Lecture 7

Cauchy-Goursat Theorem

(i.e. Cauchy's integral theorem コーシーの積分定理)

What you will learn in Lecture 7

7.1 Cauchy-Goursat Theorem

7.1.1 Simply and Multiply Connected Domains

7.1.2 Cauchy-Goursat Theorem

7.1.3 Cauchy-Goursat Theorem for Multiply Connected Domains

*7.2 Independence of Path for Contour Integral

7.1 Cauchy-Goursat Theorem

(i.e. Cauchy's integral theorem コーシーの積分定理)

In this 7.1, we shall concentrate on contour integrals, where the contour C is a **simple closed curve** with a **positive** **(counterclockwise) orientation**.

7.1.1 Simply Connected (単連結) Domains

and

Multiply Connected (多重連結) Domains

7.1.1 Simply and Multiply Connected Domains

Simply Connected (単連結) Domains

We say that **a domain D is simply connected** if **every simple closed contour C lying entirely in D can be shrunk to a point** (ポイントに縮小する) **without leaving D**. (See Figure 5.26.)

In other words, a simply connected domain **has <u>no</u> "holes" in** it.

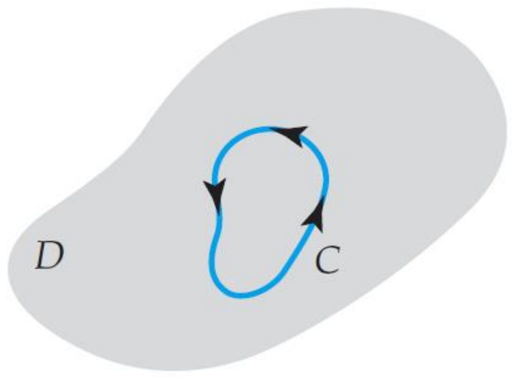

Figure 5.26 Simply connected domain D

Multiply Connected (多重連結) Domains

A domain that is not simply connected is called **a multiply connected domain**. (See Figure 5.27.)

In other words, a multiply connected domain **has "holes" in it.**

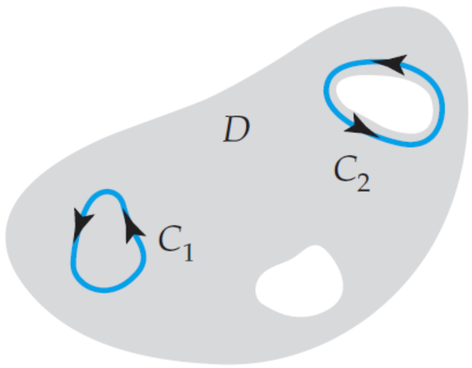

Figure 5.27 Multiply connected domain D

For example, (1)the open disk (開円板) defined by $|z| < 2$ is a simply connected domain; (2)the open circular annulus (開円環) defined by $1 < |z| < 2$ is a doubly (i.e. multiply) connected domain.

7.1.2 Cauchy-Goursat Theorem

(i.e. Cauchy's integral theorem コーシーの積分定理)

7.1.2 Cauchy-Goursat Theorem (コーシーの積分定理)

Theorem 5.4　Cauchy-Goursat Theorem (i.e. Cauchy's integral theorem コーシーの積分定理)

Suppose that a function f is **analytic (解析的)** in a **simply connected (単連結) domain** D. Then for every simple closed contour C in D, we have

$$\oint_C f(z)dz = 0$$

Because the interior (内部) of a simple closed contour is a **simply connected domain**, the **Theorem 5.4** can be **rewritten** in the slightly more practical manner:

If f is **analytic at all points within and on** a **simple closed contour** C, then

$$\oint_C f(z)dz = 0 \tag{5.3.4}$$

EXAMPLE (例題) 5.3.1 Applying the Cauchy-Goursat Theorem

Evaluate $\oint_C e^z dz$, where the contour C is shown in Figure 5.28.

Solution (解答):

Hint:

- Theorem 5.4

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check https://github.com/uoaworks/ComplexAnalysisAY2018

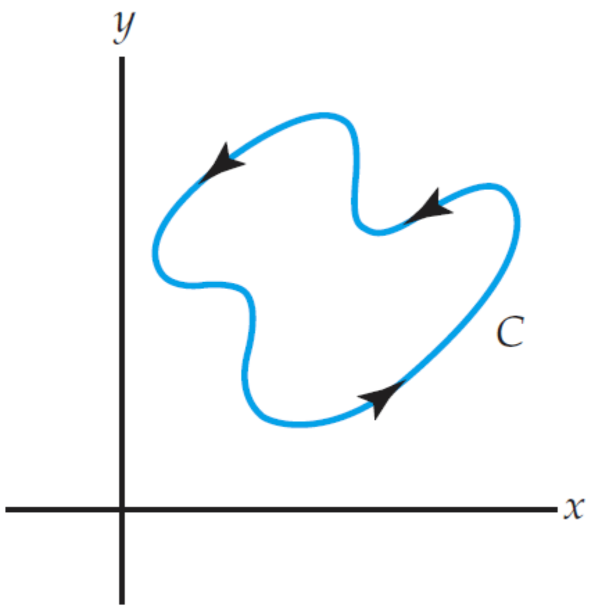

Figure 5.28 Contour for Example 5.3.1

7.1.2 Cauchy-Goursat Theorem (コーシーの積分定理)

Indeed, from Example 5.3.1, it follows that **for any simple closed contour C and any entire** function (整函数) f, such as

$$f(z) = \sin z,$$

$$f(z) = \cos z,$$

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \ \ n = 0, 1, 2, \ldots$$

we have

$$\oint_C \sin z \, dz = 0,$$

$$\oint_C \cos z \, dz = 0,$$

$$\oint_C p(z) dz = 0$$

and so on.

EXAMPLE (例題) 5.3.2 Applying the Cauchy-Goursat Theorem

Evaluate $\oint_C \frac{1}{z^2}dz$, where the contour C is the ellipse (楕円) $(x-2)^2+(y-5)^2=1$.

Solution (解答):

Hint:

- Theorem 5.4

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7.1.3 Cauchy-Goursat Theorem for Multiply Connected Domains

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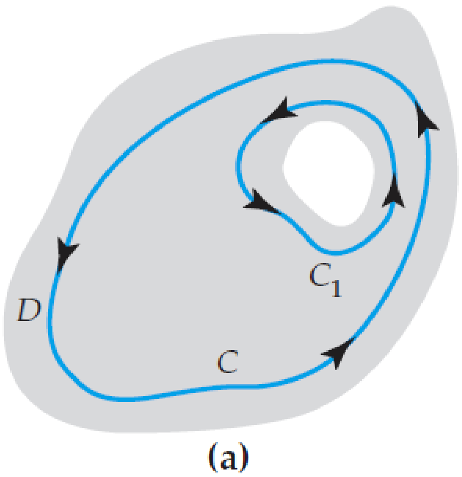

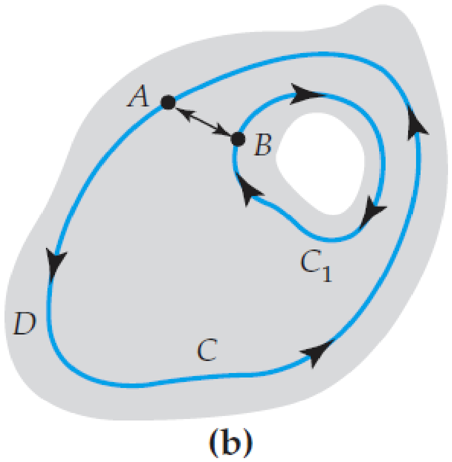

Figure 5.29 Doubly connected domain D

$$\oint_C f(z)dz + \cancel{\oint_{AB} f(z)dz} + \oint_{-C_1} f(z)dz + \cancel{\oint_{-AB} f(z)dz} = 0$$

$$\Longrightarrow \quad \oint_C f(z)dz = \oint_{C_1} f(z)dz \qquad (5.3.5)$$

The above result is sometimes called the **principle of deformation (変形) of contours** because we can think of **the contour C_1 as a continuous deformation (連続変形) of the contour C**.

In other words, (5.3.5) allows us to **evaluate an integral (積分) over a complicated (複雑な) simple closed contour C by replacing C with a contour C_1 that is more convenient (便利な)**.

7.1.3 Cauchy-Goursat Theorem for Multiply Connected Domains

EXAMPLE (例題) 5.3.3 Applying Deformation of Contours

Evaluate $\oint_C \frac{1}{z-i}dz$, where the contour C is shown in black color in Figure 5.30. (Notice that there is a point "hole" at $(1,1)$.)

Solution (解答):

Hint:

- Equation (5.3.5)
- Equation (2.2.10)
- Equation (5.2.4)

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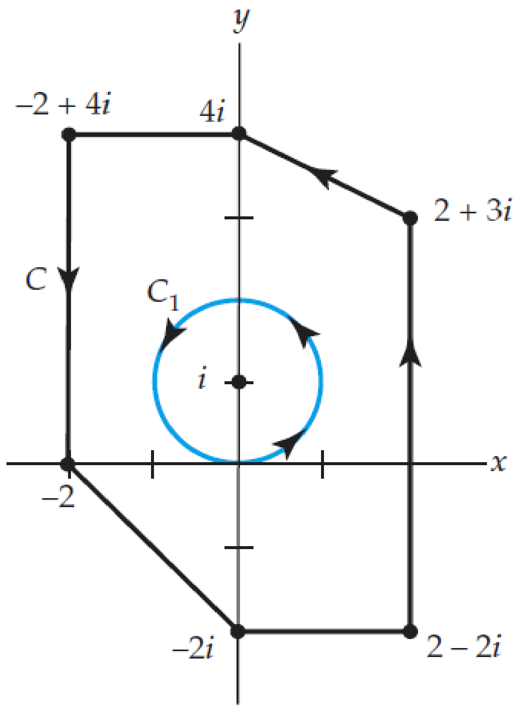

Figure 5.30 We use the simpler contour C_1 in Example 5.3.3.

7.1.3 Cauchy-Goursat Theorem for Multiply Connected Domains

Additional Point: **Common Parametric Curves in the Complex Plane**

Line

A parametrization of the line containing the points z_0 and z_1 is:

$$z(t) = z_0(1-t) + z_1 t, \qquad -\infty \leq t \leq \infty. \qquad (2.2.7)$$

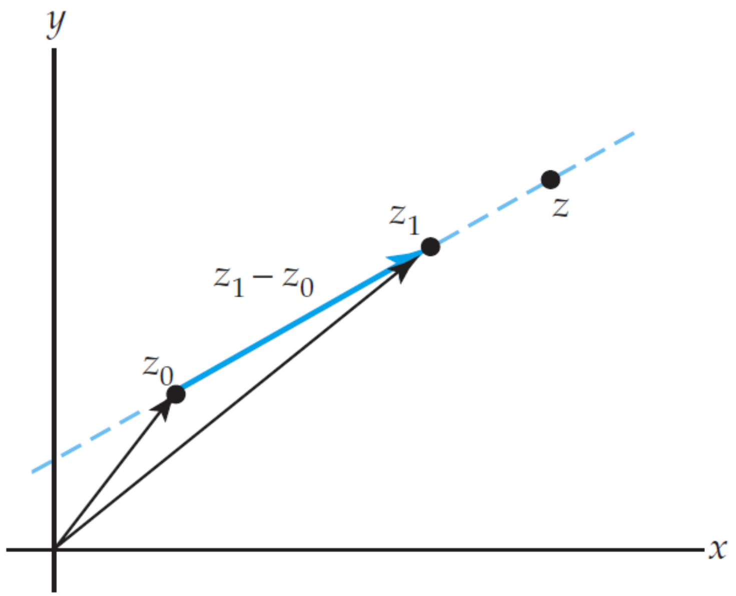

Figure 2.4 Parametrization of a line

Circle

A parametrization of the circle centered at z_0 with radius r is:

$$z(t) = z_0 + r(\cos t + i \sin t), \qquad 0 \leq t \leq 2\pi. \qquad (2.2.9)$$

In exponential notation, this parametrization is:

$$z(t) = z_0 + re^{it}, \qquad 0 \leq t \leq 2\pi. \qquad (2.2.10)$$

7.1.3 Cauchy-Goursat Theorem for Multiply Connected Domains

The result obtained in Example 5.3.3 can be generalized.

By using the principle of deformation of contours (5.3.5), it can be shown that if z_0 is **any constant complex number interior to any simple closed contour** C, then **for an integer** n we have

$$\oint_C \frac{1}{z^2} dz = \begin{cases} 2\pi i, & n = 1 \\ 0, & n \neq 1 \end{cases} \qquad (5.3.6)$$

EXAMPLE (例題) 5.3.4 Applying Formula (5.3.6)

Evaluate $\oint_C \frac{5z+7}{z^2+2z-3} dz$, where the contour C is the circle $|z-2|=2$.

Solution (解答):

Hint:

- Theorem 5.4
- Equation (5.3.6)

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Theorem 5.5 Cauchy-Goursat Theorem for Multiply Connected Domains

Suppose $C,\ C_1, \ldots, C_n$ are **simple closed curves with a positive orientation** such that $C_1, C_2, \ldots, C_n$ are interior to C but the regions interior to each C_k, $k = 1, 2, \ldots, n$, have no points in common. If f **is analytic on each contour and at each point interior to** C but exterior to all the C_k, $k = 1, 2, \ldots, n$, then

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz \qquad (5.3.8)$$

7.1.3 Cauchy-Goursat Theorem for Multiply Connected Domains

EXAMPLE (例題) 5.3.5 Applying Theorem 5.5

Evaluate $\oint_C \frac{1}{z^2+1} dz$, where the contour C is the circle $|z| = 3$.

Solution (解答):

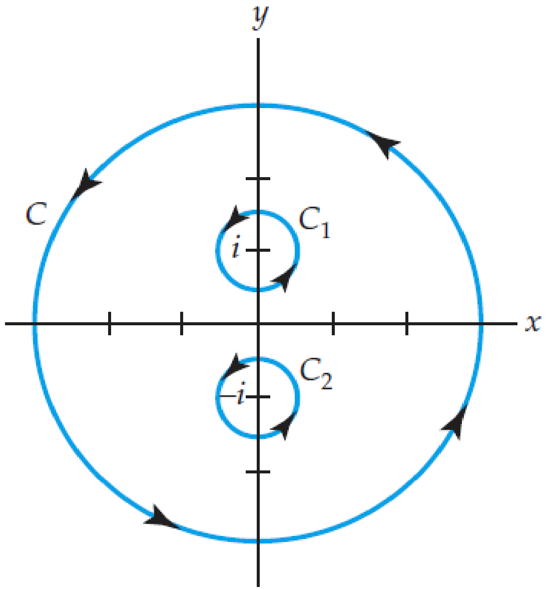

Figure 5.32 Contour for Example 5.3.5

Hint:

- Theorem 5.5
- Partial fraction decomposition (部分分数分解)

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check https://github.com/uoaworks/ComplexAnalysisAY2018

Solution (解答)(cont.):

*7.2 Independence (独立) of Path (経路) for Contour Integral

Notice: In all lecture notes, the contents marked with * are not in the scope of the final examination.

*7.2 Independence (独立) of Path (経路) for Contour Integral

There exist **Real line integrals (実·線積分)** $\int_C Pdx + Qdy$ **whose value depends only on the initial point** (始点) A **and terminal point** (終点) B **of the curve** C, and **not on** C **itself.**

In this case we say that **the line integral is independent of the path**.

For example, $\int_C ydx + xdy$ **is independent of the path**.

(1) *Can a contour integral* $\int_C f(z)dz$ ***be independent of the path?***

(2) *Is there **a complex version** of **the fundamental theorem of calculus?***

we will see that the answer to both of these questions is **YES**.

*7.2 Independence (独立) of Path (経路) for Contour Integral

Definition 5.4 Independence of the Path for Contour Integral

Let z_0 and z_1 be points in a domain D. A contour integral $\int_C f(z)dz$ **is said to be independent of the path if its value is the same for all contours** C **in** D **with initial point** z_0 **and terminal point** z_1.

*7.2 Independence (独立) of Path (経路) for Contour Integral

Now suppose, as shown in Figure 5.38, that C and C_1 are two contours lying entirely in a simply connected domain D and both with initial point z_0 and terminal point z_1.

Thus, if f is analytic in D, it follows from the Cauchy-Goursat theorem that

$$\int_C f(z)dz + \int_{-C_1} f(z)dz = 0$$

$$\Rightarrow \quad \int_C f(z)dz = \int_{C_1} f(z)dz$$

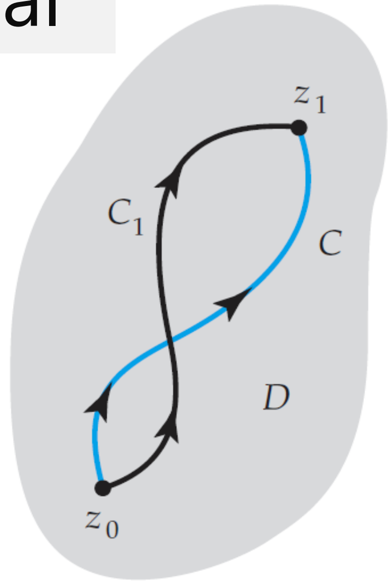

Figure 5.38 If f is analytic in D, integrals on C and C_1 are equal.

Theorem 5.6 Analyticity Implies Path Independence

Suppose that a function f is **analytic** in a **simply connected domain** D and C **is any contour in** D. Then $\int_C f(z)dz$ is **independent of the path** C.

EXAMPLE (例題) 5.4.1 Choosing a Different Path

Evaluate $\int_C 2z dz$, where the contour C is shown in blue color in Figure 5.39.

Solution (解答):

Hint:

- Theorem 5.6

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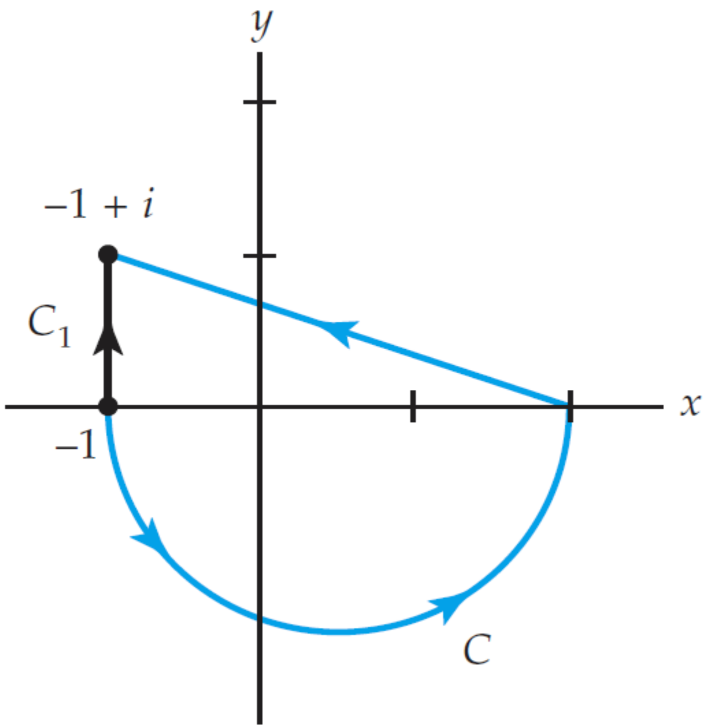

Figure 5.39 Contour for Example 5.4.1

*7.2 Independence (独立) of Path (経路) for Contour Integral

Definition 5.5 Antiderivative

Suppose that a function f is continuous on a domain D. If there exists a function F such that $F'(z) = f(z)$ for each z in D, then F is called an antiderivative of f.

For example, the function $F(z) = -\cos z$ is an antiderivative of $f(z) = \sin z$ because $F'(z) = \sin z$.

*7.2 Independence (独立) of Path (経路) for Contour Integral

Theorem 5.7 Fundamental Theorem for Contour Integrals

Suppose that a function f is continuous on a domain D and F is an antiderivative of f in D. Then for any contour C in D with initial point (始点) z_0 and terminal point (終点) z_1,

$$\int_C f(z)dz = F(z_1) - F(z_0) \tag{5.4.4}$$

EXAMPLE (例題) 5.4.2 Applying Theorem 5.7

Evaluate $\int_C 2z dz$, where the contour C is shown in color in Figure 5.39.

Solution (解答):

Hint:

- Theorem 5.7

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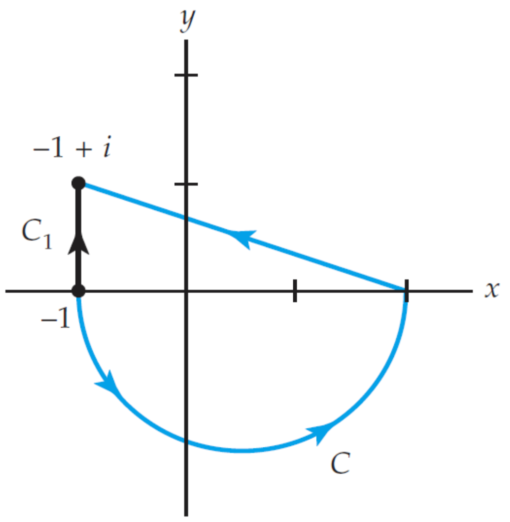

Figure 5.39 Contour for Example 5.4.1

EXAMPLE (例題) 5.4.3 Applying Theorem 5.7

Evaluate $\int_C \cos z \, dz$**, where** C is any contour with initial point $z_0 = 0$ and terminal point $z_1 = 2 + i$.

Solution (解答):

Hint:

- Theorem 5.7

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Review for Lecture 7

- Simply and Multiply Connected Domains
- Cauchy-Goursat Theorem
- Cauchy-Goursat Theorem for Multiply Connected Domains
- *Independence of Path for Contour Integral
- *Fundamental Theorem for Contour Integrals

Assignment

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: Section 5.3, 5.4, Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia