Lecture 13

Residues（留数）&

Residue Theorem （定理）Part 2

What you will learn in Lecture 13

13.1 Residues (留数) & Residue Theorem (留数定理) Part 2

13.2 Some Consequences of the Residue Theorem

13.1 Residues (留数) &

Residue Theorem (留数定理)

Part 2

13.1 Residues (留数) & Residue Theorem (留数定理) Part 2

Theorem 6.14 Residue at a Simple Pole

If f has a simple pole at $z = z0$, then

$$\text{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z) \tag{6.5.1}$$

Theorem 6.15 Residue at a Pole of Order n

If f has a pole of order n at $z = z_0$, then

$$\text{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \tag{6.5.2}$$

13.1 Residues (留数) & Residue Theorem (留数定理) Part 2

When f **is not a rational function**, calculating residues by means of (6.5.1) or (6.5.2) in Lecture 12 can sometimes be tedious.

It is possible to devise **alternative residue formulas**.

In particular, suppose a function f can be written **as a quotient** $f(z) = g(z)/h(z)$, where g and h **are analytic at** $z = z_0$.

If $g(z_0) = 0$ and if the function h **has a zero of order** 1 **at** z_0, then f **has a simple pole at** $z = z_0$ and

$$\mathrm{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)} \qquad (6.5.4)$$

13.1 Residues (留数) & Residue Theorem (留数定理) Part 2

To derive this result we shall use **the definition of a zero of order 1**, **the definition of a derivative**, and then (6.5.1).

First, since the function h **has a zero of order 1 at** z_0, we must have $h(z_0) = 0$ and $h(z_0) = 0$.

Second, by definition of the derivative given in (3.1.12) of Lecture 3 (slide 15),

$$h'(z_0) = \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{h(z)}{z - z_0}$$

We then combine the preceding two facts in the following manner in (6.5.1):

$$\operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)} = \lim_{z \to z_0} \frac{g(z)}{\dfrac{h(z)}{z - z_0}} = \frac{g(z_0)}{h'(z_0)}$$

Recall in Lecture 5

Roots of a Complex Number

Consider to find z in $z^k = w$

where z and w are complex numbers,

k is real, i.e. NOT a complex number.

then

$$z = \sqrt[k]{|w|}\left[\cos\left(\frac{\arg(w) + 2n\pi}{k}\right) + i\sin\left(\frac{\arg(w) + 2n\pi}{k}\right)\right] \quad (1.4.4)$$

where $n = 0, 1, 2, \ldots, k-1$

EXAMPLE (例題) 6.5.3 **Using (4) to Compute Residues**

The polynomial $z^4 + 1$ can be factored as $(z - z_1)(z - z_2)(z - z_3)(z - z_4)$, where z_1, z_2, z_3, and z_4 are the four distinct roots of the equation $z_4 + 1 = 0$ (or equivalently, the four fourth roots of -1). It follows from Theorem 6.13 that the function

$$\oint_C \frac{1}{z^4 + 1} dz$$

has four simple poles. By using (6.5.4), find its residues.

Solution (解答):

Hint:

- Equation (1.4.4) in Lecture 5
- Equation (6.5.4)
- Euler’s formula (1.6.6)

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check https://github.com/uoaworks/ComplexAnalysisAY2018

Solution (解答)(cont.):

Theorem 6.16 Cauchy's Residue Theorem

Let D be **a simply connected domain** and C **a simple closed contour lying entirely within** D. If a function f **is analytic on and within** C**, except at a finite number of isolated singular points** $z_1, z_2, \ldots, z_n$ **within** C, then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k) \qquad (6.5.5)$$

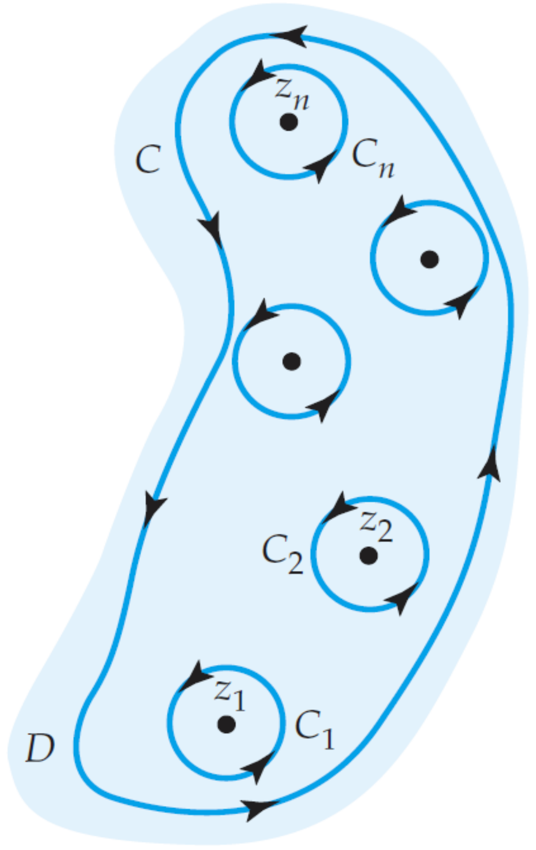

Figure 6.10 n **singular points within contour** C

Proof

Suppose $C_1, C_2, \ldots, C_n$ are circles centered at $z_1, z_2, \ldots, z_n$, respectively. Suppose further that each circle C_k has a radius r_k small enough so that $C_1, C_2, \ldots, C_n$ are mutually disjoint and are interior to the simple closed curve C.

See Figure 6.10. Now in (6.3.20) of Section 6.3 we saw that $\oint_{C_k} f(z)dz = 2\pi i \operatorname{Res}(f(z), z_k)$, and so by Theorem 5.5 we have

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k)$$

13.1 Residues (留数) & Residue Theorem (留数定理) Part 2

Replacing the complex variable s with the usual symbol z, we see that when $k - 1$, formula (6.3.8) in Lecture 11 (slide 13) for the Laurent series coefficients yields

$a_{-1} = \frac{1}{2\pi i} \oint_C \ f(z)dz$, or more important,

$$\oint_C \ f(z)dz = 2\pi i \ a_{-1} \tag{6.3.20}$$

EXAMPLE (例題) 6.5.4 Evaluation by the Residue Theorem

Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)}\,dz$, where

(a) the contour C is the rectangle defined by $x=0, x=4, y=-1, y=1$,

(b) and the contour C is the circle $|z|=2$.

Solution (解答):

Hint:

- Equation (6.5.5)
- Example 6.5.2

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EXAMPLE (例題) 6.5.5 Evaluation by the Residue Theorem

Evaluate $\oint_C \frac{2z+6}{z^2+4}\,dz$, where the contour C is the circle $|z-i|=2$.

Solution (解答):

Hint:

- Equation (6.5.5)

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EXAMPLE (例題) 6.5.6 Evaluation by the Residue Theorem

Evaluate $\oint_C \frac{e^z}{z^4+5z^3}\,dz$, where the contour C is the circle $|z| = 2$.

Solution (解答):

Hint:

- Equation (6.5.5)
- Equation (6.5.2)

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EXAMPLE (例題) 6.5.7 Evaluation by the Residue Theorem

Evaluate $\oint_C \tan z \, dz$, where the contour C is the circle $|z| = 2$.

Solution (解答):

Hint:

- The solution of cos $z = 0$
- Equation (6.5.4)

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EXAMPLE (例題) 6.5.8 Evaluation by the Residue Theorem

Evaluate $\oint_C e^{\frac{3}{z}}\,dz$, where the contour C is the circle $|z| = 1$.

Solution (解答):

Hint:

- Example 6.5.1
- Equation (6.5.1) and (6.5.2)

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*13.2 Some Consequences of the Residue Theorem

Notice: In all lectures notes, the contents marked with * are not in the scope of the final examination.

13.2 Some Consequences of the Residue Theorem

Evaluation of Real Trigonometric Integrals

Integrals of the Form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$

The basic idea here is to convert a real trigonometric integral of form (6.6.1) into a complex integral, where the contour C is the unit circle $|z| = 1$ centered at the origin.

To do this we begin with (2.2.10) of Section 2.2 to parametrize this contour by $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. We can then write

$$dz = ie^{i\theta}d\theta \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

13.2 Some Consequences of the Residue Theorem

The last two expressions follow from (4.3.2) and (4.3.3) of Lecture 5 (slide 22).

Since $dz = ie^{i\theta}\, d\theta = iz\, d\theta$ and $z^{-1} = 1/z = e^{-i\theta}$, these three quantities are equivalent to

$$d\theta = \frac{dz}{iz} \qquad \cos\theta = \frac{z+z^{-1}}{2} \qquad \sin\theta = \frac{z-z^{-1}}{2i} \tag{6.6.4}$$

The conversion of the integral in $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$ into a contour integral is accomplished by replacing, in turn, $d\theta$, $\cos\theta$, and $\sin\theta$ by the expressions in (6.6.4):

$$\oint_C F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}$$

where C is the unit circle $|z| = 1$.

***EXAMPLE (例題) 6.6.1 A Real Trigonometric Integral**

Evaluate $\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2}\, d\theta$

Solution (解答):

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Solution (解答)(cont.):

Solution (解答)(cont.):

Review for Lecture 13

- Residues (留数)
- Residue Theorem (留数定理)

Assignment

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: Section 6.5, Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia