Center manifold reduction of SDEs: Some observations

Justin Eilertsen
Tony Roberts

June 2, 2023

Reduction/comparison

Consider the toy model,

$$\begin{pmatrix}\dot{u}_1\\ \dot{u}_2\end{pmatrix} = \begin{pmatrix}-a_1 & a_2\\ a_1 & -a_2\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} + \begin{pmatrix}0\\ -\varepsilon b_2 u_2\end{pmatrix} + \sigma\begin{pmatrix}g_1 w_1\\ g_2 w_2\end{pmatrix}. \tag{1}$$

According to P&R, in the absence of flow field (angular) variation, flow field curvature, and curvature of the critical manifold, the flow converges (a càdlàg) to the projected flow,

$$\begin{pmatrix}\dot{u}_1\\ \dot{u}_2\end{pmatrix} = \Pi\begin{pmatrix}0\\ -\varepsilon b_2 u_2\end{pmatrix}\Bigg|_{U\in E^c} + \sigma\Pi\begin{pmatrix}g_1 w_1\\ g_2 w_2\end{pmatrix}\Bigg|_{u\in E^c} \tag{2}$$

where E^c is the center subspace of the limiting Jacobian (the Jacobian of the layer problem)

$$DF(u) := J = \begin{pmatrix}-a_1 & a_2\\ a_1 & -a_2\end{pmatrix}. \tag{3}$$

In our case, we have the critical manifold

$$S_0 := \{(u_1,u_2)\in\mathbb{R}^2 : -a_1u_1 + a_2u_2 = 0\}, \tag{4}$$

or equivalently, the one-dimensional center subspace

$$E^c := span\begin{pmatrix}\dfrac{a_2}{a_1}\\ \\ 1\end{pmatrix} \in T\mathbb{R}^2. \tag{5}$$

Again, according to P&R, the projection matrix, Π, is given by

$$\Pi := \frac{1}{a_1+a_2}\cdot\begin{pmatrix}a_2 & a_2\\ a_1 & a_1,\end{pmatrix} \tag{6}$$

which is nothing more than the projection onto TS_0.

Remark 1. *In their manuscript, P&R used the notation $\Pi := I - J^+J$ and referred to the pseudo inverse. Careful reading asserts that they are not referring to the Moore-Penrose pseudoinverse. Hence, the notation J^+ does imply the Moore-Penrose pseudo inverse. Recall that $I - J^+J$ is an orthogonal projector onto* $\ker J$*, which is only useful if the stable and center subspaces of J happen to be orthogonal to one another:*

$$\mathbb{R}^2 = E^c \oplus E^s,\quad \langle x,z\rangle = 0,\quad \forall x\in E^c,\;\; z\in E^s, \tag{7}$$

i.e., $E^s = (E^c)^{\perp}$. This is a rare case indeed, and we should refrain from using the notation J^+ because it might confuse the reader (I know it confused me when I first read it).

We can parameterize S_0 with u_1 *or* u_2. I'll choose to parameterize by u_1, but the choice is arbitrary. In this case, from (2) we have

$$\dot{u}_1 = -\frac{\varepsilon b_2 a_1}{a_1 + a_2} \cdot u_1 + \sigma \cdot \frac{a_2}{a_1 + a_2} \cdot (g_1 W_1 + g_2 W_2) \tag{8}$$

Let's compare with the leading order $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ terms from TR's sCMT reduction:

$$\dot{u}_1 = -\frac{\varepsilon b_2 a_1}{a_1 + a_2} \cdot u_1 + \sigma \cdot \frac{1}{a_1 + a_2} \cdot (g_1 W_1 + g_2 W_2) + \varepsilon\sigma \cdot \frac{a_1 b_2 g_1 W_2 - a_2 b_2 g_2 W_2}{(a_1 + a_2)^3}. \tag{9}$$

The differences are highlighted in red. The $\mathcal{O}(\sigma)$ terms differ in the numerator (a_2 vs 1). This needs to be resolved because the elements a_1, a_2 of the Jacobian carry units, and the units of P&R and TR differ (but this could be because I didn't correctly interpret TR's output correctly). The $\mathcal{O}(\varepsilon\sigma)$ is the interesting term that is completely missing from P&R's analysis.

Remark 2. *In JE's opinion, the additional term,*

$$\varepsilon\sigma \cdot \frac{a_1 b_2 g_1 W_2 - a_2 b_2 g_2 W_2}{(a_1 + a_2)^3}, \tag{10}$$

should be present. Interestingly, this term doesn't appear to significantly influence the long-time variance, because it ultimately amounts to an $\mathcal{O}(\varepsilon^2)$ *correction.*

This is starting to get interesting!