down from the free surface  $\zeta = 1$ ; we use the linear operator `psolv` to solve  $\partial_\zeta p' = -\text{RHS}$  such that  $p' = 0$  at  $\zeta = 1$ .

```

111 p:=p+eta*psolv(resw,zz)+restn;

```

## 9 Update stress, $u$ and $v$ from lateral momentum and surface tangential stress

### 9.1 Smagorinski large eddy stress-shear closure

Now the strain-rate tensor

$$\dot{\epsilon}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

```

112 exx:=df(u,x);
113 eyy:=df(v,y);
114 ezz:=df(w,z);
115 exz:=(df(u,z)+df(w,x))/2;
116 exy:=(df(u,y)+df(v,x))/2;
117 eyz:=(df(v,z)+df(w,y))/2;

```

Then the stress tensor for the fluid is  $\sigma_{ij} = -p\delta_{ij} + 2\rho\nu\dot{\epsilon}_{ij}$ : when the viscosity  $\nu$  is constant this models a Newtonian fluid; but here the Smagorinski closure for turbulent flow is that this eddy viscosity varies

linearly with strain-rate magnitude (analogous to a shear thickening non-Newtonian fluid). Define the magnitude  $\text{ros} = |\dot{\epsilon}|$ , the second invariant of the strain-rate tensor, where

$$|\dot{\epsilon}|^2 = \sum_{i,j} \dot{\epsilon}_{ij}^2. \quad (13)$$

```

118 rese:=exx^2+ezz^2+eyy^2+2*exz^2+2*exy^2+2*eyz^2-ros^2;
119 write length_rese:=mylength(rese);
120 ok:=if rese=0 then ok else 0;
121 ros:=ros+rese*eta*(cu+1/2)/r2*rqq;
```

Approximate the eddy viscosity at any point in the fluid as proportional to the local strain-rate magnitude,

$$\nu = c_t \eta^2 \dot{\epsilon}, \quad (14)$$

where  $c_t$  is a dimensionless constant to be chosen to fit experiments and/or observations. For whatever  $c_t$  is chosen, the deviatoric stress tensor is  $\tau_{ij} = 2\nu \dot{\epsilon}_{ij} = 2c_t \eta^2 \dot{\epsilon}_{ij}$ .

```

122 txx:=2*ct*eta^2*ros*exx;
123 tyy:=2*ct*eta^2*ros*eyy;
124 tzz:=2*ct*eta^2*ros*ezz;
125 txz:=2*ct*eta^2*ros*exz;
126 txy:=2*ct*eta^2*ros*exy;
127 tyz:=2*ct*eta^2*ros*eyz;
```

## 9.2 Compute residuals of lateral momentum

There must be no turbulent mean, tangential stress at the free surface,

$$(1 - \eta_x^2) \tau_{xz} + \eta_x (\tau_{zz} - \tau_{xx}) - \eta_y (\tau_{xy} + \eta_x \tau_{yz}) = 0 \quad \text{on } z = \eta + b \quad (15)$$

$$(1 - \eta_y^2) \tau_{yz} + \eta_y (\tau_{zz} - \tau_{yy}) - \eta_x (\tau_{xy} + \eta_y \tau_{xz}) = 0 \quad \text{on } z = \eta + b \quad (16)$$

Also, put a slip law on the mean bed to provide bed drag:

$$\vec{q}_{\text{tan}} = c_u \eta \frac{\partial \vec{q}_{\text{tan}}}{\partial \vec{n}} \quad \text{on } z = b. \quad (17)$$

for some constant  $\mathbf{c}_u \approx 11/6$  to match open channel flow observations, where the term  $\vec{q}_{\text{tan}}$  represents the velocity tangential to the curved bed and the vector  $\vec{n}$  is the unit vector normal to the bed. To solve equation (17), assume  $\vec{t}_x = \frac{1}{\sqrt{1+b_x^2}}(1, 0, b_x)$  and  $\vec{t}_y = \frac{1}{\sqrt{1+b_y^2}}(0, 1, b_y)$  are unit vectors tangential to the curved bed in the  $x$ - and  $y$ - directions. For simplicity, consider  $\vec{q}_{\text{tan}} = (\vec{t}_x \cdot \vec{q})\vec{t}_x + (\vec{t}_y \cdot \vec{q})\vec{t}_y$ . Thus, the boundary condition (17) becomes

$$\begin{aligned}\vec{t}_x \cdot \vec{q}_{\text{tan}} &= c_u \eta \frac{\partial}{\partial \vec{n}} (\vec{t}_x \cdot \vec{q}_{\text{tan}}) \quad \text{on } z = b, \\ \vec{t}_y \cdot \vec{q}_{\text{tan}} &= c_u \eta \frac{\partial}{\partial \vec{n}} (\vec{t}_y \cdot \vec{q}_{\text{tan}}) \quad \text{on } z = b,\end{aligned}$$

that is,

$$\frac{1}{\sqrt{1+b_x^2}}(u + wb_x) = \frac{c_u h}{\sqrt{1+b_x^2+b_y^2}} \frac{\partial}{\partial \vec{n}} (u + wb_x), \quad (18)$$

$$\frac{1}{\sqrt{1+b_y^2}}(v + wb_y) = \frac{c_u h}{\sqrt{1+b_x^2+b_y^2}} \frac{\partial}{\partial \vec{n}} (v + wb_y). \quad (19)$$

To get centre manifold theory support for the slow manifold model of shallow water flow, modify the surface condition (15) and (16) on the tangential stresses to have an artificial forcing proportional to the square of the local, free surface, velocity:

$$\begin{aligned}& [(1 - \eta_x^2)\tau_{xz} + \eta_x(\tau_{zz} - \tau_{xx}) - \eta_y(\tau_{xy} + \eta_x\tau_{yz})] \\ &= \frac{(1 - \gamma)\sqrt{2}c_t}{(1 + c_u)(1 + 2c_u)} u \sqrt{u^2 + v^2} \quad \text{on } z = \eta + b.\end{aligned} \quad (20)$$

$$\begin{aligned}& [(1 - \eta_y^2)\tau_{yz} + \eta_y(\tau_{zz} - \tau_{yy}) - \eta_x(\tau_{xy} + \eta_y\tau_{xz})] \\ &= \frac{(1 - \gamma)\sqrt{2}c_t}{(1 + c_u)(1 + 2c_u)} v \sqrt{u^2 + v^2} \quad \text{on } z = \eta + b.\end{aligned} \quad (21)$$

When we evaluate at  $\gamma = 1$  this artificial right-hand side becomes zero so the artificial surface conditions (20) and (21) reduces to the physical surface condition (15) and (16). However, when both the parameter  $\gamma = 0$  and the lateral derivatives are negligible,  $\partial_x = \partial_y = 0$ , then the lateral shear  $u, v \propto c_u + \zeta$  becomes a neutral mode of the dynamics. The Euler parameter

of a toy problem suggests introducing a factor  $(1 - \frac{1}{6}\gamma)$  into the left-hand side of the tangential stress boundary conditions (20) and (21) in order to improve convergence in the parameter  $\gamma$  when evaluated at the physically relevant  $\gamma = 1$ . This needs further exploration. For the moment omit such a factor.

Compute the residuals of the lateral momentum equation, an artificial tangential stress on the free surface, and the bed boundary. See that when  $\gamma = 0$  the free surface condition is effectively  $\eta\partial_z\mathbf{u} = \mathbf{u}/(1 + \mathbf{c}_u)$ , leading to our neutral mode  $\mathbf{u} \propto \mathbf{c}_u + \zeta$ , namely  $\mathbf{u} = \bar{\mathbf{u}}(\mathbf{x}, t)(\mathbf{c}_u + \zeta)/(\mathbf{c}_u + \frac{1}{2})$ ; whereas when  $\gamma = 1$  the free surface condition reduces to zero tangential stress. In the computer algebra, we expand the square root terms in conditions (18) and (19) in Taylor series.

```

128 resu:=df(u,t)+u*df(u,x)+v*df(u,y)+w*df(u,z)+df(p,x)
129      -grx-df(txx,x)-df(txy,y)-df(txz,z);
130 resv:=df(v,t)+u*df(v,x)+v*df(v,y)+w*df(v,z)+df(p,y)
131      -gry-df(tyy,y)-df(txy,x)-df(tyz,z);
132 resbu:=sub(zz=0,(-u-w*df(b,x))*(1-df(b,x)^2/2)
133      +cu*eta*(1-df(b,x)^2/2-df(b,y)^2/2)*(
134      -df(u+w*df(b,x),x)*df(b,x)
135      -df(u+w*df(b,x),y)*df(b,y)
136      +df(u+w*df(b,x),z)));
137 resbv:=sub(zz=0,(-v-w*df(b,y))*(1-df(b,y)^2/2)
138      +cu*eta*(1-df(b,x)^2/2-df(b,y)^2/2)*(
139      -df(v+w*df(b,y),x)*df(b,x)
140      -df(v+w*df(b,y),y)*df(b,y)
141      +df(v+w*df(b,y),z)));
142 write length_resu:=mylength(resu);
143 write length_resv:=mylength(resv);
144 write length_resbu:=mylength(resbu);
145 write length_resbv:=mylength(resbv);
146 ok:=if {resu,resv,resbu,resbv}={0,0,0,0} then ok else 0;

147 resuamp:=mean(u,zz)-uu(0,0);
148 resvamp:=mean(v,zz)-vv(0,0);
149 write length_resuamp:=mylength(resuamp);
150 write length_resvamp:=mylength(resvamp);
151 ok:=if {resuamp,resvamp}={0,0} then ok else 0;

```



### 9.3 Solve for updates to lateral velocity and rate-of-strain

Update the lateral fields using an as yet unknown change in the evolution for the lateral mean velocities. The lateral fields are coupled by the nonlinear stress-strain relation of the Smagorinski turbulent flow.

```
152 u:=u+(du:=resbu*(1-2*zz)/(1+2*cu)
153     +eta*(1+2*cu)*r2/(4*ct)*rqq^3*usolv(
154     +(qq^2+vv(0,0)^2)*resu
155     -uu(0,0)*vv(0,0)*resv ,zz));
156 v:=v+(dv:=resbv*(1-2*zz)/(1+2*cu)
157     +eta*(1+2*cu)*r2/(4*ct)*rqq^3*usolv(
158     +(qq^2+uu(0,0)^2)*resv
159     -uu(0,0)*vv(0,0)*resu ,zz));
160 ros:=ros+(uu(0,0)*df(du,zz)+vv(0,0)*df(dv,zz))/(r2*eta)*rqq;
```

Now use the tangential stress on the free surface to determine the evolution corrections  $gud$  and  $gvd$ . But first need to update part of the stress from changes in lateral velocity and rate-of-strain: these updates to stress and strain are indeed needed to ensure that the residual of the lateral momentum equations are satisfied.

```
161 exz:=(df(u,z)+df(w,x))/2;
162 eyz:=(df(v,z)+df(w,y))/2;
163 txz:=2*ct*eta^2*ros*exz;
164 tyz:=2*ct*eta^2*ros*eyz;
```

Then compute residuals of tangential stress equations.

```
165 resttu:=(-sub(zz=1,
166     (1-0*gamm)*((1-(etax+df(b,x))^2)*txz
167     +(etax+df(b,x))*(tzz-txx)
168     -(etay+df(b,y))*(txy+(etax+df(b,x))*tyz))
169     -(1-gamm)*r2*ct/(cu+1)/(2*cu+1)*u*qq ));
170 write length_rettu:=mylength(resttu);
171 resttv:=(-sub(zz=1,
172     (1-0*gamm)*((1-(etay+df(b,y))^2)*tyz
```

```

173      +(etay+df(b,y))*(tzz-tyy)
174      -(etax+df(b,x))*(txy+(etay+df(b,y))*txz))
175      -(1-gamm)*r2*ct/(cu+1)/(2*cu+1)*v*qq );
176 write length_resttv:=mylength(resttv);
177 ok:=if {resttu,resttv}={0,0} then ok else 0;

```

Update the lateral evolution based upon these residuals.

```

178 gu:=gu-3*(1+2*cu)*(1+cu)
179      /2/eta/(3+11*cu+12*cu^2)/(1+3*cu+3*cu^2)/(3+4*cu)
180      *(((1+5*cu+8*cu^2)*uu(0,0)^2*rqq^2
181      -(9+45*cu+80*cu^2+48*cu^3))*resttu
182      +(1+5*cu+8*cu^2)*uu(0,0)*vv(0,0)*rqq^2*resttv);
183 gv:=gv-3*(1+2*cu)*(1+cu)
184      /2/eta/(3+11*cu+12*cu^2)/(1+3*cu+3*cu^2)/(3+4*cu)
185      *(((1+5*cu+8*cu^2)*vv(0,0)^2*rqq^2
186      -(9+45*cu+80*cu^2+48*cu^3))*resttv
187      +(1+5*cu+8*cu^2)*uu(0,0)*vv(0,0)*rqq^2*resttu);

```

## 10 Postprocessing

End the iterative loop.

```

188 showtime;
189 if ok then write iter:=100000+iter;
190 end;

```

We may use these transformations to check on the dimensionality of various expressions. But not at the moment.

```

191 dims:={ h(~m)=>nh*11
192         , uu(~m)=>mu*11/tt
193         , gx=>ngx*11/tt^2 }$

```

Write out the final evolution on the slow manifold.

```

194 r2:=sqrt(2)$ %eps:=1$
195 on rounded; print_precision 4;

```

```
196 write dhdt:=length(gh);
197 write dudt:=length(gu);
198 write dvdt:=length(gv);
```

Finish.

```
199 end;
```

## A Trace prints from sample execution

```
200 d := eps
201
202 grz := gz
203
204 grx := gx*eps
205
206 gry := 0
207
208 gamm := gam*eps
209
210 ITERATION 1
211
212 length_resc := 9
213
214 length_resa := 3
215
216 length_resw := 74
217
218 length_restn := 3
219
220 length_rese := 51
221
222 length_resu := 93
223
224 length_resv := 88
225
226 length_resbu := 0
```

```
227
228 length_resbv := 0
229
230 length_restvu := 153
231
232 length_rettv := 134
233
234 Time: 250 ms
235
236 ITERATION 2
237
238 length_resc := 377
239
240 length_resa := 0
241
242 length_resw := 337
243
244 length_retn := 94
245
246 length_rese := 249
247
248 length_resu := 780
249
250 length_resv := 818
251
252 length_resbu := 0
253
254 length_resbv := 0
255
256 length_restvu := 143
257
258 length_rettv := 152
259
260 Time: 5039 ms   plus GC time: 150 ms
261
262 ITERATION 3
263
```

```
264 length_resc := 380
265
266 length_resa := 0
267
268 length_resw := 317
269
270 length_restn := 99
271
272 length_rese := 268
273
274 length_resu := 682
275
276 length_resv := 728
277
278 length_resbu := 0
279
280 length_resbv := 0
281
282 length_rettu := 143
283
284 length_rettv := 152
285
286 Time: 11660 ms   plus GC time: 410 ms
287
288 ITERATION 4
289
290 length_resc := 0
291
292 length_resa := 0
293
294 length_resw := 125
295
296 length_restn := 43
297
298 length_rese := 0
299
300 length_resu := 285
```

```
301
302 length_resv := 303
303
304 length_resbu := 0
305
306 length_resbv := 0
307
308 length_rettu := 0
309
310 length_rettv := 0
311
312 Time: 14591 ms   plus GC time: 550 ms
313
314 ITERATION 5
315
316 length_resc := 0
317
318 length_resa := 0
319
320 length_resw := 0
321
322 length_retn := 0
323
324 length_rese := 0
325
326 length_resu := 0
327
328 length_resv := 0
329
330 length_resbu := 0
331
332 length_resbv := 0
333
334 length_rettu := 0
335
336 length_rettv := 0
337
```

```

338 Time: 14489 ms   plus GC time: 550 ms
339
340 iter := 100005
341
342 dhdt := 4
343
344 dudt := 175
345
346 dvdt := 175

```

## References

- Bijvelds, M. D. J. P., Kranenburg, C. & Stelling, G. S. (1999), ‘3d numerical simulation of turbulent shallow-water in square harbor’, *Journal of Hydraulic Engineering* **125**(1), 26–31. [http://dx.doi.org/10.1061/\(ASCE\)0733-9429\(1999\)125:1\(26\)](http://dx.doi.org/10.1061/(ASCE)0733-9429(1999)125:1(26)).
- Janosi, I. M., Jan, D., Szabo, K. G. & Tel, T. (2004), ‘Turbulent drag reduction in dam-break flows’, *Experiments in Fluids* **37**, 219–229. <http://dx.doi.org/10.1007/s00348-004-0804-4>.
- Kim, T. K. (2002), A Modified Smagorinsky Subgrid Scale Model for the Large Eddy Simulation of Turbulent Flow, PhD thesis, Univeristy of California, Davis. [<http://arXiv.org/abs/math.sc/0209377>].
- Marstop, L. (2006), Subgrid-scale modelling for large-eddy simulation including scalar mixing in rotating turbulent shear flows, PhD thesis, Royal Institute of Technology KTH Mechanics SE-100 44 Stockholm, Sweden.
- Nezu, I. (2005), ‘Open-channel flow turbulence and its research prospect in the 21st century’, *J. Hydraulic Engineering* p. 229. [http://dx.doi.org/10.1061/\(ASCE\)0733-9429\(2005\)131:4\(229\)](http://dx.doi.org/10.1061/(ASCE)0733-9429(2005)131:4(229)).
- Özögökmen et al.
- Özögökmen, T. M., Iliescu, T., Fischer, P. F., Srinivasan, A. & Duan, J. (2007), ‘Large eddy simulation of stratified mixing and two-dimensional dam-break problem in a rectangular enclosed domain’, *Ocean Modelling* **16**, 106–140. <http://dx.doi.org/10.1016/j.ocemod.2006.08.006>.

Roberts, A. J. (2007*a*), Computer algebra models the inertial dynamics of a thin film flow of power law fluids and other generalised newtonian fluids, Technical report, University of Southern Queensland. <http://eprints.usq.edu.au/archive/00002010/>.  
**URL:** <http://eprints.usq.edu.au/archive/00002010/>

Roberts, A. J. (2007*b*), ‘The inertial dynamics of thin film flow of non-newtonian fluids’, *Physics Letters A* **372**, 1607–1611. <http://dx.doi.org/10.1016/j.physleta.2007.10.014>.

Roberts, A. J., Georgiev, D. J. & Strunin, D. V. (2008), Model turbulent floods with the Smagorinski large eddy closure, Technical report, <http://arxiv.org/abs/0805.3192>.