Examples of invariant manifold construction

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Instructions

- Download and install the computer algebra package *Reduce* via http://www.reduce-algebra.com

- Navigate to folder `Examples` within folder `InvariantManifold`.

- For each example of interest, start-up *Reduce* and enter the command `in_tex "filename.tex"$` where filename is the root name of the example (as listed in the following table of contents).

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1 Five representative examples

1.1 `simple3d`: Slow manifold of a basic 3D system

The basic example system to analyse for a slow manifold is

$$\dot{u}_1 = 2u_1 + u_2 + 2u_3 + u_2 u_3\,,$$
$$\dot{u}_2 = u_1 - u_2 + u_3 - u_1 u_3\,,$$
$$\dot{u}_3 = -3u_1 - u_2 - 3u_3 - u_1 u_2\,,$$

(Section 1.5 constructs its stable manifold).

Start by loading the procedure.

```
1 in_tex "../invariantManifold.tex"$
```

Execute the construction of the slow manifold for this system.

```
 2 invariantmanifold(
 3     mat(( 2*u1+u2+2*u3+u2*u3,
 4           u1-u2+u3-u1*u3,
 5           -3*u1-u2-3*u3-u1*u2 )),
 6     mat((0)),
 7     mat((1,0,-1)),
 8     mat((4,1,3)),
 9     3 )$
10 end;
```

The matrix $\begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 1 \\ -3 & -1 & -3 \end{bmatrix}$ of the linearisation about the origin has eigenvalues zero and -1 (multiplicity two). We seek the slow manifold so specify the eigenvalue zero in the second parameter to the procedure. A corresponding eigenvector is $\vec{e} = (1, 0, -1)$, and corresponding left-eigenvector is $\vec{z} = (4, 1, 3)$, as specified above. The last parameter, 3, specifies to construct the slow manifold to errors $\mathcal{O}(\varepsilon^3)$.

The procedure actually analyses the embedding system

$$\dot{u}_1 = 2u_1 + u_2 + 2u_3 + \varepsilon u_2 u_3\,,$$
$$\dot{u}_2 = u_1 - u_2 + u_3 - \varepsilon u_1 u_3\,,$$
$$\dot{u}_3 = -3u_1 - u_2 - 3u_3 - \varepsilon u_1 u_2\,.$$

Consequently, here the artificial parameter ε has a physical interpretation in that it counts the nonlinearity: a term in ε^p will be a $(p+1)$th order term in $\vec{u} = (u_1, u_2, u_3)$. Hence the specified error $\mathcal{O}(\varepsilon^3)$ is here the same as error $\mathcal{O}(|\vec{u}|^4)$ and $\mathcal{O}(|\vec{s}|^4)$.

The slow manifold The constructed slow manifold is, in terms of the parameter s_1 (to error $\mathcal{O}(\varepsilon^2)$, and reverse ordering!),

$$u_1 = -\varepsilon s_1^2 + s_1\,, \qquad u_2 = \varepsilon s_1^2\,, \qquad u_3 = \varepsilon s_1^2 - s_1\,.$$

Slow manifold ODEs On this slow manifold the evolution is

$$\dot{s}_1 = -9\varepsilon^2 s_1^3 + \varepsilon s_1^2.$$

Here the leading term in s_1^2 establishes the origin is unstable.[1]

Normals to isochrons at the slow manifold To project initial conditions onto the slow manifold, or non-autonomous forcing, or modifications of the original system, or to quantify uncertainty (Roberts 1989, 2000), use the projection defined by the derived vector

$$\vec{z}_1 = \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} 258\varepsilon^2 s_1^2 - 16\varepsilon s_1 + 4 \\ 93\varepsilon^2 s_1^2 - 9\varepsilon s_1 + 1 \\ 240\varepsilon^2 s_1^2 - 16\varepsilon s_1 + 3 \end{bmatrix}.$$

Evaluate these at $\varepsilon = 1$ to apply to the original specified system, or here just interpret ε as a way to count the order of each term.

1.2 `doubleHopfDDE`: Double Hopf interaction in a 2D DDE

Erneux (2009) [§7.2] explored an example of a laser subject to optoelectronic feedback, coded as a delay differential equation. For certain parameter values it has a two frequency Hopf bifurcation. Near Erneux's parameters $(\eta, \theta) = (3/5, 2)$, the system may be represented as

$$\dot{u}_1 = -4(1+\delta)^2\left[\tfrac{5}{8}u_2 + \tfrac{3}{8}u_2(t-\pi)\right]$$
$$\dot{u}_2 = u_1(1+u_2).$$

for small parameter δ. Due to the delay, $u_2(t-\pi)$, this system is effectively an infinite-dimensional dynamical system. Here we describe the emergent dynamics on its four-dimensional centre manifold.

The linearisation of this system at the origin has modes with frequencies $\omega = 1, 2$, and corresponding eigenvectors $(1, \mp i/\omega)e^{\pm i\omega t}$. Corresponding eigenvectors of the adjoint are $(1, \mp i\omega)e^{\pm i\omega t}$. We model the nonlinear interaction of these four modes over long times.

Start by loading the procedure.

```
11 in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameter δ.

```
12 factor s,delta,exp;
```

Execute the construction of the slow manifold for this system, where `u2(pi)` denotes the delayed variable $u_2(t-\pi)$, and where `1+small*delta` reflects that we wish to use the 'small' parameter δ to explore regimes where this factor is near the value 1.

---

[1] Then the large negative s_1^3 term *suggests* the existence of a finite amplitude equilibrium with $s_1 \approx 1/9$ (it is actually closer to $s_1 \approx 0.2$).

```
13 invariantmanifold(
14     mat(( -4*(1+small*delta)^2*(5/8*u2 +3/8*u2(pi)),
15           +u1*(1+u2) )),
16     mat(( i,-i,2*i,-2*i )),
17     mat( (1,-i), (1,+i), (1,-i/2), (1,+i/2) ),
18     mat( (1,-i), (1,+i), (1,-2*i), (1,+2*i) ),
19     3 )$
20 end;
```

The code works for errors of order higher than cubic, but is much slower: takes several minutes per iteration.

The procedure actually analyses the embedding system

$$\dot{u}_1 = -4(1 + 2\varepsilon^2\delta + \varepsilon^3\delta^2)\left[\tfrac{5}{8}u_2 + \tfrac{3}{8}u_2(t-\pi)\right]$$
$$\dot{u}_2 = u_1(1 + \varepsilon u_2).$$

The centre manifold   These give the location of the invariant manifold in terms of parameters s_j. Here, $u_1 \approx s_1 e^{it} + s_2 e^{-it} + s_3 e^{i2t} + s_4 e^{-i2t}$ so that (for real solutions) s_1, s_2 are complex conjugate amplitudes that modulate the oscillations of frequency $\omega = 1$, whereas s_3, s_4 are complex conjugate amplitudes that modulate the oscillations of frequency $\omega = 2$.

$$\begin{aligned}
u_1 = \ &e^{-it}s_4 s_1 \varepsilon\big(0.2309i - 0.04495\big) + e^{-it}s_2 + 0.1667\,e^{-4it}s_4^2\varepsilon i + \\
&0.1875\,e^{-3it}s_4 s_2 \varepsilon i + e^{-2it}s_4 + e^{-2it}s_2^2\varepsilon\big(-0.3953i - 0.1233\big) + \\
&e^{it}s_3 s_2 \varepsilon\big(-0.2309i - 0.04495\big) + e^{it}s_1 - 0.1667\,e^{4it}s_3^2\varepsilon i - \\
&0.1875\,e^{3it}s_3 s_1 \varepsilon i + e^{2it}s_3 + e^{2it}s_1^2\varepsilon\big(0.3953i - 0.1233\big)
\end{aligned}$$

$$\begin{aligned}
u_2 = \ &e^{-it}s_4 s_1 \varepsilon\big(0.04495i + 0.2309\big) + e^{-it}s_2 i - 0.1667\,e^{-4it}s_4^2\varepsilon - \\
&0.5625\,e^{-3it}s_4 s_2 \varepsilon + 0.5\,e^{-2it}s_4 i + e^{-2it}s_2^2\varepsilon\big(0.06167i - 0.1977\big) + \\
&e^{it}s_3 s_2 \varepsilon\big(-0.04495i + 0.2309\big) - e^{it}s_1 i - 0.1667\,e^{4it}s_3^2\varepsilon - \\
&0.5625\,e^{3it}s_3 s_1 \varepsilon - 0.5\,e^{2it}s_3 i + e^{2it}s_1^2\varepsilon\big(-0.06167i - 0.1977\big)
\end{aligned}$$

Centre manifold ODEs   The system evolves on the invariant manifold such that the parameters evolve according to these ODEs that characterise how the modulation of the oscillations evolve due to their nonlinear interaction.

$$\begin{aligned}
\dot{s}_1 = \ &s_4 s_3 s_1 \varepsilon^2\big(-0.03089i + 0.05032\big) + s_3 s_2 \varepsilon\big(-0.08991i - 0.03816\big) + \\
&s_2 s_1^2 \varepsilon^2\big(-0.01837i - 0.1095\big) + s_1 \delta\varepsilon^2\big(0.1526i - 0.3596\big)
\end{aligned}$$

$$\begin{aligned}
\dot{s}_2 = \ &s_4 s_3 s_2 \varepsilon^2\big(0.03089i + 0.05032\big) + s_4 s_1 \varepsilon\big(0.08991i - 0.03816\big) + \\
&s_2^2 s_1 \varepsilon^2\big(0.01837i - 0.1095\big) + s_2 \delta\varepsilon^2\big(-0.1526i - 0.3596\big)
\end{aligned}$$

$$\begin{aligned}
\dot{s}_3 = \ &s_4 s_3^2 \varepsilon^2\big(-0.0349i - 0.04111\big) + s_3 s_2 s_1 \varepsilon^2\big(-0.2499i - \\
&0.2153\big) + s_3 \delta\varepsilon^2\big(0.8376i + 0.9867\big) + s_1^2 \varepsilon\big(-0.4934i + 0.4188\big)
\end{aligned}$$

$$\begin{aligned}
\dot{s}_4 = \ &s_4^2 s_3 \varepsilon^2\big(0.0349i - 0.04111\big) + s_4 s_2 s_1 \varepsilon^2\big(0.2499i - 0.2153\big) + \\
&s_4 \delta\varepsilon^2\big(-0.8376i + 0.9867\big) + s_2^2 \varepsilon\big(0.4934i + 0.4188\big)
\end{aligned}$$

1.3   `metastable4`: Metastability in a four state Markov chain

Variable ϵ characterises the rate of exchange between metastable states u_1 and u_4 in this system ([Roberts 2015](), Exercise 5.1):

$$\dot{u}_1 = +u_2 - \epsilon u_1\,,$$
$$\dot{u}_2 = -u_2 + \epsilon(u_3 - u_2 + u_1),$$
$$\dot{u}_3 = -u_3 + \epsilon(u_4 - u_3 + u_2),$$
$$\dot{u}_4 = +u_3 - \epsilon u_4\,.$$

Start by loading the procedure.

```
21 in_tex "../invariantManifold.tex"$
```

Execute the construction of the slow manifold for this system. The explicit parameter `small`, math-name ε, gets replaced by `small^2` in the code, so in effect $\varepsilon^2 = \epsilon$.

```
22 invariantmanifold(
23     mat(( u2-small*u1,
24           -u2+small*(u1-u2+u3),
25           -u3+small*(u2-u3+u4),
26            u3-small*u4 )),
27     mat((0,0)),
28     mat((1,0,0,0),(0,0,0,1)),
29     mat((1,1,0,0),(0,0,1,1)),
30     6 )$
31 end;
```

The matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, of the linearisation about $\varepsilon = 0$, has eigenvalues 0 and -1 (both multiplicity two). We seek the slow manifold so specify the two zero eigenvalues in the second parameter to the procedure. Corresponding eigenvectors are $\vec{e}_1 = (1, 0, 0, 0)$ and $\vec{e}_2 = (0, 0, 0, 1)$. Choosing corresponding left-vector (here not an eigenvector) is $\vec{z}_1 = (1, 1, 0, 0)$ and $\vec{z}_2 = (0, 0, 1, 1)$ means that the slow manifold parameters s_1, s_2 have the physical meaning, respectively, of being the probability that the system is in states $\{1, 2\}$ and $\{3, 4\}$. The last parameter, 6, specifies to construct the slow manifold to errors $\mathcal{O}(\varepsilon^6)$, that is, errors $\mathcal{O}(\epsilon^3)$.

The slow manifold   The constructed slow manifold is, in terms of the lumped-state probability parameters s_1, s_2 (to error $\mathcal{O}(\varepsilon^2)$, and reverse ordering!),

$$u_1 = \varepsilon^4\big(-s_2 + 2s_1\big) - \varepsilon^2 s_1 + s_1\,, \quad u_3 = \varepsilon^4\big(-2s_2 + s_1\big) + \varepsilon^2 s_2\,,$$
$$u_2 = \varepsilon^4\big(s_2 - 2s_1\big) + \varepsilon^2 s_1\,, \qquad\qquad u_4 = \varepsilon^4\big(2s_2 - s_1\big) - \varepsilon^2 s_2 + s_2\,.$$

Slow manifold ODEs   On this slow manifold the evolution of the lumped-state probabilities is

$$\dot{s}_1 = \varepsilon^4\big(s_2 - s_1\big), \quad \dot{s}_2 = \varepsilon^4\big(-s_2 + s_1\big).$$

Hence here the long-term evolution is that on a time-scale of $\mathcal{O}(1/\epsilon^2)$, $\mathcal{O}(1/\epsilon^4)$, the system equilibrates between the two lumped states, that is, between $\{1,2\}$ and $\{3,4\}$.

Normals to isochrons at the slow manifold   To project initial conditions onto the slow manifold, or non-autonomous forcing, or modifications of the original system, or to quantify uncertainty (Roberts 1989, 2000), use the projection defined by the derived vector

$$\vec{z}_1 = \begin{bmatrix} z_{11} \\ z_{12} \\ z_{13} \\ z_{14} \end{bmatrix} = \begin{bmatrix} \varepsilon^4 + 1 \\ 4\varepsilon^4 - \varepsilon^2 + 1 \\ -4\varepsilon^4 + \varepsilon^2 \\ -\varepsilon^4 \end{bmatrix}, \quad \vec{z}_2 = \begin{bmatrix} z_{21} \\ z_{22} \\ z_{23} \\ z_{24} \end{bmatrix} = \begin{bmatrix} -\varepsilon^4 \\ -4\varepsilon^4 + \varepsilon^2 \\ 4\varepsilon^4 - \varepsilon^2 + 1 \\ \varepsilon^4 + 1 \end{bmatrix}.$$

Evaluate all these at $\varepsilon^2 = \epsilon$ to apply to the original specified system.

1.4 `nonlinNormModes`: Interaction of nonlinear normal modes

Renson et al. (2012) explored finite element construction of the nonlinear normal modes of a pair of coupled oscillators. Defining two new variables one of their example systems is

$$\dot{x}_1 = x_3, \qquad \dot{x}_3 = -2x_1 + x_2 - \frac{1}{2}x_1^3 + \frac{3}{10}(-x_3 + x_4),$$

$$\dot{x}_2 = x_4, \qquad \dot{x}_4 = x_1 - 2x_2 + \frac{3}{10}(x_3 - 2x_4).$$

The linearisation of this system at the origin has modes with frequencies $\omega = 1, \sqrt{3}$, and corresponding eigenvectors?? $(1, \mp i/\omega)e^{\pm i\omega t}$. Corresponding eigenvectors of the adjoint are $(1, \mp i\omega)e^{\pm i\omega t}$. We model the nonlinear interaction of these four modes over long times.

Here, the analysis constructs a full state space coordinate transformation. We find a mapping from the modulation variables $\vec{s} = (s_1, s_2, s_3, s_4)$ to the original variables $\vec{u} = (u_1, u_2, u_3, u_4)$, and find the corresponding evolution of $\vec{s}$. The modulation variables $\vec{s}$ are 'slow' because the coordinate transform uses time-dependent (rotating) basis vectors that account for the fast oscillation in $\vec{u}$. Hence the new variables $\vec{s}$ are good variables for making long-term predictions and forming understanding.

Start by loading the procedure.

```
32 in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, and the complex exponential.

```
33 factor s,exp;
```

The following code makes the linear damping to be effectively small (which then makes it `small` squared); consequently, also scale the smallness of the cubic nonlinearity to match.

```
34 invariantmanifold(
35     mat(( u3,
36            u4,
37            -2*u1 +u2 -small*u1^3/2 +small*3/10*(-u3+u4),
38            u1  -2*u2 +small*3/10*(u3 -2*u4) )),
39     mat(( i,-i,sqrt(3)*i,-sqrt(3)*i )),
40     mat( (1,1,+i,+i), (1,1,-i,-i),
41            (1,-1,i*sqrt(3),-i*sqrt(3)),
42            (1,-1,-i*sqrt(3),i*sqrt(3)) ),
43     mat( (1,1,+i,+i), (1,1,-i,-i),
44            (-i*sqrt(3),+i*sqrt(3),1,-1),
45            (+i*sqrt(3),-i*sqrt(3),1,-1) ),
46     3 )$
47 end;
```

The square root eigenvalues do not cause any trouble (although one may need to reformat the LaTeX of the cis operator). In the model, observe that $s_1 = s_2 = 0$ is invariant, as is $s_3 = s_4 = 0$. These are the nonlinear normal modes.

The procedure actually analyses the embedding system

$$\dot{u}_1 = u_3, \quad \dot{u}_3 = \varepsilon^2\big(-1/2u_1^3 - 3/10u_3 + 3/10u_4\big) - 2u_1 + u_2,$$
$$\dot{u}_2 = u_4, \quad \dot{u}_4 = \varepsilon^2\big(3/10u_3 - 3/5u_4\big) + u_1 - 2u_2.$$

The invariant manifold Here these give the reparametrisation of the state space $\vec{u}$ in terms of parameters s_j, via rotating basis vectors. Here, the coordinate transform is very complicated so I do not give the complexity. The leading approximation is, of course, the linear, errors $\mathcal{O}\big(\varepsilon^2\big)$,

$$u_1 = \mathrm{e}^{-\sqrt{3}it}s_4 + \mathrm{e}^{-it}s_2 + \mathrm{e}^{\sqrt{3}it}s_3 + \mathrm{e}^{it}s_1$$
$$u_2 = -\mathrm{e}^{-\sqrt{3}it}s_4 + \mathrm{e}^{-it}s_2 - \mathrm{e}^{\sqrt{3}it}s_3 + \mathrm{e}^{it}s_1$$
$$u_3 = -\sqrt{3}\,\mathrm{e}^{-\sqrt{3}it}s_4i - \mathrm{e}^{-it}s_2i + \sqrt{3}\,\mathrm{e}^{\sqrt{3}it}s_3i + \mathrm{e}^{it}s_1i$$
$$u_4 = \sqrt{3}\,\mathrm{e}^{-\sqrt{3}it}s_4i - \mathrm{e}^{-it}s_2i - \sqrt{3}\,\mathrm{e}^{\sqrt{3}it}s_3i + \mathrm{e}^{it}s_1i$$

Invariant manifold ODEs The system evolves according to these ODEs that characterise how the modulation of the oscillations evolve in state space due to their nonlinear interaction.

$$\dot{s}_1 = 3/4s_4s_3s_1\varepsilon^2i + 3/8s_2s_1^2\varepsilon^2i - 3/40s_1\varepsilon^2$$
$$\dot{s}_2 = -3/4s_4s_3s_2\varepsilon^2i - 3/8s_2^2s_1\varepsilon^2i - 3/40s_2\varepsilon^2$$
$$\dot{s}_3 = 1/8\sqrt{3}s_4s_3^2\varepsilon^2i + 1/4\sqrt{3}s_3s_2s_1\varepsilon^2i - 3/8s_3\varepsilon^2$$
$$\dot{s}_4 = -1/8\sqrt{3}s_4^2s_3\varepsilon^2i - 1/4\sqrt{3}s_4s_2s_1\varepsilon^2i - 3/8s_4\varepsilon^2$$

Here one can see that each oscillation decays, with a frequency shift due to a combination of nonlinear interaction and nonlinear self-interaction.

1.5 `stable3d`: Stable manifold of a basic 3D system

Let's revisit the example of Section 1.1, namely

$$\dot{u}_1 = 2u_1 + u_2 + 2u_3 + u_2u_3\,,$$
$$\dot{u}_2 = u_1 - u_2 + u_3 - u_1u_3\,,$$
$$\dot{u}_3 = -3u_1 - u_2 - 3u_3 - u_1u_2\,,$$

but here construct its 2D stable manifold.

Start by loading the procedure.

```
48 in_tex "../invariantManifold.tex"$
```

Execute the construction of the stable manifold for this system.

```
49 invariantmanifold(
50     mat(( 2*u1+u2+2*u3+u2*u3,
51           u1-u2+u3-u1*u3,
52           -3*u1-u2-3*u3-u1*u2 )),
53     mat(( -1,-1 )),
54     mat( (1,-1,-1),(0.4,1.4,-1) ),
55     mat( (1,0,1),(1,0,-1) ),
56     3 )$
57 end;
```

The matrix $\begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 1 \\ -3 & -1 & -3 \end{bmatrix}$ of the linearisation about the origin has eigenvalues 0 and -1 (multiplicity two). We seek the 2D stable manifold so specify the eigenvalue -1, twice, in the second parameter to the procedure. A corresponding eigenvector is $\vec{e}_1 = (1, -1, -1)$, and corresponding left-eigenvector is $\vec{z}_2 = (1, 0, 1)$, as specified above. We need two basis eigenvectors, but here there is only one because the other is a generalised eigenvector. We must do more work to find a generalised eigenvector is $\vec{e}_2 = (0.4, 1.4, -1)$, and a generalised left-eigenvector is $\vec{z}_2 = (1, 0, -1)$. The last parameter, 3, specifies to construct the stable manifold to errors $\mathcal{O}(\varepsilon^3)$.

Because of the generalised eigenvector, the procedure modifies the *linear* terms to a more convenient form (not necessary, just *convenient*)—see the warning in its report. So, the procedure actually analyses the embedding system

$$\dot{u}_1 = \varepsilon\big(-u_1 + u_2u_3 - u_3\big) + 3u_1 + u_2 + 3u_3\,,$$
$$\dot{u}_2 = \varepsilon\big(-u_1u_3 + u_1 + u_3\big) - u_2\,,$$
$$\dot{u}_3 = \varepsilon\big(-u_1u_2 + u_1 + u_3\big) - 4u_1 - u_2 - 4u_3\,.$$

The stable manifold The constructed stable manifold is, in terms of the parameters s_1, s_2 (to error $\mathcal{O}(\varepsilon^2)$, and reverse ordering!, and in terms of the ugly $e^{iti} = e^{-t}$ which I aim to improve soon!),

$$u_1 = \varepsilon\big(-51/25\,e^{-2t}s_2^2 - 6/5\,e^{-2t}s_2s_1 + 3\,e^{-2t}s_1^2\big) + 2/5\,e^{-t}s_2 + e^{-t}s_1\,,$$
$$u_2 = \varepsilon\big(-2/5\,e^{-2t}s_2^2 - 7/5\,e^{-2t}s_2s_1 - e^{-2t}s_1^2\big) + 7/5\,e^{-t}s_2 - e^{-t}s_1\,,$$
$$u_3 = \varepsilon\big(4\,e^{-2t}s_2^2 + 13/5\,e^{-2t}s_2s_1 - 5\,e^{-2t}s_1^2\big) - e^{-t}s_2 - e^{-t}s_1\,.$$

Observe the linear terms in $\vec{s}$ all have e^{-t}, and the quadratic terms in $\vec{s}$ all have e^{-2t}. Consequently, we could in principle write the stable manifold in terms of, say, the variables $x_j = s_j e^{-t}$ giving

$$u_1 = \varepsilon\left(-51/25x_2^2 - 6/5x_2x_1 + 3x_1^2\right) + 2/5x_2 + x_1\,,$$
$$u_2 = \varepsilon\left(-2/5x_2^2 - 7/5x_2x_1 - x_1^2\right) + 7/5x_2 - x_1\,,$$
$$u_3 = \varepsilon\left(4x_2^2 + 13/5x_2x_1 - 5x_1^2\right) - x_2 - x_1\,.$$

This would be a more usual parametrisation. But here let's remain with $\vec{s}$ and remember to interpret $\vec{s}$ as modifying the exponential decay e^{-t} on this stable manifold.

Stable manifold ODEs   On the stable manifold the evolution is

$$\dot{s}_1 = 3/5\varepsilon s_2\,, \quad \dot{s}_2 = 0\,.$$

So, s_2 is constant, and hence s_1 increases linearly. But such increase only modifies slightly the robust exponential decay, e^{-t}, on the stable manifold.

In terms of $\vec{x}$ this evolution is $\dot{x}_1 = -x_1 + \frac{3}{5}\varepsilon x_2$, $\dot{x}_2 = -x_2$.

2 Slow invariant manifolds

2.1 `simple2d`: Slow manifold of a simple 2D system

The example system to analyse is specified to be

$$\dot{u}_1 = -u_1 + u_2 - u_1^2 \,, \quad \dot{u}_2 = u_1 - u_2 + u_2^2 \,.$$

Start by loading the procedure.

```
58 in_tex "../invariantManifold.tex"$
```

Execute the construction of the slow manifold for this system.

```
59 invariantmanifold(
60     mat((-u1+u2-u1^2,u1-u2+u2^2)),
61     mat((0)),
62     mat((1,1)),
63     mat((1,1)),
64     5)$
65 end;
```

We seek the slow manifold so specify the eigenvalue zero. From the linearisation matrix $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ a corresponding eigenvector is $\vec{e} = (1,1)$, and corresponding left-eigenvector is $\vec{z} = \vec{e} = (1,1)$, as specified. The last parameter specifies to construct the slow manifold to errors $\mathcal{O}(\varepsilon^5)$.

The procedure actually analyses the embedding system

$$\dot{u}_1 = -u_1 + u_2 - \varepsilon u_1^2 \,, \quad \dot{u}_2 = u_1 - u_2 + \varepsilon u_2^2 \,.$$

So here the artificial parameter ε has a physical interpretation in that it counts the nonlinearity: a term in ε^p will be a $(p+1)$th order term in $\vec{u} = (u_1, u_2)$. Hence the specified error $\mathcal{O}(\varepsilon^5)$ is here the same as error $\mathcal{O}(|\vec{u}|^6)$.

The slow manifold   The constructed slow manifold is, in terms of the parameter s_1 (and reverse ordering!),

$$u_1 = 3/8\varepsilon^3 s_1^4 - 1/2\varepsilon s_1^2 + s_1 \,,$$
$$u_2 = -3/8\varepsilon^3 s_1^4 + 1/2\varepsilon s_1^2 + s_1 \,.$$

Slow manifold ODEs   On this slow manifold the evolution is

$$\dot{s}_1 = -3/4\varepsilon^4 s_1^5 + \varepsilon^2 s_1^3 \,:$$

here the leading term in s_1^3 indicates the origin is unstable.

Normals to isochrons at the slow manifold   To project initial conditions onto the slow manifold, or non-autonomous forcing, or modifications of the original system, or to quantify uncertainty, use the projection defined by the derived vector

$$\vec{z}_1 = \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} 3/2\varepsilon^4 s_1^4 + 3/4\varepsilon^3 s_1^3 - 1/2\varepsilon^2 s_1^2 - 1/2\varepsilon s_1 + 1/2 \\ 3/2\varepsilon^4 s_1^4 - 3/4\varepsilon^3 s_1^3 - 1/2\varepsilon^2 s_1^2 + 1/2\varepsilon s_1 + 1/2 \end{bmatrix}.$$

Evaluate these at $\varepsilon = 1$ to apply to the original specified system, or here just interpret ε as a way to count the order of each term.

2.2   `lorenz86sm`: **Slow manifold of the Lorenz 1986 atmosphere model**

In this case we construct the slow sub-centre manifold, analogous to quasi-geostrophy, in order to disentangle the slow dynamics from fast oscillations, analogous to gravity waves, in the Lorenz (1986) model. The normals to the isochrons determine 'balancing' onto the slow manifold.

$$\begin{aligned}
\dot{u}_1 &= bu_2u_5 - u_2u_3\,, \\
\dot{u}_2 &= -bu_1u_5 + u_1u_3\,, \\
\dot{u}_3 &= -u_1u_2\,, \\
\dot{u}_4 &= -u_5\,, \\
\dot{u}_5 &= bu_1u_2 + u_4\,.
\end{aligned}$$

The parameter b controls the interaction between slow and fast waves. Section 3.3 constructs its full state space normal form in order to determine the forcing of the slow modes by the mean fast waves.

Start by loading the procedure.

```
66 in_tex "../invariantManifold.tex"$
```

Group output expressions on b.

```
67 factor b;
```

Execute the construction of the slow manifold for this system.

```
68 invariantmanifold(
69     mat(( -u2*u3+b*u2*u5,
70         u1*u3-b*u1*u5,
71         -u1*u2,
72         -u5,
73         +u4+b*u1*u2 )),
74     mat(( 0,0,0 )),
75     mat( (1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0) ),
76     mat( (1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0) ),
77     4 )$
78 end;
```

The matrix of the linearisation about the origin has eigenvalues zero (multiplicity three) and $\pm i$. We seek the slow manifold so specify the eigenvalue zero (thrice) in the second parameter to the procedure. Since the system is already in linearly separated form, the slow eigenvectors are simply the three given unit vectors. The last parameter, 4, specifies to construct the slow manifold to errors $\mathcal{O}(\varepsilon^4)$, that is, to errors $\mathcal{O}(|\vec{s}|^5)$.

The procedure actually analyses the embedding system

$$
\begin{aligned}
\dot{u}_1 &= b\varepsilon u_2 u_5 - \varepsilon u_2 u_3 \,, \\
\dot{u}_2 &= -b\varepsilon u_1 u_5 + \varepsilon u_1 u_3 \,, \\
\dot{u}_3 &= -\varepsilon u_1 u_2 \,, \\
\dot{u}_4 &= -u_5 \,, \\
\dot{u}_5 &= b\varepsilon u_1 u_2 + u_4 \,.
\end{aligned}
$$

Consequently, here the artificial parameter ε has a physical interpretation in that it counts the nonlinearity: a term in ε^p will be a $(p+1)$th order term in $\vec{s}$.

The slow manifold The constructed slow manifold is, in terms of the parameters $\vec{s}$ (to errors $\mathcal{O}(\varepsilon^3)$, and reverse ordering!),

$$
\begin{aligned}
u_1 &= s_1 \,, \\
u_2 &= s_2 \,, \\
u_3 &= s_3 \,, \\
u_4 &= -b\varepsilon s_2 s_1 \,, \\
u_5 &= b\varepsilon^2\big(-s_3 s_2^2 + s_3 s_1^2\big).
\end{aligned}
$$

Slow manifold ODEs On this slow manifold the evolution is

$$
\begin{aligned}
\dot{s}_1 &= b^2\varepsilon^3\big(-s_3 s_2^3 + s_3 s_2 s_1^2\big) - \varepsilon s_3 s_2 \,, \\
\dot{s}_2 &= b^2\varepsilon^3\big(s_3 s_2^2 s_1 - s_3 s_1^3\big) + \varepsilon s_3 s_1 \,, \\
\dot{s}_3 &= -\varepsilon s_2 s_1 \,.
\end{aligned}
$$

Here the quadratic terms in s_1, s_2, s_3 is that of nonlinear slow wave oscillations. The b-terms modify these slow waves, reflecting the influence of the fast dynamics (as distinct from the effects of fast waves—these effects are quantified by Section 3.3).

Normals to isochrons at the slow manifold To project initial conditions onto the slow manifold, or non-autonomous forcing, or modifications of the original system, or to quantify uncertainty (Roberts 1989, 2000), use the projection defined by the derived

vectors

$$\vec{z}_1 = \begin{bmatrix} b^2\varepsilon^2 s_2^2 + 1 \\ b^2\varepsilon^2 s_2 s_1 \\ 0 \\ b^3\varepsilon^3\big(s_2^3 - s_2 s_1^2\big) + b\varepsilon^3\big(-s_2^3 + s_2 s_1^2\big) + b\varepsilon s_2 \\ 0 \end{bmatrix},$$

$$\vec{z}_2 = \begin{bmatrix} -b^2\varepsilon^2 s_2 s_1 \\ -b^2\varepsilon^2 s_1^2 + 1 \\ 0 \\ b^3\varepsilon^3\big(-s_2^2 s_1 + s_1^3\big) + b\varepsilon^3\big(s_2^2 s_1 - s_1^3\big) - b\varepsilon s_1 \\ 0 \end{bmatrix},$$

$$\vec{z}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4b\varepsilon^3 s_3 s_2 s_1 \\ b\varepsilon^2\big(-s_2^2 + s_1^2\big) \end{bmatrix}.$$

Evaluate these at $\varepsilon = 1$ to apply to the original specified system, or here just interpret ε as a way to count the order of each term.

3 Oscillation on an invariant manifold

3.1 `simpleosc`: Oscillatory centre manifold—separated form

Let's try complex eigenvectors. Adjoint eigenvectors `zz_` must be the eigenvectors of the complex conjugate transpose matrix.

$$\dot{u}_1 = u_2$$
$$\dot{u}_2 = -\varepsilon u_3 u_1 - u_1$$
$$\dot{u}_3 = 5\varepsilon u_1^2 - u_3$$

Start by loading the procedure.

```
79 in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j and the complex exponential

```
80 factor s,exp;
```

Execute the construction of the centre manifold for this system.

```
81 invariantmanifold(
82     mat((u2,-u1-u1*u3,-u3+5*u1^2)),
83     mat((i,-i)),
84     mat((1,+i,0),(1,-i,0)),
85     mat((1,+i,0),(1,-i,0)),
86     3)$
87 end;
```

The centre manifold   These give the location of the invariant manifold in terms of parameters s_j.

$$u_1 = \mathrm{e}^{-it}s_2 + \mathrm{e}^{it}s_1$$
$$u_2 = -\mathrm{e}^{-it}s_2 i + \mathrm{e}^{it}s_1 i$$
$$u_3 = \mathrm{e}^{-2it}s_2^2\varepsilon\big(2i+1\big) + \mathrm{e}^{2it}s_1^2\varepsilon\big(-2i+1\big) + 10s_2 s_1\varepsilon$$

Centre manifold ODEs   The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\dot{s}_1 = s_2 s_1^2 \varepsilon^2\big(11/2i+1\big)$$
$$\dot{s}_2 = s_2^2 s_1 \varepsilon^2\big(-11/2i+1\big)$$

3.2 `quasidde`: Quasi delay DE with Hopf bifurcation

Shows Hopf bifurcation as parameter α crosses 0 to oscillations with base frequency two.

$$\dot{u}_1 = -\alpha\varepsilon^2 u_3 - \varepsilon^2 u_1^3 - 2\varepsilon u_1^2 - 4u_3$$
$$\dot{u}_2 = 2u_1 - 2u_2$$
$$\dot{u}_3 = 2u_2 - 2u_3$$

for small parameter α. We code the parameter α as 'small', and observe it is consequently considered as 'small squared' because all nonlinear terms and already 'small' terms, are multiplied by another `small`.

Start by loading the procedure.

```
88  in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameter α.

```
89  factor s,exp,alpha;
```

Execute the construction of the slow manifold for this system (ignore the warning messages about `u1` declared, and then already defined, as an operator).

```
90  invariantmanifold(
91      mat(( -4*u3-small*alpha*u3-2*u1^2-small*u1^3,
92          2*u1-2*u2,
93          2*u2-2*u3 )),
94      mat((2*i,-2*i)),
95      mat((1,1/2-i/2,-i/2),(1,1/2+i/2,+i/2)),
96      mat((1,-i,-1-i),(1,+i,-1+i)),
97      3)$
98  end;
```

The centre manifold   These give the location of the invariant manifold in terms of parameters s_1, s_2 (complex conjugate for real solutions).

$$u_1 = \mathrm{e}^{-4it}s_2^2\varepsilon\big(-7/12i + 1/12\big) + \mathrm{e}^{-2it}s_2 + \mathrm{e}^{4it}s_1^2\varepsilon\big(7/12i + 1/12\big) + \mathrm{e}^{2it}s_1 - s_2 s_1 \varepsilon$$

$$u_2 = \mathrm{e}^{-4it}s_2^2\varepsilon\big(-1/12i + 1/4\big) + \mathrm{e}^{-2it}s_2\big(1/2i + 1/2\big) + \mathrm{e}^{4it}s_1^2\varepsilon\big(1/12i + 1/4\big) + \mathrm{e}^{2it}s_1\big(-1/2i + 1/2\big) - s_2 s_1 \varepsilon$$

$$u_3 = \mathrm{e}^{-4it}s_2^2\varepsilon\big(1/12i + 1/12\big) + 1/2\,\mathrm{e}^{-2it}s_2 i + \mathrm{e}^{4it}s_1^2\varepsilon\big(-1/12i + 1/12\big) - 1/2\,\mathrm{e}^{2it}s_1 i - s_2 s_1 \varepsilon$$

Centre manifold ODEs   The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\dot{s}_1 = s_2 s_1^2 \varepsilon^2\big(-16/15i - 1/5\big) + s_1 \alpha \varepsilon^2\big(1/5i + 1/10\big)$$
$$\dot{s}_2 = s_2^2 s_1 \varepsilon^2\big(16/15i - 1/5\big) + s_2 \alpha \varepsilon^2\big(-1/5i + 1/10\big)$$

Hence there is a supercritical Hopf bifurcation as parameter α increases through zero.

3.3 `lorenz86nf`: Paradoxically justify a slow manifold despite being proven to not exist

Lorenz (1986) proposed the following simple system in order to understand aspects of the quasi-geostrophic approximation in at-

mospheric dynamics.

$$\dot{u}_1 = bu_2u_5 - u_2u_3 \,,$$
$$\dot{u}_2 = -bu_1u_5 + u_1u_3 \,,$$
$$\dot{u}_3 = -u_1u_2 \,,$$
$$\dot{u}_4 = -u_5 \,,$$
$$\dot{u}_5 = bu_1u_2 + u_4 \,.$$

The parameter b controls the interaction between slow and fast dynamics. As in Section 2.2, it appears that a slow manifold of quasi-geostrophy exists and is constructible. Nonetheless, Lorenz & Krishnamurthy (1987) proved that a slow manifold cannot exist for this system!

A resolution of this apparent paradox comes via backwards theory (Roberts 2019, §2.5). There are systems exponentially close to the above Lorenz86 system (that is, asymptotically the same to all orders in $|\vec{u}|$) which do possess a slow manifold. Hence the properties that cause the non-existence are exponentially small, they are beyond all orders, and so are likely to be physically irrelevant— they are likely to be smaller than the mathematical modelling errors of the original system.

Let's see this resolution by constructing, to any specified order, a system that has a slow manifold and is close to the Lorenz86 system. We do this by constructing a coordinate transform of the 5D state space. Start by loading the procedure.

```
99 in_tex "../invariantManifold.tex"$
```

Group output expressions on b.

```
100 factor b;
```

Execute the construction of the coordinate transform for this system.

```
101 invariantmanifold(
102     mat(( -u2*u3+b*u2*u5,
103         u1*u3-b*u1*u5,
104         -u1*u2,
105         -u5,
106         +u4+b*u1*u2 )),
107     mat(( 0,0,0,i,-i )),
108     mat( (1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),
109         (0,0,0,1,-i), (0,0,0,1,+i) ),
110     mat( (1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),
111         (0,0,0,1,-i), (0,0,0,1,+i) ),
112     4 )$
113 end;
```

The matrix of the linearisation about the origin has eigenvalues zero (multiplicity three) and $\pm i$, as specified for the eigenvalues in

the second parameter to the procedure. Corresponding eigenvectors are simply the three unit vectors and the two complex eigenvectors of the fast waves. The last parameter, 4, specifies to construct the slow manifold to errors $\mathcal{O}(\varepsilon^4)$, that is, to errors $\mathcal{O}(|\vec{s}|^5)$.

The procedure actually analyses the embedding system

$$
\begin{aligned}
\dot{u}_1 &= b\varepsilon u_2 u_5 - \varepsilon u_2 u_3 \,, \\
\dot{u}_2 &= -b\varepsilon u_1 u_5 + \varepsilon u_1 u_3 \,, \\
\dot{u}_3 &= -\varepsilon u_1 u_2 \,, \\
\dot{u}_4 &= -u_5 \,, \\
\dot{u}_5 &= b\varepsilon u_1 u_2 + u_4 \,.
\end{aligned}
$$

The coordinate transform   The constructed coordinate transform is, in terms of the slow variables $\vec{s}$ and a time-dependent basis (to errors $\mathcal{O}(\varepsilon^3)$, and reverse ordering!),

$$
\begin{aligned}
u_1 &= b^2\varepsilon^2\big(-1/2\,\mathrm{e}^{-2it}s_5^2 s_1 - 1/2\,\mathrm{e}^{2it}s_4^2 s_1\big) + b\varepsilon\big(-\mathrm{e}^{-it}s_5 s_2 - \\
&\quad \mathrm{e}^{it}s_4 s_2\big) + s_1 \,, \\
u_2 &= b^2\varepsilon^2\big(-1/2\,\mathrm{e}^{-2it}s_5^2 s_2 - 1/2\,\mathrm{e}^{2it}s_4^2 s_2\big) + b\varepsilon\big(\mathrm{e}^{-it}s_5 s_1 + \\
&\quad \mathrm{e}^{it}s_4 s_1\big) + s_2 \,, \\
u_3 &= b\varepsilon^2\big(\mathrm{e}^{-it}s_5 s_2^2 i - \mathrm{e}^{-it}s_5 s_1^2 i - \mathrm{e}^{it}s_4 s_2^2 i + \mathrm{e}^{it}s_4 s_1^2 i\big) + s_3 \,, \\
u_4 &= b^2\varepsilon^2\big(1/4\,\mathrm{e}^{-it}s_5 s_2^2 - 1/4\,\mathrm{e}^{-it}s_5 s_1^2 + 1/4\,\mathrm{e}^{it}s_4 s_2^2 - \\
&\quad 1/4\,\mathrm{e}^{it}s_4 s_1^2\big) - b\varepsilon s_2 s_1 + \mathrm{e}^{-it}s_5 + \mathrm{e}^{it}s_4 \,, \\
u_5 &= b^2\varepsilon^2\big(-1/4\,\mathrm{e}^{-it}s_5 s_2^2 i + 1/4\,\mathrm{e}^{-it}s_5 s_1^2 i + 1/4\,\mathrm{e}^{it}s_4 s_2^2 i - \\
&\quad 1/4\,\mathrm{e}^{it}s_4 s_1^2 i\big) + b\varepsilon^2\big(-s_3 s_2^2 + s_3 s_1^2\big) + \mathrm{e}^{-it}s_5 i - \mathrm{e}^{it}s_4 i \,.
\end{aligned}
$$

Transformed ODEs   In the variables $\vec{s}$ the evolution is

$$
\begin{aligned}
\dot{s}_1 &= b^2\varepsilon^3\big(-s_3 s_2^3 + s_3 s_2 s_1^2\big) - \varepsilon s_3 s_2 \,, \\
\dot{s}_2 &= b^2\varepsilon^3\big(s_3 s_2^2 s_1 - s_3 s_1^3\big) + \varepsilon s_3 s_1 \,, \\
\dot{s}_3 &= {\color{red}2b^2\varepsilon^3 s_5 s_4 s_2 s_1} - \varepsilon s_2 s_1 \,, \\
\dot{s}_4 &= b^2\varepsilon^2\big(-1/2 s_4 s_2^2 i + 1/2 s_4 s_1^2 i\big) \,, \\
\dot{s}_5 &= b^2\varepsilon^2\big(1/2 s_5 s_2^2 i - 1/2 s_5 s_1^2 i\big) \,.
\end{aligned}
$$

When $s_4 = s_5 = 0$ we recover precisely the same slow manifold as constructed by Section 2.2. Hence the above system of $\vec{u} = \cdots$ and $\dot{\vec{s}} = \cdots$ together both has a slow manifold, and is $\mathcal{O}(|\vec{s}|^5)$ close to the original Lorenz86 system. Such construction can proceed to any order, and so the above closeness of a system with a slow manifold holds to all orders in $|\vec{s}|$.

Also of interest is the red term in the $\dot{s}_3$ ODE: it shows that the evolution of the slow variables, s_1, s_2, s_3, is affected by the presence of fast waves, s_4, s_5 non-zero. That is, the evolution on and off the slow manifold differ by this term (and similar higher-order terms). Users of slow models among fast waves need to be aware of this physical feature.

3.4   `stoleriu2`: **Oscillatory centre manifold among stable and unstable modes**

Consider the case Stoleriu (2012) calls $(3\pi/4, k^2/2)$.

$$\dot{u}_1 = u_2,$$
$$\dot{u}_2 = -\sigma u_3 + 1 - \cos u_1,$$
$$\dot{u}_3 = u_4,$$
$$\dot{u}_4 = (u_3 + \tfrac{1}{\sigma})\sin u_1$$

Eigenvalues are ± 1 and $\pm i$, so we find the centre manifold among stable and unstable modes.

Start by loading the procedure.

```
114 in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j and the complex exponential

```
115 factor s,exp;
```

Execute the construction of the centre manifold for Stoleriu's system. But use Taylor expansions for trigonometric functions in the ODEs, and multiply higher-orders of nonlinearity by `small` to better (not best) count and manage nonlinearities.

```
116 invariantmanifold(
117     mat(( u2,
118         sigma*u3+u1^2/2-small*u1^4/24,
119         u4,
120         (u3+1/sigma)*(u1-small*u1^3/6)
121         )),
122     mat(( i,-i )),
123     mat( (sigma,i*sigma,-1,-i),(sigma,-i*sigma,-1,+i) ),
124     mat( (+i,-1,-i*sigma,sigma),(-i,-1,+i*sigma,sigma) ),
125     3)$
126 end;
```

Code adjoint eigenvectors `zz_` that are eigenvectors of the complex conjugate transpose matrix of the linear matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 1 \\ 1/\sigma & 0 & 0 & 0 \end{bmatrix}$. Here analyse to errors $\mathcal{O}(\varepsilon^3)$.

The procedure analyses the embedding system

$$\dot{u}_1 = u_2,$$
$$\dot{u}_2 = -1/24\varepsilon^2 u_1^4 + 1/2\varepsilon u_1^2 + \sigma u_3,$$
$$\dot{u}_3 = u_4,$$
$$\dot{u}_4 = \varepsilon^2\big(-1/6\sigma^{-1}u_1^3 - 1/6u_1^3 u_3\big) + \varepsilon u_1 u_3 + \sigma^{-1}u_1$$

The centre manifold   These give the location of the invariant manifold in terms of (complex conjugate) parameters s_1, s_2.

$$u_1 = \mathrm{e}^{-it}s_2\sigma - 1/5\,\mathrm{e}^{-2it}s_2^2\varepsilon\sigma^2 + \mathrm{e}^{it}s_1\sigma - 1/5\,\mathrm{e}^{2it}s_1^2\varepsilon\sigma^2 + 2s_2s_1\varepsilon\sigma^2$$
$$u_2 = -\,\mathrm{e}^{-it}s_2i\sigma + 2/5\,\mathrm{e}^{-2it}s_2^2\varepsilon i\sigma^2 + \mathrm{e}^{it}s_1i\sigma - 2/5\,\mathrm{e}^{2it}s_1^2\varepsilon i\sigma^2$$
$$u_3 = -\,\mathrm{e}^{-it}s_2 + 3/10\,\mathrm{e}^{-2it}s_2^2\varepsilon\sigma - \mathrm{e}^{it}s_1 + 3/10\,\mathrm{e}^{2it}s_1^2\varepsilon\sigma - s_2s_1\varepsilon\sigma$$
$$u_4 = \mathrm{e}^{-it}s_2i - 3/5\,\mathrm{e}^{-2it}s_2^2\varepsilon i\sigma - \mathrm{e}^{it}s_1i + 3/5\,\mathrm{e}^{2it}s_1^2\varepsilon i\sigma$$

Centre manifold ODEs   The system evolves on the centre manifold such that the parameters evolve according to these ODEs.

$$\dot{s}_1 = -6/5s_2s_1^2\varepsilon^2 i\sigma^2$$
$$\dot{s}_2 = 6/5s_2^2s_1\varepsilon^2 i\sigma^2$$

These establish that the leading effect of the nonlinearities is to cause a frequency down-shift in the oscillations on the centre manifold. Higher-order analysis indicates the only effect is a frequency shift of the nonlinear oscillations.

4   Invariant manifolds in delay DEs

4.1   `simple1dde`: Simple DDE with a Hopf bifurcation

Model a delayed 'logistic' system in one variable with

$$\frac{du}{dt} = -(1+a)[1+u(t)]u(t-\pi/2),$$

for small parameter a. We code the parameter a as 'small', and observe it is consequently considered as 'small squared' because all nonlinear terms and already 'small' terms are multiplied by `small`.

Start by loading the procedure.

```
127 in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameter a.

```
128 factor s,exp,a;
```

Execute the construction of the slow manifold for this system (ignore the warning messages about `u1` declared, and then already defined, as an operator).

```
129 invariantmanifold(
130     mat(( -(1+small*a)*(1+u1)*u1(pi/2) )),
131     mat((i,-i)),
132     mat((1),(1)),
133     mat((1),(1)),
134     3)$
135 end;
```

The marginal modes are $e^{\pm it}$ so nominate the frequencies ± 1. The eigenvectors are just $1 \cdot e^{\pm it}$. Because for delay differential equations the time dependence $e^{\pm i\omega t}$ is an integral part of the definition of the eigenvector; hence the coded eigenvectors can be the same, as here, because they are differentiated through the time dependence $e^{\pm i\omega t}$.

The code works for orders higher than cubic, but is slow: takes about a minute per iteration.

The procedure actually analyses the embedding system

$$\frac{du}{dt} = -[1+\varepsilon u(t)]u(t-\pi/2) - \varepsilon^2 a[1+u(t)]u(t-\pi/2).$$

The centre manifold   These give the location of the invariant manifold in terms of parameters s_j.

$$u_1 = e^{-it}s_2 + e^{-2it}s_2^2\varepsilon\big(1/5i + 2/5\big) + e^{it}s_1 + e^{2it}s_1^2\varepsilon\big(-1/5i + 2/5\big)$$

Centre manifold ODEs   The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\dot{s}_1 = s_2 s_1^2 \varepsilon^2 \big(- 2/5i\pi - 12/5i - 6/5\pi + 4/5 \big)/\big(\pi^2 + 4\big) + \\ s_1 a\varepsilon^2 \big(4i + 2\pi\big)/\big(\pi^2 + 4\big)$$

$$\dot{s}_2 = s_2^2 s_1 \varepsilon^2 \big(2/5i\pi + 12/5i - 6/5\pi + 4/5\big)/\big(\pi^2 + 4\big) + s_2 a\varepsilon^2 \big(- \\ 4i + 2\pi\big)/\big(\pi^2 + 4\big)$$

4.2   `logistic1dde`: **Logistic DDE displays a Hopf bifurcation**

Form a centre manifold for the delayed 'logistic' system in one variable, for delay $\tau = 3\pi/4$, with

$$\frac{du}{dt} = -u(t) - (\sqrt{2} + a)u(t - \tau) + \mu u(t - \tau)^2 + \nu u(t - \tau)^3,$$

for and nonlinearity parameters μ and ν, and small parameter a. Numerical computation of the spectrum indicates that the system has a Hopf bifurcation as parameter a crosses zero.

We code the parameter a as 'small', and observe it is consequently considered as 'small squared' because all nonlinear terms, and already 'small' terms, are multiplied by ε (`small`).

Start by loading the procedure.

```
136 in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameters.

```
137 factor s,exp,a,mu,nu;
```

Execute the construction of the slow manifold for this system (ignore the warning messages about `u1` declared, and then already defined, as an operator).

```
138 invariantmanifold(
139     mat(( -u1-(sqrt(2)+small*a)*u1(3*pi/4)
140     +mu*u1(3*pi/4)^2 +small*nu*u1(3*pi/4)^3 )),
141     mat((i,-i)),
142     mat((1),(1)),
143     mat((1),(1)),
144     3)$
145 end;
```

The marginal modes are $e^{\pm it}$ so nominate the frequencies ± 1. The eigenvectors are just $1 \cdot e^{\pm it}$. Because for delay differential equations the time dependence $e^{\pm i\omega t}$ is an integral part of the definition of the eigenvector; hence the coded eigenvectors can be the same, as here, because they are differentiated through the time dependence $e^{\pm i\omega t}$.

The procedure actually analyses the embedding system

$$\dot{u}_1 = -a\varepsilon^2 u_1(t-\tau) + \mu\varepsilon u_1(t-\tau)^2 + \nu\varepsilon^2 u_1(t-\tau)^3 - \sqrt{2}u_1(t-\tau) - u_1.$$

The centre manifold   These give the location of the invariant manifold in terms of parameters s_j.

$$u_1 = \mathrm{e}^{-it}s_2 + \mathrm{e}^{-2it}s_2^2\mu\varepsilon\big(-0.07901i + 0.2698\big) + \mathrm{e}^{it}s_1 + \\ \mathrm{e}^{2it}s_1^2\mu\varepsilon\big(0.07901i + 0.2698\big) + 0.8284s_2s_1\mu\varepsilon$$

Centre manifold ODEs   The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\dot{s}_1 = s_2s_1^2\mu^2\varepsilon^2\big(-0.1303i - 0.5209\big) + s_2s_1^2\nu\varepsilon^2\big(-0.1262i - \\ 0.7206\big) + s_1a\varepsilon^2\big(0.04205i + 0.2402\big)$$

$$\dot{s}_2 = s_2^2s_1\mu^2\varepsilon^2\big(0.1303i - 0.5209\big) + s_2^2s_1\nu\varepsilon^2\big(0.1262i - 0.7206\big) + \\ s_2a\varepsilon^2\big(-0.04205i + 0.2402\big)$$

Hence the centre manifold model predicts a supercritical Hopf bifurcation as parameter a increases through zero.

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