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A NOVEL APPROACH TO EXACT SLOW-FAST DECOMPOSITION OF LINEAR SINGULARLY PERTURBED SYSTEMS WITH SMALL DELAYS*

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Abstract. A linear time-invariant singularly perturbed system with multiple pointwise and distributed small time delays is considered. A novel (direct) approach to exact slow-fast decomposition of this system is proposed. In contrast with the existing method, this approach uses neither a preliminary transformation of the original differential system to an integral one nor rather complicated integral manifold and operator techniques. Moreover, the approach of the present paper does not assume the exponential stability of the fast subsystem. Based on this decomposition, an exact slow-fast decomposition of the spectrum of a singularly perturbed system with a single pointwise small delay is carried out. Using the theoretical results, the stability of a multilink single-sink optical network is analyzed.

Key words. singularly perturbed system, time delay system, small delay, exact slow-fast system decomposition, exact slow-fast spectrum decomposition, multilink single-sink optical network, stability

AMS subject classifications. 34K06, 34K26, 34K20, 34K25

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1. Introduction.

Various multi–time-scale (in particular, two–time-scale) processes are adequately modeled by singularly perturbed differential systems. Such systems have been studied extensively in the literature (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8] and references therein). An important type of singularly perturbed differential system is a system with time delays of the order of a small parameter $\varepsilon > 0$ multiplying a part of the highest-order state derivatives in the system. Such systems arise in various applications (see, e.g., [9, 10, 11, 12, 13, 14, 15]). Surveys of results in this topic can be found in [3, 5, 8, 16].

To the best of our knowledge, among various analytical approaches to study of singularly perturbed systems with delays, two approaches are mainly developed in the literature. The first one uses either a Lyapunov–Krasovskii-type functional or a Razumikhin-type theorem, leading to reduction of the analysis of the original system to analysis of a set of linear matrix inequalities (see, e.g., [12, 13, 15, 17, 18, 19, 20, 21, 22] and references therein). The second approach uses a slow-fast decomposition of the original system, followed by separate analysis of two resulting slow and fast subsystems, which are single–time-scale ones and independent of each other. Based on this analysis, various properties of the original system, valid for all sufficiently small values of the parameter of singular perturbation, are deduced (see, e.g., [16, 19, 23, 24, 25, 26, 27, 28] and references therein). Although the first approach is simple enough for implementation, it is rather conservative because it does not preserve the infinite-dimensional nature of the original system. The second

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method, preserving the infinite-dimensional nature of the original system, is much less conservative.

In the present paper, we develop the second approach to analysis of singularly perturbed systems with delays, i.e., the slow-fast decomposition approach. There are two types of such a decomposition, asymptotic and exact. The asymptotic decomposition yields a set of unconnected slow and fast subsystems, which is not equivalent to the original singularly perturbed system. Due to such a decomposition, one needs to prove that some property of the decomposed system is valid also for the original one, which is a disadvantage of the asymptotic slow-fast decomposition. In contrast with the asymptotic decomposition, the exact slow-fast decomposition transforms the original system to the equivalent set of unconnected slow and fast subsystems. The latter means that both systems, original and decomposed, have the same properties, while a study of the decomposed system can be carried out separately for the slow and fast parts, which essentially simplifies analysis of the original system. This is a considerable advantage of the exact slow-fast decomposition of a singularly perturbed system.

In this paper, we deal with the exact slow-fast decomposition. For undelayed singularly perturbed systems, starting from the work [29], such a decomposition has been developed and used in numerous works (see, e.g., [2, 4, 30, 31, 32, 33, 34] and references therein). However, the exact slow-fast decomposition of singularly perturbed systems with the small delays was developed in only two works [23, 24]. In [23], the case of the system with delays only in the fast state variable was considered, and the decomposition was applied for feedback stabilization of such a system. In [24], the case of the system with a single pointwise delay and a distributed delay in both slow and fast state variables was studied, and the decomposition was applied for stability analysis of the considered system. In these papers, the exact slow-fast decomposition was carried out based on a transformation of the original differential system to an equivalent integral one, and using integral manifolds and operator techniques subject to the assumption on the exponential stability of the fast subsystem. The decomposed system was an integral system.

In the present paper, a singularly perturbed system with multiple pointwise and distributed small time delays in both slow and fast state variables is considered. A novel (direct) approach to the exact slow-fast decomposition of such a system is proposed. This approach does not require the transformation of the original system to the integral one, as well as rather complex integral manifolds and operator techniques, and the restrictive assumption on the stability of the fast subsystem. This approach proposes explicit linear transformations of the original state variables, leading to the exact slow-fast decomposition of the original system. The decomposed system remains differential. The method proposed in this paper is simpler and more convenient for applications than the method of the existing works [23, 24]. It can be effectively used in analysis of uncontrolled singularly perturbed systems with the small delays, in design of stabilizing controls for such a type of controlled systems, as well as in studying stabilizability, detectability, controllability, and observability of such systems. Two applications of the exact slow-fast decomposition are presented in the paper. The first application is an exact slow-fast decomposition of the spectrum of a singularly perturbed system with a small pointwise delay. The second application is stability and instability analysis of a multilink single-sink optical network.

The paper is organized as follows. In the next section, the problem is formulated rigorously. In section 3, by introducing a new state variable of two arguments, the original system is transformed equivalently to a new system consisting of two ordinary functional-differential equations and one partial first-order differential equation subject to a boundary condition. In this new system, one ordinary functional-differential equation is a slow mode equation, while the second ordinary functional-differential equation, as well as the partial differential equation, are fast mode equations. The rest of this section, as well as sections 4–6, are devoted to a consecutive elimination of the slow state variable from the fast mode equations and the fast state variables from the resulting slow mode equation. These eliminations are subject to proper linear transformations of the state variables, finite-dimensional for the first elimination and infinite-dimensional for the second elimination. As a result of these eliminations, a slow-fast decomposed system, equivalent to the original one, is obtained. In section 7, the case of a singularly perturbed system with a single pointwise small delay is treated. Based on the previous results, an exact slow-fast spectrum decomposition for this system is obtained. Using the theoretical results of the previous sections, a real-world problem (stability and instability of a multilink single-sink optical network) is studied in section 8. Section 9 contains conclusions. In Appendices A, B, and C we give the proofs of three important lemmas.

Notation. (1) E^n is the real n-dimensional Euclidean space, while C^n is the complex n-dimensional Euclidean space. (2) $\|\cdot\|$ is the Euclidean norm of a vector and of a matrix (real or complex). (3) $L^2[a,b;E^n]$ is the Hilbert L^2-space of functions $f(\eta):[a,b]\to E^n$, and for any $f(\eta),g(\eta)\in L^2[a,b;E^n]$, $\langle f(\eta),g(\eta)\rangle_{L^2[a,b;E^n]}$ is the inner product in this space. (4) $L^2[a,b;C^n]$ is the Hilbert L^2-space of functions $f(\eta):[a,b]\to C^n$, and for any $f(\eta),g(\eta)\in L^2[a,b;C^n]$, $\langle f(\eta),g(\eta)\rangle_{L^2[a,b;C^n]}$ is the inner product in this space. (5) $W^{1,2}[a,b;C^n]$ is the corresponding Sobolev space of functions $f(\eta):[a,b]\to C^n$. (6) $\mathcal{M}[a,b;n]$ denotes the Hilbert space of all pairs $f=(f_E,f_L(\eta))$, $f_E\in E^n$, $f_L(\eta)\in L^2[a,b;E^n]$; the inner product in this space is $\langle f,g\rangle_{\mathcal{M}[a,b;n]}=f_E^Tg_E+\langle f_L(\eta),g_L(\eta)\rangle_{L^2[a,b;E^n]}$, and the norm is $\|f\|_{\mathcal{M}[a,b;n]}=\sqrt{\langle f,f\rangle_{\mathcal{M}[a,b;n]}}$; the superscript "$T$" denotes the transposition. (7) $\mathrm{col}(x,y)$, where $x\in E^n$, $y\in E^m$, is the column block-vector of the dimension $n+m$ with the upper block x and the lower block y. (8) I_n is the identity matrix of dimension $n\times n$. (9) $V_a^b[D(\eta)]$ is the variation in the Euclidean matrix norm of the $n\times m$-matrix-valued function $D(\eta)$, $\eta\in[a,b]$.

2. Problem statement. In subsection 2.1, the initial differential system with delays is presented. Some important notions from the topics of differential systems with delays and singularly perturbed differential systems with small delays are recalled. In subsection 2.2, the initially formulated system is converted equivalently to a form more suitable for the further analysis. In subsection 2.3, objectives of the paper are formulated.

2.1. Initial system. Consider the differential system

$$\begin{aligned} dx(t)/dt &= \sum_{j=0}^{N}\Big[A_{1j}(\varepsilon)x(t-\varepsilon h_j)+A_{2j}(\varepsilon)y(t-\varepsilon h_j)\Big] \\ &\quad+\int_{-h}^{0}\Big[G_1(\eta,\varepsilon)x(t+\varepsilon\eta)+G_2(\eta,\varepsilon)y(t+\varepsilon\eta)\Big]d\eta,\quad t\geq 0, \end{aligned}\tag{2.1}$$

$$\begin{aligned} \varepsilon dy(t)/dt &= \sum_{j=0}^{N}\Big[A_{3j}(\varepsilon)x(t-\varepsilon h_j)+A_{4j}(\varepsilon)y(t-\varepsilon h_j)\Big] \\ &\quad+\int_{-h}^{0}\Big[G_3(\eta,\varepsilon)x(t+\varepsilon\eta)+G_4(\eta,\varepsilon)y(t+\varepsilon\eta)\Big]d\eta,\quad t\geq 0, \end{aligned}\tag{2.2}$$

where $x(t) \in E^n$, $y(t) \in E^m$; $\varepsilon \in (0, \varepsilon_0]$ is a small parameter, $\varepsilon_0 > 0$ is a given constant; $N \geq 0$ is a given integer; $A_{ij}(\varepsilon)$ and $G_i(\eta, \varepsilon)$ $(i = 1, \ldots, 4;\ j = 0, \ldots, N)$ are given matrices of corresponding dimensions, dependent on ε and (η, ε), respectively; and $h_0 = 0 < h_1 < \cdots < h_N = h$ are given constants.

The system (2.1)–(2.2) is subject to the initial conditions

$$x(\theta) = \varphi_x(\theta), \quad y(\theta) = \varphi_y(\theta), \quad \theta \in [-\varepsilon h, 0), \quad x(0) = x^0, \quad y(0) = y^0, \tag{2.3}$$

where $\varphi_x(\theta) \in L^2[-\varepsilon_0 h, 0; E^n]$ and $\varphi_y(\theta) \in L^2[-\varepsilon_0 h, 0; E^m]$, as well as $x^0 \in E^n$ and $y^0 \in E^m$, are given.

The system (2.1)–(2.2) is a functional-differential system. It is infinite-dimensional. The state variable of this system has the form $(z(t), z(t + \varepsilon\eta))$, $\eta \in [-h, 0]$, where $z(t) = \mathrm{col}(x(t), y(t))$, $z(t + \varepsilon\eta) = \mathrm{col}(x(t + \varepsilon\eta), y(t + \varepsilon\eta))$. For any given $t \geq 0$ and $\varepsilon > 0$, $(z(t), z(t + \varepsilon\eta)) \in \mathcal{M}[-\varepsilon h, 0; n + m]$, $(x(t), x(t + \varepsilon\eta)) \in \mathcal{M}[-\varepsilon h, 0; n]$, and $(y(t), y(t + \varepsilon\eta)) \in \mathcal{M}[-\varepsilon h, 0; m]$. The component $z(t)$ of the state variable is called its Euclidean part, while the component $z(t + \varepsilon\eta)$ is called the functional part of the state variable. More details on a functional-differential system and its state variable can be found, for instance, in [35] and references therein.

The system (2.1)–(2.2) is singularly perturbed by a small positive parameter ε (see, e.g., [3, 5, 16] and references therein). The important feature of (2.1)–(2.2) is that all the delays (pointwise and distributed) are proportional to the small parameter ε. This system is singularly perturbed not only because of the presence of the small multiplier for a part of the derivatives, but also because of the small delay (see, e.g., [8, section 18.4], [36, 37], and references therein). Equation (2.1) is called a slow mode, and the Euclidean part $x(t)$ of the state variable $(x(t), x(t + \varepsilon\eta))$ is called a slow one, while (2.2) and the entire state variable $(y(t), y(t + \varepsilon\eta))$ are called a fast mode and a fast state variable, respectively. As it will be shown below, the functional part $x(t + \varepsilon\eta)$ of the state variable $(x(t), x(t + \varepsilon\eta))$ is fast.

In what follows, we make these assumptions:

(A1) The matrix-valued functions $A_{ij}(\varepsilon)$ $(i = 1, \ldots, 4; j = 0, \ldots, N)$ are continuously differentiable for $\varepsilon \in [0, \varepsilon_0]$.

(A2) The matrix-valued functions $G_i(\eta, \varepsilon)$ $(i = 1, \ldots, 4)$ are piecewise continuous with respect to $\eta \in [-h, 0]$ for any $\varepsilon \in [0, \varepsilon_0]$, and they are continuously differentiable with respect to $\varepsilon \in [0, \varepsilon_0]$ uniformly in $\eta \in [-h, 0]$.

By virtue of the results of [38], for any given $\varepsilon \in (0, \varepsilon_0]$, the problem (2.1)–(2.2), (2.3) has the unique locally absolutely continuous solution $z(t, \varepsilon) = \mathrm{col}(x(t, \varepsilon), y(t, \varepsilon))$, $t \geq 0$.

2.2. Conversion of the system (2.1)–(2.2). Let us consider the following matrix-valued functions:

$$D_{A,i}(\eta, \varepsilon) = \begin{cases} -\sum_{j=1}^{N} A_{ij}(\varepsilon), & \eta \leq -h, \\ -\sum_{j=1}^{k} A_{ij}(\varepsilon), & -h_{k+1} < \eta \leq -h_k,\ k = 1, \ldots, N-1, \\ 0, & \eta > -h_1, \end{cases} \tag{2.4}$$

$$D_{G,i}(\eta, \varepsilon) = -\begin{cases} \int_{-h}^{0} G_i(s, \varepsilon) ds, & \eta \leq -h, \\ \int_{\eta}^{0} G_i(s, \varepsilon) ds, & -h < \eta < 0, \\ 0, & \eta \geq 0, \end{cases} \tag{2.5}$$

$$D_i(\eta, \varepsilon) = D_{A,i}(\eta, \varepsilon) + D_{G,i}(\eta, \varepsilon), \qquad i = 1, \ldots, 4,\ \eta \in [-h, 0],\ \varepsilon \in [0, \varepsilon_0]. \tag{2.6}$$

Remark 2.1. For each $\varepsilon \in [0, \varepsilon_0]$, the matrix-valued functions $D_i(\eta, \varepsilon)$ $(i = 1, \dots, 4)$ are piecewise continuous in $\eta \in [-h, 0]$ with the break points $\eta = -h_j$ $(j = 1, \dots, N)$, where these functions have finite limits from the right $D_i(-h_j + 0, \varepsilon) = \lim_{\eta \to -h_j+0} D_i(\eta, \varepsilon)$. These functions are continuous from the left at the break points $\eta = -h_j$ $(j = 1, \dots, N-1)$. Moreover, these functions are of bounded variation (in the Euclidean matrix norm) with respect to $\eta \in [-h, 0]$ uniformly in $\varepsilon \in [0, \varepsilon_0]$.

Let us denote for all $i = 1, \dots, 4$, $j = 1, \dots, N$, $\eta \in [-h_j, -h_{j-1}]$, $\varepsilon \in [0, \varepsilon_0]$ the following:

$$D_{ij}(\eta, \varepsilon) = \begin{cases} D_i(-h_j + 0, \varepsilon), & \eta = -h_j, \\ D_i(\eta, \varepsilon), & \eta \in (-h_j, -h_{j-1}]. \end{cases}$$

Remark 2.2. The functions $D_{ij}(\eta, \varepsilon)$ $(i = 1, \dots, 4; j = 1, \dots, N)$ are differentiable with respect to $\eta \in [-h_j, -h_{j-1}]$ for all $\varepsilon \in [0, \varepsilon_0]$.

Using (2.4)–(2.6), the system (2.1)–(2.2) can be rewritten equivalently as

$$\begin{aligned} dx(t)/dt = A_{10}(\varepsilon)x(t) + A_{20}(\varepsilon)y(t) + \int_{-h}^{0} [d_\eta D_1(\eta, \varepsilon)]x(t + \varepsilon\eta) \\ + \int_{-h}^{0} [d_\eta D_2(\eta, \varepsilon)]y(t + \varepsilon\eta), \quad t \geq 0, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \varepsilon dy(t)/dt = A_{30}(\varepsilon)x(t) + A_{40}(\varepsilon)y(t) + \int_{-h}^{0} [d_\eta D_3(\eta, \varepsilon)]x(t + \varepsilon\eta) \\ + \int_{-h}^{0} [d_\eta D_4(\eta, \varepsilon)]y(t + \varepsilon\eta), \quad t \geq 0, \end{aligned} \tag{2.8}$$

where the integrals in the right-hand parts are Stieltjes ones.

2.3. Objectives of the paper. Our objectives in this paper are (I) for all sufficiently small $\varepsilon > 0$, to decompose equivalently the problem (2.7)–(2.8), (2.3) into two parts, purely slow and purely fast systems with corresponding initial conditions; (II) to use the above-mentioned decomposition for an exact decomposition of the spectrum of a system of the type (2.1)–(2.2) into purely slow and purely fast parts; (III) based on the theoretical results of the paper, to analyze the asymptotic stability and the instability of a multilink single-sink optical network.

3. Transformation of system (2.7)–(2.8). In this section, two consecutive equivalent transformations of the system (2.7)–(2.8) are carried out. The first transformation (subsection 3.1) is due to the following observation. The presence of a small delay in a state variable with the slow Euclidean part of a time delay system generates an "additional" singular perturbation effect and an additional fast state variable in the system. This new fast variable is the functional part of the above-mentioned state variable with the slow Euclidean part (see, e.g., [8, section 18.4], [36, 37], and references therein). Therefore, in the exact slow-fast decomposition of system (2.7)–(2.8), we have to separate not the mode for $(x(t), x(t + \varepsilon\eta))$ (equation (2.7)) and the mode for $(y(t), y(t+\varepsilon\eta))$ (equation (2.8)), but the mode for $x(t)$ and the modes for $x(t+\varepsilon\eta)$ and $(y(t), y(t+\varepsilon\eta))$. In order to do this, the fast mode of $x(t+\varepsilon\eta)$ should be obtained. This is done in subsection 3.1 by definition of a new state variable—the functional part of the original state variable $(x(t), x(t + \varepsilon\eta))$—and deriving a fast differential equation for this state variable. The latter allows us to transform the original system (2.7)–(2.8) into an equivalent new system consisting of one slow mode and two fast modes. The second transformation (subsection 3.2) is a linear algebraic invertible transformation of the fast state variables in the new system. This transformation allows us to eliminate the slow state variable from the fast modes of the resulting system by a proper choice of its matrix-valued coefficients.

3.1. Additional fast state variable. For a given $\varepsilon \in (0, \varepsilon_0]$, let us define the function, a new state variable,

$$v(t,\eta) \triangleq x(t+\varepsilon\eta), \quad t \geq 0, \ \eta \in [-h, 0]. \tag{3.1}$$

Remark 3.1. The new state variable $v(t,\eta)$ is the functional part of the state $(x(t), x(t+\varepsilon\eta))$. Therefore, it is a function of two independent variables, time t and the independent variable η, representing the stretched delay. The latter varies in the stretched delay interval $[-h, 0]$. Thus, in contrast with the undelayed case, here we have not only two time scales, the original t and the stretched t/ε time, but also two delay scales, the original $\theta \in [-\varepsilon h, 0]$ and the stretched $\eta \in [-h, 0]$ delay.

The state variable $v(t,\eta)$ satisfies the following differential equation and boundary and initial conditions:

$$\varepsilon \partial v(t,\eta)/\partial t - \partial v(t,\eta)/\partial \eta = 0, \quad (t,\eta) \in \Omega_\varepsilon^+, \tag{3.2}$$

$$v(t, 0) = x(t), \quad t \geq 0, \tag{3.3}$$

$$v(t,\eta) = \varphi_x(t+\varepsilon\eta), \quad (t,\eta) \in \Omega_\varepsilon^-; \qquad v(t,\eta) = x^0, \quad (t,\eta) \in \Omega_\varepsilon^0, \tag{3.4}$$

where $\Omega_\varepsilon^+ \triangleq \{(t,\eta) : t \geq 0, \ \eta \in [-h,0], \ t+\varepsilon\eta > 0\}$, $\Omega_\varepsilon^- \triangleq \{(t,\eta) : t \geq 0, \ \eta \in [-h,0], \ t+\varepsilon\eta < 0\}$, and $\Omega_\varepsilon^0 \triangleq \{(t,\eta) : t \geq 0, \ \eta \in [-h,0], \ t+\varepsilon\eta = 0\}$.

Using (3.1), we can rewrite the system (2.7)–(2.8) in the form

$$\begin{aligned} dx(t)/dt = {} & A_{10}(\varepsilon)x(t) + A_{20}(\varepsilon)y(t) + \int_{-h}^{0} [d_\eta D_1(\eta,\varepsilon)]v(t,\eta) \\ & + \int_{-h}^{0} [d_\eta D_2(\eta,\varepsilon)]y(t+\varepsilon\eta), \quad t \geq 0, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \varepsilon dy(t)/dt = {} & A_{30}(\varepsilon)x(t) + A_{40}(\varepsilon)y(t) + \int_{-h}^{0} [d_\eta D_3(\eta,\varepsilon)]v(t,\eta) \\ & + \int_{-h}^{0} [d_\eta D_4(\eta,\varepsilon)]y(t+\varepsilon\eta), \quad t \geq 0. \end{aligned} \tag{3.6}$$

Remark 3.2. Due to the results of [35, 39], the system (3.2), (3.5)–(3.6), along with the boundary condition (3.3) and the initial conditions (3.4), (2.3), is equivalent to the system (2.7)–(2.8) with the initial conditions (2.3) and the relation (3.1). In the new system, the Euclidean and the functional parts of the state variable $(x(t), x(t+\varepsilon\eta))$ become separated state variables, thus increasing the Euclidean dimension of the new system.

Remark 3.3. Since the partial derivative with respect to time t in (3.2) is multiplied by the small parameter ε, the state variable $v(t,\eta)$ is fast. Thus, in the system (3.2), (3.5)–(3.6), the mode (3.5) and the state variable $x(t)$ are slow, while the modes (3.2), (3.6) and the state variables $v(t,\eta)$, $(y(t), y(t+\varepsilon\eta))$ are fast.

3.2. Fast states' transformation of the system (3.2)–(3.3), (3.5)–(3.6). Let us transform the fast state variables of (3.2)–(3.3), (3.5)–(3.6) as follows:

$$w_v(t,\eta) = v(t,\eta) - L_v(\eta,\varepsilon)x(t), \qquad (t,\eta) \in \Omega, \tag{3.7}$$

$$\begin{aligned} w_y(t) &= y(t) - L_y(\varepsilon)x(t), \qquad t \ge 0,\\ w_y(t+\varepsilon\eta) &= y(t+\varepsilon\eta) - L_y(\varepsilon)v(t,\eta), \qquad t \ge 0,\ \eta \in [-h,0], \end{aligned} \tag{3.8}$$

where $w_v(t,\eta)$ and $(w_y(t), w_y(t+\varepsilon\eta))$ are new state variables; $L_v(\eta,\varepsilon)$ and $L_y(\varepsilon)$ are some matrix-valued functions of $(\eta,\varepsilon) \in [-h,0] \times (0,\varepsilon_0]$ and $\varepsilon \in (0,\varepsilon_0]$, respectively; $L_v(\eta,\varepsilon)$ is differentiable with respect to $\eta \in [-h,0]$ for any $\varepsilon \in (0,\varepsilon_0]$; and $\Omega \stackrel{\triangle}{=} \{(t,\eta) : t \ge 0,\ \eta \in [-h,0]\}$.

Substituting (3.7) into (3.2) yields for $(t,\eta) \in \Omega_\varepsilon^+$

$$\varepsilon\frac{\partial w_v(t,\eta)}{\partial t} - \frac{\partial w_v(t,\eta)}{\partial \eta} = \left(\frac{dL_v(\eta,\varepsilon)}{d\eta}\right)x(t) - \varepsilon L_v(\eta,\varepsilon)\left(\frac{dx(t)}{dt}\right). \tag{3.9}$$

The derivative of $w_y(t)$ for $t \ge 0$ has the form

$$\frac{dw_y(t)}{dt} = \frac{dy(t)}{dt} - L_y(\varepsilon)\left(\frac{dx(t)}{dt}\right). \tag{3.10}$$

Now, we do the following. First, we eliminate $v(t,\eta)$ and $(y(t), y(t+\varepsilon\eta))$ from (3.5), using the transformations (3.7)–(3.8). Second, we substitute (3.5)–(3.6) into (3.9) and (3.10), and eliminate $v(t,\eta)$ and $(y(t), y(t+\varepsilon\eta))$ from the resulting equations, using the same transformations. As a final result, we obtain the following system of three differential equations with respect to $x(t)$, $w_v(t,\eta)$, and $(w_y(t), w_y(t+\varepsilon\eta))$:

$$\begin{aligned} \frac{dx(t)}{dt} &= F_1\Big(\varepsilon, L_v(\eta,\varepsilon), L_y(\varepsilon)\Big)x(t) + \int_{-h}^{0}\Big[d_\eta\Gamma_{12}\Big(\eta,\varepsilon,L_y(\varepsilon)\Big)\Big]w_v(t,\eta)\\ &\quad + A_{20}(\varepsilon)w_y(t) + \int_{-h}^{0}[d_\eta D_2(\eta,\varepsilon)]w_y(t+\varepsilon\eta), \qquad t \ge 0, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \varepsilon\frac{\partial w_v(t,\eta)}{\partial t} - \frac{\partial w_v(t,\eta)}{\partial \eta} &= \left\{\frac{dL_v(\eta,\varepsilon)}{d\eta} - F_2\Big(\varepsilon, L_v(\eta,\varepsilon), L_y(\varepsilon)\Big)\right\}x(t)\\ &\quad - \varepsilon L_v(\eta,\varepsilon)\left\{\int_{-h}^{0}\Big[d_\eta\Gamma_{12}\Big(\eta,\varepsilon,L_y(\varepsilon)\Big)\Big]w_v(t,\eta) + A_{20}(\varepsilon)w_y(t)\right.\\ &\quad \left. + \int_{-h}^{0}[d_\eta D_2(\eta,\varepsilon)]w_y(t+\varepsilon\eta)\right\}, \qquad (t,\eta) \in \Omega_\varepsilon^+, \end{aligned} \tag{3.12}$$

$$\begin{aligned} \frac{\varepsilon dw_y(t)}{dt} &= F_3\Big(\varepsilon, L_v(\eta,\varepsilon), L_y(\varepsilon)\Big)x(t)\\ &\quad + \int_{-h}^{0}\Big[d_\eta\Big(\Gamma_{34}\Big(\eta,\varepsilon,L_y(\varepsilon)\Big) - \varepsilon L_y(\varepsilon)\Gamma_{12}\Big(\eta,\varepsilon,L_y(\varepsilon)\Big)\Big)\Big]w_v(t,\eta)\\ &\quad + \Big(A_{40}(\varepsilon) - \varepsilon L_y(\varepsilon)A_{20}\Big)w_y(t)\\ &\quad + \int_{-h}^{0}\Big[d_\eta\Big(D_4(\eta,\varepsilon) - \varepsilon L_y(\varepsilon)D_2(\eta,\varepsilon)\Big)\Big]w_y(t+\varepsilon\eta), \qquad t \ge 0, \end{aligned} \tag{3.13}$$

where

$$F_1\Big(\varepsilon, L_v(\eta,\varepsilon), L_y(\varepsilon)\Big) \triangleq \Upsilon_{12}\Big(\varepsilon, L_y(\varepsilon)\Big) + \int_{-h}^{0}\Big[d_\eta \Gamma_{12}\Big(\eta,\varepsilon,L_y(\varepsilon)\Big)\Big] L_v(\eta,\varepsilon), \tag{3.14}$$

$$F_2\Big(\varepsilon, L_v(\eta,\varepsilon), L_y(\varepsilon)\Big) \triangleq \varepsilon L_v(\eta,\varepsilon) F_1\Big(\varepsilon, L_v(\eta,\varepsilon), L_y(\varepsilon)\Big), \tag{3.15}$$

$$F_3\Big(\varepsilon, L_v(\eta,\varepsilon), L_y(\varepsilon)\Big) \triangleq \Upsilon_{34}\Big(\varepsilon, L_y(\varepsilon)\Big) - \varepsilon L_y(\varepsilon)\Upsilon_{12}\Big(\varepsilon, L_y(\varepsilon)\Big) + \int_{-h}^{0}\Big[d_\eta\Big(\Gamma_{34}\Big(\eta,\varepsilon,L_y(\varepsilon)\Big) - \varepsilon L_y(\varepsilon)\Gamma_{12}\Big(\eta,\varepsilon,L_y(\varepsilon)\Big)\Big)\Big] L_v(\eta,\varepsilon), \tag{3.16}$$

$$\Upsilon_{k\ k+1}\Big(\varepsilon, L_y(\varepsilon)\Big) \triangleq A_{k0}(\varepsilon) + A_{k+1\ 0}(\varepsilon)L_y(\varepsilon), \qquad k = 1,3, \tag{3.17}$$
$$\Gamma_{k\ k+1}\Big(\eta,\varepsilon, L_y(\varepsilon)\Big) \triangleq D_k(\eta,\varepsilon) + D_{k+1}(\eta,\varepsilon)L_y(\varepsilon), \quad k = 1,3.$$

The system (3.11)–(3.13) is singularly perturbed. In this system, the mode (3.11) and the state $x(t)$ are slow, while the modes (3.12), (3.13) and the states $w_v(t,\eta)$, $(w_y(t), w_y(t+\varepsilon\eta))$ are fast. Moreover, due to the transformation (3.7), the boundary condition (3.3) for the system (3.2), (3.5)–(3.6) becomes for $t \geq 0$

$$w_v(t,0) = \Big(I_n - L_v(0,\varepsilon)\Big)x(t). \tag{3.18}$$

Remark 3.4. Since the transformations (3.7)–(3.8) are invertible, the system (3.11)–(3.13) with the boundary condition (3.18) and the system (3.2), (3.5)–(3.6) with the boundary condition (3.3) are equivalent to each other subject to (3.7)–(3.8).

Now, we are in position to separate the fast modes from the slow one. Such a separation is carried out in the next section.

4. Separation of the fast modes from the slow mode in the system (3.11)–(3.13). In this section, first, we choose the matrices $L_v(\eta,\varepsilon)$ and $L_y(\varepsilon)$ in such a way that the terms in the right-hand sides of (3.12)–(3.13), containing the slow state $x(t)$, vanish. Such a choice yields a system of two equations with respect to these matrices (subsection 4.1). In subsection 4.2, the existence of a solution to this system is established. Finally, based on the results of subsections 4.1 and 4.2, the singularly perturbed upper triangular system, equivalent to the system (3.11)–(3.13), is derived. In this system, the fast modes do not contain the slow state variable, while the slow mode contains the fast state variables.

4.1. Elimination of the slow state from the fast modes (3.12)–(3.13). In order to separate the fast modes from the slow mode in the system (3.11)–(3.13), we have to eliminate the slow state variable $x(t)$ from the fast modes (3.12)–(3.13). For this purpose, we choose the matrices $L_v(\eta,\varepsilon)$ and $L_y(\varepsilon)$ such that the coefficients for $x(t)$ in the right-hand sides of these equations become zero. Thus, we obtain the following system of equations with respect to $L_v(\eta,\varepsilon)$ and $L_y(\varepsilon)$:

$$\begin{aligned} &dL_v(\eta,\varepsilon)/d\eta = F_2\Big(\varepsilon, L_v(\eta,\varepsilon), L_y(\varepsilon)\Big), \\ &\mathcal{H}_4(0)L_y(\varepsilon) = -F_3\Big(\varepsilon, L_v(\eta,\varepsilon), L_y(\varepsilon)\Big) + \mathcal{H}_4(0)L_y(\varepsilon), \end{aligned} \tag{4.1}$$

where

$$\mathcal{H}_i(\varepsilon) \triangleq A_{i0}(\varepsilon) + \int_{-h}^{0} d_\eta \Big[D_i(\eta,\varepsilon)\Big] = \sum_{j=0}^{N} A_{ij}(\varepsilon) + \int_{-h}^{0} G_i(\eta,\varepsilon)d\eta, \quad i = 1,\dots,4. \tag{4.2}$$

The system (4.1) is functional-differential-algebraic. In the next subsection, we show the existence of a solution to (4.1) by constructing and justifying its asymptotic solution with respect to ε.

4.2. Asymptotic solution of system (4.1). We look for the zero-order asymptotic solution $(L_v^0(\eta), L_y^0)$ of (4.1). Equations for this asymptotic solution are obtained by formally setting $\varepsilon = 0$ in (4.1), and replacing $L_v(\eta,\varepsilon)$ and $L_y(\varepsilon)$ with $L_v^0(\eta)$ and L_y^0, respectively. Thus, by using (3.14)–(3.16), (3.18), and (4.2), the following system of equations with respect to $L_v^0(\eta)$ and L_y^0 is obtained:

$$dL_v^0(\eta)/d\eta = 0, \quad \eta \in [-h, 0], \tag{4.3}$$

$$\mathcal{H}_4(0)L_y^0 = -A_{30}(0) - \int_{-h}^{0}[d_\eta D_3(\eta,0)]L_v^0(\eta) + \int_{-h}^{0}[d_\eta D_4(\eta,0)]L_y^0\big(I_n - L_v^0(\eta)\big). \tag{4.4}$$

It is seen that any constant $n\times n$ matrix is a solution of (4.3). In order to eliminate L_y^0 from the right-hand side of (4.4), we choose

$$L_v^0(\eta) \equiv L_v^0 = I_n, \quad \eta \in [-h, 0]. \tag{4.5}$$

Substituting (4.5) into (4.4) yields $\mathcal{H}_4(0)L_y^0 = -\mathcal{H}_3(0)$. Using the following additional assumption,

(A3) $\det\mathcal{H}_4(0) \neq 0$,

we obtain the unique solution of this equation,

$$L_y^0 = -\mathcal{H}_4^{-1}(0)\mathcal{H}_3(0). \tag{4.6}$$

Thus, the terms $L_v^0(\eta)$ and L_y^0 of the zero-order asymptotic solution to (4.1) have been derived. Based on this asymptotic solution, we obtain the following lemma.

LEMMA 4.1. *Let assumptions* (A1)–(A3) *be valid. Then there exists a positive number* ε_1^* $(\varepsilon_1^* \le \varepsilon_0)$ *such that, for all* $\varepsilon \in (0, \varepsilon_1^*]$, *system* (4.1) *has a solution* $(L_v(\eta,\varepsilon), L_y(\varepsilon))$ *satisfying the condition* $L_v(-h,\varepsilon) = I_n$ *and the inequalities*

$$\|L_v(\eta,\varepsilon) - L_v^0(\eta)\| \le a_1^*\varepsilon \quad \forall \eta \in [-h,0], \qquad \|L_y(\varepsilon) - L_y^0(\varepsilon)\| \le a_1^*\varepsilon, \tag{4.7}$$

where $a_1^* > 0$ *is some constant independent of* ε.

The lemma is proven in Appendix A.

4.3. Upper triangular system for $t \ge \varepsilon h$. Due to the existence of a solution to system (4.1), the fast mode equations (3.12) and (3.13) in the system (3.11)–(3.13) become as follows for $(t,\eta) \in \Omega_\varepsilon^+$:

$$\varepsilon\frac{\partial w_v(t,\eta)}{\partial t} - \frac{\partial w_v(t,\eta)}{\partial \eta} = -\varepsilon L_v(\eta,\varepsilon)\Big\{ \int_{-h}^{0} \Big[d_\eta \Gamma_{12}\Big(\eta,\varepsilon,L_y(\varepsilon)\Big)\Big] w_v(t,\eta) + A_{20}(\varepsilon)w_y(t) + \int_{-h}^{0}[d_\eta D_2(\eta,\varepsilon)]w_y(t+\varepsilon\eta)\Big\}, \tag{4.8}$$

$$\begin{aligned}\frac{\varepsilon dw_y(t)}{dt} &= \int_{-h}^{0}\Big[d_\eta\Big(\Gamma_{34}\Big(\eta,\varepsilon,L_y(\varepsilon)\Big) - \varepsilon L_y(\varepsilon)\Gamma_{12}\Big(\eta,\varepsilon,L_y(\varepsilon)\Big)\Big)\Big]w_v(t,\eta)\\ &\quad + \Big(A_{40}(\varepsilon) - \varepsilon L_y(\varepsilon)A_{20}(\varepsilon)\Big)w_y(t)\\ &\quad + \int_{-h}^{0}\Big[d_\eta\Big(D_4(\eta,\varepsilon) - \varepsilon L_y(\varepsilon)D_2(\eta,\varepsilon)\Big)\Big]w_y(t+\varepsilon\eta).\end{aligned} \tag{4.9}$$

Using (2.3), (3.4), and (3.7)–(3.8) yields the initial conditions for (4.8) and (4.9)

$$\begin{aligned} &w_v(t,\eta) = \varphi_x(t+\varepsilon\eta) - L_v(\eta,\varepsilon)x(t), \qquad (t,\eta)\in\Omega_\varepsilon^-,\\ &w_v(t,\eta) = x^0 - L_v(\eta,\varepsilon)x(t), \qquad (t,\eta)\in\Omega_\varepsilon^0,\end{aligned} \tag{4.10}$$

$$w_y(\theta) = \varphi_y(\theta) - L_y(\varepsilon)\varphi_x(\theta), \quad \theta\in[-\varepsilon h, 0), \quad w_y(0) = y^0 - L_y(\varepsilon)x^0. \tag{4.11}$$

It is seen that the initial conditions for $w_y(\cdot)$ are independent of the slow state variable $x(t)$, while the initial conditions for $w_v(\cdot)$ and the boundary condition (3.18) depend on $x(t)$. Thus, although the fast mode differential equations (4.8)–(4.9) do not contain the slow state variable $x(t)$, a part of the initial conditions for these equations and the boundary condition (3.18) still contain this variable.

Remark 4.2. The presence of the slow state in the boundary condition (3.18) is unavoidable, because (3.18) is a direct consequence of the boundary condition (3.3). The latter is necessary and sufficient for the equivalence of the original differential system (2.7)–(2.8) and the transformed one (3.2), (3.5)–(3.6). Without (3.3), the differential systems (2.7)–(2.8) and (3.2), (3.5)–(3.6) are not equivalent to each other. However, as will be shown in sections 6 and 7, the connection (3.18) between the fast and slow states does not prevent the further complete exact separation of the slow mode from the fast one, as well as the exact slow-fast decomposition of the spectrum of the original system (2.7)–(2.8).

In order to resolve the problem of the presence of $x(t)$ in the conditions (4.10), we introduce into consideration the domain $\tilde{\Omega}_\varepsilon \overset{\triangle}{=} \{(t,\eta) : t > \varepsilon h,\ \eta\in[-h,0]\} \subset \Omega_\varepsilon^+$. In what follows, we consider the functional-differential set of equations (3.11), (4.8), (4.9), and the boundary condition (3.18) for $(t,\eta)\in\tilde{\Omega}_\varepsilon$. Also, we consider the original problem (2.7)–(2.8), (2.3) on the interval $t\in[0,\varepsilon h]$ ($\varepsilon\in(0,\varepsilon_1^*]$), where this problem is not singularly perturbed any more, and it has the unique absolutely continuous solution $x(t) = x_{\rm in}(t,\varepsilon)$, $y(t) = y_{\rm in}(t,\varepsilon)$, bounded uniformly in $\varepsilon\in(0,\varepsilon_1^*]$. Thus, the system (3.11), (4.8)–(4.9), (3.18), considered for $(t,\eta)\in\tilde{\Omega}_\varepsilon$, is subject to the following conditions for all $\varepsilon\in(0,\varepsilon_1^*]$: (3.11) is subject to the initial condition

$$x(\varepsilon h) = x_{\rm in}(\varepsilon h); \tag{4.12}$$

(4.9) is subject to the initial condition

$$w_y(t) = y_{\rm in}(t,\varepsilon) - L_y(\varepsilon)x_{\rm in}(t,\varepsilon), \qquad t\in[0,\varepsilon h]; \tag{4.13}$$

and (3.18) is considered for $t > \varepsilon h$. Moreover, the conditions (4.10), along with the transformation (3.7), determine the fast state variable $w_v(t,\eta)$ in the rectangle $\{(t,\eta) : t \in [0,\varepsilon h],\ \eta \in [-h,0]\}$ as

(4.14)
$$\begin{array}{ll} w_v(t,\eta) = \varphi_x(t+\varepsilon\eta) - L_v(\eta,\varepsilon)x_{\rm in}(t,\varepsilon), & (t,\eta) \in \Omega_\varepsilon^-, \\ w_v(t,\eta) = x^0 - L_v(\eta,\varepsilon)x_{\rm in}(t,\varepsilon), & (t,\eta) \in \Omega_\varepsilon^0, \\ w_v(t,\eta) = x_{\rm in}(t+\varepsilon\eta,\varepsilon) - L_v(\eta,\varepsilon)x_{\rm in}(t,\varepsilon), & (t,\eta) \in \{0 \le t \le \varepsilon h\} \cap \Omega_\varepsilon^+. \end{array}$$

The transformation (3.7)–(3.8) is invertible for all $(t,\eta) \in \Omega$. Using this observation, the above-mentioned equivalence of the systems (3.2)–(3.3), (3.5)–(3.6), and (2.7)–(2.8), as well as Remark 3.2 and Lemma 4.1, one directly obtains the theorem.

THEOREM 4.3. *Let assumptions* (A1)–(A3) *be valid. Let* $(L_v(\eta,\varepsilon), L_y(\varepsilon))$ *be the solution of system* (4.1) *mentioned in Lemma* 4.1. *Then, for all* $\varepsilon \in (0,\varepsilon_1^*]$, *the system* (3.11), (4.8)–(4.9), (3.18) *for* $(t,\eta) \in \tilde{\Omega}_\varepsilon$ *with the initial conditions* (4.12)–(4.13), *and the system* (2.7)–(2.8) *for* $t \in (\varepsilon h, +\infty)$ *with the initial conditions*

$$x(t) = x_{\rm in}(t), \quad y(t) = y_{\rm in}(t), \quad t \in [0,\varepsilon h], \tag{4.15}$$

are equivalent to each other subject to the relations (3.7)–(3.8) *and* (3.1) *on their state variables.*

In what follows, the system (3.11), (4.8)–(4.9), (3.18) subject to the initial conditions (4.12)–(4.13) is called the *upper triangular system* (UTS) for $t \ge \varepsilon h$. It is seen that the right-hand sides of the fast modes (4.8) and (4.9) in the UTS do not contain the slow state variable $x(t)$, while the right-hand side of the slow mode (3.11) of this system does contain the fast state variables $w_v(t,\eta)$ and $\big(w_y(t), w_y(t+\varepsilon\eta)\big)$. To eliminate these state variables from (3.11), this equation needs a proper transformation. Such a transformation is undertaken in the next section.

5. Transformation of the slow mode in the upper triangular system for $t \ge \varepsilon h$.

In subsection 5.1, we introduce some auxiliary matrix-valued functions, used in the transformation of (3.11). The transformation itself is made in subsection 5.2. This transformation is a linear functional invertible transformation, and it allows us to eliminate the fast state variables from the resulting slow mode equation.

5.1. Auxiliary matrix-valued functions.

Let us consider the following matrix-valued functions for any $\varepsilon \in (0,\varepsilon_1^*]$:

$$Q_v(\eta,\varepsilon) = \begin{cases} 0, & \eta \le -h, \\ Q_{v,k}(\eta,\varepsilon), & -h_k < \eta \le -h_{k-1},\ k = 1,\dots,N, \\ Q_{v,1}(0,\varepsilon), & \eta > 0, \end{cases} \tag{5.1}$$

$$D_{Q,v}(\eta,\varepsilon) = -\begin{cases} \int_{-h}^{0} Q_v(s,\varepsilon)ds, & \eta \le -h, \\ \int_{\eta}^{0} Q_v(s,\varepsilon)ds, & -h < \eta < 0, \\ 0, & \eta \ge 0, \end{cases} \tag{5.2}$$

$$Q_y(\eta,\varepsilon) = \begin{cases} 0, & \eta \le -h, \\ Q_{y,k}(\eta,\varepsilon), & -h_k < \eta \le -h_{k-1},\ k = 1,\dots,N, \\ Q_{y,1}(0,\varepsilon), & \eta > 0, \end{cases} \tag{5.3}$$

$$D_{Q,y}(\eta,\varepsilon) = -\begin{cases} \int_{-h}^{0} Q_y(s,\varepsilon)ds, & \eta \le -h, \\ \int_{\eta}^{0} Q_y(s,\varepsilon)ds, & -h < \eta < 0, \\ 0, & \eta \ge 0. \end{cases} \tag{5.4}$$

In (5.1) and (5.3), for any $j \in \{1, \ldots, N\}$, $Q_{v,j}(\eta, \varepsilon)$ and $Q_{y,j}(\eta, \varepsilon)$ are some $n \times n$ and $n \times m$, respectively, matrix-valued functions of $(\eta, \varepsilon) \in [-h_j, -h_{j-1}] \times (0, \varepsilon_1^*]$, differentiable with respect to $\eta \in [-h_j, -h_{j-1}]$ for each $\varepsilon \in (0, \varepsilon_1^*]$.

5.2. Transformation of (3.11). In order to eliminate the fast state variables $w_v(t, \eta)$ and $(w_y(t), w_y(t + \varepsilon\eta))$ from (3.11), we make in this equation the following transformation of the slow state variable:

$$\begin{aligned} w_x(t) = x(t) &- \varepsilon \int_{-h}^{0} [d_\eta D_{Q,v}(\eta, \varepsilon)] w_v(t, \eta) - \varepsilon P_y(\varepsilon) w_y(t) \\ &- \varepsilon \int_{-h}^{0} [d_\eta D_{Q,y}(\eta, \varepsilon)] w_y(t + \varepsilon\eta), \qquad t \geq \varepsilon h, \ \varepsilon \in (0, \varepsilon_1^*], \end{aligned} \tag{5.5}$$

where $w_x(t)$ is a new slow state variable and $P_y(\varepsilon)$ is some $n \times m$-matrix.

Let us change the variable of integration η in both integrals of (5.5) as follows: $\omega = t + \varepsilon\eta$. Then, let us differentiate both sides of the resulting equation with respect to t. After this, let us return to the original variable of integration η in all integrals, obtained after the differentiation in t. As a final result, we obtain

$$\begin{aligned} \frac{dw_x(t)}{dt} &= \frac{dx(t)}{dt} + \int_{-h}^{0} [d_\eta Q_v(\eta, \varepsilon)] w_v(\eta, \varepsilon) \\ &- \int_{-h}^{0} [d_\eta D_{Q,v}(\eta, \varepsilon)] \left(\varepsilon \frac{\partial w_v(t, \eta)}{\partial t} - \frac{\partial w_v(t, \eta)}{\partial \eta} \right) - Q_v(0, \varepsilon) w_v(t, 0) \\ &- \varepsilon P_y(\varepsilon) \left(\frac{dw_y(t)}{dt} \right) + \int_{-h}^{0} [d_\eta Q_y(\eta, \varepsilon)] w_y(t + \varepsilon\eta) - Q_y(0, \varepsilon) w_y(t), \quad t \geq \varepsilon h. \end{aligned} \tag{5.6}$$

Now, we do the following. First, we substitute (3.11) and (4.9) into (5.6). Second, we replace the value $w_v(t, 0)$ in the obtained equation with its expression (3.18). Third, we eliminate from the resulting equation the original slow state variable $x(t)$ and the expression $\varepsilon \partial w_v(t, \eta)/\partial t - \partial w_v(t, \eta)/\partial \eta$, using (5.5) and (4.8), respectively. Thus, we obtain the differential equation for $w_x(t)$. In order to write down this equation, let us introduce into consideration the set of unknown matrices,

$$\mathcal{N}(\eta, \varepsilon) \stackrel{\triangle}{=} \Big\{ Q_v(\eta, \varepsilon), P_y(\varepsilon), Q_y(\eta, \varepsilon) \Big\}, \tag{5.7}$$

and the following expressions:

$$\mathcal{F}_1\Big(\varepsilon, \mathcal{N}(\eta, \varepsilon)\Big) \stackrel{\triangle}{=} F_1\Big(\varepsilon, L_v(\eta, \varepsilon), L_y(\varepsilon)\Big) - Q_v(0, \varepsilon)\Big(I_n - L_v(0, \varepsilon)\Big), \tag{5.8}$$

$$\begin{aligned} \mathcal{F}_2\Big(\varepsilon, \mathcal{N}(\eta, \varepsilon)\Big) &\stackrel{\triangle}{=} \varepsilon \mathcal{F}_1\Big(\varepsilon, \mathcal{N}(\eta, \varepsilon)\Big) P_y(\varepsilon) + A_{20}(\varepsilon) \\ &\quad - P_y(\varepsilon)\Big(A_{40}(\varepsilon) - \varepsilon L_y(\varepsilon) A_{20}(\varepsilon)\Big) - Q_y(0, \varepsilon) \\ &\quad + \varepsilon \int_{-h}^{0} \Big[d_\eta D_{Q,v}(\eta, \varepsilon)\Big] L_v(\eta, \varepsilon) A_{20}(\varepsilon), \end{aligned} \tag{5.9}$$

$$\begin{aligned} \mathcal{F}_3\Big(\eta, \varepsilon, \mathcal{N}(\eta, \varepsilon)\Big) &\stackrel{\triangle}{=} \varepsilon \mathcal{F}_1\Big(\varepsilon, \mathcal{N}(\eta, \varepsilon)\Big) D_{Q,v}(\eta, \varepsilon) + Q_v(\eta, \varepsilon) \\ &\quad + \left\{ I_n + \varepsilon \int_{-h}^{0} [d_\eta D_{Q,v}(\eta, \varepsilon)] L_v(\eta, \varepsilon) \right\} \Gamma_{12}\Big(\eta, \varepsilon, L_y(\varepsilon)\Big) \\ &\quad - P_y(\varepsilon) \Big[\Gamma_{34}\Big(\eta, \varepsilon, L_y(\varepsilon)\Big) - \varepsilon L_y(\varepsilon) \Gamma_{12}\Big(\eta, \varepsilon, L_y(\varepsilon)\Big)\Big], \end{aligned} \tag{5.10}$$

$$\mathcal{F}_4\Big(\eta,\varepsilon,\mathcal{N}(\eta,\varepsilon)\Big) \stackrel{\triangle}{=} \varepsilon\mathcal{F}_1\Big(\varepsilon,\mathcal{N}(\eta,\varepsilon)\Big)D_{Q,y}(\eta,\varepsilon)+Q_y(\eta,\varepsilon) + \left\{I_n+\varepsilon\int_{-h}^{0}[d_\eta D_{Q,v}(\eta,\varepsilon)]L_v(\eta,\varepsilon)\right\}D_2(\eta,\varepsilon) - P_y(\varepsilon)\Big[D_4(\eta,\varepsilon)-\varepsilon L_y(\varepsilon)D_2(\eta,\varepsilon)\Big]. \tag{5.11}$$

Using (5.7)–(5.11), we represent the above-mentioned differential equation for $w_x(t)$ in the form

$$\frac{dw_x(t)}{dt} = \mathcal{F}_1\Big(\varepsilon,\mathcal{N}(\eta,\varepsilon)\Big)w_x(t)+\mathcal{F}_2\Big(\varepsilon,\mathcal{N}(\eta,\varepsilon)\Big)w_y(t) + \int_{-h}^{0}\Big[d_\eta\mathcal{F}_3\Big(\eta,\varepsilon,\mathcal{N}(\eta,\varepsilon)\Big)\Big]w_v(t,\eta)+\int_{-h}^{0}\Big[d_\eta\mathcal{F}_4\Big(\eta,\varepsilon,\mathcal{N}(\eta,\varepsilon)\Big)\Big]w_y(t+\varepsilon\eta). \tag{5.12}$$

Due to the transformation (5.5), the boundary condition (3.18) becomes

$$w_v(t,0) = \Big(I_n-L_v(0,\varepsilon)\Big)w_x(t)+\varepsilon\Big(I_n-L_v(0,\varepsilon)\Big)\bigg\{\int_{-h}^{0}\big[d_\eta D_{Q,v}(\eta,\varepsilon)\big]w_v(t,\eta)d\eta + P_y(\varepsilon)w_y(t)+\int_{-h}^{0}\big[d_\eta D_{Q,y}(\eta,\varepsilon)\big]w_y(t+\varepsilon\eta)d\eta\bigg\}, \quad t\ge\varepsilon h,\quad \varepsilon\in(0,\varepsilon_1^*]. \tag{5.13}$$

Thus, we have transformed the UTS to the equivalent new system, which is obtained from the UTS by replacing there the slow mode (3.11) with (5.12), and the boundary condition (3.18) with (5.13), while the other equations of the UTS remain the same. This new system has the same feature as the UTS. Namely, the fast modes (4.8) and (4.9) are separated from the slow one (the fast modes do not contain the slow state variable $w_x(t)$), while the slow mode (5.12) is not separated from the fast ones (the slow mode contains the fast state variables $w_v(t,\eta)$ and $(w_y(t),w_y(t+\varepsilon\eta))$). However, in contrast with the UTS, now it is possible to separate the slow mode from the fast modes, i.e., to eliminate $w_v(t,\eta)$ and $(w_y(t),w_y(t+\varepsilon\eta))$ from (5.12) by a proper choice of $\mathcal{N}(\eta,\varepsilon)$. Such a separation is made in the next section.

6. Separation of the slow mode (5.12) from the fast modes (4.8)–(4.9). In this section, first, we choose the set of unknown matrices $\mathcal{N}(\eta,\varepsilon)$ in such a way that the terms in the right-hand sides of (5.12), containing the fast states $w_v(t,\eta)$ and $(w_y(t),w_y(t+\varepsilon\eta))$, vanish. Such a choice yields a system of three equations with respect to the matrices of the set $\mathcal{N}(\eta,\varepsilon)$ (subsection 6.1). In subsection 6.2, the existence of a solution to this system is established. Finally, based on the results of subsections 6.1 and 6.2, the singularly perturbed diagonal system, equivalent to the UTS, is derived. In this system, the fast modes do not contain the slow state variable, and the slow mode does not contain the fast state variables; i.e., the slow and fast differential equations are completely separated.

6.1. Elimination of the fast state variables from (5.12). In order to eliminate the fast state variables from (5.12), and thus to separate the slow state variable $w_x(t)$ from the fast states $w_v(t,\cdot)$ and $w_y(\cdot)$, one has to choose the set $\mathcal{N}(\eta,\varepsilon)$ such that the matrix $\mathcal{F}_2(\varepsilon,\mathcal{N}(\eta,\varepsilon))$ becomes zero, while the matrices $\mathcal{F}_3(\varepsilon,\mathcal{N}(\eta,\varepsilon))$ and $\mathcal{F}_4(\varepsilon,\mathcal{N}(\eta,\varepsilon))$ become constant with respect to η. This observation, along with (5.9)–(5.11), leads to the following equations for obtaining the set $\mathcal{N}(\eta,\varepsilon)$:

$$\mathcal{F}_2\Big(\varepsilon,\mathcal{N}(\eta,\varepsilon)\Big)=0, \tag{6.1}$$

$$\mathcal{F}_l\Big(\eta,\varepsilon,\mathcal{N}(\eta,\varepsilon)\Big) = C_l(\varepsilon), \quad \eta \in [-h,0],\ l=3,4, \tag{6.2}$$

where $C_3(\varepsilon)$ and $C_4(\varepsilon)$ are unknown matrices of the dimensions $n \times n$ and $n \times m$, respectively. These matrices should satisfy the equations

$$C_3(\varepsilon) = \mathcal{F}_3\Big(\eta,\varepsilon,\mathcal{N}(\eta,\varepsilon)\Big)\Big|_{\eta=-h}, \qquad C_4(\varepsilon) = \mathcal{F}_4\Big(\eta,\varepsilon,\mathcal{N}(\eta,\varepsilon)\Big)\Big|_{\eta=-h}. \tag{6.3}$$

By substituting (6.3) into (6.2), we obtain

$$\mathcal{F}_l\Big(\eta,\varepsilon,\mathcal{N}(\eta,\varepsilon)\Big) = \mathcal{F}_l\Big(\eta,\varepsilon,\mathcal{N}(\eta,\varepsilon)\Big)\Big|_{\eta=-h}, \quad \eta \in [-h,0],\ l=3,4. \tag{6.4}$$

Note that the system (6.1), (6.4) is functional-integral-algebraic. In the next section, we show the existence of a solution to this system by constructing and justifying its asymptotic solution with respect to ε.

6.2. Asymptotic solution of the system (6.1), (6.4). We look for the zero-order asymptotic solution $\mathcal{N}^0(\eta) \stackrel{\triangle}{=} \big(Q_v^0(\eta), P_y^0, Q_y^0(\eta)\big)$ of (6.1), (6.4).

Due to (5.1) and (5.3), it is reasonable to set

$$Q_v^0(-h) = 0, \qquad Q_y^0(-h) = 0. \tag{6.5}$$

Equations for the terms of the zero-order asymptotic solution to system (6.1), (6.4) are obtained by formally setting $\varepsilon = 0$ in this system and replacing $Q_v(\eta,\varepsilon)$, $P_y(\varepsilon)$, and $Q_y(\eta,\varepsilon)$ with $Q_v^0(\eta)$, P_y^0, and $Q_y^0(\eta)$, respectively. Thus, by using Lemma 4.1 and (6.5), we obtain the following system of equations with respect to the unknown matrices $Q_v^0(\eta)$, P_y^0, $Q_y^0(\eta)$:

$$A_{20}(0) - P_y^0 A_{40}(0) - Q_y^0(0) = 0, \tag{6.6}$$

$$\begin{aligned} &\Gamma_{12}(\eta,0,L_y^0) - \Gamma_{12}(-h,0,L_y^0) + Q_v^0(\eta) \\ &\quad - P_y^0\Big(\Gamma_{34}(\eta,0,L_y^0) - \Gamma_{34}(-h,0,L_y^0)\Big) = 0, \quad \eta \in [-h,0], \end{aligned} \tag{6.7}$$

$$Q_y^0(\eta) + D_2(\eta,0) - D_2(-h,0) - P_y^0\Big(D_4(\eta,0) - D_4(-h,0)\Big) = 0, \quad \eta \in [-h,0]. \tag{6.8}$$

Setting $\eta = 0$ in (6.8) and using (2.4)–(2.6) and (4.2) yields

$$Q_y^0(0) = P_y^0\big(\mathcal{H}_4(0) - A_{40}(0)\big) - \big(\mathcal{H}_2(0) - A_{20}(0)\big). \tag{6.9}$$

Substituting (6.9) into (6.6) yields $\mathcal{H}_2(0) - P_y^0\mathcal{H}_4(0) = 0$, leading to

$$P_y^0 = \mathcal{H}_2(0)\mathcal{H}_4^{-1}(0). \tag{6.10}$$

Now, (6.7) and (6.8) yield immediately for $\eta \in [-h,0]$

$$\begin{aligned} Q_v^0(\eta) &= \Gamma_{12}(-h,0,L_y^0) - \Gamma_{12}(\eta,0,L_y^0) + P_y^0\Big(\Gamma_{34}(\eta,0,L_y^0) - \Gamma_{34}(-h,0,L_y^0)\Big), \\ Q_y^0(\eta) &= D_2(-h,0) - D_2(\eta,0) + P_y^0\Big(D_4(\eta,0) - D_4(-h,0)\Big), \end{aligned} \tag{6.11}$$

which completes the solution of (6.6)–(6.8). This solution is unique.

Remark 6.1. Due to (6.11), $Q_v^0(-h) = 0_{n\times n}$ and $Q_y^0(-h) = 0_{n\times m}$, where $0_{n\times n}$ and $0_{n\times m}$ are zero-matrices of dimensions $n \times n$ and $n \times m$, respectively. Due to Remark 2.1, the functions $Q_v^0(\eta)$ and $Q_y^0(\eta)$ are piecewise continuous in $\eta \in [-h, 0]$ with the break points $\eta = -h_j$ $(j = 1, \dots, N)$, where these functions have finite limits from the right $Q_v^0(-h_j + 0) = \lim_{\eta\to -h_j+0} Q_v^0(\eta)$ and $Q_y^0(-h_j + 0) = \lim_{\eta\to -h_j+0} Q_y^0(\eta)$, respectively. Also, these functions are continuous from the left at the break points $\eta = -h_j$ $(j = 1, \dots, N-1)$. Moreover, the matrix-valued functions

$$Q_{v,j}^0(\eta) = \begin{cases} \lim_{\eta\to -h_j+0} Q_v^0(\eta), & \eta = -h_j, \\ Q_v^0(\eta), & \eta \in (-h_j, -h_{j-1}], \end{cases}$$

and

$$Q_{y,j}^0(\eta) = \begin{cases} \lim_{\eta\to -h_j+0} Q_y^0(\eta), & \eta = -h_j, \\ Q_y^0(\eta), & \eta \in (-h_j, -h_{j-1}], \end{cases}$$

are differentiable on the interval $[-h_j, -h_{j-1}]$ for all $j = 1, \dots, N$.

Thus, we have completed the formal construction of the zero-order asymptotic solution to system (6.1), (6.4). Based on this asymptotic solution, we obtain the following lemma.

Lemma 6.2. *Let assumptions* (A1)–(A3) *be valid. Then there exists a positive number* ε_2^* $(\varepsilon_2^* \le \varepsilon_1^*)$ *such that, for all* $\varepsilon \in (0, \varepsilon_2^*]$, *system* (6.1), (6.4) *has a solution* $(Q_v(\eta,\varepsilon), P_y(\varepsilon), Q_y(\eta,\varepsilon))$, $\eta \in [-h, 0]$, *satisfying the inequalities*

$$\|Q_v(\eta,\varepsilon) - Q_v^0(\eta)\| \le a_2^*\varepsilon, \quad \|Q_y(\eta,\varepsilon) - Q_y^0(\eta)\| \le a_2^*\varepsilon \quad \forall \eta \in [-h, 0], \tag{6.12}$$

$$\|P_y(\varepsilon) - P_y^0\| \le a_2^*\varepsilon, \tag{6.13}$$

where $a_2^* > 0$ *is some constant independent of* ε.

Such a solution is unique. Moreover, the matrix-valued functions

$$Q_{v,j}(\eta,\varepsilon) = \begin{cases} \lim_{\eta\to -h_j+0} Q_v(\eta,\varepsilon), & \eta = -h_j, \\ Q_v(\eta,\varepsilon), & \eta \in (-h_j, -h_{j-1}], \end{cases}$$

and

$$Q_{y,j}(\eta,\varepsilon) = \begin{cases} \lim_{\eta\to -h_j+0} Q_y(\eta,\varepsilon), & \eta = -h_j, \\ Q_y(\eta,\varepsilon), & \eta \in (-h_j, -h_{j-1}], \end{cases}$$

are differentiable with respect to η *on the interval* $[-h_j, -h_{j-1}]$ *for all* $j = 1, \dots, N$ *and any* $\varepsilon \in (0, \varepsilon_2^*]$.

The lemma is proven in Appendix B.

6.3. Diagonal system for $t \ge \varepsilon h$. Due to the existence of solution to the system (6.1), (6.4), the slow mode equation (5.12) in the system (5.12), (4.8)–(4.9) becomes as follows for $t > \varepsilon h$:

$$dw_x(t)/dt = \mathcal{F}_1\Big(\varepsilon, \mathcal{N}(\eta,\varepsilon)\Big) w_x(t), \tag{6.14}$$

where $\mathcal{F}_1(\varepsilon, \mathcal{N}(\eta,\varepsilon))$ is given by (5.8).

Using (5.1)–(5.5) and (4.13)–(4.14), one obtains the initial condition for (6.14),

$$\begin{aligned} w_x(\varepsilon h) = x_{\mathrm{in}}(\varepsilon h,\varepsilon) &- \varepsilon \int_{-h}^{0} Q_v(\eta,\varepsilon)\big[x_{\mathrm{in}}(\varepsilon h + \varepsilon\eta,\varepsilon) - L_v(\eta,\varepsilon) x_{\mathrm{in}}(\varepsilon h,\varepsilon)\big] d\eta \\ &- \varepsilon P_y(\varepsilon)\big[y_{\mathrm{in}}(\varepsilon h,\varepsilon) - L_y(\varepsilon) x_{\mathrm{in}}(\varepsilon h,\varepsilon)\big] \\ &- \varepsilon \int_{-h}^{0} Q_y(\eta,\varepsilon)\big[y_{\mathrm{in}}(\varepsilon h + \varepsilon\eta,\varepsilon) - L_y(\varepsilon) x_{\mathrm{in}}(\varepsilon h + \varepsilon\eta,\varepsilon)\big] d\eta, \quad 0 < \varepsilon \le \varepsilon_2^*. \end{aligned} \tag{6.15}$$

Also, using (4.14), (5.1)–(5.4) and Lemma 6.2, the boundary condition (5.13) becomes

$$w_v(t,0) = \Big(I_n - L_v(0,\varepsilon)\Big)w_x(t) + \varepsilon\Big(I_n - L_v(0,\varepsilon)\Big)\bigg\{\int_{-h}^{0} Q_v(\eta,\varepsilon)w_v(t,\eta)d\eta$$
$$+ P_y(\varepsilon)w_y(t) + \int_{-h}^{0} Q_y(\eta,\varepsilon)w_y(t+\varepsilon\eta)d\eta\bigg\}, \quad t \geq \varepsilon h,\ \varepsilon \in (0,\varepsilon_2^*]. \tag{6.16}$$

The transformation (5.5) is invertible for all $t \geq \varepsilon h$. This observation, as well as Theorem 4.3 and Lemma 6.2, yields immediately the following theorem.

THEOREM 6.3. *Let assumptions* (A1)–(A3) *be valid. Let* $\big(L_v(\eta,\varepsilon), L_y(\varepsilon)\big)$ *be the solution of system* (4.1) *mentioned in Lemma* 4.1. *Let* $(Q_v(\eta,\varepsilon), P_y(\varepsilon), Q_y(\eta,\varepsilon))$ *be the solution of system* (6.1), (6.4) *mentioned in Lemma* 6.2. *Then, for all* $\varepsilon \in (0,\varepsilon_2^*]$, *the system* (6.14) *and* (4.8)–(4.9), *considered in the domain* $\tilde{\Omega}_\varepsilon$, *with the initial conditions* (6.15), (4.13) *and the boundary condition* (6.16), *and the system* (2.7)–(2.8), *considered in the interval* $(\varepsilon h, +\infty)$, *with the initial conditions* (4.15), *are equivalent to each other subject to the relations* (5.5), (3.7)–(3.8), *and* (3.1) *on their state variables.*

In what follows, the system of equations (6.14) and (4.8)–(4.9), subject to the initial conditions (6.15), (4.13) and the boundary condition (6.16), is called the *diagonal system* (DS) for $t \geq \varepsilon h$. Note that, in the DS, the slow and fast state variables are connected only by the boundary condition (6.16), which is *not* a *differential* equation. Moreover, this connection is weak, because it is neglected for $\varepsilon \to +0$. The *differential* equations (6.14) and (4.8)–(4.9) for the slow $w_x(t)$ and fast $w_v(t,\eta)$, $(w_y(t), w_y(t+\varepsilon\eta))$ state variables are completely disconnected from each other.

7. Important particular case: Single pointwise delay. In this section, the particular case of system (2.1)–(2.2) with a single pointwise delay and without a distributed delay is treated. In subsection 7.1, we briefly describe the exact slow-fast decomposition of such a system. Then, we derive a set of equations for the spectrum of the decomposed system (subsection 7.2). It is shown that this set consists of two unconnected equations. One of these equations determines the slow part of the spectrum, while the other determines its fast part. Since the decomposed system and the original one are equivalent, they have the same spectrum. Therefore, the derivation of the above-mentioned set of equations for the spectrum of the decomposed system means the exact slow-fast decomposition of the characteristic equation (spectrum equation) for the original singularly perturbed system.

We consider the case of system (2.1)–(2.2), frequently arising in applications, where

$$N = 1, \quad h_1 = h, \quad G_i(\eta,\varepsilon) \equiv 0, \quad i = 1,\dots,4. \tag{7.1}$$

7.1. Exact slow-fast decomposition. In the case (7.1), we deal directly with the original system (2.1)–(2.2), not converting it to the equivalent form (2.7)–(2.8). Namely, the system (4.1) becomes

$$dL_v(\eta,\varepsilon)/d\eta = \varepsilon L_v(\eta,\varepsilon)\Big[\Psi_{12,0}\Big(\varepsilon, L_y(\varepsilon)\Big) + \Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)L_v(-h,\varepsilon)\Big],$$

$$\mathcal{H}_4(0)L_y(\varepsilon) = \varepsilon L_y(\varepsilon)\Big[\Psi_{12,0}\Big(\varepsilon, L_y(\varepsilon)\Big) + \Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)L_v(-h,\varepsilon)\Big]$$
$$-\Big[\Psi_{34,0}\Big(\varepsilon, L_y(\varepsilon)\Big) + \Psi_{34,1}\Big(\varepsilon, L_y(\varepsilon)\Big)L_v(-h,\varepsilon)\Big] + \mathcal{H}_4(0)L_y(\varepsilon), \tag{7.2}$$

where $\mathcal{H}_i(\varepsilon)$ $(i = 1, \dots, 4)$ are given by (4.2) subject to (7.1),

$$\Psi_{k\ k+1,j}\Big(\varepsilon, L_y(\varepsilon)\Big) \triangleq A_{kj}(\varepsilon) + A_{k+1\ j}(\varepsilon)L_y(\varepsilon), \quad k = 1, 3,\ j = 0, 1. \tag{7.3}$$

The terms of the zero-order asymptotic solution to system (7.2) have the form (4.5), (4.6). The upper triangular system (3.11), (4.8)–(4.9) becomes

$$\begin{aligned}\frac{dx(t)}{dt} &= \Big[\Psi_{12,0}\Big(\varepsilon, L_y(\varepsilon)\Big) + \Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)L_v(-h,\varepsilon)\Big]x(t)\\ &+ \Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)w_v(t,-h) + A_{20}(\varepsilon)w_y(t) + A_{21}(\varepsilon)w_y(t-\varepsilon h), \quad t > \varepsilon h,\end{aligned} \tag{7.4}$$

$$\begin{aligned}\varepsilon\frac{\partial w_v(t,\eta)}{\partial t} - \frac{\partial w_v(t,\eta)}{\partial \eta} &= -\varepsilon L_v(\eta,\varepsilon)\Big[\Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)w_v(t,-h)\\ &+ A_{20}(\varepsilon)w_y(t) + A_{21}(\varepsilon)w_y(t-\varepsilon h)\Big], \quad (t,\eta) \in \tilde{\Omega}_\varepsilon,\end{aligned} \tag{7.5}$$

$$\begin{aligned}\frac{\varepsilon dw_y(t)}{dt} &= \Big[\Psi_{34,1}\Big(\varepsilon, L_y(\varepsilon)\Big) - \varepsilon L_y(\varepsilon)\Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)\Big]w_v(t,-h)\\ &+\Big(A_{40}(\varepsilon) - \varepsilon L_y(\varepsilon)A_{20}(\varepsilon)\Big)w_y(t) + \Big(A_{41}(\varepsilon) - \varepsilon L_y(\varepsilon)A_{21}(\varepsilon)\Big)w_y(t-\varepsilon h),\\ &\qquad t > \varepsilon h.\end{aligned} \tag{7.6}$$

For system (7.4)–(7.6), the transformation of variables (5.5) becomes $w_x(t) = x(t) - \varepsilon\int_{-h}^{0} Q_v(\eta,\varepsilon)w_v(t,\eta)d\eta - \varepsilon P_y(\varepsilon)w_y(t) - \varepsilon\int_{-h}^{0} Q_y(\eta,\varepsilon)w_y(t+\varepsilon\eta)d\eta$, where $t \geq \varepsilon h$; $P_y(\varepsilon)$, as in (5.5), is some $n \times m$-matrix-valued function of $\varepsilon \in (0, \varepsilon_1^*]$; $Q_v(\eta,\varepsilon)$ and $Q_y(\eta,\varepsilon)$ are some $n\times n$- and $n\times m$-matrix-valued functions of $(\eta,\varepsilon) \in [-h,0]\times(0,\varepsilon_1^*]$; however, these functions, in contrast with (5.5), are differentiable with respect to $\eta \in [-h, 0]$ for each $\varepsilon \in (0, \varepsilon_1^*]$.

The matrices $Q_v(\eta,\varepsilon)$, $P(\varepsilon)$, $Q_y(\eta,\varepsilon)$, allowing us to eliminate the fast state variables $w_v(t,\eta)$ and $(w_y(t), w_y(t+\varepsilon\eta))$ from the slow mode equation for the new slow state $w_x(t)$, satisfy the following set of equations:

$$\begin{aligned}dQ_v(\eta,\varepsilon)/d\eta &= -\varepsilon\Big[\Psi_{12,0}\Big(\varepsilon, L_y(\varepsilon)\Big) + \Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)L_v(-h,\varepsilon)\\ &\quad - Q_v(0,\varepsilon)\Big(I_n - L_v(0,\varepsilon)\Big)\Big]Q_v(\eta,\varepsilon), \qquad \eta \in [-h,0],\\ dQ_y(\eta,\varepsilon)/d\eta &= -\varepsilon\Big[\Psi_{12,0}\Big(\varepsilon, L_y(\varepsilon)\Big) + \Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)L_v(-h,\varepsilon)\\ &\quad - Q_v(0,\varepsilon)\Big(I_n - L_v(0,\varepsilon)\Big)\Big]Q_y(\eta,\varepsilon), \qquad \eta \in [-h,0],\\ Q_v(-h,\varepsilon) &= -\Big(I_n + \varepsilon\int_{-h}^{0} Q_v(\eta,\varepsilon)L_v(\eta,\varepsilon)d\eta\Big)\Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)\\ &\quad + P_y(\varepsilon)\Big[\Psi_{34,1}\Big(\varepsilon, L_y(\varepsilon)\Big) - \varepsilon L_y(\varepsilon)\Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)\Big],\\ Q_y(-h,\varepsilon) &= -\Big(I_n + \varepsilon\int_{-h}^{0} Q_v(\eta,\varepsilon)L_v(\eta,\varepsilon)d\eta\Big)A_{21}(\varepsilon)\\ &\quad + P_y(\varepsilon)\Big(A_{41}(\varepsilon) - \varepsilon L_y(\varepsilon)A_{21}(\varepsilon)\Big),\end{aligned}$$

$$
\begin{aligned}
Q_y(0,\varepsilon) = {} & \left(I_n + \varepsilon\int_{-h}^{0} Q_v(\eta,\varepsilon)L_v(\eta,\varepsilon)d\eta\right)A_{20}(\varepsilon) \\
& + \varepsilon\Big[\Psi_{12,0}\Big(\varepsilon, L_y(\varepsilon)\Big) + \Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)L_v(-h,\varepsilon) \\
& - Q_v(0,\varepsilon)\Big(I_n - L_v(0,\varepsilon)\Big)\Big]P_y(\varepsilon) - P_y(\varepsilon)\Big(A_{40}(\varepsilon) - \varepsilon L_y(\varepsilon)A_{20}(\varepsilon)\Big).
\end{aligned}
\tag{7.7}
$$

The terms of the zero-order asymptotic solution to the set (7.7) have the form (6.10) and $Q_v^0(\eta) \equiv Q_v^0 = P_y^0\Psi_{34,1}(0, L_y^0) - \Psi_{12,1}(0, L_y^0)$, $Q_y^0(\eta) \equiv Q_y^0 = P_y^0 A_{41}(0) - A_{21}(0)$, $\eta \in [-h, 0]$.

Finally, the slow mode equation (6.14) in the DS becomes

$$
\begin{aligned}
dw_x(t)/dt = {} & \Big[\Psi_{12,0}\Big(\varepsilon, L_y(\varepsilon)\Big) + \Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)L_v(-h,\varepsilon) \\
& - Q_v(0,\varepsilon)\Big(I_n - L_v(0,\varepsilon)\Big)\Big]w_x(t), \qquad t > \varepsilon h,
\end{aligned}
\tag{7.8}
$$

while the fast mode equations in the DS are (7.5)–(7.6). Moreover, this system is subject to the boundary condition (6.16).

7.2. Equations for the spectrum of the diagonal and original systems. Let us introduce into consideration the following matrix-valued functions of the small parameter ε and a complex variable λ:

$$
\begin{aligned}
\Lambda_{11}(\varepsilon,\lambda) \triangleq {} & \Psi_{12,0}\Big(\varepsilon, L_y(\varepsilon)\Big) + \Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)L_v(-h,\varepsilon) \\
& - Q_v(0,\varepsilon)\Big(I_n - L_v(0,\varepsilon)\Big) - \lambda I_n,
\end{aligned}
\tag{7.9}
$$

$$
\begin{aligned}
\Lambda_{22}\big(\varepsilon,\lambda\big) \triangleq {} & A_{40}(\varepsilon) - \varepsilon L_y(\varepsilon)A_{20}(\varepsilon) \\
& + \Big(A_{41}(\varepsilon) - \varepsilon L_y(\varepsilon)A_{21}(\varepsilon)\Big)\exp\big(-\varepsilon\lambda h\big) - \varepsilon\lambda I_m, \\
\Lambda_{23}(\varepsilon) \triangleq {} & \Psi_{34,1}\Big(\varepsilon, L_y(\varepsilon)\Big) - \varepsilon L_y(\varepsilon)\Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big),
\end{aligned}
\tag{7.10}
$$

$$
\begin{aligned}
\Lambda_{32}\big(\varepsilon,\lambda\big) & \triangleq M_v\big(-h,\varepsilon,\lambda\big)\big[A_{20}(\varepsilon) + A_{21}(\varepsilon)\exp\big(-\varepsilon\lambda h\big)\big], \\
\Lambda_{33}\big(\varepsilon,\lambda\big) & \triangleq \Big[M_v\big(-h,\varepsilon,\lambda\big)\Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big) - I_n\Big], \\
\Lambda_{34}\big(\varepsilon,\lambda\big) & \triangleq \exp\big(-\varepsilon\lambda h\big)I_n,
\end{aligned}
\tag{7.11}
$$

$$
\begin{aligned}
\Lambda_{41}(\varepsilon) \triangleq {} & \Big(I_n - L_v(0,\varepsilon)\Big), \\
\Lambda_{42}\big(\varepsilon,\lambda\big) \triangleq {} & \varepsilon\Big(I_n - L_v(0,\varepsilon)\Big)\bigg\{\int_{-h}^{0} Q_v(\eta,\varepsilon)M_v\big(\eta,\varepsilon,\lambda\big)d\eta\Big[A_{20}(\varepsilon) \\
& + A_{21}(\varepsilon)\exp\big(-\varepsilon\lambda h\big)\Big] + P_y(\varepsilon) + \int_{-h}^{0} Q_y(\eta,\varepsilon)\exp\big(\varepsilon\lambda\eta\big)d\eta\bigg\}, \\
\Lambda_{43}\big(\varepsilon,\lambda\big) \triangleq {} & \varepsilon\Big(I_n - L_v(0,\varepsilon)\Big)\int_{-h}^{0} Q_v(\eta,\varepsilon)M_v\big(\eta,\varepsilon,\lambda\big)d\eta\Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big), \\
\Lambda_{44}\big(\varepsilon,\lambda\big) \triangleq {} & \varepsilon\Big(I_n - L_v(0,\varepsilon)\Big)\int_{-h}^{0} Q_v(\eta,\varepsilon)\exp\big(\varepsilon\lambda\eta\big)d\eta - I_n,
\end{aligned}
\tag{7.12}
$$

where $M_v(\eta,\varepsilon,\lambda)$ has the form

$$M_v(\eta,\varepsilon,\lambda)=\varepsilon\int_0^{\eta}\exp\big(\varepsilon\lambda(\eta-\zeta)\big)L_v(\zeta,\varepsilon)d\zeta. \tag{7.13}$$

Based on (7.9)–(7.12), let us consider the set of two equations

$$\Delta_s(\varepsilon,\lambda)=0,\qquad \Delta_s(\varepsilon,\lambda)\stackrel{\triangle}{=}\det\Lambda_{11}(\varepsilon,\lambda), \tag{7.14}$$

$$\Delta_f(\varepsilon,\lambda)=0,\qquad \Delta_f(\varepsilon,\lambda)\stackrel{\triangle}{=}\det\begin{bmatrix}\Lambda_{22}(\varepsilon,\lambda) & \Lambda_{23}(\varepsilon) & 0_{m\times n}\\ \Lambda_{32}(\varepsilon,\lambda) & \Lambda_{33}(\varepsilon,\lambda) & \Lambda_{34}(\varepsilon,\lambda)\\ \Lambda_{42}(\varepsilon,\lambda) & \Lambda_{43}(\varepsilon,\lambda) & \Lambda_{44}(\varepsilon,\lambda)\end{bmatrix}, \tag{7.15}$$

where $0_{m\times n}$ is zero matrix of the dimension $m\times n$.

Lemma 7.1. *Let assumptions* (A1), (A3) *be valid. Then, for each* $\varepsilon\in(0,\varepsilon_2^*]$, *the spectrum of the DS* (7.8), (7.5)–(7.6), (6.16) *coincides with the set of all roots* $\lambda=\lambda(\varepsilon)$ *of the system* (7.14)–(7.15), *where* $\varepsilon_2^*>0$ *is defined in Lemma* 6.2.

The lemma is proven in Appendix C.

Remark 7.2. The first equation of system (7.14)–(7.15) is a polynomial equation of degree n (the dimension of the Euclidean slow state variables $x(t)$ and $w_x(t)$ in the original and the diagonal systems), while the second equation is a quasipolynomial equation of degree m (the dimension of the Euclidean fast state variables $y(t)$ and $w_y(t)$ in the original and the diagonal systems).

Let us analyze (7.14) and (7.15) separately.

Due to (7.9), the coefficient for λ^n (the highest degree of λ in the polynomial equation (7.14)) is $(-1)^n$, while the other coefficients are bounded with respect to $\varepsilon\in(0,\varepsilon_2^*]$. This observation directly yields the following lemma.

Lemma 7.3. *Let assumptions* (A1), (A3) *be valid. Then there exists a positive number* γ_s, *independent of* ε, *such that any root* $\lambda(\varepsilon)$ *of* (7.14) *satisfies the inequality* $|\lambda(\varepsilon)|\le\gamma_s$ *for all* $\varepsilon\in(0,\varepsilon_2^*]$.

We proceed to (7.15).

Lemma 7.4. *Let assumptions* (A1), (A3) *be valid. Then there exist a positive number* $\varepsilon_3^*\le\varepsilon_2^*$ *and a positive number* γ_f, *independent of* ε, *such that any root* $\lambda(\varepsilon)$ *of the quasipolynomial equation* (7.15) *satisfies the inequality* $|\lambda(\varepsilon)|\ge\gamma_f/\varepsilon$ *for all* $\varepsilon\in(0,\varepsilon_3^*]$.

Proof. We prove the lemma by contradiction. Namely, let us assume that the statement of the lemma is wrong. This means the existence of three sequences $\{\varepsilon_k\}$, $\{\gamma_{f,k}\}$, $\{\lambda_k\}$ satisfying the following properties: (i) $\varepsilon_k>0$ $(k=1,2,\ldots)$; (ii) $\varepsilon_k\to+0$ for $k\to+\infty$; (iii) $\gamma_{f,k}>0$ $(k=1,2,\ldots)$; (iv) $\gamma_{f,k}\to+0$ for $k\to+\infty$; (v) $\Delta_f(\varepsilon_k,\lambda_k)=0$ $(k=1,2,\ldots)$; and (vi) $|\lambda_k|<\gamma_{f,k}/\varepsilon_k$ $(k=1,2,\ldots)$.

Due to the properties (i), (iii), (iv), and (vi), $\varepsilon_k\lambda_k\to0$ for $k\to+\infty$. Using the latter, as well as (7.10)–(7.12), (7.15), and Lemmas 4.1 and 6.2, we obtain

$$\lim_{k\to+\infty}\Delta_f(\varepsilon_k,\lambda_k)=\det\begin{bmatrix}\mathcal{H}_4(0) & \Psi_{34,1}\Big(0,L_y^0\Big) & 0_{m\times n}\\ 0_{n\times m} & -I_n & I_n\\ 0_{n\times m} & 0_{n\times n} & -I_n\end{bmatrix}=\det\mathcal{H}_4(0). \tag{7.16}$$

Remember that $\mathcal{H}_4(0)$ is given by (4.2) subject to (7.1).

Equation (7.16), along with the property (v), yields $\det \mathcal{H}_4(0) = 0$, which contradicts assumption (A3). This contradiction proves the lemma. ∎

Let $\mathcal{R}_s(\varepsilon)$, $\varepsilon \in (0, \varepsilon_2^*]$, be the set of all roots of (7.14). Let $\mathcal{R}_f(\varepsilon)$, $\varepsilon \in (0, \varepsilon_3^*]$, be the set of all roots of (7.15). Finally, for a given $\varepsilon > 0$ let $\mathcal{R}(\varepsilon)$ be the spectrum of the original singularly perturbed system (2.1)–(2.2), (7.1), i.e., the set of all roots of the following quasipolynomial equation of degree $(n + m)$ with respect to λ (the characteristic equation):

(7.17)

$$\det \begin{bmatrix} A_{10}(\varepsilon) + A_{11}(\varepsilon)\exp(-\varepsilon\lambda h) - \lambda I_n & A_{20}(\varepsilon) + A_{21}(\varepsilon)\exp(-\varepsilon\lambda h) \\ \frac{A_{30}(\varepsilon)+A_{31}(\varepsilon)\exp(-\varepsilon\lambda h)}{\varepsilon} & \frac{A_{40}(\varepsilon)+A_{41}(\varepsilon)\exp(-\varepsilon\lambda h)}{\varepsilon} - \lambda I_m \end{bmatrix} = 0.$$

Now, Lemmas 7.1, 7.3, and 7.4 and the above-mentioned fact, that the original system (2.1)–(2.2), (7.1) and the corresponding DS (7.8), (7.5)–(7.6), (6.16) have the same spectrum, directly yield the following theorem.

THEOREM 7.5. *Let assumptions* (A1), (A3) *be valid. Then there exists a positive number $\varepsilon_4^* \le \varepsilon_3^*$ such that, for all $\varepsilon \in (0, \varepsilon_4^*]$, the following relations hold: $\mathcal{R}(\varepsilon) = \mathcal{R}_s(\varepsilon) \bigcup \mathcal{R}_f(\varepsilon)$, $\mathcal{R}_s(\varepsilon) \bigcap \mathcal{R}_f(\varepsilon) = \emptyset$.*

Remark 7.6. Concluding this section, we would like to note the following. Theorem 7.5 states the exact slow-fast decomposition of (7.17) for the spectrum $\mathcal{R}(\varepsilon)$ of the original system (2.1)–(2.2), (7.1) into two unconnected equations, (7.14) and (7.15). The set $\mathcal{R}_s(\varepsilon)$ of roots of (7.14) is the slow part of the spectrum $\mathcal{R}(\varepsilon)$, while the set $\mathcal{R}_f(\varepsilon)$ of roots of (7.15) is the fast part of this spectrum. Note that $\mathcal{R}_s(\varepsilon)$ coincides with the spectrum of the slow homogeneous subsystem (7.8) of the DS (7.8), (7.5)–(7.6), (6.16), while $\mathcal{R}_f(\varepsilon)$ coincides with the spectrum of the homogeneous system corresponding to the fast nonhomogeneous subsystem (7.5)–(7.6), (6.16), where $w_x(t)$ is considered as a known nonhomogeneous term. Note also that in the works [19, 40] a qualitative slow-fast decomposition of the spectrum of a singularly perturbed system with small delays was studied. In the present paper, in contrast with [19, 40], the quantitative exact slow-fast decomposition of the spectrum of a singularly perturbed system with a small pointwise delay was carried out.

8. Example: Stability analysis of a multilink single-sink optical network. In this section, we consider a real-world example, to which the results of the previous sections are applied. This example is devoted to stability analysis of the singularly perturbed system of nonlinear differential equations with a small pointwise delay, modeling a multilink single-sink optical network. In subsection 8.1, an analytical study is carried out. In subsection 8.2, a numerical illustration is presented.

8.1. Analytical study. The mathematical model of the multilink single-sink optical network is a nonlinear singularly perturbed time-invariant system with a single pointwise delay (see equations (23)–(24) of [15]). This system consists of scalar slow mode and vector fast mode equations. The delay is on the order of a small parameter $\varepsilon > 0$ multiplying the part of state derivatives in the system. In [15], the asymptotic stability of the trivial solution to this system was studied, using the linear matrix inequality (LMI) approach. A sufficient stability condition in the form of feasibility of some rather complicated and high-dimensional LMI was obtained in this work.

In the present section, sufficient conditions for the asymptotic stability, as well as the instability, of the trivial solution to the system modeling the multilink single-sink optical network are derived based on its exact slow-fast decomposition and on the exact slow-fast decomposition of its spectrum.

In the case of perfectly known coefficients, this system can be represented as

$$\begin{aligned} &dx(t)/dt = bx(t)/\big(x(t)+c\big) + \alpha^T y(t), \\ &\varepsilon dy(t)/dt = \varepsilon\gamma x(t)/\big(x(t)+c\big)^3 + \varepsilon By(t)/\big(x(t)+c\big)^2 + Cy(t-\varepsilon h), \end{aligned} \tag{8.1}$$

where $t \geq 0$; $x(t) \in E^1$, $y(t) \in E^m$; b and $c \neq 0$ are given numbers; α and γ are given m-dimensional column-vectors; B and C are given $m \times m$-matrices; and ε and h have the same sense as in (2.1)–(2.2).

In system (8.1), $x(t)$ is a shifted dynamic pricing term, while each coordinate of $y(t)$ is a shifted optical input power at the transmitter in the respective link from the transmitter to the receiver in the optical path of channels. The small positive parameter ε represents a fast dynamics of $y(t)$ and a smallness of the delay. More details on the physical and engineering sense of (8.1) can be found in [15].

We analyze the asymptotic stability of the trivial solution of system (8.1) based on the first Lyapunov method [41], i.e., on the linearization of this system. Then, the spectrum of the resulting linear system is analyzed by its exact slow-fast decomposition.

The linearization of (8.1) in a neighborhood of the trivial solution $\{x(t) \equiv 0, y(t) \equiv 0\}$, $t \geq -\varepsilon h$, yields the system

$$\begin{aligned} &dx(t)/dt = b_1 x(t) + \alpha^T y(t), \\ &\varepsilon dy(t)/dt = \varepsilon\gamma_1 x(t) + \varepsilon B_1 y(t) + Cy(t-\varepsilon h), \end{aligned} \tag{8.2}$$

where $b_1 = b/c$, $\gamma_1 = \gamma/c^3$, $B_1 = B/c^2$.

Using the transformations (3.1), (3.7)–(3.8), (5.5), system (8.2) is converted equivalently for all sufficiently small $\varepsilon > 0$ to the following diagonal system in the domain $\tilde{\Omega}_\varepsilon$:

$$\begin{aligned} &dw_x(t)/dt = \big[b_1 + \alpha^T L_y(\varepsilon) - Q_v(0,\varepsilon)\big(1 - L_v(0,\varepsilon)\big)\big] w_x(t), \\ &\varepsilon \partial w_v(t,\tau)/\partial t - \partial w_v(t,\tau)/\partial\tau = -\varepsilon L_v(\tau,\varepsilon)\alpha^T w_y(t), \\ &\varepsilon dw_y(t)/dt = CL_y(\varepsilon) w_v(t,-h) + \varepsilon\big[B_1 - L_y(\varepsilon)\alpha^T\big] w_y(t) + Cw_y(t-\varepsilon h), \end{aligned} \tag{8.3}$$

and the boundary condition (6.16).

The scalar $L_v(\tau,\varepsilon)$ and the column m-vector $L_y(\varepsilon)$, appearing in (3.7)–(3.8) and in (8.3), (6.16), satisfy the condition $L_v(-h,\varepsilon) = 1$ and the system

$$\begin{aligned} &dL_v(\eta,\varepsilon)/d\eta = \varepsilon L_v(\eta,\varepsilon)\big(b_1 + \alpha^T L_y(\varepsilon)\big), \\ &CL_y(\varepsilon) = -\varepsilon\big[\gamma_1 + B_1 L_y(\varepsilon) - L_y(\varepsilon)\big(b_1 + \alpha^T L_y(\varepsilon)\big)\big]. \end{aligned} \tag{8.4}$$

For this system, assumption (A3) becomes $\det(C) \neq 0$. Subject to this assumption, the zero-order asymptotic solution of this system is $L_v^0 = 1$, $L_y^0 = 0$, and Lemma 4.1 is valid; i.e., (8.4) has a solution for all sufficiently small $\varepsilon > 0$.

The scalar $Q_v(\tau,\varepsilon)$ and the row m-vectors $P_y(\varepsilon)$ and $Q_y(\tau,\varepsilon)$, appearing in (5.5) and in (8.3), (6.16), satisfy the following system:

(8.5)
$$
\begin{aligned}
&dQ_v(\eta,\varepsilon)/d\eta = -\varepsilon\big[b_1 + \alpha^T L_y(\varepsilon) - Q_v(0,\varepsilon)\big(1 - L_v(0,\varepsilon)\big)\big]Q_v(\eta,\varepsilon),\\
&dQ_y(\eta,\varepsilon)/d\eta = -\varepsilon\big[b_1 + \alpha^T L_y(\varepsilon) - Q_v(0,\varepsilon)\big(1 - L_v(0,\varepsilon)\big)\big]Q_y(\eta,\varepsilon),\\
&Q_v(-h,\varepsilon) = P_y(\varepsilon)CL_y(\varepsilon),\\
&Q_y(-h,\varepsilon) = P_y(\varepsilon)C,\\
&Q_y(0,\varepsilon) = \left(1 + \varepsilon\int_{-h}^{0} Q_v(\eta,\varepsilon)L_v(\eta,\varepsilon)d\eta\right)\alpha^T\\
&\qquad + \varepsilon\big[b_1 + \alpha^T L_y(\varepsilon) - Q_v(0,\varepsilon)\big(1 - L_v(0,\varepsilon)\big)\big]P_y(\varepsilon) - \varepsilon P_y(\varepsilon)[B_1 - L_y(\varepsilon)\alpha^T].
\end{aligned}
$$

The zero-order asymptotic solution of this system is $P_y^0 = \alpha^T C^{-1}$, $Q_v^0(\tau) \equiv 0$, $Q_y^0(\tau) \equiv \alpha^T$. Moreover, Lemma 6.2 is valid for (8.5); i.e., this system has the solution.

Based on the exact slow-fast decomposition of system (8.2), i.e., on its transformation to the DS, we can establish properties of the spectrum $\mathcal{R}(\varepsilon)$ of (8.2), guaranteeing the asymptotic stability of this system for all sufficiently small $\varepsilon > 0$. Namely, due to Theorem 7.5, $\mathcal{R}(\varepsilon) = \mathcal{R}_s(\varepsilon)\bigcup\mathcal{R}_f(\varepsilon)$ ($\mathcal{R}_s(\varepsilon)\bigcap\mathcal{R}_f(\varepsilon) = \emptyset$). Moreover, due to this theorem and (7.14), the slow part $\mathcal{R}_s(\varepsilon)$ of the spectrum consists of the single element

$$
\lambda_s(\varepsilon) = b_1 + \alpha^T L_y(\varepsilon) - Q_v(0,\varepsilon)\big(1 - L_v(0,\varepsilon)\big). \tag{8.6}
$$

The fast part $\mathcal{R}_f(\varepsilon)$ of the spectrum is obtained from (7.15), where

$$
\begin{aligned}
&\Lambda_{22}(\varepsilon,\lambda) = \varepsilon\big(B_1 - L_y(\varepsilon)\alpha^T\big) + C\exp\big(-\varepsilon\lambda h\big) - \varepsilon\lambda I_m,\\
&\Lambda_{23}(\varepsilon,\lambda) = CL_y(\varepsilon), \quad \Lambda_{32}(\varepsilon,\lambda) = \varepsilon M_v(-h,\varepsilon,\lambda)\alpha^T, \quad \Lambda_{33}(\varepsilon,\lambda) = -1,\\
&\Lambda_{34}(\varepsilon,\lambda) = \exp\big(-\varepsilon\lambda(\varepsilon)h\big),\\
&\Lambda_{42}(\varepsilon,\lambda) = \varepsilon\big(1 - L_v(0,\varepsilon)\big)\\
&\qquad \times\left[\int_{-h}^{0} Q_v(\eta,\varepsilon)M_v(\eta,\varepsilon,\lambda)d\eta\alpha^T + P_y(\varepsilon) + \int_{-h}^{0} Q_y(\eta,\varepsilon)\exp(\varepsilon\lambda\eta)d\eta\right],\\
&\Lambda_{43}(\varepsilon,\lambda) = 0, \quad \Lambda_{44}(\varepsilon,\lambda) = \varepsilon\big(1 - L_v(0,\varepsilon)\big)\int_{-h}^{0} Q_v(\eta,\varepsilon)\exp(\varepsilon\lambda\eta)d\eta - 1.
\end{aligned} \tag{8.7}
$$

By transformation of the unknown $\lambda = \mu/\varepsilon$ in (7.15), (8.7), formally setting the first argument $\varepsilon = 0$ and redenoting the second argument μ as $\bar{\mu}_f$ in the resulting equations, we obtain the quasipolynomial equation with respect to $\bar{\mu}_f$

$$
\det\Big(C\exp(\bar{\mu}_f h) - \bar{\mu}_f I_m\Big) = 0. \tag{8.8}
$$

Let $\bar{\mathcal{R}}_f$ be the set of all roots of (8.8).

In what follows, we assume

$$
b_1 < 0, \qquad \max_{\bar{\mu}_f\in\bar{\mathcal{R}}_f} \mathrm{Re}\big(\bar{\mu}_f\big) \stackrel{\triangle}{=} \beta_\mu < 0. \tag{8.9}
$$

Due to the first inequality in (8.9), for any given constant $0 < \kappa < 1$ there exists a positive number $\varepsilon_{s,\kappa}$ such that $\lambda_s(\varepsilon)$, given by (8.6), satisfies the inequality $\lambda_s(\varepsilon) < \kappa b_1 < 0$, $\varepsilon \in (0, \varepsilon_{s,\kappa}]$.

LEMMA 8.1. *Let the second inequality in* (8.9) *be valid. Then, for any given constant* $0 < \kappa < 1$ *there exists a positive number* $\varepsilon_{f,\kappa}$ *such that each* $\lambda_f(\varepsilon) \in \mathcal{R}_f(\varepsilon)$ *satisfies the inequality* $\mathrm{Re}\big(\lambda_f(\varepsilon)\big) < \kappa\beta_\mu/\varepsilon$, $\varepsilon \in (0, \varepsilon_{f,\kappa}]$.

Proof. We prove the lemma by contradiction. Let us assume that the statement of the lemma is wrong. This means the existence of two sequences $\{\varepsilon_k\}$ and $\{\lambda_{f,k}\}$ with the following properties: (a) $\{\varepsilon_k\}$ is positive and convergent to zero; (b) $\mathrm{Re}(\lambda_{f,k}) \geq \kappa\beta_\mu/\varepsilon_k$ $(k = 1, 2, \ldots)$; and (c) (7.15) and (8.7), determining the set $\mathcal{R}_f(\varepsilon)$, are satisfied for any pair $(\varepsilon, \lambda) = (\varepsilon_k, \lambda_{f,k})$.

Consider the sequence $\{\mu_{f,k}\}$ such that $\mu_{f,k} = \varepsilon_k\lambda_{f,k}$. The following two cases can be distinguished: (i) $\{\mu_{f,k}\}$ is bounded; (ii) $\{\mu_{f,k}\}$ is unbounded. Let us start with the first case. In this case, there exists a convergent subsequence of $\{\mu_{f,k}\}$. For the sake of simplicity (but without loss of generality), we assume that the sequence $\{\mu_{f,k}\}$ itself is such a subsequence. Let $\hat{\mu}_f = \lim_{k\to+\infty} \mu_{f,k}$. Due to the property (b), $\mathrm{Re}(\hat{\mu}_f) \geq \kappa\beta_\mu$. Substituting $(\varepsilon, \lambda) = (\varepsilon_k, \lambda_{f,k})$ into (7.15) and (8.7), calculating the limit of the resulting equalities for $k \to +\infty$, and using the zero-order asymptotic solutions of the systems (8.4) and (8.5) yields, after some rearrangement, $\det(C\exp(\hat{\mu}_f h) - \hat{\mu}_f I_m) = 0$. The latter means that $\hat{\mu}_f$ is a root of (8.8). Thus, due to the second inequality in (8.9), $\mathrm{Re}(\hat{\mu}_f) \leq \beta_\mu < 0$, which contradicts the above obtained inequality $\mathrm{Re}(\hat{\mu}_f) \geq \kappa\beta_\mu$.

Proceed to the case (ii), where the sequence $\{\mu_{f,k}\}$ is unbounded. In this case, there exists a subsequence of $\{\mu_{f,k}\}$, modules of elements of which tend to infinity. As for the case (i), we assume that $\{\mu_{f,k}\}$ itself is such a subsequence; i.e., $\lim_{k\to+\infty} |\mu_{f,k}| = +\infty$. By substituting $(\varepsilon, \lambda) = (\varepsilon_k, \lambda_{f,k})$ into (7.15), (8.7), dividing the resulting equality (7.15) by $\mu_{f,k}$, and then calculating the limit of the last equality for $k \to +\infty$, one obtains the contradiction $(-1)^m = 0$.

The contradictions obtained in the cases (i) and (ii) prove the lemma. ◻

Thus, based on the exact slow-fast decompositions of system (8.2) and its spectrum $\mathcal{R}(\varepsilon)$, we have established the following. Subject to the inequalities (8.9), for any given number $0 < \kappa < 1$ and all sufficiently small $\varepsilon > 0$, each element $\lambda \in \mathcal{R}(\varepsilon)$ satisfies the inequality $\mathrm{Re}(\lambda) < \kappa b_1$. The latter means that the trivial solution of the linearized system (8.2) is asymptotically stable. Moreover, by virtue of the first Lyapunov method [41], the trivial solution of the original nonlinear system (8.1) is asymptotically stable for all sufficiently small $\varepsilon > 0$. It is important to note that if at least one of the numbers, either b_1 or β_μ, is positive, then the trivial solution of (8.1) is unstable for all sufficiently small $\varepsilon > 0$.

Remark 8.2. Similarly to the above presented analysis of the system (8.1) with perfectly known coefficients, an uncertain version of this system can be studied. Namely, let the coefficients of (8.1) have the form $b = b_{\mathrm{nom}} + \Delta b$, $\alpha = \alpha_{\mathrm{nom}} + \Delta\alpha$, $\gamma = \gamma_{\mathrm{nom}} + \Delta\gamma$, $B = B_{\mathrm{nom}} + \Delta B$, $C = C_{\mathrm{nom}} + \Delta C$, where the values b_{nom}, α_{nom}, γ_{nom}, B_{nom}, and C_{nom} are known nominal values, while Δb, $\Delta\alpha$, $\Delta\gamma$, ΔB, and ΔC are uncertainties, satisfying the inequalities $|\Delta b| \leq \delta_b$, $\|\Delta\alpha\| \leq \delta_\alpha$, $\|\Delta\gamma\| \leq \delta_\gamma$, $\|\Delta B\| \leq \delta_B$, and $\|\Delta C\| \leq \delta_C$. In this case, the inequalities (8.9) with $b_1 = (b_{\mathrm{nom}} + \delta_b)/c$ and $C = C_{\mathrm{nom}}$ guarantee the robust asymptotic stability of the trivial solution to system (8.1) for all sufficiently small $\varepsilon > 0$ and $\delta_C > 0$. The fulfilment of either the inequality $(b_{\mathrm{nom}} - \delta_b)/c > 0$ or the inequality $\beta_\mu > 0$, where β_μ is obtained from (8.9) for $C = C_{\mathrm{nom}}$, guarantees the robust instability of the trivial solution to system (8.1) for all sufficiently small $\varepsilon > 0$ and $\delta_C > 0$.

8.2. Numerical illustration. Consider a particular case of system (8.1) with the following data: $m = 1$, $b = -2$, $c = 1$, $\alpha = 3$, $\gamma = 8$, $B = 2$, $C = -0.9$, $h = 0.6$. For these data, $b_1 = -2$, $\gamma_1 = 4$, $B_1 = 2$. Equation (8.8) becomes $-0.9\exp(\bar{\mu}_f) - \bar{\mu}_f = 0$, and all roots of this equation satisfy the inequality $\mathrm{Re}(\bar{\mu}_f) < -0.53$. Thus, both inequalities in (8.9) are valid.

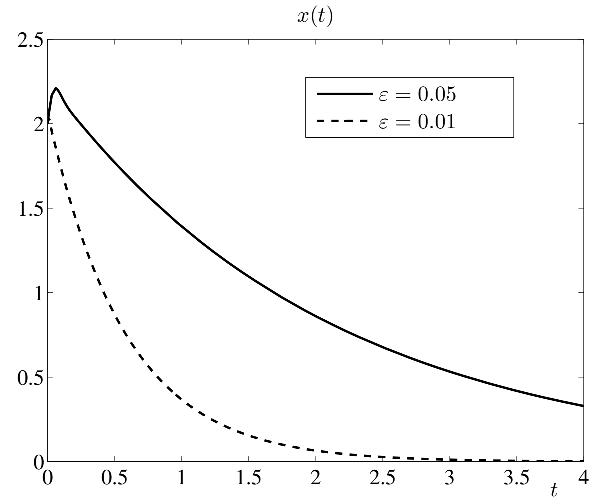

FIG. 1. *x-component of the trajectory of system* (8.2).

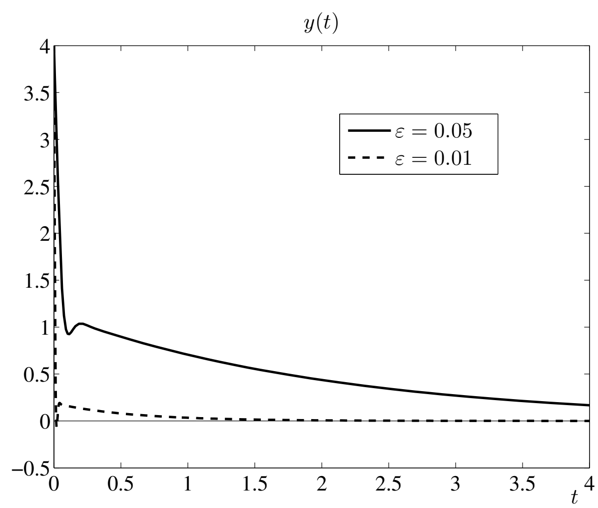

FIG. 2. *y-component of the trajectory of system* (8.2).

In Figures 1 and 2, the x- and y-components of the trajectory of the linearized system (8.2), for two different values of ε and subject to the initial conditions $x(0) = 2$, $y(\theta) = 4$, $\theta \in [-0.6\varepsilon, 0]$, are depicted. It is seen that, for both values of ε, these components tend to zero as $t \to +\infty$.

In Figures 3 and 4, the x- and y-components of the trajectory of the nonlinear system (8.1) are depicted for the same values of ε and subject to the same initial conditions as for the linearized system (8.2) (Figures 1 and 2). Due to the large initial values of the state variables, the solution of the nonlinear system (8.1) differs considerably from the solution of the linearized system (8.2) for each value of ε. Nevertheless, for these values of ε, the solution of the nonlinear system tends to its trivial solution as $t \to +\infty$, meaning that the attraction domain of the trivial solution

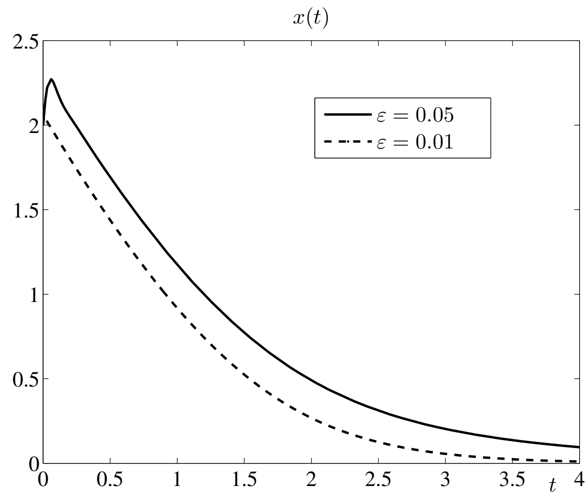

FIG. 3. *x-component of the trajectory of system* (8.1).

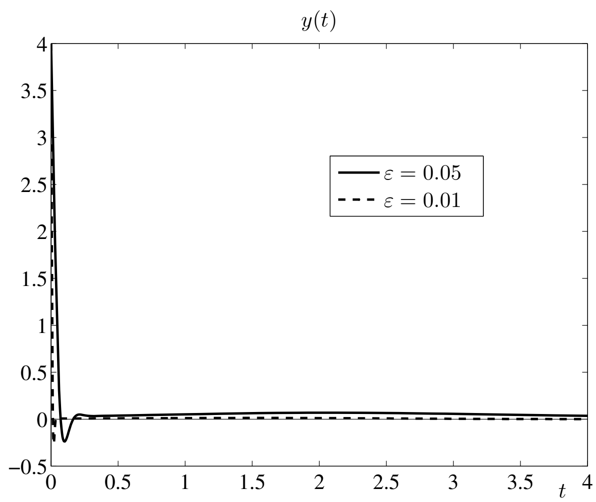

FIG. 4. *y-component of the trajectory of system* (8.1).

is extremely large.

Extensive computer simulation showed that in this numerical example, for $\varepsilon \geq 0.07$, the solutions of neither the linearized nor nonlinear systems approach the trivial solution as $t \to +\infty$.

9. Conclusions. The linear time-invariant singularly perturbed system with multiple pointwise and distributed time delays was considered. The delays are on the order of the small multiplier $\varepsilon > 0$ for a part of the derivatives in the system. A direct method of the exact slow-fast decomposition of this system was developed. This method consists of several stages. First, a new state variable, the functional part of the slow Euclidean state variable, is introduced. This new state depends on two arguments and satisfies a partial first-order differential equation with the multiplier ε

for the time derivative. Thus, this state is fast. By introduction of the new state, the original system is transformed to the equivalent system of three equations, one slow and two fast modes. Then, a linear algebraic transformation of the fast states with the matrix-valued coefficients, satisfying a proper set of functional-differential-algebraic equations, eliminates the slow state from the fast modes. A linear functional-algebraic transformation of the slow state with the matrix-valued coefficients, satisfying a proper set of functional-integral-algebraic equations, completely separates the slow and fast mode equations. It was shown that, under reasonable assumptions, solutions of both, functional-differential-algebraic and functional-integral-algebraic, sets exist and can be obtained asymptotically. Based on this slow-fast decomposition of the original singularly perturbed system of time delay equations, an exact slow-fast decomposition of its spectrum was carried out in the case of a single pointwise delay. This spectrum decomposition yields two scalar unconnected algebraic equations: The first is a polynomial equation for the slow part of the spectrum, while the second one is a quasipolynomial equation for the fast part of the spectrum. Using this spectrum decomposition, the stability of a multilink single-sink optical network was analyzed. Conditions of the asymptotic stability and the instability of the trivial solution to the corresponding nonlinear singularly perturbed time delay system were derived and illustrated by a numerical example.

Appendix A. Proof of Lemma 4.1. Let us make the following transformation of variables in system (4.1):

$$L_v(\eta,\varepsilon) = L_v^0 + \theta_v(\eta,\varepsilon), \qquad L_y(\varepsilon) = L_y^0 + \theta_y(\varepsilon), \tag{A.1}$$

where $\theta_v(\eta,\varepsilon)$ and $\theta_y(\varepsilon)$ are new unknown matrices.

Substituting (A.1) into (4.1) and using (4.2), (4.5), one obtains after some rearrangement the system of equations for $\theta_v(\eta,\varepsilon)$ and $\theta_y(\varepsilon)$

$$\begin{aligned} d\theta_v(\eta,\varepsilon)/d\eta &= H_v\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) + K_v(\eta,\varepsilon),\\ \mathcal{H}_4(0)\theta_y(\varepsilon) &= H_y\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) + K_y(\varepsilon), \end{aligned} \tag{A.2}$$

where

$$\begin{aligned} H_v\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) &\stackrel{\triangle}{=} F_2\Big(\varepsilon,L_v^0+\theta_v(\eta,\varepsilon),L_y^0+\theta_y(\varepsilon)\Big) - F_2\Big(\varepsilon,L_v^0,L_y^0\Big),\\ H_y\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) &\stackrel{\triangle}{=} -F_3\Big(\varepsilon,L_v^0+\theta_v(\eta,\varepsilon),L_y^0+\theta_y(\varepsilon)\Big)\\ &\qquad + F_3\Big(\varepsilon,L_v^0,L_y^0\Big) + \mathcal{H}_4(0)\theta_y(\varepsilon), \end{aligned} \tag{A.3}$$

$$K_v(\varepsilon) \stackrel{\triangle}{=} F_2\Big(\varepsilon,L_v^0,L_y^0\Big), \qquad K_y(\varepsilon) \stackrel{\triangle}{=} -F_3\Big(\varepsilon,L_v^0,L_y^0\Big). \tag{A.4}$$

By virtue of the algorithm for constructing the zero-order asymptotic solution $\{L_v^0(\eta),\ L_y^0\}$ to the system (4.1) (see subsection 4.1), we directly have

$$\|K_v(\varepsilon)\| \le a\varepsilon, \quad \|K_y(\varepsilon)\| \le a\varepsilon, \quad \varepsilon\in(0,\varepsilon_0], \tag{A.5}$$

where $a>0$ is some constant independent of ε.

Due to (3.15), (3.18), and (4.5), the expression for H_v in (A.3) can be rewritten as

$$H_v\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) = \varepsilon\Big[P\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) + \theta_v(\eta,\varepsilon)Q\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big)\Big], \tag{A.6}$$

where

$$P\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) \triangleq A_{20}(\varepsilon)\theta_y(\varepsilon) + \int_{-h}^{0}\Big[d_\eta\Gamma_{12}\Big(\eta,\varepsilon,L_y^0+\theta_y(\varepsilon)\Big)\Big]\theta_v(\eta,\varepsilon) + \int_{-h}^{0}[d_\eta D_2(\eta,\varepsilon)]\theta_y(\varepsilon), \tag{A.7}$$

$$Q\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) \triangleq \Upsilon_{12}\Big(\varepsilon,L_y^0+\theta_y(\varepsilon)\Big) + \int_{-h}^{0}\Big[d_\eta\Gamma_{12}\Big(\eta,\varepsilon,L_y^0+\theta_y(\varepsilon)\Big)\Big]\Big(I_n+\theta_v(\eta,\varepsilon)\Big). \tag{A.8}$$

Similarly, by using (3.16)–(3.18), (4.2), and (4.5), we can rewrite the expression for H_y from (A.3) in the form

$$H_y\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) = \big(\mathcal{H}_4(0)-\mathcal{H}_4(\varepsilon)\big)\theta_y(\varepsilon) + \varepsilon\Big[L_y^0 P\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) + \theta_y(\varepsilon)Q\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big)\Big] + R\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big), \tag{A.9}$$

where $R(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)) \triangleq -\int_{-h}^{0}[d_\eta\Gamma_{34}(\eta,\varepsilon,L_y^0+\theta_y(\varepsilon))]\theta_v(\eta,\varepsilon)$.

By virtue of assumptions (A1) and (A2) along with (4.2),

$$\|\mathcal{H}_4(0)-\mathcal{H}_4(\varepsilon)\| \le a\varepsilon, \quad \varepsilon\in(0,\varepsilon_0], \tag{A.10}$$

where $a>0$ is some constant independent of ε.

In the remainder of the proof, we look for the solution $\{\theta_v(\eta,\varepsilon),\ \theta_y(\varepsilon)\}$ of the system (A.2) satisfying the condition $\theta_v(-h,\varepsilon)=0$. Using assumption (A3), system (A.2), subject to this condition, can be rewritten in the equivalent form

$$\begin{aligned}\theta_v(\eta,\varepsilon) &= \int_{-h}^{\eta} H_v\Big(\varepsilon,\theta_v(\sigma,\varepsilon),\theta_y(\varepsilon)\Big)d\sigma + \int_{-h}^{\eta} K_v(\sigma,\varepsilon)d\sigma,\\ \theta_y(\varepsilon) &= \mathcal{H}_4^{-1}(0)H_y\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) + \mathcal{H}_4^{-1}(0)K_y(\varepsilon).\end{aligned} \tag{A.11}$$

Substituting the first equation of (A.11) into the second yields after some redenoting and rearrangement (by using (A.9)) the following system, equivalent to (A.11):

$$\begin{aligned}\theta_v(\eta,\varepsilon) &= S_v\Big(\eta,\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) + T_v(\eta,\varepsilon),\\ \theta_y(\varepsilon) &= S_y\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) + T_y(\varepsilon),\end{aligned} \tag{A.12}$$

where

$$S_v\Big(\eta,\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) \triangleq \int_{-h}^{\eta} H_v\Big(\varepsilon,\theta_v(\sigma,\varepsilon),\theta_y(\varepsilon)\Big)d\sigma, \tag{A.13}$$

$$T_v(\eta,\varepsilon) \triangleq K_v(\varepsilon)(\eta+h), \tag{A.14}$$

$$S_y\Big(\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big) \triangleq \mathcal{H}_4^{-1}(0)\Big\{H_y\Big(\varepsilon,S_v\Big(\eta,\varepsilon,\theta_v(\eta,\varepsilon),\theta_y(\varepsilon)\Big),\theta_y(\varepsilon)\Big) + \varepsilon L_y^0\int_{-h}^{0}[d_\eta D_2(\eta,\varepsilon)]\theta_y(\varepsilon)T_v(\eta,\varepsilon) - \int_{-h}^{0}[d_\eta D_4(\eta,\varepsilon)]\theta_y(\varepsilon)T_v(\eta,\varepsilon)\Big\}, \tag{A.15}$$

$$T_y(\varepsilon) \triangleq \mathcal{H}_4^{-1}(0)\Big\{\varepsilon L_y^0 \int_{-h}^{0} \Big[d_\eta \Gamma_{12}\Big(\eta,\varepsilon,L_y^0\Big)\Big] T_v(\eta,\varepsilon) - \int_{-h}^{0} \Big[d_\eta \Gamma_{34}\Big(\eta,\varepsilon,L_y^0\Big)\Big] T_v(\eta,\varepsilon) + K_y(\varepsilon)\Big\}. \tag{A.16}$$

By virtue of the first inequality in (A.5), we obtain

$$\|T_v(\eta,\varepsilon)\| \le a\varepsilon, \quad \eta \in [-h,0],\ \varepsilon \in (0,\varepsilon_0]. \tag{A.17}$$

Using Remark 2.1, (3.18), the second inequality in (A.5), and (A.17), one has

$$\|T_y(\varepsilon)\| \le a\varepsilon, \quad \varepsilon \in (0,\varepsilon_0]. \tag{A.18}$$

In (A.17)–(A.18), $a > 0$ is some constant independent of ε.

Let Θ_{vy} be the set of all pairs $\theta_{vy}(\eta) \triangleq \{\theta_v(\eta), \theta_y\}$, where $\theta_v(\eta)$ is an $n \times n$-matrix-valued function, continuous for $\eta \in [-h,0]$, while θ_y is an $m \times n$-matrix. Let $\theta_{vy}^1(\eta) = \{\theta_v^1(\eta), \theta_y^1\}$ and $\theta_{vy}^2(\eta) = \{\theta_v^2(\eta), \theta_y^2\}$ be any two elements of Θ_{vy}. Let us define the linear combination of these elements with any real coefficients α_1 and α_2 as $\alpha_1\theta_{vy}^1(\eta) + \alpha_2\theta_{vy}^2(\eta) = \{\alpha_1\theta_v^1(\eta) + \alpha_2\theta_v^2(\eta), \alpha_1\theta_y^1 + \alpha_2\theta_y^2\}$. This definition converts the set Θ_{vy} to a linear space. Further, for any $\theta_{vy}(\eta) \in \Theta_{vy}$, let us define the number

$$\|\theta_{vy}(\eta)\|_\Theta \triangleq \max_{\eta\in[-h,0]} \|\theta_v(\eta)\| + \|\theta_y\|. \tag{A.19}$$

It is directly verified that this number is a norm of an element in the linear space Θ_{vy}. Moreover, Θ_{vy} endowed with the norm (A.19) is a Banach space.

Let $\varepsilon > 0$ be a sufficiently small number. Consider a ball in Θ_{vy},

$$\mathcal{B}(c,\varepsilon) \triangleq \Big\{\theta_{vy}(\eta) \in \Theta_{vy} : \|\theta_{vy}(\eta)\|_\Theta \le c\varepsilon\Big\}, \tag{A.20}$$

where $c > 0$ is some constant independent of ε.

For the aforementioned $\varepsilon > 0$, consider the operator, given in the space Θ_{vy},

$$\mathcal{F}_{vy,\varepsilon}\Big(\theta_{vy}(\eta)\Big) \triangleq \Big\{S_v\Big(\eta,\varepsilon,\theta_v(\eta),\theta_y\Big) + T_v(\eta,\varepsilon), S_y\Big(\varepsilon,\theta_v(\eta),\theta_y\Big) + T_y(\varepsilon)\Big\}. \tag{A.21}$$

For any number $c > 0$ and any $\varepsilon \in (0,\varepsilon_0]$ this operator maps the ball (A.20) into the space Θ_{vy}. Now, we are going to show that, for a proper choice of numbers $c > 0$ and $\bar{\varepsilon} > 0$, the operator $\mathcal{F}_{vy,\varepsilon}(\theta_{vy}(\eta))$ maps the ball (A.20) into itself for any $\varepsilon \in (0,\bar{\varepsilon}]$. Namely, let us choose $c = 4a$, where $a > 0$ is the constant appearing in the inequalities (A.17)–(A.18). Further, let us estimate $H_v(\eta,\varepsilon,\theta_v(\eta),\theta_y)$ for any $\{\theta_v(\eta),\theta_y\} \in \mathcal{B}(4a,\varepsilon)$, where $\varepsilon \in (0,\varepsilon_0]$ is any fixed number.

Using (A.6)–(A.8) and (A.20), one has for $\eta \in [-h,0]$

$$\begin{aligned}
\Big\|H_v\Big(\varepsilon,\theta_v(\eta),\theta_y\Big)\Big\| &\le \varepsilon\|L_y^0\|\Big\|P\Big(\varepsilon,\theta_v(\eta),\theta_y\Big)\Big\| + 4a\varepsilon^2\Big\|Q\Big(\varepsilon,\theta_v(\eta),\theta_y\Big)\Big\|,\\
\Big\|P\Big(\varepsilon,\theta_v(\eta),\theta_y\Big)\Big\| &\le 4a\varepsilon\Big\{\|A_{20}(\varepsilon)\| + V_{-h}^0\Big[\Gamma_{12}\Big(\eta,\varepsilon,L_y^0\Big)\Big]\\
&\quad + (4a\varepsilon+1)V_{-h}^0[D_2(\eta,\varepsilon)]\Big\},\\
\Big\|Q\Big(\varepsilon,\theta_v(\eta),\theta_y\Big)\Big\| &\le \Big\|\Upsilon_{12}\Big(\varepsilon,L_y^0(\varepsilon)\Big)\Big\| + 4a\varepsilon\|A_{20}(\varepsilon)\|\\
&\quad +(4a\varepsilon+1)\Big\{V_{-h}^0\Big[\Gamma_{12}\Big(\eta,\varepsilon,L_y^0\Big)\Big] + 4a\varepsilon V_{-h}^0[D_2(\eta,\varepsilon)]\Big\}.
\end{aligned} \tag{A.22}$$

Due to assumptions (A1)–(A3), Remark 2.1, and (3.18), there exists a positive constant c_1, independent of ε, such that the following inequality is valid for all $\varepsilon \in (0, \varepsilon_0]$:

$$\begin{aligned} &\|L_y^0\|\Big\{\|A_{20}(\varepsilon)\| + V_{-h}^0\Big[\Gamma_{12}\Big(\eta, \varepsilon, L_y^0\Big)\Big] + (4a\varepsilon + 1)V_{-h}^0[D_2(\eta, \varepsilon)]\Big\} \\ &\quad + \Big\|\Upsilon_{12}\Big(\varepsilon, L_y^0(\varepsilon)\Big)\Big\| + 4a\varepsilon\|A_{20}(\varepsilon)\| \\ &\quad + (4a\varepsilon + 1)\Big\{V_{-h}^0\Big[\Gamma_{12}\Big(\eta, \varepsilon, L_y^0\Big)\Big] + 4a\varepsilon V_{-h}^0[D_2(\eta, \varepsilon)]\Big\} \le c_1. \end{aligned} \tag{A.23}$$

Based on the inequalities (A.22)–(A.23), we obtain after a simple algebra the following estimate: $\|H_v(\varepsilon, \theta_v(\eta), \theta_y)\| \le 4ac_1\varepsilon^2$, $\eta \in [-h, 0]$, $\varepsilon \in (0, \varepsilon_0]$. Using this estimate, (A.13), and the inequality (A.17), one immediately has the following inequality for all $\eta \in [-h, 0]$ and $\varepsilon \in (0, \varepsilon_0]$:

$$\Big\|S_v\Big(\eta, \varepsilon, \theta_v(\eta), \theta_y\Big) + T_v(\eta, \varepsilon)\Big\| \le 4ac_1h\varepsilon^2 + a\varepsilon = a\varepsilon(1 + 4c_1h\varepsilon). \tag{A.24}$$

Let us introduce the positive numbers $\varepsilon_1 = 1/(4c_1h)$, $\bar\varepsilon_1 = \min\{\varepsilon_0, \varepsilon_1\}$. Then, from (A.24), we obtain the following inequality for $\{\theta_v(\eta), \theta_y\} \in \mathcal{B}(4a, \varepsilon)$:

$$\max_{\eta\in[-h,0]} \Big\|S_v\Big(\eta, \varepsilon, \theta_v(\eta), \theta_y\Big) + T_v(\eta, \varepsilon)\Big\| \le 2a\varepsilon, \quad \varepsilon \in (0, \bar\varepsilon_1]. \tag{A.25}$$

Similarly to (A.25), based on (A.9), (A.15) and the inequalities (A.10), (A.18), one can prove the existence of a positive number $\bar\varepsilon_2$ ($\bar\varepsilon_2 \le \varepsilon_0$) such that for all $\varepsilon \in (0, \bar\varepsilon_2]$ and $\{\theta_v(\eta), \theta_y\} \in \mathcal{B}(4a, \varepsilon)$ the following inequality is valid:

$$\Big\|S_y\Big(\varepsilon, \theta_v(\eta), \theta_y\Big) + T_y(\varepsilon)\Big\| \le 2a\varepsilon. \tag{A.26}$$

The inequalities (A.25)–(A.26), along with the definitions (A.19)–(A.20), mean immediately that the operator $\mathcal{F}_{vy,\varepsilon}(\theta_{vy}(\eta))$ (see (A.21)) maps the ball $\mathcal{B}(4a, \varepsilon)$ into itself for any $\varepsilon \in (0, \bar\varepsilon]$, where $\bar\varepsilon = \min(\bar\varepsilon_1, \bar\varepsilon_2)$.

Now we are going to show that, for all sufficiently small $\varepsilon > 0$, the operator $\mathcal{F}_{vy,\varepsilon}(\theta_{vy}(\eta))$ satisfies the Lipschitz condition in the ball $\mathcal{B}(4a, \varepsilon)$ with the constant 1/2. For this purpose, we first estimate the difference $H_v(\varepsilon, \theta_{v,1}(\eta), \theta_{y,1}) - H_v(\varepsilon, \theta_{v,2}(\eta), \theta_{y,2})$ for any $\theta_{vy,k}(\eta) = \{\theta_{v,k}(\eta), \theta_{y,k}\} \in \mathcal{B}(4a, \varepsilon)$ ($k = 1, 2$). Namely, from (A.6), we obtain for $\eta \in [-h, 0]$, $\varepsilon \in (0, \bar\varepsilon]$

$$\begin{aligned} &\Big\|H_v\Big(\varepsilon, \theta_{v,1}(\eta), \theta_{y,1}\Big) - H_v\Big(\varepsilon, \theta_{v,2}(\eta), \theta_{y,2}\Big)\Big\| \\ &\quad \le \varepsilon\Big\|P_v\Big(\varepsilon, \theta_{v,1}(\eta), \theta_{y,1}\Big) - P_v\Big(\varepsilon, \theta_{v,2}(\eta), \theta_{y,2}\Big)\Big\| \\ &\quad + \varepsilon\Big\|\theta_{v,1}(\eta)Q_v\Big(\varepsilon, \theta_{v,1}(\eta), \theta_{y,1}\Big) - \theta_{v,2}(\eta)Q_v\Big(\varepsilon, \theta_{v,2}(\eta), \theta_{y,2}\Big)\Big\|. \end{aligned} \tag{A.27}$$

Using (A.7), Remark 2.1, and (3.18), we obtain for $\varepsilon \in (0, \bar\varepsilon]$

$$\begin{aligned} &\Big\|P_v\Big(\varepsilon, \theta_{v,1}(\eta), \theta_{y,1}\Big) - P_v\Big(\varepsilon, \theta_{v,2}(\eta), \theta_{y,2}\Big)\Big\| \le \|\theta_{vy,1}(\eta) - \theta_{vy,2}(\eta)\|_\Theta\Big\{\|A_{20}(\varepsilon)\| \\ &\qquad + (8a\varepsilon + 1)V_{-h}^0[D_2(\eta, \varepsilon)] + V_{-h}^0\Big[\Gamma_{12}\Big(\eta, \varepsilon, L_y^0\Big)\Big]\Big\}. \end{aligned} \tag{A.28}$$

Similarly, using (A.8), (3.18) and Remark 2.1 yields for $\eta \in [-h, 0]$, $\varepsilon \in (0, \bar{\varepsilon}]$

$$\begin{aligned}
&\Big\|\theta_{v,1}(\eta)Q_v\Big(\varepsilon, \theta_{v,1}(\eta), \theta_{y,1}\Big) - \theta_{v,2}(\eta)Q_v\Big(\varepsilon, \theta_{v,2}(\eta), \theta_{y,2}\Big)\Big\| \\
&\quad\le 4a\varepsilon\|\theta_{vy,1}(\eta) - \theta_{vy,2}(\eta)\|_\Theta\Big\{\|A_{20}(\varepsilon)\| + (1 + 4a\varepsilon)V_{-h}^0[D_2(\eta, \varepsilon)]\Big\} \\
&\qquad + \|\theta_{vy,1}(\eta) - \theta_{vy,2}(\eta)\|_\Theta\Big\{4a\varepsilon V_{-h}^0\Big[\Gamma_{12}\Big(\eta, \varepsilon, L_y^0\Big)\Big] \\
&\qquad\qquad + 16a^2\varepsilon^2 V_{-h}^0[D_2(\eta, \varepsilon)] + \Big\|Q_v\Big(\varepsilon, \theta_{v,2}(\eta), \theta_{y,2}\Big)\Big\|\Big\}.
\end{aligned} \tag{A.29}$$

It should be noted that, due to (A.8) and the inclusion $\theta_{vy,2}(\eta) \in \mathcal{B}(4a, \varepsilon)$, the value $\|Q_v(\varepsilon, \theta_{v,2}(\eta), \theta_{y,2})\|$ is bounded for $\varepsilon \in (0, \bar{\varepsilon}]$.

Similarly to the inequality (A.23), one obtains the existence of a positive constant c_2 independent of ε such that for all $\varepsilon \in (0, \bar{\varepsilon}]$ the following inequality is valid:

$$\begin{aligned}
&(1 + 4a\varepsilon)\Big\{\|A_{20}(\varepsilon)\| + (1 + 8a\varepsilon)V_{-h}^0[D_2(\eta, \varepsilon)] + V_{-h}^0\Big[\Gamma_{12}\Big(\eta, \varepsilon, L_{y,1}(\varepsilon)\Big)\Big]\Big\} \\
&\qquad + \Big\|Q_v\Big(\varepsilon, \theta_{v,2}(\eta), \theta_{y,2}\Big)\Big\| \le c_2.
\end{aligned} \tag{A.30}$$

The inequalities (A.27)–(A.30) directly yield the following inequality:

$$\begin{aligned}
&\Big\|H_v\Big(\varepsilon, \theta_{v,1}(\eta), \theta_{y,1}\Big) - H_v\Big(\varepsilon, \theta_{v,2}(\eta), \theta_{y,2}\Big)\Big\| \\
&\qquad \le c_2\varepsilon\|\theta_{vy,1}(\eta) - \theta_{vy,2}(\eta)\|_\Theta, \qquad \eta \in [-h, 0],\ \varepsilon \in (0, \bar{\varepsilon}].
\end{aligned} \tag{A.31}$$

This estimate, along with (A.13), leads immediately to the estimate

$$\begin{aligned}
&\max_{\eta\in[-h,0]}\Big\|S_v\Big(\eta, \varepsilon, \theta_{v,1}(\eta), \theta_{y,1}\Big) - S_v\Big(\eta, \varepsilon, \theta_{v,2}(\eta), \theta_{y,2}\Big)\Big\| \\
&\qquad \le c_2h\varepsilon\|\theta_{vy,1}(\eta) - \theta_{vy,2}(\eta)\|_\Theta, \qquad \varepsilon \in (0, \bar{\varepsilon}].
\end{aligned} \tag{A.32}$$

By introducing the positive number $\bar{\varepsilon}_3 = \min\{\bar{\varepsilon}, 1/(4c_2h)\}$, we obtain from (A.32)

$$\begin{aligned}
&\max_{\eta\in[-h,0]}\Big\|S_v\Big(\eta, \varepsilon, \theta_{v,1}(\eta), \theta_{y,1}\Big) - S_v\Big(\eta, \varepsilon, \theta_{v,2}(\eta), \theta_{y,2}\Big)\Big\| \\
&\qquad \le (1/4)\|\theta_{vy,1}(\eta) - \theta_{vy,2}(\eta)\|_\Theta, \qquad \varepsilon \in (0, \bar{\varepsilon}_3].
\end{aligned} \tag{A.33}$$

Similarly to (A.33), one can show the existence of a positive constant $\bar{\varepsilon}_4$ ($\bar{\varepsilon}_4 \le \bar{\varepsilon}$) such that, for all $\varepsilon \in (0, \bar{\varepsilon}_4]$, the following inequality is satisfied:

$$\Big\|S_y\Big(\varepsilon, \theta_{v,1}(\eta), \theta_{y,1}\Big) - S_y\Big(\varepsilon, \theta_{v,2}(\eta), \theta_{y,2}\Big)\Big\| \le (1/4)\|\theta_{vy,1}(\eta) - \theta_{vy,2}(\eta)\|_\Theta, \tag{A.34}$$

which yields, along with (A.33), (A.19), and (A.21), the following inequality for any $\varepsilon \in (0, \varepsilon_1^*]$ ($\varepsilon_1^* = \min\{\bar{\varepsilon}_3, \bar{\varepsilon}_4\}$) and any $\theta_{vy,1}(\eta), \theta_{vy,2}(\eta) \in \mathcal{B}(4a, \varepsilon)$:

$$\Big\|\mathcal{F}_{vy,\varepsilon}\Big(\theta_{vy,1}(\eta)\Big) - \mathcal{F}_{vy,\varepsilon}\Big(\theta_{vy,2}(\eta)\Big)\Big\| \le (1/2)\|\theta_{vy,1}(\eta) - \theta_{vy,2}(\eta)\|_\Theta. \tag{A.35}$$

Thus, for any $\varepsilon \in (0, \varepsilon_1^*]$, the operator $\mathcal{F}_{vy,\varepsilon}(\theta_{vy}(\eta))$ satisfies the Lipschitz condition in the ball $\mathcal{B}(4a, \varepsilon)$ with the constant $1/2$.

Now, let us show that

$$\Big\|\mathcal{F}_{vy,\varepsilon}\Big(0_\Theta\Big)\Big\|_\Theta \le 2a\varepsilon, \qquad \varepsilon \in (0, \varepsilon_1^*], \tag{A.36}$$

where $a > 0$ is the constant appearing in the inequalities (A.17)–(A.18) and in the radius of the ball $\mathcal{B}(4a, \varepsilon)$; 0_Θ is zero element of the space Θ_{vy}, i.e., $0_\Theta = \{0_{n\times n}, 0_{m\times n}\}$.

Indeed, due to (A.3), (A.13), and (A.15), we obtain $S_v(\eta, \varepsilon, 0_{n\times n}, 0_{m\times n}) \equiv 0$ and $S_y(\varepsilon, 0_{n\times n}, 0_{m\times n}) \equiv 0$. The latter, along with (A.21), yields $\mathcal{F}_{vy,\varepsilon}(0_\Theta) = \{T_v(\eta, \varepsilon), T_y(\varepsilon)\}$, $\eta \in [-h, 0]$, $\varepsilon \in (0, \varepsilon_0]$. This equation, along with (A.19), the inequalities (A.17)–(A.18), and the inequality $\varepsilon_1^* \le \varepsilon_0$, directly implies the inequality (A.36).

Thus, we have proven that, for any $\varepsilon \in (0, \varepsilon_1^*]$, the operator $\mathcal{F}_{vy,\varepsilon}(\theta_{vy}(\eta))$ maps the ball $\mathcal{B}(4a, \varepsilon)$ into itself. Moreover, the operator $\mathcal{F}_{vy,\varepsilon}(\theta_{vy}(\eta))$ satisfies the Lipschitz condition in the ball $\mathcal{B}(4a, \varepsilon)$ with the constant $1/2$, and the inequality (A.36) is valid. Now, by using the aforementioned properties of the operator $\mathcal{F}_{vy,\varepsilon}(\theta_{vy}(\eta))$, the fact that $0_\Theta \in \mathcal{B}(4a, \varepsilon)$, and the results of [42], one directly obtains the existence of the unique solution $\{\theta_v(\eta, \varepsilon), \theta_y(\varepsilon)\}$ to the set (A.12) in the ball $\mathcal{B}(4a, \varepsilon)$ for any $\varepsilon \in (0, \varepsilon_1^*]$. Moreover, by virtue of (A.13)–(A.14), one has

$$\theta_v(-h, \varepsilon) = 0, \quad \varepsilon \in (0, \varepsilon_1^*]. \tag{A.37}$$

Now, (A.1) and (A.37), and the equivalence of the set (A.12) to the problem (A.2), (A.37) directly yield all the statements of the lemma with $a_1^* = 4a$ in the inequalities (4.7).

Appendix B. Proof of Lemma 6.2. Let us make the following transformation of variables in the system (6.1), (6.4):

$$Q_v(\eta, \varepsilon) = Q_v^0(\eta) + \vartheta_v(\eta, \varepsilon), \quad Q_y(\eta, \varepsilon) = Q_y^0(\eta) + \vartheta_y(\eta, \varepsilon), \quad \eta \in [-h, 0], \tag{B.1}$$

$$P_y(\varepsilon) = P_y^0 + \vartheta(\varepsilon), \tag{B.2}$$

where $\vartheta_v(\eta, \varepsilon)$, $\vartheta(\varepsilon)$, and $\vartheta_y(\eta, \varepsilon)$ are new unknown matrices.

Equations (5.1), (5.3), (B.1) and Remark 6.1 yield $\vartheta_v(-h, \varepsilon) = 0$, $\vartheta_y(-h, \varepsilon) = 0$, $\varepsilon \in (0, \varepsilon_1^*]$.

Denote $\mathcal{N}_\vartheta(\eta, \varepsilon) \stackrel{\triangle}{=} \{\vartheta_v(\eta, \varepsilon), \vartheta(\varepsilon), \vartheta_y(\eta, \varepsilon)\}$. Thus, due to this notation and (5.7), $\mathcal{N}(\eta, \varepsilon) = \{Q_v^0(\eta) + \vartheta_v(\eta, \varepsilon), P_y^0 + \vartheta(\varepsilon), Q_y^0(\eta) + \vartheta_y(\eta, \varepsilon)\} = \mathcal{N}^0(\eta) + \mathcal{N}_\vartheta(\eta, \varepsilon)$.

By substituting (B.1)–(B.2) into (6.1), (6.4), one obtains after some rearrangement the following system of equations for $\vartheta_v(\eta, \varepsilon)$, $\vartheta(\varepsilon)$, and $\vartheta_y(\eta, \varepsilon)$:

$$\vartheta(\varepsilon)A_{40}(0) + \vartheta_y(0, \varepsilon) = \mathcal{D}_2\Big(\varepsilon, \mathcal{N}_\vartheta(\eta, \varepsilon)\Big) + \mathcal{L}_2(\varepsilon), \tag{B.3}$$

$$\begin{aligned}
&\vartheta_v(\eta, \varepsilon) - \vartheta(\varepsilon)\Big(\Gamma_{34}(\eta, 0, L_y^0) - \Gamma_{34}(-h, 0, L_y^0)\Big)\\
&= \mathcal{D}_3\Big(-h, \varepsilon, \mathcal{N}_\vartheta(\eta, \varepsilon)\Big) - \mathcal{D}_3\Big(\eta, \varepsilon, \mathcal{N}_\vartheta(\eta, \varepsilon)\Big) + \mathcal{L}_3(-h, \varepsilon) - \mathcal{L}_3(\eta, \varepsilon),
\end{aligned}$$

$$\begin{aligned}
&\vartheta_y(\eta, \varepsilon) - \vartheta(\varepsilon)\Big(D_4(\eta, 0) - D_4(-h, 0)\Big)\\
&= \mathcal{D}_4\Big(-h, \varepsilon, \mathcal{N}_\vartheta(\eta, \varepsilon)\Big) - \mathcal{D}_4\Big(\eta, \varepsilon, \mathcal{N}_\vartheta(\eta, \varepsilon)\Big) + \mathcal{L}_4(-h, \varepsilon) - \mathcal{L}_4(\eta, \varepsilon),
\end{aligned}$$

where

$$\begin{aligned}\mathcal{D}_2\Big(\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big) &= \mathcal{F}_2\Big(\varepsilon,\mathcal{N}^0(\eta)+\mathcal{N}_\vartheta(\eta,\varepsilon)\Big)+\vartheta(\varepsilon)A_{40}(0)\\ &\quad +\vartheta_y(0,\varepsilon)-\mathcal{F}_2\Big(\varepsilon,\mathcal{N}^0(\eta)\Big), \qquad \mathcal{L}_2(\varepsilon)=\mathcal{F}_2\Big(\varepsilon,\mathcal{N}^0(\eta)\Big),\\ \mathcal{D}_3\Big(\eta,\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big) &= \mathcal{F}_3\Big(\eta,\varepsilon,\mathcal{N}^0(\eta)+\mathcal{N}_\vartheta(\eta,\varepsilon)\Big)-\vartheta_v(\eta,\varepsilon)\\ &\quad +\vartheta(\varepsilon)\Gamma_{34}(\eta,0,L_y^0)-\mathcal{F}_3\Big(\eta,\varepsilon,\mathcal{N}^0(\eta)\Big), \quad \mathcal{L}_3(\eta,\varepsilon)=\mathcal{F}_3\Big(\eta,\varepsilon,\mathcal{N}^0(\eta)\Big),\\ \mathcal{D}_4\Big(\eta,\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big) &= \mathcal{F}_4\Big(\eta,\varepsilon,\mathcal{N}^0(\eta)+\mathcal{N}_\vartheta(\eta,\varepsilon)\Big)-\vartheta_y(\eta,\varepsilon)\\ &\quad +\vartheta(\varepsilon)D_4(\eta,0)-\mathcal{F}_4\Big(\eta,\varepsilon,\mathcal{N}^0(\eta)\Big), \qquad \mathcal{L}_4(\eta,\varepsilon)=\mathcal{F}_4\Big(\eta,\varepsilon,\mathcal{N}^0(\eta)\Big).\end{aligned} \tag{B.4}$$

Remark B.1. Due to (5.10)–(5.11) and Remarks 2.1, 6.1, for all $\varepsilon\in(0,\varepsilon_1^*]$ the functions $\mathcal{L}_3(\eta,\varepsilon)$ and $\mathcal{L}_4(\eta,\varepsilon)$ are piecewise continuous in $\eta\in[-h,0]$ with the break points $\eta=-h_j$ $(j=1,\dots,N)$, where these functions have finite limits from the right, $\mathcal{L}_3(-h_j+0,\varepsilon)=\lim_{\eta\to -h_j+0}\mathcal{L}_3(\eta,\varepsilon)$ and $\mathcal{L}_4(-h_j+0,\varepsilon)=\lim_{\eta\to -h_j+0}\mathcal{L}_4(\eta,\varepsilon)$, respectively. Also, these functions are continuous from the left at the break points $\eta=-h_j$ $(j=1,\dots,N-1)$. Moreover, by virtue of the algorithm for constructing the asymptotic solution $\mathcal{N}^0(\eta)$ to the system (6.1), (6.4) (see subsection 6.1), we directly have the following inequalities for all $\varepsilon\in(0,\varepsilon_1^*]$:

$$\|\mathcal{L}_2(\varepsilon)\|\le a\varepsilon, \quad \|\mathcal{L}_k(-h,\varepsilon)-\mathcal{L}_k(\eta,\varepsilon)\|\le a\varepsilon, \quad k=3,4,\ \eta\in[-h,0], \tag{B.5}$$

where the positive constant ε_1^* has been introduced in Lemma 4.1 and where $a>0$ is some constant independent of ε.

Let us denote for all $j=1,\dots,N$, $\eta\in[-h_j,-h_{j-1}]$, $\varepsilon\in(0,\varepsilon_1^*]$,

$$\mathcal{L}_{k,j}(\eta,\varepsilon)=\begin{cases}\mathcal{L}_k(-h_j+0,\varepsilon), & \eta=-h_j,\\ \mathcal{L}_k(\eta,\varepsilon), & \eta\in(-h_j,-h_{j-1}],\end{cases} \qquad k=3,4. \tag{B.6}$$

Remark B.2. Due to Remark 2.2, the functions $\mathcal{L}_{3,j}(\eta,\varepsilon)$ and $\mathcal{L}_{4,j}(\eta,\varepsilon)$ $(j=1,\dots,N)$ are differentiable with respect to $\eta\in[-h_j,-h_{j-1}]$ for all $\varepsilon\in(0,\varepsilon_1^*]$.

The system (B.3) can be transformed equivalently to the following one:

$$\begin{aligned}\vartheta(\varepsilon)&=\mathcal{S}_2\Big(\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big)+\mathcal{T}_2(\varepsilon),\\ \vartheta_v(\eta,\varepsilon)&=\mathcal{S}_3\Big(\eta,\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big)+\mathcal{T}_3(\eta,\varepsilon),\\ \vartheta_y(\eta,\varepsilon)&=\mathcal{S}_4\Big(\eta,\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big)+\mathcal{T}_4(\eta,\varepsilon),\end{aligned} \tag{B.7}$$

where

$$\begin{aligned}\mathcal{S}_2\Big(\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big) &= \Big[\mathcal{D}_2\Big(\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big)+\mathcal{D}_4\Big(0,\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big)\\ &\qquad -\mathcal{D}_4\Big(-h,\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big)\Big]\mathcal{H}_4^{-1}(0),\\ \mathcal{S}_3\Big(\eta,\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big) &= \mathcal{S}_2\Big(\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big)\Big(\Gamma_{34}(\eta,0,L_y^0)-\Gamma_{34}(-h,0,L_y^0)\Big)\\ &\qquad +\mathcal{D}_3\Big(-h,\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big)-\mathcal{D}_3\Big(\eta,\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big),\\ \mathcal{S}_4\Big(\eta,\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big) &= \mathcal{S}_2\Big(\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big)\Big(D_4(\eta,0)-D_4(-h,0)\Big)\\ &\qquad +\mathcal{D}_4\Big(-h,\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big)-\mathcal{D}_4\Big(\eta,\varepsilon,\mathcal{N}_\vartheta(\eta,\varepsilon)\Big),\end{aligned} \tag{B.8}$$

$$
\begin{aligned}
&\mathcal{T}_2(\varepsilon) = \Big[\mathcal{L}_2(\varepsilon) + \mathcal{L}_4(0,\varepsilon) - \mathcal{L}_4(-h,\varepsilon)\Big]\mathcal{H}_4^{-1}(0),\\
&\mathcal{T}_3(\eta,\varepsilon) = \mathcal{T}_2(\varepsilon)\Big(\Gamma_{34}(\eta,0,L_y^0) - \Gamma_{34}(-h,0,L_y^0)\Big) + \mathcal{L}_3(-h,\varepsilon) - \mathcal{L}_3(\eta,\varepsilon),\\
&\mathcal{T}_4(\eta,\varepsilon) = \mathcal{T}_2(\varepsilon)\Big(D_4(\eta,0) - D_4(-h,0)\Big) + \mathcal{L}_4(-h,\varepsilon) - \mathcal{L}_4(\eta,\varepsilon).
\end{aligned}
\tag{B.9}
$$

Remark B.3. By virtue of (B.9) and Remarks 2.1, 6.1, B.1, for all $\varepsilon \in (0, \varepsilon_1^*]$ the functions $\mathcal{T}_3(\eta,\varepsilon)$ and $\mathcal{T}_4(\eta,\varepsilon)$ are piecewise continuous in $\eta \in [-h, 0]$ with the break points $\eta = -h_j$ $(j = 1, \dots, N)$, where these functions have finite limits from the right, $\mathcal{T}_3(-h_j + 0, \varepsilon) = \lim_{\eta\to -h_j+0} \mathcal{T}_3(\eta,\varepsilon)$ and $\mathcal{T}_4(-h_j + 0, \varepsilon) = \lim_{\eta\to -h_j+0} \mathcal{T}_4(\eta,\varepsilon)$, respectively. Also, these functions are continuous from the left at the break points $\eta = -h_j$ $(j = 1, \dots, N-1)$. Moreover, due to the inequalities (B.5), we immediately obtain for all $\varepsilon \in (0, \varepsilon_1^*]$

$$
\|\mathcal{T}_2(\varepsilon)\| \le a\varepsilon, \qquad \|\mathcal{T}_k(\eta,\varepsilon)\| \le a\varepsilon, \quad k = 3, 4,\ \eta \in [-h, 0],
\tag{B.10}
$$

where $a > 0$ is some constant independent of ε.

Let us denote for all $j = 1, \dots, N$, $\eta \in [-h_j, -h_{j-1}]$, $\varepsilon \in (0, \varepsilon_1^*]$,

$$
\mathcal{T}_{k,j}(\eta,\varepsilon) = \begin{cases} \mathcal{T}_k(-h_j + 0, \varepsilon), & \eta = -h_j,\\ \mathcal{T}_k(\eta,\varepsilon), & \eta \in (-h_j, -h_{j-1}], \end{cases} \qquad k = 3, 4.
\tag{B.11}
$$

Remark B.4. Due to Remarks 2.2 and B.2, the functions $\mathcal{T}_{3,j}(\eta,\varepsilon)$ and $\mathcal{T}_{4,j}(\eta,\varepsilon)$ $(j = 1, \dots, N)$ are differentiable with respect to $\eta \in [-h_j, -h_{j-1}]$ for all $\varepsilon \in (0, \varepsilon_1^*]$.

Let $\mathcal{M}_{vy}$ be the set of all triplets $\vartheta_{vy}(\eta) \stackrel{\triangle}{=} \{\vartheta_v(\eta), \vartheta, \vartheta_y(\eta)\}$, where $\vartheta_v(\eta)$ and $\vartheta_y(\eta)$ are $n \times n$- and $n \times m$-matrix-valued functions, respectively, defined for $\eta \in [-h, 0]$; ϑ is an $n \times m$ matrix. Moreover, the functions $\vartheta_v(\eta)$ and $\vartheta_y(\eta)$ are piecewise continuous with the break points $\eta = -h_j$ $(j = 1, \dots, N)$, where these functions have finite limits from the right, $\vartheta_v(-h_j + 0) = \lim_{\eta\to -h_j+0} \vartheta_v(\eta)$ and $\vartheta_y(-h_j + 0) = \lim_{\eta\to -h_j+0} \vartheta_y(\eta)$, respectively. Also, these functions are continuous from the left at the break points $\eta = -h_j$ $(j = 1, \dots, N-1)$. Let $\vartheta_{vy}^1(\eta) = \{\vartheta_v^1(\eta), \vartheta^1, \vartheta_y^1(\eta)\}$ and $\theta_{vy}^2(\eta) = \{\vartheta_v^2(\eta), \vartheta^2, \vartheta_y^2(\eta)\}$ be any two elements of $\mathcal{M}_{vy}$. Let us define the linear combination of these elements with any real coefficients α_1 and α_2 as follows: $\alpha_1\vartheta_{vy}^1(\eta) + \alpha_2\vartheta_{vy}^2(\eta) = \{\alpha_1\vartheta_v^1(\eta) + \alpha_2\vartheta_v^2(\eta), \alpha_1\vartheta^1 + \alpha_2\vartheta^2, \alpha_1\vartheta_y^1(\eta) + \alpha_2\vartheta_y^2(\eta)\}$. This definition converts the set $\mathcal{M}_{vy}$ to a linear space. Further, for any $\vartheta_{vy}(\eta) \in \mathcal{M}_{vy}$, let us define the number $\|\vartheta_{vy}(\eta)\|_{\mathcal{M}} \stackrel{\triangle}{=} \sup_{\eta\in[-h,0]} \|\vartheta_v(\eta)\| + \|\vartheta\| + \sup_{\eta\in[-h,0]} \|\vartheta_y(\eta)\|$. It is verified immediately that this number is a norm of an element in the linear space $\mathcal{M}_{vy}$. Moreover, $\mathcal{M}_{vy}$, endowed with this norm, is a Banach space.

Let $\varepsilon > 0$ be a sufficiently small number. Consider the following ball in $\mathcal{M}_{vy}$: $\mathcal{B}_{\mathcal{M}}(c,\varepsilon) \stackrel{\triangle}{=} \{\vartheta_{vy}(\eta) \in \mathcal{M}_{vy} : \|\vartheta_{vy}(\eta)\|_{\mathcal{M}} \le c\varepsilon\}$, where $c > 0$ is some constant independent of ε.

For the aforementioned $\varepsilon > 0$, consider the following operator in the space $\mathcal{M}_{vy}$: $\mathcal{G}_{vy,\varepsilon}(\vartheta_{vy}(\eta)) \stackrel{\triangle}{=} \big\{\mathcal{S}_3(\eta,\varepsilon,\vartheta_{vy}(\eta)) + \mathcal{T}_3(\eta,\varepsilon), \mathcal{S}_2(\varepsilon,\vartheta_{vy}(\eta)) + \mathcal{T}_2(\varepsilon), \mathcal{S}_4(\eta,\varepsilon,\vartheta_{vy}(\eta)) + \mathcal{T}_4(\eta,\varepsilon)\big\}$. Using this operator and based on Remark B.3, one can show (similarly to the proof of Lemma 4.1 in Appendix A) the existence of a positive number ε_2^* $(\varepsilon_2^* \le \varepsilon_1^*)$ such that for any $\varepsilon \in (0, \varepsilon_2^*]$ the system (B.7) (and therefore the equivalent one (B.3)) has the unique solution $\mathcal{N}_\vartheta(\eta,\varepsilon) = \{\vartheta_v(\eta,\varepsilon), \vartheta(\varepsilon), \vartheta_y(\eta,\varepsilon)\}$ in the ball $\mathcal{B}_{\mathcal{M}}(6a,\varepsilon)$, where $a > 0$ is the constant appearing in the inequalities (B.10). Thus, due to (B.1)–(B.2), we have proven the existence of a solution $\mathcal{N}(\eta,\varepsilon) = \{Q_v(\eta,\varepsilon), P_y(\varepsilon), Q_y(\eta,\varepsilon)\}$

to the system (6.1), (6.4) for all $\varepsilon \in (0, \varepsilon_2^*]$, and the validity of the inequalities (6.12)–(6.13) for these values of ε and $a_2^* = 6a$.

Now, let us show the differentiability of the functions $Q_{v,j}(\eta, \varepsilon)$, $Q_{y,j}(\eta, \varepsilon)$ with respect to $\eta \in [-h_j, -h_{j-1}]$ for all $j \in \{1, \ldots, N\}$, $\varepsilon \in (0, \varepsilon_2^*]$.

Let us denote for all $j = 1, \ldots, N$, $\eta \in [-h_j, -h_{j-1}]$, $\varepsilon \in (0, \varepsilon_2^*]$,

$$\begin{aligned} \vartheta_{v,j}(\eta,\varepsilon) &= \begin{cases} \vartheta_v(-h_j+0,\varepsilon), & \eta = -h_j, \\ \vartheta_v(\eta,\varepsilon), & \eta \in (-h_j, -h_{j-1}], \end{cases} \\ \vartheta_{y,j}(\eta,\varepsilon) &= \begin{cases} \vartheta_y(-h_j+0,\varepsilon), & \eta = -h_j, \\ \vartheta_{y,k}(\eta,\varepsilon), & \eta \in (-h_j, -h_{j-1}], \end{cases} \end{aligned} \tag{B.12}$$

Remark B.5. Due to Remarks 2.2, B.2, and B.4, as well as (5.8)–(5.11), the expressions for $\mathcal{D}_2(\varepsilon, \mathcal{N}_\vartheta(\eta,\varepsilon))$, $\mathcal{D}_3(\eta, \varepsilon, \mathcal{N}_\vartheta(\eta,\varepsilon))$, $\mathcal{D}_4(\eta, \varepsilon, \mathcal{N}_\vartheta(\eta,\varepsilon))$ in (B.4), and the equations (B.8), the functions $\vartheta_{v,j}(\eta,\varepsilon)$ and $\vartheta_{y,j}(\eta,\varepsilon)$ $(j = 1, \ldots, N)$ are differentiable with respect to $\eta \in [-h_j, -h_{j-1}]$ for all $\varepsilon \in (0, \varepsilon_2^*]$.

This observation and (B.1) imply immediately the differentiability of the functions $Q_{v,j}(\eta,\varepsilon)$, $Q_{y,j}(\eta,\varepsilon)$ $(j = 1, \ldots, N)$. Thus, the lemma is proven.

Appendix C. Proof of Lemma 7.1. First, let us transform the DS (7.8), (7.5)–(7.6) by introducing, for any $\varepsilon \in (0, \varepsilon_2^*]$, the new state

$$z(t,\eta) \stackrel{\triangle}{=} w_y(t+\varepsilon\eta), \quad (t,\eta) \in \tilde{\Omega}_\varepsilon. \tag{C.1}$$

Remember that ε_2^* was introduced in Lemma 6.2 and then was used in Theorem 6.3.

The new state variable $z(t,\eta)$ satisfies the following differential equation and boundary condition:

$$\frac{\varepsilon \partial z(t,\eta)}{\partial t} - \frac{\partial z(t,\eta)}{\partial \eta} = 0, \quad (t,\eta) \in \tilde{\Omega}_\varepsilon, \tag{C.2}$$

$$z(t,0) = w_y(t), \quad t > \varepsilon h. \tag{C.3}$$

Using (C.2), we can rewrite (7.5)–(7.6) and the condition (6.16) in the form

$$\begin{aligned} \varepsilon \frac{\partial w_v(t,\eta)}{\partial t} - \frac{\partial w_v(t,\eta)}{\partial \eta} &= -\varepsilon L_v(\eta,\varepsilon)\Big[\Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big) w_v(t,-h) \\ &\quad + A_{20}(\varepsilon) w_y(t) + A_{21}(\varepsilon) z(t,-h)\Big], \quad (t,\eta) \in \tilde{\Omega}_\varepsilon, \end{aligned} \tag{C.4}$$

$$\begin{aligned} \frac{\varepsilon d w_y(t)}{dt} &= \Big[\Psi_{34,1}\Big(\varepsilon, L_y(\varepsilon)\Big) - \varepsilon L_y(\varepsilon)\Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)\Big] w_v(t,-h) \\ &\quad + \Big(A_{40}(\varepsilon) - \varepsilon L_y(\varepsilon) A_{20}(\varepsilon)\Big) w_y(t) + \Big(A_{41}(\varepsilon) - \varepsilon L_y(\varepsilon) A_{21}(\varepsilon)\Big) z(t,-h), \\ &\qquad t > \varepsilon h, \end{aligned} \tag{C.5}$$

$$\begin{aligned} w_v(t,0) &= \Big(I_n - L_v(0,\varepsilon)\Big) w_x(t) + \varepsilon \Big(I_n - L_v(0,\varepsilon)\Big) \Big\{ \int_{-h}^0 Q_v(\eta,\varepsilon) w_v(t,\eta) d\eta \\ &\quad + P_y(\varepsilon) w_y(t) + \int_{-h}^0 Q_y(\eta,\varepsilon) z(t,\eta) d\eta \Big\}, \quad t \geq \varepsilon h. \end{aligned} \tag{C.6}$$

Remark C.1. The new differential system (7.8), (C.2), (C.4)–(C.5) subject to the conditions (C.3) and (C.6) is equivalent to the DS (7.8), (7.5)–(7.6), (6.16). Therefore, the spectrum of the latter equals the spectrum of the former.

We proceed to deriving the equation for the spectrum of the system (7.8), (C.2)–(C.6). Let Φ_{xyvz} be the set of all subsets $\phi_{xyvz}(\eta) \triangleq \{\phi_x, \phi_y, \phi_v(\eta), \phi_z(\eta)\}$, consisting of four elements, where $\phi_x \in C^n$, $\phi_y \in C^m$, $\phi_v(\eta) \in L^2[-h, 0; C^n]$, $\phi_z(\eta) \in L^2[-h, 0; C^m]$. Let $\phi^k_{xyvz}(\eta) = \{\phi^k_x, \phi^k_y, \phi^k_v(\eta), \phi^k_z(\eta)\}$ $(k = 1, 2)$ be any two elements of Φ_{xyvz}. Let us define the linear combination of these elements with any complex coefficients α_1 and α_2 as follows: $\alpha_1\phi^1_{xyvz}(\eta) + \alpha_2\phi^2_{xyvz}(\eta) = \{\alpha_1\phi^1_x + \alpha_2\phi^2_x, \alpha_1\phi^1_y + \alpha_2\phi^2_y, \alpha_1\phi^1_v(\eta) + \alpha_2\phi^2_v(\eta), \alpha_1\phi^1_z(\eta) + \alpha_2\phi^2_z(\eta)\}$. This definition converts the set Φ_{xyvz} to a linear space. Further, for any $\phi^1_{xyvz}(\eta) \in \Phi_{xyvz}$ and $\phi^2_{xyvz}(\eta) \in \Phi_{xyvz}$, let us define the number

$$\begin{aligned} \Big\langle \phi^1_{xyvz}(\eta), \phi^2_{xyvz}(\eta) \Big\rangle_{\Phi} &\triangleq \big(\phi^1_x\big)^T \overline{\phi^2_x} + \big(\phi^1_y\big)^T \overline{\phi^2_y} \\ &+ \Big\langle \phi^1_v(\eta), \phi^2_v(\eta) \Big\rangle_{L^2[-h,0;C^n]} + \Big\langle \phi^1_z(\eta), \phi^2_z(\eta) \Big\rangle_{L^2[-h,0;C^m]}. \end{aligned} \tag{C.7}$$

One can verify immediately that this number is an inner product in the linear space Φ_{xyvz}. Moreover, this space, endowed with the inner product (C.7) and the norm induced by (C.7), is a Hilbert space. In the Hilbert space Φ_{xyvz}, we consider the linear subspace $\tilde{\Phi}_{xyvz}(\varepsilon)$, each element $\phi_{xyvz}(\eta) = \{\phi_x, \phi_y, \phi_v(\eta), \phi_z(\eta)\}$ of which satisfies the following conditions:

$$\begin{aligned} &\phi_v(\eta) \in W^{1,2}[-h, 0; C^n], \quad \phi_z(\eta) \in W^{1,2}[-h, 0; C^m], \quad \phi_z(0) = \phi_y, \\ &\phi_v(0) = \Big(I_n - L_v(0, \varepsilon)\Big)\phi_x + \varepsilon\Big(I_n - L_v(0, \varepsilon)\Big)\bigg\{ \int_{-h}^{0} Q_v(\eta, \varepsilon)\phi_v(\eta)d\eta \\ &\qquad\qquad + P_y(\varepsilon)\phi_y + \int_{-h}^{0} Q_y(\eta, \varepsilon)\phi_z(\eta)d\eta \bigg\}. \end{aligned} \tag{C.8}$$

For any $\varepsilon \in (0, \varepsilon^*_2]$, the subspace $\tilde{\Phi}_{xyvz}(\varepsilon)$ is dense in the Hilbert space Φ_{xyvz}. Let us consider the following linear operator mapping $\tilde{\Phi}_{xyvz}(\varepsilon)$ into Φ_{xyvz}:

$$g_{xyvz}(\eta) = \mathcal{A}_\varepsilon \phi_{xyvz}(\eta), \tag{C.9}$$

where $g_{xyvz}(\eta) = \{g_x, g_y, g_v(\eta), g_z(\eta)\}$ and

$$\begin{aligned} g_x = \Big[&\Psi_{12,0}\Big(\varepsilon, L_y(\varepsilon)\Big) + \Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)L_v(-h, \varepsilon) \\ &- Q_v(0, \varepsilon)\Big(I_n - L_v(0, \varepsilon)\Big)\Big]\phi_x, \end{aligned} \tag{C.10}$$

$$\begin{aligned} g_y &= (1/\varepsilon)\Big[\Psi_{34,1}\Big(\varepsilon, L_y(\varepsilon)\Big) - \varepsilon L_y(\varepsilon)\Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)\Big]\phi_v(-h) \\ &+ (1/\varepsilon)\Big(A_{40}(\varepsilon) - \varepsilon L_y(\varepsilon)A_{20}(\varepsilon)\Big)\phi_y + (1/\varepsilon)\Big(A_{41}(\varepsilon) - \varepsilon L_y(\varepsilon)A_{21}(\varepsilon)\Big)\phi_z(-h), \\ g_v(\eta) &= (1/\varepsilon)d\phi_v(\eta)/d\eta - L_v(\eta, \varepsilon)\Big[\Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)\phi_v(-h) \\ &\qquad + A_{20}(\varepsilon)\phi_y + A_{21}(\varepsilon)\phi_z(-h)\Big], \\ g_z(\eta) &= (1/\varepsilon)d\phi_z(\eta)/d\eta. \end{aligned}$$

Remark C.2. Comparing the system of equations (7.8), (C.2)–(C.6) with the definition of the operator $\mathcal{A}_\varepsilon : \tilde{\Phi}_{xyvz}(\varepsilon) \to \Phi_{xyvz}$ (see (C.9)–(C.10)), one can conclude that the set of eigenvalues of the latter coincides with the spectrum of the former for any $\varepsilon \in (0, \varepsilon_2^*]$.

Let, for any fixed $\varepsilon \in (0, \varepsilon_2^*]$, λ be an eigenvalue of $\mathcal{A}_\varepsilon$ and $\psi_{xyvz}(\eta) = \{\psi_x, \psi_y, \psi_v(\eta), \psi_z(\eta)\} \in \tilde{\Phi}_{xyvz}(\varepsilon)$ be an eigenfunction corresponding to λ. Then, λ and $\psi_{xyvz}(\eta)$ satisfy the following equations:

$$\begin{aligned}\Big[\Psi_{12,0}\Big(\varepsilon, L_y(\varepsilon)\Big) + \Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big) L_v(-h,\varepsilon) \\ - Q_v(0,\varepsilon)\Big(I_n - L_v(0,\varepsilon)\Big)\Big]\psi_x = \lambda\psi_x,\end{aligned} \tag{C.11}$$

$$\begin{aligned}\Big[\Psi_{34,1}\Big(\varepsilon, L_y(\varepsilon)\Big) - \varepsilon L_y(\varepsilon)\Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)\Big]\psi_v(-h) \\ + \Big(A_{40}(\varepsilon) - \varepsilon L_y(\varepsilon) A_{20}(\varepsilon)\Big)\psi_y \\ + \Big(A_{41}(\varepsilon) - \varepsilon L_y(\varepsilon) A_{21}(\varepsilon)\Big)\psi_z(-h) = \varepsilon\lambda\psi_y,\end{aligned} \tag{C.12}$$

$$\begin{aligned}d\psi_v(\eta)/d\eta - \varepsilon L_v(\eta,\varepsilon)\Big[\Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)\psi_v(-h) \\ + A_{20}(\varepsilon)\psi_y + A_{21}(\varepsilon)\psi_z(-h)\Big] = \varepsilon\lambda\psi_v(\eta), \quad \eta \in [-h, 0],\end{aligned} \tag{C.13}$$

$$d\psi_z(\eta)/d\eta = \varepsilon\lambda\psi_z(\eta), \quad \eta \in [-h, 0], \qquad \psi_z(0) = \psi_y, \tag{C.14}$$

$$\begin{aligned}\psi_v(0) = \Big(I_n - L_v(0,\varepsilon)\Big)\psi_x + \varepsilon\Big(I_n - L_v(0,\varepsilon)\Big)\Big\{\int_{-h}^{0} Q_v(\eta,\varepsilon)\psi_v(\eta)d\eta \\ + P_y(\varepsilon)\psi_y + \int_{-h}^{0} Q_y(\eta,\varepsilon)\psi_z(\eta)d\eta\Big\}.\end{aligned} \tag{C.15}$$

Equation (C.11) can be rewritten in the form of a linear homogeneous algebraic equation with respect to ψ_x,

$$\Lambda_{11}(\varepsilon, \lambda)\psi_x = 0, \tag{C.16}$$

where $\Lambda_{11}(\varepsilon, \lambda)$ is given by (7.9).

Solving (C.13) and the terminal-value problem (C.14) yields

$$\begin{aligned}\psi_v(\eta) = \exp\big(\varepsilon\lambda\eta\big)\psi_v(0) + M_v\big(\eta, \varepsilon, \lambda\big)\Big[\Psi_{12,1}\Big(\varepsilon, L_y(\varepsilon)\Big)\psi_v(-h) \\ + A_{20}(\varepsilon)\psi_y + A_{21}(\varepsilon)\psi_z(-h)\Big], \quad \eta \in [-h, 0],\end{aligned} \tag{C.17}$$

$$\psi_z(\eta) = \exp\big(\varepsilon\lambda\eta\big)\psi_y, \qquad \eta \in [-h, 0], \tag{C.18}$$

where $M_v\big(\eta, \varepsilon, \lambda\big)$ is introduced by (7.13).

By substituting (C.18) into (C.12), one obtains the linear homogeneous algebraic equation with respect to ψ_y and $\psi_v(-h)$,

$$\Lambda_{22}\big(\varepsilon, \lambda\big)\psi_y + \Lambda_{23}(\varepsilon)\psi_v(-h) = 0, \tag{C.19}$$

where $\Lambda_{22}(\varepsilon,\lambda)$ and $\Lambda_{23}(\varepsilon,\lambda)$ are given in (7.10).

Let us set $\eta = -h$ in (C.17) and substitute (C.18) into the resulting equation. Thus, we obtain the linear homogeneous algebraic equation with respect to ψ_y, $\psi_v(-h)$, and $\psi_v(0)$,

$$\Lambda_{32}(\varepsilon,\lambda)\psi_y + \Lambda_{33}(\varepsilon,\lambda)\psi_v(-h) + \Lambda_{34}(\varepsilon,\lambda)\psi_v(0) = 0, \tag{C.20}$$

where $\Lambda_{32}(\varepsilon,\lambda)$, $\Lambda_{33}(\varepsilon,\lambda)$, and $\Lambda_{34}(\varepsilon,\lambda)$ are given in (7.11).

Finally, by substituting (C.17) into (C.15), one obtains one more linear homogeneous algebraic equation with respect to ψ_x, ψ_y, $\psi_v(-h)$, and $\psi_v(0)$:

$$\begin{aligned} &\Lambda_{41}(\varepsilon)\psi_x + \Lambda_{42}(\varepsilon,\lambda)\psi_y \\ &\quad + \Lambda_{43}(\varepsilon,\lambda)\psi_v(-h) + \Lambda_{44}(\varepsilon,\lambda)\psi_v(0) = 0, \end{aligned} \tag{C.21}$$

where $\Lambda_{41}(\varepsilon,\lambda)$, $\Lambda_{42}(\varepsilon,\lambda)$, $\Lambda_{43}(\varepsilon,\lambda)$, and $\Lambda_{44}(\varepsilon,\lambda)$ are given in (7.12).

Thus, we have constructed the set of four vector linear homogeneous algebraic equations (C.16), (C.19), (C.20), (C.21) with respect to ψ_x, ψ_y, $\psi_v(-h)$, and $\psi_v(0)$. This set consists of $3n+m$ scalar equations with respect to $3n+m$ scalar unknowns. Moreover, the set of equations (C.16), (C.19), (C.20), (C.21), (C.17)–(C.18), with respect to ψ_x, ψ_y, $\psi_v(\eta)$, $\psi_z(\eta)$, and λ, is equivalent to the set (C.11)–(C.15). Hence, the complex number λ is an eigenvalue of the operator $\mathcal{A}_\varepsilon$ if and only if the determinant of the set (C.16), (C.19), (C.20), (C.21) equals zero, i.e., if and only if λ is a root of the equation

$$\Delta(\varepsilon,\lambda) \triangleq \det \begin{bmatrix} \Lambda_{11}(\varepsilon,\lambda) & 0_{n\times m} & 0_{n\times n} & 0_{n\times n} \\ 0_{m\times n} & \Lambda_{22}(\varepsilon,\lambda) & \Lambda_{23}(\varepsilon) & 0_{m\times n} \\ 0_{n\times n} & \Lambda_{32}(\varepsilon,\lambda) & \Lambda_{33}(\varepsilon,\lambda) & \Lambda_{34}(\varepsilon,\lambda) \\ \Lambda_{41}(\varepsilon) & \Lambda_{42}(\varepsilon,\lambda) & \Lambda_{43}(\varepsilon,\lambda) & \Lambda_{44}(\varepsilon,\lambda) \end{bmatrix} = 0. \tag{C.22}$$

Since $\Delta(\varepsilon,\lambda) = \Delta_s(\varepsilon,\lambda)\Delta_f(\varepsilon,\lambda)$, then (C.22) is equivalent to the system (7.14)–(7.15). This observation, along with Remarks C.1 and C.2, directly yields the statement of the lemma.

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REFERENCES

[1] A. B. Vasil'eva, V. F. Butuzov, and L. V. Kalachev, *The Boundary Function Method for Singular Perturbation Problems*, SIAM Stud. Appl. Math. 14, SIAM, Philadelphia, 1995.

[2] P. V. Kokotovic, H. K. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design*, Classics in Appl. Math. 25, SIAM, Philadelphia, 1999.

[3] M. G. Dmitriev and G. A. Kurina, *Singular perturbations in control problems*, Autom. Remote Control, 67 (2006), pp. 1–43, https://doi.org/10.1134/S0005117906010012.

[4] S. R. Shimjith, A. P. Tiwari, and B. Bandyopadhyay, *Modeling and Control of a Large Nuclear Reactor: A Three-Time-Scale Approach*, Springer, Berlin, 2013.

[5] Y. Zhang, D. S. Naidu, C. Cai, and Y. Zou, *Singular perturbations and time scales in control theories and applications: An overview* 2002–2012, Int. J. Inform. System Sci., 9 (2014), pp. 1–36.

[6] A. Narang-Siddarth and J. Valasek, *Nonlinear Time Scale Systems in Standard and Nonstandard Forms: Analysis and Control*, Adv. Des. Control 26, SIAM, Philadelphia, 2014.

[7] R. E. O'Malley, *Historical Developments in Singular Perturbations*, Springer, New York, 2014.

[8] C. Kuehn, *Multiple Time Scale Dynamics*, Springer, New York, 2015.

[9] P. B. Reddy and P. Sannuti, *Optimal control of a coupled-core nuclear reactor by singular perturbation method*, IEEE Trans. Automat. Control, 20 (1975), pp. 766–769, https://doi.org/10.1109/TAC.1975.1101096.

[10] M. L. Pena, *Asymptotic expansion for the initial value problem of the sunflower equation*, J. Math. Anal. Appl., 143 (1989), pp. 471–479, https://doi.org/10.1016/0022-247X(89)90053-X.

[11] C. G. Lange and R. M. Miura, *Singular perturbation analysis of boundary-value problems for differential-difference equations.* V. *Small shifts with layer behavior*, SIAM J. Appl. Math., 54 (1994), pp. 249–272, https://doi.org/10.1137/S0036139992228120.

[12] N. Stefanovic and L. Pavel, *A stability analysis with time-delay of primal-dual power control in optical networks*, Automatica J. IFAC, 45 (2009), pp. 1319–1325, https://doi.org/10.1016/j.automatica.2009.01.005.

[13] N. Stefanovic and L. Pavel, *A Lyapunov-Krasovskii stability analysis for game-theoretic based power control in optical links*, Telecommun. Syst., 47 (2011), pp. 19–33, https://doi.org/10.1007/s11235-010-9299-x.

[14] L. Pavel, *Game Theory for Control of Optical Networks*, Birkhäuser, Basel, 2012.

[15] N. Stefanovic and L. Pavel, *Robust power control of multi-link single-sink optical networks with time-delays*, Automatica J. IFAC, 49 (2013), pp. 2261–2266, https://doi.org/10.1016/j.automatica.2013.04.009.

[16] V. Y. Glizer, *L^2-stabilizability conditions for a class of nonstandard singularly perturbed functional-differential systems*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms, 16 (2009), pp. 181–213.

[17] E. Fridman, *Robust sampled-data H_∞ control of linear singularly perturbed systems*, IEEE Trans. Automat. Control, 51 (2006), pp. 470–475, https://doi.org/10.1109/TAC.2005.864194.

[18] W.-H. Chen, S.-T. Yang, X. Lu, and Y. Shen, *Exponential stability and exponential stabilization of singularly perturbed stochastic systems with time-varying delay*, Internat. J. Robust Nonlinear Control, 20 (2010), pp. 2021–2044, https://doi.org/10.1002/rnc.1564.

[19] V. Y. Glizer and E. Fridman, *Stability of singularly perturbed functional-differential systems: Spectrum analysis and LMI approaches*, IMA J. Math. Control Inform., 29 (2012), pp. 79–111, https://doi.org/10.1093/imamci/dnr027.

[20] F. Sun, L. Zhou, Q. Zhang, and Y. Shen, *Stability bound analysis and synthesis for singularly perturbed systems with time-varying delay*, Math. Prob. Eng., 2013 (2013), 517258.

[21] K.-I. Kang, K.-S. Park, and J.-T. Lim, *Exponential stability of singularly perturbed systems with time delay and uncertainties*, Internat. J. Systems Sci., 46 (2015), pp. 170–178, https://doi.org/10.1080/00200721.2013/775391.

[22] W.-H. Chenab, Z. Wang, D. Wei, and X. Lu, *Robust sampled-data H^∞ control of uncertain singularly perturbed systems using time-dependent Lyapunov functionals*, Internat. J. Systems Sci., 46 (2015), pp. 2832–2852, https://doi.org/10.1080/00200721.2014.880193.

[23] E. Fridman, *Decoupling transformation of singularly perturbed systems with small delays and its applications*, Z. Angew. Math. Mech., 76 (1996), pp. 201–204.

[24] V. Y. Glizer and E. Fridman, *H_∞ control of linear singularly perturbed systems with small state delay*, J. Math. Anal. Appl., 250 (2000), pp. 49–85, https://doi.org/10.1006/jmaa.2000.6955.

[25] Z. Baolin and M. Q. Fan, *Near optimal control for singularly perturbed systems with small time-delay*, in Proceedings of the 7th World Congress on Intelligent Control and Automation, Chongqing, IEEE, 2008, pp. 7212–7216, https://doi.org/10.1109/WCICA.2008.4594039.

[26] Z. Baolin, G. Dexin, L. Qiang, and C. Feilong, *Approximation design of composite control for singularly perturbed time-delay systems via delay compensation*, in Proceedings of the 29th Chinese Control Conference, Beijing, IEEE, 2010, pp. 1676–1680.

[27] S. Puga, M. Bonilla, S. Mondie, and M. Malabre, *Singularly perturbed control law for linear time-invariant delay SISO systems*, Bol. Soc. Mat. Mexicana, 20 (2014), pp 69–85, https://doi.org/10.1007/s40590-013-0003-5.

[28] M. Ping and M. Guoying, *H^∞ performance analysis for a class of time delayed singularly perturbed systems within finite frequency range*, in Proceedings of the 33rd Chinese Control Conference, Nanjing, IEEE, 2014, pp. 6095–6098, https://doi.org/10.1109/ChiCC.2014.6895986.

[29] K. W. Chang, *Singular perturbations of a general boundary value problem*, SIAM J. Math. Anal., 3 (1972), pp. 520–526, https://doi.org/10.1137/0503050.

[30] D. R. Smith, *Decoupling and order reduction via the Riccati transformation*, SIAM Rev., 29 (1987), pp. 91–113, https://doi.org/10.1137/1029003.

[31] M. Qureshi and Z. Gajic, *A new version of the Chang transformation*, IEEE Trans. Automat. Control, 37 (1992), pp. 800–801, https://doi.org/10.1109/9.256335.

[32] W. Su, Z. Gajic, and X. Shen, *The exact slow-fast decomposition of the algebraic Riccati equation of singularly perturbed systems*, IEEE Trans. Automat. Control, 37 (1992), pp. 1456–1459, https://doi.org/10.1109/9.159592.

[33] E. Fridman, *Exact slow-fast decomposition of the nonlinear singularly perturbed optimal control problem*, Systems Control Lett., 40 (2000), pp. 121–131, https://doi.org/10.1016/S0167-6911(00)00008-6.

[34] S. Koskie, C. Coumarbatch, and Z. Gajic, *Exact slow-fast decomposition of the singularly perturbed matrix differential Riccati equation*, Appl. Math. Comput., 216 (2010), pp. 1401–1411, https://doi.org/10.1016/j.amc.2010.02.040.

[35] M. C. Delfour, C. McCalla, and S. K. Mitter, *Stability and the infinite-time quadratic cost problem for linear hereditary differential systems*, SIAM J. Control, 13 (1975), pp. 48–88, https://doi.org/10.1137/0313004.

[36] R. E. O'Malley, Jr., *Boundary layer methods for nonlinear initial value problems*, SIAM Rev., 13 (1971), pp. 425–434, https://doi.org/10.1137/1013091.

[37] P. M. Ndiaye and M. Sorine, *Delay sensitivity of quadratic controllers: A singular perturbation approach*, SIAM J. Control Optim., 38 (2000), pp. 1655–1682, https://doi.org/10.1137/S0363012997329858.

[38] M. C. Delfour and S. K. Mitter, *Controllability, observability and optimal feedback control of affine hereditary differential systems*, SIAM J. Control, 10 (1972), pp. 298–328, https://doi.org/10.1137/0310023.

[39] R. B. Vinter and R. H. Kwong, *The infinite time quadratic control problem for linear systems with state and control delays: An evolution equation approach*, SIAM J. Control Optim., 19 (1981), pp. 139–153, https://doi.org/10.1137/0319011.

[40] V. Y. Glizer, *Blockwise estimate of the fundamental matrix of linear singularly perturbed differential systems with small delay and its application to uniform asymptotic solution*, J. Math. Anal. Appl., 278 (2003), pp. 409–433, https://doi.org/10.1016/S0022-247X(02)00715-1.

[41] J.-P. Richard, M. Dambrine, and J.-F. Lafay, *Recent advances in the study of time delay systems: Modelling, stability, structure and control*, e-STA, Sciences & Technologies de l'Automatique, 1 (2004).

[42] V. A. Trenogin, *Functional Analysis*, FIZMATLIT, Moscow, 2007 (in Russian).