Invariant manifold of your dynamical system

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Throughout and generally: the lowest order, most important, terms are near the end of each expression.

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% 129.127.149.251 Thursday 30th October 2025 08:22:42 PM
in_tex "invariantManifold.tex"$
factor small,d,g;
invariantmanifold({ x },
    mat((u2,-g*diff(3*u5-u3-diff(u1,x),x)
,u4,+g*(3*u5-u3-diff(u1,x))+d*diff(u3,x,x)
,u6,-g*(3*u5-u3-diff(u1,x))+d*diff(u5,x,x)-u7*u6
,small^9 )),
    mat((0,0,0,0,0)),
    mat((1,0,0,0,0,0,0),(0,1,0,0,0,0,0)
,(0,0,3,0,1,0,0),(0,0,0,3,0,1,0)
,(0,0,0,0,0,0,1)),
    mat((1,0,0,0,0,0,0),(0,1,0,0,0,0,0)
,(0,0,1,0,1,0,0),(0,0,0,1,0,1,0)
,(0,0,0,0,0,0,1)),
    5);
quit;
end;
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The specified dynamical system

$$\dot{u}_1 = \varepsilon u_2$$

$$\dot{u}_2 = g\varepsilon^2 \frac{\mathrm{d}^2 u_1}{\mathrm{d}x^2} + g\varepsilon\left(\frac{\mathrm{d}\, u_3}{\mathrm{d}\, x} - 3\frac{\mathrm{d}\, u_5}{\mathrm{d}\, x}\right)$$

$$\dot{u}_3 = \varepsilon(3/4u_4 + 3/4u_6) + 1/4u_4 - 3/4u_6$$

$$\dot{u}_4 = d\varepsilon^2 \frac{\mathrm{d}^2 u_3}{\mathrm{d}x^2} - g\varepsilon \frac{\mathrm{d}\, u_1}{\mathrm{d}\, x} + g(- u_3 + 3u_5)$$

$$\dot{u}_5 = \varepsilon(1/4u_4 + 1/4u_6) - 1/4u_4 + 3/4u_6$$

$$\dot{u}_6 = d\varepsilon^2 \frac{\mathrm{d}^2 u_5}{\mathrm{d}x^2} + g\varepsilon \frac{\mathrm{d}\, u_1}{\mathrm{d}\, x} + g(u_3 - 3u_5) - \varepsilon u_6 u_7$$

$$\dot{u}_7 = \varepsilon^{10} \exp(0)$$

Invariant subspace basis vectors

$$\vec{e}_1 = \{\{1,0,0,0,0,0,0\}, \exp(0)\}$$

$$\vec{e}_2 = \{\{0,1,0,0,0,0,0\}, \exp(0)\}$$

$$\vec{e}_3 = \{\{0,0,3,0,1,0,0\}, \exp(0)\}$$

$$\vec{e}_4 = \{\{0,0,0,3,0,1,0\}, \exp(0)\}$$

$$\vec{e}_5 = \{\{0,0,0,0,0,0,1\}, \exp(0)\}$$

$$\vec{z}_1 = \{\{1,0,0,0,0,0,0\}, \exp(0)\}$$

$$\vec{z}_2 = \{\{0,1,0,0,0,0,0\}, \exp(0)\}$$

$$\vec{z}_3 = \{\{0,0,1/4,0,1/4,0,0\}, \exp(0)\}$$

$$\vec{z}_4 = \{\{0,0,0,1/4,0,1/4,0\}, \exp(0)\}$$

$$\vec{z}_5 = \{\{0,0,0,0,0,0,1\}, \exp(0)\}$$

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The invariant manifold These give the location of the invariant manifold in terms of parameters s_j.

$$u_1 = O(\varepsilon^4) + s_1$$

$u_2 = O(\varepsilon^4) + s_2$

$u_3 = -1/16 g^{-1} d\varepsilon^3 \frac{d^3 s_1}{dx^3} + g^{-2} d\varepsilon^3 (3/64 \frac{d^2 s_5}{dx^2} s_4 + 3/32 \frac{d s_5}{d x} \frac{d s_4}{d x} + 3/64 \frac{d^2 s_4}{dx^2} s_5) - 1/4\varepsilon \frac{d s_1}{d x} + O(\varepsilon^4) + 3s_3 + 3/64 g^{-1}\varepsilon^3 \frac{d s_2}{d x} s_5 + 3/16 g^{-1}\varepsilon s_5 s_4 + 3/512 g^{-2}\varepsilon^3 s_5^3 s_4$

$u_4 = 3/16 g^{-1} d\varepsilon^3 \frac{d^2 s_3}{dx^2} s_5 - 1/4\varepsilon^2 \frac{d s_2}{d x} + O(\varepsilon^4) + 3s_4 - 3/64 g^{-1}\varepsilon^2 s_5^2 s_4$

$u_5 = 1/16 g^{-1} d\varepsilon^3 \frac{d^3 s_1}{dx^3} + g^{-2} d\varepsilon^3 (- 3/64 \frac{d^2 s_5}{dx^2} s_4 - 3/32 \frac{d s_5}{d x} \frac{d s_4}{d x} - 3/64 \frac{d^2 s_4}{dx^2} s_5) + 1/4\varepsilon \frac{d s_1}{d x} + O(\varepsilon^4) + s_3 - 3/64 g^{-1}\varepsilon^3 \frac{d s_2}{d x} s_5 - 3/16 g^{-1}\varepsilon s_5 s_4 - 3/512 g^{-2}\varepsilon^3 s_5^3 s_4$

$u_6 = -3/16 g^{-1} d\varepsilon^3 \frac{d^2 s_3}{dx^2} s_5 + 1/4\varepsilon^2 \frac{d s_2}{d x} + O(\varepsilon^4) + s_4 + 3/64 g^{-1}\varepsilon^2 s_5^2 s_4$

$u_7 = O(\varepsilon^4) + s_5$

Invariant manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$\dot s_1 = \varepsilon s_2 + O(\varepsilon^5)$

$\dot s_2 = -1/4 d\varepsilon^4 \frac{d^4 s_1}{dx^4} + g^{-1} d\varepsilon^4 (3/16 \frac{d^3 s_5}{dx^3} s_4 + 9/16 \frac{d^2 s_5}{dx^2} \frac{d s_4}{d x} + 9/16 \frac{d s_5}{d x} \frac{d^2 s_4}{dx^2} + 3/16 \frac{d^3 s_4}{dx^3} s_5) + \varepsilon^4 (3/16 \frac{d s_5}{d x} \frac{d s_2}{d x} + 3/16 \frac{d^2 s_2}{dx^2} s_5) + \varepsilon^2 (3/4 \frac{d s_5}{d x} s_4 + 3/4 \frac{d s_4}{d x} s_5) + O(\varepsilon^5) + g^{-1}\varepsilon^4 (9/128 \frac{d s_5}{d x} s_5^2 s_4 + 3/128 \frac{d s_4}{d x} s_5^3)$

$\dot s_3 = \varepsilon s_4 + O(\varepsilon^5)$

$\dot s_4 = d\varepsilon^2 \frac{d^2 s_3}{dx^2} + 3/64 g^{-1} d\varepsilon^4 \frac{d^2 s_3}{dx^2} s_5^2 - 1/16\varepsilon^3 \frac{d s_2}{d x} s_5 - 1/4\varepsilon s_5 s_4 + O(\varepsilon^5) - 3/256 g^{-1}\varepsilon^3 s_5^3 s_4$

$\dot s_5 = O(\varepsilon^5)$

Normals to isochrons at the slow manifold Use these vectors: to project initial conditions onto the slow manifold; to project non-autonomous forcing onto the slow evolution; to predict the consequences of modifying the original system; in uncertainty quantification to quantify effects on the model of uncertainties in the original system. The normal vector $\vec{z}_j := (z_{j1}, \ldots, z_{jn})$

$$z_{11} = 1/4\varepsilon^3\partial_{\dot{x}}^2 + O(\varepsilon^5) + 1$$

$$z_{12} = O(\varepsilon^5)$$

$$z_{13} = 1/16g^{-1}d\varepsilon^4\partial_{\dot{x}}^3 + 1/4\varepsilon^2\partial_{\dot{x}} + O(\varepsilon^5) - 3/256g^{-1}\varepsilon^4\partial_{\dot{x}}s_5^2$$

$$z_{14} = O(\varepsilon^5) - 3/32g^{-1}\varepsilon^3\partial_{\dot{x}}s_5$$

$$z_{15} = -3/16g^{-1}d\varepsilon^4\partial_{\dot{x}}^3 - 3/4\varepsilon^2\partial_{\dot{x}} + O(\varepsilon^5) + 9/256g^{-1}\varepsilon^4\partial_{\dot{x}}s_5^2$$

$$z_{16} = O(\varepsilon^5) + 3/32g^{-1}\varepsilon^3\partial_{\dot{x}}s_5$$

$$z_{17} = O(\varepsilon^5) - 3/16g^{-1}\varepsilon^3\partial_{\dot{x}}s_4$$

$$z_{21} = 3/16\varepsilon^3\partial_{\dot{x}}^2 s_5 + O(\varepsilon^5)$$

$$z_{22} = 1/4\varepsilon^3\partial_{\dot{x}}^2 + O(\varepsilon^5) + 1$$

$$z_{23} = 3/16\varepsilon^2\partial_{\dot{x}}s_5 + O(\varepsilon^5) + 3/256g^{-1}\varepsilon^4\partial_{\dot{x}}s_5^3$$

$$z_{24} = 1/16g^{-1}d\varepsilon^3\partial_{\dot{x}}^3 + 1/16\varepsilon^4\partial_{\dot{x}}^3 + 1/4\varepsilon\partial_{\dot{x}} + O(\varepsilon^5) - 3/256g^{-1}\varepsilon^3\partial_{\dot{x}}s_5^2$$

$$z_{25} = -3/16g^{-1}d\varepsilon^4\partial_{\dot{x}}^3 s_5 - 9/16\varepsilon^2\partial_{\dot{x}}s_5 + O(\varepsilon^5) - 9/256g^{-1}\varepsilon^4\partial_{\dot{x}}s_5^3$$

$$z_{26} = -3/16g^{-1}d\varepsilon^3\partial_{\dot{x}}^3 - 3/16\varepsilon^4\partial_{\dot{x}}^3 - 3/4\varepsilon\partial_{\dot{x}} + O(\varepsilon^5) - 15/256g^{-1}\varepsilon^3\partial_{\dot{x}}s_5^2$$

$$z_{27} = -3/16g^{-1}d\varepsilon^4\frac{\mathrm{d}^2 s_3}{\mathrm{d}x^2}\partial_{\dot{x}} + O(\varepsilon^5) - 3/64g^{-1}\varepsilon^3\partial_{\dot{x}}s_5 s_4$$

$$z_{31} = O(\varepsilon^5) + 1/128g^{-1}\varepsilon^4\partial_{\dot{x}}s_5^2$$

$$z_{32} = O(\varepsilon^5) + 1/64g^{-1}\varepsilon^4\partial_{\dot{x}}s_5$$

$$z_{33} = O(\varepsilon^5) + 1/4 + 1/128g^{-1}\varepsilon^3 s_5^2$$

$$z_{34} = 1/256g^{-2}d\varepsilon^4\partial_{\dot{x}}^2 s_5 + O(\varepsilon^5) + 1/64g^{-1}\varepsilon^2 s_5 + 1/2048g^{-2}\varepsilon^4 s_5^3$$

$$z_{35} = O(\varepsilon^5) + 1/4 - 3/128g^{-1}\varepsilon^3 s_5^2$$

$$z_{36} = -3/256g^{-2}d\varepsilon^4\partial_{\dot{x}}^2 s_5 + O(\varepsilon^5) - 3/64g^{-1}\varepsilon^2 s_5 - 9/2048g^{-2}\varepsilon^4 s_5^3$$

$$z_{37} = O(\varepsilon^5)$$

$$z_{41} = -1/64g^{-1}d\varepsilon^4\partial_{\dot{x}}^3 s_5 - 1/16\varepsilon^2\partial_{\dot{x}}s_5 + O(\varepsilon^5) - 5/1024g^{-1}\varepsilon^4\partial_{\dot{x}}s_5^3$$

$$z_{42} = O(\varepsilon^5) + 1/128g^{-1}\varepsilon^4\partial_{\dot{x}}s_5^2$$

$$z_{43} = 1/64g^{-1}d\varepsilon^4\partial_{\dot{x}}^2 s_5 - 1/64\varepsilon^4\partial_{\dot{x}}^2 s_5 - 1/16\varepsilon s_5 + O(\varepsilon^5) - 5/1024g^{-1}\varepsilon^3 s_5^3$$

$$z_{44} = g^{-2} d\varepsilon^4 (3/1024 \frac{\mathrm{d}^2 s_5}{\mathrm{d}x^2} s_5 + 3/512 \frac{\mathrm{d}\, s_5}{\mathrm{d}\, x} \partial_{\dot{x}} s_5 + 1/512 \partial_{\dot{x}}^2 s_5^2) + O(\varepsilon^5) + 1/4 + 1/128 g^{-1} \varepsilon^2 s_5^2 + 5/8192 g^{-2} \varepsilon^4 s_5^4$$

$$z_{45} = -3/64 g^{-1} d\varepsilon^4 \partial_{\dot{x}}^2 s_5 + 3/64 \varepsilon^4 \partial_{\dot{x}}^2 s_5 + 3/16 \varepsilon s_5 + O(\varepsilon^5) + 15/1024 g^{-1} \varepsilon^3 s_5^3$$

$$z_{46} = g^{-2} d\varepsilon^4 (3/1024 \frac{\mathrm{d}^2 s_5}{\mathrm{d}x^2} s_5 + 3/512 \frac{\mathrm{d}\, s_5}{\mathrm{d}\, x} \partial_{\dot{x}} s_5 + 3/512 \partial_{\dot{x}}^2 s_5^2) + O(\varepsilon^5) + 1/4 + 3/128 g^{-1} \varepsilon^2 s_5^2 + 21/8192 g^{-2} \varepsilon^4 s_5^4$$

$$z_{47} = g^{-2} d\varepsilon^4 (3/256 \frac{\mathrm{d}^2 s_4}{\mathrm{d}x^2} s_5 + 3/128 \frac{\mathrm{d}\, s_4}{\mathrm{d}\, x} \partial_{\dot{x}} s_5 + 3/256 \partial_{\dot{x}}^2 s_5 s_4) + O(\varepsilon^5) + 3/256 g^{-1} \varepsilon^4 \frac{\mathrm{d}\, s_2}{\mathrm{d}\, x} s_5 + 3/64 g^{-1} \varepsilon^2 s_5 s_4 + 27/4096 g^{-2} \varepsilon^4 s_5^3 s_4$$

$$z_{51} = O(\varepsilon^5)$$

$$z_{52} = O(\varepsilon^5)$$

$$z_{53} = O(\varepsilon^5)$$

$$z_{54} = O(\varepsilon^5)$$

$$z_{55} = O(\varepsilon^5)$$

$$z_{56} = O(\varepsilon^5)$$

$$z_{57} = O(\varepsilon^5) + 1$$