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Notes on the Diffusion Equation

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1 Introduction

Consider an arbitrary solution $u : \mathbb{X} \times \mathbb{T} \mapsto \mathbb{R}$ to the simple diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \quad (1)$$

A computationally feasible approach would be to first establish $|\mathbb{J}|$ discrete grid-points, $\vec{X} = [X_j]_{j \in \mathbb{J}}$, and thence partition the spatial domain \mathbb{X} into contiguous intervals $\mathbb{I}_j := (X_{j-1}, X_j]$. The coarse dynamics at the grid-points are then summarised by $\vec{U} = [U_j]_{j \in \mathbb{J}}$, where $U_j(t) = u(X_j, t)$ for all $t \in \mathbb{T}$, according to some temporal evolution

$$\dot{\vec{U}}(t) = \vec{g}(\vec{U}(t)). \quad (2)$$

Consequently, a link from the coarse dynamics \vec{U} back to the continuum dynamics u might be provided by choosing an appropriate spatial mapping of the form

$$u := u(x, \vec{U}(t)). \quad (3)$$

Under this scheme, the linear diffusion equation (1) becomes

$$\frac{\partial u}{\partial \vec{U}} \cdot \vec{g} = \frac{\partial^2 u}{\partial x^2}. \quad (4)$$

Observe that the evolution of u now has nonlinear interactions with \vec{U} .

2 Centre Manifold Approximation

The original diffusion equation (1) admits eigensolutions of the form

$$u(x, t) = e^{\lambda t \pm \sqrt{\lambda} x}, \quad (5)$$

which are physically realisable for real eigenvalues $\lambda = -k^2 \leq 0$ for corresponding eigenmode wavenumbers $\pm k$. As a consequence, the transient solutions corresponding to $\lambda < 0$ decay rapidly to the centre manifold corresponding to $\lambda = 0$.

This centre manifold can be found in practice by iteratively refining approximations to u . In particular, consider a series expansion of the form

$$u \sim \hat{u}_0 + \gamma \hat{u}_1 + \gamma^2 \hat{u}_2 + \dots, \quad (6)$$

for some parameter $0 \leq \gamma \leq 1$. Now, the constant eigensolution for $\lambda = 0$ implies a slow evolution for the coarse dynamics given by equation (2), which therefore admits a series expansion of the form

$$\dot{\vec{U}} \sim \gamma \vec{g}_1 + \gamma^2 \vec{g}_2 + \dots. \quad (7)$$

Hence, equation (4) may be decomposed at each order ℓ of the parameter γ , giving

$$\frac{\partial^2 \hat{u}_0}{\partial x^2} = 0, \quad (8)$$

$$\frac{\partial^2 \hat{u}_\ell}{\partial x^2} = \sum_{m=0}^{\ell-1} \frac{\partial \hat{u}_m}{\partial \vec{U}} \cdot \vec{g}_{\ell-m}, \quad \text{for } \ell = 1, 2, \dots. \quad (9)$$

3 Leading Approximation

The leading equation (8) admits any spatially piecewise linear function as a solution. Hence, in keeping with the discretisation imposed by the coarse dynamics, consider the linear approximation

$$\hat{u}_0 = \sum_{j \in \mathbb{J}} \chi_j (\xi_j U_j + (1 - \xi_j) U_{j-1}), \quad (10)$$

where $\chi_j(x)$ is an indicator that takes on the value 1 (or 0) inside (or outside) of the j th interval, and $\xi_j(x) = \frac{x - X_{j-1}}{X_j - X_{j-1}}$ is a linear, spatial interpolator. This

particular approximation is chosen to be continuous across the boundaries of each interval. In general, it suffices to impose a continuity condition at the right-hand end of each arbitrary j th interval, namely:

$$[u]_j := \lim_{\epsilon \rightarrow 0^+} u(X_j + \epsilon, t) - u(X_j - \epsilon, t) = 0. \quad (11)$$

Unfortunately, this linear approximation is not smooth at the interval boundaries. For convenience, consider regular grid spacings of size $X_j - X_{j-1} = H$. Then, denoting $\partial u / \partial x$ as u' , observe that

$$[\hat{u}'_1]_j = \frac{1}{H}(U_{j+1} + U_{j-1} - 2U_j) = \frac{1}{H} \delta^2 \hat{u}_0|_{X_j}, \quad (12)$$

for the centred difference $\delta u(x, t) := u(x + \frac{H}{2}, t) - u(x - \frac{H}{2}, t)$. However, this non-smoothness may be corrected at higher order by imposing a further internal boundary condition, namely

$$[u']_j = \frac{1 - \gamma}{H} \delta^2 u|_{X_j}. \quad (13)$$

Consequently, smooth approximations are found in the limit as $\gamma \rightarrow 1$.

4 Linear Eigenmode Analysis

Consider a single eigenmode of the form (5) for some fixed, non-dimensionalised wavenumber $\kappa = kH > 0$. Thus, allowing for the partitioning of \mathbb{X} , let

$$u \sim \sum_{j \in \mathbb{J}} \chi_j a_j e^{i\kappa \xi_j} + \text{c.c.}, \quad (14)$$

for arbitrary, time-varying, complex coefficients $a_j = A_j + iB_j$. We now seek the ‘spatial’ evolution from interval to interval for the given wavenumber. The continuity condition (11) implies that

$$a_{j+1} - a_j e^{i\kappa \xi_j} + \text{c.c.} = 0. \quad (15)$$

Similarly, the smoothness condition (13) implies that

$$ika_{j+1} - ika_j e^{i\kappa} + \text{c.c.} = \frac{1 - \gamma}{H} (a_{j+1} e^{i\kappa} + a_j - 2a_j e^{i\kappa}) + \text{c.c.}, \quad (16)$$

where continuity has also been invoked at the left-hand of the j th interval. In coefficient form, the update from the j th to $(j + 1)$ th segment is

$$\begin{bmatrix} 1 & 0 \\ fc & 1 - fs \end{bmatrix} \begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} = \begin{bmatrix} c & -s \\ s + f(2c - 1) & c - 2fs \end{bmatrix} \begin{bmatrix} A_j \\ B_j \end{bmatrix}, \quad (17)$$

where $c + is := e^{i\kappa}$ and $f := \frac{1-\gamma}{\kappa}$. Now, letting $a_{j+1} = \mu a_j$, the characteristic equation for the growth factor μ is

$$(1 - fs) \left[\mu^2 - 2 \frac{c - fs}{1 - fs} \mu + 1 \right] = 0, \quad (18)$$

with roots given by

$$\mu = \beta \pm \sqrt{\beta^2 - 1} \quad \text{for } \beta = \frac{c - fs}{1 - fs}. \quad (19)$$

Observe that $\beta \leq 1$ since $c = \cos \kappa \leq 1$ and $1 - fs = 1 - (1 - \gamma) \frac{\sin \kappa}{\kappa} \geq 0$. Thus, for $|\beta| < 1$, the factors are complex with magnitude $|\mu| = 1$, indicating marginally stable evolution of a_j . This includes the limiting case of $\gamma = 1$ ($f = 0$), for which $\mu = c \pm is = e^{\pm i\kappa}$. Likewise, $\mu = \pm 1$ for $\beta = \pm 1$, corresponding to $\kappa = n\pi$, $n = 0, 1, 2, \dots$. Finally, for small regions near each $\kappa = (2n + 1)\pi$, it is found that $\beta < -1$, resulting in two real factors, $\mu < -1$ and $-1 < \mu < 0$, indicating unstable (saddle) evolution. More precisely, these unstable regions occur when

$$\frac{\kappa}{2} < (1 - \gamma) \tan \frac{\kappa}{2}, \quad \kappa \neq n\pi. \quad (20)$$

Thus, at equilibrium ($\gamma = 0$) there is an initial forbidden gap $\kappa \in (0, \pi)$ adjacent to the centre manifold wavenumber $\kappa = 0$, indicating that transient solutions decay to the centre manifold at a rate of at least $\lambda = -\frac{\kappa^2}{H^2} = -\frac{\pi^2}{H^2}$. It is this gap that provides robustness to nonlinear perturbations of the system about the equilibrium.

5 Linear Dual Space

Assuming a spatially square-integrable field over \mathbb{X} , an appropriate inner product for a solution u and its dual v can be shown to obey

$$\langle u'', v \rangle = \langle u, v'' \rangle + R, \quad (21)$$

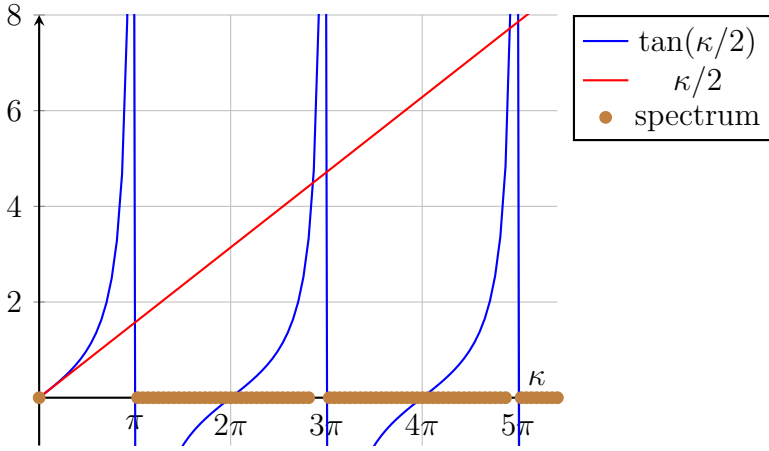


Figure 1: The equilibrium spectrum determined by the forbidding condition (20).

with a residual

$$R = \sum_{j \in \mathbb{J}} [u'v - v'u]_{X_{j-1}}^{X_j} . \quad (22)$$

Now, letting $r = u'v - v'u$, the residual becomes

$$R = r_{\bar{J}} - r_{\underline{J}} - \sum_{j=\underline{J}+1}^{\bar{J}-1} [r]_j , \quad (23)$$

for $\underline{J} = \inf \mathbb{J}$ and $\bar{J} = \sup \mathbb{J}$, if there are at least two intervals. Assuming that v also obeys conditions (11) and (13), the j -th residual jump becomes

$$\begin{aligned} [r]_j &= [u']_j V_j - [v']_j U_j \\ &= \frac{1-\gamma}{H} [(U_{j+1} + U_{j-1})V_j - (V_{j+1} + V_{j-1})U_j] . \end{aligned} \quad (24)$$

Observe that terms from adjacent interval boundaries will cancel, leaving only contriutions from the outermost boundary intervals, namely:

$$\begin{aligned} R &= u'_{\bar{J}} V_{\bar{J}} - v'_{\bar{J}} U_{\bar{J}} - u'_{\underline{J}} V_{\underline{J}} + v'_{\underline{J}} U_{\underline{J}} \\ &\quad - \frac{1-\gamma}{H} [U_{\underline{J}} V_{\underline{J}+1} - V_{\underline{J}} U_{\underline{J}+1} + U_{\bar{J}} V_{\bar{J}-1} - V_{\bar{J}} U_{\bar{J}-1}] . \end{aligned} \quad (25)$$

There are three main boundary conditions on a finite domain that lead to a zero residual:

periodic TODO;

Dirichlet Setting $u = 0$ on the boundaries corresponds to $U_{\underline{J}} = U_{\bar{J}} = 0$, giving $R = 0$ if correspondingly $v = 0$ on the boundaries.

Neumann Requiring $u' = 0$ on the boundaries corresponds to choosing $u'_{\underline{J}} = \frac{1-\gamma}{H}(U_{\underline{J}+1} - U_{\underline{J}})$ and $u'_{\bar{J}} = \frac{1-\gamma}{H}(U_{\bar{J}} - U_{\bar{J}-1})$, giving $R = 0$ if correspondingly $v' = 0$ on the boundaries.

Under any of the above three conditions, observe that $\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}v \rangle$ for $\mathcal{L} = \partial^2/\partial x^2$, and hence \mathcal{L} is self-adjoint. Furthermore, we are free to choose any dual v to satisfy $\mathcal{L}v = 0$ for convenience. In particular, we may specifically target the j -th interval for u , for each $j \in \mathbb{J}$, with $v = \hat{v}_0^{[j]}$, where

$$\hat{v}_0^{[j]} = \chi_j \xi_j + \chi_{j+1} (1 - \xi_{j+1}), \quad (26)$$

which satisfies conditions (11) and (13) for $\gamma = 0$.

6 First-order Approximation

Substituting the leading approximation (10) into the nonlinear diffusion equation (9) for $\ell = 1$ results in the first-order equation

$$\hat{u}_1'' = \sum_{j \in \mathbb{J}} \chi_j (\xi_j g_{1,j} + (1 - \xi_j) g_{1,j-1}). \quad (27)$$

Spatial integration then gives

$$\hat{u}_1' = \frac{H}{2} \sum_{j \in \mathbb{J}} \chi_j (\xi_j^2 g_{1,j} - (1 - \xi_j)^2 g_{1,j-1} + c_{1,j}), \quad (28)$$

$$\hat{u}_1 = \frac{H^2}{6} \sum_{j \in \mathbb{J}} \chi_j (\xi_j^3 g_{1,j} + (1 - \xi_j)^3 g_{1,j-1} + 3\xi_j c_{1,j} + d_{1,j}). \quad (29)$$

Recall from the chosen spatial discretisation that $u|_{X_j} = U_j$ at each grid-point. Observe this is already satisfied by \hat{u}_0 from equation (10), implying from expansion (6) that

$$\hat{u}_\ell|_{X_j} = 0 \quad \text{for } \ell = 1, 2, \dots. \quad (30)$$

Thus $[\hat{u}_\ell]_j = 0$, satisfying the continuity condition (11), and $\delta^2 \hat{u}_\ell|_{X_j} = 0$. Now, evaluating equation (29) at $\xi_j = 0$ gives $d_{1,j} = -g_{1,j-1}$, and at $\xi_j = 1$ gives $3c_{1,j} = -(g_{1,j} - g_{1,j-1})$. Thus, from equation (28), observe that

$$[\hat{u}'_1]_j = -H(1 + \frac{1}{6}\delta^2)g_{1,j}. \quad (31)$$

However, the smoothness condition (13) gives

$$[\hat{u}'_1]_j = \frac{1}{H} \delta^2 \hat{u}_1|_{X_j} - \frac{1}{H} \delta^2 \hat{u}_0|_{X_j} = -\frac{1}{H} \delta^2 U_j, \quad (32)$$

and hence

$$\left(1 + \frac{1}{6}\delta^2\right)g_{1,j} = \frac{1}{H^2}\delta^2 U_j. \quad (33)$$

This solution can also be obtained more directly by solving the dual equation $\langle \mathcal{L}\hat{u}_1, \hat{v}_0^{[j]} \rangle = 0$.