Examples of invariant manifold construction

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Instructions

- Download and install the computer algebra package *Reduce* via `http://www.reduce-algebra.com`
- Navigate to folder `Examples` within folder `InvariantManifold`.
- For each example of interest, start-up *Reduce* and enter the command `in_tex "filename.tex"$` where filename is the root name of the example (as listed in the following table of contents).

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1 Five representative examples

1.1 `simple3d`: Slow manifold of a basic 3D system

The basic example system to analyse is

$$\begin{aligned}\dot u_1 &= 2u_1 + u_2 + 2u_3 + u_2u_3\,,\\ \dot u_2 &= u_1 - u_2 + u_3 - u_1u_3\,,\\ \dot u_3 &= -3u_1 - u_2 - 3u_3 - u_1u_2\,.\end{aligned}$$

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

Execute the construction of the slow manifold for this system.

```
invariantmanifold(
    mat(( 2*u1+u2+2*u3+u2*u3,
          u1-u2+u3-u1*u3,
          -3*u1-u2-3*u3-u1*u2 )),
    mat((0)),
    mat((1,0,-1)),
    mat((4,1,3)),
    3 )$
end;
```

The matrix $\begin{bmatrix} 2 & 1 & 2\\ 1 & -1 & 1\\ -3 & -1 & -3\end{bmatrix}$ of the linearisation about the origin has eigenvalues zero and -1 (multiplicity two). We seek the slow manifold so specify the eigenvalue zero in the second parameter to the procedure. A corresponding eigenvector is $\vec e = (1,0,-1)$, and corresponding left-eigenvector is $\vec z = (4,1,3)$, as specified above. The last parameter, 3, specifies to construct the slow manifold to errors $\mathcal{O}\big(\varepsilon^3\big)$.

The procedure actually analyses the embedding system

$$\begin{aligned}\dot u_1 &= 2u_1 + u_2 + 2u_3 + \varepsilon u_2u_3\,,\\ \dot u_2 &= u_1 - u_2 + u_3 - \varepsilon u_1u_3\,,\\ \dot u_3 &= -3u_1 - u_2 - 3u_3 - \varepsilon u_1u_2\,.\end{aligned}$$

Consequently, here the artificial parameter ε has a physical interpretation in that it counts the nonlinearity: a term in ε^p will be a $(p+1)$th order term in $\vec u = (u_1,u_2,u_3)$. Hence the specified error $\mathcal{O}\big(\varepsilon^3\big)$ is here the same as error $\mathcal{O}\big(|\vec u|^4\big)$.

The slow manifold The constructed slow manifold is, in terms of the parameter s_1 (to error $\mathcal{O}\big(\varepsilon^2\big)$, and reverse ordering!),

$$u_1 = -\varepsilon s_1^2 + s_1\,, \qquad u_2 = \varepsilon s_1^2\,, \qquad u_3 = \varepsilon s_1^2 - s_1\,.$$

Slow manifold ODEs On this slow manifold the evolution is

$$\dot{s}_1 = -9\varepsilon^2 s_1^3 + \varepsilon s_1^2 .$$

Here the leading term in s_1^2 establishes the origin is unstable.[1]

Normals to isochrons at the slow manifold To project initial conditions onto the slow manifold, or non-autonomous forcing, or modifications of the original system, or to quantify uncertainty (Roberts 1989, 2000), use the projection defined by the derived vector

$$\vec z_1 = \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} 258\varepsilon^2 s_1^2 - 16\varepsilon s_1 + 4 \\ 93\varepsilon^2 s_1^2 - 9\varepsilon s_1 + 1 \\ 240\varepsilon^2 s_1^2 - 16\varepsilon s_1 + 3 \end{bmatrix} .$$

Evaluate these at $\varepsilon = 1$ to apply to the original specified system, or here just interpret ε as a way to count the order of each term.

1.2 `doubleHopfDDE`: Double Hopf interaction in a 2D DDE

Erneux (2009) [§7.2] explored an example of a laser subject to optoelectronic feedback, coded as a delay differential equation. For certain parameter values it has a two frequency Hopf bifurcation. Near Erneux's parameters $(\eta, \theta) = (3/5, 2)$, the system may be represented as

$$\begin{aligned} \dot u_1 &= -4(1+\delta)^2\left[\tfrac{5}{8}u_2 + \tfrac{3}{8}u_2(t-\pi)\right] \\ \dot u_2 &= u_1(1+u_2). \end{aligned}$$

for small parameter δ. Due to the delay, $u_2(t-\pi)$, this system is effectively an infinite-dimensional dynamical system. Here we describe the emergent dynamics on its four-dimensional centre manifold.

The linearisation of this system at the origin has modes with frequencies $\omega = 1, 2$, and corresponding eigenvectors $(1, \mp i/\omega)e^{\pm i\omega t}$. Corresponding eigenvectors of the adjoint are $(1, \mp i\omega)e^{\pm i\omega t}$. We model the nonlinear interaction of these four modes over long times.

Start by loading the procedure.

```
11 in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameter δ.

```
12 factor s,delta,exp;
```

Execute the construction of the slow manifold for this system, where `u2(pi)` denotes the delayed variable $u_2(t-\pi)$, and where `1+small*delta` reflects that we wish to use the 'small' parameter δ to explore regimes where this factor is near the value 1.

[1] Then the large negative s_1^3 term *suggests* the existence of a finite amplitude equilibrium with $s_1 \approx 1/9$ (it is actually closer to $s_1 \approx 0.2$).

```
invariantmanifold(
    mat(( -4*(1+small*delta)^2*(5/8*u2 +3/8*u2(pi)),
          +u1*(1+u2) )),
    mat(( i,-i,2*i,-2*i )),
    mat( (1,-i), (1,+i), (1,-i/2), (1,+i/2) ),
    mat( (1,-i), (1,+i), (1,-2*i), (1,+2*i) ),
    3 )$
end;
```

The code works for errors of order higher than cubic, but is much slower: takes several minutes per iteration.

The procedure actually analyses the embedding system

$$\begin{aligned}
\dot u_1 &= -4(1+2\varepsilon^2\delta+\varepsilon^3\delta^2)\big[\tfrac58 u_2+\tfrac38 u_2(t-\pi)\big]\\
\dot u_2 &= u_1(1+\varepsilon u_2).
\end{aligned}$$

The centre manifold These give the location of the invariant manifold in terms of parameters s_j. Here, $u_1 \approx s_1e^{it}+s_2e^{-it}+s_3e^{i2t}+s_4e^{-i2t}$ so that (for real solutions) s_1,s_2 are complex conjugate amplitudes that modulate the oscillations of frequency $\omega=1$, whereas s_3,s_4 are complex conjugate amplitudes that modulate the oscillations of frequency $\omega=2$.

$$\begin{aligned}
u_1 = {}& \mathrm{e}^{-it}s_4s_1\varepsilon\big(0.2309i-0.04495\big)+\mathrm{e}^{-it}s_2+0.1667\,\mathrm{e}^{-4it}s_4^2\varepsilon i+{}\\
&0.1875\,\mathrm{e}^{-3it}s_4s_2\varepsilon i+\mathrm{e}^{-2it}s_4+\mathrm{e}^{-2it}s_2^2\varepsilon\big(-0.3953i-0.1233\big)+{}\\
&\mathrm{e}^{it}s_3s_2\varepsilon\big(-0.2309i-0.04495\big)+\mathrm{e}^{it}s_1-0.1667\,\mathrm{e}^{4it}s_3^2\varepsilon i-{}\\
&0.1875\,\mathrm{e}^{3it}s_3s_1\varepsilon i+\mathrm{e}^{2it}s_3+\mathrm{e}^{2it}s_1^2\varepsilon\big(0.3953i-0.1233\big)\\
u_2 = {}& \mathrm{e}^{-it}s_4s_1\varepsilon\big(0.04495i+0.2309\big)+\mathrm{e}^{-it}s_2i-0.1667\,\mathrm{e}^{-4it}s_4^2\varepsilon-{}\\
&0.5625\,\mathrm{e}^{-3it}s_4s_2\varepsilon+0.5\,\mathrm{e}^{-2it}s_4i+\mathrm{e}^{-2it}s_2^2\varepsilon\big(0.06167i-0.1977\big)+{}\\
&\mathrm{e}^{it}s_3s_2\varepsilon\big(-0.04495i+0.2309\big)-\mathrm{e}^{it}s_1i-0.1667\,\mathrm{e}^{4it}s_3^2\varepsilon-{}\\
&0.5625\,\mathrm{e}^{3it}s_3s_1\varepsilon-0.5\,\mathrm{e}^{2it}s_3i+\mathrm{e}^{2it}s_1^2\varepsilon\big(-0.06167i-0.1977\big)
\end{aligned}$$

Centre manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs that characterise how the modulation of the oscillations evolve due to their nonlinear interaction.

$$\begin{aligned}
\dot s_1 = {}& s_4s_3s_1\varepsilon^2\big(-0.03089i+0.05032\big)+s_3s_2\varepsilon\big(-0.08991i-0.03816\big)+{}\\
&s_2s_1^2\varepsilon^2\big(-0.01837i-0.1095\big)+s_1\delta\varepsilon^2\big(0.1526i-0.3596\big)\\
\dot s_2 = {}& s_4s_3s_2\varepsilon^2\big(0.03089i+0.05032\big)+s_4s_1\varepsilon\big(0.08991i-0.03816\big)+{}\\
&s_2^2s_1\varepsilon^2\big(0.01837i-0.1095\big)+s_2\delta\varepsilon^2\big(-0.1526i-0.3596\big)\\
\dot s_3 = {}& s_4s_3^2\varepsilon^2\big(-0.0349i-0.04111\big)+s_3s_2s_1\varepsilon^2\big(-0.2499i-{}\\
&0.2153\big)+s_3\delta\varepsilon^2\big(0.8376i+0.9867\big)+s_1^2\varepsilon\big(-0.4934i+0.4188\big)\\
\dot s_4 = {}& s_4^2s_3\varepsilon^2\big(0.0349i-0.04111\big)+s_4s_2s_1\varepsilon^2\big(0.2499i-0.2153\big)+{}\\
&s_4\delta\varepsilon^2\big(-0.8376i+0.9867\big)+s_2^2\varepsilon\big(0.4934i+0.4188\big)
\end{aligned}$$

1.3 `metastable4`: Metastability in a four state Markov chain

Variable ϵ characterises the rate of exchange between metastable states u_1 and u_4 in this system.

$$\begin{aligned}
\dot u_1 &= +u_2 - \epsilon u_1\,,\\
\dot u_2 &= -u_2 + \epsilon(u_3 - u_2 + u_1),\\
\dot u_3 &= -u_3 + \epsilon(u_4 - u_3 + u_2),\\
\dot u_4 &= +u_3 - \epsilon u_4\,.
\end{aligned}$$

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

Execute the construction of the slow manifold for this system. The explicit parameter `small`, math-name ε, gets replaced by `small^2` in the code, so in effect $\varepsilon^2 = \epsilon$.

```
invariantmanifold(
    mat(( u2-small*u1,
         -u2+small*(u1-u2+u3),
         -u3+small*(u2-u3+u4),
          u3-small*u4 )),
    mat((0,0)),
    mat((1,0,0,0),(0,0,0,1)),
    mat((1,1,0,0),(0,0,1,1)),
    6 )$
end;
```

The matrix $\begin{bmatrix} 0&1&0&0\\ 0&-1&0&0\\ 0&0&-1&0\\ 0&0&1&0 \end{bmatrix}$, of the linearisation about $\varepsilon = 0$, has eigenvalues 0 and -1 (both multiplicity two). We seek the slow manifold so specify the two zero eigenvalues in the second parameter to the procedure. Corresponding eigenvectors are $\vec e_1 = (1,0,0,0)$ and $\vec e_2 = (0,0,0,1)$. Choosing corresponding left-vector (here not an eigenvector) is $\vec z_1 = (1,1,0,0)$ and $\vec z_2 = (0,0,1,1)$ means that the slow manifold parameters s_1, s_2 have the physical meaning, respectively, of being the probability that the system is in states $\{1,2\}$ and $\{3,4\}$. The last parameter, 6, specifies to construct the slow manifold to errors $\mathcal{O}\big(\varepsilon^6\big)$, that is, errors $\mathcal{O}\big(\epsilon^3\big)$.

The slow manifold The constructed slow manifold is, in terms of the lumped-state probability parameters s_1, s_2 (to error $\mathcal{O}\big(\varepsilon^2\big)$, and reverse ordering!),

$$\begin{aligned}
u_1 &= \varepsilon^4\big(-s_2 + 2s_1\big) - \varepsilon^2 s_1 + s_1\,, & u_3 &= \varepsilon^4\big(-2s_2 + s_1\big) + \varepsilon^2 s_2\,,\\
u_2 &= \varepsilon^4\big(s_2 - 2s_1\big) + \varepsilon^2 s_1\,, & u_4 &= \varepsilon^4\big(2s_2 - s_1\big) - \varepsilon^2 s_2 + s_2\,.
\end{aligned}$$

Slow manifold ODEs On this slow manifold the evolution of the lumped-state probabilities is

$$\dot s_1 = \varepsilon^4\big(s_2 - s_1\big), \quad \dot s_2 = \varepsilon^4\big(-s_2 + s_1\big).$$

Hence here the long-term evolution is that on a time-scale of $\mathcal{O}(1/\epsilon^2)$, $\mathcal{O}(1/\varepsilon^4)$, the system equilibrates between the two lumped states, that is, between $\{1, 2\}$ and $\{3, 4\}$.

Normals to isochrons at the slow manifold To project initial conditions onto the slow manifold, or non-autonomous forcing, or modifications of the original system, or to quantify uncertainty (Roberts 1989, 2000), use the projection defined by the derived vector

$$\vec z_1 = \begin{bmatrix} z_{11} \\ z_{12} \\ z_{13} \\ z_{14} \end{bmatrix} = \begin{bmatrix} \varepsilon^4 + 1 \\ 4\varepsilon^4 - \varepsilon^2 + 1 \\ -4\varepsilon^4 + \varepsilon^2 \\ -\varepsilon^4 \end{bmatrix}, \quad \vec z_2 = \begin{bmatrix} z_{21} \\ z_{22} \\ z_{23} \\ z_{24} \end{bmatrix} = \begin{bmatrix} -\varepsilon^4 \\ -4\varepsilon^4 + \varepsilon^2 \\ 4\varepsilon^4 - \varepsilon^2 + 1 \\ \varepsilon^4 + 1 \end{bmatrix}.$$

Evaluate all these at $\varepsilon^2 = \epsilon$ to apply to the original specified system.

1.4 `nonlinNormModes`: Interaction of nonlinear normal modes

Renson et al. (2012) explored finite element construction of the nonlinear normal modes of a pair of coupled oscillators. Defining two new variables one of their example systems is

$$\begin{aligned} \dot x_1 &= x_3\,, & \dot x_3 &= -2x_1 + x_2 - \frac{1}{2}x_1^3 + \frac{3}{10}(-x_3 + x_4)\,, \\ \dot x_2 &= x_4\,, & \dot x_4 &= x_1 - 2x_2 + \frac{3}{10}(x_3 - 2x_4)\,. \end{aligned}$$

The linearisation of this system at the origin has modes with frequencies $\omega = 1, \sqrt{3}$, and corresponding eigenvectors?? $(1, \mp i/\omega)e^{\pm i\omega t}$. Corresponding eigenvectors of the adjoint are $(1, \mp i\omega)e^{\pm i\omega t}$. We model the nonlinear interaction of these four modes over long times.

Here, the analysis constructs a full state space coordinate transformation. We find a mapping from the modulation variables $\vec s = (s_1, s_2, s_3, s_4)$ to the original variables $\vec u = (u_1, u_2, u_3, u_4)$, and find the corresponding evolution of $\vec s$. The modulation variables $\vec s$ are 'slow' because the coordinate transform uses time-dependent (rotating) basis vectors that account for the fast oscillation in $\vec u$. Hence the new variables $\vec s$ are good variables for making long-term predictions and forming understanding.

Start by loading the procedure.

```
32 in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, and the complex exponential.

```
33 factor s,exp;
```

The following code makes the linear damping to be effectively small (which then makes it `small` squared); consequently, also scale the smallness of the cubic nonlinearity to match.

```
invariantmanifold(
    mat(( u3,
          u4,
          -2*u1 +u2 -small*u1^3/2 +small*3/10*(-u3+u4),
          u1  -2*u2 +small*3/10*(u3 -2*u4) )),
    mat(( i,-i,sqrt(3)*i,-sqrt(3)*i )),
    mat( (1,1,+i,+i), (1,1,-i,-i),
         (1,-1,i*sqrt(3),-i*sqrt(3)),
         (1,-1,-i*sqrt(3),i*sqrt(3)) ),
    mat( (1,1,+i,+i), (1,1,-i,-i),
         (-i*sqrt(3),+i*sqrt(3),1,-1),
         (+i*sqrt(3),-i*sqrt(3),1,-1) ),
    3 )$
end;
```

The square root eigenvalues do not cause any trouble (although one may need to reformat the LaTeX of the cis operator). In the model, observe that $s_1 = s_2 = 0$ is invariant, as is $s_3 = s_4 = 0$. These are the nonlinear normal modes.

The procedure actually analyses the embedding system

$$\begin{aligned} \dot u_1 &= u_3\,, \quad \dot u_3 = \varepsilon^2\big(-1/2u_1^3 - 3/10u_3 + 3/10u_4\big) - 2u_1 + u_2\,, \\ \dot u_2 &= u_4\,, \quad \dot u_4 = \varepsilon^2\big(3/10u_3 - 3/5u_4\big) + u_1 - 2u_2\,. \end{aligned}$$

The invariant manifold Here these give the reparametrisation of the state space $\vec u$ in terms of parameters s_j, via rotating basis vectors. Here, the coordinate transform is very complicated so I do not give the complexity. The leading approximation is, of course, the linear, errors $\mathcal{O}\big(\varepsilon^2\big)$,

$$\begin{aligned} u_1 &= \mathrm{e}^{-\sqrt{3}it}s_4 + \mathrm{e}^{-it}s_2 + \mathrm{e}^{\sqrt{3}it}s_3 + \mathrm{e}^{it}s_1 \\ u_2 &= -\,\mathrm{e}^{-\sqrt{3}it}s_4 + \mathrm{e}^{-it}s_2 - \mathrm{e}^{\sqrt{3}it}s_3 + \mathrm{e}^{it}s_1 \\ u_3 &= -\sqrt{3}\,\mathrm{e}^{-\sqrt{3}it}s_4 i - \mathrm{e}^{-it}s_2 i + \sqrt{3}\,\mathrm{e}^{\sqrt{3}it}s_3 i + \mathrm{e}^{it}s_1 i \\ u_4 &= \sqrt{3}\,\mathrm{e}^{-\sqrt{3}it}s_4 i - \mathrm{e}^{-it}s_2 i - \sqrt{3}\,\mathrm{e}^{\sqrt{3}it}s_3 i + \mathrm{e}^{it}s_1 i \end{aligned}$$

Invariant manifold ODEs The system evolves according to these ODEs that characterise how the modulation of the oscillations evolve in state space due to their nonlinear interaction.

$$\begin{aligned} \dot s_1 &= 3/4s_4s_3s_1\varepsilon^2 i + 3/8s_2s_1^2\varepsilon^2 i - 3/40s_1\varepsilon^2 \\ \dot s_2 &= -3/4s_4s_3s_2\varepsilon^2 i - 3/8s_2^2s_1\varepsilon^2 i - 3/40s_2\varepsilon^2 \\ \dot s_3 &= 1/8\sqrt{3}s_4s_3^2\varepsilon^2 i + 1/4\sqrt{3}s_3s_2s_1\varepsilon^2 i - 3/8s_3\varepsilon^2 \\ \dot s_4 &= -1/8\sqrt{3}s_4^2s_3\varepsilon^2 i - 1/4\sqrt{3}s_4s_2s_1\varepsilon^2 i - 3/8s_4\varepsilon^2 \end{aligned}$$

Here one can see that each oscillation decays, with a frequency shift due to a combination of nonlinear interaction and nonlinear self-interaction.

1.5 `stable3d`: Stable manifold of a basic 3D system

Let's revisit the example of Section 1.1, namely

$$\begin{aligned}
\dot u_1 &= 2u_1 + u_2 + 2u_3 + u_2u_3\,,\\
\dot u_2 &= u_1 - u_2 + u_3 - u_1u_3\,,\\
\dot u_3 &= -3u_1 - u_2 - 3u_3 - u_1u_2\,,
\end{aligned}$$

but here construct its 2D stable manifold.

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

Execute the construction of the stable manifold for this system.

```
invariantmanifold(
    mat(( 2*u1+u2+2*u3+u2*u3,
          u1-u2+u3-u1*u3,
          -3*u1-u2-3*u3-u1*u2 )),
    mat(( -1,-1 )),
    mat( (1,-1,-1),(0.4,1.4,-1) ),
    mat( (1,0,1),(1,0,-1) ),
    3 )$
end;
```

The matrix $\begin{bmatrix} 2 & 1 & 2\\ 1 & -1 & 1\\ -3 & -1 & -3\end{bmatrix}$ of the linearisation about the origin has eigenvalues 0 and -1 (multiplicity two). We seek the 2D stable manifold so specify the eigenvalue -1, twice, in the second parameter to the procedure. A corresponding eigenvector is $\vec e_1 = (1,-1,-1)$, and corresponding left-eigenvector is $\vec z_2 = (1,0,1)$, as specified above. We need two basis eigenvectors, but here there is only one because the other is a generalised eigenvector. We must do more work to find a generalised eigenvector is $\vec e_2 = (0.4, 1.4, -1)$, and a generalised left-eigenvector is $\vec z_2 = (1,0,-1)$. The last parameter, 3, specifies to construct the stable manifold to errors $\mathcal{O}(\varepsilon^3)$.

Because of the generalised eigenvector, the procedure modifies the *linear* terms to a more convenient form (not necessary, just *convenient*)—see the warning in its report. So, the procedure actually analyses the embedding system

$$\begin{aligned}
\dot u_1 &= \varepsilon\big(- u_1 + u_2u_3 - u_3\big) + 3u_1 + u_2 + 3u_3\,,\\
\dot u_2 &= \varepsilon\big(- u_1u_3 + u_1 + u_3\big) - u_2\,,\\
\dot u_3 &= \varepsilon\big(- u_1u_2 + u_1 + u_3\big) - 4u_1 - u_2 - 4u_3\,.
\end{aligned}$$

The stable manifold The constructed stable manifold is, in terms of the parameters s_1, s_2 (to error $\mathcal{O}(\varepsilon^2)$, and reverse ordering!, and in terms of the ugly $e^{iti} = e^{-t}$ which I aim to improve soon!),

$$\begin{aligned}
u_1 &= \varepsilon\big(-51/25\,\mathrm{e}^{-2t}s_2^2 - 6/5\,\mathrm{e}^{-2t}s_2s_1 + 3\,\mathrm{e}^{-2t}s_1^2\big) + 2/5\,\mathrm{e}^{-t}s_2 + \mathrm{e}^{-t}s_1\,,\\
u_2 &= \varepsilon\big(- 2/5\,\mathrm{e}^{-2t}s_2^2 - 7/5\,\mathrm{e}^{-2t}s_2s_1 - \mathrm{e}^{-2t}s_1^2\big) + 7/5\,\mathrm{e}^{-t}s_2 - \mathrm{e}^{-t}s_1\,,\\
u_3 &= \varepsilon\big(4\,\mathrm{e}^{-2t}s_2^2 + 13/5\,\mathrm{e}^{-2t}s_2s_1 - 5\,\mathrm{e}^{-2t}s_1^2\big) - \mathrm{e}^{-t}s_2 - \mathrm{e}^{-t}s_1\,.
\end{aligned}$$

Observe the linear terms in $\vec{s}$ all have e^{-t}, and the quadratic terms in $\vec{s}$ all have e^{-2t}. Consequently, we could in principle write the stable manifold in terms of, say, the variables $x_j = s_j e^{-t}$ giving

$$\begin{aligned}
u_1 &= \varepsilon\big(- 51/25x_2^2 - 6/5x_2x_1 + 3x_1^2\big) + 2/5x_2 + x_1\,, \\
u_2 &= \varepsilon\big(- 2/5x_2^2 - 7/5x_2x_1 - x_1^2\big) + 7/5x_2 - x_1\,, \\
u_3 &= \varepsilon\big(4x_2^2 + 13/5x_2x_1 - 5x_1^2\big) - x_2 - x_1\,.
\end{aligned}$$

This would be a more usual parametrisation. But here let's remain with $\vec{s}$ and remember to interpret $\vec{s}$ as modifying the exponential decay e^{-t} on this stable manifold.

Stable manifold ODEs On the stable manifold the evolution is

$$\dot s_1 = 3/5\varepsilon s_2\,, \quad \dot s_2 = 0\,.$$

So, s_2 is constant, and hence s_1 increases linearly. But such increase only modifies slightly the robust exponential decay, e^{-t}, on the stable manifold.

In terms of $\vec{x}$ this evolution is $\dot x_1 = -x_1 + \frac{3}{5}\varepsilon x_2\,,\ \dot x_2 = -x_2\,.$

2 Slow invariant manifolds

2.1 `simple2d`: Slow manifold of a simple 2D system

The example system to analyse is specified to be

$$\dot u_1 = -u_1 + u_2 - u_1^2\,, \quad \dot u_2 = u_1 - u_2 + u_2^2\,.$$

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

Execute the construction of the slow manifold for this system.

```
invariantmanifold(
    mat((-u1+u2-u1^2,u1-u2+u2^2)),
    mat((0)),
    mat((1,1)),
    mat((1,1)),
    5)$
end;
```

We seek the slow manifold so specify the eigenvalue zero. From the linearisation matrix $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ a corresponding eigenvector is $\vec e = (1,1)$, and corresponding left-eigenvector is $\vec z = \vec e = (1,1)$, as specified. The last parameter specifies to construct the slow manifold to errors $\mathcal{O}\big(\varepsilon^5\big)$.

The procedure actually analyses the embedding system

$$\dot u_1 = -u_1 + u_2 - \varepsilon u_1^2\,, \quad \dot u_2 = u_1 - u_2 + \varepsilon u_2^2\,.$$

So here the artificial parameter ε has a physical interpretation in that it counts the nonlinearity: a term in ε^p will be a $(p+1)$th order term in $\vec u = (u_1, u_2)$. Hence the specified error $\mathcal{O}\big(\varepsilon^5\big)$ is here the same as error $\mathcal{O}\big(|\vec u|^6\big)$.

The slow manifold The constructed slow manifold is, in terms of the parameter s_1 (and reverse ordering!),

$$u_1 = 3/8\varepsilon^3 s_1^4 - 1/2\varepsilon s_1^2 + s_1\,,$$
$$u_2 = -3/8\varepsilon^3 s_1^4 + 1/2\varepsilon s_1^2 + s_1\,.$$

Slow manifold ODEs On this slow manifold the evolution is

$$\dot s_1 = -3/4\varepsilon^4 s_1^5 + \varepsilon^2 s_1^3\ :$$

here the leading term in s_1^3 indicates the origin is unstable.

Normals to isochrons at the slow manifold To project initial conditions onto the slow manifold, or non-autonomous forcing, or modifications of the original system, or to quantify uncertainty, use the projection defined by the derived vector

$$\vec{z}_1 = \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} 3/2\varepsilon^4 s_1^4 + 3/4\varepsilon^3 s_1^3 - 1/2\varepsilon^2 s_1^2 - 1/2\varepsilon s_1 + 1/2 \\ 3/2\varepsilon^4 s_1^4 - 3/4\varepsilon^3 s_1^3 - 1/2\varepsilon^2 s_1^2 + 1/2\varepsilon s_1 + 1/2 \end{bmatrix}.$$

Evaluate these at $\varepsilon = 1$ to apply to the original specified system, or here just interpret ε as a way to count the order of each term.

3 Oscillations on the invariant manifolds

3.1 `simpleosc`: Oscillatory centre manifold—separated form

Let's try complex eigenvectors. Adjoint eigenvectors `zz_` must be the eigenvectors of the complex conjugate transpose matrix.

$$\begin{aligned}\dot u_1 &= u_2\\ \dot u_2 &= -\varepsilon u_3 u_1 - u_1\\ \dot u_3 &= 5\varepsilon u_1^2 - u_3\end{aligned}$$

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j and the complex exponential

```
factor s,exp;
```

Execute the construction of the centre manifold for this system.

```
invariantmanifold(
    mat((u2,-u1-u1*u3,-u3+5*u1^2)),
    mat((i,-i)),
    mat((1,+i,0),(1,-i,0)),
    mat((1,+i,0),(1,-i,0)),
    3)$
end;
```

The centre manifold These give the location of the invariant manifold in terms of parameters s_j.

$$\begin{aligned}u_1 &= \mathrm{e}^{-it} s_2 + \mathrm{e}^{it} s_1\\ u_2 &= -\mathrm{e}^{-it} s_2 i + \mathrm{e}^{it} s_1 i\\ u_3 &= \mathrm{e}^{-2it} s_2^2 \varepsilon\big(2i+1\big) + \mathrm{e}^{2it} s_1^2 \varepsilon\big(-2i+1\big) + 10 s_2 s_1 \varepsilon\end{aligned}$$

Centre manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\begin{aligned}\dot s_1 &= s_2 s_1^2 \varepsilon^2\big(11/2 i + 1\big)\\ \dot s_2 &= s_2^2 s_1 \varepsilon^2\big(-11/2 i + 1\big)\end{aligned}$$

3.2 `quasidde`: Quasi delay DE with Hopf bifurcation

Shows Hopf bifurcation as parameter α crosses 0 to oscillations with base frequency two.

$$\begin{aligned}\dot u_1 &= -\alpha\varepsilon^2 u_3 - \varepsilon^2 u_1^3 - 2\varepsilon u_1^2 - 4u_3\\ \dot u_2 &= 2u_1 - 2u_2\\ \dot u_3 &= 2u_2 - 2u_3\end{aligned}$$

for small parameter α. We code the parameter α as 'small', and observe it is consequently considered as 'small squared' because all nonlinear terms and already 'small' terms, are multiplied by another `small`.

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameter α.

```
factor s,exp,alpha;
```

Execute the construction of the slow manifold for this system (ignore the warning messages about `u1` declared, and then already defined, as an operator).

```
invariantmanifold(
    mat(( -4*u3-small*alpha*u3-2*u1^2-small*u1^3,
        2*u1-2*u2,
        2*u2-2*u3 )),
    mat((2*i,-2*i)),
    mat((1,1/2-i/2,-i/2),(1,1/2+i/2,+i/2)),
    mat((1,-i,-1-i),(1,+i,-1+i)),
    3)$
end;
```

The centre manifold These give the location of the invariant manifold in terms of parameters s_1, s_2 (complex conjugate for real solutions).

$$\begin{aligned}
u_1 &= \mathrm{e}^{-4it} s_2^2 \varepsilon\big(-7/12i + 1/12\big) + \mathrm{e}^{-2it} s_2 + \mathrm{e}^{4it} s_1^2 \varepsilon\big(7/12i + \\
&\quad 1/12\big) + \mathrm{e}^{2it} s_1 - s_2 s_1 \varepsilon \\
u_2 &= \mathrm{e}^{-4it} s_2^2 \varepsilon\big(-1/12i + 1/4\big) + \mathrm{e}^{-2it} s_2\big(1/2i + 1/2\big) + \\
&\quad \mathrm{e}^{4it} s_1^2 \varepsilon\big(1/12i + 1/4\big) + \mathrm{e}^{2it} s_1\big(-1/2i + 1/2\big) - s_2 s_1 \varepsilon \\
u_3 &= \mathrm{e}^{-4it} s_2^2 \varepsilon\big(1/12i + 1/12\big) + 1/2\,\mathrm{e}^{-2it} s_2 i + \mathrm{e}^{4it} s_1^2 \varepsilon\big(-1/12i + \\
&\quad 1/12\big) - 1/2\,\mathrm{e}^{2it} s_1 i - s_2 s_1 \varepsilon
\end{aligned}$$

Centre manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\begin{aligned}
\dot s_1 &= s_2 s_1^2 \varepsilon^2\big(-16/15i - 1/5\big) + s_1 \alpha \varepsilon^2\big(1/5i + 1/10\big) \\
\dot s_2 &= s_2^2 s_1 \varepsilon^2\big(16/15i - 1/5\big) + s_2 \alpha \varepsilon^2\big(-1/5i + 1/10\big)
\end{aligned}$$

Hence there is a supercritical Hopf bifurcation as parameter α increases through zero.

4 Invariant manifolds in delay DEs

4.1 simple1dde: Simple DDE with a Hopf bifurcation

Model a delayed 'logistic' system in one variable with

$$\frac{du}{dt} = -(1+a)[1+u(t)]u(t-\pi/2),$$

for small parameter a. We code the parameter a as 'small', and observe it is consequently considered as 'small squared' because all nonlinear terms and already 'small' terms are multiplied by `small`.

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameter a.

```
factor s,exp,a;
```

Execute the construction of the slow manifold for this system (ignore the warning messages about `u1` declared, and then already defined, as an operator).

```
invariantmanifold(
    mat(( -(1+small*a)*(1+u1)*u1(pi/2) )),
    mat((i,-i)),
    mat((1),(1)),
    mat((1),(1)),
    3)$
end;
```

The marginal modes are $e^{\pm it}$ so nominate the frequencies ± 1. The eigenvectors are just $1 \cdot e^{\pm it}$. Because for delay differential equations the time dependence $e^{\pm i\omega t}$ is an integral part of the definition of the eigenvector; hence the coded eigenvectors can be the same, as here, because they are differentiated through the time dependence $e^{\pm i\omega t}$.

The code works for orders higher than cubic, but is slow: takes about a minute per iteration.

The procedure actually analyses the embedding system

$$\frac{du}{dt} = -[1+\varepsilon u(t)]u(t-\pi/2) - \varepsilon^2 a[1+u(t)]u(t-\pi/2).$$

The centre manifold These give the location of the invariant manifold in terms of parameters s_j.

$$u_1 = \mathrm{e}^{-it} s_2 + \mathrm{e}^{-2it} s_2^2 \varepsilon\big(1/5i+2/5\big) + \mathrm{e}^{it} s_1 + \mathrm{e}^{2it} s_1^2 \varepsilon\big(-1/5i+2/5\big)$$

Centre manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\dot s_1 = s_2 s_1^2 \varepsilon^2 \big(-2/5 i\pi - 12/5 i - 6/5\pi + 4/5 \big) / \big(\pi^2 + 4 \big) + s_1 a \varepsilon^2 \big(4i + 2\pi \big) / \big(\pi^2 + 4 \big)$$

$$\dot s_2 = s_2^2 s_1 \varepsilon^2 \big(2/5 i\pi + 12/5 i - 6/5\pi + 4/5 \big) / \big(\pi^2 + 4 \big) + s_2 a \varepsilon^2 \big(-4i + 2\pi \big) / \big(\pi^2 + 4 \big)$$

4.2 `logistic1dde`: Logistic DDE displays a Hopf bifurcation

Form a centre manifold for the delayed 'logistic' system in one variable, for delay $\tau = 3\pi/4$, with

$$\frac{du}{dt} = -u(t) - (\sqrt{2} + a)u(t-\tau) + \mu u(t-\tau)^2 + \nu u(t-\tau)^3,$$

for and nonlinearity parameters μ and ν, and small parameter a. Numerical computation of the spectrum indicates that the system has a Hopf bifurcation as parameter a crosses zero.

We code the parameter a as 'small', and observe it is consequently considered as 'small squared' because all nonlinear terms, and already 'small' terms, are multiplied by ε (`small`).

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameters.

```
factor s,exp,a,mu,nu;
```

Execute the construction of the slow manifold for this system (ignore the warning messages about `u1` declared, and then already defined, as an operator).

```
invariantmanifold(
    mat(( -u1-(sqrt(2)+small*a)*u1(3*pi/4)
    +mu*u1(3*pi/4)^2 +small*nu*u1(3*pi/4)^3 )),
    mat((i,-i)),
    mat((1),(1)),
    mat((1),(1)),
    3)$
end;
```

The marginal modes are $e^{\pm it}$ so nominate the frequencies ± 1. The eigenvectors are just $1 \cdot e^{\pm it}$. Because for delay differential equations the time dependence $e^{\pm i\omega t}$ is an integral part of the definition of the eigenvector; hence the coded eigenvectors can be the same, as here, because they are differentiated through the time dependence $e^{\pm i\omega t}$.

The procedure actually analyses the embedding system

$$\dot u_1 = -a\varepsilon^2 u_1(t-\tau) + \mu\varepsilon u_1(t-\tau)^2 + \nu\varepsilon^2 u_1(t-\tau)^3 - \sqrt{2} u_1(t-\tau) - u_1\,.$$

The centre manifold These give the location of the invariant manifold in terms of parameters s_j.

$$u_1 = \mathrm{e}^{-it} s_2 + \mathrm{e}^{-2it} s_2^2 \mu\varepsilon\big(-0.07901i + 0.2698\big) + \mathrm{e}^{it} s_1 + \mathrm{e}^{2it} s_1^2 \mu\varepsilon\big(0.07901i + 0.2698\big) + 0.8284 s_2 s_1 \mu\varepsilon$$

Centre manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\dot s_1 = s_2 s_1^2 \mu^2\varepsilon^2\big(-0.1303i - 0.5209\big) + s_2 s_1^2 \nu\varepsilon^2\big(-0.1262i - 0.7206\big) + s_1 a\varepsilon^2\big(0.04205i + 0.2402\big)$$

$$\dot s_2 = s_2^2 s_1 \mu^2\varepsilon^2\big(0.1303i - 0.5209\big) + s_2^2 s_1 \nu\varepsilon^2\big(0.1262i - 0.7206\big) + s_2 a\varepsilon^2\big(-0.04205i + 0.2402\big)$$

Hence the centre manifold model predicts a supercritical Hopf bifurcation as parameter a increases through zero.

References

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