Lecture 2

Complex Functions (複素関数)

What you will learn in Lecture 2

2.1 Complex Functions (複素関数)

2.2 Complex Function as Mappings (写像、変換)

2.3 Limits (極限) and Continuity (連続性)

2.1 Complex Functions

(複素関数)

2.1 Complex Functions (複素関数)

$$y = f(x)$$

Output $\mathbf{R}$ | Name of Function | Input $\mathbf{R}$

real-valued functions (実数値関数) of a real variable (実変数)

$$w = f(z)$$

Output $\mathbf{C}$ | Name of Function | Input $\mathbf{C}$

complex-valued function of a complex variable (複素変数)

Definition (定義) 2.1 Complex Function (複素関数)

A **complex function (複素関数)** is a function f whose **domain** (定義域) and **range** (値域) are subsets of the set $\mathbf{C}$ of complex numbers.

We denote **the domain and range of a function** f by $\mathrm{Dom}(f)$ and $\mathrm{Range}(f)$, respectively.

EXAMPLE (例題) 2.1.1 Complex Function (複素関数)

(a) Evaluate $f(z) = z^2 - (2+i)z$ when (1) $z = i$ and (2) $z = 1 + i$

(b) Evaluate $g(z) = z + 2\mathrm{Re}(z)$ when (1) $z = i$ and (2) $z = 2 - 3i$

Solution (解答):

(a) (1) $f(i) = i^2 - (2+i)(i) = -1 - 2i + 1 = -2i\,.$

(2) $f(1+i) = (1+i)^2 - (2+i)(1+i)$

$$= (1 + 2i - 1) - (2 + 2i + i - 1) = -1 - i\,.$$

(b) (1) $g(i) = i + 2\mathrm{Re}(i) = i + 2\cdot(0) = i$

(2) $g(2-3i) = 2 - 3i + 2\mathrm{Re}(2-3i) = 2 - 3i + 2\cdot(2) = 6 - 3i$

Notice: When the domain (定義域) of a complex function (複素関数) is not explicitly stated, we assume the domain (定義域) to be the set of all complex numbers z for which $f(z)$ is defined.

2.1 Complex Functions (複素関数)

Real and Imaginary Parts of a Complex Function

If $w = f(z)$ is a complex function (複素関数), then the image (値域) of a complex number $z = x + iy$ under f is a complex number $w = u + iv$.

For example, suppose we have the complex function $w = f(z) = z^2$, then

$$w = z^2 = (x + iy)^2 = (x^2 - y^2) + 2xyi \qquad (2.1.1)$$
$$= u + iv$$

It shows that, if $w = u + iv = f(z) = f(x + iy)$ is a complex function, then both u and v are real functions of the two real variables x and y, i.e.

$$w = f(z) = u(x, y) + iv(x, y) \qquad (2.1.2)$$

The functions (i.e. 実2変数関数) $u(x, y)$ and $v(x, y)$ in (2.1.2) are called the real and imaginary parts of f, respectively.

EXAMPLE (例題) 2.1.2

If $z = x + iy$, find the real and imaginary parts (実部と虚部) of the functions (a) $f(z) = z^2 - (2+i)z$ (b) $g(z) = z + 2\mathrm{Re}(z)$

Solution (解答):

(a) $f(z) = z^2 - (2+i)z = (x+iy)^2 - (2+i)(x+iy)$

$$= x^2 + 2xyi - y^2 - (2x + 2yi + ix - y)$$

$$= x^2 - 2x + y - y^2 + (2xy - x - 2y)i$$

Therefore $u(x,y) = x^2 - 2x + y - y^2 \quad v(x,y) = 2xy - x - 2y$

(b) $g(z) = z + 2\mathrm{Re}(z) = x + iy + 2\mathrm{Re}(x+iy) = x + iy + 2x = 3x + iy$

Therefore $u(x,y) = 3x \qquad v(x,y) = y$

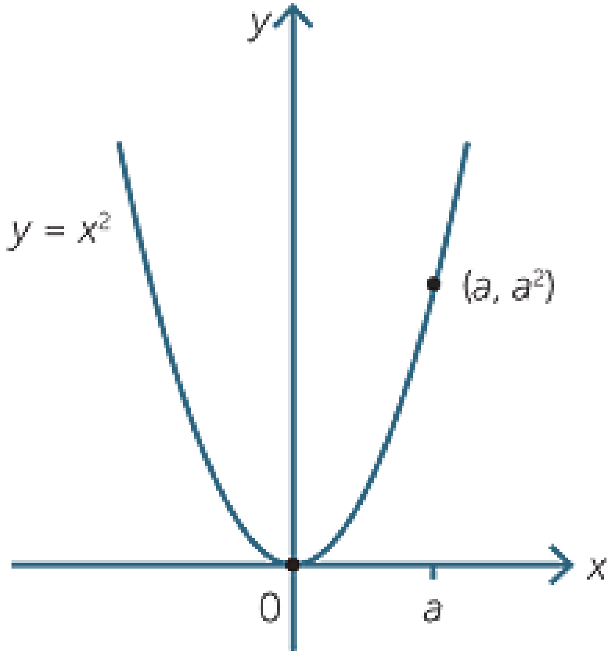
2.2 Complex Function as Mappings

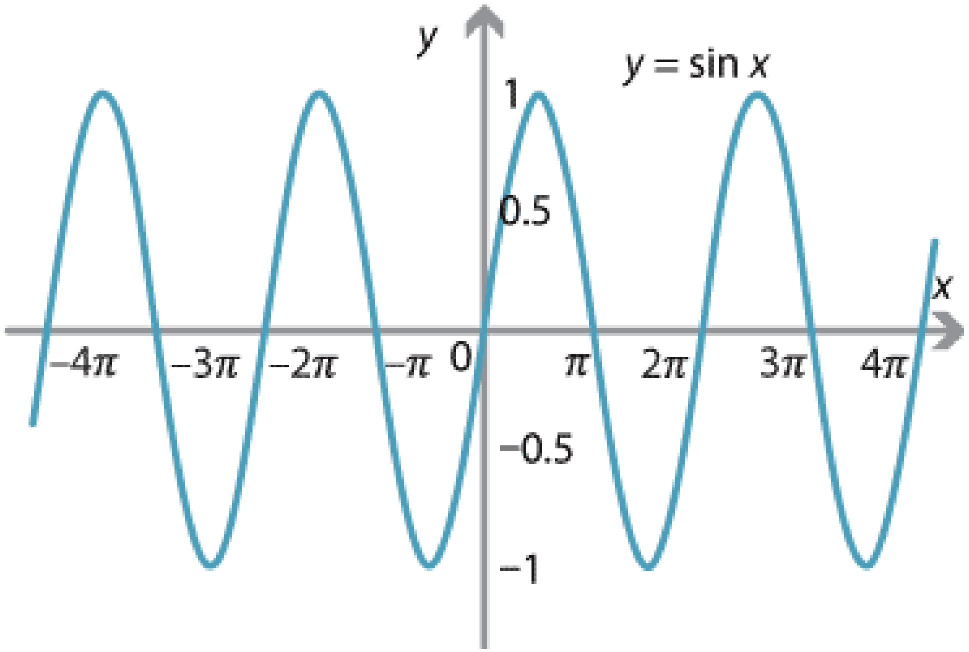
(写像、変換)

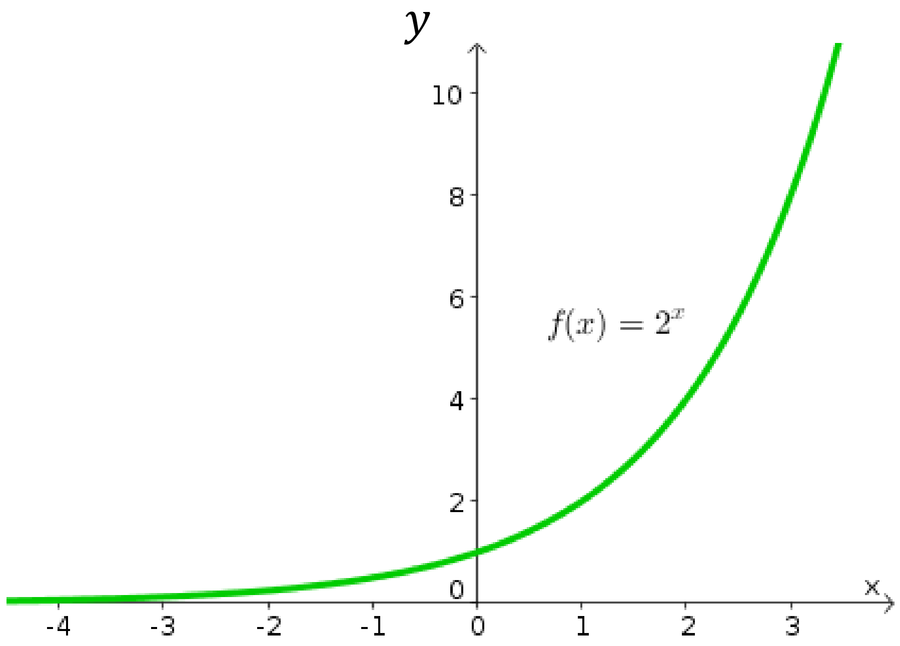
2.2 Complex Function as Mappings (写像、変換)

We can plot the graph (グラフ) of real-valued function !

Recall that in *Calculus I*, **if** $y = f(x)$ is a **real-valued function (実数値関数) of a real variable** x, then **the graph (グラフ) of** f is defined to be the set of all points $(x, f(x))$ (i.e. (x, y)) in the two-dimensional Cartesian plane (i.e. 2次元空間) (デカルト座標系、直交座標系).







2.2 Complex Function as Mappings (写像、変換)

Can we plot graph of complex function?

However, if $w = f(z)$ is a complex function, then both z and w lie in a complex plane (複素平面).

It follows that the set (集合) of all points $(z, f(z))$ (i.e. (z, w)) lies in four-dimensional space (4次元空間) (two dimensions from the input z and two dimensions from the output w).

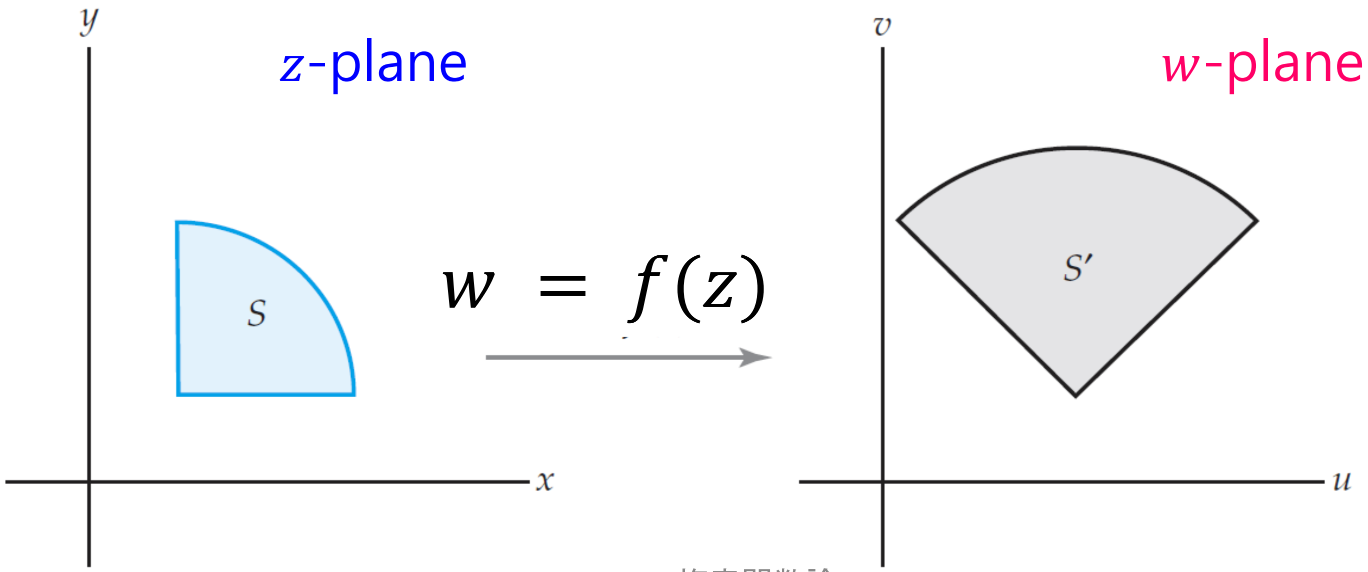
Therefore,

We cannot draw the graph of a complex function.

Instead (代わりに), we use the idea of mapping (写像、変換).

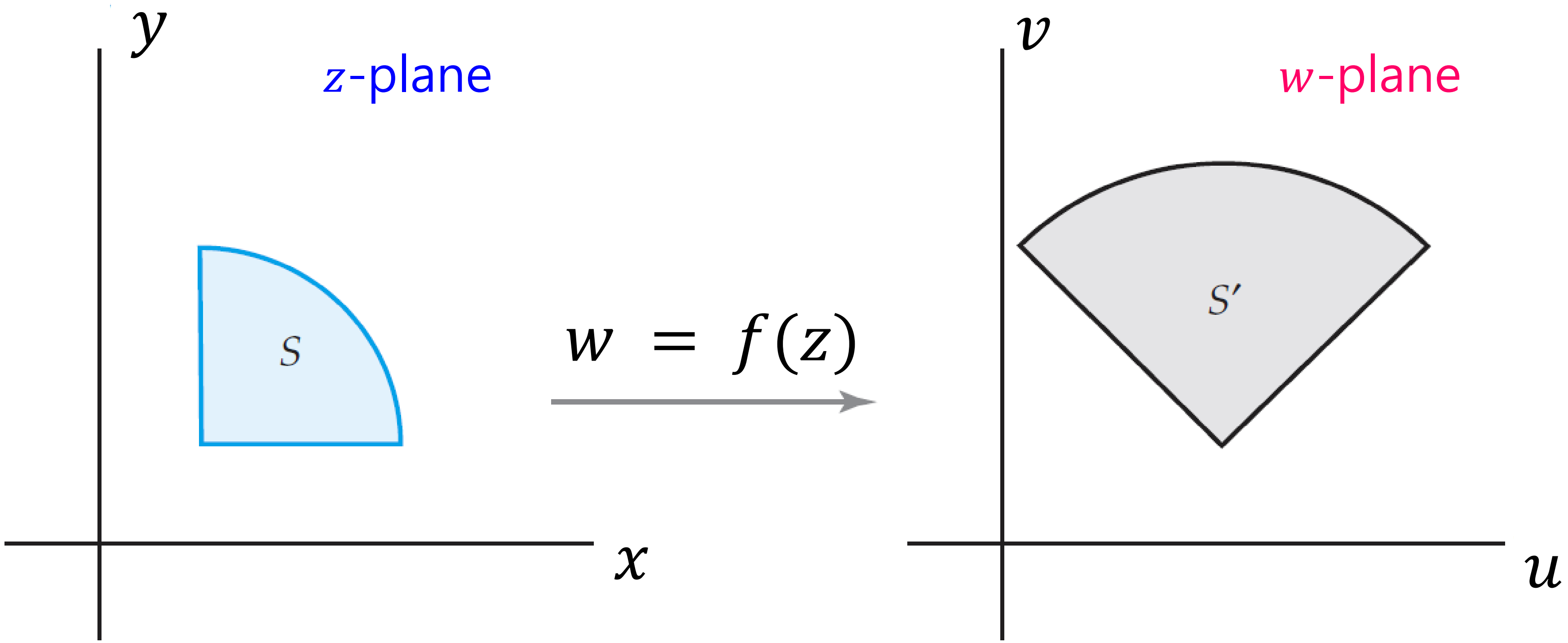
Complex Function as Mappings (写像、変換)

- Define two complex planes.
- The point z in the z-plane is associated with the unique point $w = f(z)$ in the w-plane.
- Every complex function $w = f(z)$ describes a correspondence (i.e. mapping) between points in two complex planes.



Complex Function as Mappings (写像、変換)

If $w = f(z)$ is a complex mapping and if S is a set of points in the z-plane, then we call the image (値域) of S under f as the set S′.



The set S in the z-plane The image of S' in the w-plane

Figure 2.1 The image (値域) of a set S under a mapping $w = f(z)$

EXAMPLE (例題) 2.2.1 Image of a Half-Plane under $w = iz$

Find the image of the half-plane $\mathrm{Re}(z) \geq 2$ under the complex mapping $w = f(z) = iz$ and represent the mapping graphically.

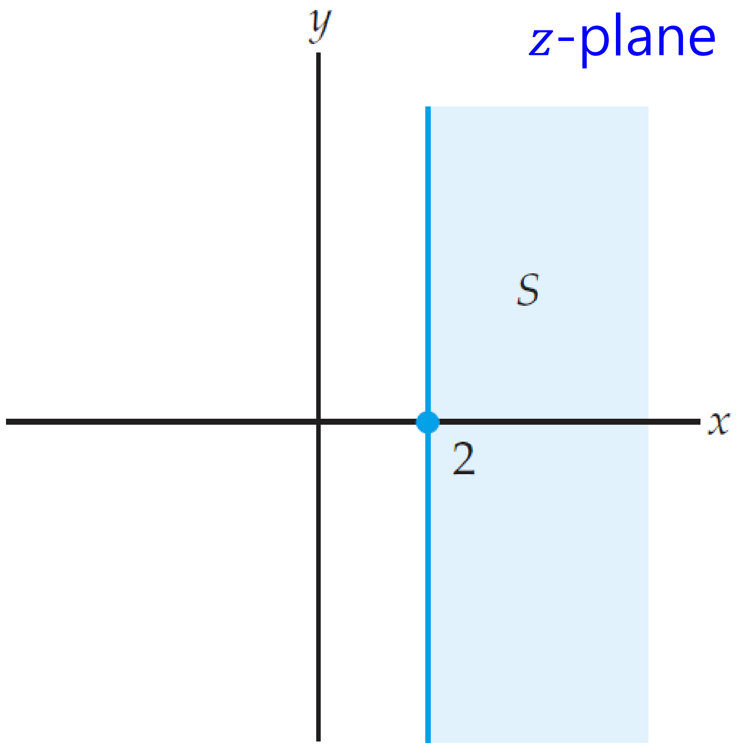
Solution (解答):

The vertical (垂直の) boundary line (境界線)

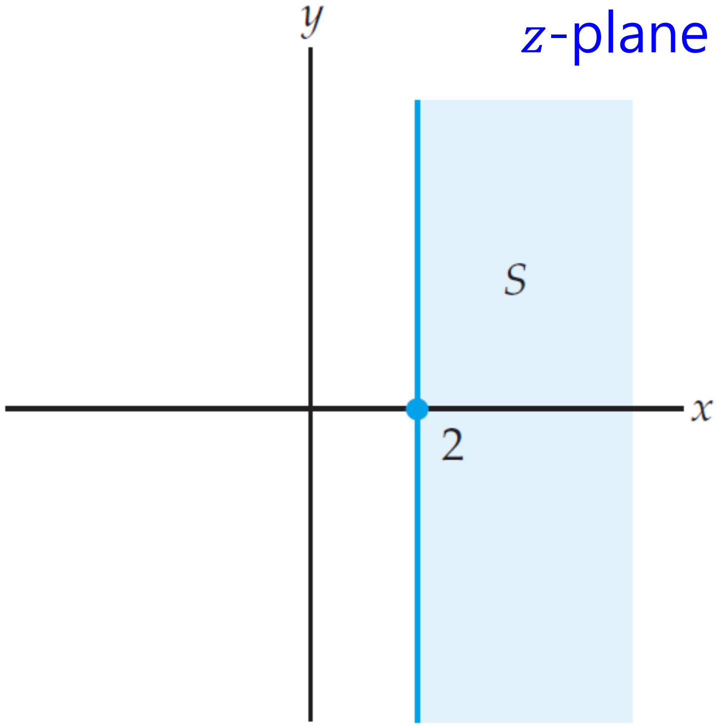
$\mathrm{Re}(z) = x = 2$ of S

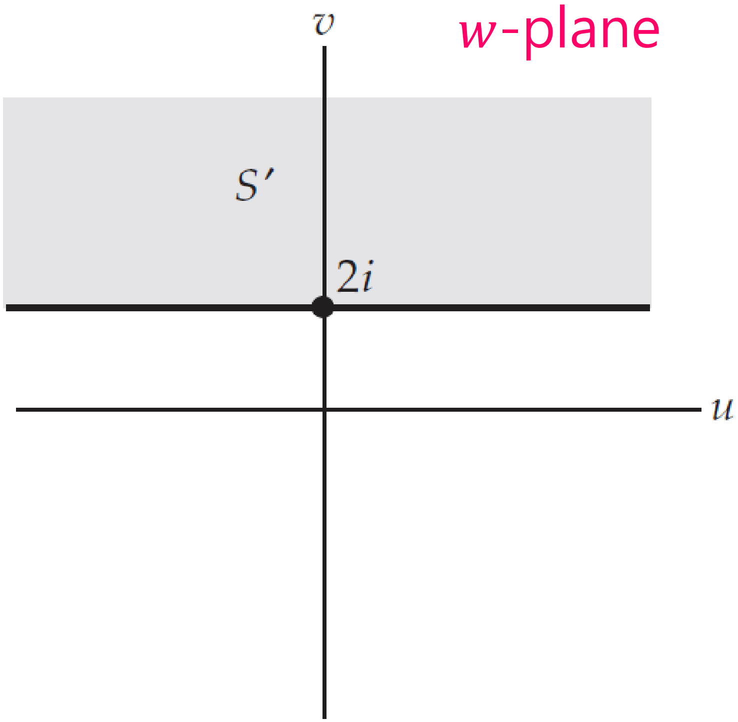
For any point z on this line we have $z = 2 + iy$ where $-\infty < y < \infty$.

$$w = f(z) = f(2 + iy) = i\,(2 + iy) = -y + 2i$$



2.2 Complex Function as Mappings (写像、変換)

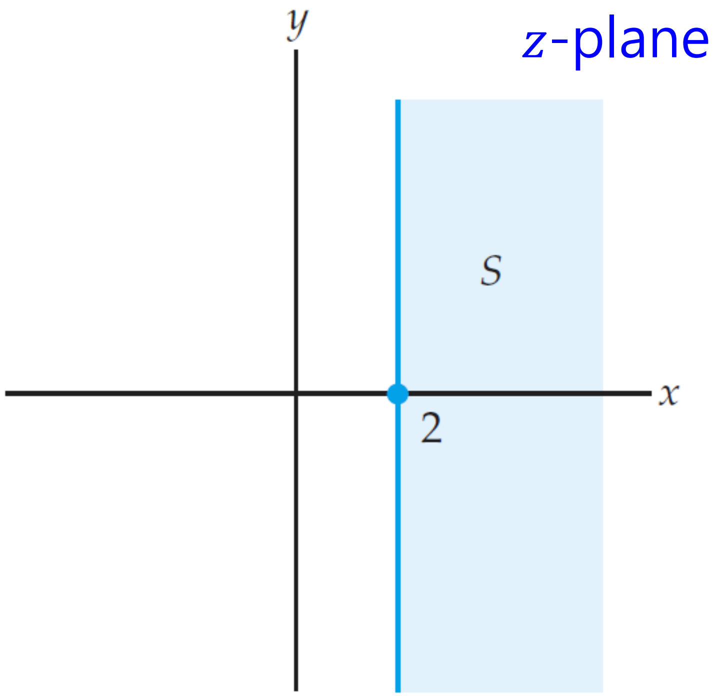


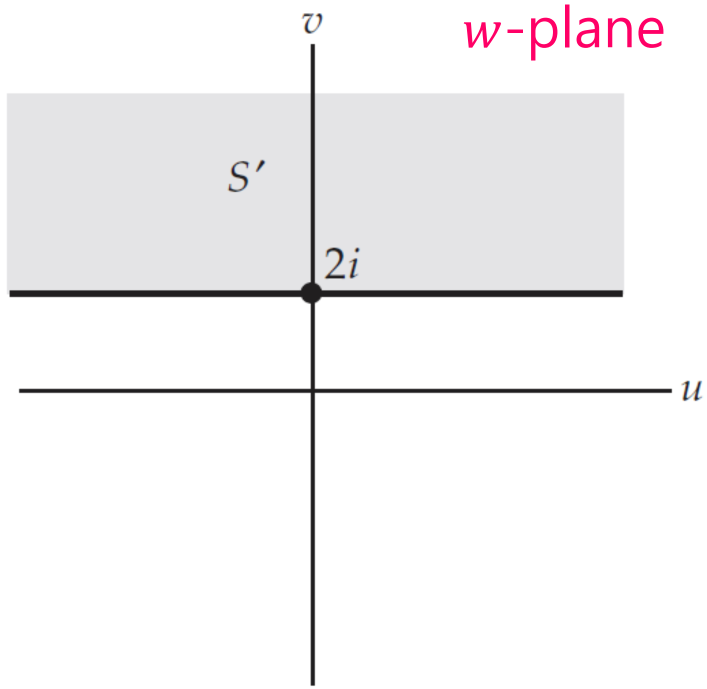


Because the set of points $w = -y + 2i$, $-\infty < y < \infty$, is the line $v = 2$ in the w-plane,

We conclude that the vertical line (垂直線) $x = 2$ in the z-plane is **mapped onto** the horizontal line (水平線) $v = 2$ in the w-plane by the **mapping** $w = f(z) = iz$.

2.2 Complex Function as Mappings (写像、変換)





The set S satisfies inequalities (不等式):

$x \geq 2$ and $-\infty < y < \infty$

$\Rightarrow$ $\mathrm{Re}(z) \geq 2$

The set S′ satisfies inequalities (不等式):

$v \geq 2$ and $-\infty < u < \infty$.

$\Rightarrow$ $\mathrm{Im}(w) \geq 2$

In summary, the half-plane $\mathrm{Re}(z) \geq 2$ shown **in blue color** of left figure **is mapped onto** the half-plane $\mathrm{Im}(w) \geq 2$ shown **in gray color** (灰色) in right figure by **the complex mapping** $w = f(z) = iz$.

2.3 Limits (極限) and Continuity (連続性)

Limit of Real Function

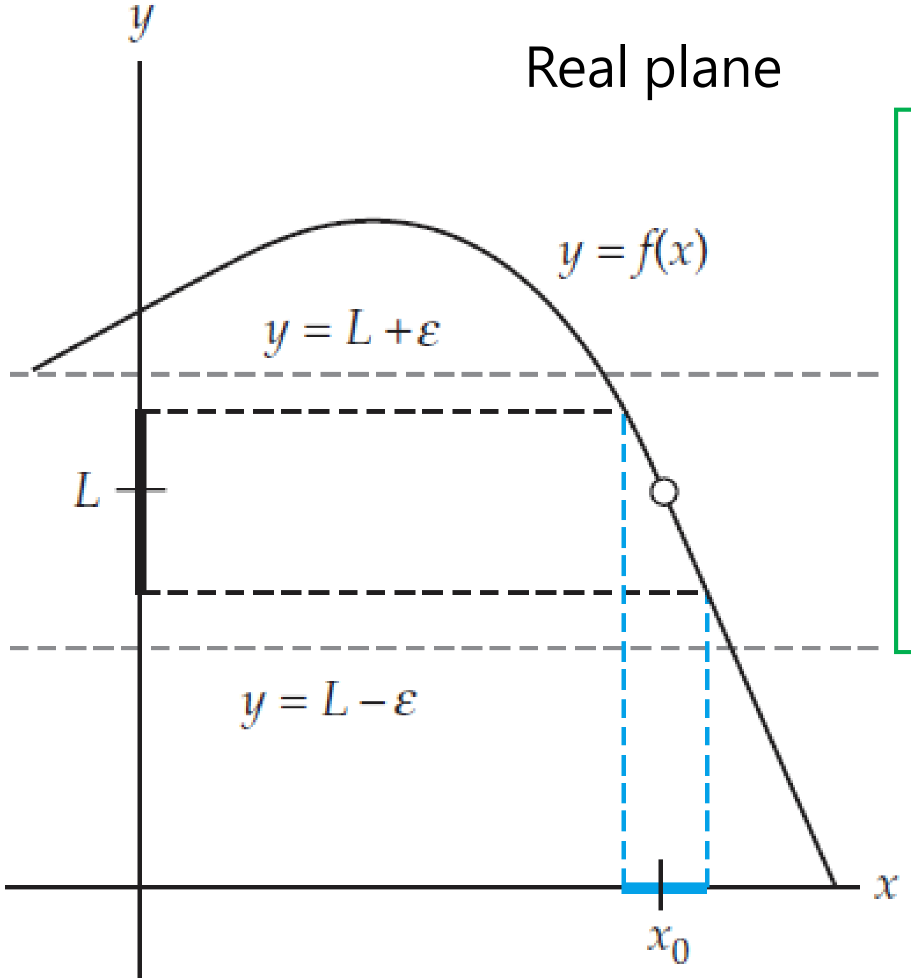


Figure 2.50 Geometric (幾何学の) meaning of a real limit.

The limit of f as x tends x_0 exists and is equal to L if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - x0| < \delta$.

$$\lim_{x \to x_0} f(x) = L$$

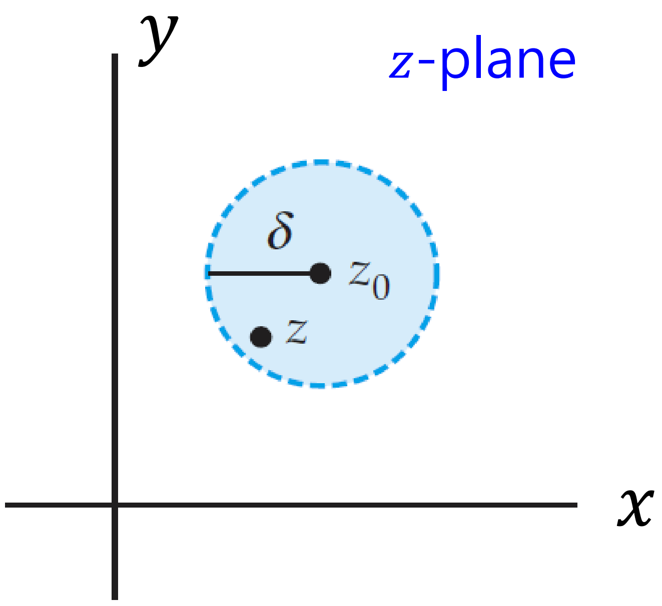
Limit of Complex Function (複素関数の極限)

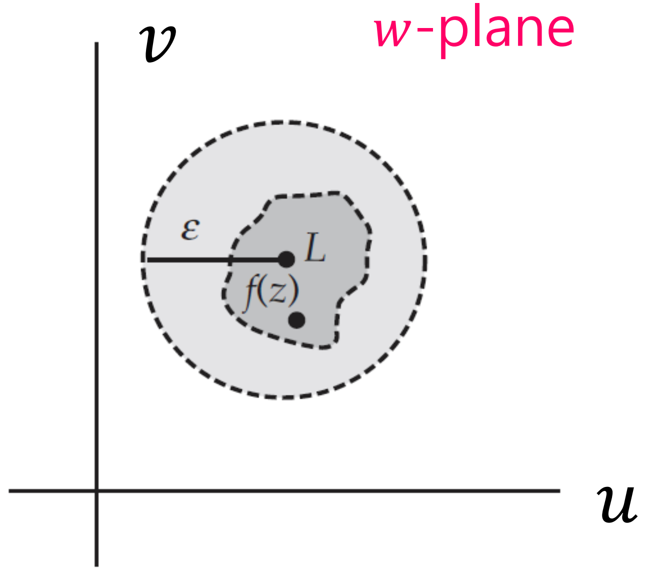
Definition (定義) 2.8
Limit of a Complex Function (複素関数の極限)

Suppose that a **complex function** f is defined in a deleted neighborhood of z_0 and suppose that L is a **complex number**. **The limit of f as z tends to z_0 exists and is equal to L, written as $\lim_{z\to z_0} f(z) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z0| < \delta$.**

in w-plane

in z-plane





2.3 Limits (極限) and Continuity (連続性)

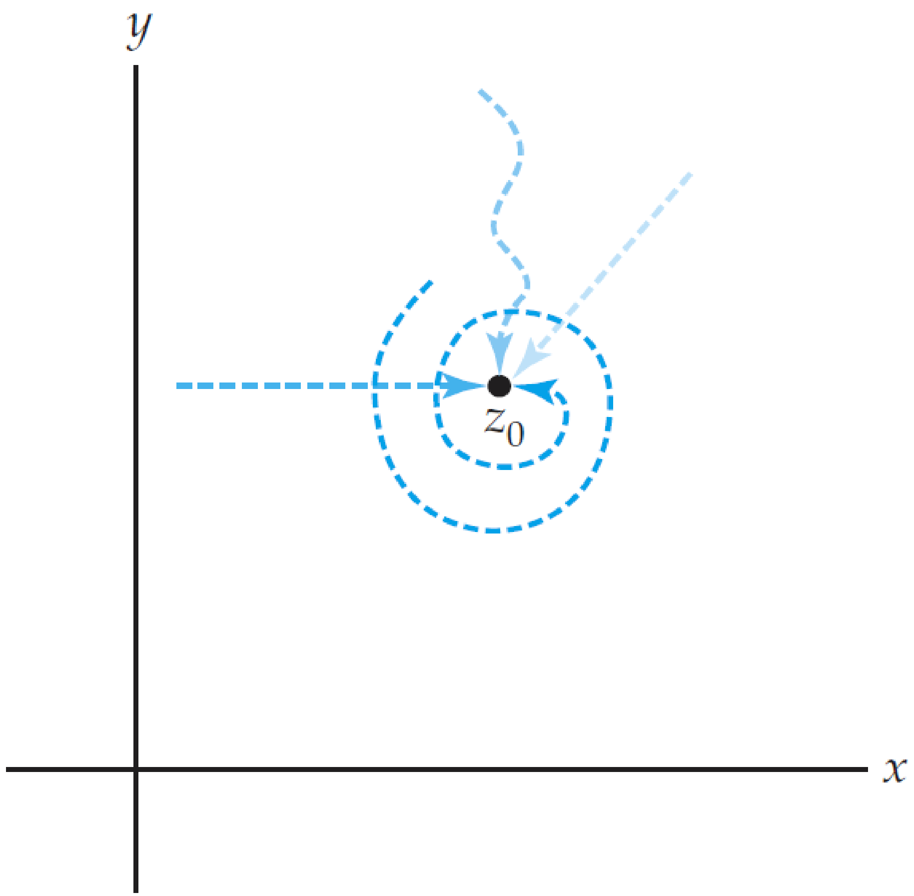


Figure 2.53 Different ways to approach z_0 in a limit.

Criterion (基準) for the Nonexistence (存在しない) of a Limit

If f approaches two complex numbers $L_1 \neq L_2$ for two different curves or paths through z_0,

then $\lim_{z \to z_0} f(z) = L$ does not exist.

EXAMPLE (例題) 2.6.1

Show that $\lim_{z\to 0}\frac{z}{\bar{z}}$ does not exist.

Solution (解答):

First, we let z approach 0 along the real axis, i.e. we consider complex numbers of the form $z=x+0i$ where the real number x is approaching 0

$$\lim_{z\to 0}\frac{z}{\bar{z}}=\lim_{x\to 0}\frac{x+0i}{x-0i}=\lim_{x\to 0}1=1$$

Second, we let z approach 0 along the imaginary axis, then $z = 0+iy$ where the real number y is approaching 0

$$\lim_{z\to 0}\frac{z}{\bar{z}}=\lim_{y\to 0}\frac{0+iy}{0-iy}=\lim_{y\to 0}(-1)=-1$$

The two limits are not same, then conclude that $\lim_{z\to 0}\frac{z}{\bar{z}}$ **does not exist.**

Theorem 2.1 Real and Imaginary Parts (実部と虚部) of a Limit

Suppose that $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$, and

$L = u_0 + iv_0$. Then $\lim_{z \to z_0} f(z) = L$ **if and only if**

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$$

EXAMPLE (例題) 2.6.3

Use Theorem 2.1 to compute $\lim_{z \to 1+i} (z^2 + i)$, where $z = x + iy$.

Solution (解答):

Since $f(z) = z^2 + i = x^2 - y^2 + (2xy + 1)i$,

Apply Theorem 2.1 with $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy + 1$, and

$z_0 = 1 + i \Rightarrow x_0 = 1, y_0 = 1$

$$u_0 = \lim_{(x,y)\to(x_0,y_0)} u(x, y) = \lim_{(x,y)\to(1,1)} (x^2 - y^2) = \lim_{(x,y)\to(1,1)} (1^2 - 1^2) = 0$$

$$v_0 = \lim_{(x,y)\to(x_0,y_0)} v(x, y) = \lim_{(x,y)\to(1,1)} (2xy + 1) = \lim_{(x,y)\to(1,1)} (2 \cdot 1 \cdot 1 + 1) = 3$$

so $L = u_0 + iv_0 = 0 + i(3) = 3i$. Therefore, $\lim_{z \to 1+i} (z^2 + i) = 3i$

Theorem 2.2 Properties (性質) of Complex Limits

Suppose that f and g are complex functions. Then $\lim_{z \to z_0} f(z) = L$ and $\lim_{z \to z_0} g(z) = M$, then

(i) $\lim_{z \to z_0} cf(z) = cL$, where c is a complex constant,

(ii) $\lim_{z \to z_0} \big(f(z) \pm g(z)\big) = L \pm M$,

(iii) $\lim_{z \to z_0} f(z) \cdot g(z) = L \cdot M$, and

(iv) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L}{M}$, provided $M \neq 0$.

2.3 **Limits** (**極限**) and **Continuity** (**連続性**)

We establish two basic complex limits:

- The complex constant (定数) function $f(z) = c$, where c is a complex constant (定数)

$$\lim_{z \to z_0} c = c \tag{2.6.15}$$

- The complex identity (恒等) function $f(z) = z$

$$\lim_{z \to z_0} z = z_0 \tag{2.6.16}$$

EXAMPLE (例題) 2.6.4

Use Theorem 2.2 and the basic limits (2.6.15) and (2.6.16) to compute the limits $\lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1}$.

Solution (解答):

By Theorem 2.2(iii) and (2.6.16),

$$\lim_{z \to i} z^2 = \lim_{z \to i} z \cdot z = \left(\lim_{z \to i} z\right) \cdot \left(\lim_{z \to i} z\right) = i \cdot i = -1$$

Similarly, $\lim_{z \to i} z^4 = i^4 = 1$

Solution (解答) (cont.):

Using these limits, Theorems 2.2(i), 2.2(ii), and (2.6.16), we obtain:

$$\lim_{z\to i}\left((3+i)z^4 - z^2 + 2z\right) = (3+i)\lim_{z\to i} z^4 - \lim_{z\to i} z^2 + 2\lim_{z\to i} z$$

$$= (3+i)\cdot(1) - (-1) + 2\cdot(i)$$

$$= 4 + 3i$$

$$\lim_{z\to i} z + 1 = 1 + i$$

Therefore, by Theorem 2.2(iv), we have:

$$\lim_{z\to i}\frac{(3+i)z^4 - z^2 + 2z}{z+1} = \frac{\lim_{z\to i}\left((3+i)z^4 - z^2 + 2z\right)}{\lim_{z\to i} z + 1} = \frac{4+3i}{1+i} = \frac{7}{2} - \frac{1}{2}i$$

Continuity (連続性) of Complex Functions

Definition 2.9 Continuity (連続性) of a Complex Function

A complex function f is continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Continuity (連続性) of Complex Functions

Criteria (基準) for Continuity (連続) at a Point

A complex function f is continuous at a point z_0 if each of the following three conditions (条件) hold (満たす):

(i) $\lim_{z \to z_0} f(z)$ exists,

(ii) f is defined at z_0, and

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$

Continuity (連続性) of Complex Functions

EXAMPLE (例題) 2.6.5 **Checking Continuity at a Point**

Consider the function $f(z) = z^2 - iz + 2$ to determine if f is continuous at the point $z_0 = 1 - i$.

Solution (解答):

From Theorem 2.2 and the limits in (2.6.15) and (2.6.16) we obtain:

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} (z^2 - iz + 2) = (1 - i)^2 - i\,(1 - i) + 2 = 1 - 3i.$$

Furthermore, for $z_0 = 1 - i$ we have:

$$f(z_0) = f(1 - i) = (1 - i)^2 - i\,(1 - i) + 2 = 1 - 3i.$$

Since $\lim_{z \to z_0} f(z) = f(z_0)$, we conclude that f is continuous at the point $z_0 = 1 - i$.

Continuity (連続性) of Complex Functions

Theorem 2.3 Real and Imaginary Parts of a Continuous Function

Suppose that $f(z) = u(x,y) + iv(x,y)$ and $z_0 = x_0 + iy_0$.

Then the complex function (複素関数) f is continuous at the point z_0 **if and only if** both real functions (実数値関数) u and v are continuous at the point (x_0, y_0).

Continuity (連続性) of Complex Functions

EXAMPLE (例題) 2.6.7 Checking Continuity Using Theorem 2.3
Show that the function $f(z) = \bar{z}$ is continuous on $\mathbf{C}$.

Solution (解答):

According to Theorem 2.3, $f(z) = \bar{z} = \overline{x + iy} = x - iy$ is continuous at $z_0 = x_0 + iy_0$ if both $u(x, y) = x$ and $v(x, y) = -y$ are continuous at (x_0, y_0).

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = x_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x, y) = -y_0$$

Because u and v are two-variable polynomial functions, we have (2.6.13) that

$$\lim_{(x,y)\to(x_0,y_0)} p(x, y) = p(x_0, y_0) \tag{2.6.13}$$

This implies that u and v are continuous at (x_0, y_0), and, therefore, that f is continuous at $z_0 = x_0 + iy_0$ by Theorem 2.3.

Since $z_0 = x_0 + iy_0$ was an arbitrary (任意の) point, we conclude that the function $f(z) = \bar{z}$ is continuous on $\mathbf{C}$.

Continuity (連続性) of Complex Functions

Theorem 2.4 Properties (性質) of Continuous Functions

If f and g are continuous at the point z_0, then the following functions are continuous at the point z_0:

(i) cf, where c is a complex constant,

(ii) $f \pm g$,

(iii) $f \cdot g$,

(iv) $\frac{f}{g}$, provided $g(z_0) \neq 0$.

Continuity (連続性) of Complex Functions

Theorem 2.5 Continuity of Polynomial Functions (多項式関数)

Polynomial functions (多項式関数) are continuous on the entire complex plane **C**.

Review for Lecture 2

- Complex Functions
- Complex Functions as Mapping
- Limit of Complex Function
- Continuity of Complex Function

Exercise

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: Section , Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia