

# Invariant manifold of your dynamical system

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Throughout and generally: the lowest order, most important, terms are near the end of each expression.

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```
% 129.127.149.251 Thursday 30th October 2025 07:21:44 PM
in_tex "invariantManifold.tex"$
factor small;
invariantmanifold({ x,y },
    mat((-diff(u1,x)-diff(u1,y)+(1+u4)*(u2-u1)
, +diff(u2,x) +(u1-2*u2+u3) +u4*(u1-u2)
, -diff(u3,x)+diff(u3,y)+(u2-u3)
,small^9 )),
    mat((0,0)),
    mat((1/3,1/3,1/3,0),(0,0,0,1)),
    mat((1,1,1,0),(0,0,0,1)),
    4);
quit;
end;
```

## The specified dynamical system

$$\dot{u}_1 = \varepsilon \left( -\frac{du_1}{dx} - \frac{du_1}{dy} - u_1 u_4 + u_2 u_4 \right) - u_1 + u_2$$

$$\dot{u}_2 = \varepsilon \left( \frac{du_2}{dx} + u_1 u_4 - u_2 u_4 \right) + u_1 - 2u_2 + u_3$$

$$\dot{u}_3 = \varepsilon \left( -\frac{d u_3}{d x} + \frac{d u_3}{d y} \right) + u_2 - u_3$$

$$\dot{u}_4 = \varepsilon^{10} \exp(0)$$

### Invariant subspace basis vectors

$$\vec{e}_1 = \{\{1/3, 1/3, 1/3, 0\}, \exp(0)\}$$

$$\vec{e}_2 = \{\{0, 0, 0, 1\}, \exp(0)\}$$

$$\vec{z}_1 = \{\{1, 1, 1, 0\}, \exp(0)\}$$

$$\vec{z}_2 = \{\{0, 0, 0, 1\}, \exp(0)\}$$

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**The invariant manifold** These give the location of the invariant manifold in terms of parameters  $s_j$ .

$$u_1 = \varepsilon^2 \left( 8/27 \frac{d^2 s_1}{d x d y} - 4/243 \frac{d^2 s_1}{d x^2} + 4/27 \frac{d s_1}{d x} s_2 + 1/27 \frac{d^2 s_1}{d y^2} + 2/9 \frac{d s_1}{d y} s_2 \right) + \varepsilon \left( -2/27 \frac{d s_1}{d x} - 1/3 \frac{d s_1}{d y} \right) + O(\varepsilon^3) + 1/3 s_1$$

$$u_2 = \varepsilon^2 \left( 8/243 \frac{d^2 s_1}{d x^2} - 2/27 \frac{d s_1}{d x} s_2 - 2/27 \frac{d^2 s_1}{d y^2} - 1/9 \frac{d s_1}{d y} s_2 \right) + 4/27 \varepsilon \frac{d s_1}{d x} + O(\varepsilon^3) + 1/3 s_1$$

$$u_3 = \varepsilon^2 \left( -8/27 \frac{d^2 s_1}{d x d y} - 4/243 \frac{d^2 s_1}{d x^2} - 2/27 \frac{d s_1}{d x} s_2 + 1/27 \frac{d^2 s_1}{d y^2} - 1/9 \frac{d s_1}{d y} s_2 \right) + \varepsilon \left( -2/27 \frac{d s_1}{d x} + 1/3 \frac{d s_1}{d y} \right) + O(\varepsilon^3) + 1/3 s_1$$

$$u_4 = O(\varepsilon^3) + s_2$$

**Invariant manifold ODEs** The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\begin{aligned}\dot{s}_1 &= \varepsilon^3 \left( -4/27 \frac{d s_2}{d x} \frac{d s_1}{d x} - 2/9 \frac{d s_2}{d x} \frac{d s_1}{d y} - 2/9 \frac{d s_2}{d y} \frac{d s_1}{d x} - 1/3 \frac{d s_2}{d y} \frac{d s_1}{d y} - \right. \\ &\quad \left. 20/27 \frac{d^3 s_1}{d x d y^2} - 4/9 \frac{d^2 s_1}{d x d y} s_2 + 16/243 \frac{d^3 s_1}{d x^3} - 4/27 \frac{d^2 s_1}{d x^2} s_2 - 1/3 \frac{d^2 s_1}{d y^2} s_2 \right) + \\ &\quad \varepsilon^2 \left( 8/27 \frac{d^2 s_1}{d x^2} + 2/3 \frac{d^2 s_1}{d y^2} \right) - 1/3 \varepsilon \frac{d s_1}{d x} + O(\varepsilon^4) \\ \dot{s}_2 &= O(\varepsilon^4)\end{aligned}$$

**Normals to isochrons at the slow manifold** Use these vectors: to project initial conditions onto the slow manifold; to project non-autonomous forcing onto the slow evolution; to predict the consequences of modifying the original system; in uncertainty quantification to quantify effects on the model of uncertainties in the original system. The normal vector  $\vec{z}_j := (z_{j1}, \dots, z_{jn})$

$$\begin{aligned}z_{11} &= \varepsilon^3 \left( -8/729 \partial_x^3 - 4/27 \partial_x^2 \partial_y + 38/27 \partial_x \partial_y^2 - 4/9 \partial_x \partial_y s_2 - 4/9 \partial_x s_2^2 + \right. \\ &\quad \left. 11/9 \partial_y^3 + 2/9 \partial_y^2 s_2 - 2/3 \partial_y s_2^2 \right) + \varepsilon^2 \left( -4/27 \partial_x^2 + 8/9 \partial_x \partial_y + 4/9 \partial_x s_2 - \right. \\ &\quad \left. 5/9 \partial_y^2 + 2/3 \partial_y s_2 \right) + \varepsilon \left( -2/9 \partial_x - \partial_y \right) + O(\varepsilon^4) + 1 \\ z_{12} &= \varepsilon^3 \left( -80/729 \partial_x^3 + 28/27 \partial_x \partial_y^2 + 8/9 \partial_x \partial_y s_2 + 2/9 \partial_x s_2^2 + 8/9 \partial_y^2 s_2 + \right. \\ &\quad \left. 1/3 \partial_y s_2^2 \right) + \varepsilon^2 \left( -2/9 \partial_x s_2 - 8/9 \partial_y^2 - 1/3 \partial_y s_2 \right) + 4/9 \varepsilon \partial_x + O(\varepsilon^4) + 1 \\ z_{13} &= \varepsilon^3 \left( -8/729 \partial_x^3 + 4/27 \partial_x^2 \partial_y + 8/27 \partial_x^2 s_2 + 38/27 \partial_x \partial_y^2 + 4/3 \partial_x \partial_y s_2 + \right. \\ &\quad \left. 2/9 \partial_x s_2^2 - 11/9 \partial_y^3 + 8/9 \partial_y^2 s_2 + 1/3 \partial_y s_2^2 \right) + \varepsilon^2 \left( -4/27 \partial_x^2 - 8/9 \partial_x \partial_y - \right. \\ &\quad \left. 2/9 \partial_x s_2 - 5/9 \partial_y^2 - 1/3 \partial_y s_2 \right) + \varepsilon \left( -2/9 \partial_x + \partial_y \right) + O(\varepsilon^4) + 1 \\ z_{14} &= \varepsilon^3 \left( 4/81 \frac{d s_1}{d x} \partial_x + 2/9 \frac{d s_1}{d x} \partial_y + 2/27 \frac{d s_1}{d y} \partial_x + 1/3 \frac{d s_1}{d y} \partial_y \right) + O(\varepsilon^4) \\ z_{21} &= O(\varepsilon^4) \\ z_{22} &= O(\varepsilon^4) \\ z_{23} &= O(\varepsilon^4) \\ z_{24} &= O(\varepsilon^4) + 1\end{aligned}$$