Many examples of normal forms for stochastic or non-autonomous differential equations

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Instructions

- Download and install the computer algebra package *Reduce*[1] via `http://www.reduce-algebra.com`
- Navigate to folder `Examples` within folder `StoNormForm`.
- For each example of interest, start-up *Reduce* and enter the command `in_tex "filename.tex"$` where filename is the root name of the example (as listed in the following table of contents).

Contents

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[1] Use the computer algebra package *Reduce* because it is both free and perhaps the fastest general purpose computer algebra system (Fateman 2003, e.g.).

1 Some representative examples

1.1 `ratodes`: Simple rational ODEs

A simple system of fast/slow ODEs in rational functions is

$$\dot x = -\frac{xy}{1+z}\,, \quad \dot y = -\frac{y}{1+2y} + x^2, \quad \dot z = 2\frac{z}{1+3x}\,. \tag{1}$$

Use `x(1)` to denote variable x, `y(1)` to denote variable y, and `z(1)` to denote z. Multiply each ODE by the denominator for the ODE and shift the nonlinear d/dt terms to the right-hand side.

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system.

```
stonormalform(
    {-x(1)*y(1)-z(1)*df(x(1),t)},
    {-y(1)+x(1)^2*(1+2*y(1))-2*y(1)*df(y(1),t)},
    {2*z(1)-3*x(1)*df(z(1),t)},
    4)$
end;
```

The procedure embeds the system as the $\varepsilon = 1$ version of the family

$$\begin{aligned}
\dot x_1 &= \varepsilon\big(-\frac{d\,x_1}{d\,t} z_1 - x_1 y_1\big)\\
\dot y_1 &= \varepsilon\big(-2\frac{d\,y_1}{d\,t} y_1 + 2x_1^2 y_1 + x_1^2\big) - y_1\\
\dot z_1 &= -3\varepsilon\frac{d\,z_1}{d\,t} x_1 + 2z_1
\end{aligned}$$

Time dependent coordinate transform

$$\begin{aligned}
z_1 &= 6\varepsilon^2 X_1 Y_1 Z_1 + Z_1\\
y_1 &= \varepsilon^2\big(2X_1^4 - 4X_1^2 Y_1^2 + 6Y_1^3\big) + \varepsilon\big(X_1^2 - 2Y_1^2\big) + Y_1\\
x_1 &= \varepsilon^2\big(2X_1^3 Y_1 - 1/2 X_1 Y_1^2 + X_1 Y_1 Z_1\big) + \varepsilon X_1 Y_1 + X_1
\end{aligned}$$

Result normal form DEs

$$\begin{aligned}
\dot Z_1 &= -54\varepsilon^3 X_1^3 Z_1 + 18\varepsilon^2 X_1^2 Z_1 - 6\varepsilon X_1 Z_1 + 2Z_1\\
\dot Y_1 &= 8\varepsilon^3 X_1^4 Y_1 + 4\varepsilon^2 X_1^2 Y_1 + 2\varepsilon X_1^2 Y_1 - Y_1\\
\dot X_1 &= \varepsilon^3\big(-2X_1^5 - 2X_1 Y_1^2 Z_1\big) - \varepsilon^2 X_1^3
\end{aligned}$$

1.2 `futureNoise`: Future noise in the transform

An interesting pair of fast/slow SDEs derived from stochastic advection/dispersion is

$$\dot x = -\sigma y w(t) \quad \text{and} \quad \dot y = -y + \sigma x w(t)\,, \tag{2}$$

where lowercase $w(t)$ denotes the formal derivative dW/dt of a Stratonovich Wiener process $W(t,\omega)$. Parameter σ controls the strength of the noise. In stochastic advection/dispersion parameter σ represents the lateral wavenumber of the concentration profile.

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system.

```
stonormalform(
    {-y(1)*w(1)},
    {-y(1)+x(1)*w(1)},
    {},
    5)$
end;
```

Being linear in x, y the nonlinear parameter ε does not appear in the analysis and results. Consequently, the procedure analyses the system as prescribed (since given w changed to σw). The interest in this example is the noise and the noise-noise interactions. As usual, the noise-noise interactions are truncated to errors $\mathcal{O}(\sigma^3)$.

Time dependent coordinate transform

$$y_1 = \sigma \mathrm{e}^{-1t}{\star}w_1\, X_1 + Y_1$$
$$x_1 = \sigma \mathrm{e}^{t}{\star}w_1\, Y_1 + X_1$$

Result normal form DEs

$$\dot Y_1 = \sigma^2 \mathrm{e}^{t}{\star}w_1\, w_1 Y_1 - Y_1$$
$$\dot X_1 = -\sigma^2 \mathrm{e}^{-1t}{\star}w_1\, w_1 X_1$$

The interesting aspect of this example is the explicit presence of non-Markovian, future time integrals, anticipation integrals, in the convolutions $\mathrm{e}^{t}{\star}w_1$. These appear in both the coordinate transform, and the evolution *off* the stochastic slow manifold. But, as guaranteed by theory, they do not appear on the stochastic slow manifold.

Further, this example could go to higher order noise-noise interactions very quickly, that is, to higher orders in σ. However, I do not compute such higher order terms in this code.

1.3 `othersFail`: Other methodologies fail

Consider, for small bifurcation parameter ϵ, the system

$$\text{slow mode } \dot x = \epsilon x + x^3 - (1-\sigma w)xy\,,$$
$$\text{fast mode } \dot y = -y + x^2 + y^2 + \sigma y w\,.$$

Deterministically, there is a bifurcation to two equilibria for small $\epsilon > 0$. The noise w affects this bifurcation somehow.

Why is this tricky? *Cross-sectional averaging* is simply projection onto the slow space $y = 0$ which predicts instability of subcritical bifurcation $\dot{x} = \epsilon x + x^3$. Whereas *adiabatic approximation*, *singular perturbation*, and *multiple scales* set $\dot{y} = 0$ whence $y \approx x^2$ and thus predict only the linear growth of $\dot{x} = \epsilon x$. Our normal form transforms get the deterministic dynamics correctly. But what happens for stochastic dynamics?

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system. Multiply a cubic terms in the x SDE in order to count orders of approximation best (since the right-hand side is multiplied by `small`). Multiply the bifurcation parameter by `small` in order to make it scale with ε^2.

```
stonormalform(
    {small*epsilon*x(1)+small*x(1)^3
        -x(1)*y(1)*(1-small*w(1))},
    {-y(1)+x(1)^2+y(1)^2+y(1)*w(1)},
    {},
    5)$
end;
```

With the above artifices, the procedure analyses the following system which reduce to the given one for $\varepsilon = 1$:

$$\begin{aligned}
\dot{x}_1 &= \sigma\varepsilon w_1 x_1 y_1 + \varepsilon^2\big(x_1^3 + x_1\epsilon\big) - \varepsilon x_1 y_1 \\
\dot{y}_1 &= \sigma w_1 y_1 + \varepsilon\big(x_1^2 + y_1^2\big) - y_1
\end{aligned}$$

Time dependent coordinate transform This transform is quite complicated, due to the noise, and involve fast-time future and history integrals. The deterministic terms at the end.

$$\begin{aligned}
y_1 = {}& \sigma\varepsilon^3\big(- 4\mathrm{e}^{t}\star\mathrm{e}^{t}\star w_1\, X_1^2 Y_1^2 + 4\mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_1\, X_1^4 - \\
& 2\mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_1\, X_1^2\epsilon + 2\mathrm{e}^{2t}\star w_1\, X_1^2 Y_1^2 - 10\mathrm{e}^{t}\star w_1\, X_1^2 Y_1^2 - \\
& 3\mathrm{e}^{t}\star w_1\, Y_1^4 + \mathrm{e}^{-1t}\star w_1\, X_1^4 + 3\mathrm{e}^{-1t}\star w_1\, X_1^2 Y_1^2 - \\
& 2\mathrm{e}^{-1t}\star w_1\, X_1^2\epsilon\big) + \sigma\varepsilon^2\big(2\mathrm{e}^{t}\star w_1\, Y_1^3 - 2\mathrm{e}^{-1t}\star w_1\, X_1^2 Y_1\big) + \sigma\varepsilon\big(- \\
& \mathrm{e}^{t}\star w_1\, Y_1^2 + \mathrm{e}^{-1t}\star w_1\, X_1^2\big) + \varepsilon^3\big(X_1^4 - 7X_1^2 Y_1^2 - 2X_1^2\epsilon - Y_1^4\big) + \\
& \varepsilon^2 Y_1^3 + \varepsilon\big(X_1^2 - Y_1^2\big) + Y_1 \\
x_1 = {}& \sigma\varepsilon^3\big(- \mathrm{e}^{3t}\star w_1\, X_1 Y_1^3 + \mathrm{e}^{2t}\star w_1\, X_1 Y_1^3 + 3\mathrm{e}^{t}\star w_1\, X_1^3 Y_1\big) + \\
& \sigma\varepsilon^2\big(\mathrm{e}^{2t}\star w_1\, X_1 Y_1^2 - \mathrm{e}^{t}\star w_1\, X_1 Y_1^2 + \mathrm{e}^{-1t}\star w_1\, X_1^3\big) + 2\varepsilon^3 X_1^3 Y_1 + \\
& \varepsilon X_1 Y_1 + X_1
\end{aligned}$$

Result normal form DEs

$$
\begin{aligned}
\dot{Y}_1 = {}& \sigma^2\varepsilon^4\big(8\mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_1\, w_1X_1^4Y_1 - 4\mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_1\, w_1X_1^2Y_1\epsilon + \\
& 6\mathrm{e}^{t}\star w_1\, w_1X_1^4Y_1 + 22\mathrm{e}^{-1t}\star w_1\, w_1X_1^4Y_1 - 4\mathrm{e}^{-1t}\star w_1\, w_1X_1^2Y_1\epsilon\big) + \\
& 2\sigma^2\varepsilon^2\mathrm{e}^{-1t}\star w_1\, w_1X_1^2Y_1 + \sigma\varepsilon^4\big(22w_1X_1^4Y_1 - 4w_1X_1^2Y_1\epsilon\big) + \\
& 2\sigma\varepsilon^2w_1X_1^2Y_1 + \sigma w_1Y_1 + \varepsilon^4\big(6X_1^4Y_1 - 4X_1^2Y_1\epsilon\big) + 4\varepsilon^2X_1^2Y_1 - Y_1 \\
\dot{X}_1 = {}& -3\sigma^2\varepsilon^4\mathrm{e}^{-1t}\star w_1\, w_1X_1^5 - 2\sigma\varepsilon^4w_1X_1^5 + \varepsilon^4\big(-X_1^5 + 2X_1^3\epsilon\big) + \varepsilon^2X_1\epsilon
\end{aligned}
$$

- As expected, $Y_1 = 0$ is the stochastic slow manifold, and is exponentially attractive (almost always) in some domain about the origin.
- The slow X_1 evolution is independent of Y_1. Deterministically ($\sigma = 0$), we predict a bifurcation to $X_1 \approx \pm\epsilon^{1/4}$. The noise appears to modify this slightly.
- The time-dependent coordinate transform maps these predictions back into the xy-plane.

1.4 `offdiagonal`: Levy area contraction: off-diagonal example

Pavliotis & Stuart (2008) assert the following system of five coupled SDEs are interesting for various parameters a and for small ϵ.

$$
\begin{aligned}
dx_1 &= \epsilon y_1\, dt\,, \\
dx_2 &= \epsilon y_2\, dt\,, \\
dx_3 &= \epsilon(x_1y_2 - x_2y_1)dt\,, \\
dy_1 &= (-y_1 - ay_2)dt + dW_1\,, \\
dy_2 &= (+ay_1 - y_2)dt + dW_2\,.
\end{aligned}
$$

This stochastic system has two noise sources. We treat $W_i(t,\omega)$ as Stratonovich Wiener processes. Use `x(i)` to denote variable x_i, `y(i)` to denote variable y_i, and `w(i)` to denote noise dW_i/dt.

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

It is convenient to factor written results on the two given parameters ϵ, a:

```
factor epsilon,a;
```

Execute the construction of a normal form for this system. A coding is to specify the system as given: specify the slow SDEs via a three component list; and the fast stable SDEs via a two component list.

```
stonormalform(
    {epsilon*y(1),
     epsilon*y(2),
     epsilon*(x(1)*y(2)-x(2)*y(1))},
    {-y(1)-a*y(2)+w(1),
     -y(2)+a*y(1)+w(2)},
```

```
    {},
    4)$
end;
```

Now the approach can only analyse systems which are linearly diagonalised, but this system has two off-diagonal terms in the $\vec{y}$-SDEs (terms that cause oscillations in $\vec{y}$ with frequency a as $\vec{y}$ decays in magnitude like e^{-t}). In order to make some sort of progress, the procedure is brutal with such off-diagonal terms. *Anything linear and off-diagonal is multiplied by the parameter* `small` *and so is treated as asymptotically small.* When it does so, it gives the warning message

```
***** Warning ****
Off diagonal linear terms in y- or z- equations
assumed small.  Answers are rubbish if not
asymptotically appropriate.
```

As the message says, the results may consequently be rubbish.

Here then, the procedure analyses the following system which reduce to the given one for $\varepsilon = 1$:

$$\begin{aligned}
\dot{x}_1 &= \epsilon\varepsilon y_1 \\
\dot{x}_2 &= \epsilon\varepsilon y_2 \\
\dot{x}_3 &= \epsilon\varepsilon\big(-x_2y_1 + x_1y_2\big) \\
\dot{y}_1 &= -a\varepsilon y_2 + \sigma w_1 - y_1 \\
\dot{y}_2 &= a\varepsilon y_1 + \sigma w_2 - y_2
\end{aligned}$$

That is, the code treats the frequency parameter a as small, and so the results are appropriate only for small a, as well as only for small ϵ.

If one really needs to analyse non-small a, then more sophisticated code has to be developed.

Time dependent coordinate transform This transform is quite complicated, due to the noise, and involve fast-time future and history integrals.

$$\begin{aligned}
y_1 &= -a^2\sigma\varepsilon^2 e^{-1t}\star e^{-1t}\star e^{-1t}\star w_1 \; - a\sigma\varepsilon e^{-1t}\star e^{-1t}\star w_2 \; + \\
&\quad \sigma e^{-1t}\star w_1 \; + Y_1 \\
y_2 &= -a^2\sigma\varepsilon^2 e^{-1t}\star e^{-1t}\star e^{-1t}\star w_2 \; + a\sigma\varepsilon e^{-1t}\star e^{-1t}\star w_1 \; + \\
&\quad \sigma e^{-1t}\star w_2 \; + Y_2 \\
x_1 &= a\epsilon\sigma\varepsilon^2\big(e^{-1t}\star e^{-1t}\star w_2 \; + e^{-1t}\star w_2\big) + a\epsilon\varepsilon^2 Y_2 - \epsilon\sigma\varepsilon e^{-1t}\star w_1 \; - \\
&\quad \epsilon\varepsilon Y_1 + X_1 \\
x_2 &= a\epsilon\sigma\varepsilon^2\big(-e^{-1t}\star e^{-1t}\star w_1 \; - e^{-1t}\star w_1\big) - a\epsilon\varepsilon^2 Y_1 \; - \\
&\quad \epsilon\sigma\varepsilon e^{-1t}\star w_2 \; - \epsilon\varepsilon Y_2 + X_2 \\
x_3 &= a\epsilon\sigma\varepsilon^2\big(-e^{-1t}\star e^{-1t}\star w_2\, X_2 - e^{-1t}\star e^{-1t}\star w_1\, X_1 \; - \\
&\quad e^{-1t}\star w_2\, X_2 - e^{-1t}\star w_1\, X_1\big) + a\epsilon\varepsilon^2\big(-X_2Y_2 - X_1Y_1\big) + \\
&\quad \epsilon^2\sigma\varepsilon^2\big(e^{t}\star w_2\, Y_1 - e^{1t}\star w_1\, Y_2\big) + \epsilon\sigma\varepsilon\big(-e^{-1t}\star w_2\, X_1 \; + \\
&\quad e^{-1t}\star w_1\, X_2\big) + \epsilon\varepsilon\big(X_2Y_1 - X_1Y_2\big) + X_3
\end{aligned}$$

Result normal form DEs

$$\begin{aligned}
\dot Y_1 &= -a\varepsilon Y_2 - Y_1\\
\dot Y_2 &= a\varepsilon Y_1 - Y_2\\
\dot X_1 &= -a^2\epsilon\sigma\varepsilon^3 w_1 - a\epsilon\sigma\varepsilon^2 w_2 + \epsilon\sigma\varepsilon w_1\\
\dot X_2 &= -a^2\epsilon\sigma\varepsilon^3 w_2 + a\epsilon\sigma\varepsilon^2 w_1 + \epsilon\sigma\varepsilon w_2\\
\dot X_3 &= a^2\epsilon\sigma\varepsilon^3\big(- w_2 X_1 + w_1 X_2\big) + a\epsilon^2\sigma^2\varepsilon^3\big(\mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_2\; w_2 +\\
&\qquad \mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_1\; w_1 + \mathrm{e}^{-1t}\star w_2\, w_2 + \mathrm{e}^{-1t}\star w_1\, w_1\big) +\\
&\qquad a\epsilon\sigma\varepsilon^2\big(w_2 X_2 + w_1 X_1\big) + \epsilon^2\sigma^2\varepsilon^2\big(\mathrm{e}^{-1t}\star w_2\, w_1 - \mathrm{e}^{-1t}\star w_1\, w_2\big) +\\
&\qquad \epsilon\sigma\varepsilon\big(w_2 X_1 - w_1 X_2\big)
\end{aligned}$$

- As expected, $\vec Y = 0$ is the stochastic slow manifold, and is exponentially attractive (almost always) in some domain about the origin.
- The slow $\vec X$ evolution is independent of $\vec Y$: X_1, X_2 undergo a correlated 'slow' random walk; whereas X_3 is dominantly some multiplicative random walk.
- The time-dependent coordinate transform maps these predictions back into the $\vec x, \vec y$-space.

1.5 `jordanForm`: the Jordan form of position-momentum variables

Suppose $x(t)$ is the spatial position of some particle, and you want to analyse the 'mechanical' system of SDEs

$$\ddot x = -xy \quad\text{and}\quad \dot y = -2y + x^2 + \dot x^2 + \sigma w(t)\,,$$

where $w(t)$ denotes the formal derivative dW/dt of a Stratonovich Wiener process $W(t,\omega)$, or some other time-dependent forcing, called noise. Introduce position and velocity variables $x_1 = x$ and $x_2 = \dot x$, and also $y_1 = y$ to convert to the system of three coupled first-order SDEs

$$\begin{aligned}
\dot x_1 &= x_2\,,\\
\dot x_2 &= -x_1 y_1\,,\\
\dot y_1 &= -2y_1 + x_1^2 + x_2^2 + \sigma w(t)\,.
\end{aligned}$$

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system.

```
stonormalform(
    { x(2)/small,
      -x(1)*y(1) },
    { -2*y(1)+x(1)^2+x(2)^2+w(y) },
    {},
    3 )$
end;
```

Why divide `x(2)` by `small`? A possible coding is to specify the system as given, but recall that the slow SDEs are always multiplied by `small` thus changing the first SDE to $\dot{x}_1 = \varepsilon x_2$ and hence changing the relation between position and velocity—this would be OK if x_2 was viewed as momentum *and* the particle had large mass. But what if really do we want x_2 to be velocity. Fortunately, the coded iteration scheme works for systems with linear part in Jordan form, but one has to code the system as follows. Namely, divide the off-diagonal term of the Jordan form by `small` to cancel out the procedure's brutal multiplication by `small`.

Then the coded procedure reports that it analyses the following system which not only reduces to the given one for $\varepsilon = 1$, but also preserves the physical relation between position x_1 and velocity x_2:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\varepsilon x_1 y_1 \\
\dot{y}_1 &= \sigma w_y + \varepsilon\big(x_2^2 + x_1^2\big) - 2y_1
\end{aligned}$$

Further, here ε counts the order of nonlinearity so truncating to errors $\mathcal{O}\big(\varepsilon^3\big)$ is the same as truncating to errors $\mathcal{O}\big(|(\vec{x}, y)|^4\big)$.

The cost of preserving the physical relation between position x_1 and velocity x_2 is that more iterations are needed in the construction.

Time dependent coordinate transform This transform is quite complicated, due to the noise, and involve fast-time future and history integrals.

$$\begin{aligned}
y_1 &= \sigma \mathrm{e}^{-2t}{\star}w_y + \varepsilon\big(3/4X_2^2 - 1/2X_2X_1 + 1/2X_1^2\big) + Y_1 \\
x_1 &= \sigma\varepsilon\big(- 1/4\mathrm{e}^{-2t}{\star}w_y\, X_2 - 1/4\mathrm{e}^{-2t}{\star}w_y\, X_1\big) + \varepsilon\big(- 1/4X_2Y_1 - \\
&\quad 1/4X_1Y_1\big) + X_1 \\
x_2 &= \sigma\varepsilon\big(1/4\mathrm{e}^{-2t}{\star}w_y\, X_2 + 1/2\mathrm{e}^{-2t}{\star}w_y\, X_1\big) + \varepsilon\big(1/4X_2Y_1 + \\
&\quad 1/2X_1Y_1\big) + X_2
\end{aligned}$$

Result normal form DEs

$$\begin{aligned}
\dot{Y}_1 &= \varepsilon^2\big(1/2X_2^2Y_1 + 1/2X_2X_1Y_1 - 1/2X_1^2Y_1\big) - 2Y_1 \\
\dot{X}_1 &= \sigma^2\varepsilon^2\big(- 3/64\mathrm{e}^{-2t}{\star}w_y\, w_yX_2 - 3/32\mathrm{e}^{-2t}{\star}w_y\, w_yX_1\big) + \\
&\quad \sigma\varepsilon\big(1/4w_yX_2 + 1/4w_yX_1\big) + X_2 \\
\dot{X}_2 &= \sigma^2\varepsilon^2\big(3/32\mathrm{e}^{-2t}{\star}w_y\, w_yX_2 + 1/8\mathrm{e}^{-2t}{\star}w_y\, w_yX_1\big) + \sigma\varepsilon\big(- \\
&\quad 1/4w_yX_2 - 1/2w_yX_1\big) + \varepsilon^2\big(- 3/4X_2^2X_1 + 1/2X_2X_1^2 - 1/2X_1^3\big)
\end{aligned}$$

- As expected, $Y_1 = 0$ is the stochastic slow manifold, and is exponentially attractive (almost always) in some domain about the origin.
- As expected, the slow $\vec{X}$ evolution is independent of Y_1: X_2 is approximately a 'velocity' variable for 'position' X_1, and shows some nonlinear noise affected dynamics.

- The time-dependent coordinate transform maps these predictions back into the $\vec x, y$-space. Observe that $\vec X$ are not precisely the physical position-veloxity $\vec x$, but instead are affected by nonlinearity, and the noise, and their interaction.

1.6 `slowOsc`: Radek's slow oscillation with fast noise

Consider Radek's system

$$\dot x = -\epsilon x z\,, \quad \dot y = +\epsilon y z \quad \text{and} \quad \dot z = -(z-1) + \sigma w(t)\,.$$

In this linear system, x, y oscillate with 'frequency' ϵz. But $z(t)$ is an Ornstein–Uhlenbeck process with mean one. What are the dynamics?

Transform to our standard form via

$$x = x_1\,, \quad y = x_2 \quad \text{and} \quad z = 1 + y_1\,.$$

Then start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system.

```
factor x;
stonormalform(
    { -x(2)*(1+y(1)),
      x(1)*(1+y(1)) },
    { -y(1)+w(1) },
    {},
    4 )$
end;
```

With the above input the procedure analyses the following system:

$$\begin{aligned}
\dot x_1 &= x_2\varepsilon\big(-y_1 - 1\big)\\
\dot x_2 &= x_1\varepsilon\big(y_1 + 1\big)\\
\dot y_1 &= \sigma w_1 - y_1
\end{aligned}$$

This is precisely the original system, but with variables changed as above, and with parameter $\varepsilon = \epsilon$ (here we use the procedure's multiplication by ε to incorporate Radek's ϵ).

Time dependent coordinate transform

$$y_1 = \sigma \mathrm{e}^{-1t}\star w_1 + Y_1$$
$$x_1 = -\sigma\varepsilon^2 \mathrm{e}^{-1t}\star w_1\, X_1 Y_1 + \sigma\varepsilon \mathrm{e}^{-1t}\star w_1\, X_2 - 1/2\varepsilon^2 X_1 Y_1^2 + \varepsilon X_2 Y_1 + X_1$$
$$x_2 = -\sigma\varepsilon^2 \mathrm{e}^{-1t}\star w_1\, X_2 Y_1 - \sigma\varepsilon \mathrm{e}^{-1t}\star w_1\, X_1 - 1/2\varepsilon^2 X_2 Y_1^2 - \varepsilon X_1 Y_1 + X_2$$

Result normal form DEs In such linear systems, the following normal form is straightforward.

$$\begin{aligned}
\dot Y_1 &= -Y_1 \\
\dot X_1 &= -\sigma\varepsilon w_1 X_2 - \varepsilon X_2 \\
\dot X_2 &= \sigma\varepsilon w_1 X_1 + \varepsilon X_1
\end{aligned}$$

- As expected, $Y_1 = 0$ is the emergent stochastic slow manifold.
- The slow $\vec X$ evolution clearly oscillates in (X_1, X_2), $X_j \propto e^{i\theta}$, with phase angle $\theta = \varepsilon(t + \sigma W(t,\omega))$, recalling $W = \int w\,dt$. This phase grows linearly with a superposed random walk.
- The time-dependent coordinate transform maps these predictions back into the $\vec x, y_1$-plane, and thence to the original xyz-space.
- In this system, higher-order terms in ε only affect the coordinate transform, they do not change the evolution of $\vec X$.

1.7 `linearHyper`: simple linear hyperbolic noisy system

The procedure also analyses hyperbolic systems, and recovers the classic stochastic/non-autonomous results guaranteed by the Hartman–Grobman Theorem. Consider the following linear SDEs with one stable variable, and one unstable variable:

$$\begin{aligned}
\dot y_1 &= -y_1 + \sigma w_1 z_1 \\
\dot z_1 &= z_1 + \sigma w_1 y_1
\end{aligned}$$

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system: the parameter σ is automatically inserted by the procedure.

```
stonormalform(
    {},
    { -y(1)+z(1)*w(1) },
    { +z(1)+y(1)*w(1) },
    3 )$
end;
```

Time dependent coordinate transform This simply mixes Y, Z a little depending upon the noise.

$$\begin{aligned}
z_1 &= -\sigma \mathrm{e}^{2t}\star w_1\, Y_1 + Z_1 \\
y_1 &= \sigma \mathrm{e}^{-2t}\star w_1\, Z_1 + Y_1
\end{aligned}$$

Result normal form DEs In such linear systems the normal form is straightforward, as follows.

$$\dot Z_1 = \sigma^2 e^{-2t}\star w_1\, w_1 Z_1 + Z_1$$
$$\dot Y_1 = -\sigma^2 e^{2t}\star w_1\, w_1 Y_1 - Y_1$$

The Y, Z variables are decoupled. Their evolution retains effects from noise-noise interactions: Z from the past history; and Y from future anticipation.

1.8 `foliateHyper`: Duan's hyperbolic system for foliation

To illustrate a stochastic/non-autonomous Hartman–Grobman Theorem, Sun et al. (2011) used the following simple hyperbolic system with one stable variable, and one unstable variable:

$$\dot y_1 = -y_1 + \sigma w_1 y_1$$
$$\dot z_1 = z_1 + y_1^2 + \sigma w_1 z_1$$

The stable y-dynamics is simply an Ornstein–Uhlenbeck process, independent of $z(t)$. The unstable z-dynamics is similar, but with a quadratic forcing by the stable variable y. Let's unfold this effect.

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system: the parameter σ is automatically inserted by the procedure.

```
stonormalform(
    {},
    { -y(1)+y(1)*w(1) },
    { +z(1)+y(1)^2+z(1)*w(1) },
    9 )$
end;
```

In the procedure, the y_1^2 term is automatically multiplied by ε, and so, in the results, ε counts the order of nonlinearity of each term. We analyse to high-order, errors $\mathcal{O}(\varepsilon^9, \sigma^3)$, because the results are simple.

Time dependent coordinate transform To decouple the stochastic dynamics, we just need to stochastically 'bend' the z-variable. This bending forms a stochastic foliation of the system.

$$z_1 = -1/3\sigma\varepsilon e^{3t}\star w_1\, Y_1^2 - 1/3\varepsilon Y_1^2 + Z_1$$
$$y_1 = Y_1$$

Result normal form DEs The normal form dynamics is linear and decoupled, as per Hartman–Grobman, namely

$$\dot Z_1 = \sigma w_1 Z_1 + Z_1 \qquad \dot Y_1 = \sigma w_1 Y_1 - Y_1$$

1.9 `monahanFour`: Monahan's fourth example with 'three' time scales

Monahan & Culina (2011) discuss stochastic averaging and give four examples in an appendix which we also analyse—here we analyse the fourth example. They really need this approach as "a large separation often does not exist in atmosphere or ocean dynamics" between the fast and slow time scales.

Monahan & Culina (2011) comment that this, their fourth example, a linear system, has three time scales. But I do not see these time scales, I only see varying strength interactions. They consider

$$\frac{dx}{dt} = -x + \frac{a}{\sqrt{\tau}}y \quad\text{and}\quad \frac{dy}{dt} = \frac{1}{\sqrt{\tau}}x - \frac{1}{\tau}y + \frac{b}{\sqrt{\tau}}\dot W.$$

Let's rescale time, $t = \tau t'$ so that $d/dt = \frac{1}{\tau}d/dt'$ and $\dot W = \frac{1}{\sqrt{\tau}}dW/dt'$. Then, dropping dashes, the SDE system is

$$\frac{dx}{dt} = -\tau x + a\sqrt{\tau}y \quad\text{and}\quad \frac{dy}{dt} = \sqrt{\tau}x - y + b\dot W.$$

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system. Using the default inbuilt parametrisation of noise by `sigma` to represent parameter b, and using `small` in the x-SDE so that it counts the number of small $\sqrt{\tau}$, code these as the following.

```
factor tau,yy,y,w,ou;
stonormalform(
    { sqrt(tau)*a*y(1)-small*tau*x(1) },
    { sqrt(tau)*x(1)-y(1)+w(1) },
    {},
    4 )$
end;
```

The procedure reports that it analyses the following family

$$\dot x_1 = \sqrt{\tau}y_1\varepsilon a - \varepsilon^2\tau x_1 \qquad \dot y_1 = w_1\sigma - y_1 + \sqrt{\tau}\varepsilon x_1$$

in which we indeed see ε only in the grouping $\varepsilon\sqrt{\tau}$.

Time dependent coordinate transform This is linear as the system is linear.

$$y_1 = -e^{-1t}\star e^{-1t}\star w_1\,\sigma\varepsilon^2\tau a - e^{-1t}\star w_1\,\sigma\varepsilon^2\tau a + e^{-1t}\star w_1\,\sigma + Y_1 + \sqrt{\tau}\varepsilon X_1$$

$$x_1 = -\sqrt{\tau}e^{-1t}\star w_1\,\sigma\varepsilon a - \sqrt{\tau}Y_1\varepsilon a + X_1$$

Result normal form DEs The normal form dynamics is linear and decoupled, as per Hartman–Grobman, namely

$$\begin{aligned}
\dot{Y}_1 &= -Y_1\varepsilon^2\tau a - Y_1 \\
\dot{X}_1 &= w_1\sigma\varepsilon^3\tau\sqrt{\tau}\big(-2a^2 + a\big) + \sqrt{\tau}w_1\sigma\varepsilon a + \varepsilon^2\tau\big(X_1 a - X_1\big)
\end{aligned}$$

Monahan & Culina (2011) derive the last two terms in the X-equation, but not the first as it is too small for their averaging analysis. They comment that $a > 1$ is some sort of difficulty, presumably because X grows when $a > 1$: but here we have no problem with $a > 1$, especially as the decay rate to the stochastic slow manifold, the Y-SDE, is $(1 + \tau a)$ which gets stronger with increasing parameter a.

1.10 `monahanOne`: Monahan's first is a simple nonlinear example

Monahan & Culina (2011) discuss stochastic averaging and give four examples in an appendix which we also analyse—here we analyse the first example. They really need this approach as "a large separation often does not exist in atmosphere or ocean dynamics" between the fast and slow time scales.

With small scale-separation parameter τ, Monahan & Culina (2011) first consider the example

$$\frac{dx}{dt} = -x + \Sigma(x)y \quad\text{and}\quad \frac{dy}{dt} = -\frac{1}{\tau}y + \frac{1}{\sqrt{\tau}}\dot{W},$$

for general smooth functions $\Sigma(x)$. Rescale time, $t = \tau t'$ so that $d/dt = \frac{1}{\tau}d/dt'$ and $\dot{W} = \frac{1}{\sqrt{\tau}}dW/dt'$. Then, dropping dashes, the SDE is

$$\frac{dx}{dt} = -\tau x + \tau\Sigma(x)y \quad\text{and}\quad \frac{dy}{dt} = -y + \dot{W}.$$

Start by loading the procedure.

```
in_tex "../stoNormForm.tex"$
```

Execute the construction of a normal form for this system. Here let's restrict the general function $\Sigma(x)$ to the rational form $\Sigma(x) := (a_0 + a_1x + a_2x^2)/(1 + b_1x + b_2x^2)$. Code this form as the following (multiply through by the denominator).

```
factor tau,yy,y,w,ou;
operator a; defindex a(down);
operator b; defindex b(down);
stonormalform(
    { -tau*x(1)*(1+b(1)*x(1)+b(2)*x(1)^2)
      -(b(1)*x(1)+b(2)*x(1)^2)*df(x(1),t)
      +tau*y(1)*(a(0)+a(1)*x(1)+a(2)*x(1)^2) },
    { -y(1)+w(1) },
    {},
    3 )$
end;
```

The procedure reports that it analyses the following family

$$\dot x_1 = \sqrt{\tau} y_1 \varepsilon a - \varepsilon^2 \tau x_1 \qquad \dot y_1 = w_1 \sigma - y_1 + \sqrt{\tau} \varepsilon x_1$$

in which we indeed see ε only in the grouping $\varepsilon\sqrt{\tau}$.

Time dependent coordinate transform This is linear as the system is linear.

$$y_1 = -\mathrm{e}^{-1t}\star\mathrm{e}^{-1t}\star w_1\, \sigma\varepsilon^2\tau a - \mathrm{e}^{-1t}\star w_1\, \sigma\varepsilon^2\tau a + \mathrm{e}^{-1t}\star w_1\, \sigma + Y_1 + \sqrt{\tau}\varepsilon X_1$$
$$x_1 = -\sqrt{\tau}\mathrm{e}^{-1t}\star w_1\, \sigma\varepsilon a - \sqrt{\tau} Y_1 \varepsilon a + X_1$$

Result normal form DEs The normal form dynamics is linear and decoupled, as per Hartman–Grobman, namely

$$\dot Y_1 = -Y_1\varepsilon^2\tau a - Y_1$$
$$\dot X_1 = w_1\sigma\varepsilon^3\tau\sqrt{\tau}\big(-2a^2 + a\big) + \sqrt{\tau} w_1 \sigma\varepsilon a + \varepsilon^2\tau\big(X_1 a - X_1\big)$$

Monahan & Culina (2011) derive the last two terms in the X-equation, but not the first as it is too small for their averaging analysis. They comment that $a > 1$ is some sort of difficulty, presumably because X grows when $a > 1$: but here we have no problem with $a > 1$, especially as the decay rate to the stochastic slow manifold, the Y-SDE, is $(1 + \tau a)$ which gets stronger with increasing parameter a.

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