Examples of invariant manifold construction

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1 Slow invariant manifolds

1.1 Slow manifold of a simple 2D system

The example system to analyse is specified to be

$$\dot u_1 = -u_1 + u_2 - u_1^2\,, \quad \dot u_2 = u_1 - u_2 + u_2^2\,.$$

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

Execute the construction of the slow manifold for this system.

```
invariantmanifold(
    mat((-u1+u2-u1^2,u1-u2+u2^2)),
    mat((0)),
    mat((1,1)),
    mat((1,1)),
    5)$
end;
```

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We seek the slow manifold so specify the eigenvalue zero. From the linearisation matrix $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ a corresponding eigenvector is $\vec e = (1,1)$, and corresponding left-eigenvector is $\vec z = \vec e = (1,1)$, as specified. The last parameter specifies to construct the slow manifold to errors $\mathcal{O}(\varepsilon^5)$.

The procedure actually analyses the embedding system

$$\dot u_1 = -u_1 + u_2 - \varepsilon u_1^2\,, \quad \dot u_2 = u_1 - u_2 + \varepsilon u_2^2\,.$$

So here the artificial parameter ε has a physical interpretation in that it counts the nonlinearity: a term in ε^p will be a $(p+1)$th order term in $\vec u = (u_1, u_2)$. Hence the specified error $\mathcal{O}(\varepsilon^5)$ is here the same as error $\mathcal{O}(|\vec u|^6)$.

The slow manifold The constructed slow manifold is, in terms of the parameter s_1 (and reverse ordering!),

$$\begin{aligned} u_1 &= 3/8\varepsilon^3 s_1^4 - 1/2\varepsilon s_1^2 + s_1\,, \\ u_2 &= -3/8\varepsilon^3 s_1^4 + 1/2\varepsilon s_1^2 + s_1\,. \end{aligned}$$

Slow manifold ODEs On this slow manifold the evolution is

$$\dot s_1 = -3/4\varepsilon^4 s_1^5 + \varepsilon^2 s_1^3\ :$$

here the leading term in s_1^3 indicates the origin is unstable.

Normals to isochrons at the slow manifold To project initial conditions onto the slow manifold, or non-autonomous forcing, or modifications of the original system, or to quantify uncertainty, use the projection defined by the derived vector

$$\vec z_1 = \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} 3/2\varepsilon^4 s_1^4 + 3/4\varepsilon^3 s_1^3 - 1/2\varepsilon^2 s_1^2 - 1/2\varepsilon s_1 + 1/2 \\ 3/2\varepsilon^4 s_1^4 - 3/4\varepsilon^3 s_1^3 - 1/2\varepsilon^2 s_1^2 + 1/2\varepsilon s_1 + 1/2 \end{bmatrix}.$$

Evaluate these at $\varepsilon = 1$ to apply to the original specified system, or here just interpret ε as a way to count the order of each term.

2 Oscillations on the invariant manifolds

2.1 Oscillatory centre manifold—separated form

Let's try complex eigenvectors. Adjoint eigenvectors `zz_` must be the eigenvectors of the complex conjugate transpose matrix.

$$\begin{aligned} \dot u_1 &= u_2 \\ \dot u_2 &= -\varepsilon u_3 u_1 - u_1 \\ \dot u_3 &= 5\varepsilon u_1^2 - u_3 \end{aligned}$$

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j and the complex exponential

```
factor s,cis;
```

Execute the construction of the centre manifold for this system.

```
invariantmanifold(
    mat((u2,-u1-u1*u3,-u3+5*u1^2)),
    mat((i,-i)),
    mat((1,+i,0),(1,-i,0)),
    mat((1,+i,0),(1,-i,0)),
    3)$
end;
```

The centre manifold These give the location of the invariant manifold in terms of parameters s_j.

$$
\begin{aligned}
u_1 &= e^{-ti}s_2 + e^{ti}s_1 \\
u_2 &= -e^{-ti}s_2 i + e^{ti}s_1 i \\
u_3 &= e^{-2ti}s_2^2\varepsilon\big(2i+1\big) + e^{2ti}s_1^2\varepsilon\big(-2i+1\big) + 10s_2s_1\varepsilon
\end{aligned}
$$

Centre manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$
\begin{aligned}
\dot s_1 &= s_2s_1^2\varepsilon^2\big(11/2i+1\big) \\
\dot s_2 &= s_2^2s_1\varepsilon^2\big(-11/2i+1\big)
\end{aligned}
$$

2.2 Quasi delay DE with Hopf bifurcation

Shows Hopf bifurcation as parameter α crosses 0 to oscillations with base frequency two.

$$
\begin{aligned}
\dot u_1 &= -\alpha\varepsilon^2u_3 - \varepsilon^2u_1^3 - 2\varepsilon u_1^2 - 4u_3 \\
\dot u_2 &= 2u_1 - 2u_2 \\
\dot u_3 &= 2u_2 - 2u_3
\end{aligned}
$$

for small parameter α. We code the parameter α as 'small', and observe it is consequently considered as 'small squared' because all nonlinear terms and already 'small' terms, are multiplied by another `small`.

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameter α.

```
factor s,cis,alpha;
```

Execute the construction of the slow manifold for this system (ignore the warning messages about `u1` declared, and then already defined, as an operator).

```
invariantmanifold(
    mat(( -4*u3-small*alpha*u3-2*u1^2-small*u1^3,
        2*u1-2*u2,
        2*u2-2*u3 )),
    mat((2*i,-2*i)),
    mat((1,1/2-i/2,-i/2),(1,1/2+i/2,+i/2)),
    mat((1,-i,-1-i),(1,+i,-1+i)),
    3)$
end;
```

The centre manifold These give the location of the invariant manifold in terms of parameters s_1, s_2 (complex conjugate for real solutions).

$$\begin{aligned}
u_1 &= e^{-4ti}s_2^2\varepsilon\big(-7/12i+1/12\big) + e^{-2ti}s_2 \\
&\quad + e^{4ti}s_1^2\varepsilon\big(7/12i+1/12\big) + e^{2ti}s_1 - s_2s_1\varepsilon \\
u_2 &= e^{-4ti}s_2^2\varepsilon\big(-1/12i+1/4\big) + e^{-2ti}s_2\big(1/2i+1/2\big) \\
&\quad + e^{4ti}s_1^2\varepsilon\big(1/12i+1/4\big) + e^{2ti}s_1\big(-1/2i+1/2\big) - s_2s_1\varepsilon \\
u_3 &= e^{-4ti}s_2^2\varepsilon\big(1/12i+1/12\big) + 1/2\,e^{-2ti}s_2i \\
&\quad + e^{4ti}s_1^2\varepsilon\big(-1/12i+1/12\big) - 1/2\,e^{2ti}s_1i - s_2s_1\varepsilon
\end{aligned}$$

Centre manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\begin{aligned}
\dot s_1 &= s_2s_1^2\varepsilon^2\big(-16/15i-1/5\big) + s_1\alpha\varepsilon^2\big(1/5i+1/10\big) \\
\dot s_2 &= s_2^2s_1\varepsilon^2\big(16/15i-1/5\big) + s_2\alpha\varepsilon^2\big(-1/5i+1/10\big)
\end{aligned}$$

Hence there is a supercritical Hopf bifurcation as parameter α increases through zero.

3 Invariant manifolds in delay DEs

3.1 Simple DDE with a Hopf bifurcation

Model a delayed 'logistic' system in one variable with

$$\frac{du}{dt} = -(1+a)[1+u(t)]u(t-\pi/2),$$

for small parameter a. We code the parameter a as 'small', and observe it is consequently considered as 'small squared' because all nonlinear terms and already 'small' terms are multiplied by `small`.

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameter a.

```
factor s,cis,a;
```

Execute the construction of the slow manifold for this system (ignore the warning messages about `u1` declared, and then already defined, as an operator).

```
invariantmanifold(
    mat(( -(1+small*a)*(1+u1)*u1(pi/2) )),
    mat((i,-i)),
    mat((1),(1)),
    mat((1),(1)),
    3)$
end;
```

The marginal modes are $e^{\pm it}$ so nominate the frequencies ± 1. The eigenvectors are just $1\cdot e^{\pm it}$. Because for delay differential equations the time dependence $e^{\pm i\omega t}$ is an integral part of the definition of the eigenvector; hence the coded eigenvectors can be the same, as here, because they are differentiated through the time dependence $e^{\pm i\omega t}$.

The code works for orders higher than cubic, but is slow: takes about a minute per iteration.

The procedure actually analyses the embedding system

$$\frac{du}{dt} = -[1+\varepsilon u(t)]u(t-\pi/2) - \varepsilon^2 a[1+u(t)]u(t-\pi/2).$$

The centre manifold These give the location of the invariant manifold in terms of parameters s_j.

$$\begin{aligned} u_1 = {} & e^{-2ti}s_2^2\varepsilon\big(1/5i+2/5\big) + e^{-ti}s_2 \\ & + e^{2ti}s_1^2\varepsilon\big(-1/5i+2/5\big) + e^{ti}s_1 \end{aligned}$$

Centre manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\begin{aligned} \dot s_1 &= s_2 s_1^2\varepsilon^2\big(-2/5i\pi - 12/5i - 6/5\pi + 4/5\big)/\big(\pi^2+4\big) \\ &\quad + s_1 a\varepsilon^2\big(4i+2\pi\big)/\big(\pi^2+4\big) \\ \dot s_2 &= s_2^2 s_1\varepsilon^2\big(2/5i\pi + 12/5i - 6/5\pi + 4/5\big)/\big(\pi^2+4\big) \\ &\quad + s_2 a\varepsilon^2\big(-4i+2\pi\big)/\big(\pi^2+4\big) \end{aligned}$$

3.2 Logistic DDE displays a Hopf bifurcation

Form a centre manifold for the delayed 'logistic' system in one variable, for delay $\tau = 3\pi/4$, with

$$\frac{du}{dt} = -u(t) - (\sqrt{2}+a)u(t-\tau) + \mu u(t-\tau)^2 + \nu u(t-\tau)^3,$$

for and nonlinearity parameters μ and ν, and small parameter a. Numerical computation of the spectrum indicates that the system has a Hopf bifurcation as parameter a crosses zero.

We code the parameter a as 'small', and observe it is consequently considered as 'small squared' because all nonlinear terms, and already 'small' terms, are multiplied by ε (`small`).

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameters.

```
factor s,cis,a,mu,nu;
```

Execute the construction of the slow manifold for this system (ignore the warning messages about `u1` declared, and then already defined, as an operator).

```
invariantmanifold(
    mat(( -u1-(sqrt(2)+small*a)*u1(3*pi/4)
    +mu*u1(3*pi/4)^2 +small*nu*u1(3*pi/4)^3 )),
    mat((i,-i)),
    mat((1),(1)),
    mat((1),(1)),
    3)$
end;
```

The marginal modes are $e^{\pm it}$ so nominate the frequencies ± 1. The eigenvectors are just $1 \cdot e^{\pm it}$. Because for delay differential equations the time dependence $e^{\pm i\omega t}$ is an integral part of the definition of the eigenvector; hence the coded eigenvectors can be the same, as here, because they are differentiated through the time dependence $e^{\pm i\omega t}$.

The procedure actually analyses the embedding system

$$\dot{u}_1 = -a\varepsilon^2 u_1(t-\tau) + \mu\varepsilon u_1(t-\tau)^2 + \nu\varepsilon^2 u_1(t-\tau)^3 - \sqrt{2}u_1(t-\tau) - u_1\,.$$

The centre manifold These give the location of the invariant manifold in terms of parameters s_j.

$$\begin{aligned} u_1 &= e^{-2ti} s_2^2 \mu\varepsilon\big(0.2698 - 0.07901\, i\big) + e^{-ti} s_2 \\ &\quad + e^{2ti} s_1^2 \mu\varepsilon(0.2698 + 0.07901\, i) + e^{ti} s_1 \\ &\quad + s_2 s_1 \mu\varepsilon 0.8284 \end{aligned}$$

Centre manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\begin{aligned}
\dot s_1 &= s_2 s_1^2 \mu^2 \varepsilon^2(-0.5209 - 0.1303\, i) \\
&+ s_2 s_1^2 \nu \varepsilon^2(-0.7206 - 0.1262\, i) \\
&+ s_1 a \varepsilon^2(0.2402 + 0.04205\, i) \\
\dot s_2 &= s_2^2 s_1 \mu^2 \varepsilon^2(-0.5209 + 0.1303\, i) \\
&+ s_2^2 s_1 \nu \varepsilon^2(-0.7206 + 0.1262\, i) \\
&+ s_2 a \varepsilon^2(0.2402 - 0.04205\, i)
\end{aligned}$$

Hence the centre manifold model predicts a supercritical Hopf bifurcation as parameter a increases through zero.