Lecture 11

Laurent Series（ローラン級数）

What you will learn in Lecture 11

11.1 Laurent Series（ローラン級数）

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Singular Points (特異点)

In general, a point z at which a complex function $w = f(z)$ **fails (失敗する) to be analytic** is called a **singular point (特異点)** of f.

For example, the complex numbers $z = 2i$ and $z = -2i$ are **singularities (特異点)** of the function $f(z) = z/(z^2 + 4)$ because f is **discontinuous (不連続の)** at each of these points.

In this Lecture, we will be concerned with a new kind of "power series" expansion (べき級数展開) of f about an **isolated singularity** z_0. This new series will involve **both negative** and **nonnegative integer powers of** $z - z_0$.

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Suppose that $z = z_0$ is a **singularity** of a complex function f.

Isolated Singularity (孤立特異点)

The point $z = z_0$ is said to be an **isolated singularity** of the function f if there exists **some deleted neighborhood**, or **punctured open disk**, $0 < |z - z_0| < R$ of z_0 throughout which f is analytic.

For example, we have just seen that $z = 2i$ and $z = -2i$ are singularities of $f(z) = z/(z^2 + 4)$. Both $2i$ and $-2i$ are isolated singularities since f is analytic at every point in the neighborhood defined by $|z - 2i| < 1$, except at $z = 2i$, and at every point in the neighborhood defined by $|z - (-2i)| < 1$, except at $z = -2i$.

In other words, f is analytic in the deleted neighborhoods $0 < |z - 2i| < 1$ and $0 < |z + 2i| < 1$.

Nonisolated Singularity

We say that a singular point $z = z_0$ of a function f is **nonisolated** if every neighborhood of z_0 contains at least one singularity of f other than z_0.

For example, the branch point $z = 0$ is a nonisolated singularity of $\mathrm{Ln}\, z$ since every neighborhood of $z = 0$ contains points on the negative real axis. (Check Page 32 of Lecture 4 Notes)

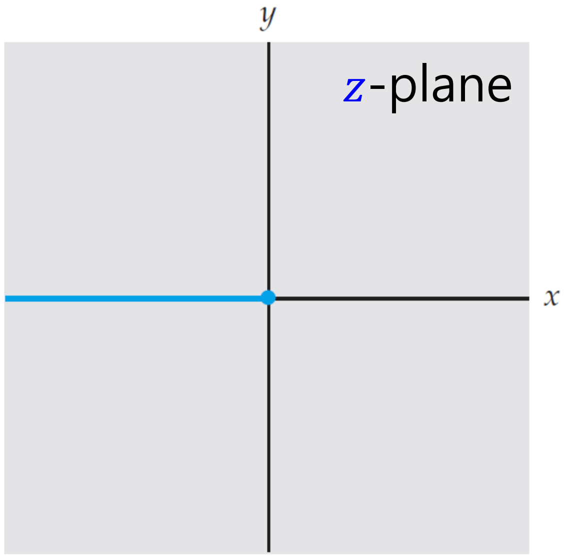

Figure 4.6 $f_1(z)$ defines on domain in gray color

$$f_1(z) = \log_e |z| + i\,\mathrm{Arg}(z),\ -\pi < \mathrm{Arg}(z) < \pi \qquad (4.1.19)$$

Here, $f_1(z)$ is $\mathrm{Ln}\, z$ except $\mathrm{Arg}(z) = \pi$

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A New Kind of Series

If $z = z_0$ is a singularity of a function f, then certainly f **cannot be expanded in a power series with z_0 as its center**.

However, about **an isolated singularity** $z = z_0$, it is possible to **represent** f **by a series involving both negative and nonnegative integer powers of** $z - z_0$; that is,

$$f(z) = \cdots + \underbrace{\frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)}}_{\text{Negative integer powers}} + \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots}_{\text{Nonnegative integer powers}} \qquad (6.3.1)$$

As an example of (6.3.1), let us consider the function $f(z) = \frac{1}{z-1}$. As can be seen, the point $z = 1$ is an **isolated singularity** of f and consequently the function cannot be expanded in a Taylor series centered at that point.

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Nevertheless, f can expanded in a series of the form given in (6.3.1) that is valid for all z near 1:

$$f(z) = \cdots + \frac{0}{(z-1)^2} + \frac{1}{(z-1)} + 0 + 0 \cdot (z-1) + 0 \cdot (z-1)^2 + \cdots \tag{6.3.2}$$

The series representation in (6.3.2) is valid for $0 < |z-1| < \infty$.

Using summation notation, we can write (6.3.1) as the sum of two series

$$f(z) = \boxed{\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}} + \boxed{\sum_{k=0}^{\infty} a_k (z-z_0)^k} \tag{6.3.3}$$

The two series on the right-hand side in (6.3.3) are given special names.

(1) The part with negative powers of $z - z_0$, that is,

$$\boxed{\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}} = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k} \tag{6.3.4}$$

is called <u>**the principal part of the series** (6.3.1)</u> and **will converge for** $|1/(z-z_0)| < r^*$ or equivalently for $|z-z_0| > 1/r^* = r$.

(2) The part consisting of the **nonnegative powers** of $z - z_0$,

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \tag{6.3.5}$$

is called **the analytic part of the series** (6.3.1) and will converge for $|z - z_0| < R$.

Hence, **the sum of (6.3.4) and (6.3.5)** converges **when** z satisfies both $|z - z_0| > r$ and $|z - z_0| < R$, that is, **when** z **is a point in an annular domain defined by** $r < |z - z_0| < R$.

By summing over negative and nonnegative integers, (6.3.1) can be written compactly as

$$\boxed{f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k}$$

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Laurent series

A series representation of a function f that has the form given in (6.3.1) is called a **Laurent series** or **a Laurent expansion of** f about z_0 **on the annulus** $r < |z - z_0| < R$.

EXAMPLE (例題) 6.3.1 Laurent Series of the Form (6.3.1)

Expand the function $f(z)=\dfrac{\sin z}{z^4}$ with the isolated singularity $z=0$ as Laurent Series.

Solution (解答):

The function $f(z)=\dfrac{\sin z}{z^4}$ is not analytic at **the isolated singularity** $z=0$ and hence cannot be expanded in a Maclaurin series.

However, $\sin z$ is **an entire function**, and from (6.2.13) of Lecture 10, we know that its Maclaurin series,

$$\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\frac{z^9}{9!}-\cdots$$

converges for $|z|<\infty$.

Solution (解答)(cont.):

By dividing this power series by z^4 we obtain a series for f with negative and positive integer powers of z:

$$f(z) = \frac{\sin z}{z^4} = \underbrace{\frac{1}{z^3} - \frac{1}{3!\,z}}_{\text{principal part}} + \underbrace{\frac{z}{5!} - \frac{z^3}{7!} + \frac{z^5}{9!}}_{\text{analytic part}} - \cdots \qquad (6.3.6)$$

The analytic part of the series in (6.3.6) converges for $|z| < \infty$. The principal part is valid to converge for $|z| > 0$. Thus (6.3.6) converges for all z except at $z = 0$; that is, the series representation is valid for $0 < |z| < \infty$.

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Theorem 6.10 Laurent's Theorem

Let f be analytic within the annular domain D defined by $r < |z - z_0| < R$. Then f has the series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \qquad (6.3.7)$$

valid for $r < |z - z_0| < R$. The coefficients a_k are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds \qquad k = 0, \pm 1, \pm 2, \ldots \qquad (6.3.8)$$

where C is a simple closed curve that lies entirely within D and has z_0 in its interior. See Figure 6.6.

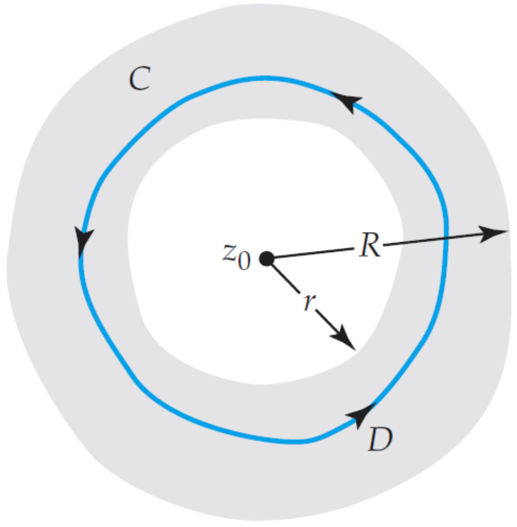

Figure 6.6 Contour for Theorem 6.10

Notice:
In the case when $a_{-k} = 0$ for $k = 1, 2, 3, \ldots$, the principal part (6.3.4) is zero and the Laurent series (6.3.7) reduces to a **Taylor series**.
Therefore, **a Laurent expansion** can be regarded as **a generalization of a Taylor series**.

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The annular (環状) domain in Theorem 6.10 defined by $r < |z - z_0| < R$ **do not must have** the "ring (円環)" shape. Here are some other possible annular domains:

(i) $r = 0$, R is finite,
(ii) $r \neq 0$, $R = \infty$,
(iii) $r = 0$, $R = \infty$.

- In (i), the series converges in annular domain defined by an $0 < |z - z_0| < R$. **This is the interior of the circle** $|z - z_0| = R$ except the point z_0; in other words, the domain is a punctured open disk.
- In (ii), the annular domain is defined by $r < |z - z_0|$ and **consists of all points exterior to the circle** $|z - z_0| = r$.
- In (iii), the domain is defined by $0 < |z - z_0|$. **This represents the entire complex plane except the point** z_0. The Laurent series in (6.3.2) and (6.3.6) are valid on this last type of domain.

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The integral formula in (6.3.8) for the coefficients of a Laurent series are rarely used in practice. As a consequence, finding the Laurent series of a function in a specified annular domain is generally not an easy task.

In many instances we can obtain a desired Laurent series either by employing a known power series expansion of a function (as we did in Example 6.3.1) or by creative manipulation of geometric series (as we did in Example 6.2.2 of Lecture 10).

The next example once again illustrates the use of geometric series.

EXAMPLE (例題) 6.3.2 **Four Laurent Expansions**

Expand $f(z)=\frac{1}{z(z-1)}$ in a Laurent series valid for the following annular domains.

(a) $0<|z|<1$ (b) $1<|z|$ (c) $0<|z-1|<1$ (d) $1<|z-1|$

Solution (解答):

The four specified annular domains are shown in Figure 6.8.

The black dots in each figure represent the two isolated singularities, $z=0$ and $z=1$, of f.

In (a) and (b), the center is $z=0$, we want to represent f in a series involving only negative and nonnegative integer powers of z,

whereas in (c) and (d), the center is $z=1$, we want to represent f in a series involving negative and nonnegative integer powers of $z-1$.

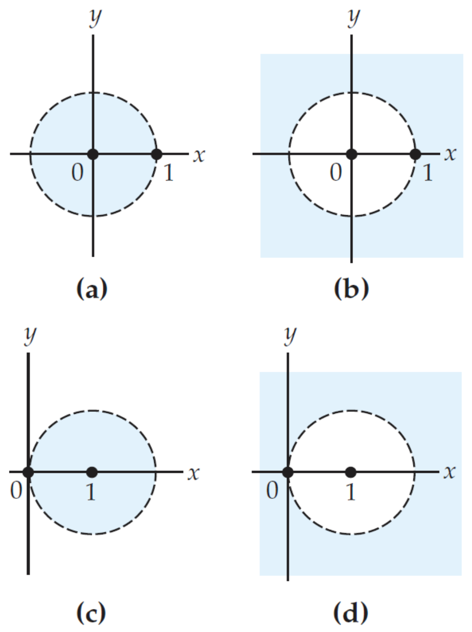

Figure 6.8 Annular domains for Example 6.3.2

Solution (解答)(cont.):

(a) By writing

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z}\frac{1}{1-z}$$

we can use geometric series (6.1.6) of Lecture 9 to write $\frac{1}{1-z}$ as a series:

$$f(z) = -\frac{1}{z}[1 + z + z^2 + z^3 + \cdots]$$

The infinite series in the brackets converges for $|z| < 1$, but after we multiply this expression by $1/z$, we see $z \neq 0$, thus the resulting series

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots$$

converges for $0 < |z| < 1$.

Solution (解答)(cont.):

(b) To obtain a series that converges for $1 < |z|$, we start by constructing a series that converges for $|1/z| < 1$. To this end we write the given function f as

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z^2 - z} = \frac{1}{z^2}\frac{1}{1-\frac{1}{z}}$$

and again use the geometric series (6.1.6) of Lecture 9 with z replaced by $1/z$:

$$f(z) = \frac{1}{z^2}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$

The series in the brackets converges for $|1/z| < 1$ or equivalently for $1 < |z|$.

Thus the required Laurent series is

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \cdots$$

Solution (解答)(cont.):

(c) This is basically the same problem as in part (a), except that we want all powers of $z-1$. To that end, we add and subtract 1 in the denominator and use (6.1.7) of Lecture 9 with z replace by $z-1$:

$$f(z)=\frac{1}{z(z-1)}=\frac{1}{(1-1+z)(z-1)}=\frac{1}{1-1+z}\frac{1}{z-1}$$

$$=\frac{1}{z-1}\frac{1}{1+(z-1)}$$

$$=\frac{1}{z-1}[1-(z-1)+(z-1)^2-(z-1)^3+\cdots]$$

$$=\frac{1}{z-1}-1+(z-1)-(z-1)^2+\cdots$$

The requirement that $z\neq 1$ is equivalent to $z<1$ and $z>1$, i.e. $0<|z-1|$, and the geometric series in brackets converges for $|z-1|<1$. Thus the last series converges for z satisfying $0<|z-1|$ and $|z-1|<1$, that is, for $0<|z-1|<1$.

Solution (解答)(cont.):

(d) Proceeding as in part (b), we write

$$f(z)=\frac{1}{z(z-1)}=\frac{1}{z-1}\frac{1}{1+(z-1)}=\frac{1}{(z-1)^2}\frac{1}{\frac{1}{z-1}+\frac{z-1}{z-1}}$$

$$=\frac{1}{(z-1)^2}\frac{1}{1+\frac{1}{z-1}}$$

$$=\frac{1}{(z-1)^2}\left[1-\frac{1}{z-1}+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\cdots\right]$$

$$=\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\frac{1}{(z-1)^4}-\frac{1}{(z-1)^5}+\cdots$$

Because the geometric series within the brackets converges for $|1/(z-1)|<1$, the final series converges for $1<|z-1|$.

EXAMPLE (例題) 6.3.3 Laurent Expansions

Expand $f(z) = \frac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for the following annular domains.

(a) $0 < |z-1| < 2$ (b) $0 < |z-3| < 2$

Solution (解答):

(a) As in parts (c) and (d) of Example 6.3.2, we note that the center is $z = 1$, then we want only powers of $z-1$ and so we need to express $z-3$ in terms of $z-1$.This can be done by writing

$$f(z) = \frac{1}{(z-1)^2}\frac{1}{(z-3)} = \frac{1}{(z-1)^2}\frac{1}{-2+(z-1)} = \frac{-1}{2(z-1)^2}\frac{1}{1-\frac{z-1}{2}}$$

and then using the geometric series (6.1.6) of Lecture 9 with the symbol z replaced by $(z-1)/2$,

$$f(z) = \frac{-1}{2(z-1)^2}\left[1 + \frac{z-1}{2} + \frac{(z-1)^2}{2^2} + \frac{(z-1)^3}{2^3} + \cdots\right]$$

$$= -\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)} - \frac{1}{8} - \frac{1}{16}(z-1) - \cdots$$

Solution (解答)(cont.):

(b) To obtain powers of $z-3$, we write $z-1=2+(z-3)$ and

$$f(z)=\frac{1}{(z-1)^2}\frac{1}{(z-3)}=\frac{1}{z-3}\frac{1}{(2+z-3)^2}=\frac{1}{z-3}\frac{1}{2^2\cdot\left(1+\frac{z-3}{2}\right)^2}=\frac{1}{4(z-3)}\left[1+\frac{z-3}{2}\right]^{-2}$$

At this point we can obtain a power series for $\left[1+\frac{z-3}{2}\right]^{-2}$ by using the binomial series

$$f(z)=\frac{1}{4(z-3)}\left[1+\frac{(-2)}{1!}\left(\frac{z-3}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{z-3}{2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{z-3}{2}\right)^3+\cdots\right]$$

The binomial series in the brackets is valid for $|(z-3)/2|<1$ or $|z-3|<2$.

Multiplying this series by $\frac{1}{4(z-3)}$ gives a Laurent series that is valid for $0<|z-3|<2$:

$$f(z)=\frac{1}{4(z-3)}-\frac{1}{4}+\frac{3}{16}(z-3)-\frac{1}{8}(z-3)^2+\cdots$$

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binomial series (https://en.wikipedia.org/wiki/Binomial_series)：

For a real number α, the binomial series $(1+z)^\alpha=1+\alpha z+\frac{\alpha(\alpha-1)}{2!}z^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3+\cdots$ is valid for $|z|<1$.

EXAMPLE (例題) 6.3.4 A Laurent Expansion

Expand $f(z)=\frac{8z+1}{z(1-z)}$ in a Laurent series valid for $0<|z|<1$.

Solution (解答):

By partial fractions (部分分数) we can rewrite f as

$$f(z)=\frac{8z+1}{z(1-z)}=\frac{1-z+9z}{z(1-z)}=\frac{1}{z}+\frac{9}{1-z}$$

Then by the geometric series (6.1.6) of Lecture 9,

$$\frac{9}{1-z}=9+9z+9z^2+\cdots$$

The foregoing geometric series converges for $|z|<1$, but after we add the term $1/z$ to it, the resulting Laurent series

$$f(z)=\frac{1}{z}+9+9z+9z^2+\cdots$$

is valid for $0<|z|<1$.

EXAMPLE (例題) 6.3.5 A Laurent Expansion

Expand $f(z) = -\frac{1}{z(1-z)}$ in a Laurent series valid for $1 < |z-2| < 2$.

Solution (解答):

- The specified annular domain is shown in Figure 6.9.
- The center of this domain, $z = 2$, is the point of analyticity of the function f.
- Our goal now is to find two series involving integer powers of z– 2, one converging for $1 < |z-2|$ and the other converging for $|z-2| < 2$.

As in the last example by decomposing f into partial fractions:

$$f(z) = -\frac{1}{z(1-z)} = \frac{z-(z-1)}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1} = f_1(z) + f_2(z) \qquad (6.3.17)$$

Now $f_1(z) = -\frac{1}{z} = -\frac{1}{2+z-2} = -\frac{1}{2}\frac{1}{1+\frac{z-2}{2}}$

$$= -\frac{1}{2}\left[1 - \frac{z-2}{2} + \frac{(z-1)^2}{2^2} - \frac{(z-2)^3}{2^3} + \cdots\right]$$

$$= -\frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots$$

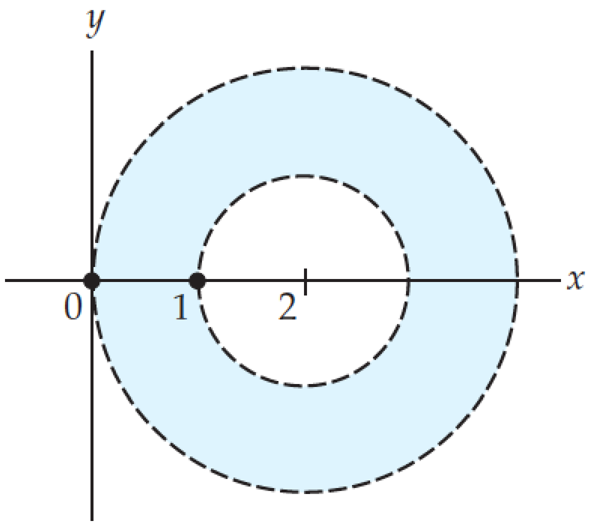

Figure 6.9 Annular domain for Example 6.1.5

Solution (解答)(cont.):

This series converges for $|(z-2)/2| < 1$ or $|z-2| < 2$.

Furthermore,

$$f_2(z) = \frac{1}{z-1} = \frac{1}{1+z-2} = \frac{1}{z-2}\frac{1}{1+\frac{1}{z-2}}$$

$$= \frac{1}{z-2}\left[1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \cdots\right]$$

$$= \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} + \cdots$$

converges for $|1/(z-2)| < 1$ or $1 < |z-2|$.

Substituting these two results in (6.3.17) then gives

$$f(z) = \cdots - \frac{1}{(z-2)^4} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{1}{z-2} - \frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-1)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots$$

This representation is valid for z satisfying $|z-2| < 2$ and $1 < |z-2|$; in other words, for $1 < |z-2| < 2$.

EXAMPLE (例題) 6.3.6 A Laurent Expansion

Expand $f(z) = e^{\frac{3}{z}}$ in a Laurent series valid for $0 < |z| < \infty$.

Solution (解答):

From (6.2.12) of Lecture 10 we know that for all finite z, that is, $|z| < \infty$,

$$e^{z} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

We obtain the Laurent series for f by simply replacing z in (6.3.18) by $\frac{3}{z}$, $z \neq 0$,

$$e^{\frac{3}{z}} = 1 + \frac{3}{z} + \frac{3^2}{2!\,z^2} + \frac{3^3}{3!\,z^3} + \cdots \tag{6.3.19}$$

This series (6.3.19) is valid for $z \neq 0$, that is, for $0 < |z| < \infty$.

Remarks

(i) Replacing the complex variable s with the usual symbol z , we see that when $k - 1$, formula (6.3.8) for the Laurent series coefficients yields $\mathrm{a}_{-1} = \frac{1}{2\pi i}\oint_C \ f(z)dz$, or more important,

$$\oint_C \ f(z)dz = 2\pi i a_{-1}$$

(ii) Regardless how a Laurent expansion of a function f is obtained in a specified annular domain it is the Laurent series; that is, the series we obtain is unique.

Review for Lecture 11

- Laurent Series

Assignment

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: Section 6.3, Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia