Many examples of normal forms for stochastic or non-autonomous differential equations

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Instructions

- Download and install the computer algebra package *Reduce*[1] via `http://www.reduce-algebra.com`
- Navigate to folder `Examples` within folder `StoNormForm`.
- For each example of interest, start-up *Reduce* and enter the command `in_tex "filename.tex"$` where filename is the root name of the example (as listed in the following table of contents).

Contents

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[1] Use the computer algebra package *Reduce* because it is both free and perhaps the fastest general purpose computer algebra system (Fateman 2003, e.g.).

1 ratodes: Simple rational ODEs

A simple system of fast/slow ODEs in rational functions is

$$\dot x = -\frac{xy}{1+z}\,, \quad \dot y = -\frac{y}{1+2y} + x^2, \quad \dot z = 2\frac{z}{1+3x}\,. \tag{1}$$

Use `x(1)` to denote variable x, `y(1)` to denote variable y, and `z(1)` to denote z. Multiply each ODE by the denominator for the ODE and shift the nonlinear d/dt terms to the right-hand side.

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system.

```
stonormalform(
    {-x(1)*y(1)-z(1)*df(x(1),t)},
    {-y(1)+x(1)^2*(1+2*y(1))-2*y(1)*df(y(1),t)},
    {2*z(1)-3*x(1)*df(z(1),t)},
    4)$
end;
```

The procedure embeds the system as the $\varepsilon = 1$ version of the family

$$\begin{aligned}
\dot x_1 &= \varepsilon\big(-\frac{d\,x_1}{d\,t} z_1 - x_1 y_1\big)\\
\dot y_1 &= \varepsilon\big(-2\frac{d\,y_1}{d\,t} y_1 + 2x_1^2 y_1 + x_1^2\big) - y_1\\
\dot z_1 &= -3\varepsilon\frac{d\,z_1}{d\,t} x_1 + 2z_1
\end{aligned}$$

Time dependent coordinate transform

$$\begin{aligned}
z_1 &= 6\varepsilon^2 X_1 Y_1 Z_1 + Z_1\\
y_1 &= \varepsilon^2\big(2X_1^4 - 4X_1^2Y_1^2 + 6Y_1^3\big) + \varepsilon\big(X_1^2 - 2Y_1^2\big) + Y_1\\
x_1 &= \varepsilon^2\big(2X_1^3Y_1 - 1/2X_1Y_1^2 + X_1Y_1Z_1\big) + \varepsilon X_1 Y_1 + X_1
\end{aligned}$$

Result normal form DEs

$$\dot Z_1 = -54\varepsilon^3 X_1^3 Z_1 + 18\varepsilon^2 X_1^2 Z_1 - 6\varepsilon X_1 Z_1 + 2Z_1$$
$$\dot Y_1 = 8\varepsilon^3 X_1^4 Y_1 + 4\varepsilon^2 X_1^2 Y_1 + 2\varepsilon X_1^2 Y_1 - Y_1$$
$$\dot X_1 = \varepsilon^3\big(- 2X_1^5 - 2X_1 Y_1^2 Z_1\big) - \varepsilon^2 X_1^3$$

2 `futureNoise`: Future noise in the transform

An interesting pair of fast/slow SDEs derived from stochastic advection/dispersion is

$$\dot x = -\sigma y w(t) \quad \text{and} \quad \dot y = -y + \sigma x w(t)\,, \tag{2}$$

where lowercase $w(t)$ denotes the formal derivative dW/dt of a Stratonovich Wiener process $W(t,\omega)$. Parameter σ controls the strength of the noise. In stochastic advection/dispersion parameter σ represents the lateral wavenumber of the concentration profile.

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system.

```
stonormalform(
    {-y(1)*w(1)},
    {-y(1)+x(1)*w(1)},
    {},
    5)$
end;
```

Being linear in x,y the nonlinear parameter ε does not appear in the analysis and results. Consequently, the procedure analyses the system as prescribed (since given w changed to σw). The interest in this example is the noise and the noise-noise interactions. As usual, the noise-noise interactions are truncated to errors $\mathcal{O}\big(\sigma^3\big)$.

Time dependent coordinate transform

$$y_1 = \sigma \mathrm{e}^{-1t}{\star}w_1\, X_1 + Y_1$$
$$x_1 = \sigma \mathrm{e}^{t}{\star}w_1\, Y_1 + X_1$$

Result normal form DEs

$$\dot Y_1 = \sigma^2 \mathrm{e}^{t}{\star}w_1\, w_1 Y_1 - Y_1$$
$$\dot X_1 = -\sigma^2 \mathrm{e}^{-1t}{\star}w_1\, w_1 X_1$$

The interesting aspect of this example is the explicit presence of non-Markovian, future time integrals, anticipation integrals, in the convolutions $\mathrm{e}^{t}{\star}w_1$. These appear in both the coordinate transform, and the evolution *off* the stochastic slow manifold. But, as guaranteed by theory, they do not appear on the stochastic slow manifold.

Further, this example could go to higher order noise-noise interactions very quickly, that is, to higher orders in σ. However, I do not compute such higher order terms in this code.

3 othersFail: Other methodologies fail

Consider, for small bifurcation parameter ϵ, the system

$$\begin{aligned}\text{slow mode } \dot x &= \epsilon x + x^3 - (1-\sigma w)xy\,,\\ \text{fast mode } \dot y &= -y + x^2 + y^2 + \sigma y w\,.\end{aligned}$$

Deterministically, there is a bifurcation to two equilibria for small $\epsilon > 0$. The noise w affects this bifurcation somehow.

Why is this tricky? *Cross-sectional averaging* is simply projection onto the slow space $y = 0$ which predicts instability of subcritical bifurcation $\dot x = \epsilon x + x^3$. Whereas *adiabatic approximation*, *singular perturbation*, and *multiple scales* set $\dot y = 0$ whence $y \approx x^2$ and thus predict only the linear growth of $\dot x = \epsilon x$. Our normal form transforms get the deterministic dynamics correctly. But what happens for stochastic dynamics?

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system. Multiply a cubic terms in the x SDE in order to count orders of approximation best (since the right-hand side is multiplied by `small`). Multiply the bifurcation parameter by `small` in order to make it scale with ε^2.

```
stonormalform(
    {small*epsilon*x(1)+small*x(1)^3
        -x(1)*y(1)*(1-small*w(1))},
    {-y(1)+x(1)^2+y(1)^2+y(1)*w(1)},
    {},
    5)$
end;
```

With the above artifices, the procedure analyses the following system which reduce to the given one for $\varepsilon = 1$:

$$\begin{aligned}\dot x_1 &= \sigma\varepsilon w_1 x_1 y_1 + \varepsilon^2\big(x_1^3 + x_1\epsilon\big) - \varepsilon x_1 y_1\\ \dot y_1 &= \sigma w_1 y_1 + \varepsilon\big(x_1^2 + y_1^2\big) - y_1\end{aligned}$$

Time dependent coordinate transform This transform is quite complicated, due to the noise, and involve fast-time future and history integrals. The deterministic terms at the end.

$$y_1 = \sigma\varepsilon^3\big(-4e^{t}\star e^{t}\star w_1\ X_1^2Y_1^2 + 4e^{-1t}\star e^{-1t}\star w_1\ X_1^4 - 2e^{-1t}\star e^{-1t}\star w_1\ X_1^2\epsilon + 2e^{2t}\star w_1\ X_1^2Y_1^2 - 10e^{t}\star w_1\ X_1^2Y_1^2 - 3e^{t}\star w_1\ Y_1^4 + e^{-1t}\star w_1\ X_1^4 + 3e^{-1t}\star w_1\ X_1^2Y_1^2 - 2e^{-1t}\star w_1\ X_1^2\epsilon\big) + \sigma\varepsilon^2\big(2e^{t}\star w_1\ Y_1^3 - 2e^{-1t}\star w_1\ X_1^2Y_1\big) + \sigma\varepsilon\big(-e^{t}\star w_1\ Y_1^2 + e^{-1t}\star w_1\ X_1^2\big) + \varepsilon^3\big(X_1^4 - 7X_1^2Y_1^2 - 2X_1^2\epsilon - Y_1^4\big) + \varepsilon^2Y_1^3 + \varepsilon\big(X_1^2 - Y_1^2\big) + Y_1$$

$$x_1 = \sigma\varepsilon^3\big(-e^{3t}\star w_1\ X_1Y_1^3 + e^{2t}\star w_1\ X_1Y_1^3 + 3e^{t}\star w_1\ X_1^3Y_1\big) + \sigma\varepsilon^2\big(e^{2t}\star w_1\ X_1Y_1^2 - e^{t}\star w_1\ X_1Y_1^2 + e^{-1t}\star w_1\ X_1^3\big) + 2\varepsilon^3X_1^3Y_1 + \varepsilon X_1Y_1 + X_1$$

Result normal form DEs

$$\dot Y_1 = \sigma^2\varepsilon^4\big(8e^{-1t}\star e^{-1t}\star w_1\ w_1X_1^4Y_1 - 4e^{-1t}\star e^{-1t}\star w_1\ w_1X_1^2Y_1\epsilon + 6e^{t}\star w_1\ w_1X_1^4Y_1 + 22e^{-1t}\star w_1\ w_1X_1^4Y_1 - 4e^{-1t}\star w_1\ w_1X_1^2Y_1\epsilon\big) + 2\sigma^2\varepsilon^2e^{-1t}\star w_1\ w_1X_1^2Y_1 + \sigma\varepsilon^4\big(22w_1X_1^4Y_1 - 4w_1X_1^2Y_1\epsilon\big) + 2\sigma\varepsilon^2w_1X_1^2Y_1 + \sigma w_1Y_1 + \varepsilon^4\big(6X_1^4Y_1 - 4X_1^2Y_1\epsilon\big) + 4\varepsilon^2X_1^2Y_1 - Y_1$$

$$\dot X_1 = -3\sigma^2\varepsilon^4e^{-1t}\star w_1\ w_1X_1^5 - 2\sigma\varepsilon^4w_1X_1^5 + \varepsilon^4\big(-X_1^5 + 2X_1^3\epsilon\big) + \varepsilon^2X_1\epsilon$$

- As expected, $Y_1 = 0$ is the stochastic slow manifold, and is exponentially attractive (almost always) in some domain about the origin.
- The slow X_1 evolution is independent of Y_1. Deterministically ($\sigma = 0$), we predict a bifurcation to $X_1 \approx \pm\epsilon^{1/4}$. The noise appears to modify this slightly.
- The time-dependent coordinate transform maps these predictions back into the xy-plane.

4 `offdiagonal`: Levy area contraction: off-diagonal example

Pavliotis & Stuart (2008) assert the following system of five coupled SDEs are interesting for various parameters a and for small ϵ.

$$\begin{aligned} dx_1 &= \epsilon y_1\,dt\,,\\ dx_2 &= \epsilon y_2\,dt\,,\\ dx_3 &= \epsilon(x_1y_2 - x_2y_1)dt\,,\\ dy_1 &= (-y_1 - ay_2)dt + dW_1\,,\\ dy_2 &= (+ay_1 - y_2)dt + dW_2\,. \end{aligned}$$

This stochastic system has two noise sources. We treat $W_i(t,\omega)$ as Stratonovich Wiener processes. Use `x(i)` to denote variable x_i, `y(i)` to denote variable y_i, and `w(i)` to denote noise dW_i/dt.

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

It is convenient to factor written results on the two given parameters ϵ, a:

```
factor epsilon,a;
```

Execute the construction of a normal form for this system. A coding is to specify the system as given: specify the slow SDEs via a three component list; and the fast stable SDEs via a two component list.

```
stonormalform(
    {epsilon*y(1),
     epsilon*y(2),
     epsilon*(x(1)*y(2)-x(2)*y(1))},
    {-y(1)-a*y(2)+w(1),
     -y(2)+a*y(1)+w(2)},
    {},
    4)$
end;
```

Now the approach can only analyse systems which are linearly diagonalised, but this system has two off-diagonal terms in the $\vec y$-SDEs (terms that cause oscillations in $\vec y$ with frequency a as $\vec y$ decays in magnitude like e^{-t}). In order to make some sort of progress, the procedure is brutal with such off-diagonal terms. *Anything linear and off-diagonal is multiplied by the parameter `small` and so is treated as asymptotically small.* When it does so, it gives the warning message

```
***** Warning ****
Off diagonal linear terms in y- or z- equations
assumed small.  Answers are rubbish if not
asymptotically appropriate.
```

As the message says, the results may consequently be rubbish.

Here then, the procedure analyses the following system which reduce to the given one for $\varepsilon = 1$:

$$\begin{aligned}
\dot x_1 &= \epsilon\varepsilon y_1\\
\dot x_2 &= \epsilon\varepsilon y_2\\
\dot x_3 &= \epsilon\varepsilon\big(-x_2y_1 + x_1y_2\big)\\
\dot y_1 &= -a\varepsilon y_2 + \sigma w_1 - y_1\\
\dot y_2 &= a\varepsilon y_1 + \sigma w_2 - y_2
\end{aligned}$$

That is, the code treats the frequency parameter a as small, and so the results are appropriate only for small a, as well as only for small ϵ.

If one really needs to analyse non-small a, then more sophisticated code has to be developed.

Time dependent coordinate transform This transform is quite complicated, due to the noise, and involve fast-time future and history integrals.

$$y_1 = -a^2\sigma\varepsilon^2 e^{-1t}\star e^{-1t}\star e^{-1t}\star w_1 \; - a\sigma\varepsilon e^{-1t}\star e^{-1t}\star w_2 \; + \sigma e^{-1t}\star w_1 \; + Y_1$$
$$y_2 = -a^2\sigma\varepsilon^2 e^{-1t}\star e^{-1t}\star e^{-1t}\star w_2 \; + a\sigma\varepsilon e^{-1t}\star e^{-1t}\star w_1 \; + \sigma e^{-1t}\star w_2 \; + Y_2$$
$$x_1 = a\epsilon\sigma\varepsilon^2\big(e^{-1t}\star e^{-1t}\star w_2 \; + e^{-1t}\star w_2\big) + a\epsilon\varepsilon^2 Y_2 - \epsilon\sigma\varepsilon e^{-1t}\star w_1 \; - \epsilon\varepsilon Y_1 + X_1$$
$$x_2 = a\epsilon\sigma\varepsilon^2\big(- e^{-1t}\star e^{-1t}\star w_1 \; - e^{-1t}\star w_1\big) - a\epsilon\varepsilon^2 Y_1 - \epsilon\sigma\varepsilon e^{-1t}\star w_2 \; - \epsilon\varepsilon Y_2 + X_2$$
$$x_3 = a\epsilon\sigma\varepsilon^2\big(- e^{-1t}\star e^{-1t}\star w_2 \; X_2 - e^{-1t}\star e^{-1t}\star w_1 \; X_1 - e^{-1t}\star w_2 \, X_2 - e^{-1t}\star w_1 \, X_1\big) + a\epsilon\varepsilon^2\big(- X_2Y_2 - X_1Y_1\big) + \epsilon^2\sigma\varepsilon^2\big(e^{t}\star w_2 \, Y_1 - e^{1t}\star w_1 \, Y_2\big) + \epsilon\sigma\varepsilon\big(- e^{-1t}\star w_2 \, X_1 + e^{-1t}\star w_1 \, X_2\big) + \epsilon\varepsilon\big(X_2Y_1 - X_1Y_2\big) + X_3$$

Result normal form DEs

$$\dot Y_1 = -a\varepsilon Y_2 - Y_1$$
$$\dot Y_2 = a\varepsilon Y_1 - Y_2$$
$$\dot X_1 = -a^2\epsilon\sigma\varepsilon^3 w_1 - a\epsilon\sigma\varepsilon^2 w_2 + \epsilon\sigma\varepsilon w_1$$
$$\dot X_2 = -a^2\epsilon\sigma\varepsilon^3 w_2 + a\epsilon\sigma\varepsilon^2 w_1 + \epsilon\sigma\varepsilon w_2$$
$$\dot X_3 = a^2\epsilon\sigma\varepsilon^3\big(- w_2X_1 + w_1X_2\big) + a\epsilon^2\sigma^2\varepsilon^3\big(e^{-1t}\star e^{-1t}\star w_2 \; w_2 + e^{-1t}\star e^{-1t}\star w_1 \; w_1 + e^{-1t}\star w_2 \, w_2 + e^{-1t}\star w_1 \, w_1\big) + a\epsilon\sigma\varepsilon^2\big(w_2X_2 + w_1X_1\big) + \epsilon^2\sigma^2\varepsilon^2\big(e^{-1t}\star w_2 \, w_1 - e^{-1t}\star w_1 \, w_2\big) + \epsilon\sigma\varepsilon\big(w_2X_1 - w_1X_2\big)$$

- As expected, $\vec Y = 0$ is the stochastic slow manifold, and is exponentially attractive (almost always) in some domain about the origin.
- The slow $\vec X$ evolution is independent of $\vec Y$: X_1, X_2 undergo a correlated 'slow' random walk; whereas X_3 is dominantly some multiplicative random walk.
- The time-dependent coordinate transform maps these predictions back into the $\vec x, \vec y$-space.

5 jordanForm: the Jordan form of position-momentum variables

Suppose $x(t)$ is the spatial position of some particle, and you want to analyse the 'mechanical' system of SDEs

$$\ddot x = -xy \quad \text{and} \quad \dot y = -2y + x^2 + \dot x^2 + \sigma w(t)\,,$$

where $w(t)$ denotes the formal derivative dW/dt of a Stratonovich Wiener process $W(t,\omega)$, or some other time-dependent forcing, called noise. Introduce position and velocity variables $x_1 = x$ and $x_2 = \dot x$, and also $y_1 = y$ to convert to the system of three coupled first-order SDEs

$$\dot x_1 = x_2\,,$$

$$\dot x_2 = -x_1 y_1\,,$$
$$\dot y_1 = -2y_1 + x_1^2 + x_2^2 + \sigma w(t)\,.$$

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system.

```
stonormalform(
    { x(2)/small,
      -x(1)*y(1) },
    { -2*y(1)+x(1)^2+x(2)^2+w(y) },
    {},
    3 )$
end;
```

Why divide `x(2)` by `small`? A possible coding is to specify the system as given, but recall that the slow SDEs are always multiplied by `small` thus changing the first SDE to $\dot x_1 = \varepsilon x_2$ and hence changing the relation between position and velocity—this would be OK if x_2 was viewed as momentum *and* the particle had large mass. But what if really do we want x_2 to be velocity. Fortunately, the coded iteration scheme works for systems with linear part in Jordan form, but one has to code the system as follows. Namely, divide the off-diagonal term of the Jordan form by `small` to cancel out the procedure's brutal multiplication by `small`.

Then the coded procedure reports that it analyses the following system which not only reduces to the given one for $\varepsilon = 1$, but also preserves the physical relation between position x_1 and velocity x_2:

$$\dot x_1 = x_2$$
$$\dot x_2 = -\varepsilon x_1 y_1$$
$$\dot y_1 = \sigma w_y + \varepsilon\big(x_2^2 + x_1^2\big) - 2y_1$$

Further, here ε counts the order of nonlinearity so truncating to errors $\mathcal{O}\big(\varepsilon^3\big)$ is the same as truncating to errors $\mathcal{O}\big(|(\vec x, y)|^4\big)$.

The cost of preserving the physical relation between position x_1 and velocity x_2 is that more iterations are needed in the construction.

Time dependent coordinate transform This transform is quite complicated, due to the noise, and involve fast-time future and history integrals.

$$y_1 = \sigma \mathrm{e}^{-2t}{\star}w_y + \varepsilon\big(3/4X_2^2 - 1/2X_2X_1 + 1/2X_1^2\big) + Y_1$$
$$x_1 = \sigma\varepsilon\big(-1/4\mathrm{e}^{-2t}{\star}w_y\,X_2 - 1/4\mathrm{e}^{-2t}{\star}w_y\,X_1\big) + \varepsilon\big(-1/4X_2Y_1 - 1/4X_1Y_1\big) + X_1$$
$$x_2 = \sigma\varepsilon\big(1/4\mathrm{e}^{-2t}{\star}w_y\,X_2 + 1/2\mathrm{e}^{-2t}{\star}w_y\,X_1\big) + \varepsilon\big(1/4X_2Y_1 + 1/2X_1Y_1\big) + X_2$$

Result normal form DEs

$$\dot Y_1 = \varepsilon^2\big(1/2X_2^2Y_1 + 1/2X_2X_1Y_1 - 1/2X_1^2Y_1\big) - 2Y_1$$
$$\dot X_1 = \sigma^2\varepsilon^2\big(- 3/64\mathrm{e}^{-2t}\star w_y\, w_yX_2 - 3/32\mathrm{e}^{-2t}\star w_y\, w_yX_1\big) + \sigma\varepsilon\big(1/4w_yX_2 + 1/4w_yX_1\big) + X_2$$
$$\dot X_2 = \sigma^2\varepsilon^2\big(3/32\mathrm{e}^{-2t}\star w_y\, w_yX_2 + 1/8\mathrm{e}^{-2t}\star w_y\, w_yX_1\big) + \sigma\varepsilon\big(- 1/4w_yX_2 - 1/2w_yX_1\big) + \varepsilon^2\big(- 3/4X_2^2X_1 + 1/2X_2X_1^2 - 1/2X_1^3\big)$$

- As expected, $Y_1 = 0$ is the stochastic slow manifold, and is exponentially attractive (almost always) in some domain about the origin.
- As expected, the slow $\vec X$ evolution is independent of Y_1: X_2 is approximately a 'velocity' variable for 'position' X_1, and shows some nonlinear noise affected dynamics.
- The time-dependent coordinate transform maps these predictions back into the $\vec x, y$-space. Observe that $\vec X$ are not precisely the physical position-veloxity $\vec x$, but instead are affected by nonlinearity, and the noise, and their interaction.

6 slowOsc: Radek's slow oscillation with fast noise

Consider Radek's system

$$\dot x = -\epsilon xz\,, \quad \dot y = +\epsilon yz \quad \text{and} \quad \dot z = -(z-1) + \sigma w(t)\,.$$

In this linear system, x, y oscillate with 'frequency' ϵz. But $z(t)$ is an Ornstein–Uhlenbeck process with mean one. What are the dynamics?

Transform to our standard form via

$$x = x_1\,, \quad y = x_2 \quad \text{and} \quad z = 1 + y_1\,.$$

Then start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system.

```
factor x;
stonormalform(
    { -x(2)*(1+y(1)),
      x(1)*(1+y(1)) },
    { -y(1)+w(1) },
    {},
    4 )$
end;
```

With the above input the procedure analyses the following system:

$$\dot x_1 = x_2\varepsilon\big(- y_1 - 1\big)$$

$$\dot{x}_2 = x_1\varepsilon\big(y_1 + 1\big)$$
$$\dot{y}_1 = \sigma w_1 - y_1$$

This is precisely the original system, but with variables changed as above, and with parameter $\varepsilon = \epsilon$ (here we use the procedure's multiplication by ε to incorporate Radek's ϵ).

Time dependent coordinate transform

$$y_1 = \sigma \mathrm{e}^{-1t}\star w_1 + Y_1$$
$$x_1 = -\sigma\varepsilon^2 \mathrm{e}^{-1t}\star w_1\, X_1Y_1 + \sigma\varepsilon \mathrm{e}^{-1t}\star w_1\, X_2 - 1/2\varepsilon^2 X_1Y_1^2 + \varepsilon X_2Y_1 + X_1$$
$$x_2 = -\sigma\varepsilon^2 \mathrm{e}^{-1t}\star w_1\, X_2Y_1 - \sigma\varepsilon \mathrm{e}^{-1t}\star w_1\, X_1 - 1/2\varepsilon^2 X_2Y_1^2 - \varepsilon X_1Y_1 + X_2$$

Result normal form DEs In such linear systems, the following normal form is straightforward.

$$\dot{Y}_1 = -Y_1$$
$$\dot{X}_1 = -\sigma\varepsilon w_1 X_2 - \varepsilon X_2$$
$$\dot{X}_2 = \sigma\varepsilon w_1 X_1 + \varepsilon X_1$$

- As expected, $Y_1 = 0$ is the emergent stochastic slow manifold.
- The slow $\vec{X}$ evolution clearly oscillates in (X_1, X_2), $X_j \propto e^{i\theta}$, with phase angle $\theta = \varepsilon(t + \sigma W(t,\omega))$, recalling $W = \int w\,dt$. This phase grows linearly with a superposed random walk.
- The time-dependent coordinate transform maps these predictions back into the $\vec{x}, y_1$-plane, and thence to the original xyz-space.
- In this system, higher-order terms in ε only affect the coordinate transform, they do not change the evolution of $\vec{X}$.

7 `linearHyper`: simple linear hyperbolic noisy system

The procedure also analyses hyperbolic systems, and recovers the classic stochastic/non-autonomous results guaranteed by the Hartman–Grobman Theorem. Consider the following linear SDEs with one stable variable, and one unstable variable:

$$\dot{y}_1 = -y_1 + \sigma w_1 z_1$$
$$\dot{z}_1 = z_1 + \sigma w_1 y_1$$

Start by loading the procedure.

```
55 in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system: the parameter σ is automatically inserted by the procedure.

```
stonormalform(
    {},
    { -y(1)+z(1)*w(1) },
    { +z(1)+y(1)*w(1) },
    3 )$
end;
```

Time dependent coordinate transform This simply mixes Y, Z a little depending upon the noise.

$$z_1 = -\sigma e^{2t}\star w_1\, Y_1 + Z_1$$
$$y_1 = \sigma e^{-2t}\star w_1\, Z_1 + Y_1$$

Result normal form DEs In such linear systems the normal form is straightforward, as follows.

$$\dot Z_1 = \sigma^2 e^{-2t}\star w_1\, w_1 Z_1 + Z_1$$
$$\dot Y_1 = -\sigma^2 e^{2t}\star w_1\, w_1 Y_1 - Y_1$$

The Y, Z variables are decoupled. Their evolution retains effects from noise-noise interactions: Z from the past history; and Y from future anticipation.

8 foliateHyper: Duan's hyperbolic system for foliation

To illustrate a stochastic/non-autonomous Hartman–Grobman Theorem, Sun et al. (2011) used the following simple hyperbolic system with one stable variable, and one unstable variable:

$$\dot y_1 = -y_1 + \sigma w_1 y_1$$
$$\dot z_1 = z_1 + y_1^2 + \sigma w_1 z_1$$

The stable y-dynamics is simply an Ornstein–Uhlenbeck process, independent of $z(t)$. The unstable z-dynamics is similar, but with a quadratic forcing by the stable variable y. Let's unfold this effect.

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system: the parameter σ is automatically inserted by the procedure.

```
stonormalform(
    {},
    { -y(1)+y(1)*w(1) },
    { +z(1)+y(1)^2+z(1)*w(1) },
    9 )$
end;
```

In the procedure, the y_1^2 term is automatically multiplied by ε, and so, in the results, ε counts the order of nonlinearity of each term. We analyse to high-order, errors $\mathcal{O}(\varepsilon^9, \sigma^3)$, because the results are simple.

Time dependent coordinate transform To decouple the stochastic dynamics, we just need to stochastically 'bend' the z-variable. This bending forms a stochastic foliation of the system.

$$\begin{aligned} z_1 &= -1/3\sigma\varepsilon e^{3t}\star w_1 Y_1^2 - 1/3\varepsilon Y_1^2 + Z_1 \\ y_1 &= Y_1 \end{aligned}$$

Result normal form DEs The normal form dynamics is linear and decoupled, as per Hartman–Grobman, namely

$$\dot{Z}_1 = \sigma w_1 Z_1 + Z_1 \qquad \dot{Y}_1 = \sigma w_1 Y_1 - Y_1$$

9 monahanFive: Monahan's five examples

Monahan & Culina (2011) discuss stochastic averaging and give several examples in the body and an appendix, of which we analyse five. They really need this approach as "a large separation often does not exist in atmosphere or ocean dynamics" between the fast and slow time scales.

9.1 Example four: 'three' time scales

Monahan & Culina (2011) comment that this, their fourth example, a linear system, has three time scales. But I do not see these time scales, I only see varying strength interactions. They consider

$$\frac{dx}{dt} = -x + \frac{a}{\sqrt{\tau}}y \quad \text{and} \quad \frac{dy}{dt} = \frac{1}{\sqrt{\tau}}x - \frac{1}{\tau}y + \frac{b}{\sqrt{\tau}}\dot{W}.$$

Let's rescale time, $t = \tau t'$ so that $d/dt = \frac{1}{\tau}d/dt'$ and $\dot{W} = \frac{1}{\sqrt{\tau}}dW/dt'$. Then, dropping dashes, the SDE system is

$$\frac{dx}{dt} = -\tau x + a\sqrt{\tau}y \quad \text{and} \quad \frac{dy}{dt} = \sqrt{\tau}x - y + b\dot{W}.$$

Start by loading the procedure.

```
69 in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system. Using the default inbuilt parametrisation of noise by `sigma` to represent parameter b, and using `small` in the x-SDE so that it counts the number of small $\sqrt{\tau}$, code these as the following.

```
factor tau,yy,y,w,ou;
stonormalform(
    { sqrt(tau)*a*y(1)-small*tau*x(1) },
    { sqrt(tau)*x(1)-y(1)+w(1) },
    {},
    4 )$
%end; % optionally end examples here
```

The procedure reports that it analyses the following family

$$\dot x_1 = \sqrt{\tau} y_1 \varepsilon a - \varepsilon^2 \tau x_1 \qquad \dot y_1 = w_1 \sigma - y_1 + \sqrt{\tau}\varepsilon x_1$$

in which we indeed see ε only in the grouping $\varepsilon\sqrt{\tau}$.

Time dependent coordinate transform This is linear as the system is linear.

$$y_1 = -e^{-1t}\star e^{-1t}\star w_1\,\sigma\varepsilon^2\tau a - e^{-1t}\star w_1\,\sigma\varepsilon^2\tau a + e^{-1t}\star w_1\,\sigma + Y_1 + \sqrt{\tau}\varepsilon X_1$$
$$x_1 = -\sqrt{\tau} e^{-1t}\star w_1\,\sigma\varepsilon a - \sqrt{\tau} Y_1 \varepsilon a + X_1$$

Result normal form DEs The normal form dynamics is linear and decoupled, as per Hartman–Grobman, namely

$$\dot Y_1 = -Y_1 \varepsilon^2 \tau a - Y_1$$
$$\dot X_1 = w_1 \sigma \varepsilon^3 \tau \sqrt{\tau}\big(-2a^2 + a\big) + \sqrt{\tau} w_1 \sigma \varepsilon a + \varepsilon^2 \tau \big(X_1 a - X_1\big)$$

Monahan & Culina (2011) derive the last two terms in the X-equation, but not the first as it is too small for their averaging analysis. They comment that $a > 1$ is some sort of difficulty, presumably because X grows when $a > 1$: but here we have no problem with $a > 1$, especially as the decay rate to the stochastic slow manifold, the Y-SDE, is $(1 + \tau a)$ which gets stronger with increasing parameter a.

9.2 Example one: simple rational nonlinear

With 'small' scale-separation parameter τ, Monahan & Culina (2011) first consider the example

$$\frac{dx}{dt} = -x + \Sigma(x) y \quad \text{and} \quad \frac{dy}{dt} = -\frac{1}{\tau} y + \frac{1}{\sqrt{\tau}} \dot W,$$

for general smooth functions $\Sigma(x)$. Rescale time, $t = \tau t'$ so that $d/dt = \frac{1}{\tau} d/dt'$ and $\dot W = \frac{1}{\sqrt{\tau}} dW/dt'$. Then, dropping dashes, the SDE is

$$\frac{dx}{dt} = -\tau x + \tau \Sigma(x) y \quad \text{and} \quad \frac{dy}{dt} = -y + \dot W.$$

The `stonormalform` procedure is already loaded. Write a message saying we are now analysing the next system.

```
write "**** Example One of Monahan (2011) ****";
```

Execute the construction of a normal form for this system. But let's restrict the general function $\Sigma(x)$ to the rational form $\Sigma(x) := (a_0 + a_1x + a_2x^2)/(1 + b_1x + b_2x^2)$. Code this form as the following (after multiplying through by the denominator).

```
factor df;
operator a; defindex a(down);
operator b; defindex b(down);
stonormalform(
    { -tau*x(1)*(1+b(1)*x(1)+b(2)*x(1)^2)
      -df(x(1),t)*(b(1)*x(1)+b(2)*x(1)^2)
      +tau*y(1)*(a(0)+a(1)*x(1)+a(2)*x(1)^2) },
    { -y(1)+w(1) },
    {},
    3 )$
%end; % optionally end examples here
```

The procedure reports that it analyses the following family

$$\dot{x}_1 = \frac{d x_1}{d t}\varepsilon\big(-b_2x_1^2 - b_1x_1\big) + y_1\varepsilon\tau\big(a_2x_1^2 + a_1x_1 + a_0\big) + \varepsilon\tau\big(-b_2x_1^3 - b_1x_1^2 - x_1\big)$$

$$\dot{y}_1 = w_1\sigma - y_1$$

so evaluate the results at $\varepsilon = 1$ to compare with the modelling of Monahan & Culina (2011).

Time dependent coordinate transform

$$y_1 = \mathrm{e}^{-1t}\star w_1\,\sigma + Y_1$$

$$x_1 = \mathrm{e}^{-1t}\star w_1\,\sigma\varepsilon\tau\big(-a_2X_1^2 - a_1X_1 - a_0\big) + Y_1\varepsilon\tau\big(-a_2X_1^2 - a_1X_1 - a_0\big) + X_1$$

Result normal form DEs

$$\dot{Y}_1 = -Y_1$$

$$\dot{X}_1 = w_1\sigma\varepsilon^2\tau^2\big(a_2b_2X_1^4 - a_2X_1^2 + 2a_1b_2X_1^3 + a_1b_1X_1^2 + 3a_0b_2X_1^2 + 2a_0b_1X_1 + a_0\big) + w_1\sigma\varepsilon^2\tau\big(-a_2b_2X_1^4 - a_2b_1X_1^3 - a_1b_2X_1^3 - a_1b_1X_1^2 - a_0b_2X_1^2 - a_0b_1X_1\big) + w_1\sigma\varepsilon\tau\big(a_2X_1^2 + a_1X_1 + a_0\big) + \varepsilon^2\tau\big(b_2^2X_1^5 + 2b_2b_1X_1^4 + b_2X_1^3 + b_1^2X_1^3 + b_1X_1^2\big) + \varepsilon\tau\big(-b_2X_1^3 - b_1X_1^2 - X_1\big)$$

Monahan & Culina (2011) derive some of this X equation. The other terms here are higher order terms that become significant at finite parameter values. For example, the next correction to their analysis, $w_1\tau^2(-3a_4X_1^4 - 2a_3X_1^3 - a_2X_1^2 + a_0)$, is probably derivable as $\tau^2(\Sigma - x\Sigma')\dot{W}$ (when rescaled).

9.3 Example three: several fast stable modes

Monahan & Culina (2011) third considered the example

$$\frac{dx}{dt} = -x + \Sigma(x)\|\vec y\| \quad \text{and} \quad \frac{d\vec y}{dt} = -\frac{1}{\tau}\vec y + \sqrt{\frac{2}{\tau}}\dot{\vec W},$$

for general smooth functions $\Sigma(x)$, and 'small' scale separation parameter τ. As before, rescale time, $t = \tau t'$ so that $d/dt = \frac{1}{\tau}d/dt'$ and $\dot W = \frac{1}{\sqrt\tau}dW/dt'$. Here I also cheat: they have $\|\vec y\|$ in the slow equation; but $\|\vec y\|$ is not a smooth multinomial and so my generic procedure cannot apply; instead I replace $\|\vec y\|$ with $\|\vec y\|^2$ which has the same symmetry but is multinomial. Then, upon the rescaling of time, and dropping dashes, the SDE is

$$\frac{dx}{dt} = -\tau x + \tau\Sigma(x)\|\vec y\|^2 \quad \text{and} \quad \frac{d\vec y}{dt} = -\vec y + \sigma\dot{\vec W}.$$

The `stonormalform` procedure is already loaded. Write a message saying we are now analysing the next system.

```
write "**** Example Three of Monahan (2011) ****";
```

Restrict analysis to the general quartic $\Sigma(x) := a_0 + a_1x + \cdots + a_4x^4$, and so code the system as the following (the generic program automatically inserts the σ in the noise). Currently restrict to just a two component $\vec y$ as I do not see any reason for any more, and Monahan & Culina (2011) do not appear to specify.

```
stonormalform(
    { -tau*x(1)+tau*(y(1)^2+y(2)^2)
      *(a(0)+a(1)*x(1)+a(2)*x(1)^2
      +a(3)*x(1)^3+a(4)*x(1)^4) },
    { -y(1)+w(1),
      -y(2)+w(2) },
    {},
    3 )$
%end; % optionally end examples here
```

The procedure reports that it analyses the following family

$$\begin{aligned}\dot x_1 &= y_2^2\varepsilon\tau\big(a_4x_1^4 + a_3x_1^3 + a_2x_1^2 + a_1x_1 + a_0\big) + y_1^2\varepsilon\tau\big(a_4x_1^4 + a_3x_1^3 + {}\\ &\quad a_2x_1^2 + a_1x_1 + a_0\big) - \varepsilon\tau x_1\\ \dot y_1 &= w_1\sigma - y_1\\ \dot y_2 &= w_2\sigma - y_2\end{aligned}$$

in which we see ε only in the grouping $\varepsilon\tau$, so truncating to errors $\mathcal O\big(\varepsilon^3\big)$ is the same as to errors $\mathcal O\big(\tau^3\big)$.

Time dependent coordinate transform

$$y_1 = \mathrm{e}^{-1t}\star w_1\,\sigma + Y_1$$

$$\begin{aligned}
y_2 &= \mathrm{e}^{-1t}\star w_2\,\sigma + Y_2\\
x_1 &= \mathrm{e}^{t}\star w_2\,Y_2\sigma\varepsilon\tau\big(- a_4X_1^4 - a_3X_1^3 - a_2X_1^2 - a_1X_1 - a_0\big) + \\
&\quad \mathrm{e}^{-1t}\star w_2\,Y_2\sigma\varepsilon\tau\big(- a_4X_1^4 - a_3X_1^3 - a_2X_1^2 - a_1X_1 - a_0\big) + \\
&\quad \mathrm{e}^{t}\star w_1\,Y_1\sigma\varepsilon\tau\big(- a_4X_1^4 - a_3X_1^3 - a_2X_1^2 - a_1X_1 - a_0\big) + \\
&\quad \mathrm{e}^{-1t}\star w_1\,Y_1\sigma\varepsilon\tau\big(-a_4X_1^4-a_3X_1^3-a_2X_1^2-a_1X_1-a_0\big)+Y_2^2\varepsilon\tau\big(- \\
&\quad 1/2a_4X_1^4-1/2a_3X_1^3-1/2a_2X_1^2-1/2a_1X_1-1/2a_0\big)+Y_1^2\varepsilon\tau\big(- \\
&\quad 1/2a_4X_1^4 - 1/2a_3X_1^3 - 1/2a_2X_1^2 - 1/2a_1X_1 - 1/2a_0\big) + X_1
\end{aligned}$$

The complicated form of x_1 only reflects the transient effects of the decaying $\vec{Y}$: once $\vec{Y} \to 0$, then $x_1 = X_1$.

Result normal form DEs

$$\begin{aligned}
\dot{Y}_1 &= -Y_1\\
\dot{Y}_2 &= -Y_2\\
\dot{X}_1 &= \mathrm{e}^{-1t}\star w_2\,w_2\sigma^2\varepsilon^2\tau^2\big(- 3/2a_4X_1^4 - a_3X_1^3 - 1/2a_2X_1^2 + 1/2a_0\big) + \\
&\quad \mathrm{e}^{-1t}\star w_2\,w_2\sigma^2\varepsilon\tau\big(a_4X_1^4 + a_3X_1^3 + a_2X_1^2 + a_1X_1 + a_0\big) + \\
&\quad \mathrm{e}^{-1t}\star w_1\,w_1\sigma^2\varepsilon^2\tau^2\big(- 3/2a_4X_1^4 - a_3X_1^3 - 1/2a_2X_1^2 + 1/2a_0\big) + \\
&\quad \mathrm{e}^{-1t}\star w_1\,w_1\sigma^2\varepsilon\tau\big(a_4X_1^4 + a_3X_1^3 + a_2X_1^2 + a_1X_1 + a_0\big) - \varepsilon\tau X_1
\end{aligned}$$

These show the decay of $\vec{Y}$, and that the irreducible noise-noise interactions are the only modifications to the slow decay of X_1, and hence of x_1.

9.4 Example two: irregular slow manifold

Monahan & Culina (2011) second consider the example SDEs

$$\frac{dx}{dt} = x - x^3 + \Sigma(x)y \quad\text{and}\quad \frac{dy}{dt} = -\frac{1}{x\tau}y + \frac{1}{\sqrt{\tau}}\dot{W},$$

for general smooth functions $\Sigma(x)$, and 'small' scale separation parameter τ. Since the y-dynamics are exponentially unstable for negative x, we restrict attention to $x > 0$. Even for positive x the system is singular as $x \to 0$ so the slow manifold is irregular in some sense (although 'singular' in a good way in that the scale separation between fast and slow becomes infinite). Let's be more sophisticated in rescaling time: let's choose the new fast time t' so that $dt = x\tau\,dt'$; that is, $t' = \int(x\tau)^{-1}dt$ which would not be explicitly known until after a solution $x(t')$ is found. I presume that the noise then transforms as $\dot{W} = \frac{1}{\sqrt{x\tau}}dW/dt'$ (needs checking). Then, dropping dashes, the SDEs are

$$\frac{dx}{dt} = \tau\left[x^2 - x^4 + x\Sigma(x)y\right] \quad\text{and}\quad \frac{dy}{dt} = -y + \sqrt{x}\dot{W}.$$

The $\sqrt{x}$ is a problem in my generic procedure as it requires multinomial systems, so transform to $x = x_1^2$ (*not* the usual $x = x_1$) so that $2x_1dx_1 = dx$. Then the SDE system becomes

$$\frac{dx_1}{dt} = \frac{1}{2}\tau\left[x_1^3 - x_1^7 + x_1\Sigma(x_1^2)y\right] \quad\text{and}\quad \frac{dy}{dt} = -y + x_1\dot{W}.$$

The stonormalform procedure is already loaded. Write a message saying we are now analysing the next system.

```
write "**** Example Two of Monahan (2011) ****";
```

Restricting to the general linear $\Sigma(x) := a_0 + a_1 x$, code the SDE system as the following (remember $\mathtt{x(1)} = x_1 = \sqrt{x}$).

```
stonormalform(
    { 1/2*tau*( x(1)^3 -x(1)^7
      +x(1)*(a(0)+a(1)*x(1)^2)*y(1) ) },
    { -y(1) +x(1)*w(1) },
    {},
    3 )$
%end; % optionally end examples here
```

The procedure reports that it analyses the following family

$$\dot{x}_1 = y_1 \varepsilon\tau\big(1/2a_1x_1^3 + 1/2a_0x_1\big) + \varepsilon\tau\big(- 1/2x_1^7 + 1/2x_1^3\big)$$
$$\dot{y}_1 = w_1\sigma x_1 - y_1$$

Again, usefully, the artificial ε only occurs in the combination $\varepsilon\tau$ and so just counts the number of factors of τ in each term. That is, errors $\mathcal{O}\big(\varepsilon^3\big)$ is the same as errors $\mathcal{O}\big(\tau^3\big)$.

Time dependent coordinate transform Straightforwardly

$$y_1 = \mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_1\ \sigma\varepsilon\tau\big(1/2X_1^7 - 1/2X_1^3\big) + \mathrm{e}^{-1t}\star w_1\,\sigma X_1 + Y_1$$
$$x_1 = \mathrm{e}^{-1t}\star w_1\,\sigma\varepsilon\tau\big(- 1/2a_1X_1^4 - 1/2a_0X_1^2\big) + Y_1\varepsilon\tau\big(- 1/2a_1X_1^3 - 1/2a_0X_1\big) + X_1$$

Result normal form DEs As expected, $Y_1 = 0$ is a stochastic slow manifold, that is almost surely emergent in some domain.

$$\dot{Y}_1 = \mathrm{e}^{t}\star w_1\,w_1Y_1\sigma^2\varepsilon^2\tau^2\big(1/4a_1^2X_1^6 + 1/2a_1a_0X_1^4 + 1/4a_0^2X_1^2\big) + \mathrm{e}^{-1t}\star w_1\,w_1Y_1\sigma^2\varepsilon^2\tau^2\big(3/4a_1^2X_1^6 + a_1a_0X_1^4 + 1/4a_0^2X_1^2\big) + w_1Y_1\sigma\varepsilon^2\tau^2\big(- a_1X_1^9 - 3/2a_0X_1^7 + 1/2a_0X_1^3\big) + w_1Y_1\sigma\varepsilon\tau\big(- 1/2a_1X_1^3 - 1/2a_0X_1\big) - Y_1$$
$$\dot{X}_1 = \mathrm{e}^{-1t}\star w_1\,w_1\sigma^2\varepsilon^2\tau^2\big(- 1/4a_1^2X_1^7 - 1/2a_1a_0X_1^5 - 1/4a_0^2X_1^3\big) + w_1\sigma\varepsilon^2\tau^2\big(a_1X_1^{10} + 3/2a_0X_1^8 - 1/2a_0X_1^4\big) + w_1\sigma\varepsilon\tau\big(1/2a_1X_1^4 + 1/2a_0X_1^2\big) + \varepsilon\tau\big(- 1/2X_1^7 + 1/2X_1^3\big)$$

Using just the leading order terms for $\dot{X}_1$, the terms linear in τ, and recalling $X_1 \approx x_1 = \sqrt{x}$, the last SDE gives the model

$$\frac{dx}{dt'} \approx \tau\left[x^2 - x^4 + x^{3/2}\Sigma(x)\sigma\frac{dW}{dt'}\right].$$

But recall that $dt' = dt/(x\tau)$ (although one should be more careful as $X_1 \approx \sqrt{x}$, not exact equality) and similarly $dW/dt' = \sqrt{x\tau}\dot{W}$ so that this model becomes

$$\frac{dx}{dt} \approx x - x^3 + \sqrt{\tau}x\Sigma(x)\sigma\frac{dW}{dt}.$$

This agrees with the Stratonovich part of (A28) by Monahan & Culina (2011). But again, the above derivation has the systematic higher order corrections that are needed for finite scale separation τ.

9.5 Idealised Stommel-like model of meridional overturning circulation

Monahan & Culina (2011) also analyse the Idealised Stommel-like model, for small scale-separation parameter τ,

$$\begin{aligned}\frac{dx}{dt} &= \mu - |y-x|x + \sigma_A \dot W_1\,,\\ \frac{dy}{dt} &= +\frac{1}{\tau}(1-y) - |y-x|y + \sqrt{\frac{2}{\tau}}\sigma_M \dot W_2\,.\end{aligned}$$

The mod-functions do not fit into my generic computer algebra so replace them with squares to preserve the symmetry. As before, rescale time, $t=\tau t'$ so that $d/dt = \frac{1}{\tau}d/dt'$ and $\dot W_j = \frac{1}{\sqrt{\tau}}dW_j/dt'$. Since for small τ, the fast variable y is strongly attracted to one, change the reference point for y by setting $y = 1 + y_1(t)$. Then the SDEs becomes akin to

$$\begin{aligned}\frac{dx}{dt'} &= \epsilon^2\left[\mu - (1+y_1-x)^2 x\right] + \epsilon\sigma_A \frac{dW_1}{dt'}\,,\\ \frac{dy_1}{dt'} &= -y_1 - \epsilon^2(1+y_1-x)^2(1+y_1) + \sqrt{2}\sigma_M\frac{dW_2}{dt'}\,.\end{aligned}$$

The `stonormalform` procedure is already loaded. Write a message saying we are now analysing the next system.

```
write "**** Stommel-like model of Monahan (2011) ****";
factor rho;
```

Let $\rho := \sigma_A/(\sqrt{2}\sigma_M)$, use the inbuilt $\sigma := \sqrt{2}\sigma_M$, and invoke `small` to correctly count the number of small $\sqrt{\tau}$s in the analysis. Code the above dynamics as the following.

```
stonormalform(
    { small*tau*(mu-(1+y(1)-x(1))^2*x(1))
      +small*sqrt(tau)*rho*w(1) },
    { -y(1)-small*tau*(1+y(1)-x(1))^2*(1+y(1))+w(2) },
    {},
    4 )$
end; % finish here if not before
```

The procedure reports that it analyses the following family, an expended version of the prescribed system,

$$\begin{aligned}\dot x_1 &= \sqrt{\tau}w_1\rho\sigma\varepsilon - y_1^2\varepsilon^2\tau x_1 + y_1\varepsilon^2\tau\big(2x_1^2 - 2x_1\big) + \varepsilon^2\tau\big(-x_1^3 + \\ &\quad 2x_1^2 - x_1 + \mu\big)\\ \dot y_1 &= w_2\sigma - y_1^3\varepsilon^2\tau + y_1^2\varepsilon^2\tau\big(2x_1 - 3\big) + y_1\varepsilon^2\tau\big(-x_1^2 + 4x_1 - 3\big) - \\ &\quad y_1 + \varepsilon^2\tau\big(-x_1^2 + 2x_1 - 1\big)\end{aligned}$$

Again the artificial ε only occurs in the combination $\varepsilon^2\tau$ and so just counts the number of factors of τ in each term. That is, errors $\mathcal{O}\big(\varepsilon^4\big)$ is the same as errors $\mathcal{O}\big(\tau^2\big)$.

Time dependent coordinate transform Straightforward but complicated in detail:

$$y_1 = \mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_2\,\sigma\varepsilon^2\tau\big(-X_1^2+4X_1-3\big)+3/2\mathrm{e}^{t}\star w_2\,Y_1^2\sigma\varepsilon^2\tau+3/2\mathrm{e}^{-1t}\star w_2\,Y_1^2\sigma\varepsilon^2\tau+\mathrm{e}^{-1t}\star w_2\,Y_1\sigma\varepsilon^2\tau\big(-4X_1+6\big)+\mathrm{e}^{-1t}\star w_2\,\sigma+1/2Y_1^3\varepsilon^2\tau+Y_1^2\varepsilon^2\tau\big(-2X_1+3\big)+Y_1+\varepsilon^2\tau\big(-X_1^2+2X_1-1\big)$$

$$x_1 = \mathrm{e}^{t}\star w_2\,Y_1\sigma\varepsilon^2\tau X_1+\mathrm{e}^{-1t}\star w_2\,Y_1\sigma\varepsilon^2\tau X_1+\mathrm{e}^{-1t}\star w_2\,\sigma\varepsilon^2\tau\big(-2X_1^2+2X_1\big)+1/2Y_1^2\varepsilon^2\tau X_1+Y_1\varepsilon^2\tau\big(-2X_1^2+2X_1\big)+X_1$$

Result normal form DEs As expected, $Y_1=0$ is a stochastic slow manifold which is almost surely emergent:

$$\dot Y_1 = -3\mathrm{e}^{-1t}\star w_2\,w_2Y_1\sigma^2\varepsilon^2\tau+4\sqrt{\tau}\mathrm{e}^{-1t}\star w_2\,w_1Y_1\rho\sigma^2\varepsilon^3\tau+w_2Y_1\sigma\varepsilon^2\tau\big(4X_1-6\big)+Y_1\varepsilon^2\tau\big(-X_1^2+4X_1-3\big)-Y_1$$

$$\dot X_1 = -\mathrm{e}^{-1t}\star w_2\,w_2\sigma^2\varepsilon^2\tau X_1+\mathrm{e}^{-1t}\star w_2\,w_1\rho\sigma^2\varepsilon^3\tau^{3/2}\big(4X_1-2\big)+w_2\sigma\varepsilon^2\tau\big(2X_1^2-2X_1\big)+\sqrt{\tau}w_1\rho\sigma\varepsilon+\varepsilon^2\tau\big(-X_1^3+2X_1^2-X_1+\mu\big)$$

Deterministically, this model has multiple equilibria for small μ, but only one equilibria for $\mu>4/27$, at finite amplitude. The noise $\dot W_1$ causes transitions between such multiple equilibria, and the multiplicative noise $\dot W_2$ contributes as well. But the same order of smallness is the *first* term in the X_1 SDE above which is a quadratic noise-noise interaction that has a mean drift effect which should enhance the stability of the small x equilibrium.

10 majdaTriad: Majda's two triad models

Majda et al. (2002) investigated averaging in two 3D SDE systems. Let's compare with their stochastic normal form.

10.1 Multiplicative triad model

The multiplicative triad model of Majda et al. (2002) consists of three modes, v_1, v_2 and v_3. These evolve in time according to

$$\frac{dv_1}{dt}=b_1v_2v_3\,,\quad \frac{dv_2}{dt}=b_2v_1v_3\,,\quad \frac{dv_3}{dt}=-v_3+b_3v_1v_2+\sigma\dot W,$$

where b_j and σ are some constants, and the noise forces the third mode. Here I have already scaled the equations so that the rate of decay of the third mode is one. Thus on long time scales we expect the third mode to be essentially negligible and the system to be modelled by the relatively slow evolution of the first two modes.

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

The system uses parameters b_j so define

```
operator b; defindex b(down);
```

Execute the construction of a normal form for this system using $x_j = v_j$ and $y_1 = v_3$.

```
factor yy;
stonormalform(
    { b(1)*x(2)*y(1),
      b(2)*x(1)*y(1) },
    { -y(1)+b(3)*x(1)*x(2)+w(3) },
    {},
    4 )$
%end; % optionally end examples here
```

The procedure reports that it analyses the following family

$$\dot{x}_1 = \varepsilon b_1 x_2 y_1 \qquad \dot{x}_2 = \varepsilon b_2 x_1 y_1 \qquad \dot{y}_1 = w_3\sigma + \varepsilon b_3 x_2 x_1 - y_1$$

Here, ε counts the order of nonlinearity so that errors $\mathcal{O}\big(\varepsilon^4\big)$ are errors $\mathcal{O}\big(|\vec{v}|^5 + \sigma^5\big)$ (due to the noise driving fluctuations of size σ).

Time dependent coordinate transform Straightforwardly,

$$\begin{aligned}
y_1 &= \mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_3\ \sigma\varepsilon^2\big(-b_3b_2X_1^2 - b_3b_1X_2^2\big) + \mathrm{e}^{-1t}\star w_3\,\sigma\varepsilon^2\big(- \\
&\quad b_3b_2X_1^2 - b_3b_1X_2^2\big) + \mathrm{e}^{-1t}\star w_3\,\sigma + Y_1 + \varepsilon b_3X_2X_1 \\
x_1 &= \mathrm{e}^{-1t}\star w_3\,Y_1\sigma\varepsilon^2 b_2b_1X_1 - \mathrm{e}^{-1t}\star w_3\,\sigma\varepsilon b_1X_2 + 1/2Y_1^2\varepsilon^2b_2b_1X_1 - \\
&\quad Y_1\varepsilon b_1X_2 + X_1 \\
x_2 &= \mathrm{e}^{-1t}\star w_3\,Y_1\sigma\varepsilon^2 b_2b_1X_2 - \mathrm{e}^{-1t}\star w_3\,\sigma\varepsilon b_2X_1 + 1/2Y_1^2\varepsilon^2b_2b_1X_2 - \\
&\quad Y_1\varepsilon b_2X_1 + X_2
\end{aligned}$$

Result normal form DEs As expected, $Y_1 = 0$ is the emergent (almost always) stochastic slow manifold.

$$\begin{aligned}
\dot{Y}_1 &= 4w_3Y_1\sigma\varepsilon^3b_3b_2b_1X_2X_1 + Y_1\varepsilon^2\big(-b_3b_2X_1^2 - b_3b_1X_2^2\big) - Y_1 \\
\dot{X}_1 &= w_3\sigma\varepsilon^3\big(-2b_3b_2b_1X_2X_1^2 - 2b_3b_1^2X_2^3\big) + w_3\sigma\varepsilon b_1X_2 + \varepsilon^2b_3b_1X_2^2X_1 \\
\dot{X}_2 &= w_3\sigma\varepsilon^3\big(-2b_3b_2^2X_1^3 - 2b_3b_2b_1X_2^2X_1\big) + w_3\sigma\varepsilon b_2X_1 + \varepsilon^2b_3b_2X_2X_1^2
\end{aligned}$$

Majda et al. (2002) predicts, their equation (52), the two leading order terms in the deterministic part and the linear noise part. I suspect their first term in each equation is an Ito version of my Stratonovich modelling. All higher order terms are missed by their averaging, but easily constructed here by increasing the argument 4 to the procedure.

10.2 Additive triad model

The additive triad model of Majda et al. (2002) consists of three modes, v_1, v_2 and v_3, as before. However, these now evolve in time according to

$$\frac{dv_1}{dt} = b_1v_2v_3\,,$$

$$
\begin{aligned}
\frac{dv_2}{dt} &= -v_2 + b_2 v_1 v_3 + \sigma_2 \dot{W}_2\,,\\
\frac{dv_3}{dt} &= -v_3 + b_3 v_1 v_2 + \sigma_3 \dot{W}_3\,,
\end{aligned}
$$

where b_j and σ_j are some constants, and there is independent stochastic forcing of the second and third modes. Here I have already scaled the equations so that the rate of decay of *both* the second and third mode is one.[2] Thus on long time scales we expect the second and third modes to be essentially negligible and the system to be modelled by the relatively slow evolution of the first mode. This section constructs the stochastic normal form of its centre manifold model as the basis for a model over long time scales with new noise processes.

The procedure `stonormalform` is already loaded. Write a message saying we are now analysing the next system.

```
write "**** Additive Triad system of Majda (2002) ****";
```

Execute the construction of a normal form for this system using $x_1 = v_1$, $y_j = v_{j+1}$, and $b_{j1}\sigma = \sigma_j$.

```
stonormalform(
    { b(1)*y(2)*y(1) },
    { -y(1)+b(2)*x(1)*y(2)+b(21)*w(2),
      -y(2)+b(3)*x(1)*y(1)+b(31)*w(3) },
    {},
    3 )$
end; % finish here if not before
```

The procedure reports that it analyses the following family

$$
\begin{aligned}
\dot{x}_1 &= \varepsilon b_1 y_2 y_1\\
\dot{y}_1 &= \sigma b_{21} w_2 + \varepsilon b_2 x_1 y_2 - y_1\\
\dot{y}_2 &= \sigma b_{31} w_3 + \varepsilon b_3 x_1 y_1 - y_2
\end{aligned}
$$

Here, ε counts the order of nonlinearity so that the errors $\mathcal{O}\big(\varepsilon^3\big)$ are errors $\mathcal{O}\big(|\vec{v}|^4 + \sigma^4\big)$ (due to the noise driving fluctuations of size σ).

Time dependent coordinate transform Straightforwardly,

$$
\begin{aligned}
y_1 &= Y_1 + \sigma\varepsilon b_{31} b_2 \mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_3\ X_1 + \sigma b_{21}\mathrm{e}^{-1t}\star w_2\\
y_2 &= Y_2 + \sigma\varepsilon b_{21} b_3 \mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_2\ X_1 + \sigma b_{31}\mathrm{e}^{-1t}\star w_3\\
x_1 &= -1/2 Y_2 Y_1 \varepsilon b_1 + Y_2\sigma\varepsilon\big(-1/2 b_{21} b_1 \mathrm{e}^{t}\star w_2 - 1/2 b_{21} b_1 \mathrm{e}^{-1t}\star w_2\big) +\\
&\quad Y_1\sigma\varepsilon\big(-1/2 b_{31} b_1 \mathrm{e}^{t}\star w_3 - 1/2 b_{31} b_1 \mathrm{e}^{-1t}\star w_3\big) + X_1
\end{aligned}
$$

[2] In contrast, Majda et al. (2002) set the two modes to have different decay rates. Do not expect much difference in using the same decay rate, it is just more convenient that the memory convolutions are then identical for the two modes rather than being different. Having the decay rates the same is also closer to my expected application to spatial problems.

Result normal form DEs As expected, $Y_1 = Y_2 = 0$ is the emergent (almost always) stochastic slow manifold. Unusually, on this slow manifold $x_1 = X_1$ exactly (to at least the next few orders).

$$\begin{aligned}\dot Y_1 &= Y_2\sigma^2\varepsilon^2\big(- 1/2b_{31}b_{21}b_2b_1\mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_3\ w_2 - \\&\quad 1/2b_{31}b_{21}b_2b_1\mathrm{e}^{-1t}\star w_3\, w_2 - 1/2b_{31}b_{21}b_2b_1\mathrm{e}^{-1t}\star w_2\, w_3\big) + \\&\quad Y_2\varepsilon b_2X_1 + Y_1\sigma^2\varepsilon^2\big(- 1/2b_{31}^2b_2b_1\mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_3\ w_3 - \\&\quad 1/2b_{31}^2b_2b_1\mathrm{e}^{-1t}\star w_3\, w_3\big) - Y_1\\ \dot Y_2 &= Y_2\sigma^2\varepsilon^2\big(- 1/2b_{21}^2b_3b_1\mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_2\ w_2 - \\&\quad 1/2b_{21}^2b_3b_1\mathrm{e}^{-1t}\star w_2\, w_2\big) - Y_2 + Y_1\sigma^2\varepsilon^2\big(- \\&\quad 1/2b_{31}b_{21}b_3b_1\mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_2\ w_3 - 1/2b_{31}b_{21}b_3b_1\mathrm{e}^{-1t}\star w_3\, w_2 - \\&\quad 1/2b_{31}b_{21}b_3b_1\mathrm{e}^{-1t}\star w_2\, w_3\big) + Y_1\varepsilon b_3X_1\\ \dot X_1 &= \sigma^2\varepsilon^2\big(1/2b_{31}^2b_2b_1\mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_3\ w_3X_1 + \\&\quad 1/2b_{31}^2b_2b_1\mathrm{e}^{-1t}\star w_3\, w_3X_1 + 1/2b_{21}^2b_3b_1\mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_2\ w_2X_1 + \\&\quad 1/2b_{21}^2b_3b_1\mathrm{e}^{-1t}\star w_2\, w_2X_1\big) + \sigma^2\varepsilon\big(1/2b_{31}b_{21}b_1\mathrm{e}^{-1t}\star w_3\, w_2 + \\&\quad 1/2b_{31}b_{21}b_1\mathrm{e}^{-1t}\star w_2\, w_3\big)\end{aligned}$$

The only terms in the model for $\dot X_1$ are the quadratic noise-noise interaction terms. Majda et al. (2002) recognise the last, σ^2 term, but not the first, $X_1\sigma^2$ term. They represent the last as a mean drift and independent noise (the mean drift comes from the Ito representation of the above Stratonovich noise-noise interaction).

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