Calculus II

Class Information

Lecture (14): Tuesday, Friday

Exercise (7): Wednesday

Grades
- 20% Assignment (>2/3)
- 80% Examination
- Attendance (>2/3)

Office hours: After each class

Textbook: Calculus, 6th Edition, James Stewart

Reading materials: MIT Open Course 18-02SC Multivariable Calculus

Prerequisites

All enrolled students must have taken *Calculus I*, *Linear Algebra*

What we will cover

Full syllabus on course website

(Mainly 2 and 3 variables, but it is able to expand to more variables)

Part 1.Multivariable calculus - differentials

Parametrized curves, Partial differentials, Differentiation of composite functions, chain rule, Local minimum/maximum of a function, Implicit functions

Part 2. Multivariable calculus - integrals

Double Integral, Triple Integral, Area, Volume

Part 3. Differential equations

Part 4. Sequence and series

Convergence, approximated functions

Differentials of Multivariable **Integrals of Multivariable** **Differential Equations** **Sequence and Series** **Review**

Calculus II

2018年6月

日	月	火	水	木	金	土
27	28	29	30	31	1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30
1	2	3	4	5	6	7

Calculus II

Lecture 1

Parametrized Curves

What you will learn in Lecture 1

I. How to **GRAPH EQUATIONS** in 3D-Space with **2-variables**

II. **VECTORS** and the **Geometry** of 3D-Space

III. **Parametrized Curves**

1.1 Reminder of Calculus I

Calculus I ≈ One-Variable Calculus

One-Variable Function

relationship, called **function**

$$y = f(x)$$

Independent variable

dependent variable

One-Variable Calculus -- derivative of f

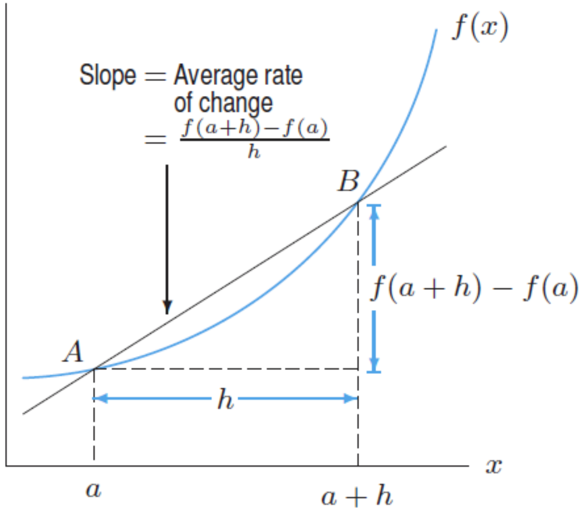
One-Variable Function

relationship, called function

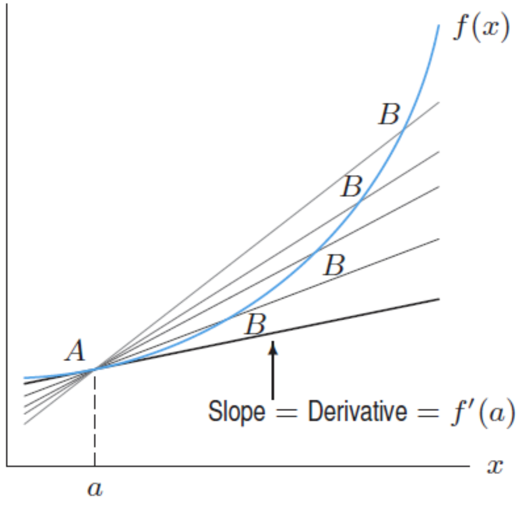
$y = f(x)$

Independent variable

dependent variable



Visualizing the average rate of change of f



Visualizing the instantaneous rate of change of f

The **derivative of f at a**, written $f'(a)$, is defined as

$$\begin{array}{c}\text{Rate of change} \\ \text{of } f \text{ at } a\end{array} = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If the limit exists, then f is said to be **differentiable at a**.

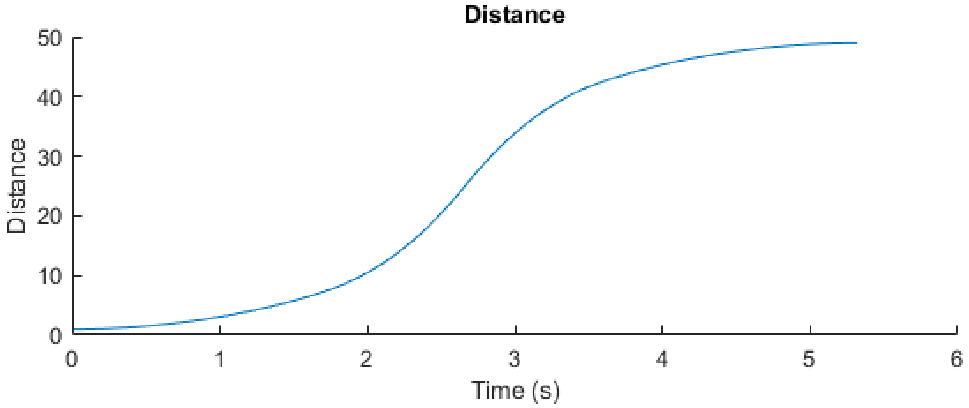
1.1 Reminder of Calculus I

One-Variable Calculus -- derivative of f

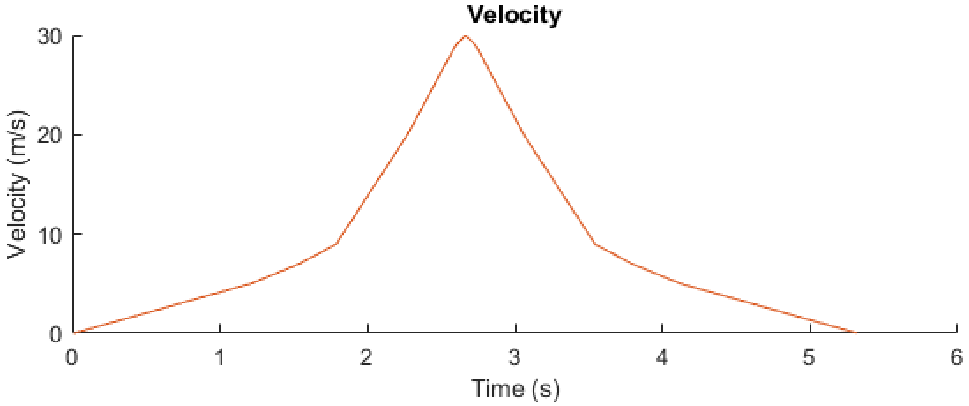
Q: How do we measure the instantaneous rate of change for distance with regard to time?

One-Variable Function $y = f(x)$ $\Longrightarrow$ $r = v_0 + vt$ Assume that $v_0 = 0$

Distance



Velocity



What you will learn in Lecture 1

I. How to **GRAPH EQUATIONS** in 3D-Space with **2-variables**

II. **VECTORS** and the **Geometry** of 3D-Space

III. Parametrized Curves

Goal of multivariable calculus:

Tools to handle problems with function of several variables.

1.2 Function of Several Variables

(We have One-Variable in *Calculus I*)

1.2 Function of Several Variables

Notation

One-Variable Function

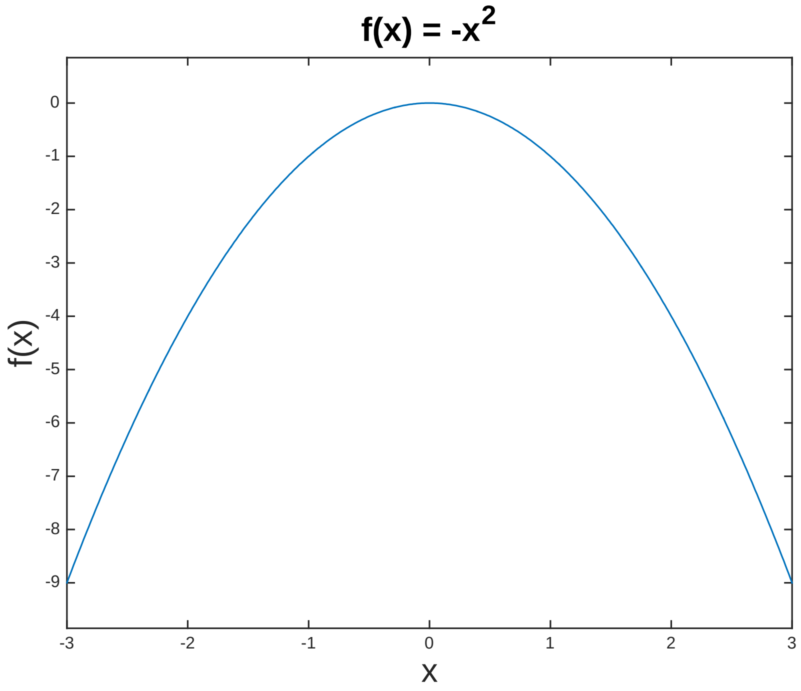
$$y = f(x)$$

$$z = f(x, y)$$

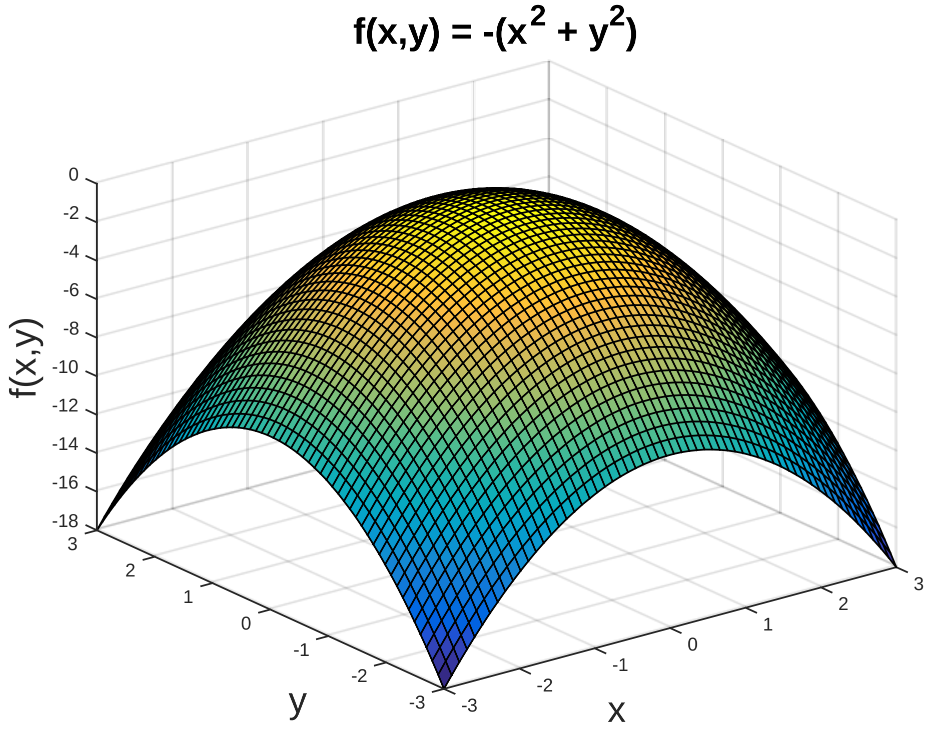
Two-Variable Function

Three-Variable Function

⋮



One-variable problem

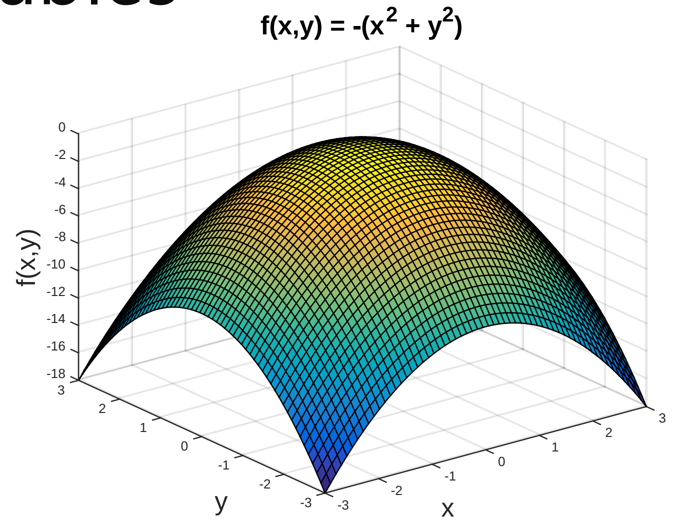


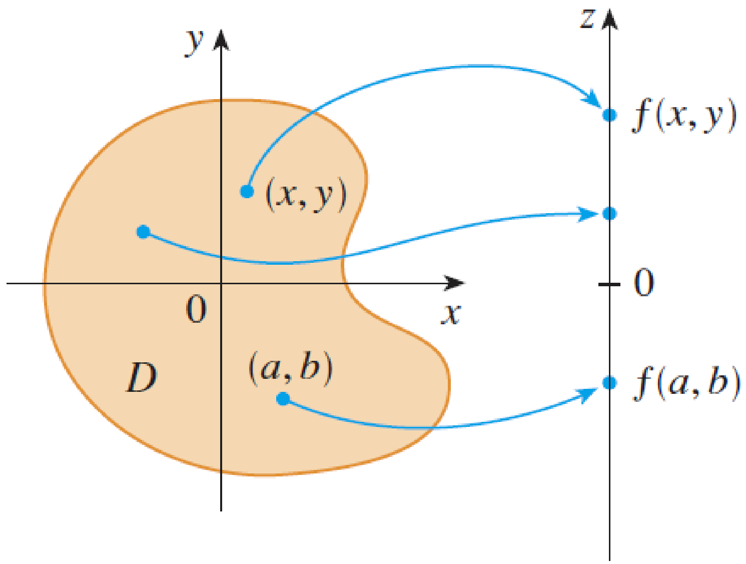
Two-variable problem

Three-variable problem

⋮

1.2 Function of Several Variables





Two-Variable Function

relationship, called **function**

$$z = f(x, y)$$

Independent variables

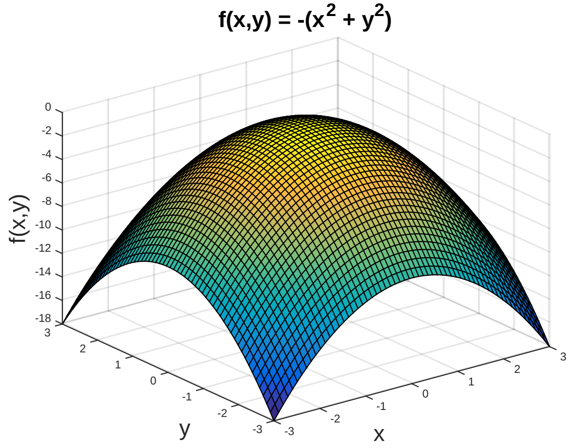
dependent variable

1.2 Function of Several Variables

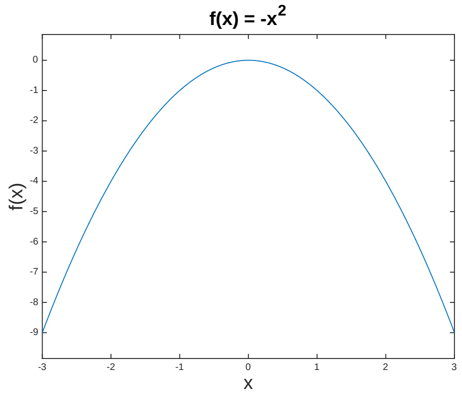
Definition of $\mathbb{R}^3$

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$.

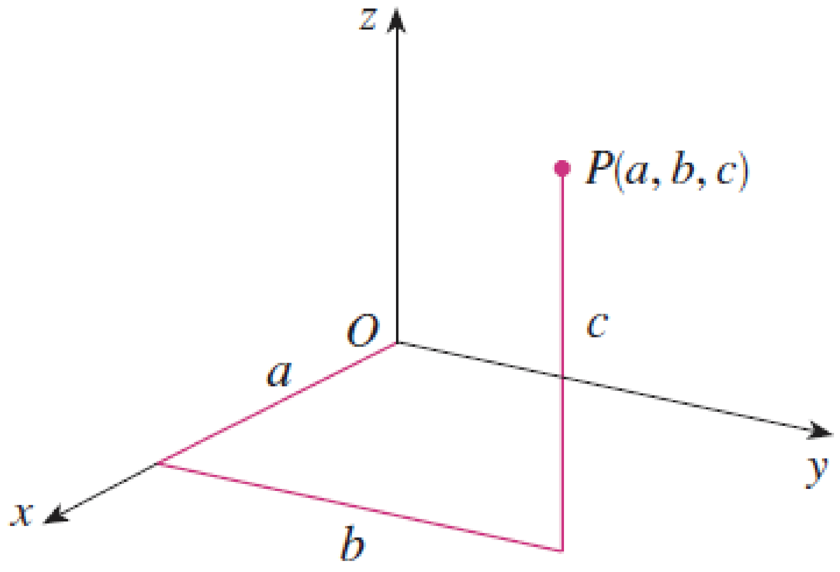
In 3D, an equation in x, y, and z represents a ***surface*** in $\mathbb{R}^3$.



In 2D, the graph of an equation involving x and y is a ***curve*** in $\mathbb{R}^2$.

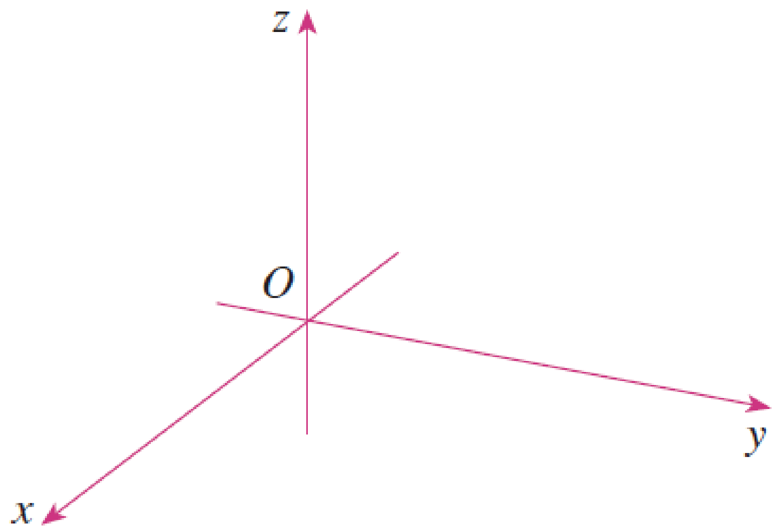


1.3 Graph of function with 2-Variables

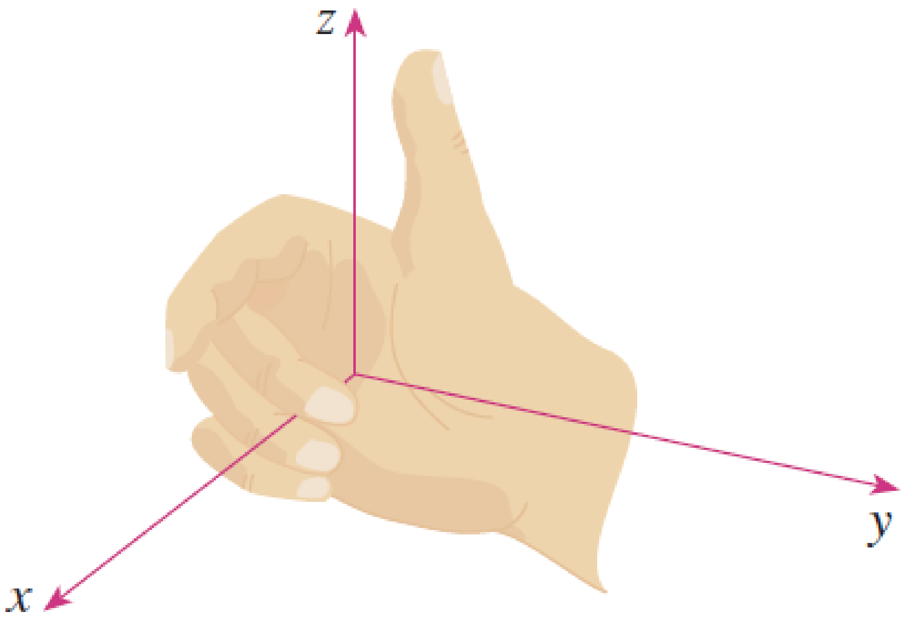


1.3 Graph of function with 2-Variables

1.3.1 Coordinate axes in space



Coordinate axes



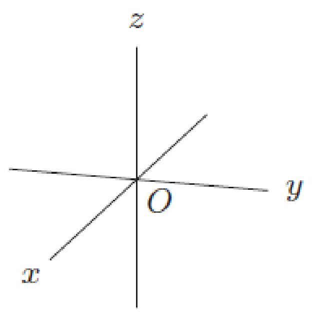
Right-hand rule

1.3 Graph of function with 2-Variables

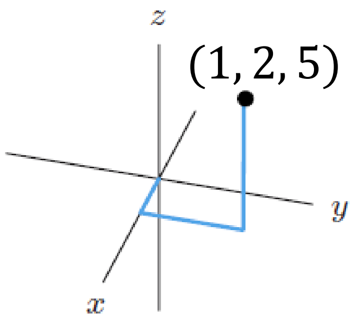
1.3.2 How to graph? By Point Table!

$z = f(x, y)$

Two-Variable Function



Coordinate axes in **3D-space**



The point $(1,2,5)$ in **3D-space**

Point Table of $f(x, y) = x^2 + y^2$

x \ y	-3	-2	-1	0	1	2	3
-3	18	13	10	9	10	13	18
-2	13	8	5	4	5	8	13
-1	10	5	2	1	2	5	10
0	9	4	1	0	1	4	9
1	10	5	2	1	2	5	10
2	13	8	5	4	5	8	13
3	18	13	10	9	10	13	18

Q: in the table, $x = 0, y = 0, z = ?$

Relationship between $z = f(x, y)$ and (x, y)

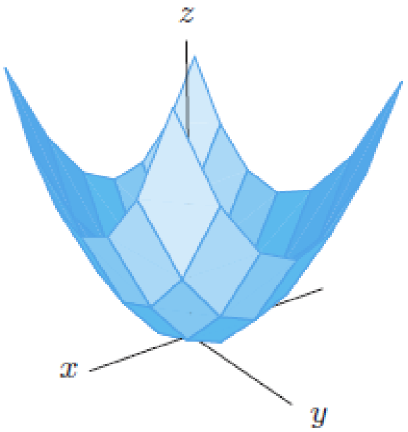
dependent variable Independent variable

1.3 Graph of function with 2-Variables

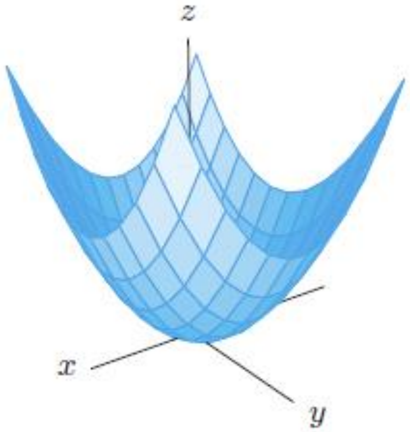
1.3.2 How to graph? By Point Table!

$z = f(x, y)$

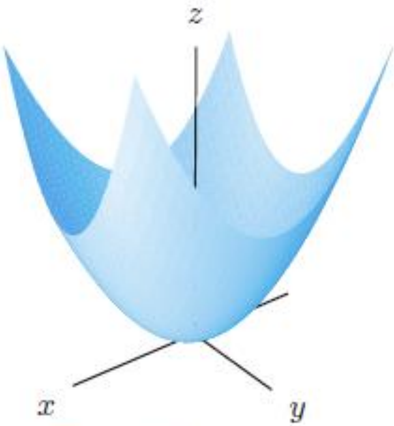
Two-Variable Function



Wire frame picture of $f(x, y) = x^2 + y^2$ for $-3 \le x \le 3$, $-3 \le y \le 3$



Wire frame picture of $f(x, y) = x^2 + y^2$ with more points plotted



Graph of $f(x, y) = x^2 + y^2$ for $-3 \le x \le 3$, $-3 \le y \le 3$

Point Table of $f(x, y) = x^2 + y^2$

x \ y	-3	-2	-1	0	1	2	3
-3	18	13	10	9	10	13	18
-2	13	8	5	4	5	8	13
-1	10	5	2	1	2	5	10
0	9	4	1	0	1	4	9
1	10	5	2	1	2	5	10
2	13	8	5	4	5	8	13
3	18	13	10	9	10	13	18

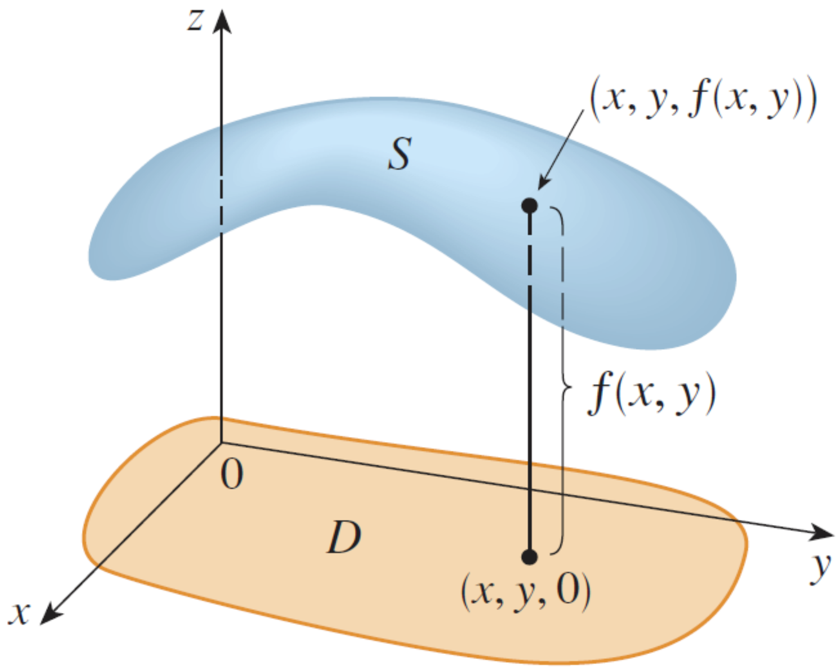
Relationship between $z = f(x, y)$ and (x, y)

1.3 Graph of function with 2-Variables

1.3.3 Definition: Graph of function $\boldsymbol{z = f(x, y)}$

$z = f(x, y)$

Two-Variable Function



$(x, y, 0) \in D$ $(x, y) \in D$

Definition: If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$

1.3 Graph of function with 2-Variables

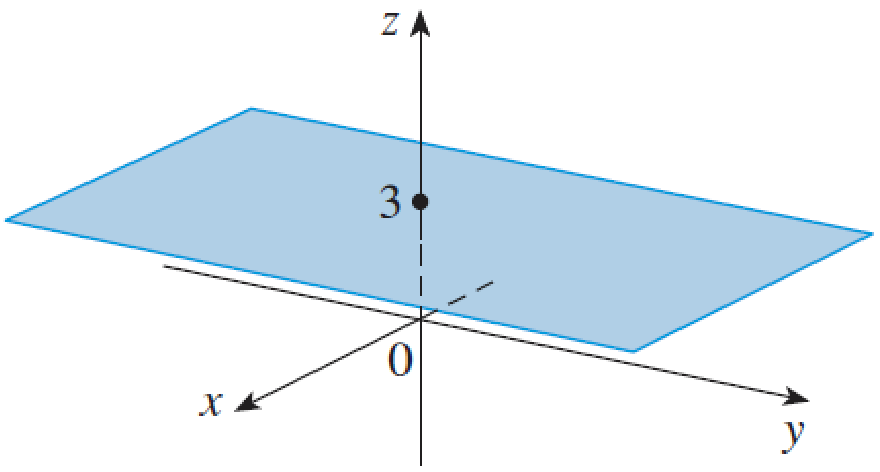
1.3.4 Plane in 3D-Space | Case 1: Naïve Plane

Example: What surfaces in $\mathbb{R}^3$ are represented by the following equations?

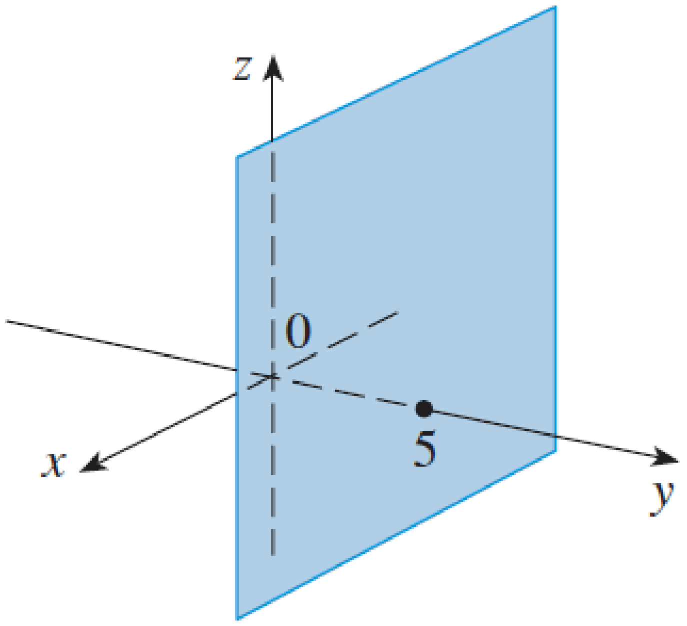
(1) $z = 3$

(2) $y = 5$

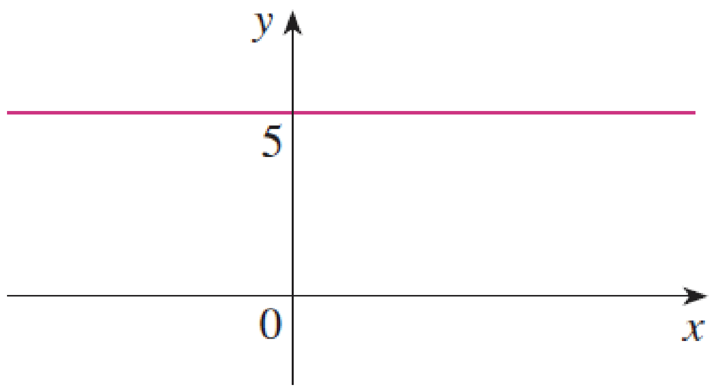
(3) $y = 5$ in $\mathbb{R}^2$



(a) $z = 3$, a plane in $\mathbb{R}^3$



(b) $y = 5$, a plane in $\mathbb{R}^3$

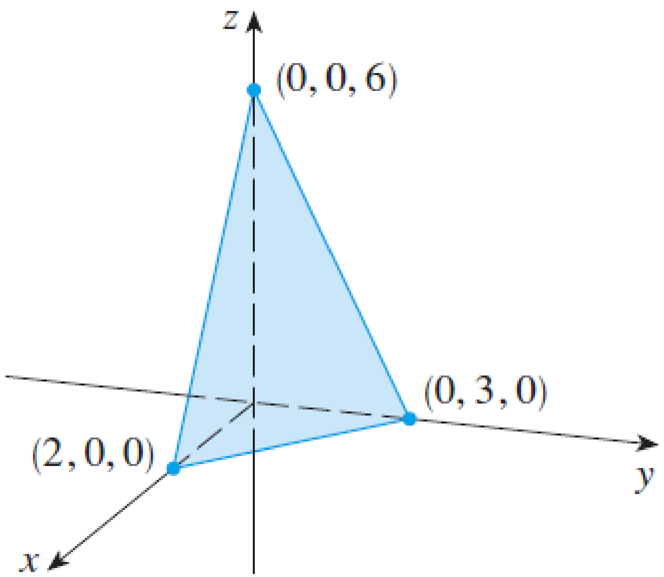


(c) $y = 5$, a line in $\mathbb{R}^2$

1.3 Graph of function with 2-Variables

1.3.4 Plane in 3D-Space | Case 2: General Plane

Example: Sketch the graph of the function $z = f(x, y) = 6 - 3x - 2y$.



General Plane — Linear function in 3D-Space

$$f(x, y) = ax + by + c$$

$$z = ax + by + c$$

$$ax + by - z + c = 0$$

Why? We will discuss later after Vector …

What you will learn in Lecture 1

I. How to **GRAPH EQUATIONS** in 3D-Space with **2-variables**

II. **VECTORS** and the **Geometry** of 3D-Space

III. Parametrized Curves

*So far, we basically only use **the points (in table)** to understand several variables in 3D-Space.*

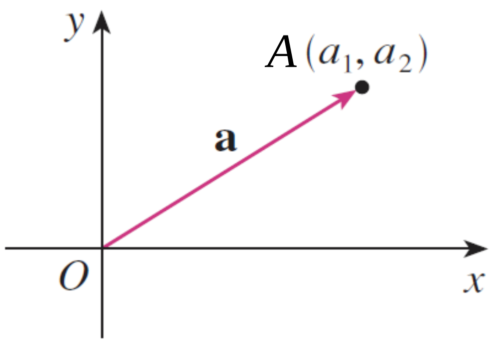
*Let's try **another way** to see:*

1.4 Vectors and The Geometry of Space

1.4 Vectors and The Geometry of Space

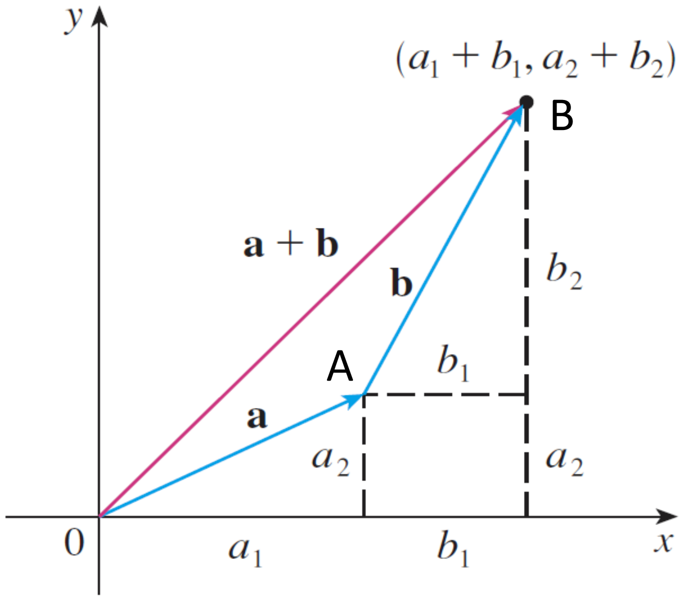
1.4.1 What is Vector?

The term **vector** is used by scientists to indicate a quantity that has **both magnitude** and **direction**.



$\mathbf{a} = \langle a_1, a_2 \rangle$

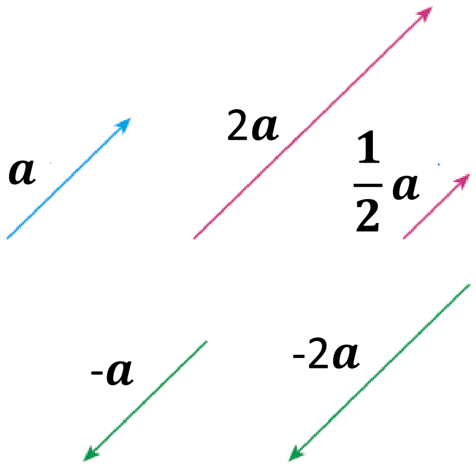
Vector ≅ {**magnitude**, **direction**}



initial point (the tail) and **terminal point** (the tip)

$$\overrightarrow{OA} = \boldsymbol{a} = \langle a_1, a_2 \rangle$$

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$



Scalar multiples of **v**

1.4 Vectors and The Geometry of Space

1.4.2 Vector with Components

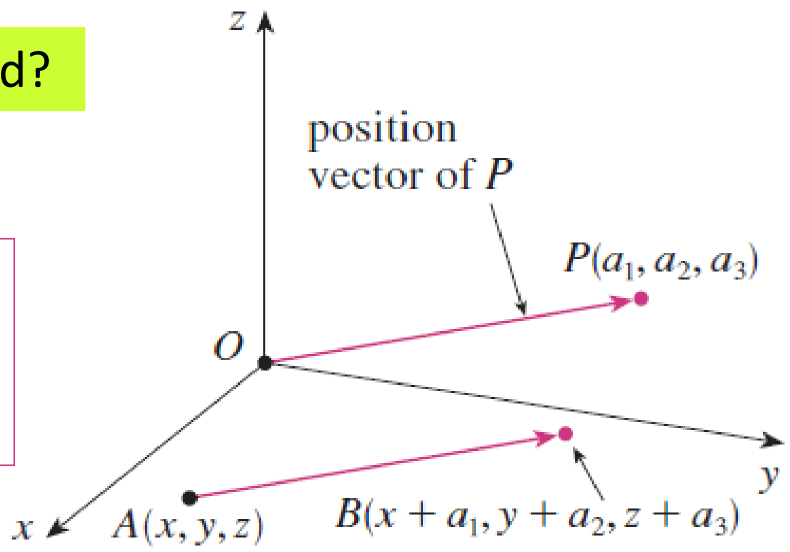
$\boldsymbol{a} = \overrightarrow{AB} = \langle a_1, a_2 \rangle$

Example: Find the vector represented by the directed line segment with initial point A(2,-3,4) and terminal point B(-2,1,1).

Hint: Compare each component of two Points, how much it changed?

1 Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is

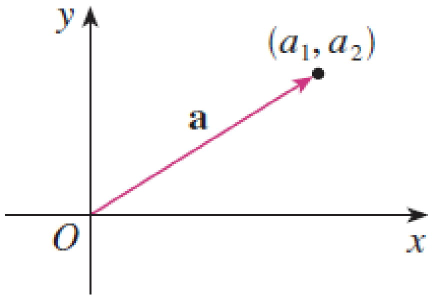
$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$



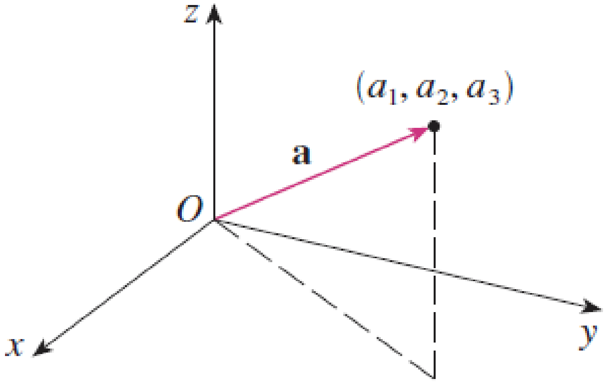
Representations of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

1.4 Vectors and The Geometry of Space

1.4.2 Vector Length (Distance between two points)



$\mathbf{a} = \langle a_1, a_2 \rangle$



$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

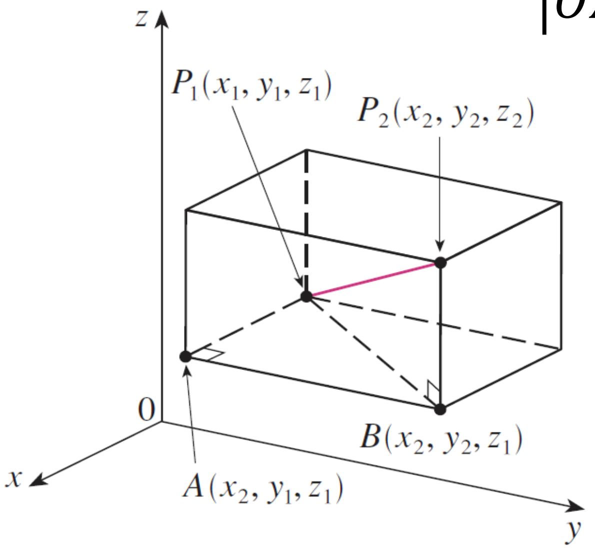
The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

Vector initialized from origin: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$$|\overrightarrow{OA}| = \sqrt{(a_1 - 0)^2 + (a_2 - 0)^2 + (a_3 - 0)^2}$$



Vector initialized from some free point P_1 in space:

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

1.4 Vectors and The Geometry of Space

1.4.2 Vector Length (Distance between two points)

Example: The vector length (distance) from the point $P(2,-1,7)$ to the point $Q(1,-3,5)$ is

Solution:

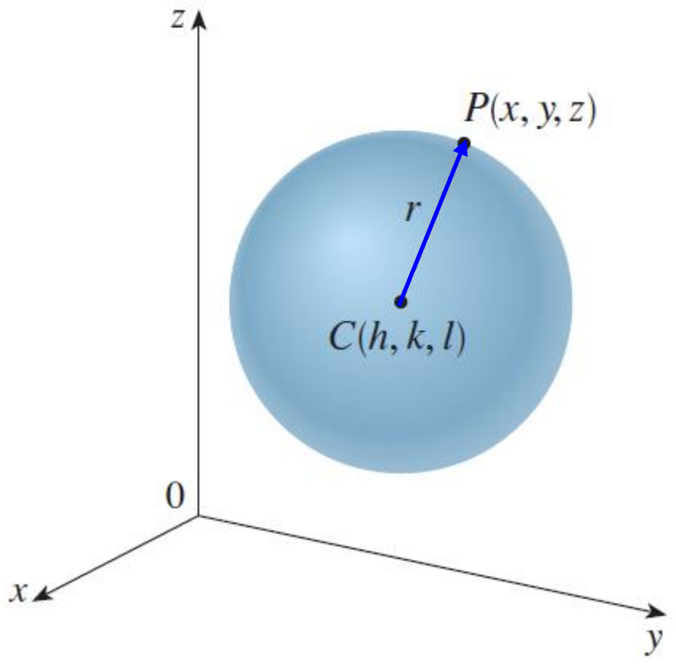
$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$$\left|\overrightarrow{PQ}\right| = \sqrt{(1-2)^2+(-3-(-1))^2+(5-7)^2} = \sqrt{1+4+4} = 3$$

1.4 Vectors and The Geometry of Space

1.4.3 Equation of Sphere (derived from Vector Length)

Example: Find an equation of a sphere with radius r and center $C(h, k, l)$.



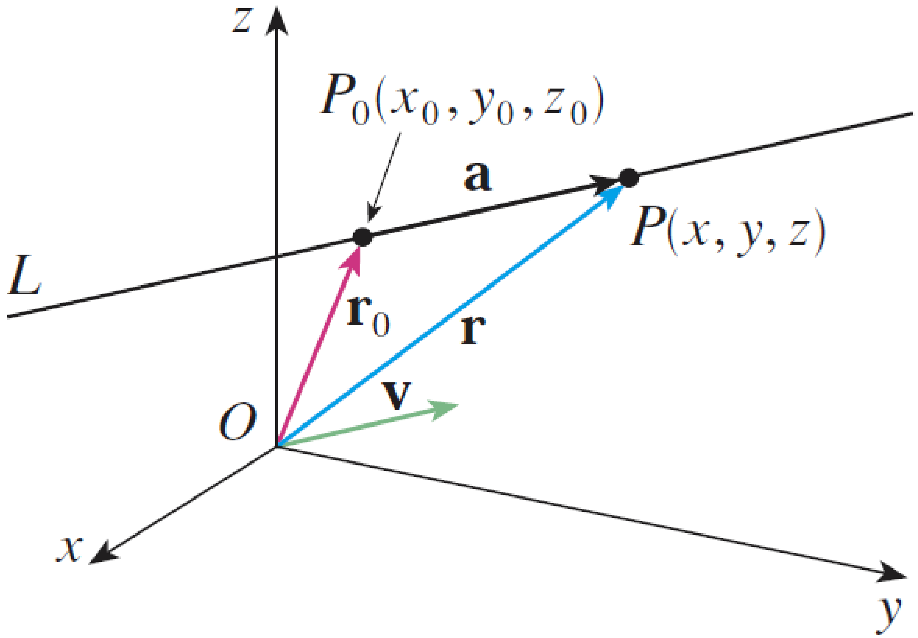
From vector length (distance), we derived the sphere equation.

$$\left|\overrightarrow{CP}\right| = r$$

$$(x-h)^2+(y-k)^2+(z-l)^2 = r^2$$

1.4 Vectors and The Geometry of Space

1.4.5 Line in 3D-Space



A Line L in 3D-Space is determined by

(1) A point on Line L

(2) Direction of L

$$\boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}}$$

(Notice: The Figure 1 in Section 13.5 (Page 830) of the textbook (6th edition) is incorrect. Here uses the Figure from 7th edition.)

1.4 Vectors and The Geometry of Space

1.4.5 Line in 3D-Space – Parametric Equation

Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:

$$\boxed{x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct}$$

where $t \in \mathbb{R}$. These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$. Each value of the parameter t gives a point (x, y, z) on L.

1.4 Vectors and The Geometry of Space

1.4.6 Plane in 3D-Space

Dot Product Scalar product

DEFINITION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

THEOREM If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

Proof: Page 816

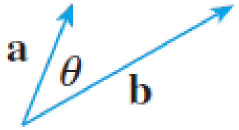
1.4 Vectors and The Geometry of Space

1.4.6 Plane in 3D-Space

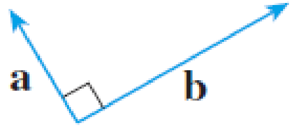
Dot Product – What it can do?

COROLLARY If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

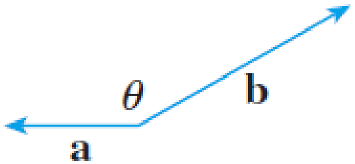


$\mathbf{a} \cdot \mathbf{b} > 0$



$\mathbf{a} \cdot \mathbf{b} = 0$

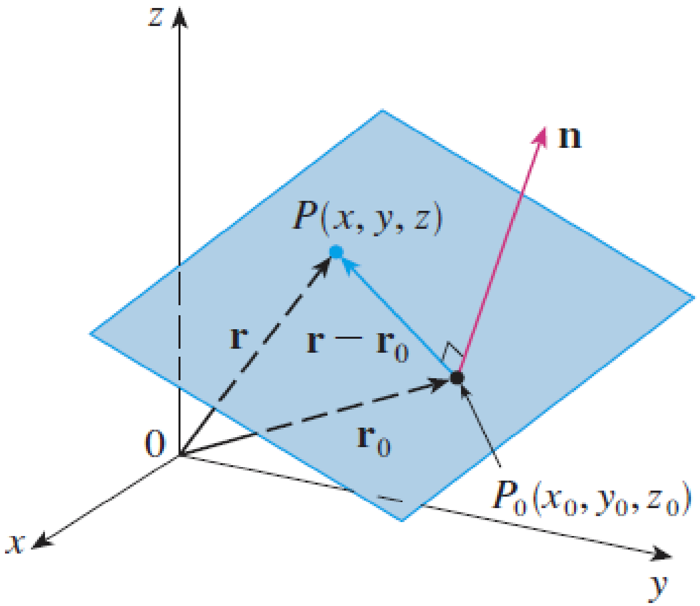
Two vectors $\mathbf{a}$ and $\mathbf{b}$ are $\boxed{\text{orthogonal}}$ if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.



$\mathbf{a} \cdot \mathbf{b} < 0$

Helpful to Find Normal Vector.

1.4.6 Plane in 3D-Space



A Plane M in 3D-Space is determined by

(1) A point on plane M

(2) Direction of Normal Vector of M

$$\boxed{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0}$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\Downarrow$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

$$\boxed{ax + by + cz + d = 0}$$

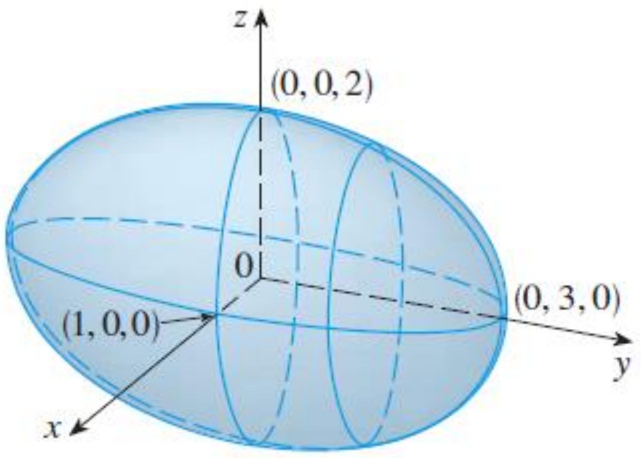
1.4 Vectors and The Geometry of Space

1.4.7 Quadric Surfaces

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

Example: Sketch the quadric surface with $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1.$

These curves of intersection of the surface are called **traces** (or cross-sections) of the surface.



The ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

What you will learn in Lecture 1

I. How to **GRAPH EQUATIONS** in 3D-Space with **2-variables**

II. **VECTORS** and the **Geometry** of 3D-Space

III. Parametrized Curves

1.5 Parametrized Curves

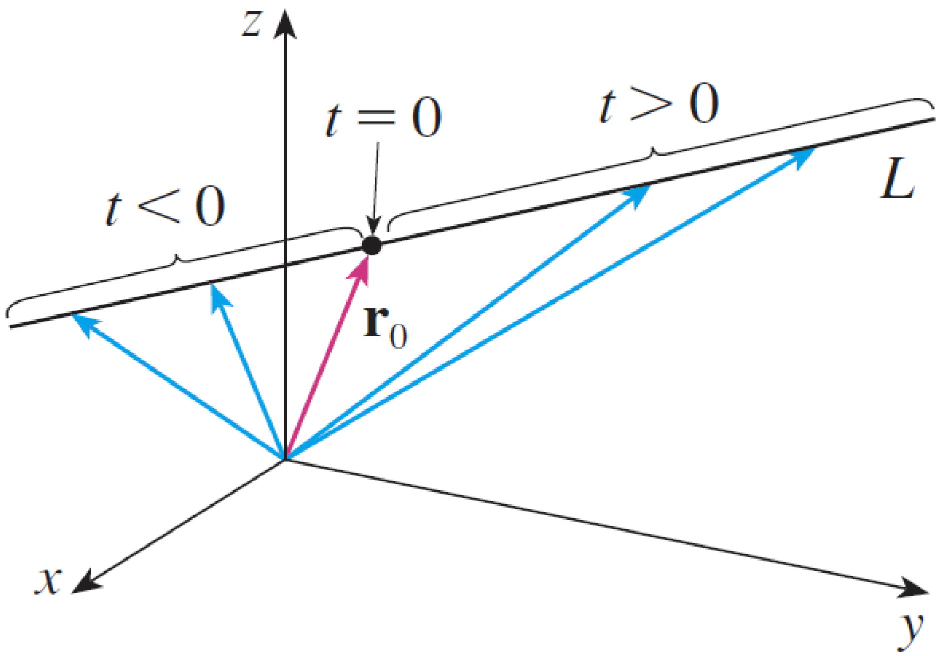
1.5 Parametrized Curves

1.5.1 Parametric Equations in Three Dimensions

A curve in the plane may be parameterized by the equations of the form

$$x = x(t), y = y(t), z = z(t)$$

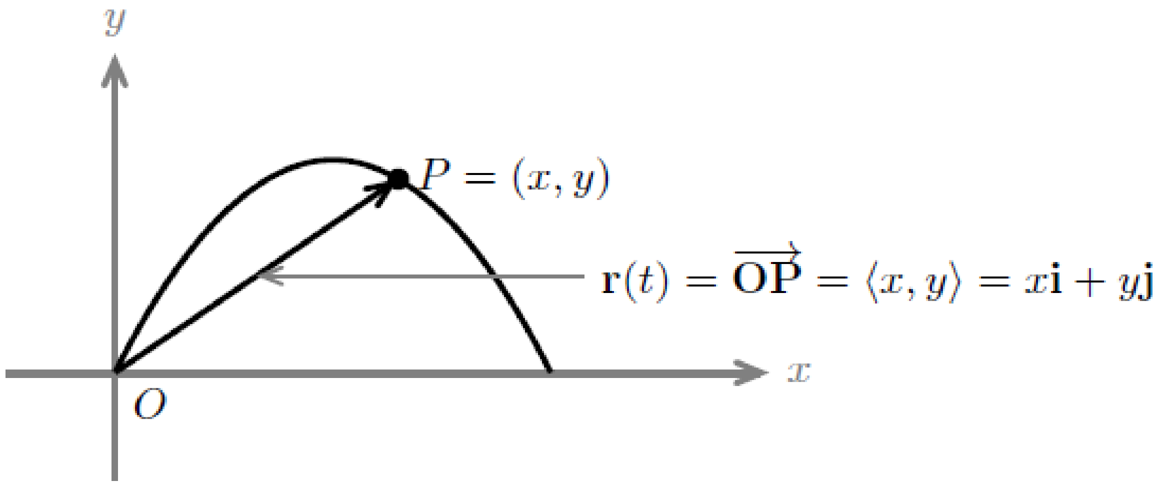
As the parameter t changes, the point (x, y) traces out the curve with motion.



1.5 Parametrized Curves

1.5.2 Parametric Equations in Three Dimensions

Example 1: Parametric equations for a rocket from the origin. Its initial x-velocity is $v_{0,x}$, and its initial y-velocity is $v_{0,y}$. (Page 877)



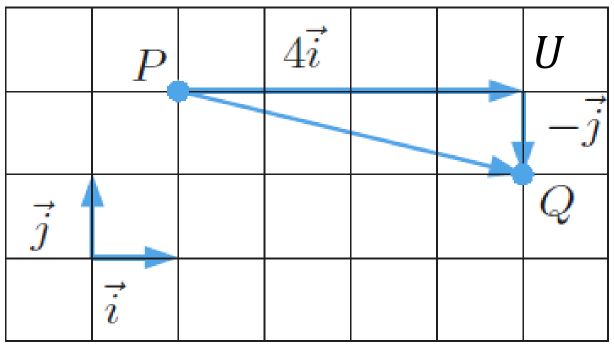
Trajectory

1.5.2 Parametric Equations in Three Dimensions

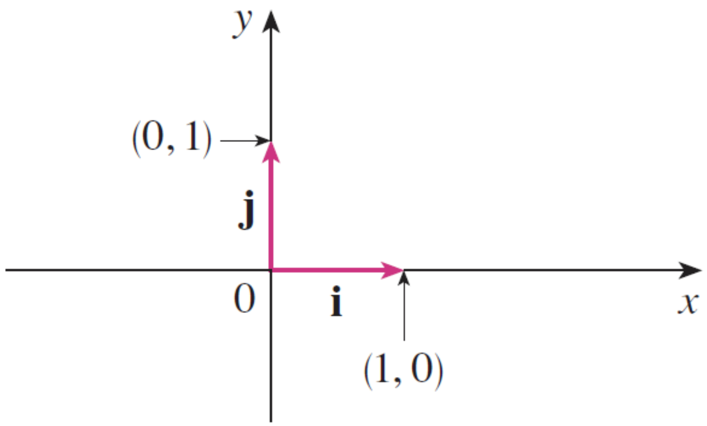
Additional knowledge of Vectors: Standard basis vectors

Standard basis vectors.

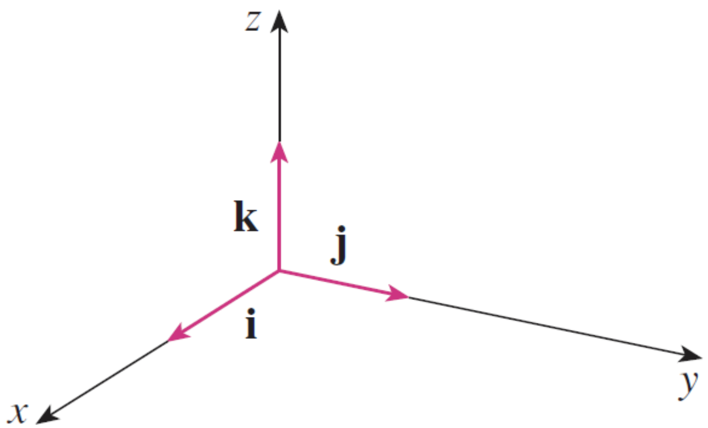
$$\mathbf{i} = \langle 1, 0, 0\rangle \qquad \mathbf{j} = \langle 0, 1, 0\rangle \qquad \mathbf{k} = \langle 0, 0, 1\rangle$$



The displacement vector from P to Q is $4\vec{i} - \vec{j}$



(a)

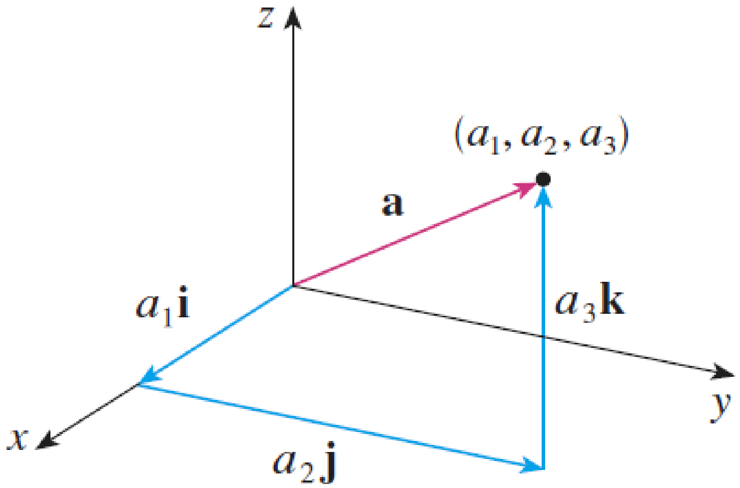


(b)

1.5 Parametrized Curves

1.5.2 Parametric Equations in Three Dimensions

Additional knowledge of Vectors:
Standard basis vectors



If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$

$$= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle$$

$$\boxed{\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}}$$

Thus any vector in V_3 can be expressed in terms of **i**, **j**, and **k**. For instance,

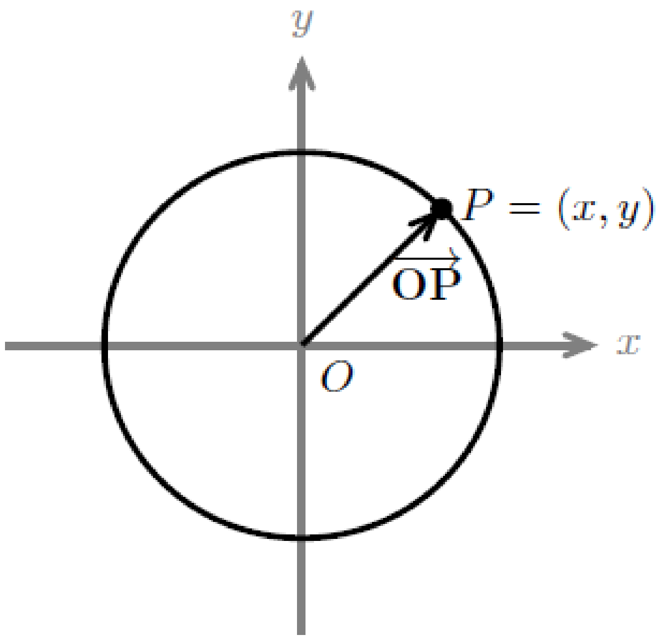
$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

1.5 Parametrized Curves

1.5.2 Parametric Equations in Three Dimensions

Example 2: Circle. Consider the parametric curve in the plane. (Page 658)

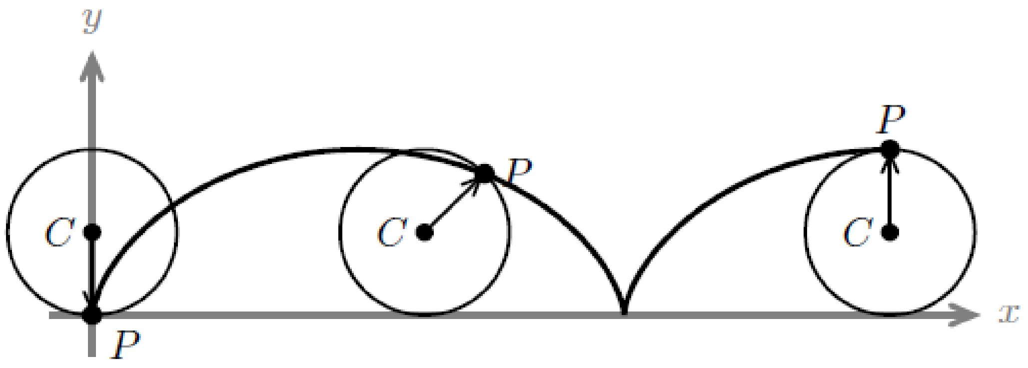
$$x(t) = a\cos t, y(t) = a\sin t.$$

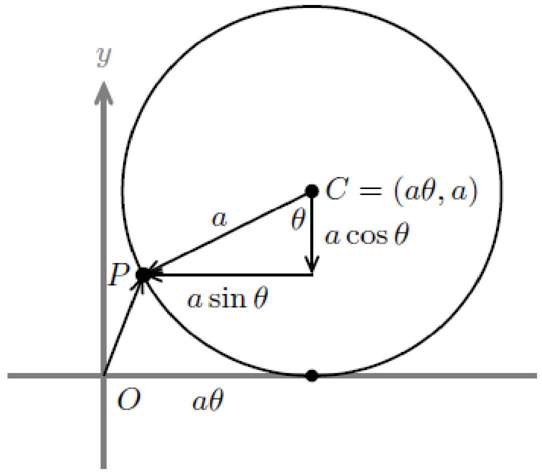


1.5.2 Parametric Equations in Three Dimensions

Example 3: Cycloid (Page 660)

The cycloid is the trajectory of a point on a circle that is rolling without slipping along the x-axis.
To be specific, we'll follow the point P that starts at the origin.

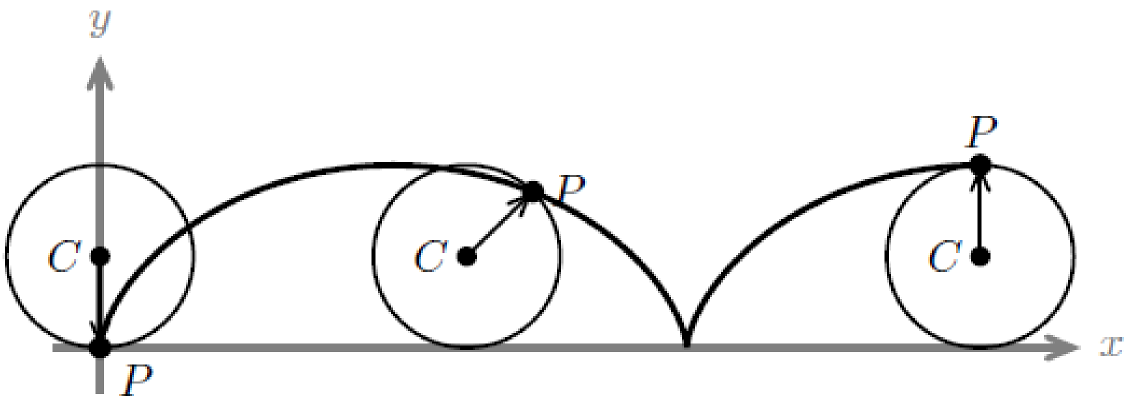


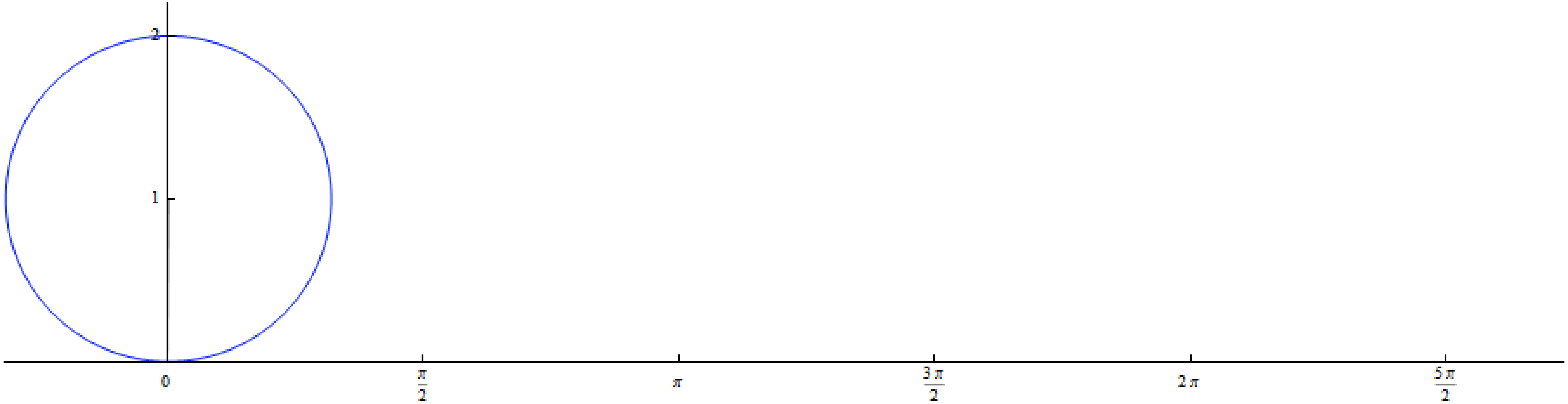


1.5 Parametrized Curves

1.5.2 Parametric Equations in Three Dimensions

Example 3: Cycloid (Page 660)





Review for Lecture 1

- Graph of function in 3D-Space with 2-Variables

  3D coordinate system, $\mathbb{R}^3$, Point Table

- Vector, Vector Length, Sphere Equation, Standard basis vector

- Equation of Line, Plane, Quadric Surface

- Normal Vector (Dot Product)

- Parametrized Curves (Parameter t; Use ‘Vector view’ to see it)

Assignment 1

github.com/uoaworks

References

- Textbook: Section 15.1, 13.1-13.3, 13.5, 13.6, 11.1
- MIT 18.02SC: MIT18_02SC_notes_8, MIT18_02SC_notes_9
- Calculus Single & Multivariable, 6th Editon, Hugues Hallett
  (Basically, figures used)