Lecture 11

Laurent Series（ローラン級数）

What you will learn in Lecture 11

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Singular Points (特異点)

In general, a point z at which a complex function $w = f(z)$ **fails (失敗する) to be analytic** is called a **singular point (特異点)** of f.

For example, the complex numbers $z = 2i$ and $z = -2i$ are **singularities (特異点)** of the function $f(z) = z/(z^2 + 4)$ because f is **discontinuous (不連続の)** at each of these points.

In this section we will be concerned with a new kind of "power series" expansion (べき級数展開) of f about an **isolated singularity** z_0. This new series will involve **negative** as well as **nonnegative integer powers of** $z - z_0$.

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Suppose that $z = z_0$ is a **singularity** of a complex function f.

Isolated Singularity (孤立特異点)

The point $z = z_0$ is said to be **an isolated singularity** of the function f if there exists **some deleted neighborhood**, or **punctured open disk**, $0 < |z - z_0| < R$ of z_0 throughout which f is analytic.

For example, we have just seen that $z = 2i$ and $z = -2i$ are singularities of $f(z) = z/(z^2 + 4)$. Both $2i$ and $-2i$ are isolated singularities since f is analytic at every point in the neighborhood defined by $|z - 2i| < 1$, except at $z = 2i$, and at every point in the neighborhood defined by $|z - (-2i)| < 1$, except at $z = -2i$.

In other words, f is analytic in the deleted neighborhoods $0 < |z - 2i| < 1$ and $0 < |z + 2i| < 1$.

Nonisolated Singularity

We say that a singular point $z = z_0$ of a function f is **nonisolated** if every neighborhood of z_0 contains at least one singularity of f other than z_0.

For example, the branch point $z = 0$ is a nonisolated singularity of $\mathrm{Ln}\, z$ since every neighborhood of $z = 0$ contains points on the negative real axis. (Check Page 32 of Lecture 4 Notes)

$$f_1(z) = \log_e |z| + i\,\mathrm{Arg}(z),\ -\pi < \mathrm{Arg}(z) < \pi \qquad (4.1.19)$$

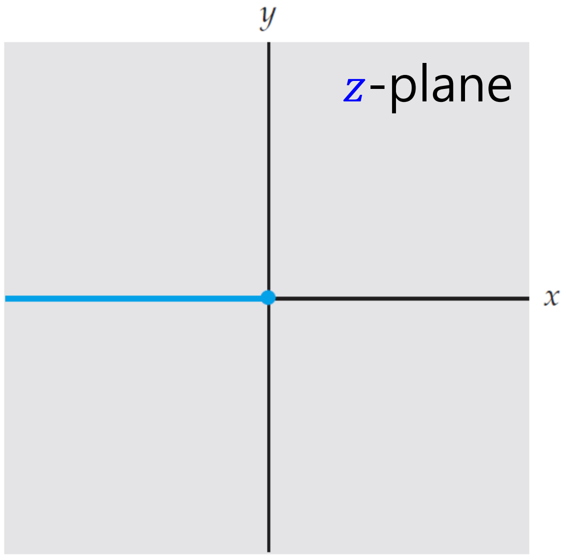

Figure 4.6 $f_1(z)$ defines on domain in gray color

Here, $f_1(z)$ is $\mathrm{Ln}\, z$ except $\mathrm{Arg}(z) = \pi$

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A New Kind of Series

If $z = z_0$ is a singularity of a function f, then certainly f **cannot be expanded in a power series with z_0 as its center**.

However, about **an isolated singularity** $z = z_0$, it is possible to **represent f by a series involving both negative and nonnegative integer powers of** $z - z_0$; that is,

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \qquad (6.3.1)$$

As an example of (6.3.1), let us consider the function $f(z) = \frac{1}{z-1}$. As can be seen, the point $z = 1$ is an **isolated singularity** of f and consequently the function cannot be expanded in a Taylor series centered at that point.

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Nevertheless, f can expanded in a series of the form given in (6.3.1) that is valid for all z near 1:

$$f(z) = \cdots + \frac{0}{(z-1)^2} + \frac{1}{(z-1)} + 0 + 0 \cdot (z-1) + 0 \cdot (z-1)^2 + \cdots \tag{6.3.2}$$

The series representation in (6.3.2) is valid for $0 < |z-1| < \infty$.

Using summation notation, we can write (6.3.1) as the sum of two series

$$f(z) = \boxed{\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}} + \boxed{\sum_{k=0}^{\infty} a_k (z-z_0)^k} \tag{6.3.3}$$

The two series on the right-hand side in (6.3.3) are given special names.

(1) The part with negative powers of $z - z_0$, that is,

$$\boxed{\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}} = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k} \tag{6.3.4}$$

is called **the principal part of the series** (6.3.1) and **will converge for** $|1/(z-z_0)| < r^*$ or equivalently for $|z-z_0| > 1/r^* = r$.

(2) The part consisting of the **nonnegative powers** of $z - z_0$,

$$\boxed{\sum_{k=0}^{\infty} a_k (z - z_0)^k} \tag{6.3.5}$$

is called **the analytic part of the series** (6.3.1) and will converge for $|z - z_0| < R$.

Hence, **the sum of (6.3.4) and (6.3.5)** converges **when** z satisfies both $|z - z_0| > r$ and $|z - z_0| < R$, that is, **when** z **is a point in an annular domain defined by** $r < |z - z_0| < R$.

By summing over negative and nonnegative integers, (6.3.1) can be written compactly as

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

Laurent series（ローラン級数）

A series representation of a function f that has the form given in (6.3.1) is called a **Laurent series** or **a Laurent expansion of** f about z_0 **on the annulus** $r < |z - z_0| < R$.

EXAMPLE (例題) 6.3.1 Laurent Series of the Form (6.3.1)

Expand the function $f(z) = \dfrac{\sin z}{z^4}$ with the isolated singularity $z = 0$ as Laurent Series.

Solution (解答):

Hint:

- Maclaurin Series
- $\sin z$ is entire function

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check https://github.com/uoaworks/ComplexAnalysisAY2018

11.1 Laurent Series（ローラン級数）

Theorem 6.10 Laurent's Theorem

Let f be analytic within the annular domain D defined by $r < |z - z_0| < R$. Then f has the series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \qquad (6.3.7)$$

valid for $r < |z - z_0| < R$. The coefficients a_k are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds \qquad k = 0, \pm 1, \pm 2, \dots \qquad (6.3.8)$$

where C is a simple closed curve that lies entirely within D and has z_0 in its interior. See Figure 6.6.

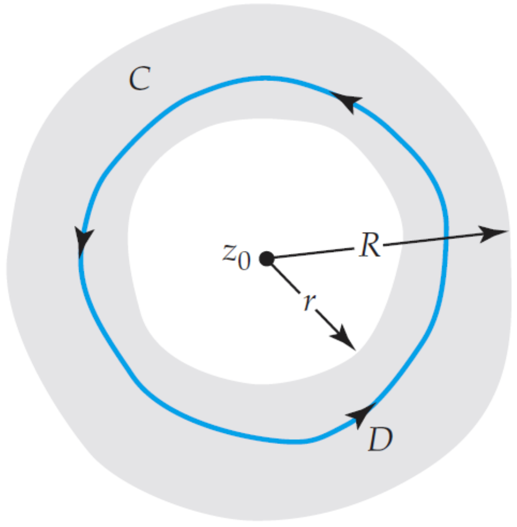

Figure 6.6 Contour for Theorem 6.10

Notice:
In the case when $a_{-k} = 0$ for $k = 1, 2, 3, \dots$, the principal part (6.3.4) is zero and the Laurent series (6.3.7) reduces to a **Taylor series**.
Therefore, **a Laurent expansion** can be regarded as **a generalization of a Taylor series**.

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The annular (環状) domain in Theorem 6.10 defined by $r < |z - z_0| < R$ **do not must have** the "ring (円環)" shape. **Here are some other possible annular domains**:

(i) $r = 0$, R is finite,
(ii) $r \neq 0$, $R = \infty$,
(iii) $r = 0$, $R = \infty$.

- In (i), the series converges in annular domain defined by an $0 < |z - z_0| < R$. **This is the interior of the circle** $|z - z_0| = R$ except the point z_0; in other words, the domain is a punctured open disk.
- In (ii), the annular domain is defined by $r < |z - z_0|$ and **consists of all points exterior to the circle** $|z - z_0| = r$.
- In (iii), the domain is defined by $0 < |z - z_0|$. **This represents the entire complex plane except the point** z_0. The Laurent series in (6.3.2) and (6.3.6) are valid on this last type of domain.

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The integral formula in (6.3.8) for the coefficients of a Laurent series are rarely used in practice. **As a consequence, finding the Laurent series of a function in a specified annular domain is generally not an easy task.**

In many instances we can obtain a desired Laurent series either by employing a known power series expansion of a function (as we did in Example 6.3.1) or by creative manipulation of geometric series (as we did in Example 6.2.2 of Lecture 10).

The next example once again illustrates the use of geometric series.

EXAMPLE (例題) 6.3.2 **Four Laurent Expansions**

Expand $f(z) = \frac{1}{z(z-1)}$ in a Laurent series valid for the following annular domains.

(a) $0 < |z| < 1$ (b) $1 < |z|$ (c) $0 < |z-1| < 1$ (d) $1 < |z-1|$

Solution (解答):

Hint:

- Geometric Series (6.1.6) of Lecture 9
- (6.1.7) of Lecture 9

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check https://github.com/uoaworks/ComplexAnalysisAY2018

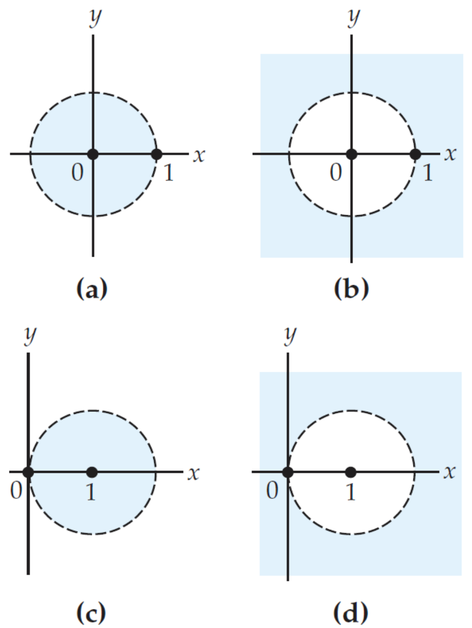

Figure 6.8 Annular domains for Example 2

EXAMPLE (例題) 6.3.3 Laurent Expansions

Expand $f(z) = \frac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for the following annular domains.

(a) $0 < |z-1| < 2$ (b) $0 < |z-3| < 2$

Solution (解答):

Hint:

- Geometric Series (6.1.6) of Lecture 9
- Example 6.3.2

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EXAMPLE (例題) 6.3.4 A Laurent Expansion

Expand $f(z) = \frac{8z+1}{z(1-z)}$ in a Laurent series valid for $0 < |z| < 1$.

Solution (解答):

Hint:

- Geometric Series (6.1.6) of Lecture 9

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EXAMPLE (例題) 6.3.5 A Laurent Expansion

Expand $f(z) = \frac{1}{z(1-z)}$ in a Laurent series valid for $1 < |z - 2| < 2$.

Solution (解答):

Hint:

- Find two series involving integer powers of $z-2$, one converging for $1 < |z - 2|$ and the other converging for $|z - 2| < 2$.

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check https://github.com/uoaworks/ComplexAnalysisAY2018

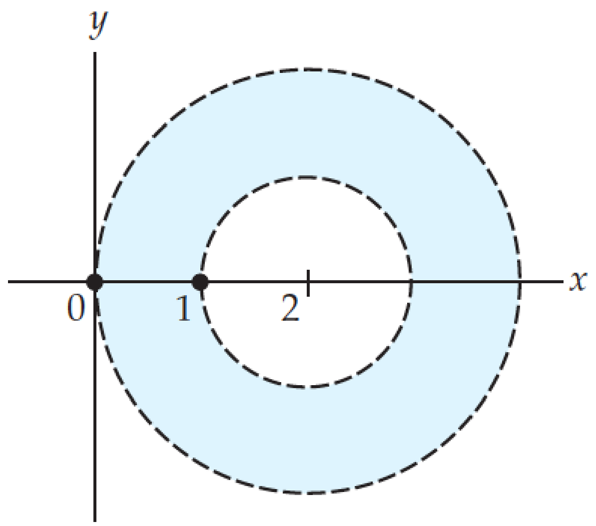

Figure 6.9 Annular domain for Example 6.1.5

EXAMPLE (例題) 6.3.6 **A Laurent Expansion**

Expand $f(z) = e^{\frac{3}{z}}$ in a Laurent series valid for $0 < |z| < \infty$.

Solution (解答):

Hint:

- (6.2.12) of Lecture 10

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Remarks

(i) Replacing the complex variable s with the usual symbol z , we see that when $k-1$, formula (6.3.8) for the Laurent series coefficients yields $\mathrm{a}_{-1} = \frac{1}{2\pi i}\oint_C f(z)dz$, or more important,

$$\oint_C f(z)dz = 2\pi i a_{-1}$$

(ii) Regardless how a Laurent expansion of a function f is obtained in a specified annular domain it is the Laurent series; that is, the series we obtain is unique.

Review for Lecture 11

- Laurent Series

Assignment

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: Section 6.3, Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia