Centre (slow) manifold of the example of §7.1 by Lee & Othmer (2010)

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Rescale the system (84) to the fast time so their $\epsilon = \varepsilon^2$ here. For simplicity set $k_i = 1$ for all i. Then $\epsilon = 0$ and $\vec{c} = (\sqrt{C_2}, C_2, C_3)$ is a global manifold of equilibria. So with $C_1 = \sqrt{C_2}$ change to local variables $\vec{u}$ by $c_i(t) = C_i + u_i(t)$. My website models the dynamics (creates the slow manifold) with the following input:

```
ff_:=tp mat((-2*((sqrt(c2)+u1)^2-(c2+u2))
,((sqrt(c2)+u1)^2-(c2+u2))
+small*((c3+u3)-(c2+u2))
,0-small*((c3+u3)-(c2+u2))));
freqm_:=mat((0,0));
ee_:=tp mat((1/2/sqrt(c2),1,0),(0,0,1));
zz_:=tp mat((0,1,0),(0,0,1));
toosmall:=3;
```

Throughout and generally: the lowest order, most important, terms are near the end of each expression.

The specified dynamical system

$$\dot{u}_1 = -2\varepsilon u_1^2 - 4\sqrt{C_2}u_1 + 2u_2$$

$$\dot{u}_2 = \varepsilon^2\big(- e^{0i}C_2 + e^{0i}C_3 - u_2 + u_3\big) + \varepsilon u_1^2 + 2\sqrt{C_2}u_1 - u_2$$

$$\dot{u}_3 = \varepsilon^2\big(e^{0i}C_2 - e^{0i}C_3 + u_2 - u_3\big)$$

Centre subspace basis vectors

$\vec{e}_1 = \{\{1/2\sqrt{C_2}C_2^{-1}, 1, 0\}, e^{0i}\}$

$\vec{e}_2 = \{\{0, 0, 1\}, e^{0i}\}$

$\vec{z}_1 = \{\{0, 1, 0\}, e^{0i}\}$

$\vec{z}_2 = \{\{0, 0, 1\}, e^{0i}\}$

The centre manifold These give the location of the centre manifold in terms of parameters s_j.

$u_1 = -1/8\sqrt{C_2}\varepsilon s_1^2 C_2^{-2} + 1/2\sqrt{C_2}s_1 C_2^{-1}$

$u_2 = s_1$

$u_3 = s_2$

That is,

$$
\begin{aligned}
c_2 &= C_2 + u_2 = C_2 + s_1, \\
c_3 &= C_3 + u_3 = C_3 + s_2, \\
c_1 &= C_1 + u_1 \approx \sqrt{C_2} + \tfrac{1}{2}s_1/\sqrt{C_2} - \varepsilon\tfrac{1}{8}s_1^2/C_2^{3/2} \approx \sqrt{C_2 + s_1}\,.
\end{aligned}
$$

Centre manifold ODEs The system evolves on the centre manifold such that the parameters evolve according to these ODEs.

$\dot{s}_1 = \varepsilon^2\big(-4\sqrt{C_2}s_2 + 4\sqrt{C_2}s_1 + 4\sqrt{C_2}C_2 - 4\sqrt{C_2}C_3 + 16s_2C_2 - 16s_1C_2 - 16C_2^2 + 16C_2C_3\big)/\big(16C_2 - 1\big)$

$\dot{s}_2 = \varepsilon^2\big(-s_2 + s_1 + C_2 - C_3\big)$

To match the result (92), consider the 2nd evolution equation first:

$$
\begin{aligned}
\dot{c}_3 &= (C_3 + s_2)^{\cdot} = \dot{s}_2 = \varepsilon^2(-s_2 + s_1 + C_2 - C_3) \\
&= \epsilon[(C_2 + s_1) - (C_3 + s_2)] = \epsilon(c_2 - c_3).
\end{aligned}
$$

Similarly, but in brief, the first evolution equation goves

$$\begin{aligned}
\dot c_2 &= (C_2+s_1)^{\cdot} = \dot s_1 \\
&= \varepsilon^2\big(-4\sqrt{C_2}s_2+4\sqrt{C_2}s_1+4\sqrt{C_2}C_2-4\sqrt{C_2}C_3 \\
&\quad +16s_2C_2-16s_1C_2-16C_2^2+16C_2C_3\big)/\big(16C_2-1\big) \\
&\equiv \epsilon\big(4\sqrt{c_2}c_2-4\sqrt{c_2}c_3+16s_2c_2-16c_2^2+16c_2c_3\big)/\big(16c_2-1\big) \\
&= \epsilon 4(c_3-c_2)\sqrt{c_2}/(4\sqrt{c_2}+1)
\end{aligned}$$

which also matches (92).

The projection of initial conditions arise from the following, but I have not cancelled the common factors to see how the match.

Normals to isochrons at the slow manifold Use these vectors: to project initial conditions onto the slow manifold; to project non-autonomous forcing onto the slow evolution; to predict the consequences of modifying the original system; in uncertainty quantification to quantify effects on the model of uncertainties in the original system. The normal vector $\vec z_j := (z_{j1},\ldots,z_{jn}\ )$

$z_{11}=\varepsilon^2\big(-3072\sqrt{C_2}s_1^2C_2+28\sqrt{C_2}s_1^2C_2^{-1}-1/4\sqrt{C_2}s_1^2C_2^{-2}-192\sqrt{C_2}s_1^2+8192\sqrt{C_2}C_2^3+2560\sqrt{C_2}C_2^2-160\sqrt{C_2}C_2-2\sqrt{C_2}+2048s_1^2C_2-128s_1^2-8192C_2^3+32C_2\big)/\big(65536C_2^4-16384C_2^3+1536C_2^2-64C_2+1\big)+\varepsilon\big(\sqrt{C_2}s_1C_2^{-1}+16\sqrt{C_2}s_1-8s_1\big)/\big(256C_2^2-32C_2+1\big)+\big(-2\sqrt{C_2}+8C_2\big)/\big(16C_2-1\big)$

$z_{12}=\varepsilon^2\big(-6144\sqrt{C_2}s_1^2C_2+56\sqrt{C_2}s_1^2C_2^{-1}-1/2\sqrt{C_2}s_1^2C_2^{-2}-384\sqrt{C_2}s_1^2+6144\sqrt{C_2}C_2^2-256\sqrt{C_2}C_2-8\sqrt{C_2}+4096s_1^2C_2-256s_1^2-8192C_2^3-1024C_2^2+96C_2\big)/\big(65536C_2^4-16384C_2^3+1536C_2^2-64C_2+1\big)+\varepsilon\big(2\sqrt{C_2}s_1C_2^{-1}+32\sqrt{C_2}s_1-16s_1\big)/\big(256C_2^2-32C_2+1\big)+\big(-4\sqrt{C_2}+16C_2\big)/\big(16C_2-1\big)$

$z_{13}=\varepsilon^2\big(-8\sqrt{C_2}+16C_2+1\big)/\big(256C_2^2-32C_2+1\big)$

$z_{21}=\varepsilon^2\big(-32\sqrt{C_2}C_2-2\sqrt{C_2}+16C_2\big)/\big(256C_2^2-32C_2+1\big)$

$z_{22}=\varepsilon^2\big(-8\sqrt{C_2}+16C_2+1\big)/\big(256C_2^2-32C_2+1\big)$

$z_{23}=1$