Many diverse examples of invariant manifold construction

A. J. Roberts[*]

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Instructions

- Download and install the computer algebra package *Reduce* via `http://www.reduce-algebra.com`
- Navigate to folder `Examples` within folder `InvariantManifold`.
- For each example of interest, start-up *Reduce* and enter the command `in_tex "filename.tex"$` where filename is the root name of the example (as listed in the following table of contents).

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[*] School of Mathematical Sciences, University of Adelaide, South Australia 5005, Australia. `https://profajroberts.github.io/`

1 Five representative examples

1.1 simple3d: Slow manifold of a basic 3D system

The basic example system to analyse for a slow manifold is

$$\begin{aligned}\dot u_1 &= 2u_1 + u_2 + 2u_3 + u_2u_3\,,\\ \dot u_2 &= u_1 - u_2 + u_3 - u_1u_3\,,\\ \dot u_3 &= -3u_1 - u_2 - 3u_3 - u_1u_2\,,\end{aligned}$$

(Section 1.5 constructs its stable manifold).

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

Execute the construction of the slow manifold for this system.

```
invariantmanifold(
    mat(( 2*u1+u2+2*u3+u2*u3,
          u1-u2+u3-u1*u3,
          -3*u1-u2-3*u3-u1*u2 )),
    mat((0)),
    mat((1,0,-1)),
    mat((4,1,3)),
    3 )$
end;
```

The matrix $\begin{bmatrix} 2 & 1 & 2\\ 1 & -1 & 1\\ -3 & -1 & -3\end{bmatrix}$ of the linearisation about the origin has eigenvalues zero and -1 (multiplicity two). We seek the slow manifold so specify the eigenvalue zero in the second parameter to the procedure. A corresponding eigenvector is $\vec e = (1, 0, -1)$, and corresponding left-eigenvector is $\vec z = (4, 1, 3)$, as specified above. The last parameter, 3, specifies to construct the slow manifold to errors $\mathcal{O}\big(\varepsilon^3\big)$.

The procedure actually analyses the embedding system

$$\begin{aligned}\dot u_1 &= 2u_1 + u_2 + 2u_3 + \varepsilon u_2u_3\,,\\ \dot u_2 &= u_1 - u_2 + u_3 - \varepsilon u_1u_3\,,\\ \dot u_3 &= -3u_1 - u_2 - 3u_3 - \varepsilon u_1u_2\,.\end{aligned}$$

Consequently, here the artificial parameter ε has a physical interpretation in that it counts the nonlinearity: a term in ε^p will be a $(p+1)$th order term in $\vec u = (u_1, u_2, u_3)$. Hence the specified error $\mathcal{O}\big(\varepsilon^3\big)$ is here the same as error $\mathcal{O}\big(|\vec u|^4\big)$ and $\mathcal{O}\big(|\vec s|^4\big)$.

The slow manifold The constructed slow manifold is, in terms of the parameter s_1 (to error $\mathcal{O}\big(\varepsilon^2\big)$, and reverse ordering!),

$$u_1 = -\varepsilon s_1^2 + s_1\,, \qquad u_2 = \varepsilon s_1^2\,, \qquad u_3 = \varepsilon s_1^2 - s_1\,.$$

Slow manifold ODEs On this slow manifold the evolution is

$$\dot{s}_1 = -9\varepsilon^2 s_1^3 + \varepsilon s_1^2\,.$$

Here the leading term in s_1^2 establishes the origin is unstable.[1]

Normals to isochrons at the slow manifold To project initial conditions onto the slow manifold, or non-autonomous forcing, or modifications of the original system, or to quantify uncertainty (Roberts 1989, 2000), use the projection defined by the derived vector

$$\vec{z}_1 = \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} 258\varepsilon^2 s_1^2 - 16\varepsilon s_1 + 4 \\ 93\varepsilon^2 s_1^2 - 9\varepsilon s_1 + 1 \\ 240\varepsilon^2 s_1^2 - 16\varepsilon s_1 + 3 \end{bmatrix}.$$

Evaluate these at $\varepsilon = 1$ to apply to the original specified system, or here just interpret ε as a way to count the order of each term.

1.2 `doubleHopfDDE`: Double Hopf interaction in a 2D DDE

Erneux (2009) [§7.2] explored an example of a laser subject to optoelectronic feedback, coded as a delay differential equation. For certain parameter values it has a two frequency Hopf bifurcation. Near Erneux's parameters $(\eta, \theta) = (3/5, 2)$, the system may be represented as

$$\begin{aligned} \dot{u}_1 &= -4(1+\delta)^2\left[\tfrac{5}{8}u_2 + \tfrac{3}{8}u_2(t-\pi)\right] \\ \dot{u}_2 &= u_1(1+u_2). \end{aligned}$$

for small parameter δ. Due to the delay, $u_2(t-\pi)$, this system is effectively an infinite-dimensional dynamical system. Here we describe the emergent dynamics on its four-dimensional centre manifold.

The linearisation of this system at the origin has modes with frequencies $\omega = 1, 2$, and corresponding eigenvectors $(1, \mp i/\omega)e^{\pm i\omega t}$. Corresponding eigenvectors of the adjoint are $(1, \mp i\omega)e^{\pm i\omega t}$. We model the nonlinear interaction of these four modes over long times.

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

But turn off `gcd` as it wrecks this code for some unknown reason.

```
off gcd,ezgcd;
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameter δ.

```
factor s,delta,exp;
```

[1] Then the large negative s_1^3 term *suggests* the existence of a finite amplitude equilibrium with $s_1 \approx 1/9$ (it is actually closer to $s_1 \approx 0.2$).

Execute the construction of the slow manifold for this system, where `u2(pi)` denotes the delayed variable $u_2(t-\pi)$, and where `1+small*delta` reflects that we wish to use the 'small' parameter δ to explore regimes where this factor is near the value 1.

```
invariantmanifold(
    mat(( -4*(1+small*delta)^2*(5/8*u2 +3/8*u2(pi)),
          +u1*(1+u2) )),
    mat(( i,-i,2*i,-2*i )),
    mat( (1,-i), (1,+i), (1,-i/2), (1,+i/2) ),
    mat( (1,-i), (1,+i), (1,-2*i), (1,+2*i) ),
    3 )$
end;
```

The code works for errors of order higher than cubic, but is much slower: takes several minutes per iteration.

The procedure actually analyses the embedding system

$$\begin{aligned}\dot u_1 &= -4(1+2\varepsilon^2\delta+\varepsilon^3\delta^2)\big[\tfrac58 u_2+\tfrac38 u_2(t-\pi)\big]\\ \dot u_2 &= u_1(1+\varepsilon u_2).\end{aligned}$$

The centre manifold These give the location of the invariant manifold in terms of parameters s_j. Here, $u_1 \approx s_1e^{it}+s_2e^{-it}+s_3e^{i2t}+s_4e^{-i2t}$ so that (for real solutions) s_1, s_2 are complex conjugate amplitudes that modulate the oscillations of frequency $\omega=1$, whereas s_3, s_4 are complex conjugate amplitudes that modulate the oscillations of frequency $\omega=2$.

$$\begin{aligned}u_1 = {} & \mathrm{e}^{-it}s_4s_1\varepsilon\big(0.2309i-0.04495\big)+\mathrm{e}^{-it}s_2+0.1667\,\mathrm{e}^{-4it}s_4^2\varepsilon i+\\ & 0.1875\,\mathrm{e}^{-3it}s_4s_2\varepsilon i+\mathrm{e}^{-2it}s_4+\mathrm{e}^{-2it}s_2^2\varepsilon\big(-0.3953i-0.1233\big)+\\ & \mathrm{e}^{it}s_3s_2\varepsilon\big(-0.2309i-0.04495\big)+\mathrm{e}^{it}s_1-0.1667\,\mathrm{e}^{4it}s_3^2\varepsilon i-\\ & 0.1875\,\mathrm{e}^{3it}s_3s_1\varepsilon i+\mathrm{e}^{2it}s_3+\mathrm{e}^{2it}s_1^2\varepsilon\big(0.3953i-0.1233\big)\end{aligned}$$

$$\begin{aligned}u_2 = {} & \mathrm{e}^{-it}s_4s_1\varepsilon\big(0.04495i+0.2309\big)+\mathrm{e}^{-it}s_2i-0.1667\,\mathrm{e}^{-4it}s_4^2\varepsilon-\\ & 0.5625\,\mathrm{e}^{-3it}s_4s_2\varepsilon+0.5\,\mathrm{e}^{-2it}s_4i+\mathrm{e}^{-2it}s_2^2\varepsilon\big(0.06167i-0.1977\big)+\\ & \mathrm{e}^{it}s_3s_2\varepsilon\big(-0.04495i+0.2309\big)-\mathrm{e}^{it}s_1i-0.1667\,\mathrm{e}^{4it}s_3^2\varepsilon-\\ & 0.5625\,\mathrm{e}^{3it}s_3s_1\varepsilon-0.5\,\mathrm{e}^{2it}s_3i+\mathrm{e}^{2it}s_1^2\varepsilon\big(-0.06167i-0.1977\big)\end{aligned}$$

Centre manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs that characterise how the modulation of the oscillations evolve due to their nonlinear interaction.

$$\begin{aligned}\dot s_1 = {} & s_4s_3s_1\varepsilon^2\big(-0.03089i+0.05032\big)+s_3s_2\varepsilon\big(-0.08991i-0.03816\big)+\\ & s_2s_1^2\varepsilon^2\big(-0.01837i-0.1095\big)+s_1\delta\varepsilon^2\big(0.1526i-0.3596\big)\end{aligned}$$

$$\begin{aligned}\dot s_2 = {} & s_4s_3s_2\varepsilon^2\big(0.03089i+0.05032\big)+s_4s_1\varepsilon\big(0.08991i-0.03816\big)+\\ & s_2^2s_1\varepsilon^2\big(0.01837i-0.1095\big)+s_2\delta\varepsilon^2\big(-0.1526i-0.3596\big)\end{aligned}$$

$$\begin{aligned}\dot s_3 = {} & s_4s_3^2\varepsilon^2\big(-0.0349i-0.04111\big)+s_3s_2s_1\varepsilon^2\big(-0.2499i-\\ & 0.2153\big)+s_3\delta\varepsilon^2\big(0.8376i+0.9867\big)+s_1^2\varepsilon\big(-0.4934i+0.4188\big)\end{aligned}$$

$$\begin{aligned}\dot s_4 = {} & s_4^2s_3\varepsilon^2\big(0.0349i-0.04111\big)+s_4s_2s_1\varepsilon^2\big(0.2499i-0.2153\big)+\\ & s_4\delta\varepsilon^2\big(-0.8376i+0.9867\big)+s_2^2\varepsilon\big(0.4934i+0.4188\big)\end{aligned}$$

1.3 `metastable4`: Metastability in a four state Markov chain

Variable ϵ characterises the rate of exchange between metastable states u_1 and u_4 in this system (Roberts 2015, Exercise 5.1):

$$\begin{aligned}
\dot u_1 &= +u_2 - \epsilon u_1\,,\\
\dot u_2 &= -u_2 + \epsilon(u_3 - u_2 + u_1),\\
\dot u_3 &= -u_3 + \epsilon(u_4 - u_3 + u_2),\\
\dot u_4 &= +u_3 - \epsilon u_4\,.
\end{aligned}$$

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

Execute the construction of the slow manifold for this system. The explicit parameter `small`, math-name ε, gets replaced by `small^2` in the code, so in effect $\varepsilon^2 = \epsilon$.

```
invariantmanifold(
    mat(( u2-small*u1,
         -u2+small*(u1-u2+u3),
         -u3+small*(u2-u3+u4),
          u3-small*u4 )),
    mat((0,0)),
    mat((1,0,0,0),(0,0,0,1)),
    mat((1,1,0,0),(0,0,1,1)),
    6 )$
end;
```

The matrix $\begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$, of the linearisation about $\varepsilon = 0$, has eigenvalues 0 and -1 (both multiplicity two). We seek the slow manifold so specify the two zero eigenvalues in the second parameter to the procedure. Corresponding eigenvectors are $\vec e_1 = (1,0,0,0)$ and $\vec e_2 = (0,0,0,1)$. Choosing corresponding left-vector (here not an eigenvector) is $\vec z_1 = (1,1,0,0)$ and $\vec z_2 = (0,0,1,1)$ means that the slow manifold parameters s_1, s_2 have the physical meaning, respectively, of being the probability that the system is in states $\{1,2\}$ and $\{3,4\}$. The last parameter, 6, specifies to construct the slow manifold to errors $\mathcal{O}(\varepsilon^6)$, that is, errors $\mathcal{O}(\epsilon^3)$.

The slow manifold The constructed slow manifold is, in terms of the lumped-state probability parameters s_1, s_2 (to error $\mathcal{O}(\varepsilon^2)$, and reverse ordering!),

$$u_1 = \varepsilon^4\big(-s_2 + 2s_1\big) - \varepsilon^2 s_1 + s_1\,, \quad u_3 = \varepsilon^4\big(-2s_2 + s_1\big) + \varepsilon^2 s_2\,,$$
$$u_2 = \varepsilon^4\big(s_2 - 2s_1\big) + \varepsilon^2 s_1\,, \quad u_4 = \varepsilon^4\big(2s_2 - s_1\big) - \varepsilon^2 s_2 + s_2\,.$$

Slow manifold ODEs On this slow manifold the evolution of the lumped-state probabilities is

$$\dot s_1 = \varepsilon^4\big(s_2 - s_1\big), \quad \dot s_2 = \varepsilon^4\big(-s_2 + s_1\big).$$

Hence here the long-term evolution is that on a time-scale of $\mathcal{O}\big(1/\epsilon^2\big)$, $\mathcal{O}\big(1/\varepsilon^4\big)$, the system equilibrates between the two lumped states, that is, between $\{1,2\}$ and $\{3,4\}$.

Normals to isochrons at the slow manifold To project initial conditions onto the slow manifold, or non-autonomous forcing, or modifications of the original system, or to quantify uncertainty (Roberts 1989, 2000), use the projection defined by the derived vector

$$\vec z_1 = \begin{bmatrix} z_{11} \\ z_{12} \\ z_{13} \\ z_{14} \end{bmatrix} = \begin{bmatrix} \varepsilon^4+1 \\ 4\varepsilon^4-\varepsilon^2+1 \\ -4\varepsilon^4+\varepsilon^2 \\ -\varepsilon^4 \end{bmatrix}, \quad \vec z_2 = \begin{bmatrix} z_{21} \\ z_{22} \\ z_{23} \\ z_{24} \end{bmatrix} = \begin{bmatrix} -\varepsilon^4 \\ -4\varepsilon^4+\varepsilon^2 \\ 4\varepsilon^4-\varepsilon^2+1 \\ \varepsilon^4+1 \end{bmatrix}.$$

Evaluate all these at $\varepsilon^2=\epsilon$ to apply to the original specified system.

1.4 `nonlinNormModes`: Interaction of nonlinear normal modes

Renson et al. (2012) explored finite element construction of the nonlinear normal modes of a pair of coupled oscillators. Defining two new variables one of their example systems is

$$\begin{aligned} \dot x_1 &= x_3\,, & \dot x_3 &= -2x_1+x_2-\frac12 x_1^3+\frac{3}{10}(-x_3+x_4)\,, \\ \dot x_2 &= x_4\,, & \dot x_4 &= x_1-2x_2+\frac{3}{10}(x_3-2x_4)\,. \end{aligned}$$

The linearisation of this system at the origin has modes with frequencies $\omega=1,\sqrt3$, corresponding eigenvalues $\lambda=\pm i,\pm i\sqrt3$, and corresponding eigenvectors?? $(1,1,\pm i\omega,\pm i\omega)$. Corresponding eigenvectors of the adjoint are $(1,1,\pm i,\pm i)$ and $(\mp i\omega,\pm i\omega,1,-1)$. We model the nonlinear interaction of these four modes over long times.

Here, the analysis constructs a full state space coordinate transformation. We find a mapping from the modulation variables $\vec s=(s_1,s_2,s_3,s_4)$ to the original variables $\vec u=(u_1,u_2,u_3,u_4)$, and find the corresponding evolution of $\vec s$. The modulation variables $\vec s$ are 'slow' because the coordinate transform uses time-dependent (rotating) basis vectors that account for the fast oscillation in $\vec u$. Hence the new variables $\vec s$ are good variables for making long-term predictions and forming understanding.

Start by loading the procedure.

```
33 in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, and the complex exponential.

```
34 factor small;
```

The following code makes the linear damping to be effectively small (which then makes it `small` squared); consequently, also scale the smallness of the cubic nonlinearity to match.

```
invariantmanifold(
    mat(( u3,
          u4,
          -2*u1 +u2 -small*u1^3/2 +small*3/10*(-u3+u4),
          u1  -2*u2 +small*3/10*(u3 -2*u4) )),
    mat(( i,-i,sqrt(3)*i,-sqrt(3)*i )),
    mat( (1,1,+i,+i), (1,1,-i,-i),
         (1,-1,i*sqrt(3),-i*sqrt(3)),
         (1,-1,-i*sqrt(3),i*sqrt(3)) ),
    mat( (1,1,+i,+i), (1,1,-i,-i),
         (-i*sqrt(3),+i*sqrt(3),1,-1),
         (+i*sqrt(3),-i*sqrt(3),1,-1) ),
    3 )$
end;
```

The square root eigenvalues do not cause any trouble (although one may need to reformat the LaTeX of the exp operator). In the model, observe that $s_1 = s_2 = 0$ is invariant, as is $s_3 = s_4 = 0$. These are the nonlinear normal modes.

The procedure actually analyses the embedding system

$$\begin{aligned} \dot u_1 &= u_3\,, \quad \dot u_3 = \varepsilon^2\big(-1/2u_1^3 - 3/10u_3 + 3/10u_4\big) - 2u_1 + u_2\,, \\ \dot u_2 &= u_4\,, \quad \dot u_4 = \varepsilon^2\big(3/10u_3 - 3/5u_4\big) + u_1 - 2u_2\,. \end{aligned}$$

The invariant manifold Here these give the reparametrisation of the state space $\vec u$ in terms of parameters s_j, via rotating basis vectors. Here, the coordinate transform is very complicated so I do not give the complexity. The leading approximation is, of course, the linear, errors $\mathcal{O}\big(\varepsilon^2\big)$,

$$\begin{aligned} u_1 &= \mathrm{e}^{-\sqrt{3}it} s_4 + \mathrm{e}^{-it} s_2 + \mathrm{e}^{\sqrt{3}it} s_3 + \mathrm{e}^{it} s_1 \\ u_2 &= -\mathrm{e}^{-\sqrt{3}it} s_4 + \mathrm{e}^{-it} s_2 - \mathrm{e}^{\sqrt{3}it} s_3 + \mathrm{e}^{it} s_1 \\ u_3 &= -\sqrt{3}\,\mathrm{e}^{-\sqrt{3}it} s_4 i - \mathrm{e}^{-it} s_2 i + \sqrt{3}\,\mathrm{e}^{\sqrt{3}it} s_3 i + \mathrm{e}^{it} s_1 i \\ u_4 &= \sqrt{3}\,\mathrm{e}^{-\sqrt{3}it} s_4 i - \mathrm{e}^{-it} s_2 i - \sqrt{3}\,\mathrm{e}^{\sqrt{3}it} s_3 i + \mathrm{e}^{it} s_1 i \end{aligned}$$

Invariant manifold ODEs The system evolves according to these ODEs that characterise how the modulation of the oscillations evolve in state space due to their nonlinear interaction.

$$\begin{aligned} \dot s_1 &= 3/4 s_4 s_3 s_1 \varepsilon^2 i + 3/8 s_2 s_1^2 \varepsilon^2 i - 3/40 s_1 \varepsilon^2 \\ \dot s_2 &= -3/4 s_4 s_3 s_2 \varepsilon^2 i - 3/8 s_2^2 s_1 \varepsilon^2 i - 3/40 s_2 \varepsilon^2 \\ \dot s_3 &= 1/8\sqrt{3} s_4 s_3^2 \varepsilon^2 i + 1/4\sqrt{3} s_3 s_2 s_1 \varepsilon^2 i - 3/8 s_3 \varepsilon^2 \\ \dot s_4 &= -1/8\sqrt{3} s_4^2 s_3 \varepsilon^2 i - 1/4\sqrt{3} s_4 s_2 s_1 \varepsilon^2 i - 3/8 s_4 \varepsilon^2 \end{aligned}$$

Here one can see that each oscillation decays, with a frequency shift due to a combination of nonlinear interaction and nonlinear self-interaction.

1.5 `stable3d`: Stable manifold of a basic 3D system

Let's revisit the example of Section 1.1, namely

$$\begin{aligned}
\dot u_1 &= 2u_1 + u_2 + 2u_3 + u_2u_3\,, \\
\dot u_2 &= u_1 - u_2 + u_3 - u_1u_3\,, \\
\dot u_3 &= -3u_1 - u_2 - 3u_3 - u_1u_2\,,
\end{aligned}$$

but here construct its 2D stable manifold.

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

Execute the construction of the stable manifold for this system.

```
invariantmanifold(
    mat(( 2*u1+u2+2*u3+u2*u3,
          u1-u2+u3-u1*u3,
          -3*u1-u2-3*u3-u1*u2 )),
    mat(( -1,-1 )),
    mat( (1,-1,-1),(0.4,1.4,-1) ),
    mat( (1,0,1),(1,0,-1) ),
    3 )$
end;
```

The matrix $\begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 1 \\ -3 & -1 & -3 \end{bmatrix}$ of the linearisation about the origin has eigenvalues 0 and -1 (multiplicity two). We seek the 2D stable manifold so specify the eigenvalue -1, twice, in the second parameter to the procedure. A corresponding eigenvector is $\vec e_1 = (1,-1,-1)$, and corresponding left-eigenvector is $\vec z_2 = (1,0,1)$, as specified above. We need two basis eigenvectors, but here there is only one because the other is a generalised eigenvector. We must do more work to find a generalised eigenvector is $\vec e_2 = (0.4, 1.4, -1)$, and a generalised left-eigenvector is $\vec z_2 = (1,0,-1)$. The last parameter, 3, specifies to construct the stable manifold to errors $\mathcal{O}(\varepsilon^3)$.

Because of the generalised eigenvector, the procedure modifies the *linear* terms to a more convenient form (not necessary, just *convenient*)—see the warning in its report. So, the procedure actually analyses the embedding system

$$\begin{aligned}
\dot u_1 &= \varepsilon\big(- u_1 + u_2u_3 - u_3\big) + 3u_1 + u_2 + 3u_3\,, \\
\dot u_2 &= \varepsilon\big(- u_1u_3 + u_1 + u_3\big) - u_2\,, \\
\dot u_3 &= \varepsilon\big(- u_1u_2 + u_1 + u_3\big) - 4u_1 - u_2 - 4u_3\,.
\end{aligned}$$

The stable manifold The constructed stable manifold is, in terms of the parameters s_1, s_2 (to error $\mathcal{O}(\varepsilon^2)$, and reverse ordering!,

$$\begin{aligned}
u_1 &= \varepsilon\big(-51/25\,\mathrm{e}^{-2t}s_2^2 - 6/5\,\mathrm{e}^{-2t}s_2s_1 + 3\,\mathrm{e}^{-2t}s_1^2\big) + 2/5\,\mathrm{e}^{-t}s_2 + \mathrm{e}^{-t}s_1\,, \\
u_2 &= \varepsilon\big(- 2/5\,\mathrm{e}^{-2t}s_2^2 - 7/5\,\mathrm{e}^{-2t}s_2s_1 - \mathrm{e}^{-2t}s_1^2\big) + 7/5\,\mathrm{e}^{-t}s_2 - \mathrm{e}^{-t}s_1\,,
\end{aligned}$$

$$u_3 = \varepsilon\big(4\,\mathrm{e}^{-2t}s_2^2 + 13/5\,\mathrm{e}^{-2t}s_2s_1 - 5\,\mathrm{e}^{-2t}s_1^2\big) - \mathrm{e}^{-t}s_2 - \mathrm{e}^{-t}s_1\,.$$

Observe the linear terms in $\vec s$ all have e^{-t}, and the quadratic terms in $\vec s$ all have e^{-2t}. Consequently, we could in principle write the stable manifold in terms of, say, the variables $x_j = s_je^{-t}$ giving

$$\begin{aligned}
u_1 &= \varepsilon\big(- 51/25x_2^2 - 6/5x_2x_1 + 3x_1^2\big) + 2/5x_2 + x_1\,,\\
u_2 &= \varepsilon\big(- 2/5x_2^2 - 7/5x_2x_1 - x_1^2\big) + 7/5x_2 - x_1\,,\\
u_3 &= \varepsilon\big(4x_2^2 + 13/5x_2x_1 - 5x_1^2\big) - x_2 - x_1\,.
\end{aligned}$$

This would be a more usual parametrisation. But here let's remain with $\vec s$ and remember to interpret $\vec s$ as modifying the exponential decay e^{-t} on this stable manifold.

Stable manifold ODEs On the stable manifold the evolution is

$$\dot s_1 = 3/5\varepsilon s_2\,,\quad \dot s_2 = 0\,.$$

So, s_2 is constant, and hence s_1 increases linearly. But such increase only modifies slightly the robust exponential decay, e^{-t}, on the stable manifold.

In terms of $\vec x$ this evolution is $\dot x_1 = -x_1 + \frac{3}{5}\varepsilon x_2\,,\ \dot x_2 = -x_2\,.$

2 Harmonically forced systems

2.1 `marcusYamabe`: Discover Marcus–Yamabe instability

In nonautonomous systems, such as $\dot{\vec u} = L(t)\vec u$, just because eigenvalues of $L(t)$ have real-part negative, for all t, does not mean that all solutions $\vec u(t)$ decay. Here consider the Marcus–Yamabe system (Chicone 2006, p.197)

$$\frac{d\vec u}{dt} = L(t)\vec u \quad \text{for } L := \begin{bmatrix} -1+\frac32\varepsilon^2\cos^2 t & 1-\frac32\varepsilon^2\sin t\,\cos t \\ -1-\frac32\varepsilon^2\sin t\,\cos t & -1+\frac32\varepsilon^2\sin^2 t\end{bmatrix}. \tag{1}$$

For example, for $\varepsilon = 1$, the eigenvalues of $L(t)$ are $\frac14(-1\pm\sqrt7 i)$ (independent of time). Despite the eigenvalues having negative real-part, there are growing solutions $\vec u = (-\cos t, \sin t)e^{t/2}$.

Here analyse the system with the late-2022 version of `invariantManifold.tex` that caters for sinusoidal non-autonomous coefficients and forcing.

```
in_tex "../invariantManifold.tex"$
factor small;
```

Encode the system with `small` $= \varepsilon$. We find instability predicted when $\frac32\varepsilon^2 > 1$; that is, $|\varepsilon| > 0.8165$; for example, $\varepsilon = 1$ as commented above. Then the induced growth of complex amplitudes s_1 and s_2 overcomes the e^{-t} decay that is in $u_1 = e^{(-1+i)t}s_1 + e^{(-1-i)t}s_2$.

```
invariantmanifold(
    mat((-u1+u2 +small*( 3/2*cos(t)^2*u1 -3/2*cos(t)*sin(t)*u2),
         -u1-u2 +small*(-3/2*cos(t)*sin(t)*u1 +3/2*sin(t)^2*u2)
         )),
    mat(( -1+i, -1-i )),
    mat( (1,i), (1,-i) ),
    mat( (1,i), (1,-i) ),
    9)$
end;
```

The function finds the following exact time-dependent transformation of this linear system. These parameterise state space in terms of s_j:

$$u_1 = \mathrm{e}^{-it-t}s_2 + \mathrm{e}^{it-t}s_1 + O\big(\varepsilon^8\big)$$
$$u_2 = -\,\mathrm{e}^{-it-t}s_2 i + \mathrm{e}^{it-t}s_1 i + O\big(\varepsilon^8\big)$$

Then the system evolves in state space such that the parameters evolve according to these ODEs.

$$\dot s_1 = \varepsilon^2\big(3/4s_2 + 3/4s_1\big) + O\big(\varepsilon^9\big)$$
$$\dot s_2 = \varepsilon^2\big(3/4s_2 + 3/4s_1\big) + O\big(\varepsilon^9\big)$$

The eigenvalues of the above system are $\lambda = 0, \frac32\varepsilon^2$. Hence the net growth of $\vec u$ is at rate $-1+\frac32\varepsilon^2$; for example, at the unstable rate $+1/2$ when $\varepsilon = 1$.

2.2 `forcedNonlinNormMode`: harmonically forced nonlinear normal mode

Renson et al. (2012) explored finite element construction of the nonlinear normal modes of a pair of coupled oscillators. Let's apply periodic forcing to their example, Section 1.4, both direct and parametric. For example, here derive the effect on the mode with frequency one. Defining two new variables one of their example systems is

$$\begin{aligned} \dot x_1 &= x_3\,, \quad & \dot x_3 &= -2x_1 + x_2 - \frac{1}{2}x_1^3 + \frac{3}{10}(-x_3 + x_4) + f_1\cos t\,, \\ \dot x_2 &= x_4\,, \quad & \dot x_4 &= x_1 - 2x_2 + \frac{3}{10}(x_3 - 2x_4)f_2\sin(t/2). \end{aligned}$$

where f_1 is the strength of the direct forcing, and f_2 is the strength of the parametric oscillation in the last ODE. The linearisation of this system at the origin has modes with frequencies $\omega = 1, \sqrt{3}$, corresponding eigenvalues $\lambda = \pm i, \pm i\sqrt{3}$, and corresponding eigenvectors $(1, 1, \pm i\omega, \pm i\omega)$. Corresponding eigenvectors of the adjoint are $(1, 1, \pm i, \pm i)$ and $(\mp i\omega, \pm i\omega, 1, -1)$. We model the nonlinear forced dynamics of the frequency one mode.

Here, the analysis constructs a nonlinear normal mode, time-dependent, coordinate transformation. We find a time-dependent mapping from the modulation variables $\vec s = (s_1, s_2)$ to the original variables $\vec u = (u_1, u_2, u_3, u_4)$, and find the corresponding evolution of $\vec s$. The modulation variables $\vec s$ are 'slow' because the coordinate transform uses time-dependent (rotating) basis vectors that account for the fast oscillation in $\vec u$. Hence the new variables $\vec s$ are good variables for making long-term predictions and forming understanding.

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, and the complex exponential.

```
factor f_1,f_2,small;
```

The following code makes the linear damping to be effectively small (which then makes it `small` squared); consequently, also scale the smallness of the cubic nonlinearity to match.

```
invariantmanifold(
    mat(( u3, u4,
    -2*u1+u2-small*u1^3/2+small*3/10*(-u3+u4)
        +small*f_1*sin(t),
    u1-2*u2+small*3/10*(u3-2*u4)*f_2*cos(t/2) )),
    mat(( i,-i )),
    mat( (1,1,+i,+i), (1,1,-i,-i) ),
    mat( (1,1,+i,+i), (1,1,-i,-i) ),
    5 )$
```

```
end;
```

In the derived ODEs for the modulation of the frequency one mode, see that the direct forcing drives effects first seen in terms linear in f_1. However, the parametric forcing drives effects quadratic in f_2 and so our higher-order, systematic, analysis is required.

The procedure actually analyses the embedding system

$$\begin{aligned}
&\dot u_1 = u_3\,, \qquad \dot u_2 = u_4\,,\\
&\dot u_3 = f_1\varepsilon^2\big(\tfrac12 e^{-it}i - \tfrac12 e^{it}i\big) + \varepsilon^2\big(-\tfrac12 u_1^3 - \tfrac{3}{10}u_3 + \tfrac{3}{10}u_4\big)\\
&\qquad - 2u_1 + u_2\,,\\
&\dot u_4 = f_2\varepsilon^2\big(\tfrac{3}{20} e^{-it/2}u_3 - \tfrac{3}{10} e^{-it/2}u_4 + \tfrac{3}{20} e^{it/2}u_3 - \tfrac{3}{10} e^{it/2}u_4\big)\\
&\qquad + u_1 - 2u_2\,.
\end{aligned}$$

The invariant manifold Here these give the reparametrisation of the state space $\vec u$ in terms of parameters s_j, via rotating basis vectors. Here, the coordinate transform is very complicated so I do not give the complexity. The leading approximation is, of course, the linear, errors $\mathcal{O}(\varepsilon^2)$,

$$\begin{aligned}
u_1 &= +e^{-it}s_2 + e^{it}s_1 + O\big(\varepsilon^2\big)\\
u_2 &= +e^{-it}s_2 + e^{it}s_1 + O\big(\varepsilon^2\big)\\
u_3 &= -e^{-it}s_2 i + e^{it}s_1 i + O\big(\varepsilon^2\big)\\
u_4 &= -e^{-it}s_2 i + e^{it}s_1 i + O\big(\varepsilon^2\big)
\end{aligned}$$

Invariant manifold ODEs The system evolves according to these ODEs that characterise how the modulation of the oscillations evolve in state space due to nonlinearity and the forcing.

$$\begin{aligned}
\dot s_1 &= f_1\varepsilon^4\big(\tfrac{9}{64}s_2s_1 - \tfrac{9}{128}s_1^2 + \tfrac{3}{160}i\big) - \tfrac18 f_1\varepsilon^2 + \tfrac{93}{5500}f_2^2\varepsilon^4 s_1 i\\
&\quad + \varepsilon^4\big(-\tfrac{155}{256}s_2^2s_1^3 i + \tfrac{9}{160}s_2s_1^2\big) + \tfrac38\varepsilon^2 s_2s_1^2 i + O\big(\varepsilon^5\big)\\
\dot s_2 &= f_1\varepsilon^4\big(-\tfrac{9}{128}s_2^2 + \tfrac{9}{64}s_2s_1 - \tfrac{3}{160}i\big) - \tfrac18 f_1\varepsilon^2 - \tfrac{93}{5500}f_2^2\varepsilon^4 s_2 i\\
&\quad + \varepsilon^4\big(\tfrac{155}{256}s_2^3s_1^2 i + \tfrac{9}{160}s_2^2s_1\big) - \tfrac38\varepsilon^2 s_2^2s_1 i + O\big(\varepsilon^5\big)
\end{aligned}$$

The second lines of these ODEs are the terms from the nonautonomous part of the system. The first line are the terms induced by the harmonic forcing. The parametric oscillation just induces an $\mathcal{O}\big(f_2^2\big)$ frequency shift. The direct harmonic forcing induces a direct $\mathcal{O}\big(f_1\big)$ forcing of the amplitudes s_j.

2.3 `oscForcedChain`: harmonically forced chain of oscillations

Mingwu Li et al. (2022) discussed the following system of the forcing of a small chain of coupled oscillators. To analyse, first load the function.

```
in_tex "../invariantManifold.tex"$
factor small,i;
```

Then encode the ODEs

$$\begin{aligned}
&\ddot x_1 + x_1 + c_1\dot x_1 + k(x_1-x_2)^3 = f_1\cos\omega t\,,\\
&\ddot x_2 + x_2 + c_2\dot x_2 + k[(x_2-x_1)^3+(x_2-x_3)^3] = 0\,,\\
&\ddot x_3 + x_3 + c_3\dot x_3 + k(x_3-x_2)^3 = 0\,,
\end{aligned}$$

as follows with $x_1=u_1$, $\dot x_1=u_2$, $x_2=u_3$, $\dot x_2=u_4$, $x_3=u_5$, $\dot x_3=u_6$,

```
odes:=mat((u2, -u1-k*(u1-u3)^3             -c_1*u2+f
           ,u4, -u3-k*(u3-u1)^3-k*(u3-u5)^3-c_2*u4
           ,u6, -u5-k*(u5-u3)^3             -c_3*u6
           ));
```

The procedure introduces the ordering parameter ε to actually analyse the following system:

$$\begin{aligned}
&\ddot x_1 + x_1 + \varepsilon\left\{c_1\dot x_1 + k(x_1-x_2)^3 - f_1\cos\omega t\right\} = 0\,,\\
&\ddot x_2 + x_2 + \varepsilon\left\{c_2\dot x_2 + k[(x_2-x_1)^3+(x_2-x_3)^3]\right\} = 0\,,\\
&\ddot x_3 + x_3 + \varepsilon\left\{c_3\dot x_3 + k(x_3-x_2)^3\right\} = 0\,,
\end{aligned}$$

Set parameters nearly as in Mingwu Li et al. (2022), but let c_1 remain variable.

```
k:=2/10;
f:=f_1*cos(w*t);
factor f_1;
c_2:=2/10; c_3:=3/10; %c_1:=1/10;
```

Set forcing frequency to one for simplicity;

```
w:=1;
```

2.3.1 Time dependent reparametrisation of entire state space

Each of the three oscillators have identical frequency of one, and the damping is numerically small, so get the procedure to treat as small by giving unperturbed eigenvalues and eigenvectors.

```
invariantmanifold(odes,
    mat(( i,-i,i,-i,i,-i)),
    mat( (1,+i,0,0,0,0), (1,-i,0,0,0,0),
         (0,0,1,+i,0,0), (0,0,1,-i,0,0),
         (0,0,0,0,1,+i), (0,0,0,0,1,-i)
        ),
    mat( (1,+i,0,0,0,0), (1,-i,0,0,0,0),
         (0,0,1,+i,0,0), (0,0,1,-i,0,0),
         (0,0,0,0,1,+i), (0,0,0,0,1,-i)
        ),
    2 )$
```

The state space These give the location in state space in terms of parameters s_j.

$$\begin{aligned}
u_1 &= \mathrm{e}^{-it}s_2 + \mathrm{e}^{it}s_1 + O\big(\varepsilon\big)\\
u_2 &= i\big(- \mathrm{e}^{-it}s_2 + \mathrm{e}^{it}s_1\big) + O\big(\varepsilon\big)\\
u_3 &= \mathrm{e}^{-it}s_4 + \mathrm{e}^{it}s_3 + O\big(\varepsilon\big)\\
u_4 &= i\big(- \mathrm{e}^{-it}s_4 + \mathrm{e}^{it}s_3\big) + O\big(\varepsilon\big)\\
u_5 &= \mathrm{e}^{-it}s_6 + \mathrm{e}^{it}s_5 + O\big(\varepsilon\big)\\
u_6 &= i\big(- \mathrm{e}^{-it}s_6 + \mathrm{e}^{it}s_5\big) + O\big(\varepsilon\big)
\end{aligned}$$

State space ODEs The system evolves such that the parameters evolve according to these ODEs. They show forcing, weak damping, nonlinear interaction among the modulation of the three modes.

$$\begin{aligned}
\dot s_1 &= -\tfrac{1}{4}f_1 i\varepsilon + i\varepsilon\big(- \tfrac{3}{10}s_4s_3^2 + \tfrac{3}{5}s_4s_3s_1 - \tfrac{3}{10}s_4s_1^2 + \tfrac{3}{10}s_3^2s_2\\
&\quad - \tfrac{3}{5}s_3s_2s_1 + \tfrac{3}{10}s_2s_1^2\big) - \tfrac{1}{2}\varepsilon s_1c_1 + O\big(\varepsilon^2\big)\\
\dot s_2 &= \tfrac{1}{4}f_1 i\varepsilon + i\varepsilon\big(\tfrac{3}{10}s_4^2s_3 - \tfrac{3}{10}s_4^2s_1 - \tfrac{3}{5}s_4s_3s_2 + \tfrac{3}{5}s_4s_2s_1\\
&\quad + \tfrac{3}{10}s_3s_2^2 - \tfrac{3}{10}s_2^2s_1\big) - \tfrac{1}{2}\varepsilon s_2c_1 + O\big(\varepsilon^2\big)\\
\dot s_3 &= i\varepsilon\big(- \tfrac{3}{10}s_6s_5^2 + \tfrac{3}{5}s_6s_5s_3 - \tfrac{3}{10}s_6s_3^2 + \tfrac{3}{10}s_5^2s_4 - \tfrac{3}{5}s_5s_4s_3\\
&\quad + \tfrac{3}{5}s_4s_3^2 - \tfrac{3}{5}s_4s_3s_1 + \tfrac{3}{10}s_4s_1^2 - \tfrac{3}{10}s_3^2s_2 + 3/5s_3s_2s_1\\
&\quad - \tfrac{3}{10}s_2s_1^2\big) - \tfrac{1}{10}\varepsilon s_3 + O\big(\varepsilon^2\big)\\
\dot s_4 &= i\varepsilon\big(\tfrac{3}{10}s_6^2s_5 - \tfrac{3}{10}s_6^2s_3 - \tfrac{3}{5}s_6s_5s_4 + \tfrac{3}{5}s_6s_4s_3 + \tfrac{3}{10}s_5s_4^2\\
&\quad - \tfrac{3}{5}s_4^2s_3 + \tfrac{3}{10}s_4^2s_1 + \tfrac{3}{5}s_4s_3s_2 - \tfrac{3}{5}s_4s_2s_1 - 3/10s_3s_2^2\\
&\quad + \tfrac{3}{10}s_2^2s_1\big) - \tfrac{1}{10}\varepsilon s_4 + O\big(\varepsilon^2\big)\\
\dot s_5 &= i\varepsilon\big(\tfrac{3}{10}s_6s_5^2 - \tfrac{3}{5}s_6s_5s_3 + \tfrac{3}{10}s_6s_3^2 - \tfrac{3}{10}s_5^2s_4 + \tfrac{3}{5}s_5s_4s_3\\
&\quad - \tfrac{3}{10}s_4s_3^2\big) - \tfrac{3}{20}\varepsilon s_5 + O\big(\varepsilon^2\big)\\
\dot s_6 &= i\varepsilon\big(- \tfrac{3}{10}s_6^2s_5 + \tfrac{3}{10}s_6^2s_3 + \tfrac{3}{5}s_6s_5s_4 - \tfrac{3}{5}s_6s_4s_3 - \tfrac{3}{10}s_5s_4^2\\
&\quad + \tfrac{3}{10}s_4^2s_3\big) - \tfrac{3}{20}\varepsilon s_6 + O\big(\varepsilon^2\big)
\end{aligned}$$

2.3.2 Emergent mode at general frequency

Here suppose damping c_1 is significantly smaller than the other damping. Hence here consider the first oscillator as dominantly the 'master' mode. First, analyse for general frequency ω by 'clearing' `w`.

```
clear w;
invariantmanifold(odes,
    mat(( i,-i)),
    mat( (1,+i,0,0,0,0), (1,-i,0,0,0,0)
        ),
    mat( (1,+i,0,0,0,0), (1,-i,0,0,0,0)
        ),
    2 )$
```

The invariant manifold These give the location of the invariant manifold in terms of parameters s_j. There are divisions by $(1-\omega^2)$ flagging resonance when the forcing is at the resonant frequency.

$$
\begin{aligned}
u_1 &= f_1\varepsilon\big(-\tfrac{1}{2}\,\mathrm{e}^{-itw}-\tfrac{1}{2}\,\mathrm{e}^{itw}\big)/\big(w^2-1\big)+i\varepsilon\big(\tfrac{1}{4}\,\mathrm{e}^{-it}s_2c_1-1/4\,\mathrm{e}^{it}s_1c_1\big)\\
&\quad+\varepsilon\big(-\tfrac{3}{20}\,\mathrm{e}^{-it}s_2^2s_1+\tfrac{1}{40}\,\mathrm{e}^{-3it}s_2^3-\tfrac{3}{20}\,\mathrm{e}^{it}s_2s_1^2+\tfrac{1}{40}\,\mathrm{e}^{3it}s_1^3\big)\\
&\quad+\mathrm{e}^{-it}s_2+\mathrm{e}^{it}s_1+O\big(\varepsilon^2\big)\\
u_2 &= f_1i\varepsilon\big(\tfrac{1}{2}\,\mathrm{e}^{-itw}w-\tfrac{1}{2}\,\mathrm{e}^{itw}w\big)/\big(w^2-1\big)+i\varepsilon\big(-\tfrac{3}{20}\,\mathrm{e}^{-it}s_2^2s_1\\
&\quad-\tfrac{3}{40}\,\mathrm{e}^{-3it}s_2^3+\tfrac{3}{20}\,\mathrm{e}^{it}s_2s_1^2+\tfrac{3}{40}\,\mathrm{e}^{3it}s_1^3\big)+i\big(-\mathrm{e}^{-it}s_2+\mathrm{e}^{it}s_1\big)\\
&\quad+\varepsilon\big(-\tfrac{1}{4}\,\mathrm{e}^{-it}s_2c_1-\tfrac{1}{4}\,\mathrm{e}^{it}s_1c_1\big)+O\big(\varepsilon^2\big)\\
u_3 &= i\varepsilon\big(3\,\mathrm{e}^{-it}s_2^2s_1+\tfrac{3}{1609}\,\mathrm{e}^{-3it}s_2^3-3\,\mathrm{e}^{it}s_2s_1^2-\tfrac{3}{1609}\,\mathrm{e}^{3it}s_1^3\big)\\
&\quad+\varepsilon\big(-\tfrac{40}{1609}\,\mathrm{e}^{-3it}s_2^3-\tfrac{40}{1609}\,\mathrm{e}^{3it}s_1^3\big)+O\big(\varepsilon^2\big)\\
u_4 &= i\varepsilon\big(\tfrac{120}{1609}\,\mathrm{e}^{-3it}s_2^3-\tfrac{120}{1609}\,\mathrm{e}^{3it}s_1^3\big)\\
&\quad+\varepsilon\big(3\,\mathrm{e}^{-it}s_2^2s_1+\tfrac{9}{1609}\,\mathrm{e}^{-3it}s_2^3+3\,\mathrm{e}^{it}s_2s_1^2+\tfrac{9}{1609}\,\mathrm{e}^{3it}s_1^3\big)+O\big(\varepsilon^2\big)\\
u_5 &= O\big(\varepsilon^2\big)\\
u_6 &= O\big(\varepsilon^2\big)
\end{aligned}
$$

Invariant manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs. With general forcing frequency the algorithm does not detect resonance here, it only shows up as above divisions by $(1-\omega^2)$. These evolution equations only show nonlinear frequency modification and the damping at rate $c_1/2$.

$$
\begin{aligned}
\dot s_1 &= \tfrac{3}{10}i\varepsilon s_2s_1^2-\tfrac{1}{2}\varepsilon s_1c_1+O\big(\varepsilon^2\big)\\
\dot s_2 &= -\tfrac{3}{10}i\varepsilon s_2^2s_1-\tfrac{1}{2}\varepsilon s_2c_1+O\big(\varepsilon^2\big)
\end{aligned}
$$

2.3.3 Emergent mode at frequency one

So here, set forcing frequency back to one and re-analyse.

```
w:=1;
invariantmanifold(odes,
    mat(( i,-i)),
    mat( (1,+i,0,0,0,0), (1,-i,0,0,0,0)
        ),
    mat( (1,+i,0,0,0,0), (1,-i,0,0,0,0)
        ),
    2 )$
```

Invariant manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs. They show the forcing directly pumps the mode.

$$
\begin{aligned}
\dot s_1 &= -\tfrac{1}{4}f_1i\varepsilon+\tfrac{3}{10}i\varepsilon s_2s_1^2-\tfrac{1}{2}\varepsilon s_1c_1+O\big(\varepsilon^2\big)\\
\dot s_2 &= \tfrac{1}{4}f_1i\varepsilon-\tfrac{3}{10}i\varepsilon s_2^2s_1-\tfrac{1}{2}\varepsilon s_2c_1+O\big(\varepsilon^2\big)
\end{aligned}
$$

2.3.4 Emergent mode near frequency one

For forcing $f \propto \cos[(1+\omega')t]$ for small ω' we write $f := a\cos t - b\sin t$ where $a = f_1\cos\omega' t$ and $b = f_1\sin\omega' t$. Then $da/dt = -\omega' b$ and $db/dt = +\omega' a$ so code these relations, and truncate independently in small ω'.

```
f:=a*cos(t)-b*sin(t);
depend a,t; depend b,t;
let { df(a,t)=>-wd*b, df(b,t)=>wd*a, wd^3=>0 };
```

Construct the invariant manifold for detuned forcing and find little difference, just a slowly modulating forcing through $a(t), b(t)$.

```
invariantmanifold(odes,
    mat(( i,-i)),
    mat( (1,+i,0,0,0,0), (1,-i,0,0,0,0)
        ),
    mat( (1,+i,0,0,0,0), (1,-i,0,0,0,0)
        ),
    2 )$
```

Invariant manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\dot s_1 = i\varepsilon\big(\tfrac{3}{10}s_2 s_1^2 - \tfrac14 a\big) + \varepsilon\big(-\tfrac12 s_1 c_1 + \tfrac14 b\big) + O\big(\varepsilon^2, \omega'^3\big)$$
$$\dot s_2 = i\varepsilon\big(-\tfrac{3}{10}s_2^2 s_1 + \tfrac14 a\big) + \varepsilon\big(-\tfrac12 s_2 c_1 + \tfrac14 b\big) + O\big(\varepsilon^2, \omega'^3\big)$$

Finish the script

```
end;
```

3 Slow invariant manifolds

Also see Sections 1.1 and 1.3.

3.1 `simple2d`: Slow manifold of a simple 2D system

The example system to analyse is specified to be

$$\dot u_1 = -u_1 + u_2 - u_1^2\,, \quad \dot u_2 = u_1 - u_2 + u_2^2\,.$$

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

Execute the construction of the slow manifold for this system.

```
invariantmanifold(
    mat((-u1+u2-u1^2,u1-u2+u2^2)),
    mat((0)),
    mat((1,1)),
    mat((1,1)),
    5)$
end;
```

We seek the slow manifold so specify the eigenvalue zero. From the linearisation matrix $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ a corresponding eigenvector is $\vec e = (1,1)$, and corresponding left-eigenvector is $\vec z = \vec e = (1,1)$, as specified. The last parameter specifies to construct the slow manifold to errors $\mathcal{O}\big(\varepsilon^5\big)$.

The procedure actually analyses the embedding system

$$\dot u_1 = -u_1 + u_2 - \varepsilon u_1^2\,, \quad \dot u_2 = u_1 - u_2 + \varepsilon u_2^2\,.$$

So here the artificial parameter ε has a physical interpretation in that it counts the nonlinearity: a term in ε^p will be a $(p+1)$th order term in $\vec u = (u_1, u_2)$. Hence the specified error $\mathcal{O}\big(\varepsilon^5\big)$ is here the same as error $\mathcal{O}\big(|\vec u|^6\big)$.

The slow manifold The constructed slow manifold is, in terms of the parameter s_1 (and reverse ordering!),

$$\begin{aligned} u_1 &= 3/8\varepsilon^3 s_1^4 - 1/2\varepsilon s_1^2 + s_1\,, \\ u_2 &= -3/8\varepsilon^3 s_1^4 + 1/2\varepsilon s_1^2 + s_1\,. \end{aligned}$$

Slow manifold ODEs On this slow manifold the evolution is

$$\dot s_1 = -3/4\varepsilon^4 s_1^5 + \varepsilon^2 s_1^3\ :$$

here the leading term in s_1^3 indicates the origin is unstable.

Normals to isochrons at the slow manifold To project initial conditions onto the slow manifold, or non-autonomous forcing, or modifications of the original system, or to quantify uncertainty, use the projection defined by the derived vector

$$\vec z_1 = \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} 3/2\varepsilon^4 s_1^4 + 3/4\varepsilon^3 s_1^3 - 1/2\varepsilon^2 s_1^2 - 1/2\varepsilon s_1 + 1/2 \\ 3/2\varepsilon^4 s_1^4 - 3/4\varepsilon^3 s_1^3 - 1/2\varepsilon^2 s_1^2 + 1/2\varepsilon s_1 + 1/2 \end{bmatrix}.$$

Evaluate these at $\varepsilon = 1$ to apply to the original specified system, or here just interpret ε as a way to count the order of each term.

3.2 `lorenz86sm`: Slow manifold of the Lorenz 1986 atmosphere model

In this case we construct the slow sub-centre manifold, analogous to quasi-geostrophy, in order to disentangle the slow dynamics from fast oscillations, analogous to gravity waves, in the Lorenz (1986) model. The normals to the isochrons determine ‘balancing’ onto the slow manifold.

$$\begin{aligned} \dot u_1 &= bu_2u_5 - u_2u_3\,, \\ \dot u_2 &= -bu_1u_5 + u_1u_3\,, \\ \dot u_3 &= -u_1u_2\,, \\ \dot u_4 &= -u_5\,, \\ \dot u_5 &= bu_1u_2 + u_4\,. \end{aligned}$$

The parameter b controls the interaction between slow and fast waves. Section 4.3 constructs its full state space normal form in order to determine the forcing of the slow modes by the mean fast waves.

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

Group output expressions on b.

```
factor b;
```

Execute the construction of the slow manifold for this system.

```
invariantmanifold(
    mat(( -u2*u3+b*u2*u5,
        u1*u3-b*u1*u5,
        -u1*u2,
        -u5,
        +u4+b*u1*u2 )),
    mat(( 0,0,0 )),
    mat( (1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0) ),
    mat( (1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0) ),
    4 )$
end;
```

The matrix of the linearisation about the origin has eigenvalues zero (multiplicity three) and $\pm i$. We seek the slow manifold so specify the eigenvalue zero (thrice) in the second parameter to the procedure. Since the system is already in linearly separated form, the slow eigenvectors are simply the three given unit vectors. The last parameter, 4, specifies to construct the slow manifold to errors $\mathcal{O}(\varepsilon^4)$, that is, to errors $\mathcal{O}(|\vec s|^5)$.

The procedure actually analyses the embedding system

$$\begin{aligned}
\dot u_1 &= b\varepsilon u_2u_5 - \varepsilon u_2u_3\,,\\
\dot u_2 &= -b\varepsilon u_1u_5 + \varepsilon u_1u_3\,,\\
\dot u_3 &= -\varepsilon u_1u_2\,,\\
\dot u_4 &= -u_5\,,\\
\dot u_5 &= b\varepsilon u_1u_2 + u_4\,.
\end{aligned}$$

Consequently, here the artificial parameter ε has a physical interpretation in that it counts the nonlinearity: a term in ε^p will be a $(p+1)$th order term in $\vec s$.

The slow manifold The constructed slow manifold is, in terms of the parameters $\vec s$ (to errors $\mathcal{O}(\varepsilon^3)$, and reverse ordering!),

$$\begin{aligned}
u_1 &= s_1\,,\\
u_2 &= s_2\,,\\
u_3 &= s_3\,,\\
u_4 &= -b\varepsilon s_2s_1\,,\\
u_5 &= b\varepsilon^2\big(-s_3s_2^2 + s_3s_1^2\big).
\end{aligned}$$

Slow manifold ODEs On this slow manifold the evolution is

$$\begin{aligned}
\dot s_1 &= b^2\varepsilon^3\big(-s_3s_2^3 + s_3s_2s_1^2\big) - \varepsilon s_3s_2\,,\\
\dot s_2 &= b^2\varepsilon^3\big(s_3s_2^2s_1 - s_3s_1^3\big) + \varepsilon s_3s_1\,,\\
\dot s_3 &= -\varepsilon s_2s_1\,.
\end{aligned}$$

Here the quadratic terms in s_1, s_2, s_3 is that of nonlinear slow wave oscillations. The b-terms modify these slow waves, reflecting the influence of the fast dynamics (as distinct from the effects of fast waves—these effects are quantified by Section 4.3).

Normals to isochrons at the slow manifold To project initial conditions onto the slow manifold, or non-autonomous forcing, or modifications of the original system, or to quantify uncertainty (Roberts 1989, 2000), use the projection defined by the derived vectors

$$\vec z_1 = \begin{bmatrix} b^2\varepsilon^2s_2^2+1\\ b^2\varepsilon^2s_2s_1\\ 0\\ b^3\varepsilon^3\big(s_2^3-s_2s_1^2\big)+b\varepsilon^3\big(-s_2^3+s_2s_1^2\big)+b\varepsilon s_2\\ 0\end{bmatrix},$$

$$\vec z_2 = \begin{bmatrix} -b^2\varepsilon^2 s_2 s_1 \\ -b^2\varepsilon^2 s_1^2 + 1 \\ 0 \\ b^3\varepsilon^3\big(- s_2^2 s_1 + s_1^3\big) + b\varepsilon^3\big(s_2^2 s_1 - s_1^3\big) - b\varepsilon s_1 \\ 0 \end{bmatrix},$$

$$\vec z_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4b\varepsilon^3 s_3 s_2 s_1 \\ b\varepsilon^2\big(- s_2^2 + s_1^2\big) \end{bmatrix}.$$

Evaluate these at $\varepsilon = 1$ to apply to the original specified system, or here just interpret ε as a way to count the order of each term.

4 Oscillation in a centre manifold

Also see Sections 1.4 and 2.2.

4.1 `simpleosc`: Oscillatory centre manifold—separated form

Let's try complex eigenvectors. Adjoint eigenvectors `zz_` must be the eigenvectors of the complex conjugate transpose matrix.

$$\begin{aligned}
\dot u_1 &= u_2\\
\dot u_2 &= -\varepsilon u_3 u_1 - u_1\\
\dot u_3 &= 5\varepsilon u_1^2 - u_3
\end{aligned}$$

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j and the complex exponential

```
factor s,exp;
```

Execute the construction of the centre manifold for this system.

```
invariantmanifold(
    mat((u2,-u1-u1*u3,-u3+5*u1^2)),
    mat((i,-i)),
    mat((1,+i,0),(1,-i,0)),
    mat((1,+i,0),(1,-i,0)),
    3)$
end;
```

The centre manifold These give the location of the invariant manifold in terms of parameters s_j.

$$\begin{aligned}
u_1 &= \mathrm{e}^{-it} s_2 + \mathrm{e}^{it} s_1\\
u_2 &= -\mathrm{e}^{-it} s_2 i + \mathrm{e}^{it} s_1 i\\
u_3 &= \mathrm{e}^{-2it} s_2^2 \varepsilon \big(2i+1\big) + \mathrm{e}^{2it} s_1^2 \varepsilon \big(-2i+1\big) + 10 s_2 s_1 \varepsilon
\end{aligned}$$

Centre manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\begin{aligned}
\dot s_1 &= s_2 s_1^2 \varepsilon^2 \big(11/2i+1\big)\\
\dot s_2 &= s_2^2 s_1 \varepsilon^2 \big(-11/2i+1\big)
\end{aligned}$$

4.2 `quasidde`: Quasi-delay DE with Hopf bifurcation

Shows Hopf bifurcation as parameter α crosses 0 to oscillations with base frequency two.

$$\dot u_1 = -\alpha\varepsilon^2 u_3 - \varepsilon^2 u_1^3 - 2\varepsilon u_1^2 - 4u_3$$

$$\dot u_2 = 2u_1 - 2u_2$$
$$\dot u_3 = 2u_2 - 2u_3$$

for small parameter α. We code the parameter α as 'small', and observe it is consequently considered as 'small squared' because all nonlinear terms and already 'small' terms, are multiplied by another `small`.

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameter α.

```
factor s,exp,alpha;
```

Execute the construction of the slow manifold for this system (ignore the warning messages about `u1` declared, and then already defined, as an operator).

```
invariantmanifold(
    mat(( -4*u3-small*alpha*u3-2*u1^2-small*u1^3,
        2*u1-2*u2,
        2*u2-2*u3 )),
    mat((2*i,-2*i)),
    mat((1,1/2-i/2,-i/2),(1,1/2+i/2,+i/2)),
    mat((1,-i,-1-i),(1,+i,-1+i)),
    3)$
end;
```

The centre manifold These give the location of the invariant manifold in terms of parameters s_1, s_2 (complex conjugate for real solutions).

$$\begin{aligned} u_1 &= e^{-4it} s_2^2 \varepsilon \big(-7/12i + 1/12 \big) + e^{-2it} s_2 + e^{4it} s_1^2 \varepsilon \big(7/12i + \\ &\quad 1/12 \big) + e^{2it} s_1 - s_2 s_1 \varepsilon \\ u_2 &= e^{-4it} s_2^2 \varepsilon \big(-1/12i + 1/4 \big) + e^{-2it} s_2 \big(1/2i + 1/2 \big) + \\ &\quad e^{4it} s_1^2 \varepsilon \big(1/12i + 1/4 \big) + e^{2it} s_1 \big(-1/2i + 1/2 \big) - s_2 s_1 \varepsilon \\ u_3 &= e^{-4it} s_2^2 \varepsilon \big(1/12i + 1/12 \big) + 1/2\, e^{-2it} s_2 i + e^{4it} s_1^2 \varepsilon \big(-1/12i + \\ &\quad 1/12 \big) - 1/2\, e^{2it} s_1 i - s_2 s_1 \varepsilon \end{aligned}$$

Centre manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\dot s_1 = s_2 s_1^2 \varepsilon^2 \big(-16/15i - 1/5 \big) + s_1 \alpha \varepsilon^2 \big(1/5i + 1/10 \big)$$
$$\dot s_2 = s_2^2 s_1 \varepsilon^2 \big(16/15i - 1/5 \big) + s_2 \alpha \varepsilon^2 \big(-1/5i + 1/10 \big)$$

Hence there is a supercritical Hopf bifurcation as parameter α increases through zero.

4.3 `lorenz86nf`: Paradoxically justify a slow manifold despite being proven to not exist

Lorenz (1986) proposed the following simple system in order to understand aspects of the quasi-geostrophic approximation in atmospheric dynamics.

$$\begin{aligned}
\dot u_1 &= bu_2u_5 - u_2u_3\,,\\
\dot u_2 &= -bu_1u_5 + u_1u_3\,,\\
\dot u_3 &= -u_1u_2\,,\\
\dot u_4 &= -u_5\,,\\
\dot u_5 &= bu_1u_2 + u_4\,.
\end{aligned}$$

The parameter b controls the interaction between slow and fast dynamics. As in Section 3.2, it appears that a slow manifold of quasi-geostrophy exists and is constructible. Nonetheless, Lorenz & Krishnamurthy (1987) proved that a slow manifold cannot exist for this system!

A resolution of this apparent paradox comes via backwards theory (Roberts 2022, §2.5). There are systems exponentially close to the above Lorenz86 system (that is, asymptotically the same to all orders in $|\vec u|$) which do possess a slow manifold. Hence the properties that cause the non-existence are exponentially small, they are beyond all orders, and so are likely to be physically irrelevant—they are likely to be smaller than the mathematical modelling errors of the original system.

Let's see this resolution by constructing, to any specified order, a system that has a slow manifold and is close to the Lorenz86 system. We do this by constructing a coordinate transform of the 5D state space. Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

Group output expressions on b.

```
factor b;
```

Execute the construction of the coordinate transform for this system.

```
invariantmanifold(
    mat(( -u2*u3+b*u2*u5,
        u1*u3-b*u1*u5,
        -u1*u2,
        -u5,
        +u4+b*u1*u2 )),
    mat(( 0,0,0,i,-i )),
    mat( (1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),
        (0,0,0,1,-i), (0,0,0,1,+i) ),
    mat( (1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),
        (0,0,0,1,-i), (0,0,0,1,+i) ),
```

```
      4 )$
end;
```

The matrix of the linearisation about the origin has eigenvalues zero (multiplicity three) and $\pm i$, as specified for the eigenvalues in the second parameter to the procedure. Corresponding eigenvectors are simply the three unit vectors and the two complex eigenvectors of the fast waves. The last parameter, 4, specifies to construct the slow manifold to errors $\mathcal{O}\big(\varepsilon^4\big)$, that is, to errors $\mathcal{O}\big(|\vec s|^5\big)$.

The procedure actually analyses the embedding system

$$\begin{aligned}
\dot u_1 &= b\varepsilon u_2u_5 - \varepsilon u_2u_3\,,\\
\dot u_2 &= -b\varepsilon u_1u_5 + \varepsilon u_1u_3\,,\\
\dot u_3 &= -\varepsilon u_1u_2\,,\\
\dot u_4 &= -u_5\,,\\
\dot u_5 &= b\varepsilon u_1u_2 + u_4\,.
\end{aligned}$$

The coordinate transform The constructed coordinate transform is, in terms of the slow variables $\vec s$ and a time-dependent basis (to errors $\mathcal{O}\big(\varepsilon^3\big)$, and reverse ordering!),

$$\begin{aligned}
u_1 &= b^2\varepsilon^2\big(-1/2\,\mathrm{e}^{-2it}s_5^2s_1 - 1/2\,\mathrm{e}^{2it}s_4^2s_1\big) + b\varepsilon\big(-\mathrm{e}^{-it}s_5s_2 - {}\\
&\qquad \mathrm{e}^{it}s_4s_2\big) + s_1\,,\\
u_2 &= b^2\varepsilon^2\big(-1/2\,\mathrm{e}^{-2it}s_5^2s_2 - 1/2\,\mathrm{e}^{2it}s_4^2s_2\big) + b\varepsilon\big(\mathrm{e}^{-it}s_5s_1 + {}\\
&\qquad \mathrm{e}^{it}s_4s_1\big) + s_2\,,\\
u_3 &= b\varepsilon^2\big(\mathrm{e}^{-it}s_5s_2^2i - \mathrm{e}^{-it}s_5s_1^2i - \mathrm{e}^{it}s_4s_2^2i + \mathrm{e}^{it}s_4s_1^2i\big) + s_3\,,\\
u_4 &= b^2\varepsilon^2\big(1/4\,\mathrm{e}^{-it}s_5s_2^2 - 1/4\,\mathrm{e}^{-it}s_5s_1^2 + 1/4\,\mathrm{e}^{it}s_4s_2^2 - {}\\
&\qquad 1/4\,\mathrm{e}^{it}s_4s_1^2\big) - b\varepsilon s_2s_1 + \mathrm{e}^{-it}s_5 + \mathrm{e}^{it}s_4\,,\\
u_5 &= b^2\varepsilon^2\big(-1/4\,\mathrm{e}^{-it}s_5s_2^2i + 1/4\,\mathrm{e}^{-it}s_5s_1^2i + 1/4\,\mathrm{e}^{it}s_4s_2^2i - {}\\
&\qquad 1/4\,\mathrm{e}^{it}s_4s_1^2i\big) + b\varepsilon^2\big(-s_3s_2^2 + s_3s_1^2\big) + \mathrm{e}^{-it}s_5i - \mathrm{e}^{it}s_4i\,.
\end{aligned}$$

Transformed ODEs In the variables $\vec s$ the evolution is

$$\begin{aligned}
\dot s_1 &= b^2\varepsilon^3\big(-s_3s_2^3 + s_3s_2s_1^2\big) - \varepsilon s_3s_2\,,\\
\dot s_2 &= b^2\varepsilon^3\big(s_3s_2^2s_1 - s_3s_1^3\big) + \varepsilon s_3s_1\,,\\
\dot s_3 &= 2b^2\varepsilon^3s_5s_4s_2s_1 - \varepsilon s_2s_1\,,\\
\dot s_4 &= b^2\varepsilon^2\big(-1/2s_4s_2^2i + 1/2s_4s_1^2i\big)\,,\\
\dot s_5 &= b^2\varepsilon^2\big(1/2s_5s_2^2i - 1/2s_5s_1^2i\big)\,.
\end{aligned}$$

When $s_4 = s_5 = 0$ we recover precisely the same slow manifold as constructed by Section 3.2. Hence the above system of $\vec u = \cdots$ and $\dot{\vec s} = \cdots$ together both has a slow manifold, and is $\mathcal{O}\big(|\vec s|^5\big)$ close to the original Lorenz86 system. Such construction can proceed to any order, and so the above closeness of a system with a slow manifold holds to all orders in $|\vec s|$.

Also of interest is the red term in the $\dot s_3$ ODE: it shows that the evolution of the slow variables, s_1, s_2, s_3, is affected by the presence of fast waves, s_4, s_5 non-zero. That is, the evolution on and off the slow manifold differ by this term (and similar higher-order terms). Users of slow models among fast waves need to be aware of this physical feature.

4.4 `stoleriu2`: Oscillatory centre manifold among stable and unstable modes

Consider the case Stoleriu (2012) calls $(3\pi/4, k^2/2)$.

$$\begin{aligned}
\dot u_1 &= u_2\,,\\
\dot u_2 &= -\sigma u_3 + 1 - \cos u_1\,,\\
\dot u_3 &= u_4\,,\\
\dot u_4 &= (u_3 + \tfrac{1}{\sigma})\sin u_1
\end{aligned}$$

Eigenvalues are ± 1 and $\pm i$, so we find the centre manifold among stable and unstable modes.

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j and the complex exponential

```
factor s,exp;
```

Execute the construction of the centre manifold for Stoleriu's system. But use Taylor expansions for trigonometric functions in the ODEs, and multiply higher-orders of nonlinearity by `small` to better (not best) count and manage nonlinearities.

```
invariantmanifold(
    mat(( u2,
        sigma*u3+u1^2/2-small*u1^4/24,
        u4,
        (u3+1/sigma)*(u1-small*u1^3/6)
      )),
    mat(( i,-i )),
    mat( (sigma,i*sigma,-1,-i),(sigma,-i*sigma,-1,+i) ),
    mat( (+i,-1,-i*sigma,sigma),(-i,-1,+i*sigma,sigma) ),
    3)$
end;
```

Code adjoint eigenvectors `zz_` that are eigenvectors of the complex conjugate transpose matrix of the linear matrix $\begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & \sigma & 0\\ 0 & 0 & 0 & 1\\ 1/\sigma & 0 & 0 & 0\end{bmatrix}$. Here analyse to errors $\mathcal{O}(\varepsilon^3)$.

The procedure analyses the embedding system

$$\dot u_1 = u_2\,,$$

$$\begin{aligned}
\dot u_2 &= -1/24\varepsilon^2 u_1^4 + 1/2\varepsilon u_1^2 + \sigma u_3\,,\\
\dot u_3 &= u_4\,,\\
\dot u_4 &= \varepsilon^2\big(-1/6\sigma^{-1}u_1^3 - 1/6u_1^3u_3\big) + \varepsilon u_1u_3 + \sigma^{-1}u_1
\end{aligned}$$

The centre manifold These give the location of the invariant manifold in terms of (complex conjugate) parameters s_1, s_2.

$$\begin{aligned}
u_1 &= \mathrm{e}^{-it}s_2\sigma - 1/5\,\mathrm{e}^{-2it}s_2^2\varepsilon\sigma^2 + \mathrm{e}^{it}s_1\sigma - 1/5\,\mathrm{e}^{2it}s_1^2\varepsilon\sigma^2 + 2s_2s_1\varepsilon\sigma^2\\
u_2 &= -\mathrm{e}^{-it}s_2i\sigma + 2/5\,\mathrm{e}^{-2it}s_2^2\varepsilon i\sigma^2 + \mathrm{e}^{it}s_1i\sigma - 2/5\,\mathrm{e}^{2it}s_1^2\varepsilon i\sigma^2\\
u_3 &= -\mathrm{e}^{-it}s_2 + 3/10\,\mathrm{e}^{-2it}s_2^2\varepsilon\sigma - \mathrm{e}^{it}s_1 + 3/10\,\mathrm{e}^{2it}s_1^2\varepsilon\sigma - s_2s_1\varepsilon\sigma\\
u_4 &= \mathrm{e}^{-it}s_2i - 3/5\,\mathrm{e}^{-2it}s_2^2\varepsilon i\sigma - \mathrm{e}^{it}s_1i + 3/5\,\mathrm{e}^{2it}s_1^2\varepsilon i\sigma
\end{aligned}$$

Centre manifold ODEs The system evolves on the centre manifold such that the parameters evolve according to these ODEs.

$$\begin{aligned}
\dot s_1 &= -6/5s_2s_1^2\varepsilon^2 i\sigma^2\\
\dot s_2 &= 6/5s_2^2s_1\varepsilon^2 i\sigma^2
\end{aligned}$$

These establish that the leading effect of the nonlinearities is to cause a frequency down-shift in the oscillations on the centre manifold. Higher-order analysis indicates the only effect is a frequency shift of the nonlinear oscillations.

4.5 `bauer2021`: Rephrase phase-averaging as nonlinear normal modes

Bauer et al. (2021) introduced a *higher order phase averaging method* for nonlinear oscillatory systems. Here we construct cognate high-order approximations by constructing the modulation of the nonlinear normal modes in the system. Their example system (3.2) may be rewritten in variables $\vec u(t)$ as

$$\begin{aligned}
\dot u_1 &= \omega_R u_2\,, & \dot u_2 &= -\omega_R u_1 + \frac{\lambda}{\omega_R}u_1u_5\,,\\
\dot u_3 &= \omega_R u_4\,, & \dot u_4 &= -\omega_R u_3 + \frac{\lambda}{\omega_R}u_3u_5\,,\\
\dot u_5 &= \omega_Z u_6\,, & \dot u_6 &= -\omega_Z u_5 + \frac{\lambda}{\omega_z}(u_1^2 + u_3^2).
\end{aligned}$$

Bauer et al. (2021), their §4, chose base frequencies $\omega_R = \pi$ and $\omega_Z = 2\pi$ so we do so also.

The linearisation at the origin then has the following modes:

- eigenvalues $\pm i\pi$ with corresponding eigenvectors proportional to $(1, \pm i, 0, 0, 0, 0)$ and $(0, 0, 1, \pm i, 0, 0)$;
- eigenvalues $\pm 2i\pi$ with corresponding eigenvector proportional to $(0, 0, 0, 0, 1, \pm i)$.

We model the nonlinear interaction of these six modes over long times—these are the nonlinear normal modes. The analysis constructs a full state space coordinate transformation mapping from the complex-valued modulation variables $\vec{s}=(s_1,\ldots,s_6)$ to the original variables $\vec{u}=(u_1,\ldots,u_6)$, and find the corresponding evolution of $\vec{s}$. The modulation variables $\vec{s}$ are 'slow' because the coordinate transform uses time-dependent (rotating) basis vectors that account for the fast oscillation in $\vec{u}$. Hence the new variables $\vec{s}$ are good variables for making long-term predictions and forming understanding.

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms depending upon real or imaginary coefficient, and factor out π.

```
factor pi,i;
```

The following procedure call constructs the time-dependent coordinate transform for this system.

```
invariantmanifold(
    mat((pi*u2,-pi*u1+u1*u5/pi
        ,pi*u4,-pi*u3+u3*u5/pi
        ,2*pi*u6,-2*pi*u5+(u1^2+u3^2)/pi/2 )),
    mat((pi*i,-pi*i,pi*i,-pi*i,2*pi*i,-2*pi*i)),
    mat((1,+i,0,0,0,0),(1,-i,0,0,0,0)
       ,(0,0,1,+i,0,0),(0,0,1,-i,0,0)
       ,(0,0,0,0,1,+i),(0,0,0,0,1,-i)),
    mat((1,+i,0,0,0,0),(1,-i,0,0,0,0)
       ,(0,0,1,+i,0,0),(0,0,1,-i,0,0)
       ,(0,0,0,0,1,+i),(0,0,0,0,1,-i)),
    3 )$
end;
```

The procedure then actually analyses the embedding system

$$\begin{aligned}
\dot u_1 &= \pi u_2 & \dot u_2 &= -\pi u_1 + \pi^{-1}\varepsilon u_1 u_5\\
\dot u_3 &= \pi u_4 & \dot u_4 &= -\pi u_3 + \pi^{-1}\varepsilon u_3 u_5\\
\dot u_5 &= 2\pi u_6 & \dot u_6 &= -2\pi u_5 + \pi^{-1}\varepsilon\big(1/2u_1^2 + 1/2u_3^2\big)
\end{aligned}$$

Hence the procedure's artificial parameter ε is precisely the physical parameter λ of Bauer et al. (2021). As specified, the construction is here done to errors $\mathcal{O}\big(\varepsilon^3\big)$.

The invariant manifold Here these give the reparametrisation of the state space $\vec{u}$ in terms of modulation variables s_j, via rotating basis vectors.

$$u_1 = \mathrm{e}^{-i\pi t}s_2 + \mathrm{e}^{i\pi t}s_1 + \pi^{-2}\varepsilon\big(1/4\,\mathrm{e}^{-i\pi t}s_6s_1 - 1/8\,\mathrm{e}^{-3i\pi t}s_6s_2 + 1/4\,\mathrm{e}^{i\pi t}s_5s_2 - 1/8\,\mathrm{e}^{3i\pi t}s_5s_1\big)$$

$$u_2 = i\big(-\mathrm{e}^{-i\pi t}s_2 + \mathrm{e}^{i\pi t}s_1\big) + \pi^{-2}i\varepsilon\big(1/4\,\mathrm{e}^{-i\pi t}s_6s_1 + 3/8\,\mathrm{e}^{-3i\pi t}s_6s_2 - 1/4\,\mathrm{e}^{i\pi t}s_5s_2 - 3/8\,\mathrm{e}^{3i\pi t}s_5s_1\big)$$

$$u_3 = \mathrm{e}^{-i\pi t}s_4 + \mathrm{e}^{i\pi t}s_3 + \pi^{-2}\varepsilon\big(1/4\,\mathrm{e}^{-i\pi t}s_6s_3 - 1/8\,\mathrm{e}^{-3i\pi t}s_6s_4 + 1/4\,\mathrm{e}^{i\pi t}s_5s_4 - 1/8\,\mathrm{e}^{3i\pi t}s_5s_3\big)$$

$$u_4 = i\big(-\mathrm{e}^{-i\pi t}s_4 + \mathrm{e}^{i\pi t}s_3\big) + \pi^{-2}i\varepsilon\big(1/4\,\mathrm{e}^{-i\pi t}s_6s_3 + 3/8\,\mathrm{e}^{-3i\pi t}s_6s_4 - 1/4\,\mathrm{e}^{i\pi t}s_5s_4 - 3/8\,\mathrm{e}^{3i\pi t}s_5s_3\big)$$

$$u_5 = \mathrm{e}^{-2i\pi t}s_6 + \mathrm{e}^{2i\pi t}s_5 + \pi^{-2}\varepsilon\big(1/16\,\mathrm{e}^{-2i\pi t}s_4^2 + 1/16\,\mathrm{e}^{-2i\pi t}s_2^2 + 1/16\,\mathrm{e}^{2i\pi t}s_3^2 + 1/16\,\mathrm{e}^{2i\pi t}s_1^2 + 1/2s_4s_3 + 1/2s_2s_1\big)$$

$$u_6 = i\big(-\mathrm{e}^{-2i\pi t}s_6 + \mathrm{e}^{2i\pi t}s_5\big) + \pi^{-2}i\varepsilon\big(1/16\,\mathrm{e}^{-2i\pi t}s_4^2 + 1/16\,\mathrm{e}^{-2i\pi t}s_2^2 - 1/16\,\mathrm{e}^{2i\pi t}s_3^2 - 1/16\,\mathrm{e}^{2i\pi t}s_1^2\big)$$

Invariant manifold ODEs The system evolves according to these ODEs that characterise how the modulation of the oscillations evolve in state space due to their nonlinear interaction.

$$\dot s_1 = -1/2\pi^{-1}i\varepsilon s_5s_2 + \pi^{-3}i\varepsilon^2\big(-1/16s_6s_5s_1 - 1/4s_4s_3s_1 - 1/32s_3^2s_2 - 9/32s_2s_1^2\big)$$

$$\dot s_2 = 1/2\pi^{-1}i\varepsilon s_6s_1 + \pi^{-3}i\varepsilon^2\big(1/16s_6s_5s_2 + 1/32s_4^2s_1 + 1/4s_4s_3s_2 + 9/32s_2^2s_1\big)$$

$$\dot s_3 = -1/2\pi^{-1}i\varepsilon s_5s_4 + \pi^{-3}i\varepsilon^2\big(-1/16s_6s_5s_3 - 9/32s_4s_3^2 - 1/32s_4s_1^2 - 1/4s_3s_2s_1\big)$$

$$\dot s_4 = 1/2\pi^{-1}i\varepsilon s_6s_3 + \pi^{-3}i\varepsilon^2\big(1/16s_6s_5s_4 + 9/32s_4^2s_3 + 1/4s_4s_2s_1 + 1/32s_3s_2^2\big)$$

$$\dot s_5 = \pi^{-1}i\varepsilon\big(-1/4s_3^2 - 1/4s_1^2\big) + \pi^{-3}i\varepsilon^2\big(-1/16s_5s_4s_3 - 1/16s_5s_2s_1\big)$$

$$\dot s_6 = \pi^{-1}i\varepsilon\big(1/4s_4^2 + 1/4s_2^2\big) + \pi^{-3}i\varepsilon^2\big(1/16s_6s_4s_3 + 1/16s_6s_2s_1\big)$$

These all preserve complex conjugation, and so preserve reality. All coefficients are pure imaginary, so the dominant effect of the modulation is to modify the frequency of the oscillations. Amplitude modifications arise due to the phase relationship between the modes.

5 Stable invariant manifolds

Also see Section 1.5.

5.1 `stable2d`: Stable manifold of a 2D system

Let's construct the 1D stable manifold of the system, for small bifurcation parameter ϵ,

$$\begin{aligned}\dot u_1 &= -\tfrac{1}{2}u_1 - u_2 - u_1^2u_2\,,\\ \dot u_2 &= -u_1 - 2u_2 + \epsilon u_2 - u_2^2\,.\end{aligned}$$

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

Execute the construction of the stable manifold for this system.

```
invariantmanifold(
    mat(( -u1/2-u2-small*u1^2*u2,
        -u1-2*u2+small*epsilon*u2-u2^2 )),
    mat(( -5/2 )),
    mat( (1,2) ),
    mat( (1,2) ),
    5 )$
end;
```

The matrix $\begin{bmatrix}-\frac{1}{2} & -1\\ -1 & -2\end{bmatrix}$ of the linearisation about the origin has eigenvalues 0 and $-5/2$. We seek the 1D stable manifold so specify the eigenvalue $-5/2$ in the second parameter to the procedure. Due to symmetry, corresponding eigenvectors are $\vec e_1 = \vec z_1 = (1,2)$ in the third and fourth parameter. The last parameter, 5, specifies to construct the stable manifold to errors $\mathcal{O}(\varepsilon^5)$.

To consistently count the orders of the nonlinearities we multiply the cubic term by `small`. To treat parameter ϵ as small, we also multiply it by `small` so it becomes effectively a second-order order-parameter (useful for pitchfork bifurcations). So, the procedure actually analyses the embedding system

$$\begin{aligned}\dot u_1 &= -\varepsilon^2u_1^2u_2 - 1/2u_1 - u_2\,,\\ \dot u_2 &= \varepsilon^2\epsilon u_2 - \varepsilon u_2^2 - u_1 - 2u_2\,.\end{aligned}$$

The stable manifold The constructed stable manifold is, in terms of the parameter s_1 (to error $\mathcal{O}(\varepsilon^4)$, and reverse ordering!, and in terms of the ugly $\mathrm{e}^{(-5t}/2) = e^{-5t/2}$ which needs fixing sometime!),

$$\begin{aligned}u_1 = {}&\varepsilon^3\big(53152/140625\,\mathrm{e}^{-10t}s_1^4 + 88/625\,\mathrm{e}^{-5t}s_1^2\epsilon\big) + {}\\ &\varepsilon^2\big(838/1875\,\mathrm{e}^{(-15t}/2)s_1^3 + 8/25\,\mathrm{e}^{(-5t}/2)s_1\epsilon\big) + {}\\ &8/25\varepsilon\,\mathrm{e}^{-5t}s_1^2 + \mathrm{e}^{(-5t}/2)s_1\,,\end{aligned}$$

$$\begin{aligned} u_2 = {}& \varepsilon^3\big(122444/140625\,\mathrm{e}^{-10t}s_1^4 + 76/625\,\mathrm{e}^{-5t}s_1^2\epsilon\big) + \\ & \varepsilon^2\big(2116/1875\,\mathrm{e}^{\left(-15t\right.}/2\big)s_1^3 - 4/25\,\mathrm{e}^{\left(-5t\right.}/2\big)s_1\epsilon\big) + \\ & 36/25\varepsilon\,\mathrm{e}^{-5t}s_1^2 + 2\,\mathrm{e}^{\left(-5t\right.}/2\big)s_1\,. \end{aligned}$$

Observe the linear terms in s_1 all have $e^{-5t/2}$, and the quadratic terms in s_1 all have e^{-5t}, and so on. Consequently, we could in principle write the stable manifold in terms of, say, the variables $x_1 = s_1e^{-5t/2}$ giving

$$\begin{aligned} u_1 = {}& \varepsilon^3\big(53152/140625x_1^4 + 88/625x_1^2\epsilon\big) + \varepsilon^2\big(838/1875x_1^3 + \\ & 8/25x_1\epsilon\big) + 8/25\varepsilon x_1^2 + x_1\,, \\ u_2 = {}& \varepsilon^3\big(122444/140625x_1^4 + 76/625x_1^2\epsilon\big) + \varepsilon^2\big(2116/1875x_1^3 - \\ & 4/25x_1\epsilon\big) + 36/25\varepsilon x_1^2 + 2x_1\,. \end{aligned}$$

This would be a more usual parametrisation. But here let's remain with s_1 and remember to interpret s_1 as modifying the exponential decay $e^{-5t/2}$ on this stable manifold.

Stable manifold ODEs On the stable manifold the evolution is

$$\dot s_1 = -8/125\varepsilon^4 s_1\epsilon^2 + 4/5\varepsilon^2 s_1\epsilon\,.$$

That the ODE for s_1 is linear is a consequence of the Hartmann-Grobman Theorem. It just reflects that the decay-rate of the stable mode varies with parameter ϵ: evidently, the decay rate is approximately $-\frac{5}{2} + \frac{4}{5}\epsilon - \frac{8}{125}\epsilon^2$.

6 Invariant manifolds in delay DEs

Also see Section 1.2

6.1 `simple1dde`: Simple DDE with a Hopf bifurcation

Model a delayed 'logistic' system in one variable with

$$\frac{du}{dt} = -(1+a)[1+u(t)]u(t-\pi/2),$$

for small parameter a. We code the parameter a as 'small', and observe it is consequently considered as 'small squared' because all nonlinear terms and already 'small' terms are multiplied by `small`.

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameter a.

```
factor s,exp,a;
```

Execute the construction of the slow manifold for this system (ignore the warning messages about `u1` declared, and then already defined, as an operator).

```
invariantmanifold(
    mat(( -(1+small*a)*(1+u1)*u1(pi/2) )),
    mat((i,-i)),
    mat((1),(1)),
    mat((1),(1)),
    3)$
end;
```

The marginal modes are $e^{\pm it}$ so nominate the frequencies ± 1. The eigenvectors are just $1 \cdot e^{\pm it}$. Because for delay differential equations the time dependence $e^{\pm i\omega t}$ is an integral part of the definition of the eigenvector; hence the coded eigenvectors can be the same, as here, because they are differentiated through the time dependence $e^{\pm i\omega t}$.

The code works for orders higher than cubic, but is slow: takes about a minute per iteration.

The procedure actually analyses the embedding system

$$\frac{du}{dt} = -[1+\varepsilon u(t)]u(t-\pi/2) - \varepsilon^2 a[1+u(t)]u(t-\pi/2).$$

The centre manifold These give the location of the invariant manifold in terms of parameters s_j.

$$u_1 = \mathrm{e}^{-it}s_2 + \mathrm{e}^{-2it}s_2^2\varepsilon\big(1/5i+2/5\big) + \mathrm{e}^{it}s_1 + \mathrm{e}^{2it}s_1^2\varepsilon\big(-1/5i+2/5\big)$$

Centre manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\dot s_1 = s_2 s_1^2\varepsilon^2\big(- 2/5i\pi - 12/5i - 6/5\pi + 4/5\big)/\big(\pi^2 + 4\big) + s_1 a\varepsilon^2\big(4i + 2\pi\big)/\big(\pi^2 + 4\big)$$

$$\dot s_2 = s_2^2 s_1\varepsilon^2\big(2/5i\pi + 12/5i - 6/5\pi + 4/5\big)/\big(\pi^2 + 4\big) + s_2 a\varepsilon^2\big(- 4i + 2\pi\big)/\big(\pi^2 + 4\big)$$

6.2 `logistic1dde`: Logistic DDE displays a Hopf bifurcation

Form a centre manifold for the delayed 'logistic' system in one variable, for delay $\tau = 3\pi/4$, with

$$\frac{du}{dt} = -u(t) - (\sqrt{2} + a)u(t-\tau) + \mu u(t-\tau)^2 + \nu u(t-\tau)^3,$$

for and nonlinearity parameters μ and ν, and small parameter a. Numerical computation of the spectrum indicates that the system has a Hopf bifurcation as parameter a crosses zero.

We code the parameter a as 'small', and observe it is consequently considered as 'small squared' because all nonlinear terms, and already 'small' terms, are multiplied by ε (`small`).

Start by loading the procedure.

```
in_tex "../invariantManifold.tex"$
```

In the printed output, group terms with like powers of amplitudes s_j, the complex exponential, and the parameters.

```
factor s,exp,a,mu,nu;
```

Execute the construction of the slow manifold for this system (ignore the warning messages about `u1` declared, and then already defined, as an operator).

```
invariantmanifold(
    mat(( -u1-(sqrt(2)+small*a)*u1(3*pi/4)
    +mu*u1(3*pi/4)^2 +small*nu*u1(3*pi/4)^3 )),
    mat((i,-i)),
    mat((1),(1)),
    mat((1),(1)),
    3)$
end;
```

The marginal modes are $e^{\pm it}$ so nominate the frequencies ± 1. The eigenvectors are just $1 \cdot e^{\pm it}$. Because for delay differential equations the time dependence $e^{\pm i\omega t}$ is an integral part of the definition of the eigenvector; hence the coded eigenvectors can be the same, as here, because they are differentiated through the time dependence $e^{\pm i\omega t}$.

The procedure actually analyses the embedding system

$$\dot u_1 = -a\varepsilon^2 u_1(t-\tau) + \mu\varepsilon u_1(t-\tau)^2 + \nu\varepsilon^2 u_1(t-\tau)^3 - \sqrt{2}u_1(t-\tau) - u_1\,.$$

The centre manifold These give the location of the invariant manifold in terms of parameters s_j.

$$u_1 = \mathrm{e}^{-it} s_2 + \mathrm{e}^{-2it} s_2^2 \mu\varepsilon\big(-0.07901i + 0.2698\big) + \mathrm{e}^{it} s_1 + \mathrm{e}^{2it} s_1^2 \mu\varepsilon\big(0.07901i + 0.2698\big) + 0.8284 s_2 s_1 \mu\varepsilon$$

Centre manifold ODEs The system evolves on the invariant manifold such that the parameters evolve according to these ODEs.

$$\dot{s}_1 = s_2 s_1^2 \mu^2 \varepsilon^2 \big(-0.1303i - 0.5209\big) + s_2 s_1^2 \nu \varepsilon^2 \big(-0.1262i - 0.7206\big) + s_1 a \varepsilon^2 \big(0.04205i + 0.2402\big)$$

$$\dot{s}_2 = s_2^2 s_1 \mu^2 \varepsilon^2 \big(0.1303i - 0.5209\big) + s_2^2 s_1 \nu \varepsilon^2 \big(0.1262i - 0.7206\big) + s_2 a \varepsilon^2 \big(-0.04205i + 0.2402\big)$$

Hence the centre manifold model predicts a supercritical Hopf bifurcation as parameter a increases through zero.

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