



Calculus **II**

Class Information

Lecture (14): Tuesday, Friday



Exercise (7): Wednesday



Grades
20% Assignment ($> 2/3$)
80% Examination
Attendance ($> 2/3$)

Office hours: After each class

Textbook: Calculus, 6th Edition, James Stewart

Reading materials: MIT Open Course 18-02SC Multivariable Calculus

Prerequisites

All enrolled students must have taken *Calculus I, Linear Algebra*

What we will cover

Full syllabus on course website

(Mainly 2 and 3 variables, but it is able to expand to more variables)

Part 1. Multivariable calculus - differentials

Parametrized curves, Partial differentials, Differentiation of composite functions,
chain rule, Local minimum/maximum of a function, Implicit functions

Part 2. Multivariable calculus - integrals

Double Integral, Triple Integral, Area, Volume

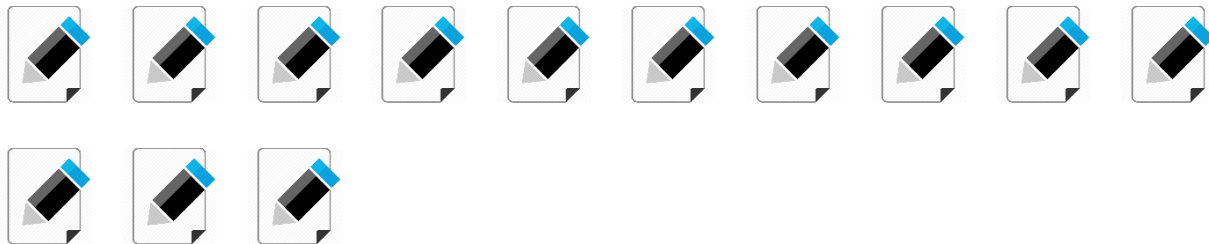
Part 3. Differential equations

Part 4. Sequence and series

Convergence, approximated functions



Calculus **II**



2018年6月						
日	月	火	水	木	金	土
27	28	29	30	31	1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30
1	2	3	4	5	6	7

Calculus II



Lecture 1

Parametrized Curves

What you will learn in Lecture 1

I. How to GRAPH EQUATIONS in 3D-Space with 2-variables

II. VECTORS and the Geometry of 3D-Space

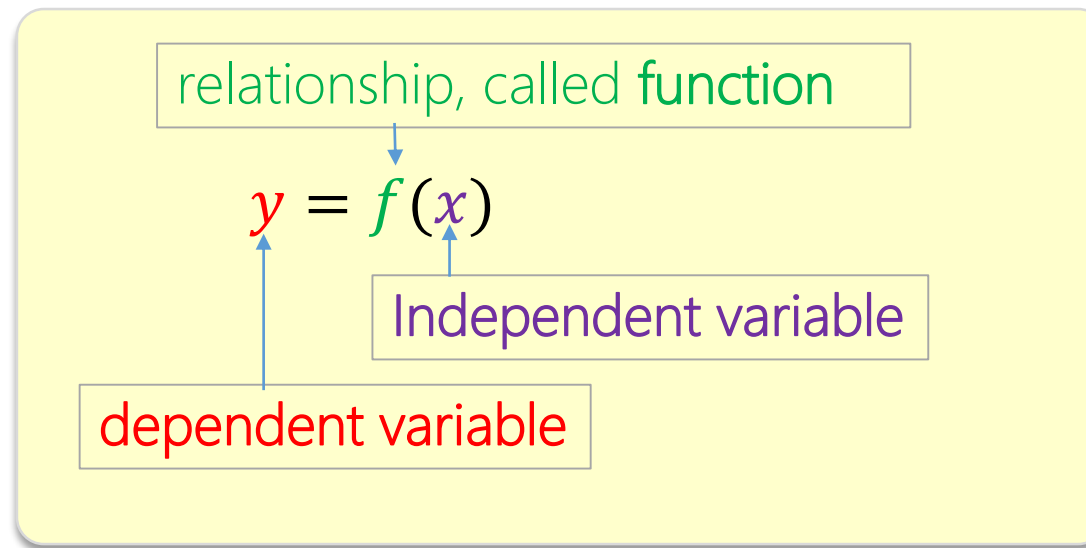
III. Parametrized Curves

1.1 Reminder of Calculus I

1.1 Reminder of Calculus I

Calculus I \approx One-Variable Calculus

One-Variable Function



1.1 Reminder of Calculus I

One-Variable Calculus -- derivative of f

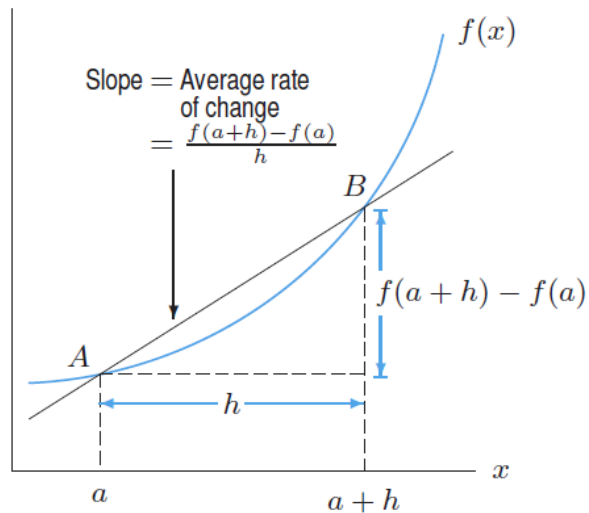
One-Variable Function

relationship, called function

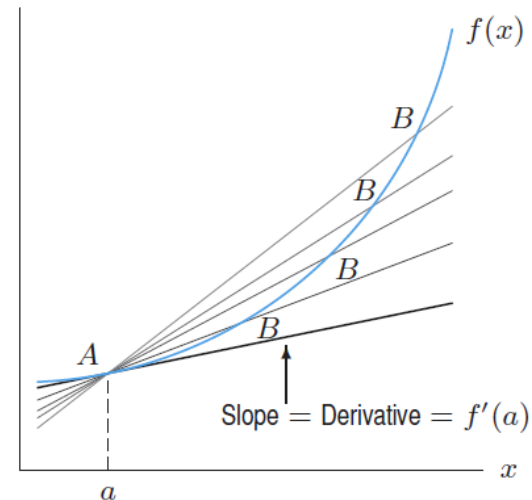
$$y = f(x)$$

Independent variable

dependent variable



Visualizing the average rate of change of f



Visualizing the instantaneous rate of change of f

The **derivative of f at a** , written $f'(a)$, is defined as

$$\begin{array}{l} \text{Rate of change} \\ \text{of } f \text{ at } a \end{array} = f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

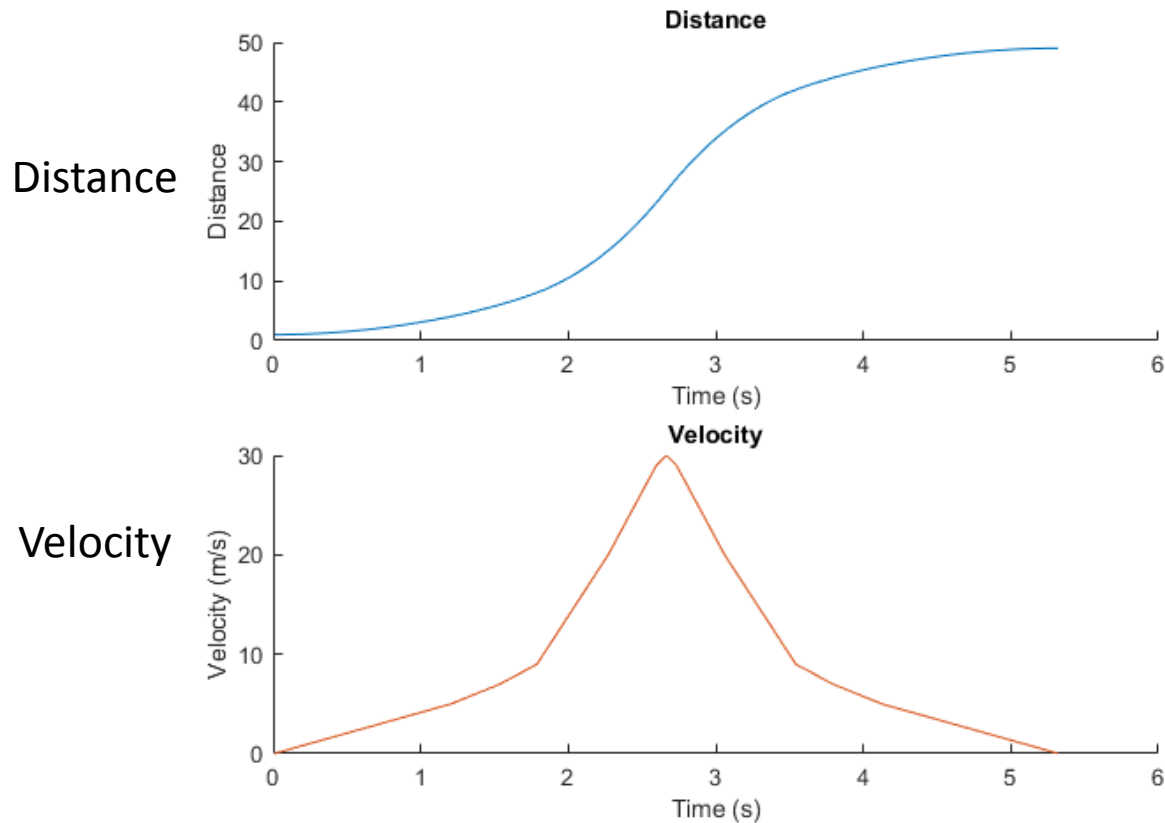
If the limit exists, then f is said to be **differentiable at a** .

1.1 Reminder of Calculus I

One-Variable Calculus -- derivative of f

Q: How do we measure the **instantaneous rate of change** for distance with regard to time?

One-Variable Function $y = f(x) \Rightarrow r = v_0 + vt$ Assume that $v_0 = 0$



What you will learn in Lecture 1

I. How to GRAPH EQUATIONS in 3D-Space with 2-variables

II. VECTORS and the Geometry of 3D-Space

III. Parametrized Curves

Goal of multivariable calculus:

Tools to handle problems with function of several variables.

1.2 Function of Several Variables

(We have One-Variable in *Calculus I*)

1.2 Function of Several Variables

Notation

One-Variable Function

$y = f(x)$

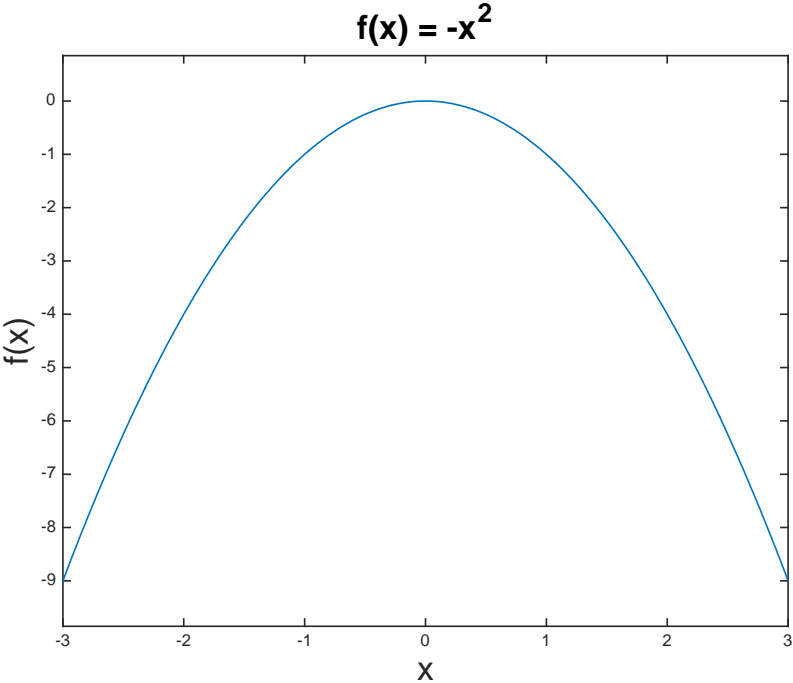


$z = f(x, y)$

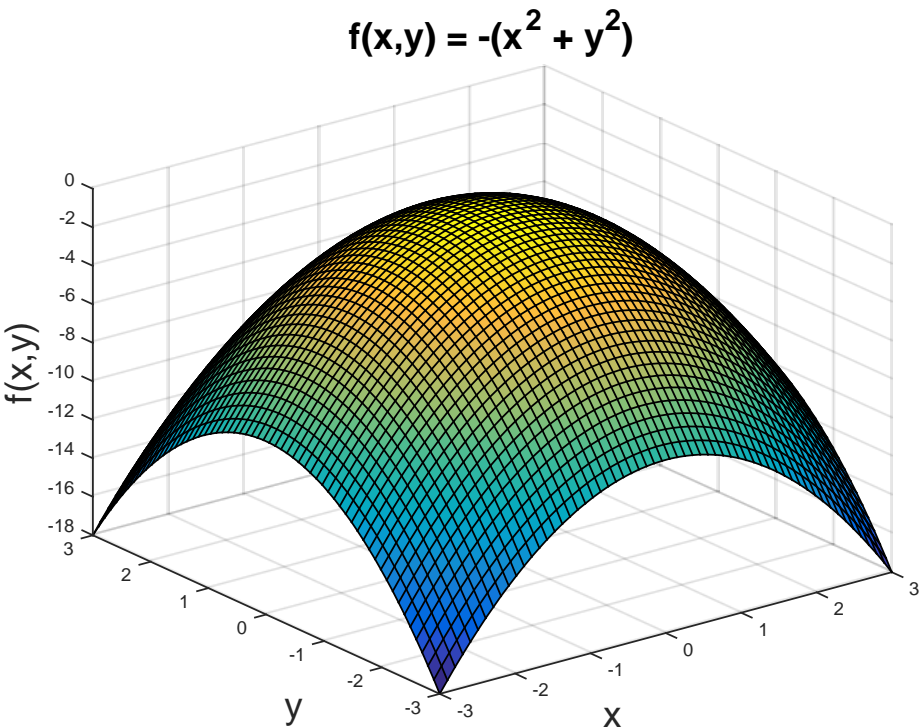
Two-Variable Function

Three-Variable Function

⋮



One-variable problem

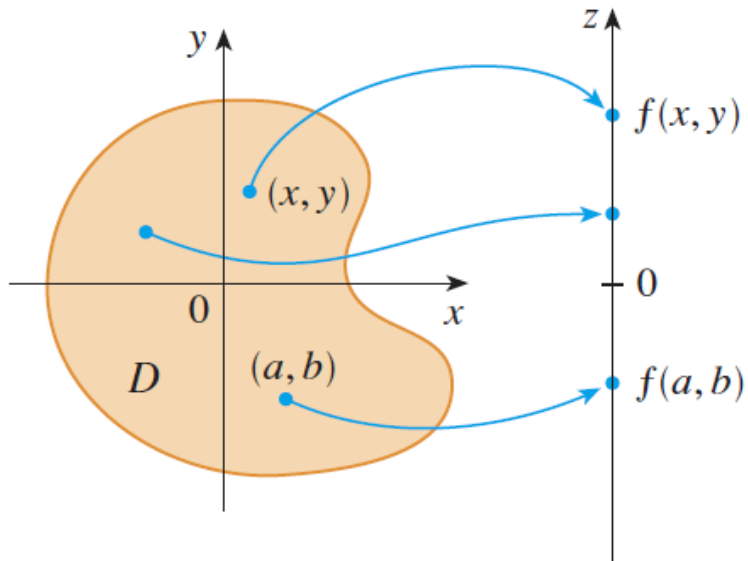
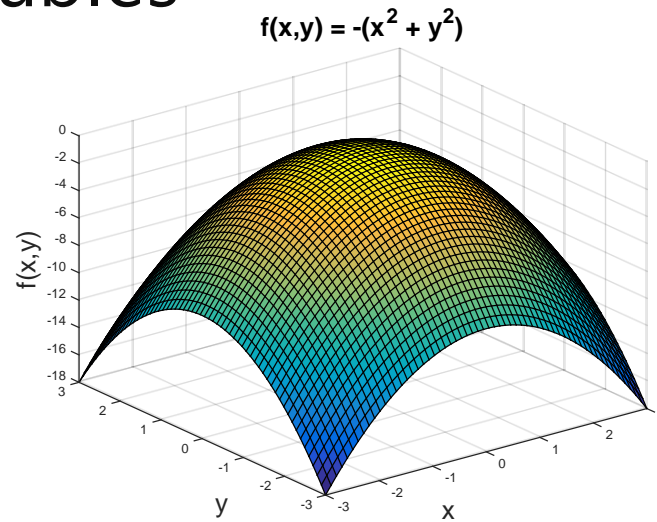


Two-variable problem

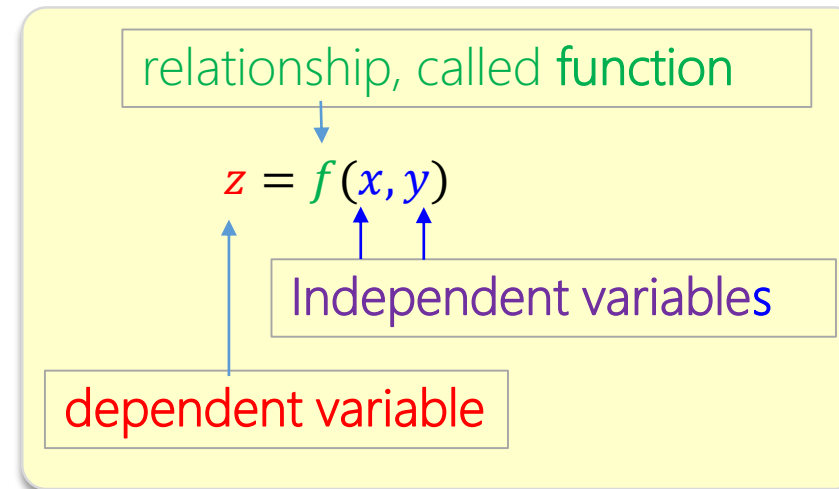
Three-variable problem

⋮

1.2 Function of Several Variables



Two-Variable Function



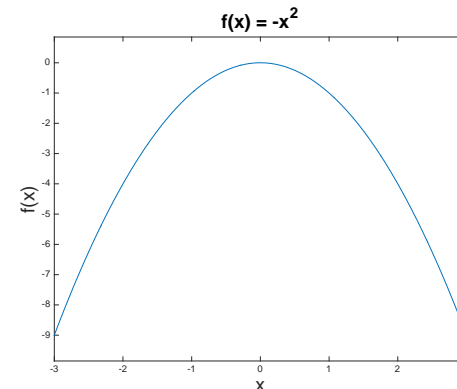
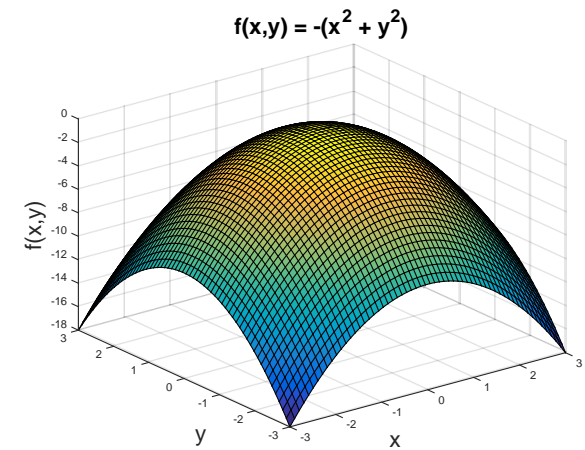
1.2 Function of **Several** Variables

Definition of \mathbb{R}^3

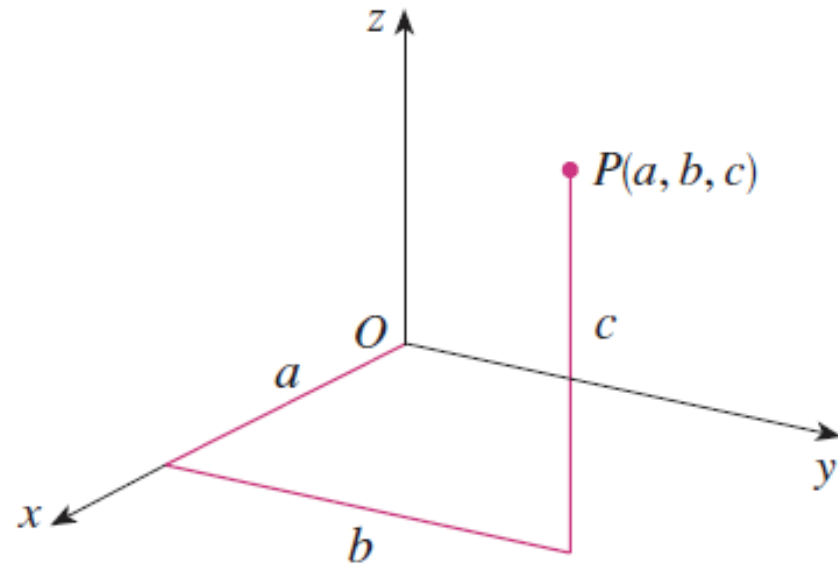
The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ is the set of all ordered **triples** of real numbers and is denoted by \mathbb{R}^3 .

In **3D**, an equation in **x , y , and z** represents a ***surface*** in \mathbb{R}^3 .

In **2D**, the graph of an equation involving **x and y** is a ***curve*** in \mathbb{R}^2 .

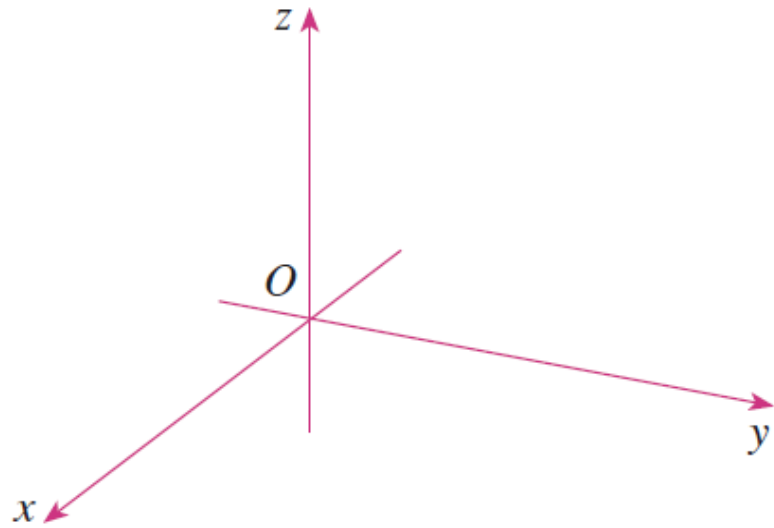


1.3 Graph of function with **2**-Variables

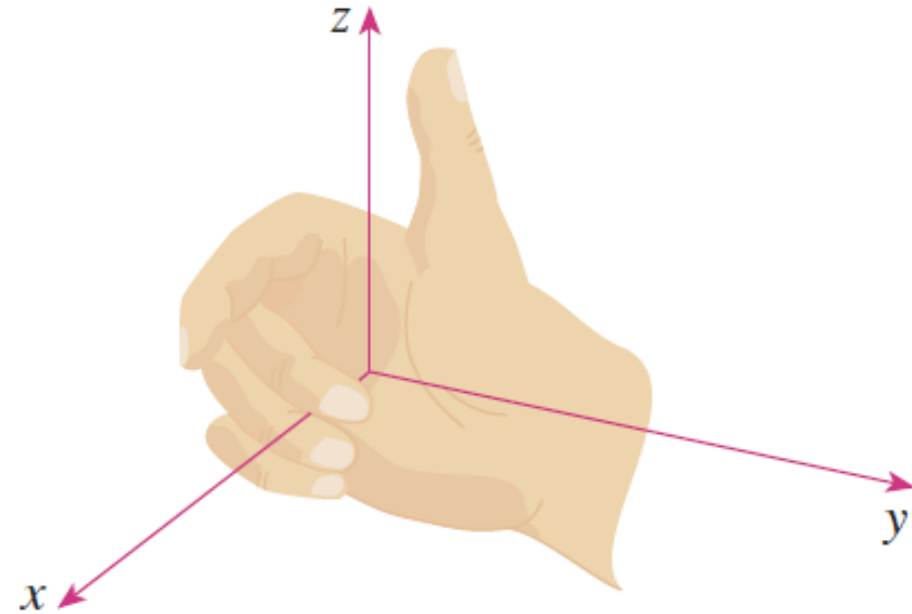


1.3 Graph of function with 2-Variables

1.3.1 Coordinate axes in space



Coordinate axes



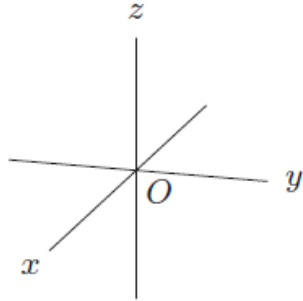
Right-hand rule

1.3 Graph of function with 2-Variables

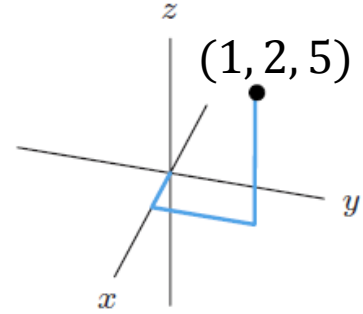
1.3.2 How to graph? By Point Table!

$$z = f(x, y)$$

Two-Variable Function



Coordinate axes in 3D-space



The point (1,2,5) in 3D-space

Point Table of $f(x, y) = x^2 + y^2$

		y						
		-3	-2	-1	0	1	2	3
x	-3	18	13	10	9	10	13	18
	-2	13	8	5	4	5	8	13
	-1	10	5	2	1	2	5	10
	0	9	4	1	0	1	4	9
	1	10	5	2	1	2	5	10
	2	13	8	5	4	5	8	13
	3	18	13	10	9	10	13	18

Q: in the table,
 $x = 0, y = 0, z = ?$

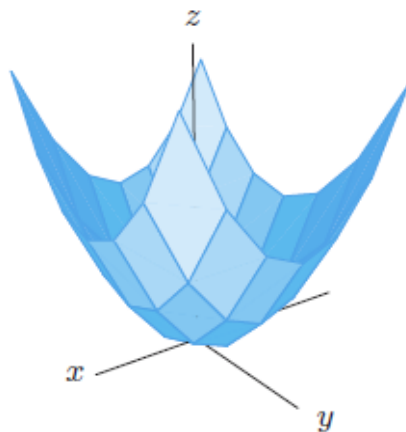
Relationship between $z = f(x, y)$ and (x, y)
dependent variable Independent variable

1.3 Graph of function with 2-Variables

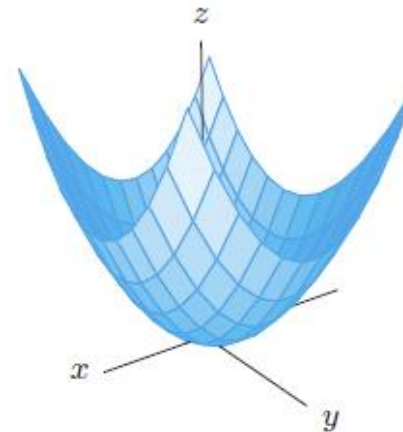
1.3.2 How to graph? By Point Table!

$$z = f(x, y)$$

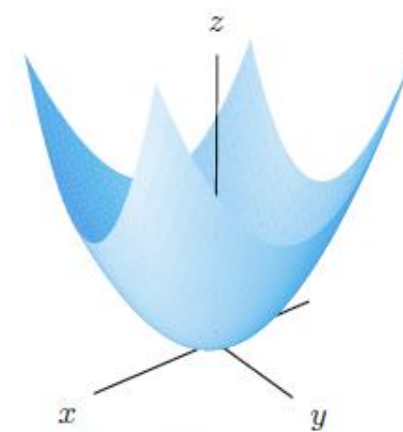
Two-Variable Function



Wire frame
picture of $f(x, y) = x^2 + y^2$
for $-3 \leq x \leq 3, -3 \leq y \leq 3$



Wire frame
picture of $f(x, y) = x^2 + y^2$
with more points plotted



Graph of
 $f(x, y) = x^2 + y^2$ for
 $-3 \leq x \leq 3, -3 \leq y \leq 3$

Point Table of $f(x, y) = x^2 + y^2$

		y						
		-3	-2	-1	0	1	2	3
x	-3	18	13	10	9	10	13	18
	-2	13	8	5	4	5	8	13
	-1	10	5	2	1	2	5	10
	0	9	4	1	0	1	4	9
	1	10	5	2	1	2	5	10
	2	13	8	5	4	5	8	13
	3	18	13	10	9	10	13	18

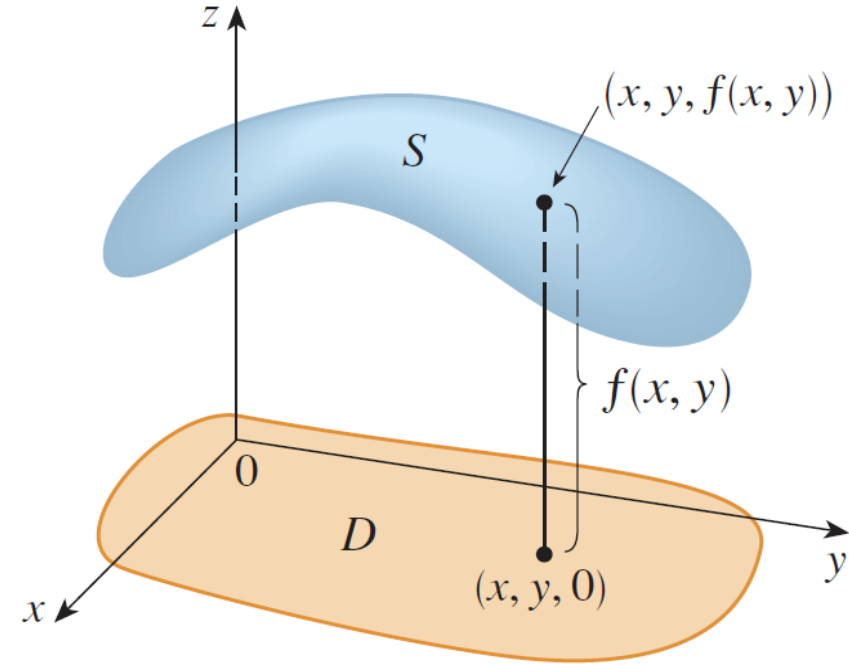
Relationship between $z = f(x, y)$ and (x, y)

1.3 Graph of function with 2-Variables

1.3.3 Definition: Graph of function $z = f(x, y)$

$$z = f(x, y)$$

Two-Variable Function



$$(x, y, 0) \in D \quad (x, y) \in D$$

Definition: If f is a function of two variables with domain D , then the graph of f is the set of all points (x, y, z) in \mathbb{R}^3 such that $z = f(x, y)$

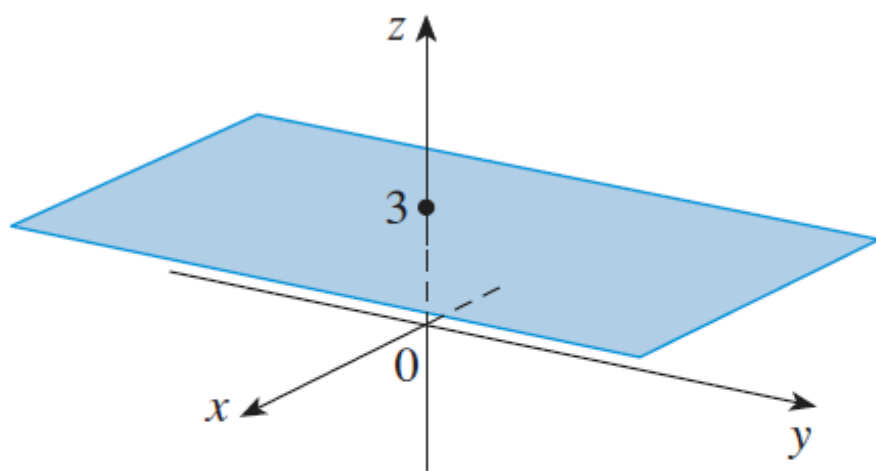
1.3 Graph of function with 2-Variables

1.3.4 Plane in 3D-Space

Case 1: Naïve Plane

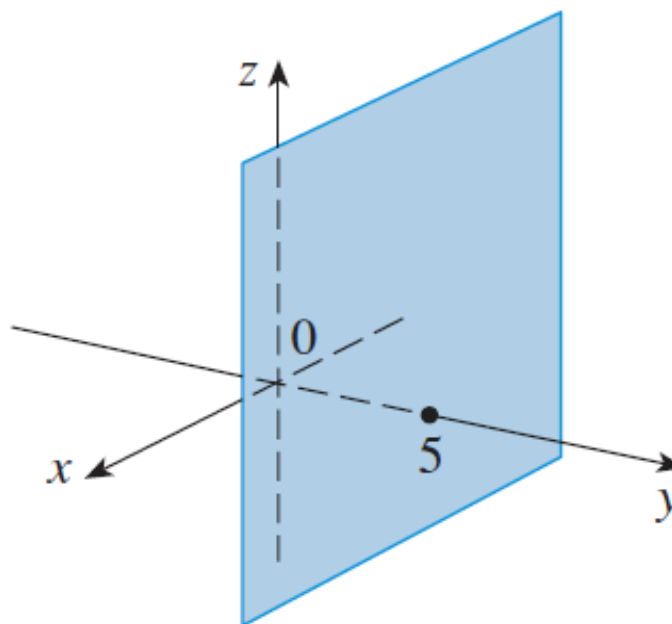
Example: What surfaces in \mathbb{R}^3 are represented by the following equations?

(1) $z = 3$



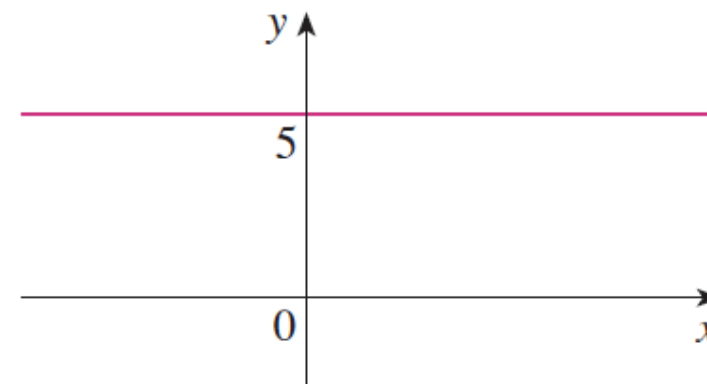
(a) $z = 3$, a plane in \mathbb{R}^3

(2) $y = 5$



(b) $y = 5$, a plane in \mathbb{R}^3

(3) $y = 5$ in \mathbb{R}^2



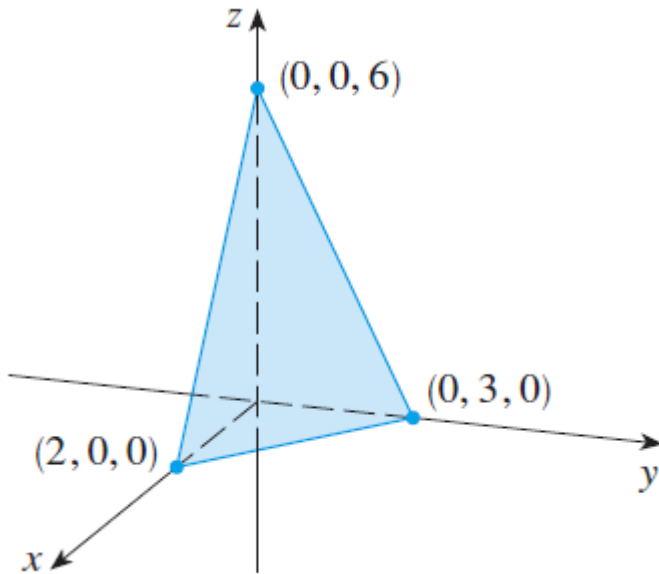
(c) $y = 5$, a line in \mathbb{R}^2

1.3 Graph of function with 2-Variables

1.3.4 Plane in 3D-Space

Case 2: General Plane

Example: Sketch the graph of the function $z = f(x, y) = 6 - 3x - 2y$.



General Plane

Linear function in 3D-Space

$$f(x, y) = ax + by + c$$

$$z = ax + by + c$$

$$ax + by - z + c = 0$$

Why? We will discuss later after Vector ...

What you will learn in Lecture 1

I. How to **GRAPH EQUATIONS** in 3D-Space with 2-variables

II. **VECTORS** and the Geometry of 3D-Space

III. Parametrized Curves

*So far, we basically only use **the points (in table)** to understand **several variables** in 3D-Space.*

*Let's try **another way** to see:*

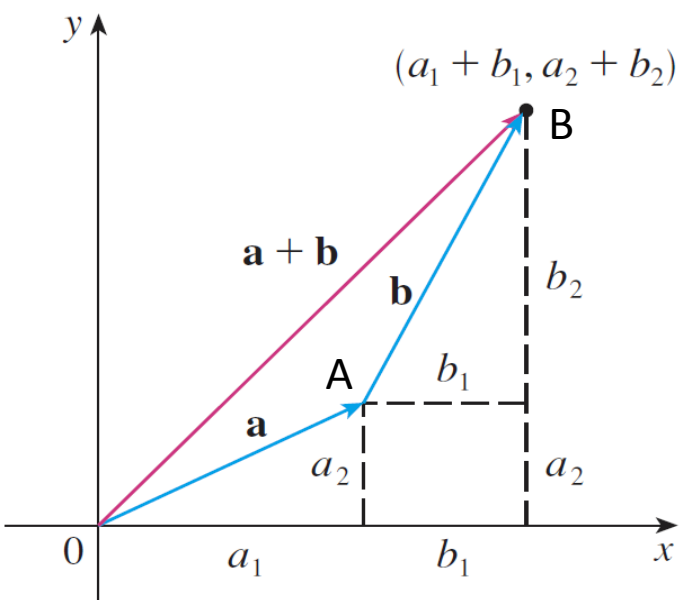
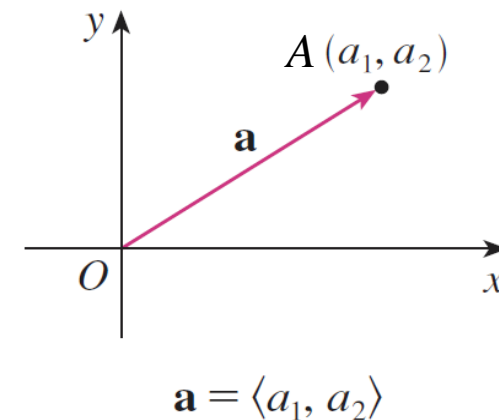
1.4 Vectors and The Geometry of Space

1.4 Vectors and The Geometry of Space

1.4.1 What is Vector?

The term **vector** is used by scientists to indicate a quantity that has both **magnitude** and **direction**.

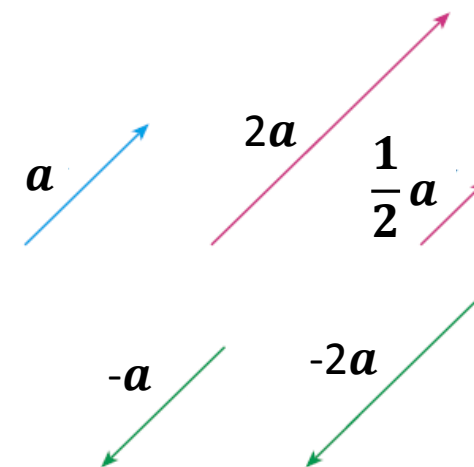
Vector \cong {**magnitude**, **direction**}



initial point (the tail) and **terminal point** (the tip)

$$\overrightarrow{OA} = \mathbf{a} = \langle a_1, a_2 \rangle$$

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$



Scalar multiples of \mathbf{v}

1.4 Vectors and The Geometry of Space

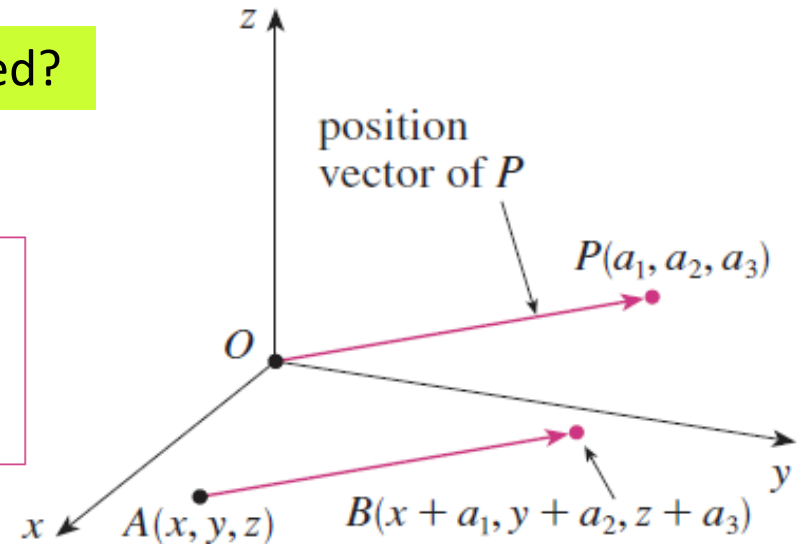
1.4.2 Vector with Components $\mathbf{a} = \overrightarrow{AB} = \langle a_1, a_2 \rangle$

Example: Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

Hint: Compare each component of two Points, how much it changed?

I Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector \mathbf{a} with representation \overrightarrow{AB} is

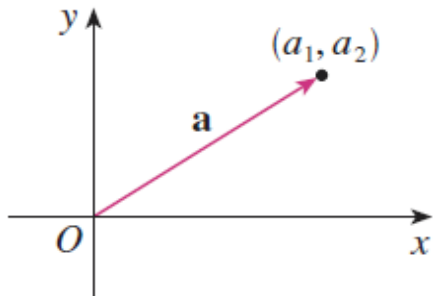
$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$



Representations of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

1.4 Vectors and The Geometry of Space

1.4.2 Vector Length (Distance between two points)



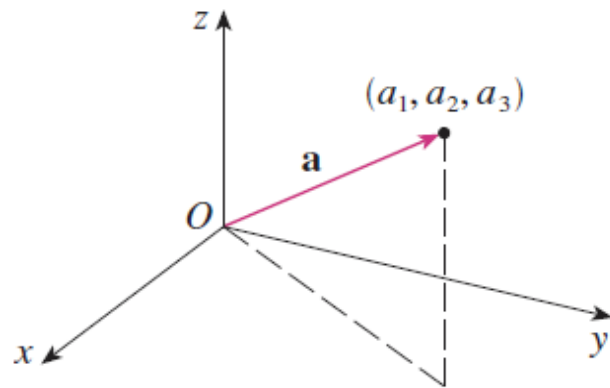
$$\mathbf{a} = \langle a_1, a_2 \rangle$$

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

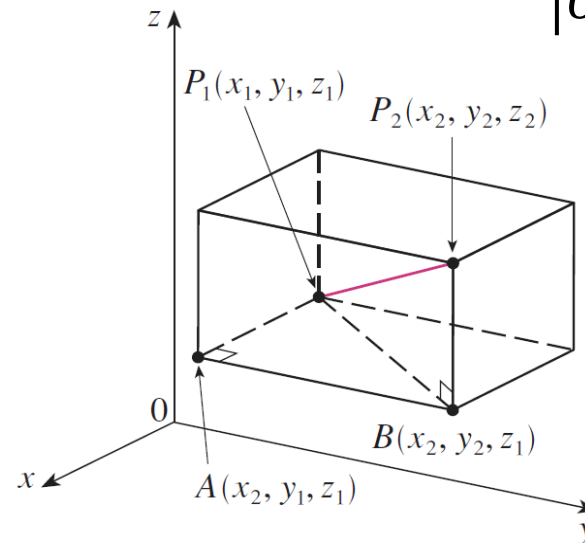
The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

Vector initialized from origin: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$



$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

$$|\overrightarrow{OA}| = \sqrt{(a_1 - 0)^2 + (a_2 - 0)^2 + (a_3 - 0)^2}$$



Vector initialized from some free point P_1 in space:

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

1.4 Vectors and The Geometry of Space

1.4.2 Vector Length (Distance between two points)

Example: The vector length (distance) from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$ is

Solution:

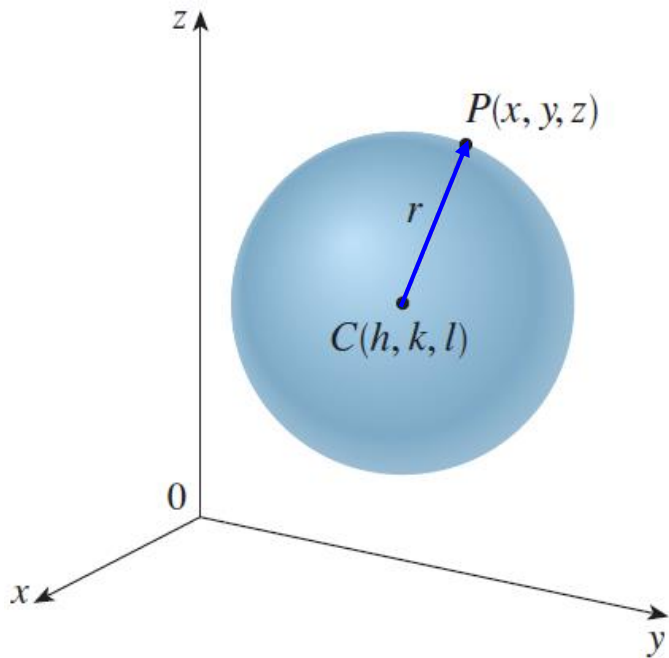
$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$|\overrightarrow{PQ}| = \sqrt{(1 - 2)^2 + (-3 - (-1))^2 + (5 - 7)^2} = \sqrt{1 + 4 + 4} = 3$$

1.4 Vectors and The Geometry of Space

1.4.3 Equation of Sphere (derived from Vector Length)

Example: Find an equation of a sphere with radius r and center $C(h, k, l)$.



From vector length (distance), we derived the sphere equation.

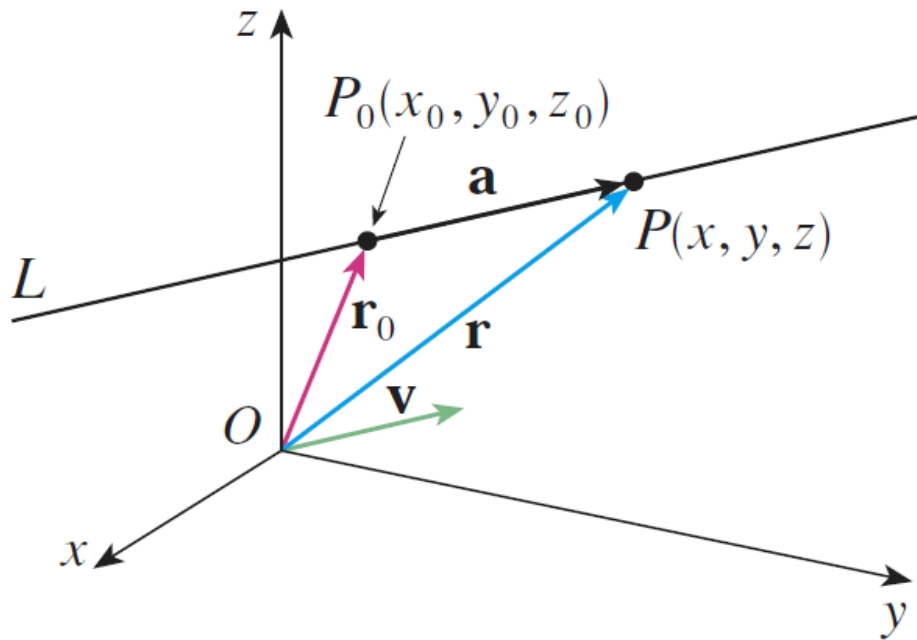
$$|\overrightarrow{CP}| = r$$

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

1.4 Vectors and The Geometry of Space



1.4.5 Line in 3D-Space



(Notice: The Figure 1 in Section 13.5 (Page 830) of the textbook (6th edition) is incorrect. Here uses the Figure from 7th edition.)

A Line L in 3D-Space is determined by

(1) A point on Line L

(2) Direction of L

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

1.4 Vectors and The Geometry of Space



1.4.5 Line in 3D-Space – Parametric Equation

Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

where $t \in \mathbb{R}$. These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$. Each value of the parameter t gives a point (x, y, z) on L .

1.4 Vectors and The Geometry of Space



1.4.6 Plane in 3D-Space

Dot Product Scalar product

DEFINITION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of \mathbf{a} and \mathbf{b} is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

THEOREM If θ is the angle between the vectors \mathbf{a} and \mathbf{b} , then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

Proof: Page 816

1.4 Vectors and The Geometry of Space

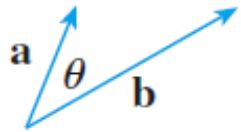


1.4.6 Plane in 3D-Space

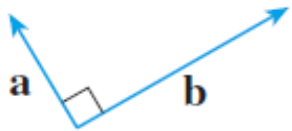
Dot Product – What it can do?

COROLLARY If θ is the angle between the nonzero vectors \mathbf{a} and \mathbf{b} , then

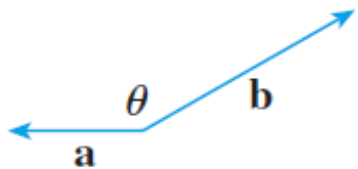
$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$



$$\mathbf{a} \cdot \mathbf{b} > 0$$



$$\mathbf{a} \cdot \mathbf{b} = 0$$



$$\mathbf{a} \cdot \mathbf{b} < 0$$

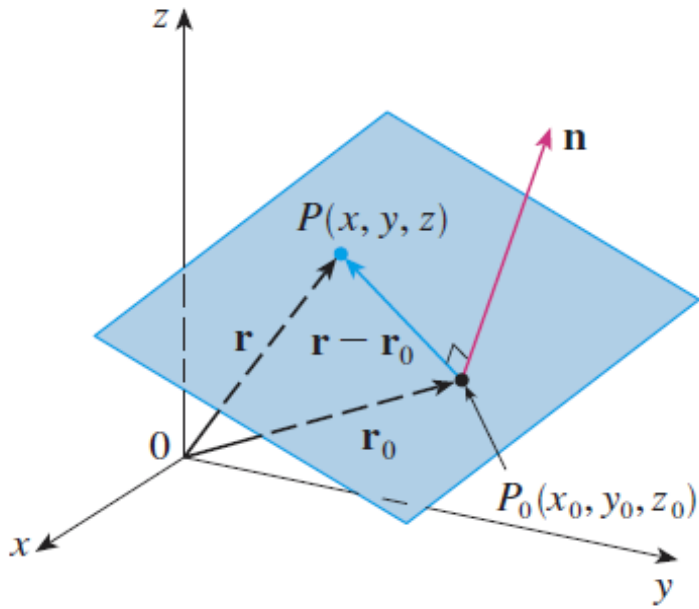
Two vectors \mathbf{a} and \mathbf{b} are **orthogonal** if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Helpful to Find Normal Vector.

1.4 Vectors and The Geometry of Space



1.4.6 Plane in 3D-Space



A Plane M in 3D-Space is determined by

- (1) A point on plane M
- (2) Direction of Normal Vector of M

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$



$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz + d = 0$$

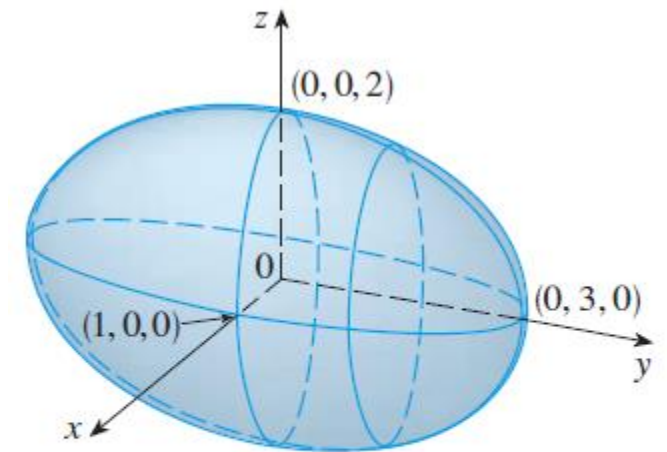
1.4 Vectors and The Geometry of Space

1.4.7 Quadric Surfaces

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

Example: Sketch the quadric surface with $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$.

These curves of intersection of the surface are called **traces** (or cross-sections) of the surface.



The ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

What you will learn in Lecture 1

I. How to GRAPH EQUATIONS in 3D-Space with 2-variables

II. VECTORS and the Geometry of 3D-Space

III. Parametrized Curves

1.5 Parametrized Curves

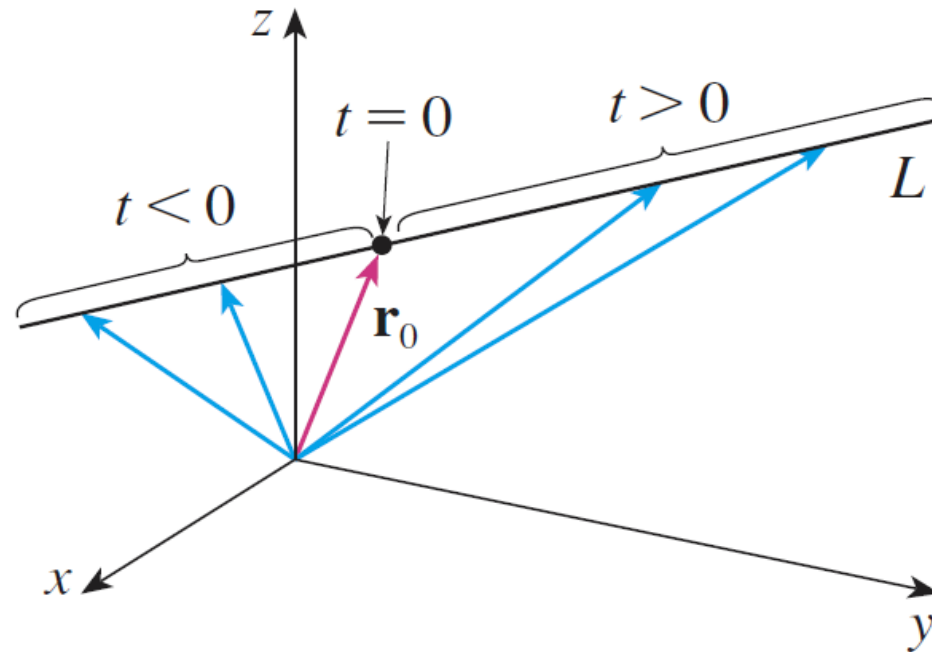
1.5 Parametrized Curves

1.5.1 Parametric Equations in Three Dimensions

A curve in the plane may be parameterized by the equations of the form

$$x = x(t), y = y(t), z = z(t)$$

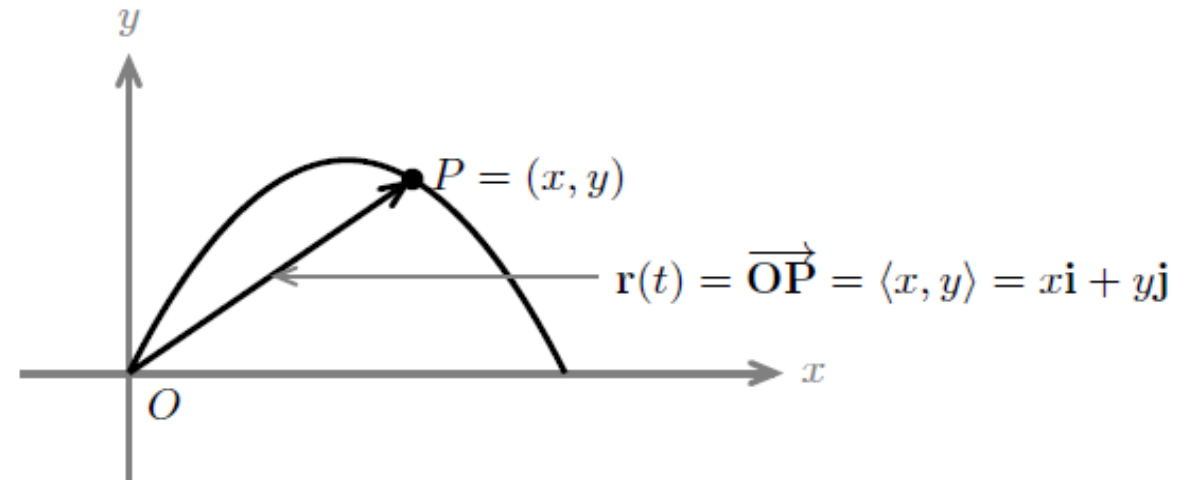
As the parameter t changes, the point (x, y) traces out the curve with motion.



1.5 Parametrized Curves

1.5.2 Parametric Equations in Three Dimensions

Example 1: Parametric equations for a rocket from the origin. Its initial x-velocity is $v_{0,x}$, and its initial y-velocity is $v_{0,y}$. (Page 877)



Trajectory

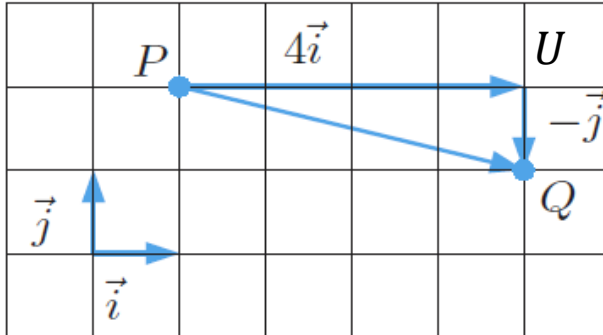
1.5 Parametrized Curves

1.5.2 Parametric Equations in Three Dimensions

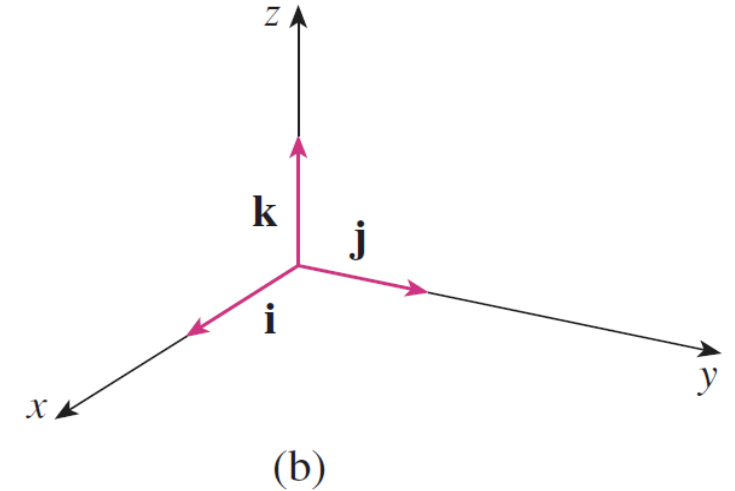
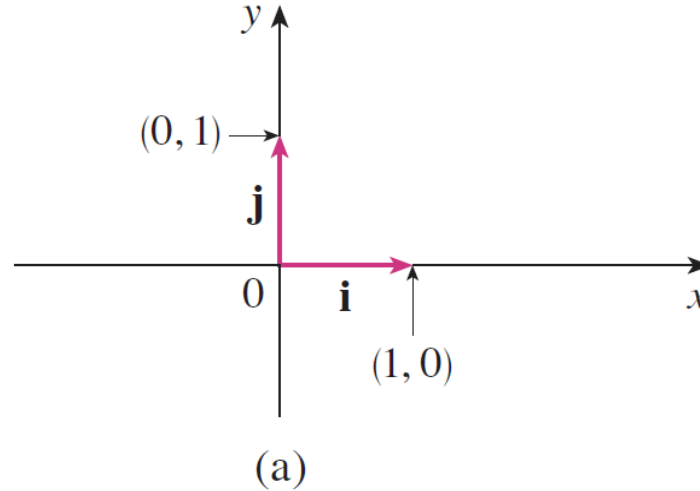
Additional knowledge of Vectors:
Standard basis vectors

Standard basis vectors.

$$\mathbf{i} = \langle 1, 0, 0 \rangle \quad \mathbf{j} = \langle 0, 1, 0 \rangle \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$



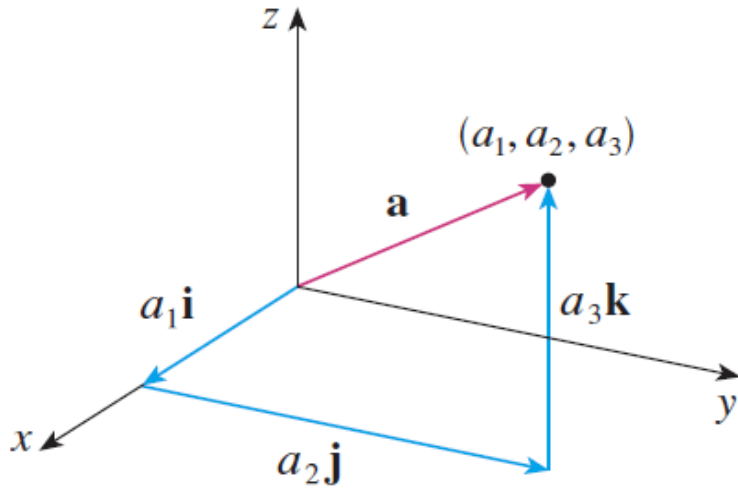
The displacement
vector from P to Q is $4\vec{i} - \vec{j}$



1.5 Parametrized Curves

1.5.2 Parametric Equations in Three Dimensions

Additional knowledge of Vectors:
Standard basis vectors



If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\begin{aligned}\mathbf{a} &= \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle\end{aligned}$$

$$\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$$

Thus any vector in V_3 can be expressed in terms of \mathbf{i} , \mathbf{j} , and \mathbf{k} . For instance,

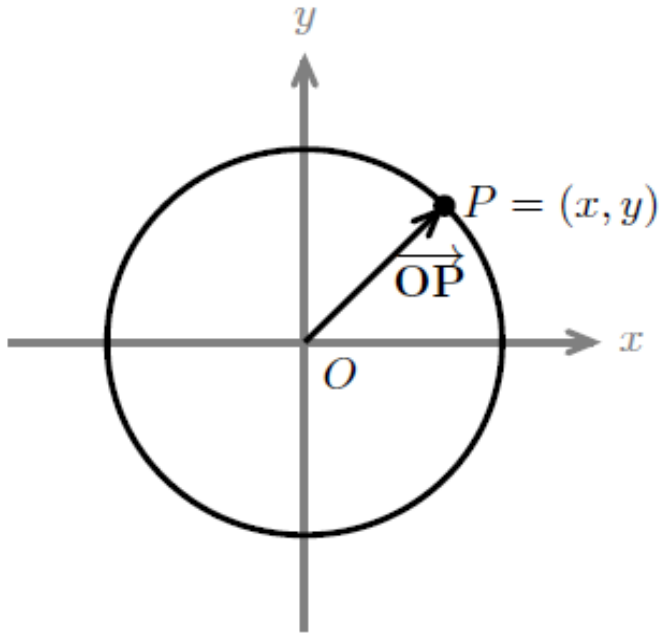
$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

1.5 Parametrized Curves

1.5.2 Parametric Equations in Three Dimensions

Example 2: Circle. Consider the parametric curve in the plane. (Page 658)

$$x(t) = a \cos t, y(t) = a \sin t.$$

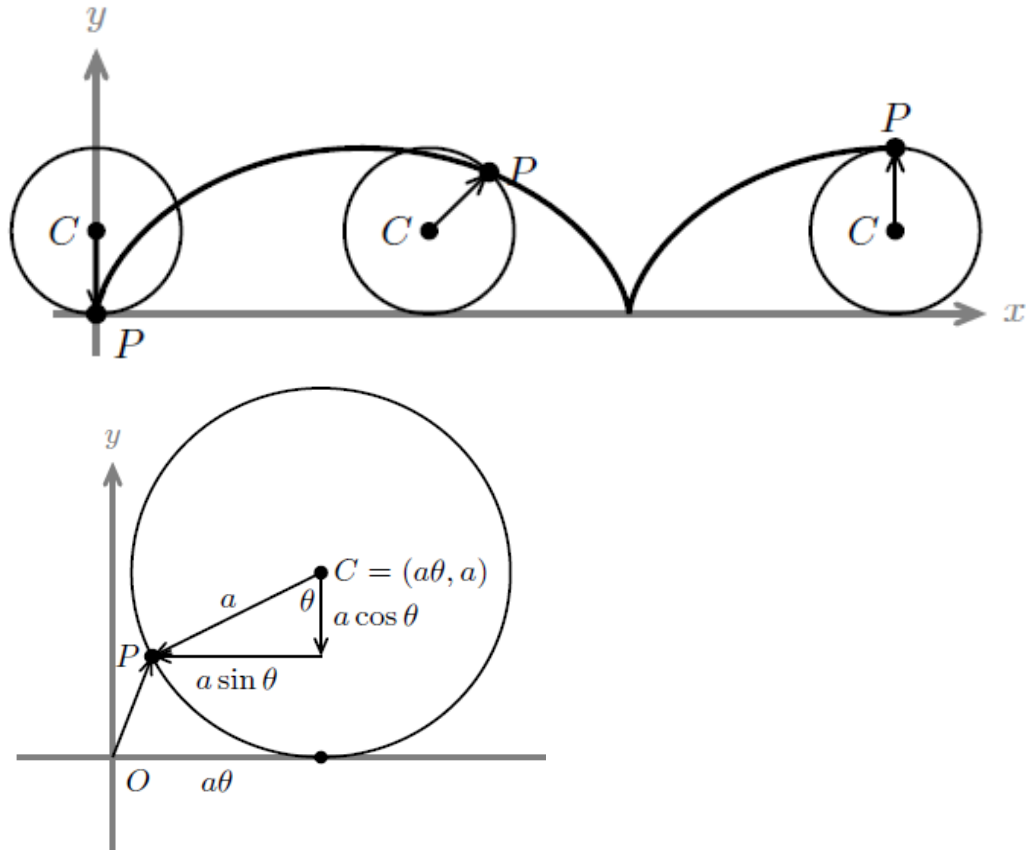


1.5 Parametrized Curves

1.5.2 Parametric Equations in Three Dimensions

Example 3: Cycloid (Page 660)

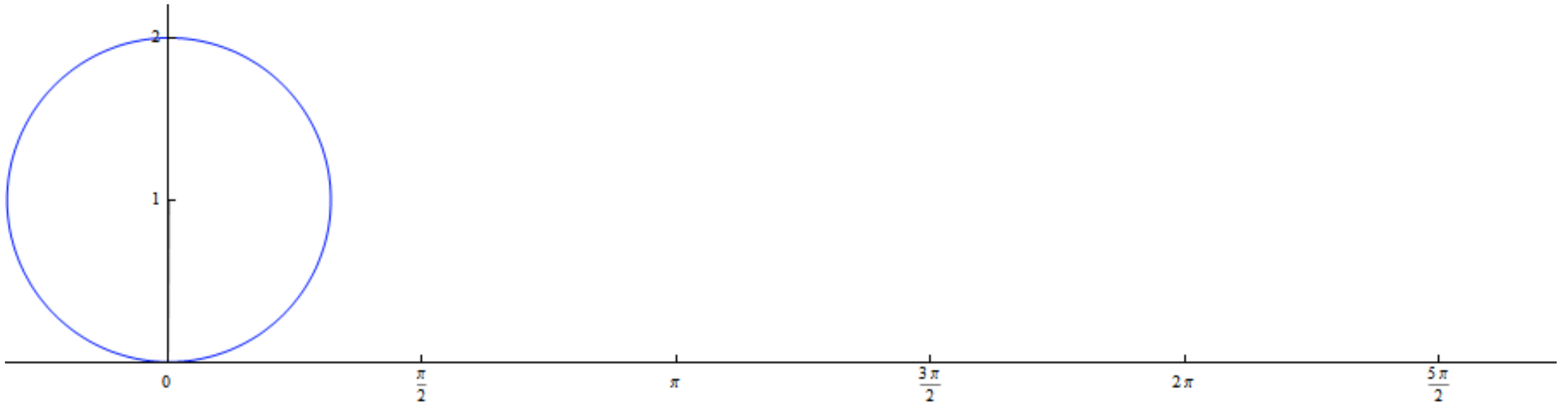
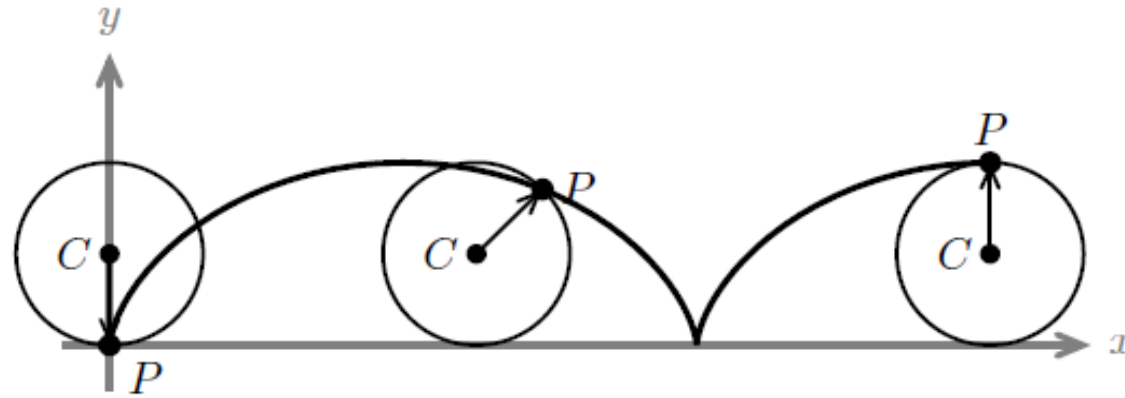
The cycloid is the trajectory of a point on a circle that is rolling without slipping along the x-axis. To be specific, we'll follow the point P that starts at the origin.



1.5 Parametrized Curves

1.5.2 Parametric Equations in Three Dimensions

Example 3: Cycloid (Page 660)



Review for Lecture 1

- Graph of function in 3D-Space with 2-Variables
3D coordinate system, \mathbb{R}^3 , Point Table
- Vector, Vector Length, Sphere Equation, Standard basis vector
- Equation of Line, Plane, Quadric Surface
- Normal Vector (Dot Product)
- Parametrized Curves (Parameter t ; Use 'Vector view' to see it)

Assignment 1



github.com/uoaworks

References

- Textbook: Section 15.1, 13.1-13.3, 13.5, 13.6, 11.1
- MIT 18.02SC: MIT18_02SC_notes_8, MIT18_02SC_notes_9
- Calculus Single & Multivariable, 6th Edition, Hughes Hallett
(Basically, figures used)