Lecture 4

Harmonic Functions (調和関数)

Exponential and Logarithmic Functions

(指数関数 と 対数関数)

What you will learn in Lecture 4

4.1 Harmonic Functions (調和関数)

4.2 Elementary Functions (初等関数) 1:

Exponential and Logarithmic Functions (指数関数と対数関数)

4.1 Harmonic Functions (調和関数)

4.1 Harmonic Functions (調和関数)

Laplace's Equation (ラプラス方程式)

The **second-order partial differential equation (2階偏微分方程式)**

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{3.3.1}$$

is called **Laplace's equation (ラプラス方程式)** in two independent variables x and y.

(The sum of the two second partial derivatives in (3.3.1) is denoted by $\nabla^2 \phi$ and is called the Laplacian of ϕ. Laplace's equation is then abbreviated as $\nabla^2 \phi = 0$.)

Definition 3.3 Harmonic Functions (調和関数)

A real-valued function ϕ of two real variables x and y that has <u>**continuous (連続)**</u> <u>**first and second-order partial derivatives (1と2階偏微分) in a domain D**</u> and <u>satisfies **Laplace's equation**</u> is said to be **harmonic** in D.

Harmonic Functions (調和関数)

Theorem 3.7 Analyticity (解析性) and Harmonic Functions (調和関数)

Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D.

then $u(x,y)$ is harmonic in D

and $v(x,y)$ is harmonic in D.

Proof: The Page 160 of Textbook

Harmonic Functions (調和関数)

EXAMPLE (例題) 3.3.1 Harmonic Functions
Show that the real and imaginary parts of function $f(z) = z^2$, where $z = x + iy$, are harmonic in $\mathbb{C}$.

Solution (解答):

The function $f(z) = z^2 = x^2 - y^2 + 2xyi$ is **entire** (**i.e. 整函数**).

Then the function $f(z) = z^2 = x^2 - y^2 + 2xyi$ is **analytic at every point** z in the complex plane.

According to Theorem 3.7,
The functions $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ are necessarily **harmonic** in the complex plane, i.e. in $\mathbb{C}$.

4.1 Harmonic Functions (調和関数)

Harmonic Conjugate Functions (共役調和関数)

Now suppose $u(x, y)$ is a given real function that is **harmonic** in D;

⇩

find another real harmonic function $v(x, y)$ so that u and v satisfy the **Cauchy-Riemann equations** throughout the domain D;

⇩

then this function $v(x, y)$ is called a **harmonic conjugate function (共役調和関数)** of $u(x, y)$.

⇩

By combining the functions as $u(x, y) + iv(x, y) = f(z)$, we obtain a function $f(z)$ that is **analytic** in D.

Harmonic Conjugate Functions (共役調和関数)

EXAMPLE (例題) 3.3.2 Harmonic Conjugate Function

(a) Verify that the function $u(x, y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane.

(b) Find the harmonic conjugate function of $u(x, y)$.

Solution (解答):

(a) From the first and second-order partial derivatives

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = -6xy - 5 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -6x$$

we see that u **satisfies Laplace's equation**: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$

Therefore, according to the Definition 3.3, $u(x, y)$ is **harmonic** in **C**.

4.1 Harmonic Functions (調和関数)

Solution (解答)(cont.):

(b) Since the **conjugate harmonic function** v must **satisfy the Cauchy-Riemann equations** $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ then we must have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(-6xy - 5) = 6xy + 5 \qquad (3.3.3)$$

Compute the partial integration of $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2$ **with respect to** y gives $v(x,y) = 3x^2y - y^3 + h(x)$.

From this $v(x,y)$, we compute the partial derivative with respect to x as $\frac{\partial v}{\partial x} = 6xy + h'(x)$.

Compare this $\frac{\partial v}{\partial x}$ with the second equation in (3.3.3), we can obtain $h'(x) = 5$, and so $h(x) = 5x + c$, where c is a real constant.

Therefore, the harmonic conjugate function of $u(x,y)$ is $v(x,y) = 3x^2y - y^3 + 5x + c$.

4.2 Elementary Functions (初等関数) 1:

Exponential and Logarithmic Functions

(指数関数と対数関数)

Complex Exponential Function (複素指数函数)

Suppose we know the fact that $e^{\alpha+\beta} = e^{\alpha}e^{\beta}$, where α and β are complex numbers.

Definition 4.1 Complex Exponential Function (複素指数函数)

The function e^z (where $z = x + iy$) defined by

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

i.e. $$e^z = e^x \cos y + i e^x \sin y \tag{4.1.1}$$

is called the **complex exponential function**.

Notice: this definition agrees with the real exponential function, i.e. if z is real number, then $z = x + 0i$, and Definition 4.1 gives:

$$e^{x+i0} = e^x(\cos 0 + i\sin 0) = e^x(1 + i0) = e^x$$

Complex Exponential Function (複素指数函数)

Theorem 4.1 Analyticity (解析性) of e^z

The **exponential function** e^z is **entire** and **its derivative is given by**:

$$\frac{d}{dz}e^z = e^z \tag{4.1.3}$$

Proof: The Page 177 of Textbook

Complex Exponential Function (複素指数函数)

EXAMPLE (例題) 4.1.1 Derivatives of Exponential Functions

Find the derivative of each of the following functions:

(a) $iz^4(z^2 - e^z)$ and (b) $e^{z^2-(1+i)z+3}$

Solution (解答):

(a) Using Equation (4.1.3) and the product rule (積の微分法則) (3.1.4) in Lecture 3:

Product Rule (積の法則): $$\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z) \tag{3.1.4}$$

$$\frac{d}{dz}[iz^4(z^2 - e^z)] = iz^4(2z - e^z) + i4z^3(z^2 - e^z)$$
$$= i6z^5 - iz^4e^z - i4z^3e^z$$

(b) Using Equation (4.1.3) and the chain rule (連鎖律) (3.1.6) in Lecture 3:

Chain Rule (連鎖律): $$\frac{d}{dz}f(g(z)) = f'\big(g(z)\big)g'(z) \tag{3.1.6}$$

$$\frac{d}{dz}\left[e^{z^2-(1+i)z+3}\right] = e^{z^2-(1+i)z+3} \cdot \big(2z - (1+i)\big) = e^{z^2-(1+i)z+3} \cdot (2z - 1 - i)$$

Complex Exponential Function (複素指数函数)

Theorem 4.2 Properties (性質) of e^z

If z_1 and z_2 are complex numbers, then

(i) $e^0 = 1$

(ii) $e^{z_1} e^{z_2} = e^{z_1 + z_2}$

(iii) $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

(iv) $(e^{z_1})^n = e^{n z_1}, n = 0, \pm 1, \pm 2, \ldots$

(v) $|e^z| = e^{\mathrm{Re}(z)}, \ \arg(e^z) = \mathrm{Im}(z)$

(vi) $\overline{e^z} = e^{\bar{z}}$

(vii) $e^z \neq 0$, for all $z \in \mathbf{C}$

4.2 Exponential and Logarithmic Functions (指数関数と対数関数)

Complex Exponential Function (複素指数函数)

Modulus (複素数の絶対値) and Argument (偏角)

We have the complex number $w = f(z) = e^z$ in polar form $re^{i\theta}$:

$$w = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = r(\cos\theta + i\sin\theta)$$

then we see that the modulus $r = e^x$ and the argument $\theta = y + 2n\pi$, for $n = 0, \pm 1, \pm 2, \ldots$

Modulus $$|e^z| = r = e^x = e^{\mathrm{Re}(z)} \tag{4.1.4}$$

Argument $$\arg(e^z) = \theta = y + 2n\pi = \mathrm{Im}(z) \quad \text{for } n = 0, \pm 1, \pm 2, \ldots \tag{4.1.5}$$

Conjugate (複素共役)

Because $\cos(-y) = \cos y$ $\sin(-y) = -\sin y$

$$\overline{e^z} = e^x \cos y - ie^x \sin y = e^x \cos(-y) + ie^x \sin(-y) = e^{x-iy} = e^{\bar{z}} \tag{4.1.6}$$

Nonzero (非ゼロ)

From (4.1.4), we know $|e^z| > 0$ because $e^x > 0$ for all $x \in \mathbf{R}$. Then it implies $e^z \neq 0$, for all $z \in \mathbf{C}$.

4.2 Exponential and Logarithmic Functions (指数関数と対数関数)

Complex Exponential Function (複素指数函数)

Periodicity (周期性)

$$\boxed{e^{z+2\pi i} = e^z}$$

The complex exponential function e^z is periodic with a pure imaginary period (純虚数周期) $2\pi i$.

This is because, by (4.1.1) and Theorem 4.2(ii) ,

we have $e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos 2\pi + i\sin 2\pi) = e^z$

Notice that $e^{z+4\pi i} = e^{(z+2\pi i)+2\pi i} = e^{z+2\pi i} = e^z$

By repeating this process we find that

$e^{z+2n\pi i} = e^z$ for $n = 0, \pm 1, \pm 2, \ldots$

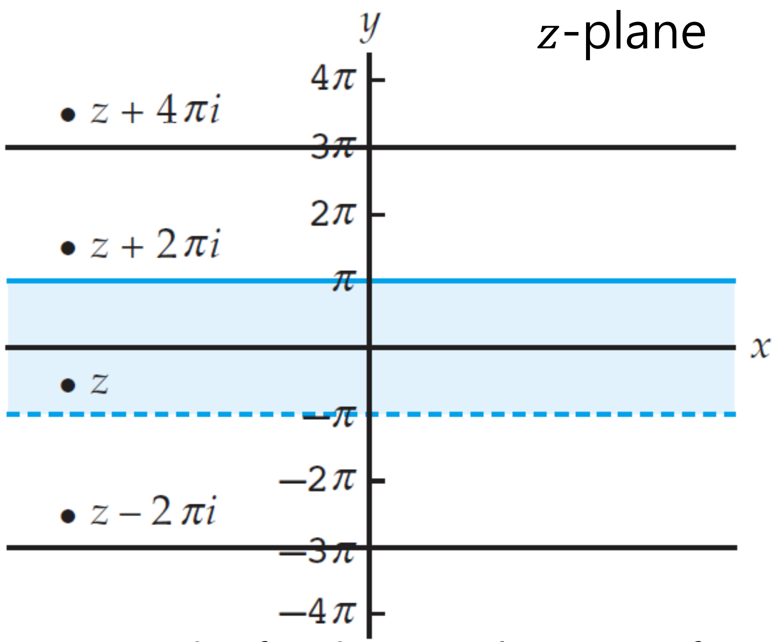


Figure 4.1 The fundamental region of e^z

4.2 Exponential and Logarithmic Functions (指数関数と対数関数)

Complex Exponential Function (複素指数函数)

Fundamental Region of the complex exponential function

- Because $e^{z+2n\pi i} = e^z$ for $n = 0, \pm 1, \pm 2, \ldots$

thus there are many points in the z-plane, for example, $z - 2\pi i, z + 4\pi i, z + 6\pi i$, ...will correspond to the same single point $w = e^z$ in the w-plane,

i.e. the complex exponential function $w = f(z) = e^z$ is not one-to-one (一対一) mapping from z-plane to w-plane.

- **We divide the complex plane into horizontal strips.**
- The **infinite horizontal (水平な) strip** defined by:

$$-\infty < x < \infty, -\pi < y \leq \pi$$

is called the **fundamental region (基本領域)** of the complex exponential function e^z.

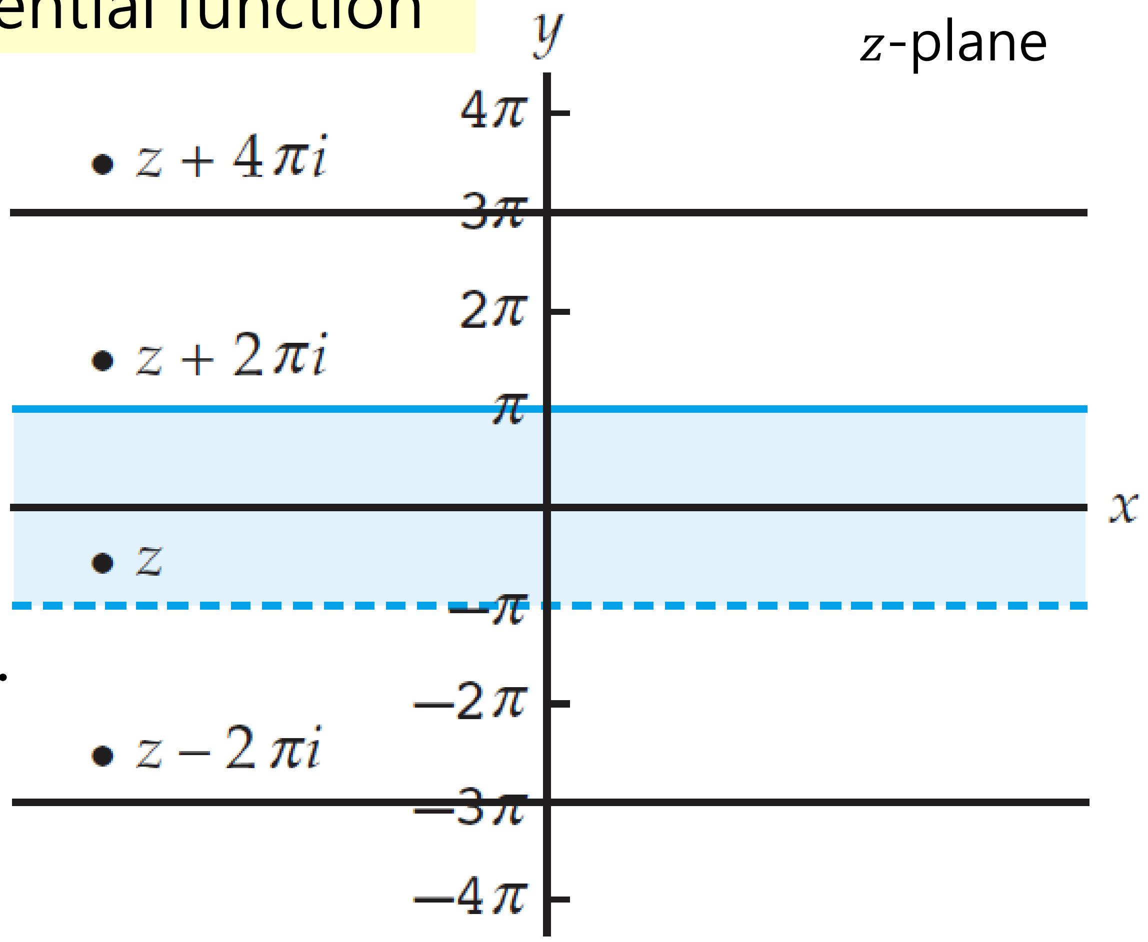
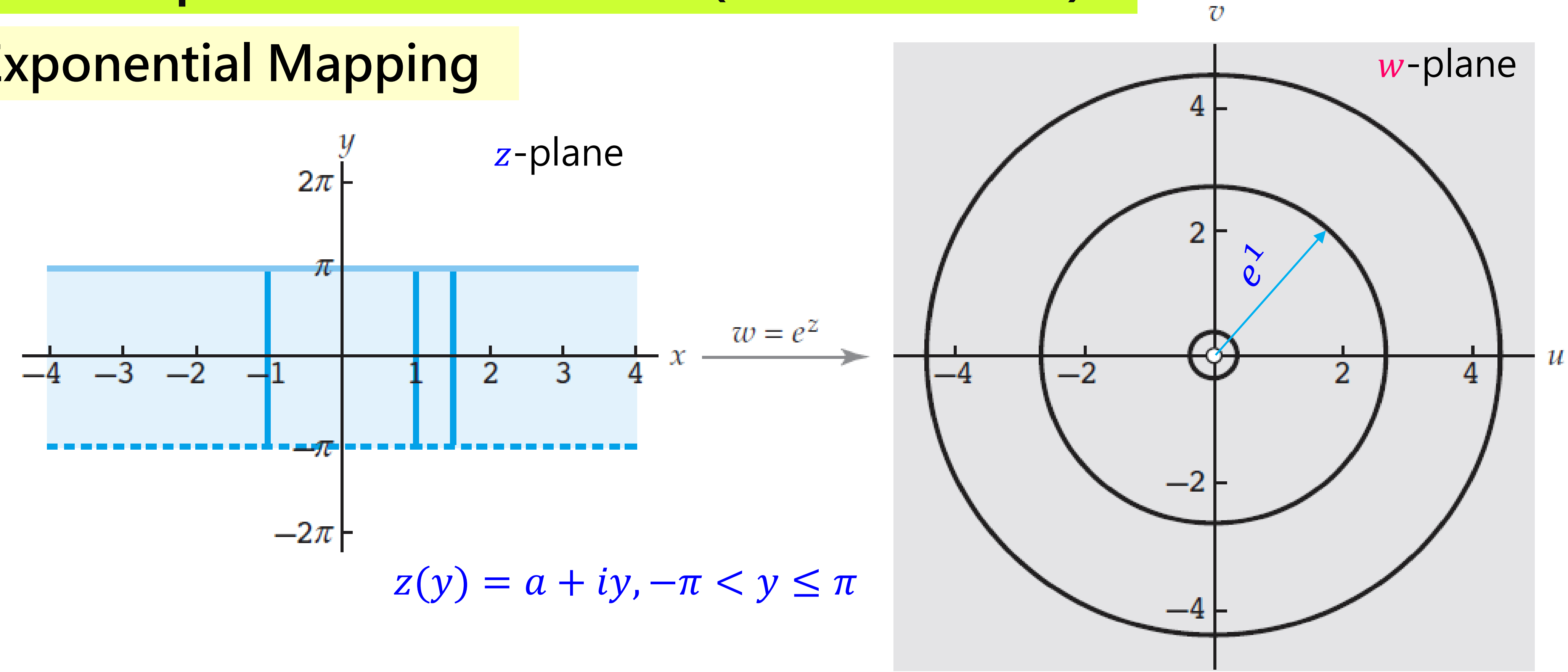


Figure 4.1 The fundamental region of e^z

Complex Exponential Function (複素指数函数)

*The Exponential Mapping



$z(y) = a + iy, -\pi < y \leq \pi$

(a) Vertical line segments in the fundamental region (b) Images of the line segments in (a) are circles (円).

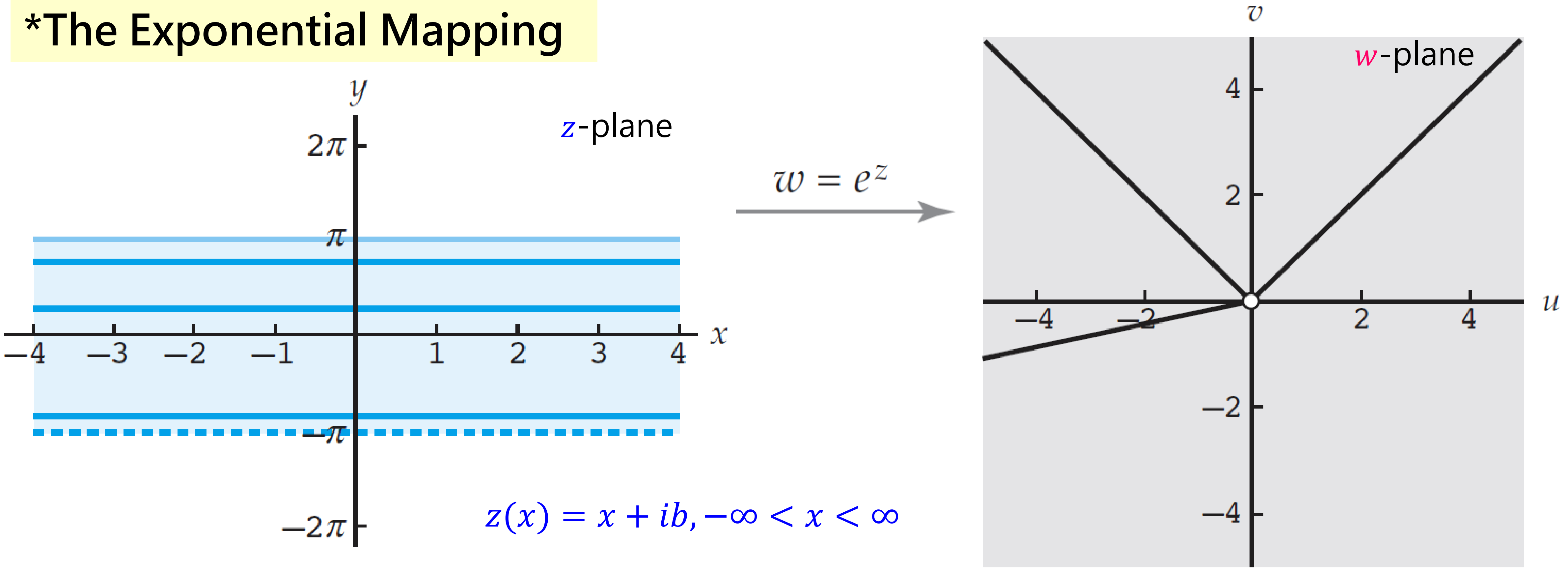
Figure 4.2 The image (i.e. mapping result) of the fundamental region under $w = f(z) = e^z$ for the vertical line segments

Notice: In all lectures, the contents marked with * are not in the scope of the final examination.

4.2 Exponential and Logarithmic Functions (指数関数と対数関数)

Complex Exponential Function (複素指数函数)

*The Exponential Mapping



(a) Horizontal lines in the fundamental region

(b) Images of the lines in (a) are rays (半直線) .

Figure 4.3 The mapping $w = f(z) = e^z$ for the horizontal lines

Complex Exponential Function (複素指数函数)

*Exponential Mapping Properties

(i) $w = e^z$ maps the **fundamental region** (**基本領域**) $-\infty < x < \infty$, $-\pi < y \le \pi$, onto the set $|w| > 0$, i.e. the points that satisfies $|w| \neq 0$.

(Recall Theorem 4.2(vii), Nonzero property.)

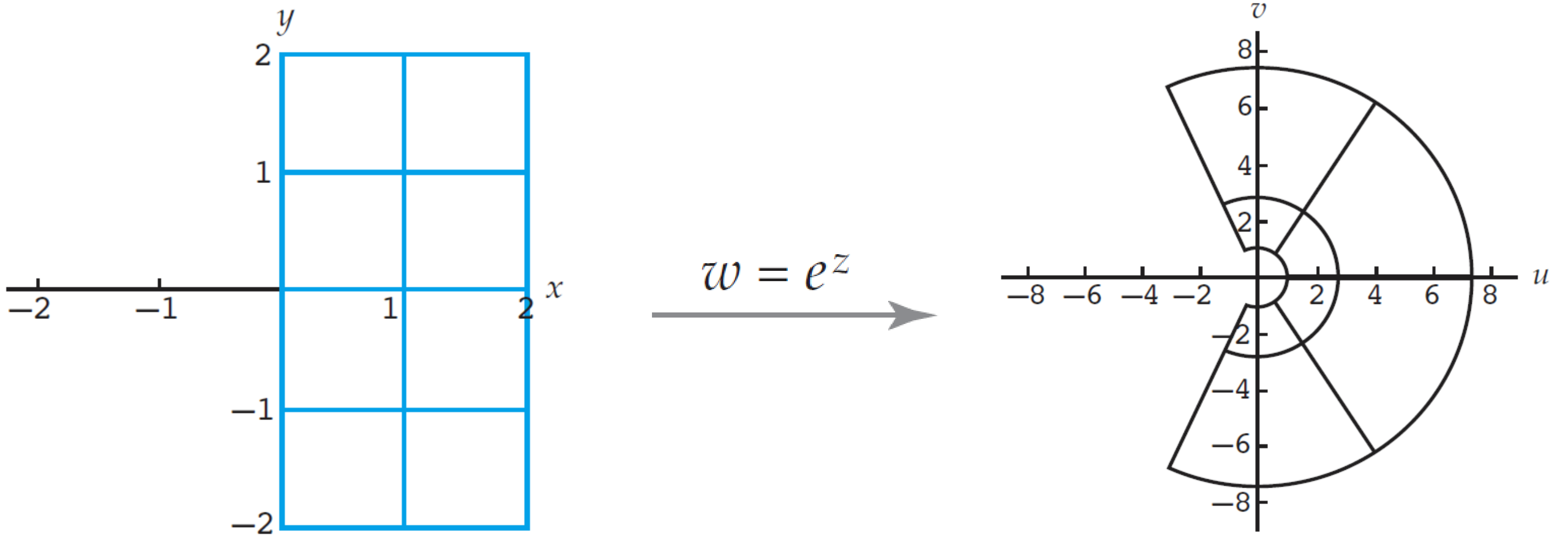
(ii) $w = e^z$ **maps** the **vertical line segment** (**垂直の線分**) $x = a$, $-\pi < y \le \pi$, **onto** the **circle** (**円**) $|w| = e^a$, where a is a real number.

(iii) $w = e^z$ **maps** the **horizontal line** (**水平線**) $y = b$, $-\infty < x < \infty$, **onto** the **ray** (**半直線**) $\arg(w) = b$.

Complex Exponential Function (複素指数函数)

***EXAMPLE (例題) 4.1.2 Exponential Mapping of a Grid**

Find the image of the grid shown in Figure 4.4(a) under $w = f(z) = e^z$.



(a) The grid (i.e. Line segments) in z-plane

(b) Image of the grid in w-plane

Figure 4.4 The mapping $w=f(z)=e^z$

Read more in the Page 181 of Textbook.

Complex Logarithmic Functions (複素対数関数)

In **real** domain, the **natural logarithm function** $\ln x$ is often defined as an **inverse function (逆関数)** of **the real exponential function** e^x.

From now on, we will use the alternative notation $\log_e x$ to represent the real natural logarithm function.

- The **real exponential function** e^x is **one-to-one (一対一) on its domain** $\mathbf{R}$,
- **But the complex exponential function** e^z **is NOT a one-to-one function on its domain** $\mathbf{C}$, because there are **infinitely (無限に) many arguments (偏角)** of z.

Complex Logarithmic Functions (複素対数関数)

If $e^{w} = z$, then $w = \log_e |z| + i \arg(z)$ (4.1.10)

Because of the Periodicity (周期性), there are infinitely (無限に) many arguments (偏角) of z, thus **(4.1.10) gives infinitely many solutions** w to the equation $e^{w} = z$.

The set of solutions given by (4.1.10) defines a multiple-valued function as:

Definition 4.2 Complex Logarithm Function (複素対数関数)

The multi-valued function $\ln z$ (where $z = x + iy$) defined by

$$\ln z = \log_e |z| + i \arg(z) \tag{4.1.11}$$

is called the **complex logarithm**.

Notice: We use the lowercase (小文字) first letter for symbol $\ln z$.

Complex Logarithmic Functions (複素対数関数)

EXAMPLE (例題) 4.1.3 Solving Exponential Equations

Find all complex solutions to each of the following equations.

(a)$e^w = i$ (b)$e^w = 1 + i$ (c)$e^w = -2$

Solution (解答):

Because $\lim_{a \to \infty} \arctan(a) = \frac{\pi}{2}$

(a) For $e^w = z = i$, we have $|i| = 1$ and $\arg(i) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}) = \frac{\pi}{2} + 2n\pi$.

Thus, from (4.1.11) we obtain:

$$w = \ln i = \log_e |i| + i \arg(i)$$

$$= \log_e 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = 0 + i\left(\frac{\pi}{2} + 2n\pi\right) = \frac{(4n+1)\pi}{2} i \quad n = 0, \pm 1, \pm 2, \dots$$

Therefore, each of the values $w = \cdots, -\frac{3\pi}{2} i,\ \frac{\pi}{2} i,\ \frac{5\pi}{2} i, \dots$ satisfies the equation $e^w = i$.

Complex Logarithmic Functions (複素対数関数)

Solution (解答)(cont.):

(b) For $z = 1 + i$, we have $|1 + i| = \sqrt{2}$ and $\arg(1 + i) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}) = \frac{\pi}{4} + 2n\pi$.

Thus, from (4.1.11) we obtain:

$$w = \ln(1 + i) = \log_e |1 + i| + i \arg(1 + i)$$
$$= \log_e \sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right) i$$
$$= \frac{1}{2}\log_e 2 + \frac{(8n + 1)\pi}{4} i \qquad n = 0, \pm 1, \pm 2, \ldots$$

(c) For $z = -2$, we have $|-2| = 2$ and $\arg(-2) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}) = \pi + 2n\pi$.

Thus, from (4.1.11) we obtain:

$$w = \ln(-2) = \log_e |-2| + i \arg(-2)$$
$$= \log_e 2 + i(\pi + 2n\pi) i$$
$$= \log_e 2 + (2n + 1)\pi i \qquad n = 0, \pm 1, \pm 2, \ldots$$

Complex Logarithmic Functions (複素対数関数)

Theorem 4.3 Algebraic Properties of $\ln z$

If z_1 and z_2 are nonzero complex numbers and n is an integer, then

(i) $$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

(ii) $$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$$

(iii) $$\ln z_1^n = n \ln z_1$$

Complex Logarithmic Functions (複素対数関数)

Definition 4.3 Principal Value (主值) of the Complex Logarithm

The multi-valued function $\mathrm{Ln}\, z$ (where $z = x + iy$) defined by

$$\mathrm{Ln}\, z = \log_e |z| + i\, \mathrm{Arg}(z),\ -\pi < \mathrm{Arg}(z) \leq \pi \qquad \text{(4.1.14 and 4.1.15)}$$

is called **the principal value (主值) of the complex logarithm**.

Notice: We use the uppercase (大文字) first letter for $\mathrm{Ln}\, z$ here!

Complex Logarithmic Functions (複素対数関数)

EXAMPLE (例題) 4.1.4 Principal Value of the Complex Logarithm

Compute the principal value of the complex logarithm $\mathrm{Ln}\, z$ for

(a) $z = i$ (b) $z = 1 + i$ (c) $z = -2$

Solution (解答):

(a) For $e^w = z = i$, we have $|i| = 1$ and $\mathrm{Arg}(i) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}) = \frac{\pi}{2} \cancel{+ 2n\pi}$

Thus, from (4.1.11) we obtain:

$$w = \mathrm{Ln}\, i = \log_e |i| + i\, \mathrm{Arg}(i)$$

$$= \log_e 1 + \frac{\pi}{2} i = 0 + \frac{\pi}{2} i = \frac{\pi}{2} i \qquad \cancel{n = 0, \pm 1, \pm 2, \ldots}$$

Complex Logarithmic Functions (複素対数関数)

Solution (解答)(cont.):

(b) For $e^w = z = 1 + i$, we have $|1 + i| = \sqrt{2}$ and $\mathrm{Arg}(i) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}) = \frac{\pi}{4}$.

Thus, from (4.1.11) we obtain:

$$w = \mathrm{Ln}\, i = \log_e |1 + i| + i\,\mathrm{Arg}(1 * i)$$

$$= \log_e \sqrt{2} + \frac{\pi}{4} i = \frac{1}{2} \log_e 2 + \frac{\pi}{4} i \approx 0.3466 + 0.7854i$$

(c) For $e^w = z = -2$, we have $|-2| = 2$ and $\mathrm{Arg}(i) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}) = \pi$.

Thus, from (4.1.11) we obtain:

$$w = \mathrm{Ln}\, i = \log_e |-2| + i\,\mathrm{Arg}(i)$$

$$= \log_e 2 + \pi i \approx 0.6931 + 3.1416i$$

Complex Logarithmic Functions (複素対数関数)

$\mathrm{Ln}\, z$ as an Inverse Function (逆関数) of e^z

Follows from (4.1.10) that

$$e^{\mathrm{Ln}\, z} = z \text{ for all } z \neq 0. \tag{4.1.16}$$

If the complex exponential function $f(z) = e^z$ is defined on the fundamental region $-\infty < x < \infty, -\pi < y \leq \pi$,

then f is **one-to-one (一対一)** and the **inverse function (逆関数)** of f is the **principal value of the complex logarithm** $f^{-1}(z) = \mathrm{Ln}\, z$.

Complex Logarithmic Functions (複素対数関数)

Recall that

Theorem 2.3 Real and Imaginary Parts of a Continuous Function

Suppose that $f(z) = u(x,y) + iv(x,y)$ and $z_0 = x_0 + iy_0$.
Then the complex function (複素関数) f is continuous at the point z_0 **if and only if** both real functions (実数値関数) u and v are continuous at the point (x_0, y_0).

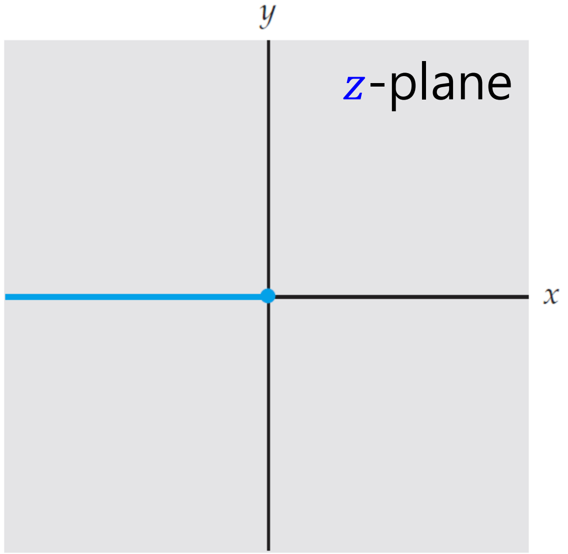


Figure 4.6 $f_1(z)$ defines on domain in gray color

The principal value of the complex logarithm function

$$\operatorname{Ln} z = \log_e |z| + i\,\operatorname{Arg}(z), \quad -\pi < \operatorname{Arg}(z) \le \pi$$

Real part $\quad u(x,y) = \log_e |z| = \log_e \sqrt{x^2+y^2}$ is **continuous at all points** in the plane **except** $(0,0)$

Imaginary part $\quad v(x,y) = \operatorname{Arg}(z)$ is **continuous on the domain** $|z| > 0,\ -\pi < \operatorname{Arg}(z) < \pi$

Therefore, $\operatorname{Ln} z$ is **a continuous function on the domain** $|z| > 0,\ -\pi < \operatorname{Arg}(z) < \pi$

We give this new function a name by "principal branch of the complex logarithm function"

$$f_1(z) = \log_e |z| + i\,\operatorname{Arg}(z), \quad -\pi < \operatorname{Arg}(z) < \pi \qquad (4.1.19)$$

Here, $f_1(z)$ is $\operatorname{Ln} z$ except $\operatorname{Arg}(z) = \pi$

Complex Logarithmic Functions (複素対数関数)

Theorem 4.4 Analyticity of the Principal Branch of $\ln z$

The principal branch f_1 of the complex logarithm defined by (4.1.19) is **an analytic function** and **its derivative is given by**:

$$f_1'(z) = \frac{1}{z} \tag{4.1.20}$$

The theorem 4.4 implies that $\mathrm{Ln}\, z$ is differentiable in the domain $|z| > 0$, $-\pi < \mathrm{Arg}(z) < \pi$, and its derivative is given by $f_1'(z)$.

That is, if $|z| > 0$, $-\pi < \mathrm{Arg}(z) < \pi$ then

$$\boxed{\frac{d}{dx}\mathrm{Ln}\, z = \frac{1}{z}} \tag{4.1.21}$$

Complex Logarithmic Functions (複素対数関数)

EXAMPLE (例題) 4.1.5 Derivatives of Logarithmic Functions

Find the derivatives of the following functions in an appropriate domain:

(a) $z \operatorname{Ln} z$ and *(b) $\operatorname{Ln}(z+1)$

Solution (解答):

(a) The function $z\operatorname{Ln} z$ is differentiable at all points where both of the functions z and $\operatorname{Ln} z$ are differentiable.

Because z is entire (整函数) and $\operatorname{Ln} z$ is differentiable on the domain given in (4.1.19), as $|z| > 0$, $-\pi < \operatorname{Arg}(z) < \pi$, it follows that $z\operatorname{Ln} z$ is differentiable on the domain defined by $|z| > 0$, $-\pi < \operatorname{Arg}(z) < \pi$

In this domain, the derivative is given by the product rule (積の微分法則) (3.1.4) of Lecture 3 and (4.1.21):

$$\frac{d}{dx}[\mathrm{zLn}\, z] = z \cdot \frac{1}{z} + 1 \cdot \operatorname{Ln} z = 1 + \operatorname{Ln} z$$

Complex Logarithmic Functions (複素対数関数)

Solution (解答)(cont.):

*(b) The function $\mathrm{Ln}\,(z+1)$ is a composition ($f(g(z))$) of the functions $\mathrm{Ln}\,z$ and $z+1$.

Because $z+1$ is entire (整函数) and $\mathrm{Ln}\,z$ is differentiable on the domain given in (4.1.19), as $|z|>0$, $-\pi<\mathrm{Arg}(z)<\pi$, it follows from the chain rule (連鎖律) that $\mathrm{Ln}\,(z+1)$ is differentiable at all points $g=z+1$ such that $|g|>0$, $-\pi<\mathrm{Arg}(g)<\pi$

Recall the domain $|g|>0$, $-\pi<\mathrm{Arg}(g)<\pi$ for $f_1(g)$ in (4.1.19) as figure 4.6, we obtain the domain for $g=z+1$ as figure T1. (Notice: The g here is equivalent to the original "z" in (4.1.19).)

The equation $z=g-1$ defines a linear mapping of the g-plane onto the z-plane given by translation (i.e. shift along x-axis) by -1.

In this domain in z-plane, the derivative is given by the chain rule (連鎖律) (3.1.6) of Lecture 3 and (4.1.21):

$$\frac{d}{dx}[\mathrm{Ln}\,(z+1)]=\frac{1}{z+1}\cdot 1=\frac{1}{z+1}$$

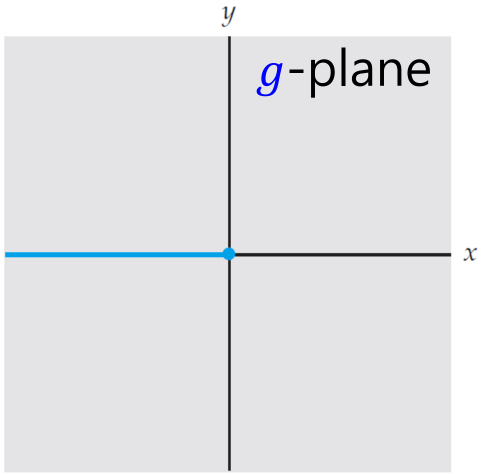


Figure T1. The defined domain in g-plane with gray color

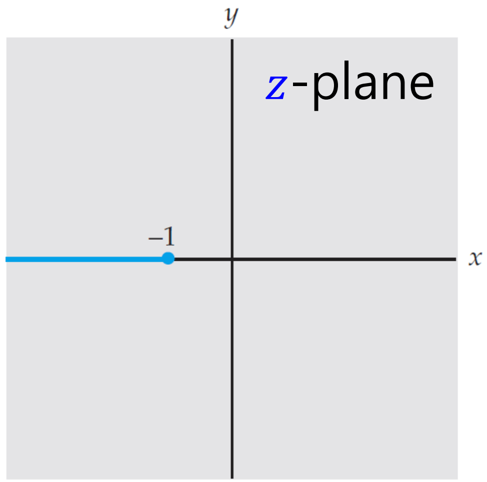


Figure 4.7 $\mathrm{Ln}(z+1)$ is not differentiable on the ray shown in blue color.

Review for Lecture 4

- Harmonic Functions
- Exponential Functions
- Exponential Mapping
- Logarithmic Functions
- The principal value of the Logarithmic Functions
- Analyticity of the Principal Branch of $\ln z$

Exercise

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: Section , Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia

Appendix (付録)

Elementary function:

https://en.wikipedia.org/wiki/Elementary_function

初等関数とは:

http://www.cc.miyazaki-u.ac.jp/yazaki/teaching/di/di-function.pdf

Appendix (付録)

Horizontal strip とは: