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A HOMEWORK EXERCISE IN FINITE ELEMENTS

GILBERT STRANG

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, U.S.A.

SUMMARY

We suggest an exercise which illustrates many of the fundamental ideas of finite elements, and is nevertheless easily programmed and quickly solved. It is governed by a partial differential equation on a circular region, but by maintaining symmetry the stiffness matrix becomes small and tridiagonal. The accuracy of linear elements can be studied in detail, and a straightforward extension leads to the eigenvalue problem, elements of higher degree, numerical integration, isoparametric elements, and a simple example in plasticity.

This note deals, not with a sophisticated new element or a subtle application, but only with the most familiar element and the simplest of all partial differential equations. My object is to propose a problem so elementary that it can be assigned at a very early stage of a course on finite elements. For such a purpose, it has to be extremely flexible; the discrete equations can be assembled with pencil and paper, and solved either on a pocket calculator or as a very inexpensive exercise on the computer. In the latter case one can vary the number of elements, or the loading, or the degree of the polynomial shape functions, and study the effect on the complexity of programming and on the accuracy. It is a chance, often desperately needed, for the student to take some real initiative.

The governing equation ought to come from a problem in statics, and we choose

$$-u_{xx}-u_{yy} = 4 \text{ on the disk } x^2+y^2 \leqslant 1$$

This describes the deflection of a circular membrane under a uniform load, and the boundary will be fixed:

$$u = 0 \text{ on the circle } x^2+y^2 = 1$$

The first step in the exercise is to solve for u. Of course the reader may object that this is exceedingly easy, which it is, or that it only involves an ordinary differential equation in the radius r: the θ co-ordinate disappears because everything is rotationally symmetric. In fact, this objection is exactly our main point: if the finite element approximation preserves that symmetry, then we shall have *a two-dimensional problem in which the number of degrees of freedom* (and the consequent work and expense) *is no greater than in one dimension.* It is in that way that we can keep the exercise small enough for pencil and paper, and for plenty of computer runs.

To repeat, we will not transform to polar co-ordinates and discover that $u = 1-r^2$. Of course it is entirely proper to introduce the whole finite element technique with a problem in one variable. But our object is to take the next step, and work with *linear elements on triangles.*

The first requirement is to triangulate the circle. We do so in a rotationally symmetric way, starting with N concentric circles of radius $r_0 = 1 > r_1 > r_2 > \ldots > r_N = 0$. On the circumference of each circle we place M equally spaced nodes. Among many possibilities, we propose

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the following arrangement: on the circles of radius $r_0, r_2, r_4 \ldots$ let the nodes fall at *even* multiples of the angle $\alpha = \pi/M$, and on the circles of radius $r_1, r_3 \ldots$ let them fall at *odd* multiples $\theta = \alpha, 3\alpha, \ldots, (2M-1)\alpha$. (See Figure 1.) Then there is a natural way to join the M nodes on one circle

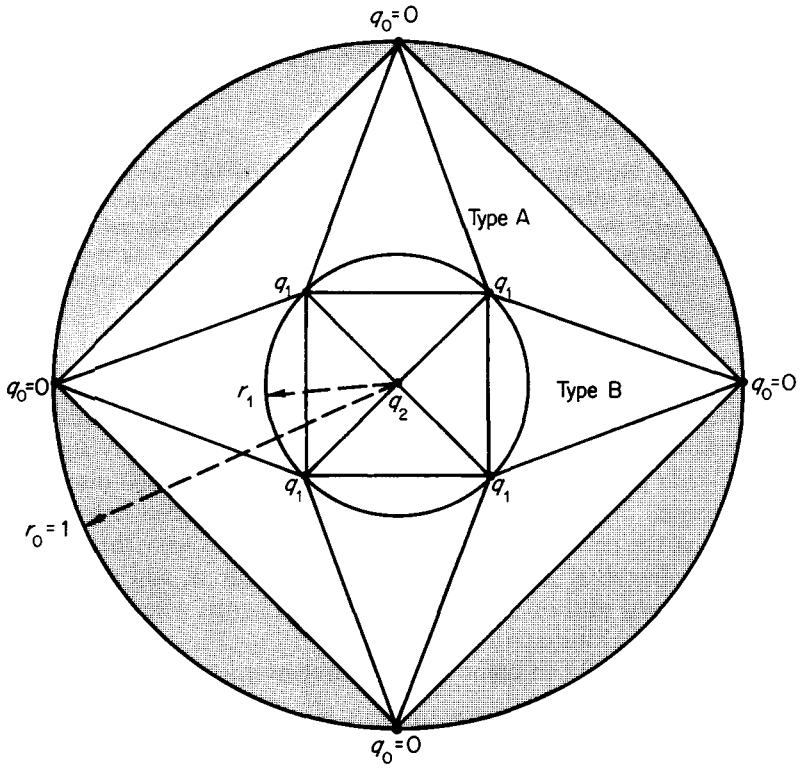

Figure 1. Symmetric triangulation of a circle: $N = 2$, $M = 4$

to the M nodes on the next, creating $2M$ triangles in each of the annular 'rings'. Half the triangles point inward (Type A) and half point outward (Type B); the only exception is the innermost circle, whose M triangles are all of Type A. Since the angles at the centre are very small if M is large, one side problem is to study their effect on the condition of the stiffness matrix and on the accuracy of the approximation; according to the theory, a single small angle in any element ought not to harm the accuracy.

This construction would seem to produce MN different nodes, in addition to the one at the centre. The order of the stiffness matrix would then be $MN+1$—or slightly less because $u = 0$ is specified at the M nodes on the outer circle. Nevertheless, you see immediately that in fact *there are only N unknowns.* On each circle the approximate solution must take the same value at every node, since a rotation through 2α leaves the whole problem unchanged. Therefore we introduce the notation $q_1, \ldots, q_N$ for the nodal values on the circles of radius $r_1, \ldots, r_N$; on the outer circle we set $q_0 = 0$ because of the boundary condition.

The next step in the exercise is to compute the element stiffness matrices, and to assemble them into a global matrix $\mathbf{K}$ of order N. In principle, that means the computation over every triangle of $\iint (v_x^2 + v_y^2)\,dx\,dy$, where v is the linear function taking specified values at the

vertices of the triangle. The student soon discovers, however, that for each ring there are only two calculations, one for each type of triangle. They both give a simple exercise in the introduction of local co-ordinates. For a triangle of Type A in the ring $r_i \leqslant r \leqslant r_{i-1}$, the shape function v takes the value q_{i-1} at both of the outer vertices. By linearity it keeps this value along the edge which connects them. Since it jumps to the value q_i at the other node, its slope is easy to find. In fact, with the triangle oriented as in Figure 2, v depends only on x:

$$v = \frac{q_{i-1}-q_i}{r_{i-1}\cos\alpha - r_i}x + \text{constant}$$

Now the element integral is easy:

$$\iint (v_x^2+v_y^2)\,dx\,dy = \text{area}\cdot\text{slope}^2 = c_i(q_{i-1}-q_i)^2$$

Because the same unknown appeared at two nodes, the usual 3 by 3 record of this element integral contracts into 2 by 2:

$$c_i(q_{i-1}-q_i)^2 = [q_{i-1} \quad q_i]\begin{bmatrix} c_i & -c_i \\ -c_i & c_i \end{bmatrix}\begin{bmatrix} q_{i-1} \\ q_i \end{bmatrix}$$

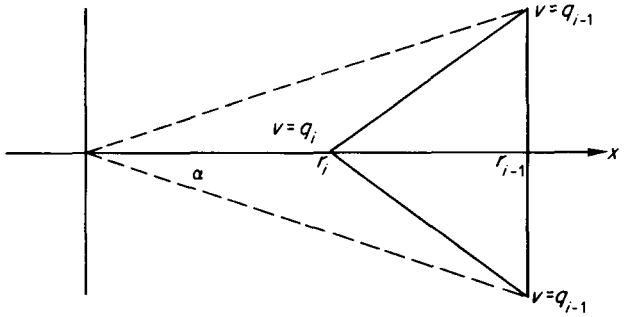

Figure 2. A triangle of Type A, oriented along the x-axis

The contribution from a triangle of Type B will look exactly the same, with a different constant c_i'. (In the innermost circle, the formulas should produce $c_N' = 0$.) Then, since there are M identical pairs of triangles around the ring, their total contribution is just

$$q^{\mathrm{T}}k_i q = [q_{i-1} \quad q_i]\begin{bmatrix} C_i & -C_i \\ -C_i & C_i \end{bmatrix}\begin{bmatrix} q_{i-1} \\ q_i \end{bmatrix}, \qquad C_i = Mc_i + Mc_i'$$

Note that k_i is symmetric and positive semidefinite, as it should be. In the outer circle, $q_0 = 0$ is fixed by the boundary condition and we have only $C_1 q_1^2$.

Now for assembly. The global matrix $\mathbf{K}$ will be tridiagonal, because q_i is linked only to q_{i-1} and q_{i+1}. It will also be symmetric positive definite, and it is assembled according to

$$\mathbf{K} = \begin{bmatrix} C_1+C_2 & -C_2 & & \\ -C_2 & C_2+C_3 & -C_3 & \\ & -C_3 & \cdot & \cdot \\ & & \cdot & C_{N-1}+C_N \end{bmatrix}$$

The entries depend on α and on the r_i; it is natural to start with equally spaced rings.

The next step is to compute the load vector **F**, coming from the right-hand side 4 in the original differential equation. To do that absolutely systematically, we should integrate over each triangle:

$$\iint 4v \, dx \, dy = \begin{cases} d_i q_{i-1} + e_i q_i & \text{for Type A in ring } i \\ d_i' q_{i-1} + e_i' q_i & \text{for Type B in ring } i \end{cases}$$

Summing over all the triangles, the coefficient of each particular q_i is $F_i = M(e_i + e_i' + d_{i+1} + d_{i+1}')$ —reflecting the contributions from the two rings which are influenced by the given q_i.

The student should do all that, but ultimately—because a constant load is so important a special case—he should recognize that it is really the *areas* of the triangles which appear in these coefficients. For a given node, the coefficient F_i will normally be the total area of the triangles which contain that node (times 4, the value of the load). In our case, the M identical configurations produce the F_i given above. One way or the other, we arrive at **F**.

Now the total energy in the trial function v is

$$I(v) = \iint (\tfrac{1}{2}v_x^2 + \tfrac{1}{2}v_y^2 - fv) \, dx \, dy = \tfrac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q} - \mathbf{q}^T\mathbf{F}$$

The rules ask us to minimize this quadratic, and therefore to solve $\mathbf{K}\mathbf{q} = \mathbf{F}$; the solution vector $\mathbf{q} = (q_1, \ldots, q_N)$ will give the nodal values in our finite element approximation to the solution $u = 1 - r^2$. At this point the computer is likely to enter, but hopefully not the whole strength of the computation centre; it is an essential part of the exercise to recognize the special simplicity of a tridiagonal matrix. In fact, a pocket calculator is perhaps a better way to begin, using small values of M and N. With the differences $q_{i-1} - q_i$ as new unknowns, even a closed formula may be possible—but much less instructive than numerical values which can be compared with $u = 1 - r_i^2$. A point to examine is the separate dependence on M and N, which can make the triangles very thin in one direction or the other. It would be useful to study the effect on the condition number $\lambda_{\max}/\lambda_{\min}$ of **K**, and then on the round-off error itself. (Note: a large value of N can 'flip' the triangles of Type A, when $r_{i-1}\cos\alpha - r_i$ becomes negative and the geometry degenerates—but very possibly without spoiling the solution!) Certainly the error at points such as the centre should be graphed, to recognize that it decreases like $1/N^2$ when M is kept proportional to N.

There is also the error in the slope. Mathematically the natural measure of length is the strain energy $\iint v_x^2 + v_y^2$, and there are two essential points:

(i) the approximation is always too 'stiff,' so the strain energy in u exceeds the strain energy in the approximate solution U—even though the total energies satisfy $I(u) < I(U)$ (see Reference 1, p. 40).

(ii) there is a 'right-angle' in function space between the approximation U and the error $u - U$, with u as the hypotenuse, so the energy in the error should agree with

$$\iint [(u-U)_x^2 + (u-U)_y^2] = \iint (u_x^2 + u_y^2) - \iint (U_x^2 + U_y^2)$$

Again the theory predicts a $1/N^2$ decay with linear elements; the error in the slopes is of order $1/N$ (assuming $M \sim N$) and this error is squared in computing the strain energy.

The basic exercise is now complete, and we summarize the main steps:

1. Find the true solution u.
2. Establish the triangulation.
3. Compute the element integrals.

4. Assemble **K** and **F**.
5. Solve $\mathbf{Kq} = \mathbf{F}$.
6. Compute the errors in function values and slopes, and study their dependence on M and N.

For convenience we include a few experimental values, whose accuracy we do not guarantee (despite the fact that the homework grade was A). The circles were equally spaced and the value given is q_N, the approximation to the true value $u = 1$ at the centre:

	$M = 4$	$M = 8$	$M = 16$	$M = 32$
$N = 2$	0·50	0·96	1·07	1·10
$N = 4$	?	0·84	0·998	1·026

At $N = 100$, the errors are of course very different:

$$U = 0{\cdot}998 \text{ at } M = 100, \quad 0{\cdot}9998 \text{ at } M = 200, \quad 1{\cdot}0001 \text{ at } M = 1000$$

It is natural to think about the limiting problem as $M \to \infty$.

EXTENSIONS OF THE EXERCISE

I believe that the preceding problem will lead to fruitful discussions and comparisons: it can be shared by a whole class. Later in the term it is natural to ask for separate presentations of individual work, and in many cases that means preliminary research on a thesis. But the exercise itself also suggests some questions, going more deeply into the technique of finite elements, which can serve a valuable purpose:

The eigenvalue problem

With the same linear elements, the Rayleigh–Ritz approximation to $-u_{xx} - u_{yy} = \lambda u$ leads as always to a discrete eigenvalue problem $\mathbf{Kx} = \lambda_h \mathbf{Mx}$. Both matrices are tridiagonal; **K** is the stiffness matrix as before, and the *mass matrix* **M** comes from the undifferentiated terms: $\mathbf{q}^T\mathbf{Mq} = \iint v^2 \, dx \, dy$, which is the denominator in the Rayleigh quotient for λ.

Determining the exact frequency λ will give some experience with Bessel functions; computing λ_h will require a decision among the alternative approaches to the generalized eigenvalue problem (with two matrices instead of one). The conversion to $\mathbf{M}^{-1}\mathbf{K}$ or to $\mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}$ gives a full matrix, and I think that the merits of subspace iteration or determinant search—which stay with sparse matrices—should be recognized.

Elements of higher degree

For the original problem with solution $u = 1 - r^2$, and for the same triangulation, it is not difficult to introduce elements of degree two. There are six nodes in each triangle—the three vertices, the midpoints of the two equal edges, and the remaining midpoint. The vertex unknowns are denoted by q_{i-1} and q_i as before, the two matching midpoints share a common parameter y_i, and the third midpoint contributes a further unknown—z_{i-1} or z_i, depending on whether the triangle is of Type A or B. The element stiffness matrices are of order four, and after assembly **K** may have nine non-zero entries along a row.

One feature of the errors in this case is that they come *entirely from the approximation of the domain.* The true solution $1 - x^2 - y^2$ can be reproduced exactly by quadratic shape functions,

but the change from a circle to a polygon introduces an error in the boundary condition: along the flat boundary we have not only $q_0 = 0$ which is right, but also $z_0 = 0$ which is wrong. The theory in Reference 1 predicts an error of order h^2 in the function values as well as the gradients in the interior; there should be a boundary layer in which the error in slopes is much larger. I hope it is there.

The other new feature deserves separate mention:

The use of numerical integration

Both for the mass matrices in the eigenvalue problem and for the stiffness matrices introduced above, the integrands are of degree two. Stricly speaking, the use of numerical quadrature is not yet necessary, but it is certainly convenient; it would be instructive to do the element integrals both ways. One possible formula, exact for P of degree two, is

$$\int_T \int P \, dx \, dy = \frac{\text{area of } T}{3} \sum_{i=1}^{3} P(\text{midpoint } i)$$

Bilinear and isoparametric elements

Suppose we change from triangles to quadrilaterals (or really trapezoids) which have two edges going out along rays and the other two cutting across as chords of the circles $r = r_{i-1}$ and $r = r_i$. Then, since these are not rectangles, the bilinear elements $a+bx+cy+dxy$ would ordinarily be discontinuous across element boundaries (and therefore non-conforming). In fact the symmetry in θ produces an exceptional situation, in which these elements are linear and do conform. Therefore they may be used as is. But there is also the opportunity to introduce the simplest isoparametric elements, by mapping the trapezoids onto squares and computing first the Jacobians and then the element integrals. I suspect that the symmetry will introduce an 'artificial' identity between the results for bilinear elements and the isoparametric calculations, which would be misleading but nevertheless extremely useful as a check. Normally, of course, exact integration is impossible and experimenting with different quadrature formulas and their effects is an important exercise (with the elements kept large).

More general loadings: $-u_{xx}-u_{yy} = f(x, y)$

We can either retain symmetry, for example with a point load at the centre, or abandon it and move toward an exercise which is fully two-dimensional. In the former case, since a point-loaded membrane is mathematically illegal (the deflection would include a $\log r$ term of infinite strain energy) there should be substantial difficulty with the element integrals at the centre. On the other hand a line load along a ray θ = constant or a circle r = constant should be legitimate—and should give some initial experience with singularities.

Plasticity and the St. Venant torsion problem

We come finally to a non-linear problem, using the original triangulation but with u constrained by a yield condition $u_x^2+u_y^2 \leqslant k^2$. The problem arises from the twisting of a long circular rod,[2] and with sufficient twist some annular region $R \leqslant r \leqslant 1$ will become plastic; there the stress function is $u = k(1-r)$, and the yield condition is satisfied exactly, but the differential equation is not. Within the ring $r \leqslant R$, we again have the elastic equilibrium $-u_{xx}-u_{yy} = 4$; now $u = k(1-R)$ is the boundary condition on this ring. The 'free boundary' $r = R$ is located by requiring that the slope $\partial u/\partial r$ at the boundary of the elastic region must

match the slope $-k$ in the plastic region; this occurs at $R = k/2$. All this comes from the so-called 'sandhill analogy,' which gives $u \leqslant k(1-r)$ as an alternative inequality constraint in this special problem; in the plastic region this constraint is active—equality holds—and in the elastic region there is inequality and the differential equation takes over.

In the discrete problem, with linear elements and symmetry in θ, the finite element approximation u_h is determined by the following property: it still minimizes the total energy

$$I(v) = \iint \tfrac{1}{2}v_x^2 + \tfrac{1}{2}v_y^2 - fv = \tfrac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q} - \mathbf{q}^T\mathbf{F},$$

but only over those trial functions which satisfy the yield condition. If we use the second and simpler form $q_i \leqslant k(1-r_i)$ of this condition, the minimization becomes very illuminating: the equations $\mathbf{K}\mathbf{q} = \mathbf{F}$ are satisfied near the centre, but at a certain point $r = r_i$—the *discrete free boundary*—the solution would exceed the constraint and is therefore forced into the plastic range. From that point on $q_i = k(1-r_i)$. The problem is really one of quadratic programming, minimizing a quadratic objective function under linear constaints.

We end with one brief note about the continuous problem itself: the idea of allowing the elastic–plastic boundary to appear 'spontaneously' in the minimization, rather than making it an explicit part of the problem statement, has led to substantial progress in the existence theory for plasticity. The problems left unresolved at the end of Reference 2 are in many cases settled, for example in the variational inequalities of Duvaut–Lions[3] and the incremental plasticity theory of Johnson.[4] But the actual calculation of this solution, whose existence is now proved but was perhaps not seriously doubted, remains as difficult and expensive as ever—and finite elements remain our most powerful tool.

I hope that by developing effective courses in the subject we can make that tool better understood and better employed. That is the goal of this paper. I suppose it could be regarded as a rather belated appendix to my book with Fix, but of course I cannot prevent the reader from regarding it instead as an appendix to a completely different book on finite elements! Nor would I want to.

ACKNOWLEDGEMENT

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