Lecture 6

Complex Integrals（複素積分）

What you will learn in Lecture 6

*6.1 Review of Real Line Integral (実·線積分)

6.2 Complex Integral (複素積分)

Real Line Integral

(実·線積分)

in the Cartesian Plane

Complex Integral

(複素積分)

in the Complex Plane

*6.1 Review of Real Line Integral

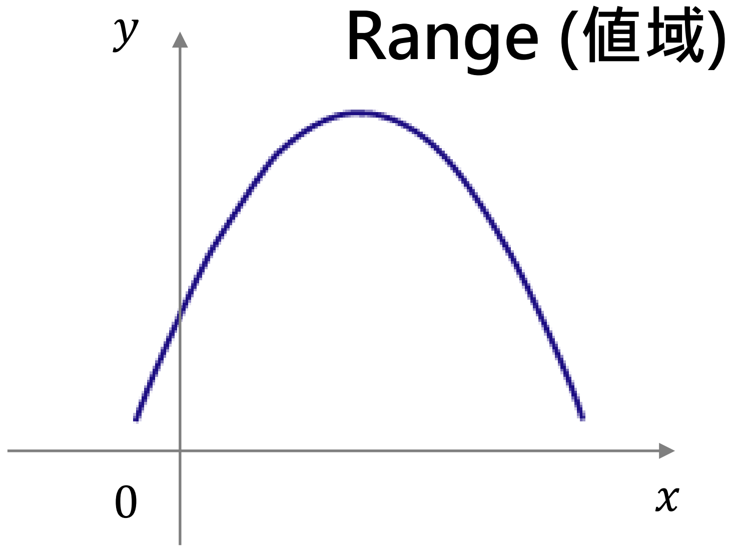
(実·線積分)

Notice: In all lecture notes, the contents marked with * are not in the scope of the final examination.

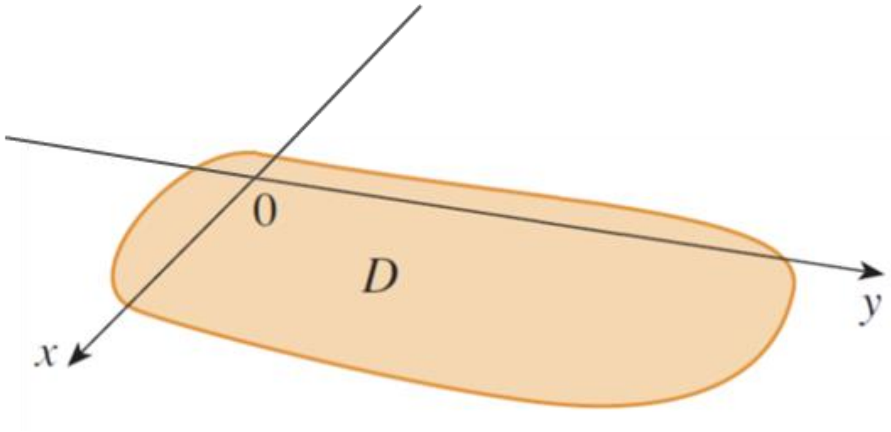
*6.1 Review of Real Line Integral (実·線積分)

Domain (定義域) $x \in \mathbf{R}$

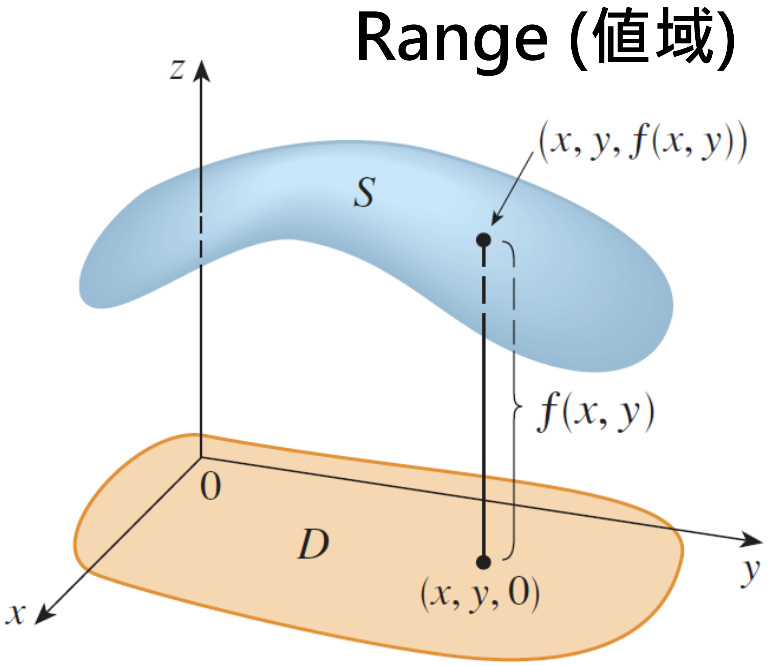
$(x, f(x))$



Domain (定義域) $(x, y) \in \mathbf{D}$



$(x, y, f(x, y))$



*6.1 Review of Real Line Integral (実·線積分)

One-Variable Calculus -- Definite integral (定積分) of f

$$\int_a^b f(x)dx = \lim_{\|\Delta x\|\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{\|\Delta x\|\to 0} [f(x_k^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x]$$

$$Area(面積) = \int_a^b f(x)dx$$

One-Variable Function

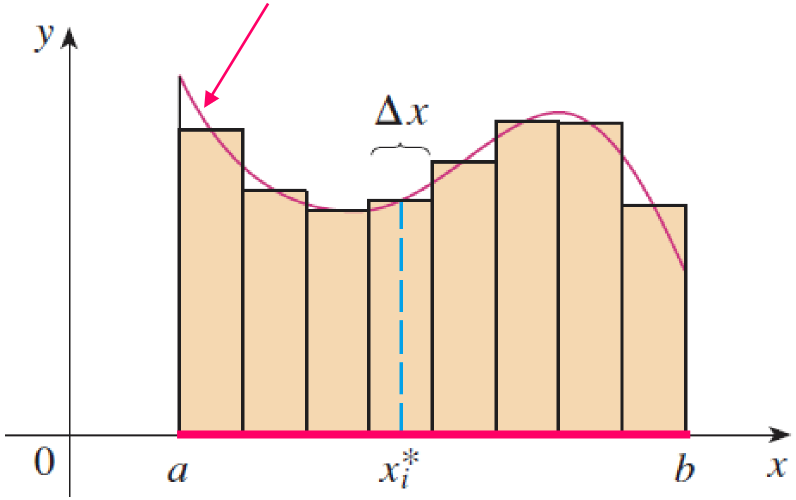
relationship, called function

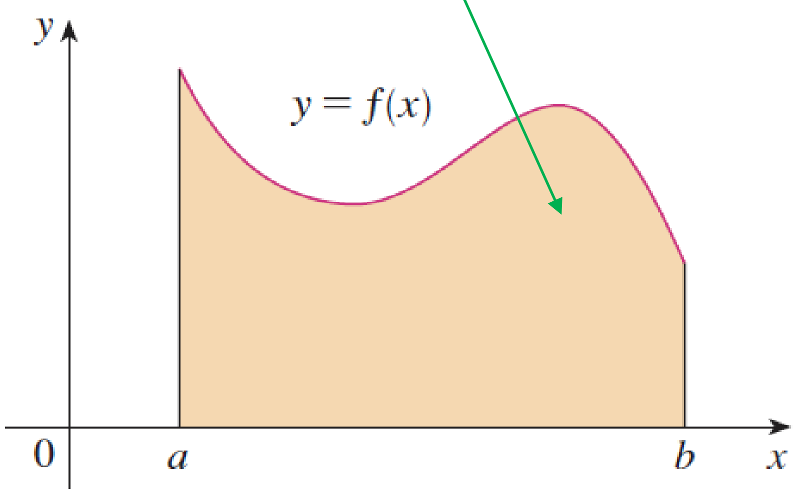
$$y = f(x)$$

Independent variable

dependent variable

$$y = f(x)$$



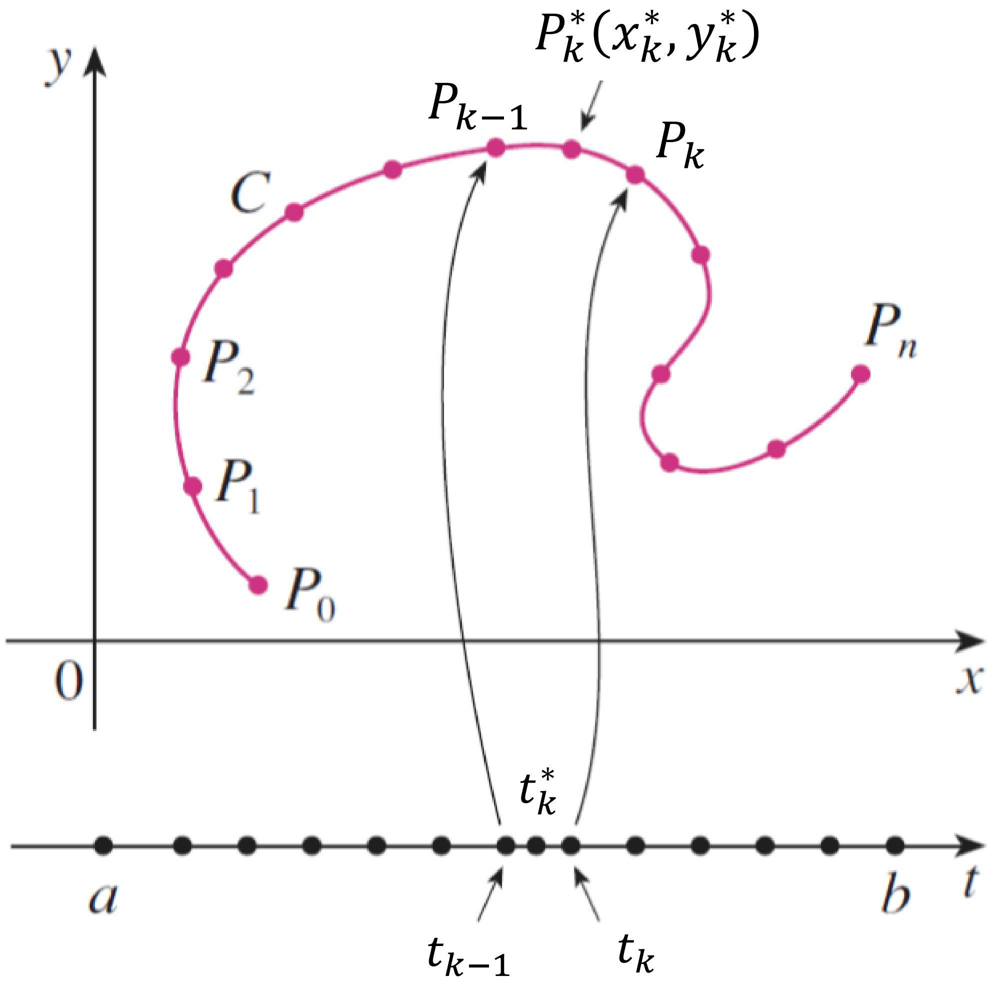


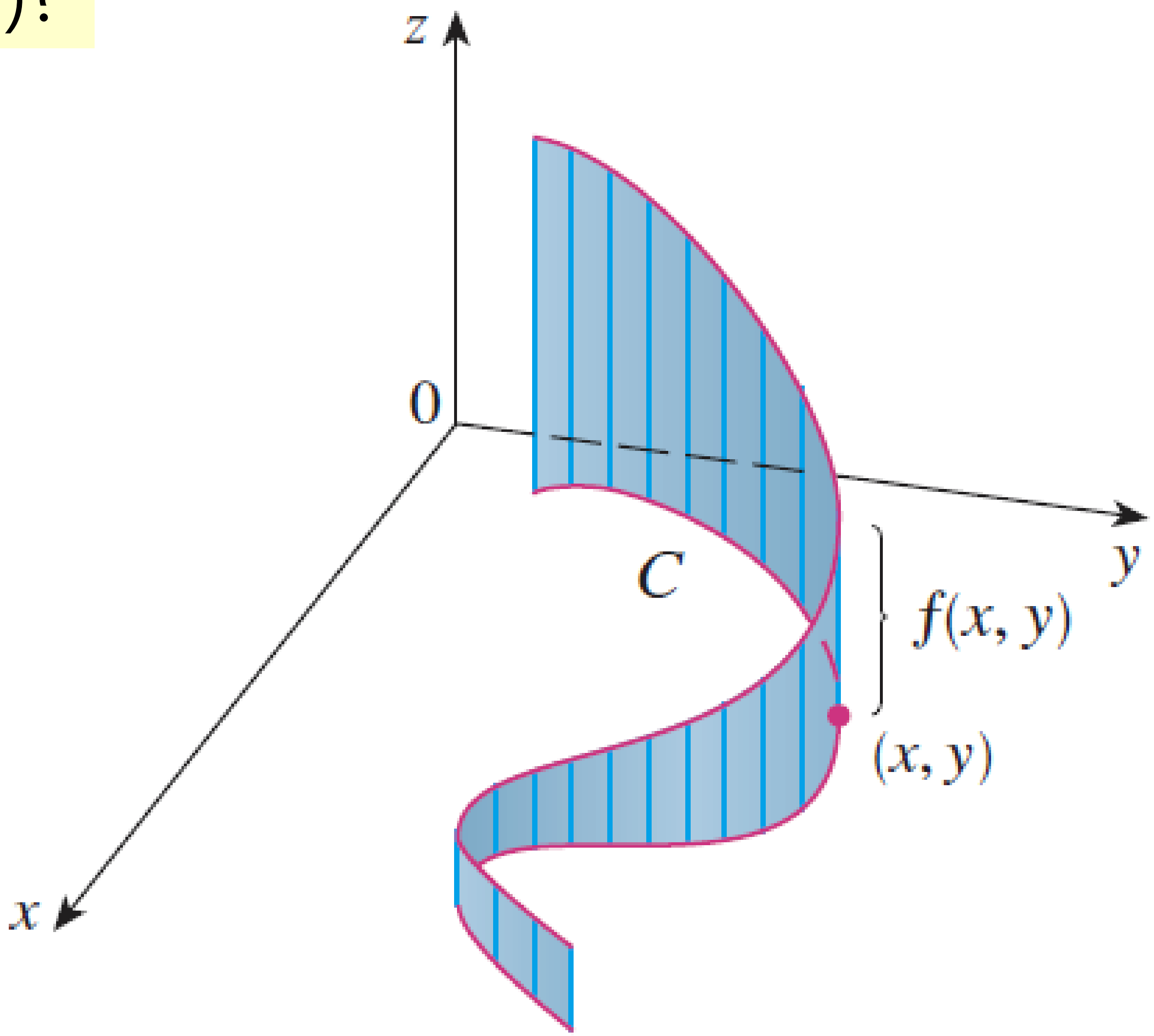
$$\Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \Delta x,$$

i.e. equal interval (等間隔)

*6.1 Review of Real Line Integral (実·線積分)

Recall: what is Line integral (線積分)?





*6.1 Review of Real Line Integral (実·線積分)

Line Integral (線積分)

If f is defined on a smooth (滑らか) or piecewise-smooth (区分的滑らか) curve C, then **the line integral of f along C** is

$$\int_C f(x,y)ds = \lim_{\|\Delta s_{max}\| \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta s_k$$

if this limits exists. (Here the norm $\|\Delta s_{max}\|$ defines the length of the longest subinterval (部分区間))

How to compute Line Integral?

By introducing **Arc length (弧長)** $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$, we have

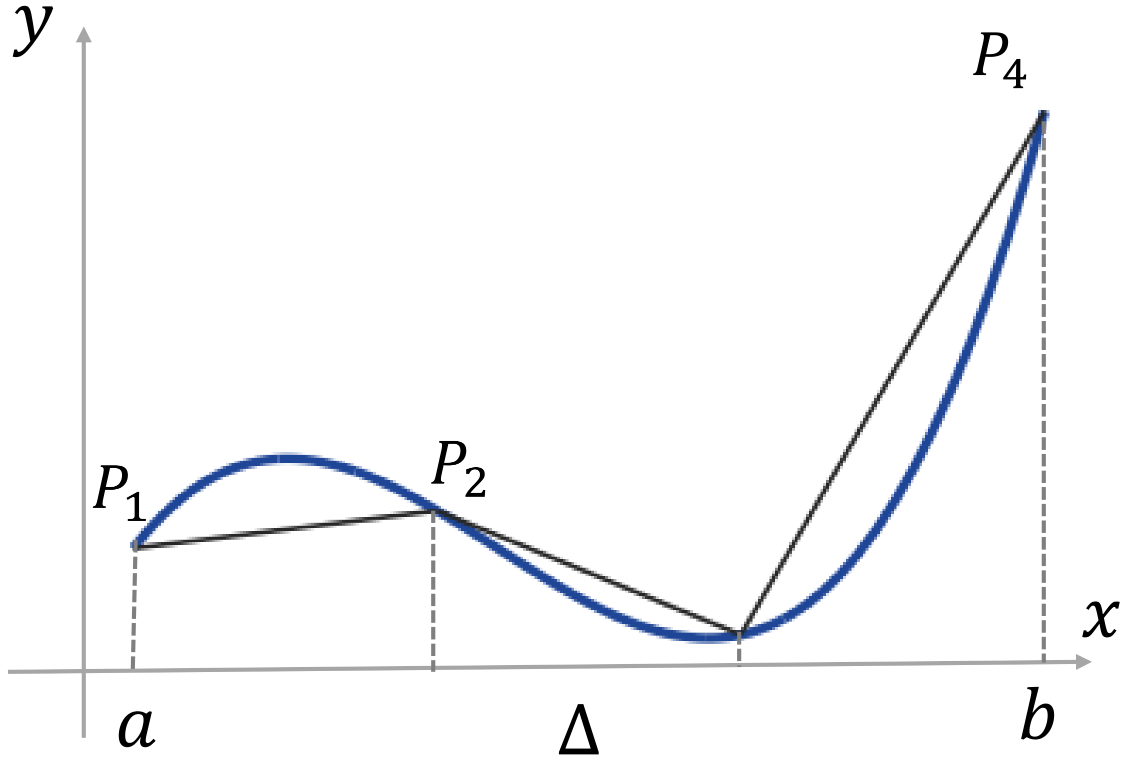
$$\boxed{\int_C f(x,y)ds = \int_C f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

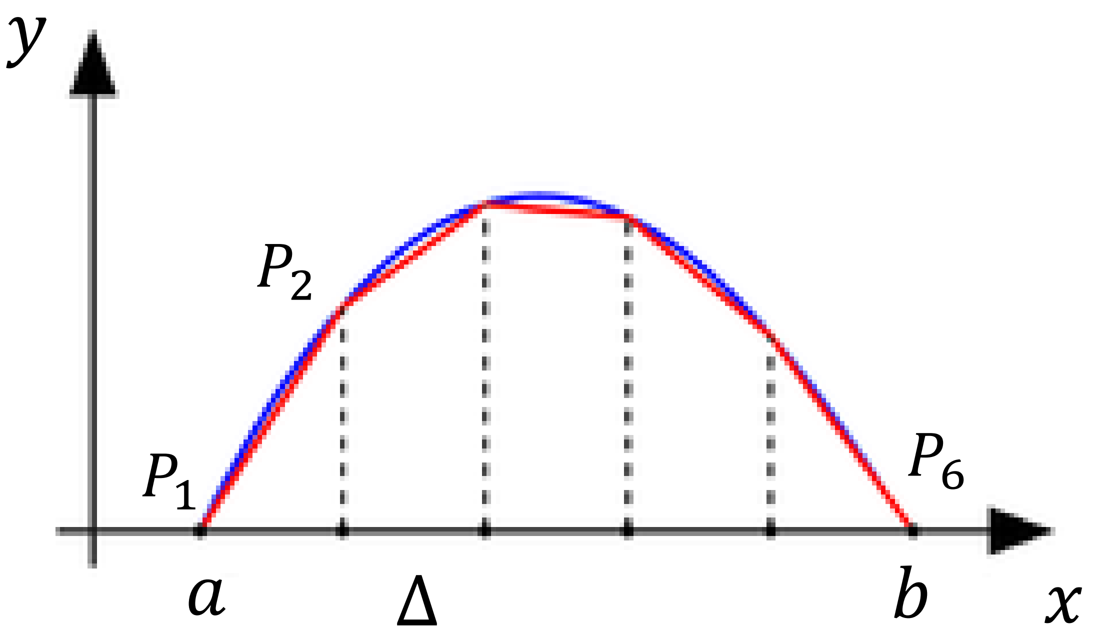
*6.1 Review of Real Line Integral (実·線積分)

Arc Length (弧長)

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

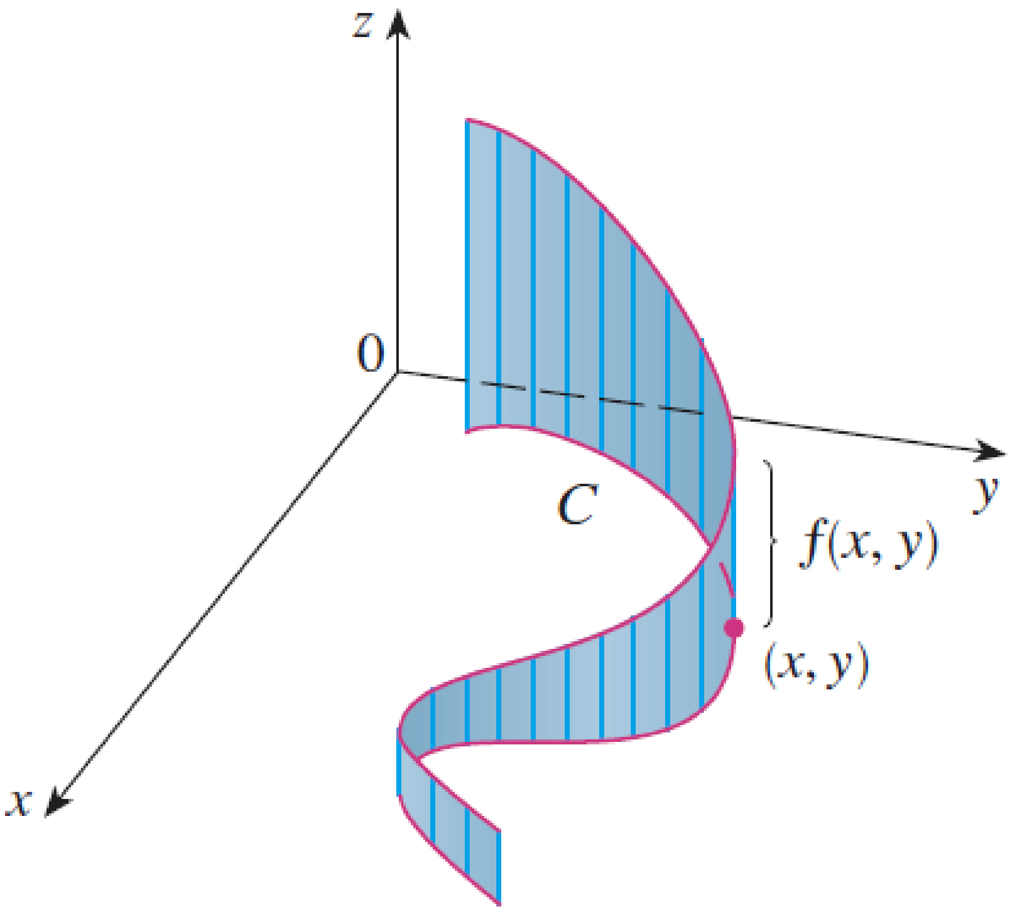
$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$





*6.1 Review of Real Line Integral (実·線積分)

Insight (洞察) of Line Integrals



In fact, if $f(x,y) > 0$,

$\int_C f(x,y)ds$ represents the area (面積) of one side of the "curtain".

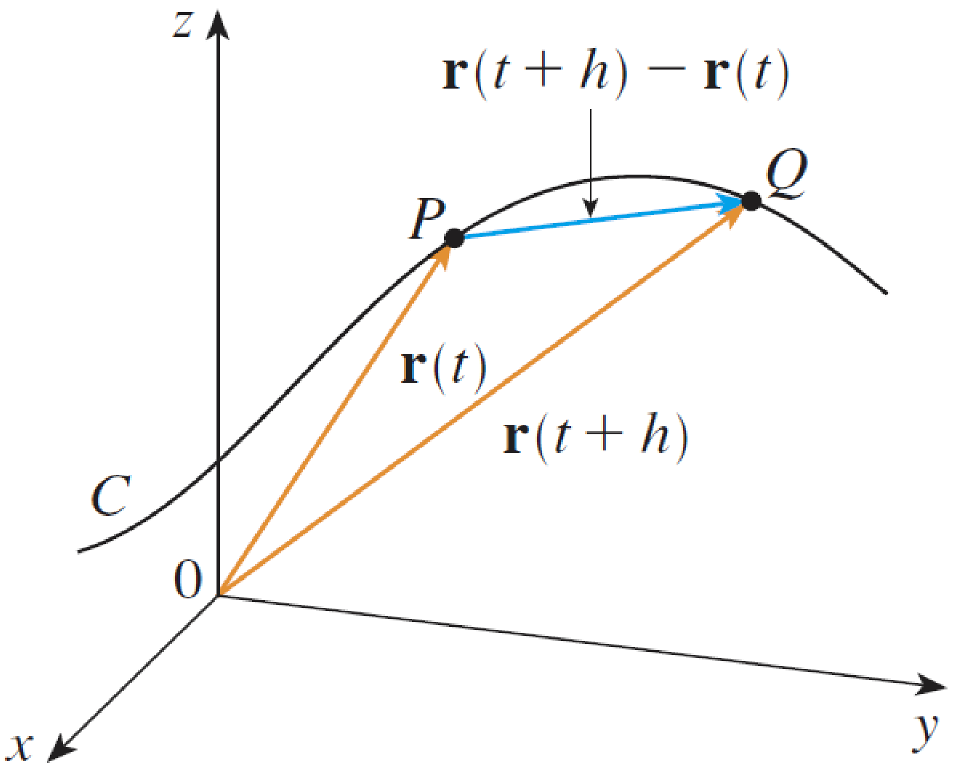
6.2 Complex Integral (複素積分)

6.2 Complex Integral (複素積分)

Q: How to represent curves in the complex plane?

6.2 Complex Integral (複素積分)

Parametrization (パラメータ表示)
of Real curve (実·曲線)



Parametrization (パラメータ表示)
of Complex curve (複素·曲線)

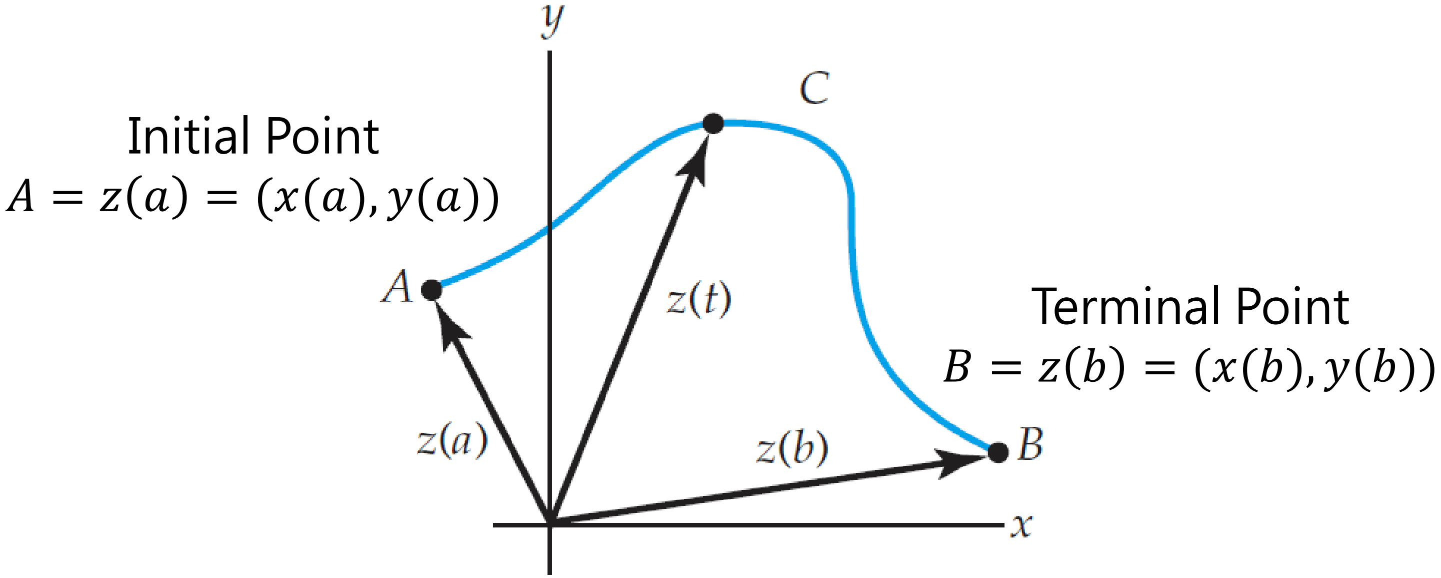
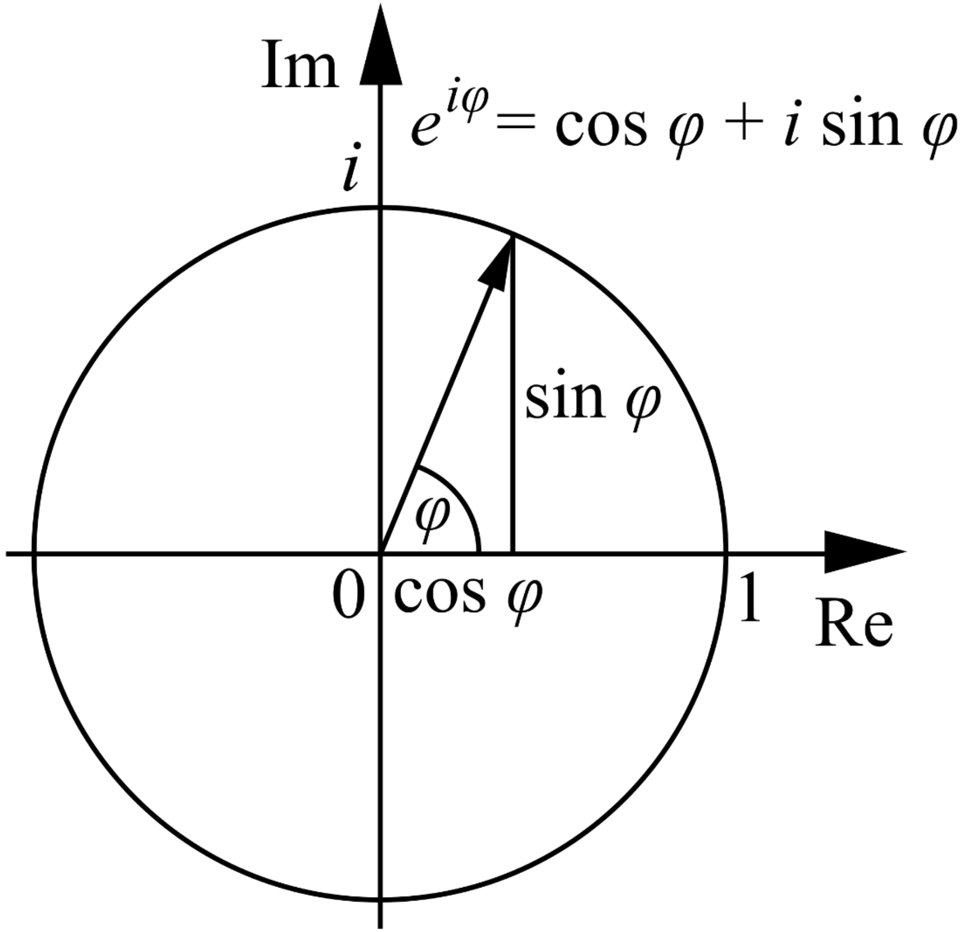


Figure 5.15 $z(t) = x(t) + iy(t)$ as a position vector

$$z(t) = x(t) + iy(t), \mathrm{a} \le \mathrm{t} \le \mathrm{b} \tag{5.2.1}$$

The points z on the curve C is expressed by a complex-valued function of a real variable t. This is called a parametrization of C.

6.2 Complex Integral (複素積分)



Parametrization (パラメータ表示) of **Complex** curve (複素·曲線)

$$z(t) = x(t) + iy(t), \mathrm{a} \le \mathrm{t} \le \mathrm{b}$$

For example,

if $x(t) = \cos t\,, y(t) = \sin t\,, 0 \le t \le 2\pi$

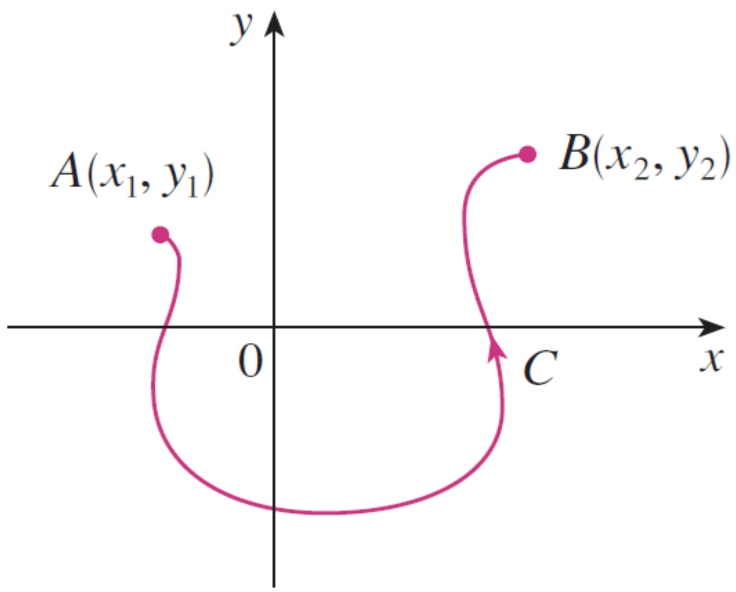
we will have

$$z(t) = \cos t + i\sin t\,, 0 \le t \le 2\pi$$

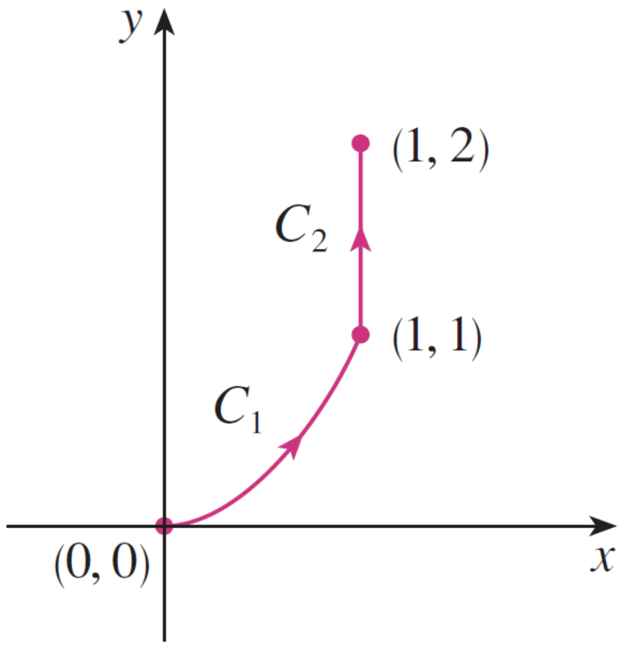
which is a parametrization of the circle C.

6.2 Complex Integral (複素積分)

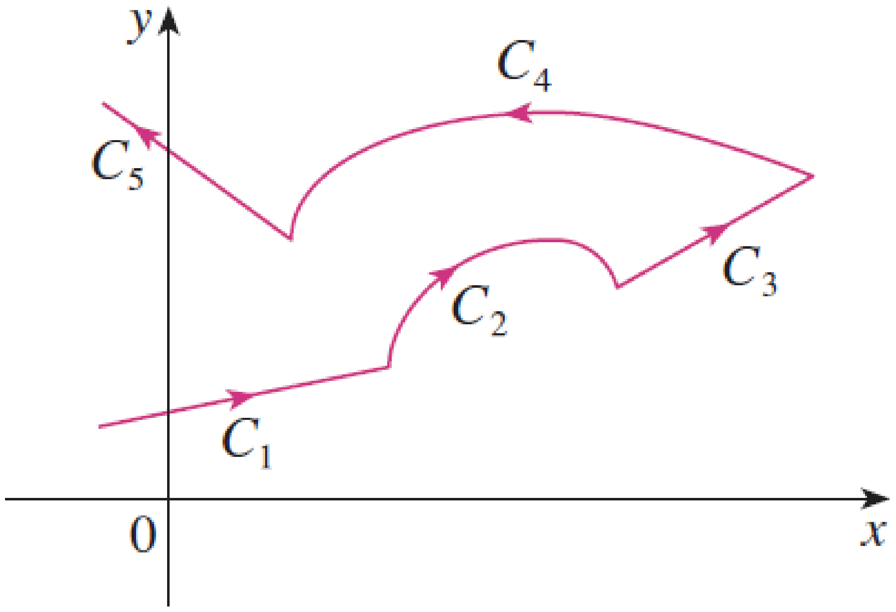
Piecewise-smooth (区分的滑らか) curve



A smooth curve C



A piecewise-smooth curve $C = C_1 \cup C_2$



A piecewise-smooth curve $C = C_1 \cup C_2 \cup \cdots \cup C_5$

6.2 Complex Integral (複素積分)

Suppose the derivative of

$$z(t) = x(t) + iy(t), \mathrm{a} \leq \mathrm{t} \leq \mathrm{b} \quad (5.2.1)$$

is

$$z'(t) = x'(t) + iy'(t)$$

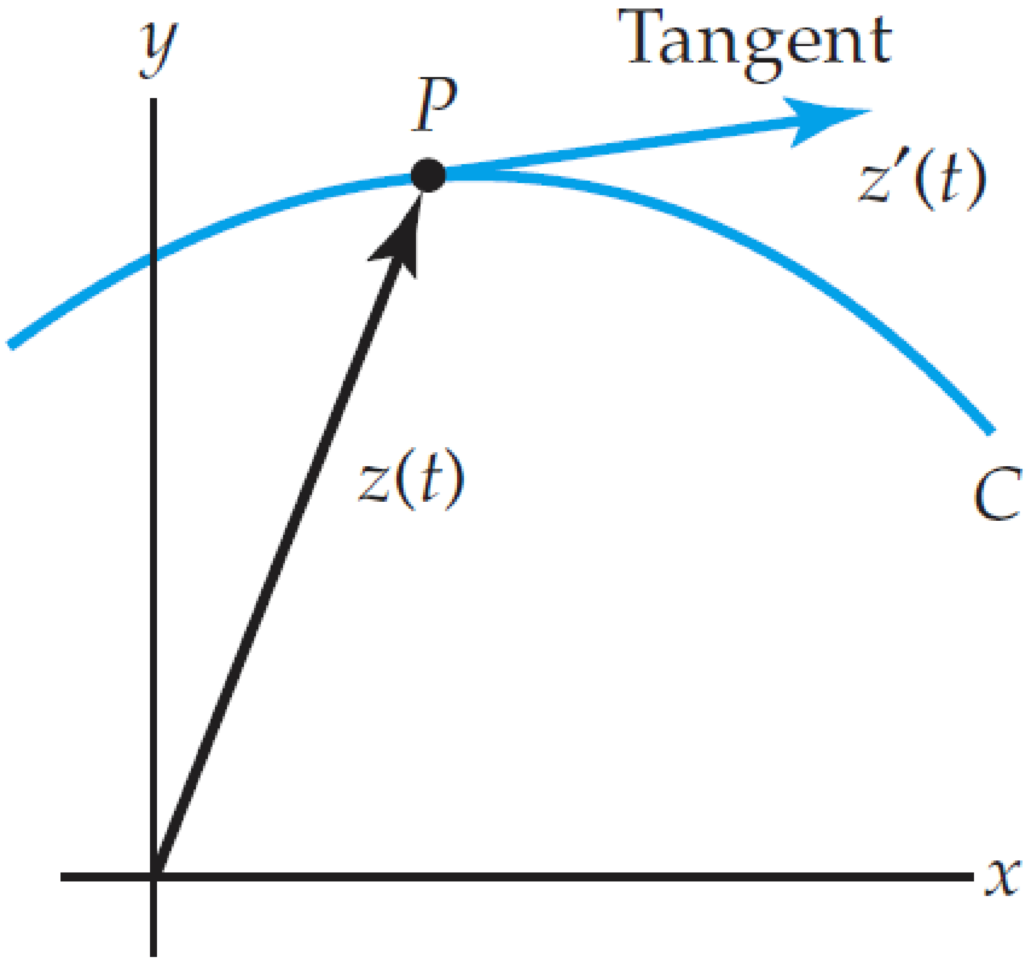


Figure 5.16 $z'(t) = x'(t) + iy'(t)$ as a tangent vector (接ベクトル)

6.2 Complex Integral (複素積分)

Smooth (滑らか) Curve

A curve C in the complex plane is called smooth if $z'(t)$ is continuous and NEVER zero in the interval $a \le t \le b$.

In other words, a smooth curve have NO sharp corners or Cusps (尖点).

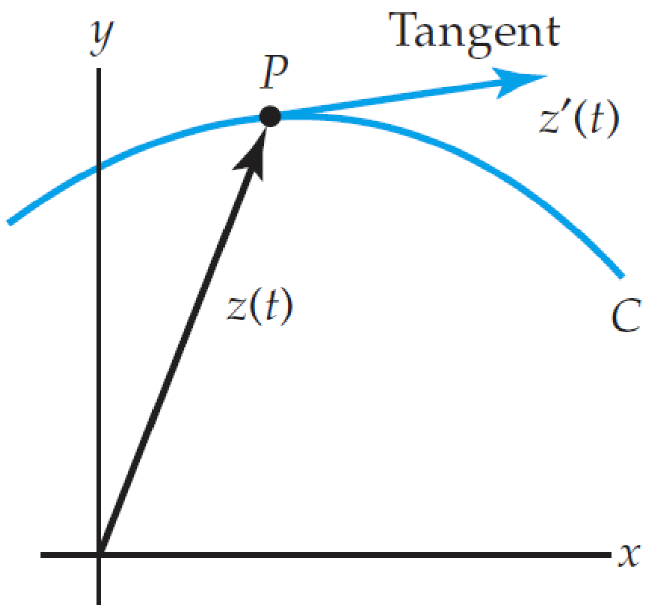


Figure 5.16 $z'(t) = x'(t) + iy'(t)$ as a tangent vector (接ベクトル)

Piecewise-smooth (区分的滑らか) Curve

A piecewise smooth curve C is continuous EXCEPT possibly at the points where the component smooth curves $C_1, C_2, \ldots, C_n$ are joined together.

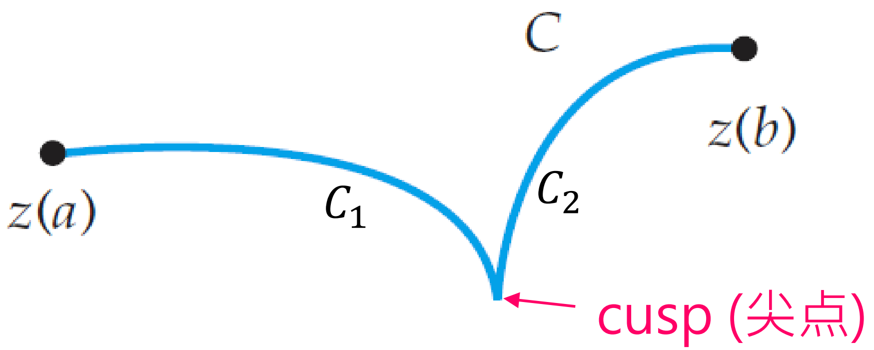


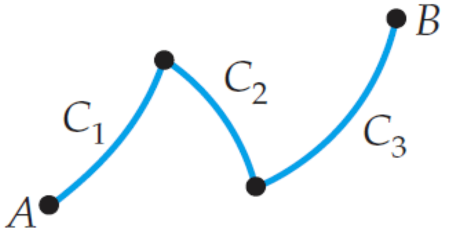
Figure 5.17 Curve C is not smooth because it has a cusp

6.2 Complex Integral (複素積分)

Simple (単一) Curve

A curve C in the complex plane is said to be a simple if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$, except possibly for initial point $t = a$ and terminal point $t = b$.

Smooth, simple, not closed



Piecewise smooth, simple, not closed

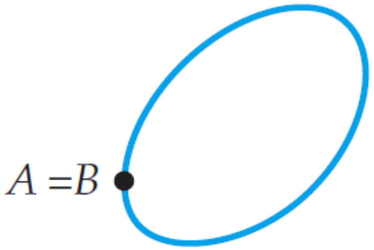
Closed (閉) Curve

C is a **closed curve** if $z(a) = z(b)$.

Simple Closed Curve (単一閉曲線)

C is a **simple closed curve** if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$ and $z(a) = z(b)$.

Smooth, closed, not simple



Smooth, simple, closed

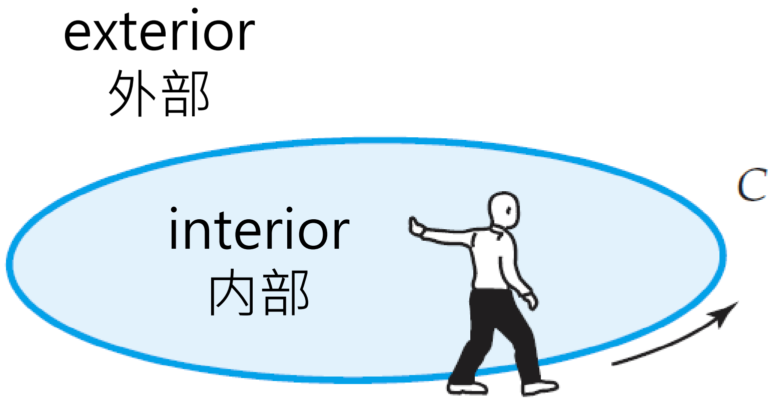
6.2 Complex Integral (複素積分)

Contour

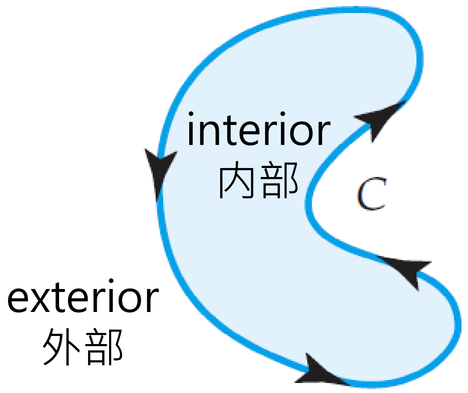
A piecewise smooth curve C is called a **Contour** or Path.

Direction (向き) on a Contour

- We define the **positive direction** on a contour C corresponding to increasing values of the parameter t.
- Roughly, for a simple closed curve C, the **positive direction** is the **counterclockwise (左回りの) direction** or the direction that **a person must walk on C and keep the interior (内部) of C at the left hand**.
- The **negative direction** on a contour C is the direction **opposite (反対の) the positive direction**.
- **Notice**: If C has positive direction, **then** its opposite curve can be denoted by $-C$.



(a) Positive direction



(b) Positive direction

Figure 5.18 Interior of each curve is at the left hand

6.2 Complex Integral (複素積分)

Note: Find more explanations in Page 247 ~ 250 of the textbook.

Definition 5.3 Complex Integral (複素積分)

An integral of a function $f(z)$ defined by

$$\int_C f(z)\,dz = \lim_{\|\Delta z_{max}\|\to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k \qquad (5.2.2)$$

is called a **complex integral**, where z is a complex number, and $f(z)$ is defined on a contour C (積分路). (Here the norm $\|\Delta z_{max}\|$ defines the length of the longest subinterval (部分区間))

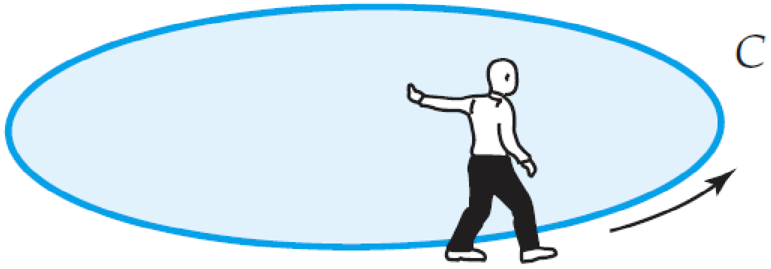
- If the limit in (5.2.2) exists, then $f(z)$ is said to be **integrable (可積分の)** on C.
- The **limit exists** whenever **if** $f(z)$ is **continuous at all points on** C, where C is either **smooth or piecewise smooth**.
- The Complex Integral $\int_C f(z)dz$ has a more common name: Contour Integral.

6.2 Complex Integral (複素積分)

- Specially, we will use the notation

$$\oint_C f(z)dz$$

as a **complex integral around a positively oriented closed curve** C.



Closed curve C with Positive direction

6.2 Complex Integral (複素積分)

Integral for Complex-Valued Function of a Real Variable

Example

If t represents a real variable, then the output of the function $f(t) = (2t+i)^2$ is a complex number. For $t = 2$,

$$f(2) = (2\cdot 2+i)^2 = 16 + 8i + i2 = 15 + 8i.$$

In general, if f_1 and f_2 are real-valued functions of a real variable t (that is, real functions), then $f(t) = f_1(t) + if_2(t)$ is a complex-valued function of a real variable t.

When we consider the interval $0 \le t \le 1$,

$$\int_0^1 (2t+i)^2\,dt = \int_0^1 (4t^2 - 1 + i4t)\,dt = \int_0^1 (4t^2-1)\,dt + i\int_0^1 4t\,dt = \left(\frac{4}{3}\cdot t^3 - t\right)\Big|_0^1 + i\cdot 2t^2\Big|_0^1 = \frac{1}{3} + 2i$$

Then we can define **the integral of the complex-valued function** $f(t) = f_1(t) + if_2(t)$ **on interval** $\mathrm{a} \le t \le b$ **as**

$$\boxed{\int_a^b f(t)\,dt = \int_a^b f_1(t)\,dt + i\int_a^b f_2(t)\,dt} \qquad (5.2.4)$$

Theorem 5.1 How to Compute a Complex Integral (i.e. Complex Integral)

If f is continuous on a smooth curve C given by the parametrization (パラメータ表示) $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, then

$$\int_C f(z)\,dz = \int_a^b f\big(z(t)\big)z'(t)\,dt \tag{5.2.11}$$

where

$$f\big(z(t)\big)z'(t) = f\big(z(t)\big)[x'(t) + iy'(t)] = \big[u\big(x(t), y(t)\big) + iv\big(x(t), y(t)\big)\big][x'(t) + iy'(t)]$$

Recall $z(t)$ in Equation (5.2.1)

EXAMPLE (例題) 5.2.1 **Evaluating a Contour Integral**

Evaluate $\int_C \bar{z}\,dz$, where C is given by $z(t) = x(t) + iy(t)$, $x(t) = 3t$, $y(t) = t^2$, $-1 \leq t \leq 4$.

Solution (解答):

$\because \quad z(t) = x(t) + iy(t) = 3t + it^2$

$\therefore \quad f\big(z(t)\big) = \overline{z(t)} = x(t) - iy(t) = 3t - it^2$ and $z'(t) = x'(t) + iy'(t) = 3 + i2t$

Then by Equation (5.2.11)

$$\int_C \bar{z}\,dz = \int_{-1}^{4} f\big(z(t)\big)z'(t)dt = \int_{-1}^{4} (3t - it^2)(3 + i2t)dt = \int_{-1}^{4} (2t^3 + 9t + 3t^2 i)dt$$

By using Equation (5.2.4) in this Lecture, we have

$$\int_C \bar{z}\,dz = \int_{-1}^{4} (2t^3 + 9t)dt + i\int_{-1}^{4} 3t^2 dt = \left(2\cdot\frac{1}{4}\cdot t^4 + 9\cdot\frac{1}{2}\cdot t^2\right)\Bigg|_{-1}^{4} + i\cdot t^3\Big|_{-1}^{4} = 195 + 65i$$

EXAMPLE (例題) 5.2.2 **Evaluating a Contour Integral**

Evaluate $\oint_C \frac{1}{z} dz$, where C is the circle $x(t) = \cos t$, $y(t) = \sin t$, $0 \le t \le 2\pi$.

Solution (解答):

$$\because \quad z(t) = \cos t + i \sin t = e^{it}$$

$$\therefore \quad f\big(z(t)\big) = \frac{1}{z(t)} = \frac{1}{e^{it}} = e^{-it} \quad \text{and} \quad z'(t) = ie^{it}$$

Then by Equation (5.2.11)

$$\oint_C \frac{1}{z} dz = \int_0^{2\pi} f\big(z(t)\big) z'(t) dt = \int_0^{2\pi} \left(e^{-it}\right) ie^{it} dt$$

$$= i \int_0^{2\pi} e^{-it+it} dt = i \int_0^{2\pi} e^0 dt = i \int_0^{2\pi} dt = i \cdot t \Big|_0^{2\pi} = 2\pi i$$

Theorem 5.2 Properties of Contour Integrals

Suppose **the functions** f and g are **continuous in a domain** D, and C is a **smooth or piecewise smooth curve in** D. Then

(i) $\int_C qf(z)\,dz = q\int_C f(z)\,dz$, where q is a complex constant.

(ii) $$\int_C [f(z)+g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$$

(iii) $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$, where C consists of the smooth curves C_1 and C_2 joined end to end.

(iv) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$, where – C denotes the curve having the opposite Orientation (向き) of C.

EXAMPLE (例題) 5.2.3 C is a Piecewise Smooth Curve

Evaluate $\int_C (x^2 + iy^2)\,dz$, where C is the contour shown in Figure 5.20.

Solution (解答):

$f(z) = x^2 + iy^2$ From Theorem 5.2(iii), we have

$$\int_C (x^2 + iy^2)\,dz = \int_{C_1} (x^2 + iy^2)\,dz + \int_{C_2} (x^2 + iy^2)\,dz$$

From the Figure 5.20, we know the curves

① C_1 is $y = x$, when $0 \le x < 1$

Therefore, $(x(t), y(t))$ becomes $(x(t), x(t))$, it makes sense that we can directly use x as parameter, then

$z(x) = x + ix$ $z'(x) = 1 + i$

$f\big(z(x)\big) = x^2 + ix^2$

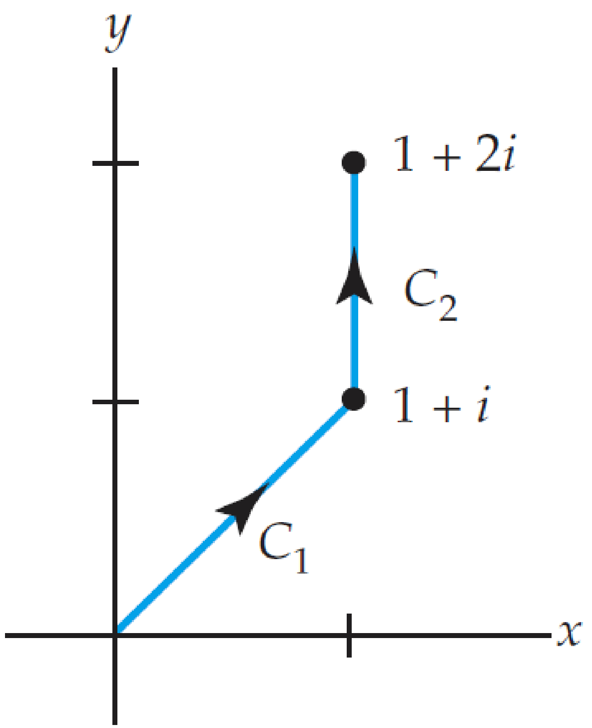


Figure 5.20 Contour $C = C_1 \cup C_2$ is piecewise-smooth

Solution (解答)(cont.):

Then by Equation (5.2.11)

$$\int_{C_1} (x^2+iy^2)\,dz = \int_0^1 f\big(z(x)\big)z'(x)\,dx = \int_0^1 (x^2+ix^2)(1+i\,)\,dx$$

$$= (1+i\,)^2 \int_0^1 x^2\,dx = (1+i\,)^2 \cdot \frac{1}{3} \cdot t^3 \Big|_0^1 = \frac{2}{3}i$$

② C_2 is $x=1$, when $1 \le y \le 2$

Therefore, $(x(t), y(t))$ becomes $\big(1, y(t)\big)$, it makes sense that we can directly use y as parameter, then

$z(y) = 1+iy \quad z'(y) = 0+i = i \quad f\big(z(y)\big) = 1+iy^2$

Then by Equation (5.2.11)

$$\int_{C_1} (x^2+iy^2)\,dz = \int_1^2 f\big(z(y)\big)z'(y)\,dy = \int_1^2 (1+iy^2)i\,dy = \underbrace{-\int_1^2 y^2\,dy + i\int_1^2 1\,dy}_{\text{by Equation (5.2.4)}} = -\frac{7}{3}+i$$

Combining the results of ① and ②, we have $\int_C (x^2+iy^2)\,dz = \frac{2}{3}i + \left(-\frac{7}{3}+\mathrm{i}\right) = -\frac{7}{3}+\frac{5}{3}i$

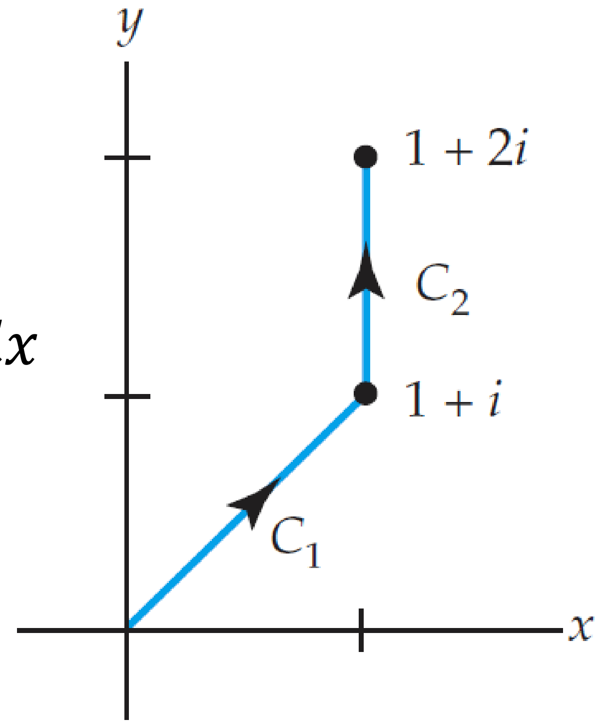


Figure 5.20 Contour $C = C_1 \cup C_2$ is piecewise-smooth

*Theorem 5.3 A Bounding (界) Theorem

If f is continuous on a smooth curve C and if modulus $|f(z)| \leq M$ for all z on C, Then $\left|\int_C f(z)\,dz\right| \leq ML$, where L is the length of C.

***EXAMPLE (例題) 5.2.4 A Bound for a Contour Integral**

Find an upper bound (上界) for the absolute value of $\oint_C \frac{e^z}{z+1}dz$, where C is the circle $|z| = 4$.

Solution (解答):

First, the length L (circumference (円周)) of the circle with radius (半径) $r = 4$ is $L = 2\pi r = 8\pi$.
Recall the inequality (1.2.7) $|z_1 + z_2| \geq |z_1| - |z_2|$ in Page 35 of the Lecture 1 Slides,
all points z on the circle that $|z + 1| \geq |z| - 1 = 4 - 1 = 3$

$$\left|\frac{e^z}{z+1}\right| \leq \frac{|e^z|}{|z| - 1} = \frac{|e^z|}{3}$$

Because $|e^z| = |e^x(\cos y + i \sin y)| = \sqrt{(e^x \cos y)^2 + (e^x \sin y)^2} = \sqrt{(e^x)^2(\cos^2 x + \sin^2 x)} = e^x$

Then for points on the circle $|z| = 4$, the maximum that $x = Re(z)$ can be is 4, therefore

$$\left|\frac{e^z}{z+1}\right| \leq \frac{e^4}{3}$$

From the Theorem 5.3, we have $$\left|\int_C \frac{e^z}{z+1}dz\right| \leq M \cdot L = \frac{e^4}{3} \cdot 8\pi = \frac{8\pi\, e^4}{3}$$

Review for Lecture 6

- Real Line Integral
- Piecewise Smooth Curve
- Simple, Closed Curve
- Complex Integral (Contour Integral)
- How to Compute Complex Integral

Assignment

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: The Section 5.1 and 5.2 of Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Calculus, 6th Edition, James Stewart, Thomas Brooks/Cole, 2009

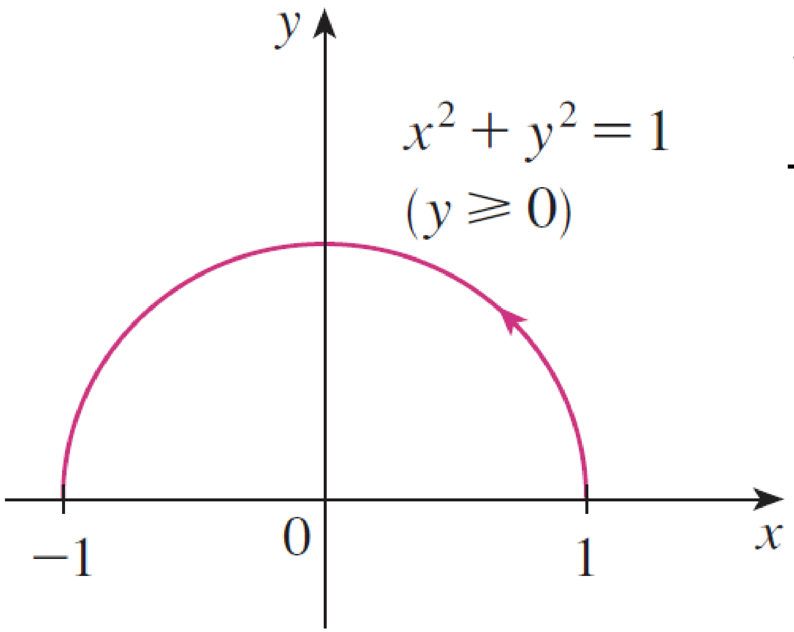
[3] Wikipedia

*6.1 Review of Real Line Integral (実·線積分)

EXAMPLE (例題) Recall Real Line Integral in *Calculus II (微積分 II)*

Evaluate $\int_C (2+x^2y)\,ds$, where C is the upper half of the unit circle $x^2+y^2=1$

Solution (解答):



Recall that the unit circle can be parametrized by

$$x=\cos t,\ y=\sin t$$

And the upper half of the circle is described by the parameter interval $0\le t\le \pi$

Therefore, from the formula in Page 14 of this lecture note, we have

$$\int_C (2+x^2y)ds=\int_0^\pi (2+\cos^2 t\sin t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

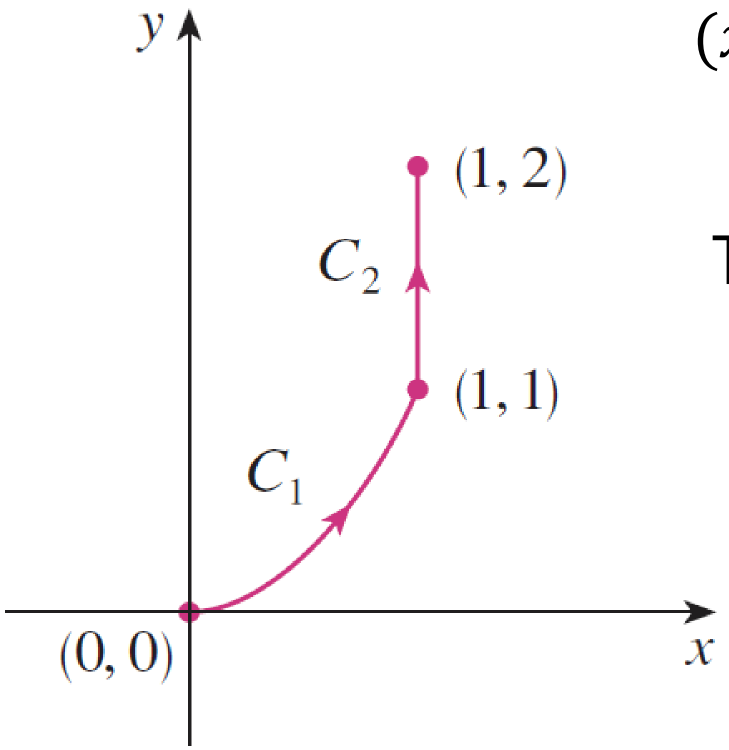
$$=\int_0^\pi (2+\cos^2 t\sin t)\sqrt{(-\sin t)^2+\cos^2 t}\,dt$$

$$=\int_0^\pi (2+\cos^2 t\sin t)=\left[2t-\frac{\cos^3 t}{3}\right]_0^\pi=2\pi+\frac{2}{3}$$

EXAMPLE (例題) Recall Real Line Integral in *Calculus II (微積分 II)*

Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

Solution (解答):



① Because from C_1 we know y is a function of x, i.e. the domain (x, y) becomes (x, x^2), so we can use x as the parameter, then

$$x = x, \quad y = x^2, \quad 0 \le x \le 1$$

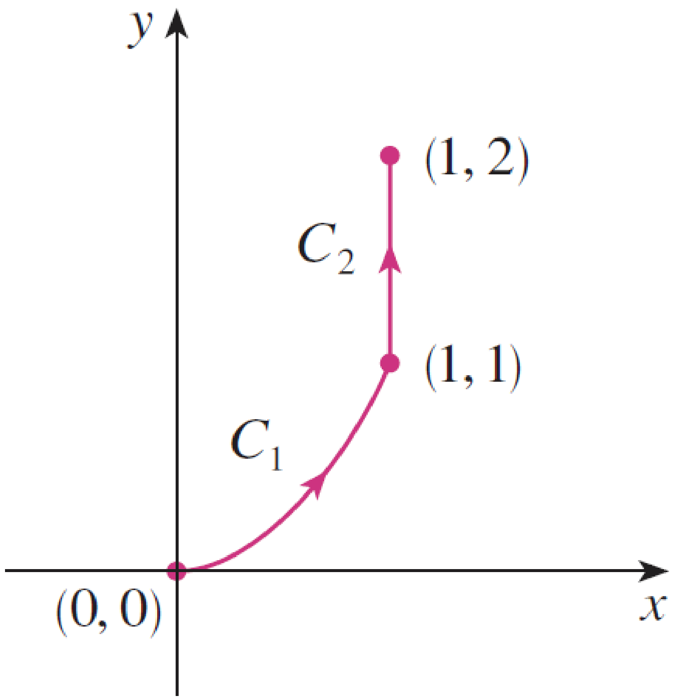
Therefore

$$\int_{C_1} 2x ds = \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 2x\sqrt{1 + (2x)^2}dx$$

$$= \left[\frac{1}{4}\cdot\frac{2}{3}(1 + 4x^2)^{\frac{3}{2}}\right]_0^1 = \frac{5\sqrt{5} - 1}{6}$$

*6.1 Review of Real Line Integral (実·線積分)

Solution (解答)(cont.):



② Because from C_2 we see a vertical line segment, so we can use y as the parameter, then

$$x = 1, \quad y = y, \quad 1 \le y \le 2$$

Therefore

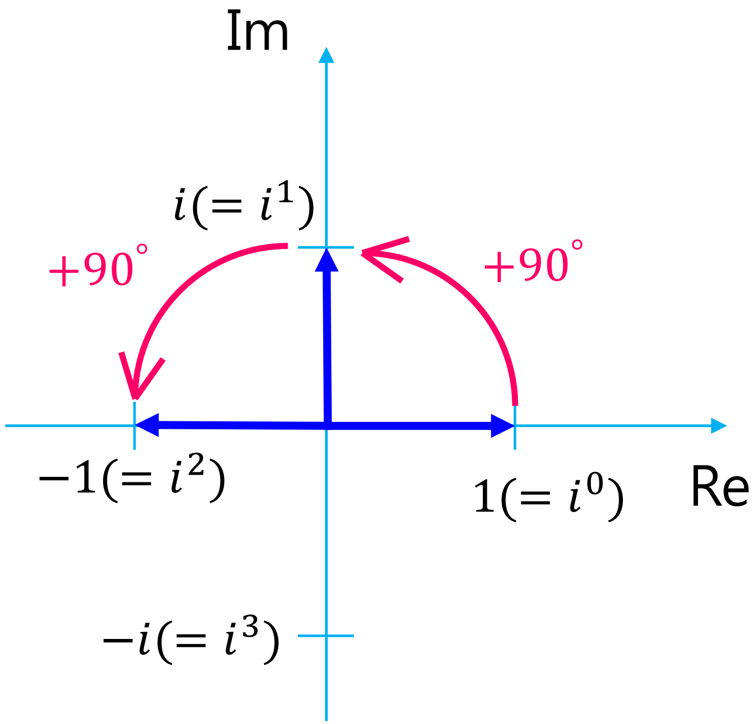
$$\int_{C_2} 2x ds = \int_1^2 2 \cdot 1 \cdot \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\, dy$$

$$= \int_1^2 2\sqrt{0+1}dy = \int_1^2 2dy = 2$$

$$\int_C 2x ds = \int_{C_1} 2x ds + \int_{C_2} 2x ds = \frac{5\sqrt{5}-1}{6} + 2$$

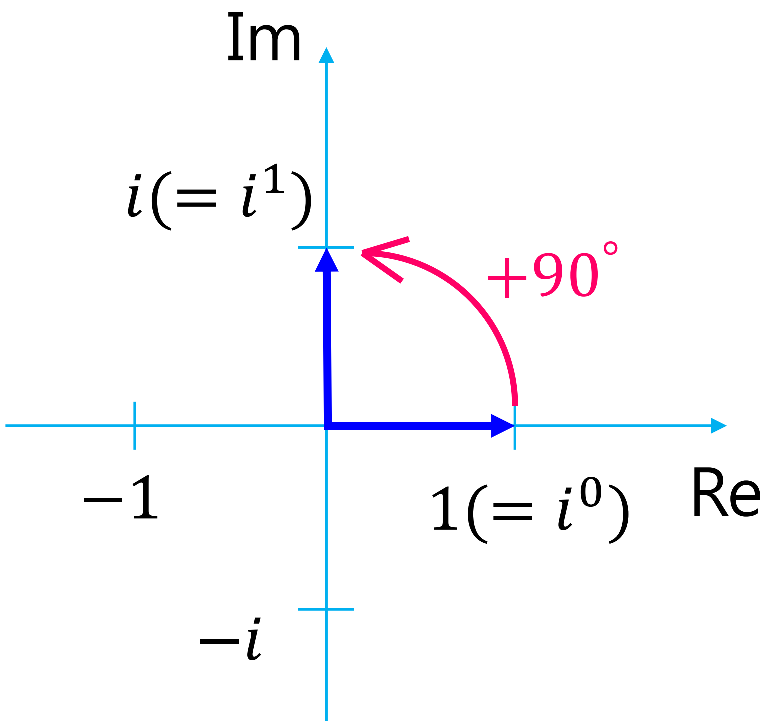
Application 2 of Complex Number :

Rotation



Application 2 of Complex Number : **Rotation**

$$i^0 \cdot i = i$$

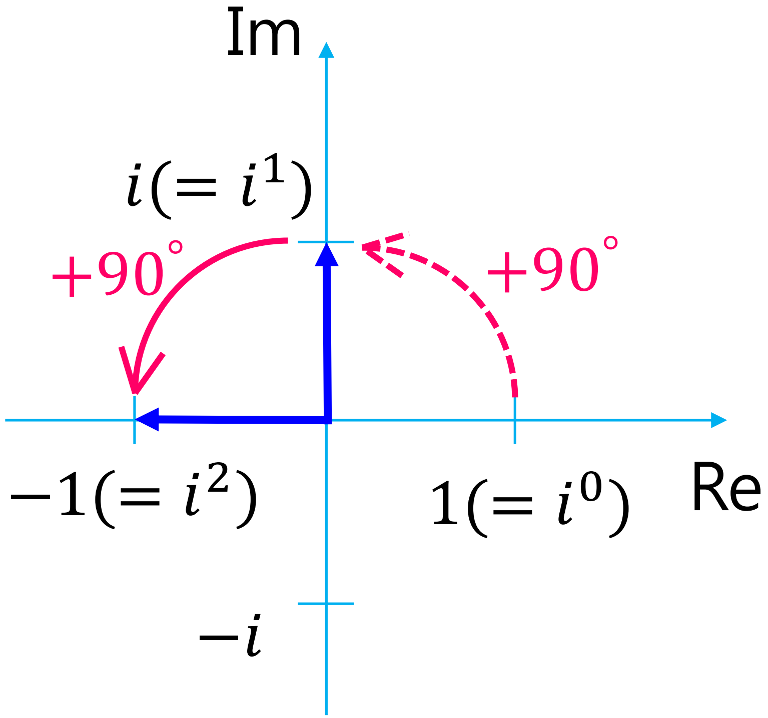


$\theta = 0^\circ$

$\theta = 90^\circ$

Application 2 of Complex Number : **Rotation**

$$i^0 \cdot i \cdot i = i^2 = -1$$

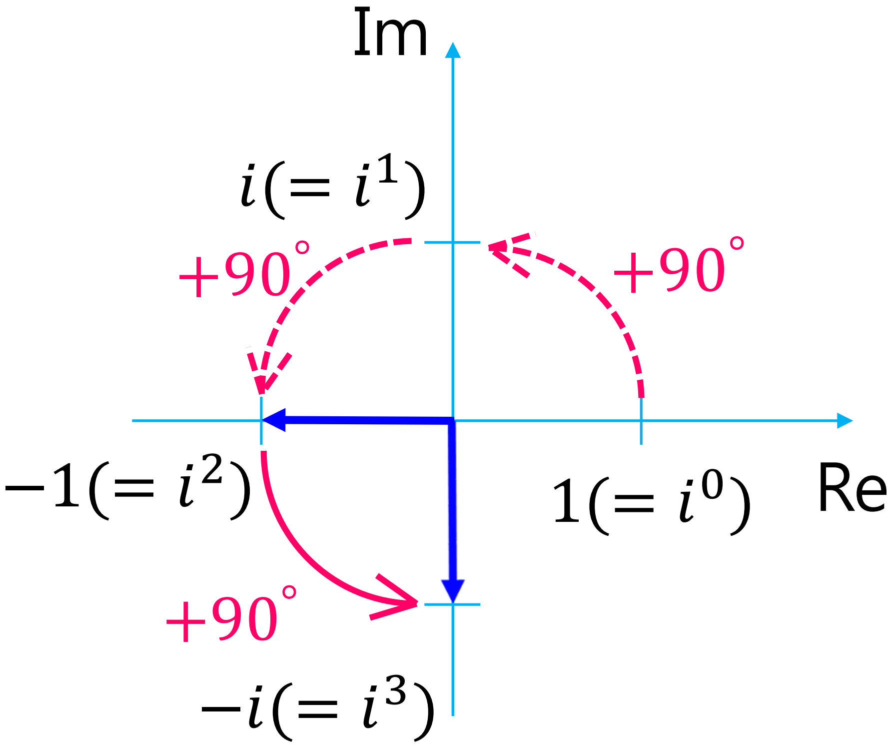


$\theta = 90^\circ$

$\theta = 180^\circ$

Application 2 of Complex Number : **Rotation**

$$i^0 \cdot i \cdot i \cdot i = i^3 = -i$$

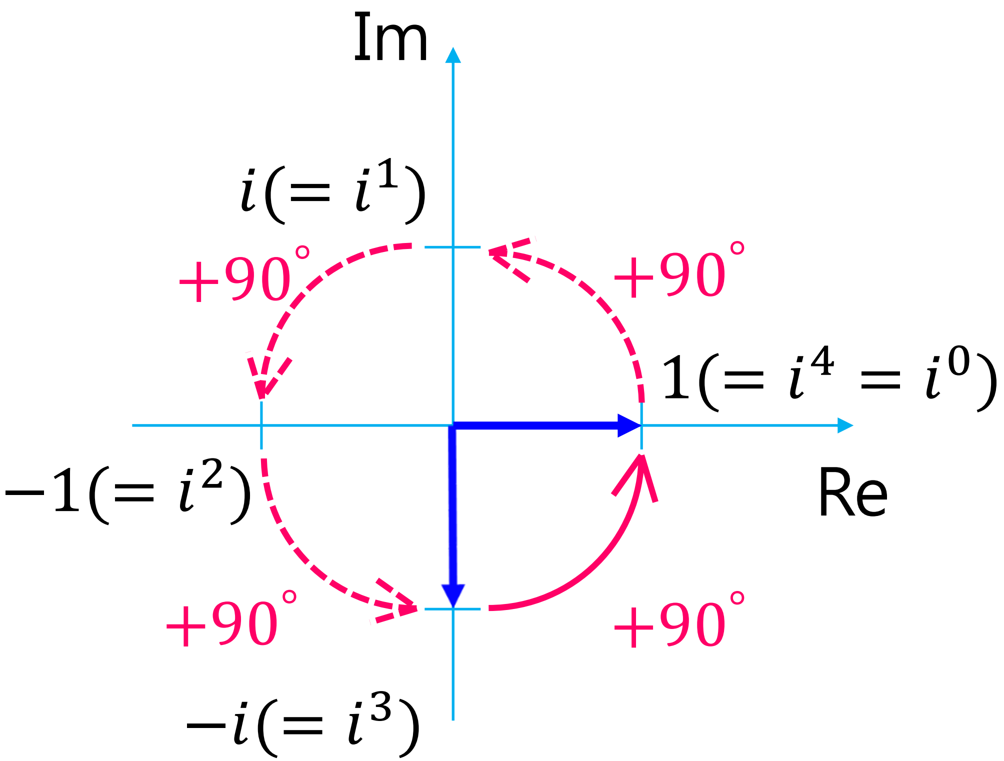


$\theta = 180^\circ$

$\theta = 270^\circ$

Application 2 of Complex Number : **Rotation**

$$i^0 \cdot i \cdot i \cdot i \cdot i = i^4 = i^0 = 1$$



$\theta = 270^\circ$

$\theta = 360^\circ$

Application 2 of Complex Number : **Rotation**

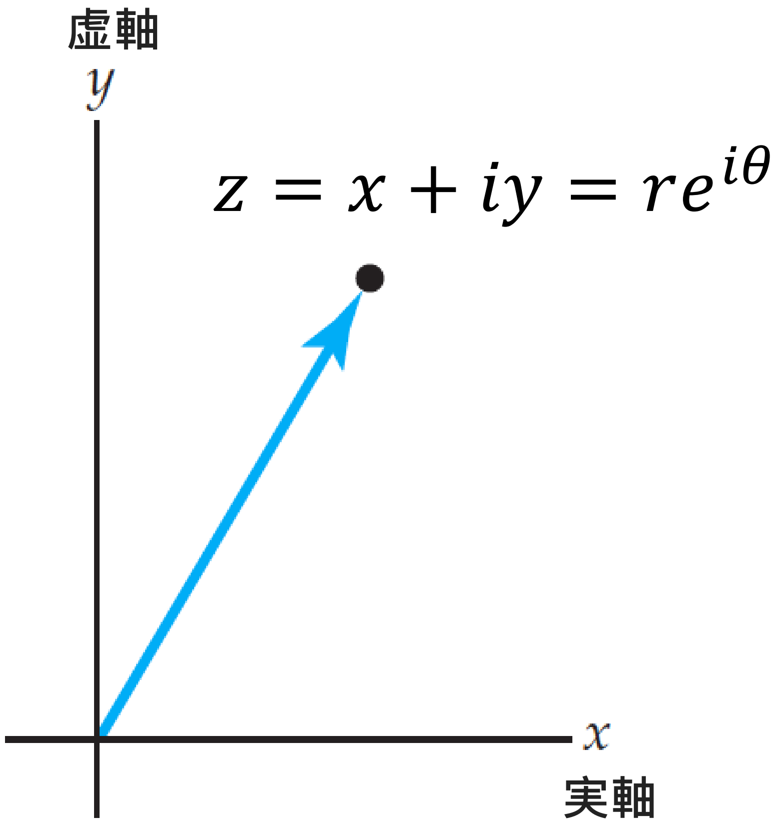


Figure 1.2 A vector z