Lecture 5

- Roots of a Complex Number
- Power Functions (複素冪函数)
- Trigonometric Functions (複素三角関数)
- Hyperbolic Functions (複素双曲線関数)

What you will learn in Lecture 5

5.1 Roots (根) of a Complex Number

5.2 (Complex) Elementary Functions (複素初等関数) 2:

5.2.1 (Complex) Power Functions (複素冪函数)

5.2.2 (Complex) Trigonometric Functions (複素三角関数)

5.2.3 (Complex) Hyperbolic Functions (複素双曲線関数)

5.1 Roots (根) of a Complex Number

Roots of a Complex Number

Consider to find z in $z^k = w$

where z and w are complex numbers,

k is real, i.e. NOT a complex number.

then

$$z = \sqrt[k]{|w|}\left[\cos\left(\frac{\arg(w) + 2n\pi}{k}\right) + i\sin\left(\frac{\arg(w) + 2n\pi}{k}\right)\right] \quad (1.4.4)$$

where $n = 0, 1, 2, \ldots, k-1$

Additional Point

Quadratic Formula (2次方程式の解の公式)

Suppose a, b, c, x are real.

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Two solutions when $b^2 - 4ac \neq 0$

Suppose a, b, c, z are complex.

$$az^2 + bz + c = 0 \Rightarrow z = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad (1.6.3)$$

Still **two solutions** because of the results of $\sqrt{b^2 - 4ac}$ when $b^2 - 4ac \neq 0$

Additional Point

Quadratic Formula (2次方程式の解の公式)

EXAMPLE (例題) 1.6.1 Solve the quadratic equation $z^2+(1-i)z-3i=0$.

Solution (解答):

From Equation (1.6.3), with $a=1$, $b=1-i$, and $c=-3i$ we have

$$z=\frac{-(1-i)+\sqrt{(1-i)^2-4\cdot 1\cdot(-3i)}}{2} \quad =\frac{1}{2}\left[-1+i+\sqrt{10i}\right]$$

To compute $\sqrt{10i}=(10i)^{\frac{1}{2}}$, we follow the Equation (1.4.4) in this Lecture, and obtain

$\left|(10i)^{\frac{1}{2}}\right|=\sqrt{10}$, $\arg\left((10i)^{\frac{1}{2}}\right)=\frac{\pi}{2}$, $k=2$, $n=0$ and $n=1$.

Therefore, $(10i)^{\frac{1}{2}}\Big|_{n=0}=\sqrt{10}\left(\cos\frac{\frac{\pi}{2}+2\cdot 0\cdot\pi}{2}+i\sin\frac{\frac{\pi}{2}+2\cdot 0\cdot\pi}{2}\right)=\sqrt{10}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$$=\sqrt{10}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=\sqrt{5}+i\sqrt{5}$$

$$\cancel{(10i)^{\frac{1}{2}}\Big|_{n=1}=\sqrt{10}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)}$$

Here $\frac{5\pi}{4}>\pi$, if we consider the principle value that satisfies $-\pi<\arg\left((10i)^{\frac{1}{2}}\right)\leq\pi$, then delete.

5.2.1 (Complex) Power Functions (複素冪函数)

5.2.1 (Complex) Power Functions (複素冪函数)

Complex Power Function (複素冪函数)

Suppose we know $z = e^{\ln z}$, for all nonzero **complex numbers** z,

(Because $\ln z = \ln e^{\ln z} = \ln z$)

then we have

Definition 4.4 Complex Power Function (複素冪函数)

If α is a complex number and $z \neq 0$, then the **multi-valued**

Complex Power Function (複素冪函数) is defined to be:

$$z^{\alpha} = e^{\alpha \ln z} \tag{4.2.1}$$

EXAMPLE (例題) 4.2.1 Complex Power Function

Find the values of the given complex power: (a) i^{2i} (b) $(1+i)^i$

Solution (解答):

(a) In part (a) of Example 4.1.3 in Lecture 4, we know

$$\ln i = \frac{(4n+1)\pi}{2} i \quad \text{, for } n = 0, \pm 1, \pm 2, \ldots$$

Thus, by identifying $z = i$ and $\alpha = 2i$ in Equation (4.2.1) we obtain:

$$i^{2i} = e^{2i \ln i} = e^{2i \cdot \frac{(4n+1)\pi}{2} i} = e^{-(4n+1)\pi} \quad \text{, for } n = 0, \pm 1, \pm 2, \ldots$$

For example, when $n = -1$, 0, and 1, the values of i^{2i} are 12391.6, 0.0432, and 1.507×10^{-7}, respectively.

Solution (解答)(cont.):

(**b**) From part (b) of Example 4.1.3 in Lecture 4, we know

$$\ln(1+i)=\frac{1}{2}\log_e 2+\frac{(8n+1)\pi}{4}i$$

Thus, by identifying $z=1+i$ and $\alpha=i$ in Equation (4.2.1) we obtain:

$$(1+i)^i=e^{i\ln(1+i)}=e^{i\cdot\left[\frac{1}{2}\log_e 2+\frac{(8n+1)\pi}{4}i\right]}\quad\text{, for } n=0,\pm1,\pm2,\dots$$

$$=e^{-\frac{(8n+1)\pi}{4}+\frac{i}{2}\log_e 2}\quad\text{, for } n=0,\pm1,\pm2,\dots$$

Properties (性質) of Complex Power Functions

Complex powers defined by (4.2.1) satisfy the following properties that are analogous to (類似する) properties of real powers:

$$\text{(i)}\quad z^{\alpha_1} z^{\alpha_2} = z^{\alpha_1+\alpha_2}$$

$$\text{(ii)}\quad \frac{z^{\alpha_1}}{z^{\alpha_2}} = z^{\alpha_1-\alpha_2} \qquad (4.2.5)$$

$$\text{(iii)}\quad (z^{\alpha})^n = z^{n\alpha},\ n = 0, \pm 1, \pm 2, \ldots$$

Definition 4.5 Principal Value (主値) of a Complex Power Function

If α is a complex number and $z \neq 0$, then the function defined by

$$z^{\alpha} = e^{\alpha \operatorname{Ln} z} \tag{4.2.6}$$

is called **the principal value of the complex power function** z^{α}.

EXAMPLE (例題) 4.2.2 Principal Value of a Complex Power Function

Find the principal value of each complex power: (a) $(-3)^{\frac{i}{\pi}}$ (b) $(2i)^{1-i}$

Solution (解答):

(a) For $z = -3$, we have $|z| = 3$ and $\mathrm{Arg}(-3) = \pi$, and so $\mathrm{Ln}\,(-3) = \log_e 3 + i\pi$ by Equation (4.1.14) in Lecture 4.

Thus, by identifying $z = -3$ and $\alpha = \frac{i}{\pi}$ in (4.2.6), we obtain:

$$(-3)^{\frac{i}{\pi}} = e^{\frac{i}{\pi}\mathrm{Ln}\,(-3)} = e^{\frac{i}{\pi}(\log_e 3 + i\pi)} = e^{-1+\frac{i\log_e 3}{\pi}} = e^{-1}\, e^{\frac{i\log_e 3}{\pi}}$$

$$= e^{-1}\left(\cos\frac{\log_e 3}{\pi} + i\sin\frac{\log_e 3}{\pi}\right)$$

$$\approx 0.3456 + 0.1260i$$

Solution (解答)(cont.):

(a) For $z = 2i$, we have $|z| = 2$ and $\mathrm{Arg}(z) = \frac{\pi}{2}$, and so $\mathrm{Ln}\,(2i) = \log_e 2 + i\,\frac{\pi}{2}$ by (4.1.14) in Lecture 4.

Thus, by identifying $z = 2i$ and $\alpha = 1 - i$ in (4.2.6), we obtain:

$$(2i)^{1-i} = e^{(1-i)\,\mathrm{Ln}\,2i} = e^{(1-i)\left(\log_e 2 + i\frac{\pi}{2}\right)} = e^{\log_e 2 + \frac{\pi}{2} - i\left(\log_e 2 - \frac{\pi}{2}\right)}$$

$$= e^{\log_e 2 + \frac{\pi}{2}} e^{-i\left(\log_e 2 - \frac{\pi}{2}\right)}$$

$$= e^{\log_e 2 + \frac{\pi}{2}}\left[\cos\left(\log_e 2 - \frac{\pi}{2}\right) - i\sin\left(\log_e 2 - \frac{\pi}{2}\right)\right]$$

$$\approx 6.1474 + 7.4008i$$

5.2.1 (Complex) Power Functions (複素冪函数)

Analyticity (解析性) of z^α

Since the function z^α is **continuous on the entire complex plane**, and since the function $\mathrm{Ln}\, z$ is **continuous on** the domain $|z| > 0, -\pi < \mathrm{Arg}(z) < \pi$, it follows that z^α is **continuous on** the domain $|z| > 0, -\pi < \mathrm{Arg}(z) < \pi$.

$$f_1(z) = e^{\alpha(\log_e |z| + i\,\mathrm{Arg}(z))},\ -\pi < \mathrm{Arg}(z) < \pi \qquad (4.2.7)$$

This is called the **principal branch of the complex power** z^α.

Therefore, on the domain $|z| > 0,\ -\pi < \mathrm{Arg}(z) < \pi$, the **principal value of the complex power** z^α is **differentiable** and

$$\boxed{\frac{d}{dz} z^\alpha = n z^{\alpha - 1},\ \text{where } \alpha \text{ is a complex number.}} \qquad (4.2.9)$$

EXAMPLE (例題) 4.2.3 Derivative of a Complex Power Function

Find the derivative of the principal value z^i at the point $z = 1 + i$.

Solution (解答):

(a) Because the point $z = 1 + i$ is in the domain $|z| > 0, -\pi < \mathrm{Arg}(z) < \pi$,

then from Equation (4.2.9) $\frac{d}{dz} z^i = i z^{i-1}$ we have

$$\left. \frac{d}{dz} z^i \right|_{z=1+i} = \left. i z^{i-1} \right|_{z=1+i} = i(1+i)^{i-1}$$

We can use (4.2.5) $z^{\alpha_1} z^{\alpha_2} = z^{\alpha_1 + \alpha_2}$ to rewrite this value as:

$$i(1+i)^{i-1} = i(1+i)^i (1+i)^{-1} = i(1+i)^i \frac{1}{1+i}$$

$$= \frac{i(1+i)}{(1+i)(1-i)} (1+i)^i = \frac{1+i}{2} (1+i)^i$$

Solution (解答)(cont.):

Moreover, from part (b) of Example 4.2.1 with $n = 0$, the principal value of $(1+i)^i$ is:

$$(1+i)^i = e^{\frac{i}{2}\log_e 2 - \frac{(8n+1)\pi}{4}}, \text{ for } n = 0 \text{ because } -\pi < \mathrm{Arg}(z) < \pi$$

$$= e^{-\frac{\pi}{4}+\frac{i}{2}\log_e 2}$$

Above all, we have

$$\left.\frac{d}{dz}z^i\right|_{z=1+i} = \frac{1+i}{2}(1+i)^i = \frac{1+i}{2}e^{-\frac{\pi}{4}+\frac{i}{2}\log_e 2} = \frac{1+i}{2}e^{-\frac{\pi}{4}}e^{\frac{i}{2}\log_e 2}$$

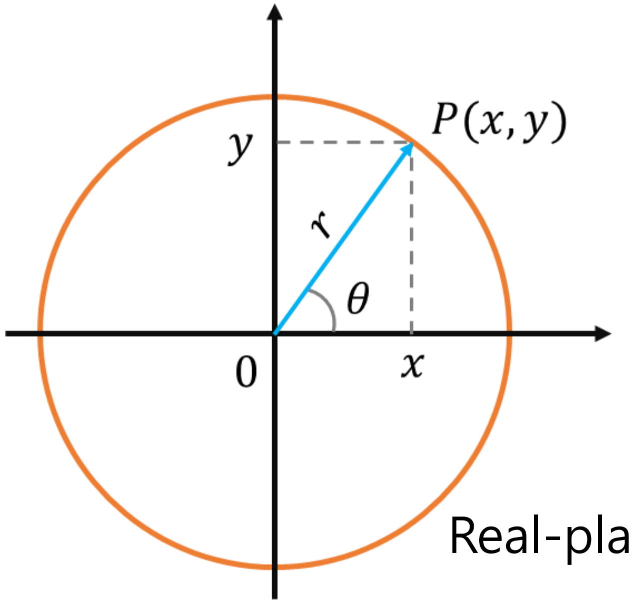
$$= \frac{1+i}{2}e^{-\frac{\pi}{4}}\left[\cos\left(\frac{1}{2}\log_e 2\right) - i\sin\left(\frac{1}{2}\log_e 2\right)\right]$$

$$\approx 0.1370 + 0.2919i$$

5.2.2 (Complex) Trigonometric Functions

(複素三角関数)

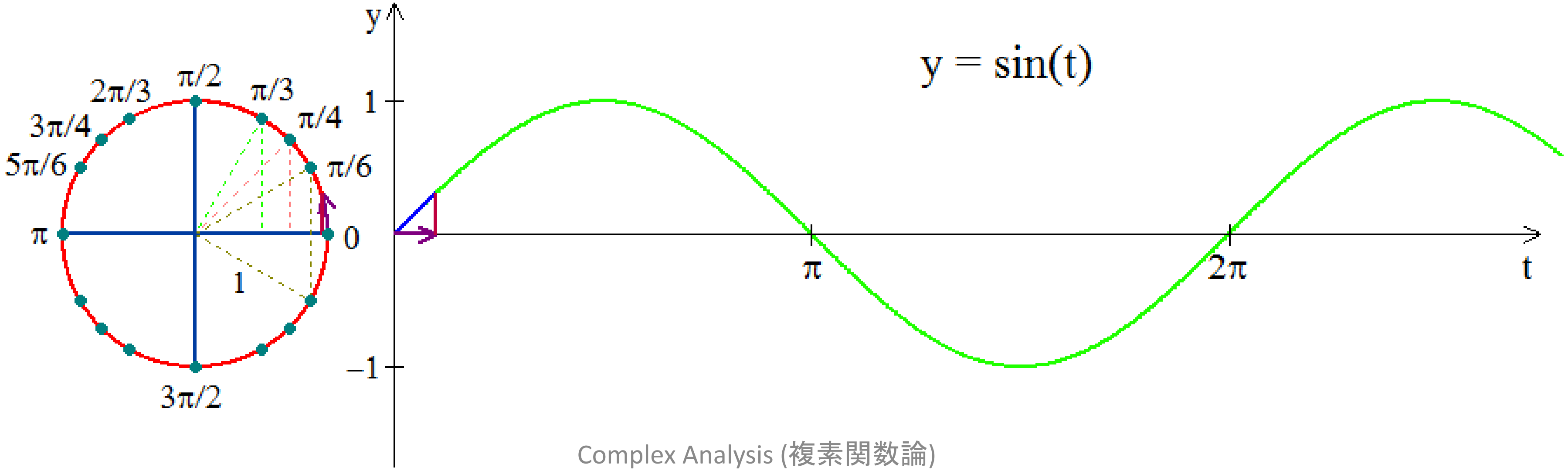
Recall Real Trigonometric Functions (実三角関数)



Real-plane

$$\sin\theta = \frac{y}{r} \qquad \tan\theta = \frac{y}{x} \qquad \sec\theta = \frac{r}{x}$$

$$\cos\theta = \frac{x}{r} \qquad \cot\theta = \frac{x}{y} \qquad \csc\theta = \frac{r}{y}$$



5.2.2 (Complex) Trigonometric Functions(複素三角関数)

Recall that in **MA04** ***Calculus II (微積分 II)***,

we have **Taylor Series** (テイラー級数) :

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

5.2.2 (Complex) Trigonometric Functions(複素三角関数)

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$ where x is a real number

Replace x with ix, then

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$$= 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots$$ because $i^2 = -1$

$$= 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

i.e. $e^{ix} = \cos x + i \sin x$ **Euler's Formula !**

5.2.2 (Complex) Trigonometric Functions(複素三角関数)

Leonhard Euler
レオンハルト・オイラー
(Switzerland) (1707~1783)

① $e^{ix} = \cos x + i \sin x$ **Euler's Formula**

② $e^{-ix} = \cos(-x) + i \sin(-x)$

$= \cos x - i \sin x$ (4.3.1)

① + ② $e^{ix} + e^{-ix} = 2\cos x$ ⇨ $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ (4.3.2)

① − ② $e^{ix} - e^{-ix} = i2\sin x$ ⇨ $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ (4.3.3)

Definition 4.6 Complex Sine and Cosine Functions (複素正弦関数と複素余弦関数)

The complex sine and cosine functions are defined by:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{4.3.4}$$

Analogous to (類似する) real trigonometric functions (実三角関数), we can define the complex tangent (複素正接), cotangent (複素余接), secant (複素正割), and cosecant (複素余割) functions by using the complex sine (複素正弦) and cosine (複素余弦):

$$\tan z = \frac{\sin z}{\cos z},\ \cot z = \frac{\cos z}{\sin z},\ \sec z = \frac{1}{\cos z} \text{ and } \csc z = \frac{1}{\sin z} \tag{4.3.5}$$

EXAMPLE (例題) 4.3.1 Values of Complex Trigonometric Functions

Express the value of the given trigonometric function in the form $a + ib$.

(a) $\cos i$ (b) $\sin(2+i)$ (c) $\tan(\pi - 2i)$

Solution (解答):

(a) By Equation (4.3.4),

$$\cos i = \frac{e^{i \cdot i} + e^{-i \cdot i}}{2} = \frac{e^{-1} + e}{2} \approx 1.5431$$

(b) By Equation (4.3.4),

$$\sin(2+i) = \frac{e^{i(2+i)} - e^{-i(2+i)}}{2i} = \frac{e^{-1+2i} - e^{1-2i}}{2i} = \frac{e^{-1}e^{2i} - e \cdot e^{-2i}}{2i}$$

$$= \frac{e^{-1}(\cos 2 + i \sin 2) - e(\cos(-2) + i \sin(-2))}{2i}$$

$$\approx 1.4031 - 0.4891i$$

Solution (解答)(cont.):

(c) By Equation (4.3.5) and (4.3.4),

$$\tan(\pi - 2i) = \frac{\sin(\pi - 2i)}{\cos(\pi - 2i)} = \frac{\dfrac{e^{i(\pi-2i)} - e^{-i(\pi-2i)}}{2i}}{\dfrac{e^{i(\pi-2i)} + e^{-i(\pi-2i)}}{2}}$$

$$= \frac{e^{i(\pi-2i)} - e^{-i(\pi-2i)}}{\left(e^{i(\pi-2i)} + e^{-i(\pi-2i)}\right)i} = \frac{\left(e^{2}e^{i\pi} - e^{-2}e^{-i\pi}\right)i}{\left(e^{2}e^{i\pi} + e^{-2}e^{-i\pi}\right)i \cdot i}$$

$$= -\frac{\left(e^{2}(\cos\pi + i\sin\pi) - e^{-2}(\cos\pi - i\sin\pi)\right)i}{\left(e^{2}(\cos\pi + i\sin\pi) + e^{-2}(\cos\pi - i\sin\pi)\right)}$$

$$= -\frac{-(e^{2} - e^{-2})i}{-(e^{2} + e^{-2})} \approx -0.9640i$$

Complex Trigonometric Identities (複素三角恒等式)

(i) $\sin(-z) = -\sin z \qquad \cos(-z) = \cos z$ Parity (偶奇性) (4.3.6)

(ii) $\sin^2 z + \cos^2 z = 1$ (4.3.7)

(iii) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$ (4.3.8)

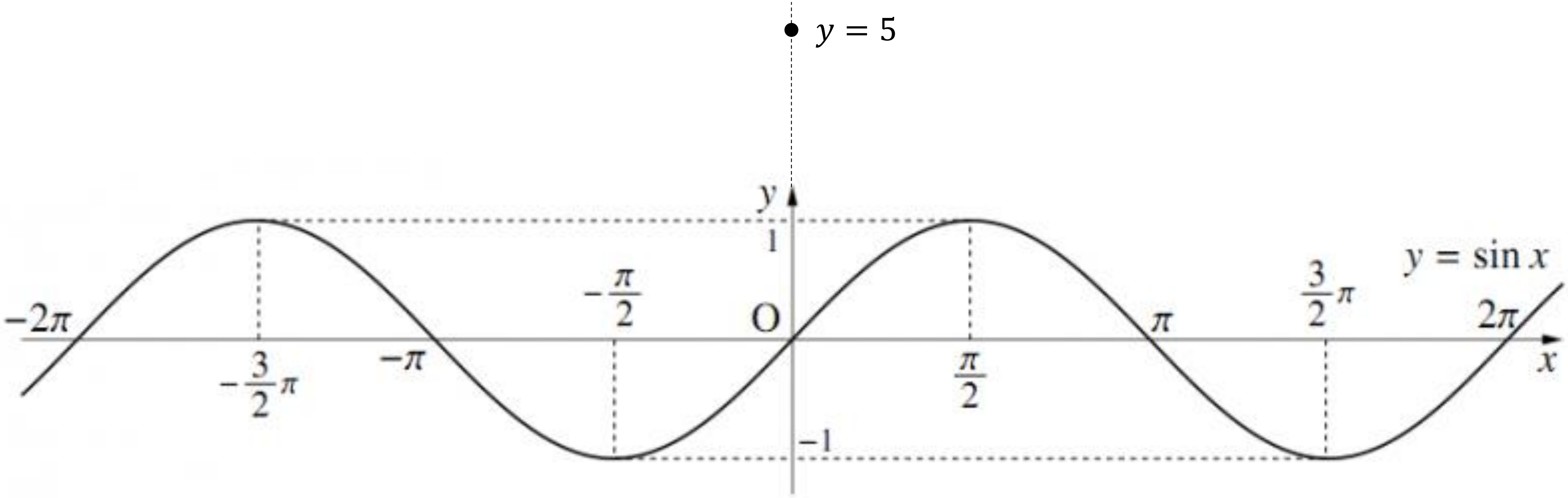
(iv) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$ (4.3.9)

(v) $\sin 2z = 2 \sin z \cos z \qquad \cos 2z = \cos^2 z - \sin^2 z$ (4.3.10)

(vi) $\sin(z + 2\pi) = \sin z \qquad \cos(z + 2\pi) = \cos z$ Periodicity (周期性) (4.3.11)

5.2.2 (Complex) Trigonometric Functions(複素三角関数)

Recall the $\sin x$, where $x \in \mathbf{R}$ (i.e. x is a real number (実数).)



Notice: We cannot find a solution for $\sin x = 5$,

when $x \in \mathbf{R}$, because $|\sin x| \leq 1$.

EXAMPLE (例題) 4.3.2 Solving Complex Trigonometric Equations
Find all solutions to the equation $\sin z = 5$, where $z \in \mathbf{C}$.

Solution (解答):

By Definition 4.6,

$$\sin z = \boxed{\frac{e^{iz} - e^{-iz}}{2i} = 5}$$

Multiplying this equation by e^{iz}

$$\frac{e^{iz}\left(e^{iz} - e^{-iz}\right)}{2i} = 5e^{iz}$$

$$e^{iz+iz} - e^{iz-iz} = 10ie^{iz}$$

$$e^{i2z} - e^{0} = 10ie^{iz}$$

$$e^{i2z} - 10ie^{iz} - 1 = 0$$

$$\left(e^{iz}\right)^2 - 10i\left(e^{iz}\right) - 1 = 0$$

From the quadratic formula (1.6.3) in this Lecture that the solutions of $az^2 + bz + cz = 0$ are given by

$$e^{iz} = \frac{-(-10i) + \sqrt{(-10i)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$$

$$= \frac{10i + \sqrt{-96}}{2}$$

$$= 5i \pm 2\sqrt{6}i$$

$$= \left(5 \pm 2\sqrt{6}\right)i$$

Solution (解答)(cont.):

- For $e^{iz} = (5 + 2\sqrt{6})i$

$$iz = \ln[(5+2\sqrt{6})i]$$

$$z = -i\ln[(5+2\sqrt{6})i]$$

$$= -i[\log_e|(5+2\sqrt{6})i| + i\arg[(5+2\sqrt{6})i]]$$

Because $(5+2\sqrt{6})i$ is a pure imaginary number (純虚数) and $5+2\sqrt{6} > 0$, we have $\arg[(5+2\sqrt{6})i] = \frac{\pi}{2} + 2n\pi$

$$= -i\left[\log_e(5+2\sqrt{6}) + i\left(\frac{\pi}{2}+2n\pi\right)\right] \quad \text{, for } n = 0, \pm1, \pm2, \ldots$$

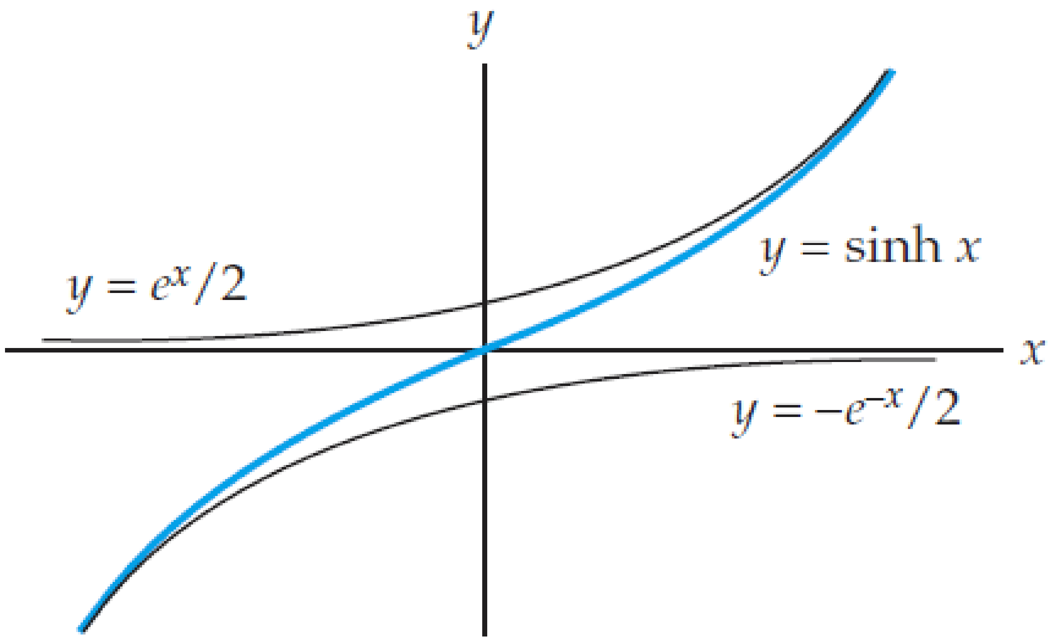
$$= \frac{(4n+1)\pi}{2} - i\log_e(5+2\sqrt{6}) \quad \text{, for } n = 0, \pm1, \pm2, \ldots$$

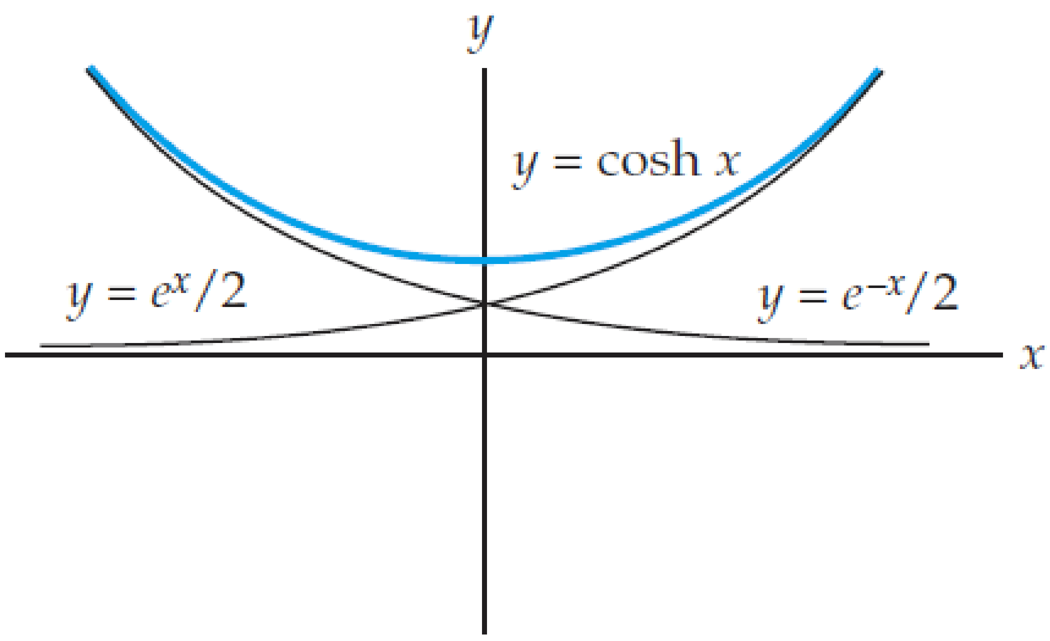
- For $e^{iz} = (5 - 2\sqrt{6})i$, similarly we have

$$z = \frac{(4n+1)\pi}{2} - i\log_e(5-2\sqrt{6}) \quad \text{, for } n = 0, \pm1, \pm2, \ldots$$

5.2.2 (Complex) Trigonometric Functions(複素三角関数)

Recall the real hyperbolic functions (実双曲線関数).





(a) $y = \sinh x = \dfrac{e^{x} - e^{-x}}{2}$ (b) $y = \cosh x = \dfrac{e^{x} + e^{-x}}{2}$

Figure 4.11 The real hyperbolic functions, $x \in \mathbf{R}$

Modulus (複素数の絶対値) of Sine and Cosine Functions

$$\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i}$$

$$= \sin x\left(\frac{e^{y} + e^{-y}}{2}\right) + i\cos x\left(\frac{e^{y} - e^{-y}}{2}\right)$$

$$= \sin x \cosh y + i \cos x \sinh y \qquad \text{where } z = x + iy \qquad (4.3.16)$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y \qquad (4.3.17)$$

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}$$

$$= \sqrt{\sin^2 x\,(1 + \sinh^2 y\,) + \cos^2 x \sinh^2 y} \qquad \text{Because } \cosh^2 y = 1 + \sinh^2 y$$

$$= \sqrt{\sin^2 x + (\sin^2 x + \cos^2 x)\sinh^2 y}$$

$$= \sqrt{\sin^2 x + \sinh^2 y} \qquad \text{Because } \sin^2 x + \cos^2 x = 1 \qquad (4.3.18)$$

$$|\cos z| = \sqrt{\cos^2 x + \sinh^2 y} \qquad (4.3.19)$$

Zeros

$\sin z = 0$ if and only if $z = n\pi$, for $n = 0, \pm 1, \pm 2, \ldots$

$\cos z = 0$ if and only if $z = \frac{(2n+1)\pi}{2}$, for $n = 0, \pm 1, \pm 2, \ldots$

EXAMPLE (例題) **Find the solution to the equation $\sin z = 0$.**

Solution (解答):

- Method 1: Use the similar way in Example 4.3.2.
- Method 2: Recall the Zero in Lecture 1 (Slide Page 23), a complex number is equal to 0 if and only if its modulus is 0, then

$$|\sin z| = 0$$

$$\sqrt{\sin^2 x + \sinh^2 y} = 0 \Rightarrow \begin{cases} \sin^2 x = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi, \text{ for } n = 0, \pm 1, \pm 2, \ldots \\ \sinh^2 y = 0 \Rightarrow \sinh y = 0 \Rightarrow y = 0 \quad \text{According to the Figure 4.11(a)} \end{cases}$$

Therefore, $z = x + iy = n\pi + i0 = n\pi$, for $n = 0, \pm 1, \pm 2, \ldots$

Derivatives of Complex Trigonometric Functions

$$\frac{d}{dx}\sin z = \cos z \qquad \frac{d}{dx}\cos z = -\sin z$$

$$\frac{d}{dx}\tan z = \sec^2 z \qquad \frac{d}{dx}\cot z = -\csc^2 z$$

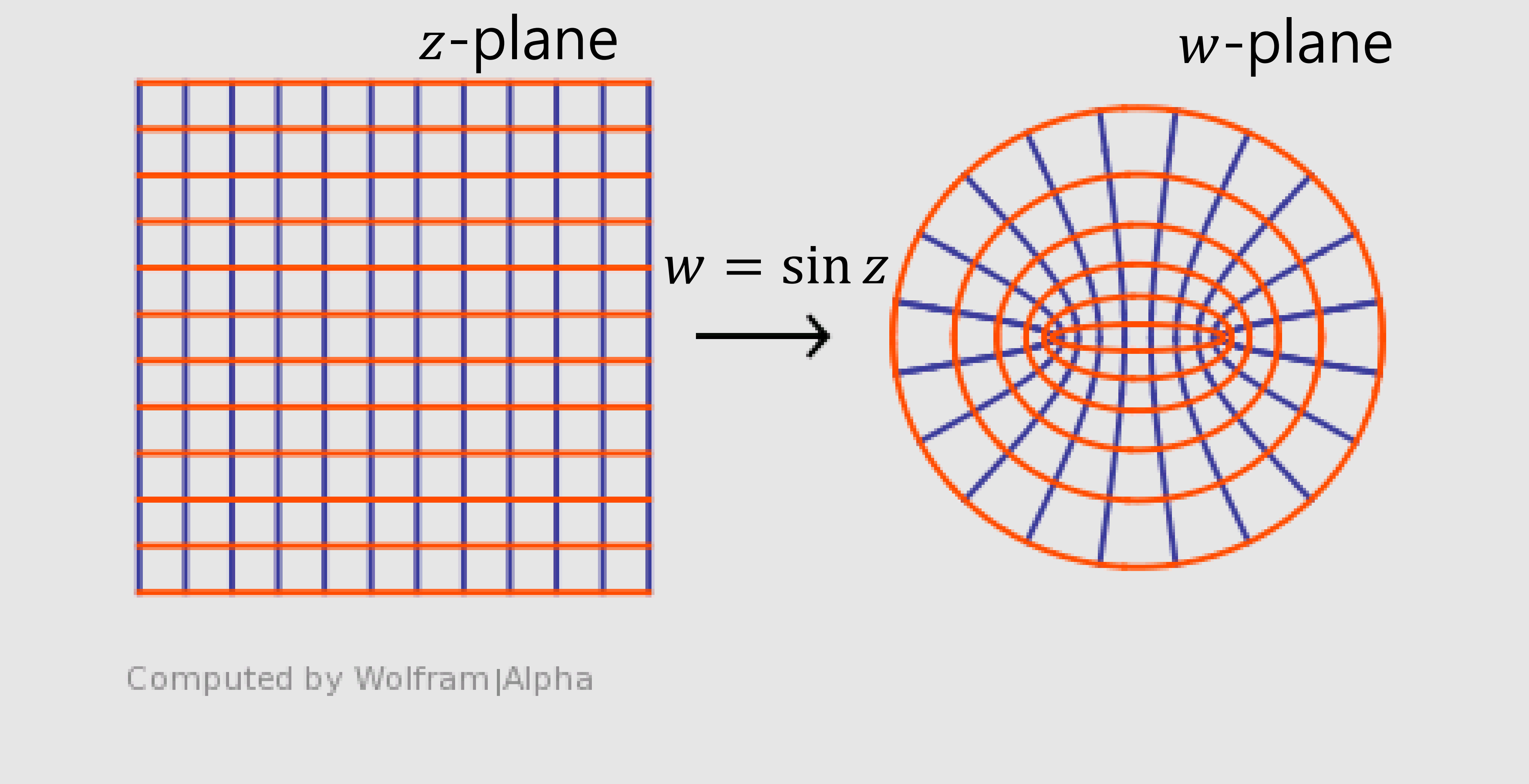
$$\frac{d}{dx}\sec z = \sec z \tan z \qquad \frac{d}{dx}\csc z = -\csc z \cot z$$

Proof for $\frac{d}{dz}\sin z$ **:**

$$\frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

- The $\sin z$ and $\cos z$ are entire (整函数).
- But $\tan z$, $\cot z$, $\sec z$, and $\csc z$ are only analytic (解析的) at those points where the denominator (分母) is nonzero (非ゼロ).

Trigonometric Mapping



Inverse Trigonometric Functions (複素数の逆三角関数)

Definition 4.8 (Complex) Inverse Sine Function (複素逆正弦関数)

The multiple-valued function $\sin^{-1} z$ defined by

$$\sin^{-1} z = -i \ln\left[iz + (1 - z^2)^{1/2}\right] \tag{4.4.3}$$

is called the **(complex) inverse sine**.

EXAMPLE (例題) 4.4.1 Values of Complex Inverse Sine

Find all values of $\sin^{-1}\sqrt{5}$.

Solution (解答):

By Equation (4.4.3),

$$\sin^{-1}\sqrt{5} = -i\ln\left[i\sqrt{5} + \left(1-\left(\sqrt{5}\right)^2\right)^{1/2}\right]$$

$$= -i\ln\left[i\sqrt{5} + (-4)^{1/2}\right]$$

$$= -i\ln\left[i\sqrt{5} \pm 2i\right]$$

$$= -i\ln\left[\left(\sqrt{5} \pm 2\right)i\right]$$

Similar to the Example 4.3.2,

$$\ln\left[\left(\sqrt{5} \pm 2\right)i\right] = \log_e\left(\sqrt{5} \pm 2\right) + i\left(\frac{\pi}{2} + 2n\pi\right) \quad , n = 0, \pm 1, \pm 2, \ldots$$

Solution (解答)(cont.):

$$\because \quad \log_e\left(\sqrt{5}-2\right)=\log_e\frac{\left(\sqrt{5}+2\right)\left(\sqrt{5}-2\right)}{\sqrt{5}+2}=\log_e\frac{1}{\sqrt{5}+2}=\log_e 1-\log_e\left(\sqrt{5}+2\right)=0-\log_e\left(\sqrt{5}+2\right)$$

$$\therefore \quad \ln\left[\left(\sqrt{5}\pm 2\right)i\right]=\log_e\left(\sqrt{5}\pm 2\right)+i\left(\frac{\pi}{2}+2n\pi\right)=\pm\log_e\left(\sqrt{5}+2\right)+i\left(\frac{\pi}{2}+2n\pi\right)$$

Therefore,

$$\begin{aligned}\sin^{-1}\sqrt{5}&=-i\ln\left[\left(\sqrt{5}\pm 2\right)i\right]\\&=-i\left[\pm\log_e\left(\sqrt{5}+2\right)+i\left(\frac{\pi}{2}+2n\pi\right)\right]\\&=\frac{(4n+1)\pi}{2}\pm i\log_e\left(\sqrt{5}+2\right)\qquad , n=0,\pm 1,\pm 2,\ldots\end{aligned}$$

Definition 4.9 (Complex) Inverse Cosine and Tangent Function

The multiple-valued function $\cos^{-1} z$ defined by

$$\cos^{-1} z = -i \ln\left[z + i(1 - z^2)^{1/2}\right] \tag{4.4.4}$$

is called the **(complex) inverse cosine**.

The multiple-valued function $\tan^{-1} z$ defined by

$$\tan^{-1} z = \frac{i}{2} \ln\left(\frac{i+z}{i-z}\right) \tag{4.4.5}$$

is called the **(complex) inverse tangent**.

*5.2.3 (Complex) Hyperbolic Functions

(複素双曲線関数)

*5.2.3 (Complex) Hyperbolic Functions (複素双曲線関数)

Definition 4.7 Complex Hyperbolic Sine and Cosine

The complex hyperbolic sine and hyperbolic cosine functions are defined by:

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2} \qquad (4.3.25)$$

The complex hyperbolic **tangent, cotangent, secant, and cosecant** are defined in terms of $\sinh z$ and $\cosh z$.

$$\tanh z = \frac{\sinh z}{\cosh z}, \ \coth z = \frac{\cosh z}{\sinh z}, \ \operatorname{sech} z = \frac{1}{\cosh z} \ \text{and} \ \operatorname{csch} z = \frac{1}{\sinh z} \qquad (4.3.26)$$

*Derivatives of Complex Hyperbolic Functions

$$\frac{d}{dx}\sinh z = \cosh z \qquad \frac{d}{dx}\cosh z = \sinh z$$

$$\frac{d}{dx}\tanh z = \operatorname{sech}^2 z \qquad \frac{d}{dx}\coth z = -\operatorname{csch}^2 z$$

$$\frac{d}{dx}\operatorname{sech} z = -\operatorname{sech} z \tanh z \qquad \frac{d}{dx}\operatorname{csch} z = -\operatorname{csch} z \coth z$$

Review for Lecture 5

- Roots () of a Complex Number
- Quadratic Formula (2次方程式の解の公式)
- (Complex) Power Functions (複素冪函数)
- (Complex) Trigonometric Functions (複素三角関数)
- (Complex) Hyperbolic Functions (複素双曲線関数)

Assignment

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials:

(1) The Section 4.2, 4.3 and 4.4 of Textbook

(2) 複素三角関数 ～ 単位円の束縛を超えて

http://taketo1024.hateblo.jp/entry/complex-trigonometric

References

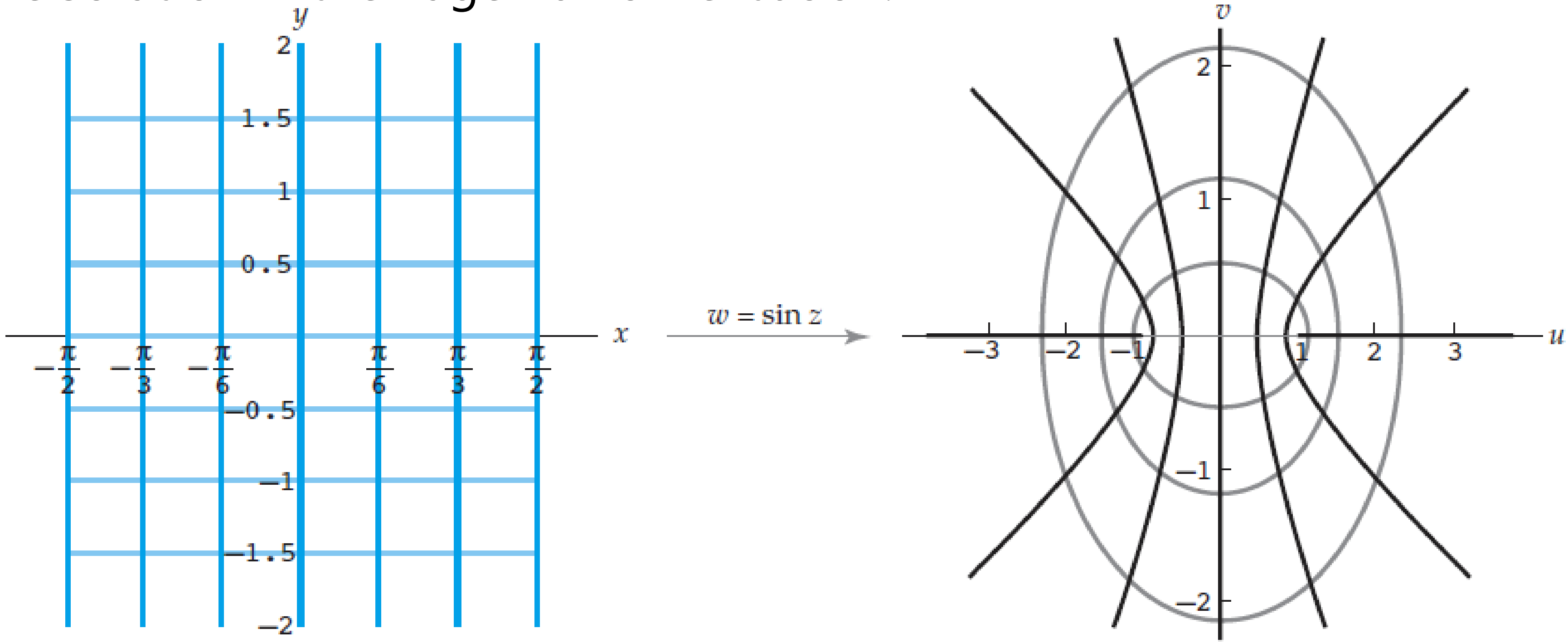
[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia

***EXAMPLE (例題) 4.3.3 The Mapping $w = \sin z$**

Describe the image of the region $-\pi/2 \le x \le \pi/2, -\infty < y < \infty$, under the complex mapping $w = \sin z$.

Hint: Read the solution in the Page 207 of Textbook.



(a) The region $-\pi/2 \le x \le \pi/2$ (b) Image of the region in (a)

Figure 4.12 The Mapping $w = \sin z$

Appendix (付録)

5.2.2 (Complex) Trigonometric Functions(複素三角関数)

*Derivatives of Branches $\sin^{-1} z$, $\cos^{-1} z$ and $\tan^{-1} z$

$$\frac{d}{dx}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}} \qquad (4.4.7)$$

$$\frac{d}{dx}\cos^{-1} z = -\frac{1}{(1-z^2)^{1/2}} \qquad (4.4.8)$$

$$\frac{d}{dx}\tan^{-1} z = \frac{1}{1+z^2} \qquad (4.4.9)$$

Hint: Read the Section “Branches and Analyticity” in the Page 217 of Textbook.

Appendix (付録)

*5.2.3 (Complex) Hyperbolic Functions (複素双曲線関数)

Relation To Sine and Cosine

(i) $$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} = i\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = i \sin z$$

$$\left.\begin{aligned} &\sin z = -i \sinh(iz) \\ \text{(ii)}\quad &\cos z = \cosh(iz) \end{aligned}\right\} \qquad (4.3.27)$$

$$\left.\begin{aligned} \text{(iii)}\quad &\sinh z = -i \sin(iz) \\ \text{(iv)}\quad &\cosh z = \cos(iz) \end{aligned}\right\} \qquad (4.3.28)$$

(v) $$\tan(iz) = \frac{\sin(iz)}{\cos(iz)} = \frac{i \sinh z}{\cosh z} = i \tanh z$$

Appendix (付録)

*5.2.3 (Complex) Hyperbolic Functions (複素双曲線関数)

Properties of Complex Hyperbolic Functions

(i) $\sinh(-z) = -\sinh z \qquad \cosh(-z) = \cosh z$ (4.3.29)

(ii) $\cosh^2 z - \sinh^2 z = 1$ (4.3.30)

(iii) $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$ (4.3.31)

(iv) $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$ (4.3.32)

Appendix (付録)

*5.2.3 (Complex) Hyperbolic Functions (複素双曲線関数)

***EXAMPLE (例題) 4.3.4 A Hyperbolic Identity**

Verify that $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

Solution (解答):

By Equation (4.3.28), $\cosh(z_1 + z_2) = \cos(iz_1 + iz_2)$, and so by the trigonometric identity (4.3.9) and additional (4.3.27) and (4.3.28), we have

$$\begin{aligned}\cosh(z_1 + z_2) &= \cos(iz_1 + iz_2)\\ &= \cos iz_1 \cos iz_2 - \sin iz_1 \sin iz_2\\ &= \cos iz_1 \cos iz_2 + (-i \sin iz_1)(-i \sin iz_2)\\ &= \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2\end{aligned}$$

Appendix (付録)

We know the trigonometric identity

$$\cos x \cdot \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

Because $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ then we have

$$\begin{aligned}
\cos x \cdot \cos y &= \frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^{iy} + e^{-iy}}{2} \\
&= \frac{1}{2} \cdot \frac{e^{i(x+y)} + e^{i(x-y)} + e^{i(-x+y)} + e^{i(-x-y)}}{2} \\
&= \frac{1}{2}\left(\underbrace{\frac{e^{i(x+y)} + e^{-i(x+y)}}{2}}_{\cos(x+y)} + \underbrace{\frac{e^{i(x-y)} + e^{-i(x-y)}}{2}}_{\cos(x-y)}\right).
\end{aligned}$$