Lecture 7

Cauchy-Goursat Theorem

(i.e. Cauchy's integral theorem コーシーの積分定理)

What you will learn in Lecture 7

7.1 Cauchy-Goursat Theorem

7.1.1 Simply and Multiply Connected Domains

7.1.2 Cauchy-Goursat Theorem

7.1.3 Cauchy-Goursat Theorem for Multiply Connected Domains

*7.2 Independence of Path for Contour Integral

7.1 Cauchy-Goursat Theorem

(i.e. Cauchy's integral theorem コーシーの積分定理)

In this 7.1, we shall concentrate on contour integrals, where the contour C is a **simple closed curve** with a **positive** (**counterclockwise**) **orientation**.

7.1.1 Simply Connected (単連結) Domains

and

Multiply Connected (多重連結) Domains

7.1.1 Simply and Multiply Connected Domains

Simply Connected (単連結) Domains

We say that **a domain D is simply connected** if **every simple closed contour C lying entirely in D can be shrunk to a point** (ポイントに縮小する) **without leaving D**. (See Figure 5.26.)

In other words, a simply connected domain **has no "holes" in** it.

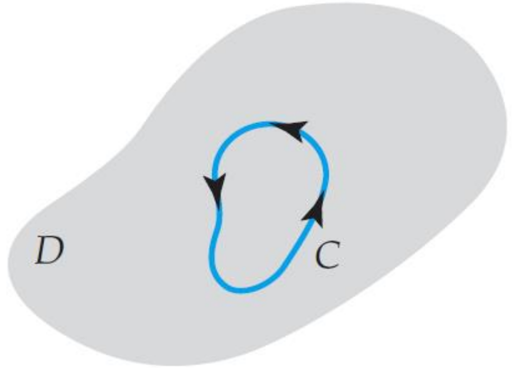


Figure 5.26 Simply connected domain D

Multiply Connected (多重連結) Domains

A domain that is not simply connected is called **a multiply connected domain**. (See Figure 5.27.)

In other words, a multiply connected domain **has "holes" in it.**

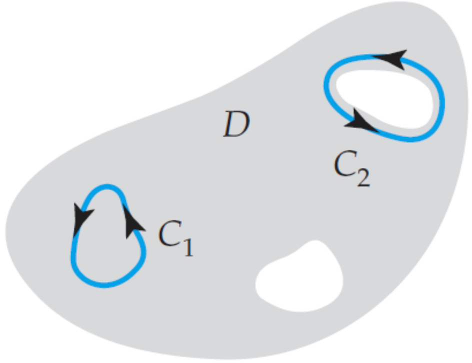


Figure 5.27 Multiply connected domain D

For example, (1)the open disk (開円板) defined by $|z| < 2$ is a simply connected domain; (2)the open circular annulus (開円環) defined by $1 < |z| < 2$ is a doubly (i.e. multiply) connected domain.

7.1.2 Cauchy-Goursat Theorem

(i.e. Cauchy's integral theorem コーシーの積分定理)

7.1.2 Cauchy-Goursat Theorem (コーシーの積分定理)

Theorem 5.4　Cauchy-Goursat Theorem (i.e. Cauchy's integral theorem コーシーの積分定理)

Suppose that a function f is **analytic (解析的)** in a **simply connected (単連結) domain** D. Then for every simple closed contour C in D, we have

$$\oint_C f(z)dz = 0$$

Because the interior (内部) of a simple closed contour is a **simply connected domain**, the **Theorem 5.4** can be **rewritten** in the slightly more practical manner:

If f is **analytic at all points within and on** a **simple closed contour** C, then

$$\oint_C f(z)dz = 0 \tag{5.3.4}$$

EXAMPLE (例題) 5.3.1 Applying the Cauchy-Goursat Theorem

Evaluate $\oint_C e^z dz$, where the contour C is shown in Figure 5.28.

Solution (解答):

The function $f(z) = e^z$ is entire (整函数) and consequently is analytic at all points within and on the simple closed contour C.

Then from the Cauchy-Goursat Theorem given in (5.3.4) that

$$\oint_C e^z dz = 0$$

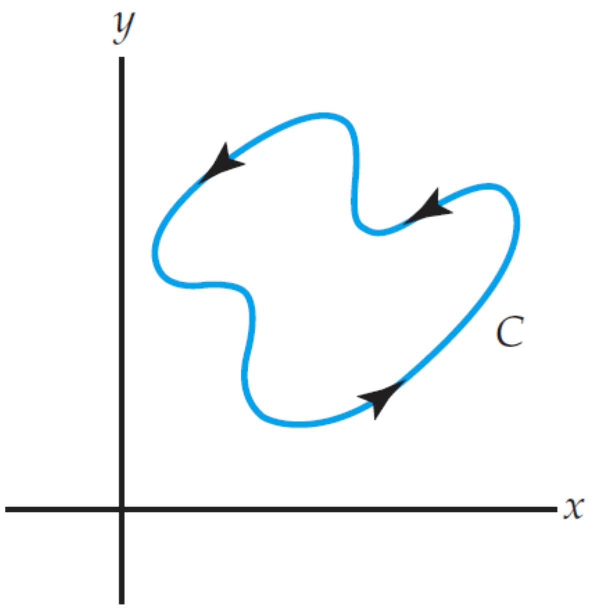


Figure 5.28 Contour for Example 5.3.1

7.1.2 Cauchy-Goursat Theorem (コーシーの積分定理)

Indeed, from Example 5.3.1, it follows that **for any simple closed contour C and any entire** function (整函数) f, such as

$$f(z) = \sin z,$$

$$f(z) = \cos z,$$

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,\ \ n = 0, 1, 2, \dots$$

we have

$$\oint_C \sin z\, dz = 0,$$

$$\oint_C \cos z\, dz = 0,$$

$$\oint_C p(z) dz = 0$$

and so on.

EXAMPLE (例題) 5.3.2 Applying the Cauchy-Goursat Theorem

Evaluate $\oint_C \frac{1}{z^2}dz$, where the contour C is the ellipse (楕円) $(x-2)^2+(y-5)^2=1$.

Solution (解答):

The rational function $\oint_C \frac{1}{z^2}dz$ is analytic everywhere except at $z=0$.

But $z=0$ is not interior to or on the simple closed elliptical contour C.

Thus, from the Cauchy-Goursat Theorem given in (5.3.4) we have that

$$\oint_C \frac{1}{z^2}dz = 0$$

7.1.3 Cauchy-Goursat Theorem for Multiply Connected Domains

7.1.3 Cauchy-Goursat Theorem for Multiply Connected Domains

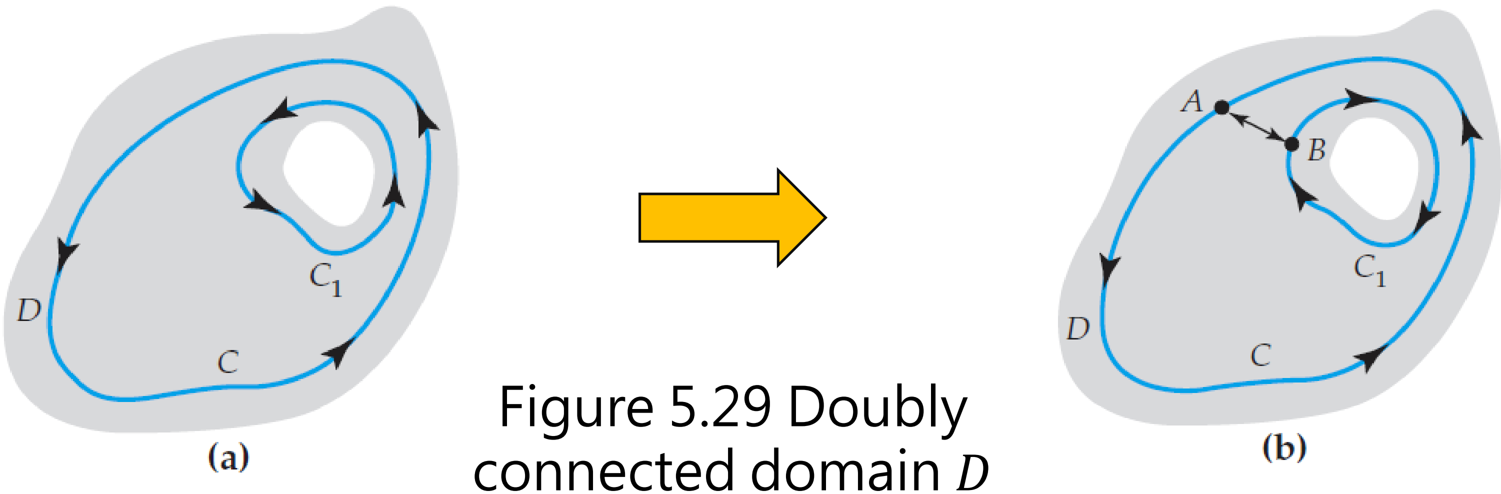


Figure 5.29 Doubly connected domain D

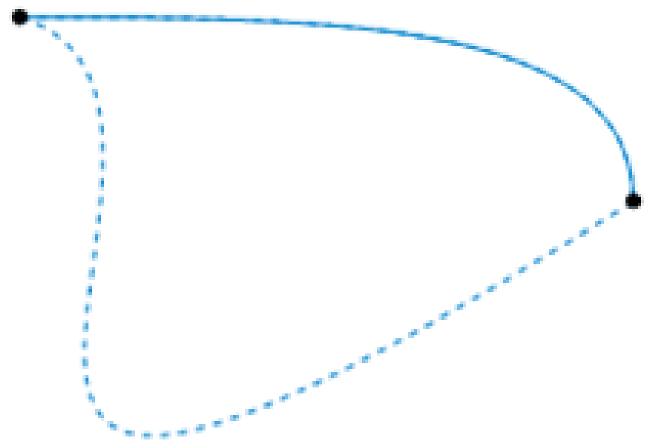
$$\oint_C f(z)dz + \cancel{\oint_{AB} f(z)dz} + \oint_{-C_1} f(z)dz + \cancel{\oint_{-AB} f(z)dz} = 0$$

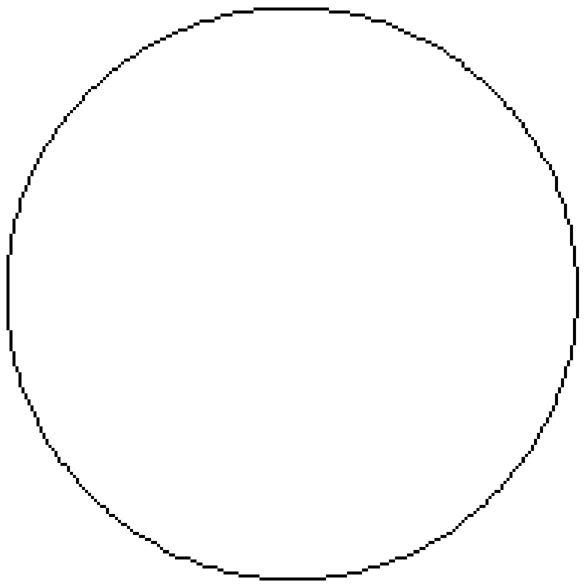
$$\Longrightarrow \quad \oint_C f(z)dz = \oint_{C_1} f(z)dz \qquad (5.3.5)$$

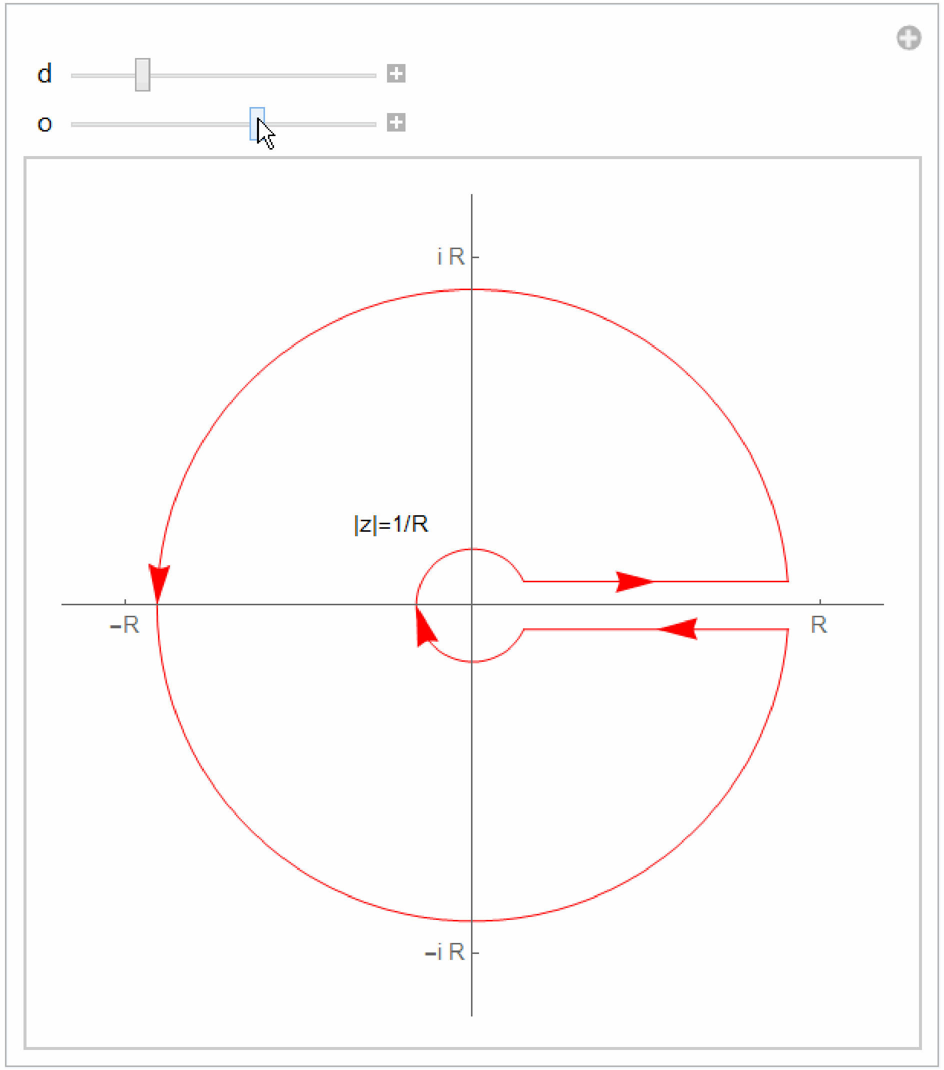
The above result is sometimes called the **principle of deformation (変形) of contours** because we can think of **the contour C_1 as a continuous deformation (連続変形) of the contour** C.

In other words, (5.3.5) allows us to **evaluate an integral (積分) over a complicated (複雑な) simple closed contour C by replacing C with a contour C_1 that is more convenient (便利な)**.

7.1.3 Cauchy-Goursat Theorem for Multiply Connected Domains







Continuous deformation (連続変形) of a contour

EXAMPLE (例題) 5.3.3 Applying Deformation of Contours

Evaluate $\oint_C \frac{1}{z-i}\,dz$, where the contour C is shown in black color in Figure 5.30. (Notice that there is a point "hole" at $(1, 1)$.)

Solution (解答):

From (5.3.5), we choose the more convenient circular contour C_1 drawn in blue color in the Figure 5.30.

By taking the radius (半径) of the circle to be $r = 1$, we are guaranteed (保証される) that C_1 lies within C. In other words, C_1 is the circle $|z - i| = 1$, which from (2.2.10) of Section 2.2 can be parametrized by $z = i + e^{it}$, $0 \le t \le 2\pi$. Thus $z - i = e^{it}$ and $dz = ie^{it}dt$, we obtain

$$\oint_C \frac{1}{z-i}\,dz = \oint_{C_1} \frac{1}{z-i}\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = i\int_0^{2\pi} dt = 2\pi i$$

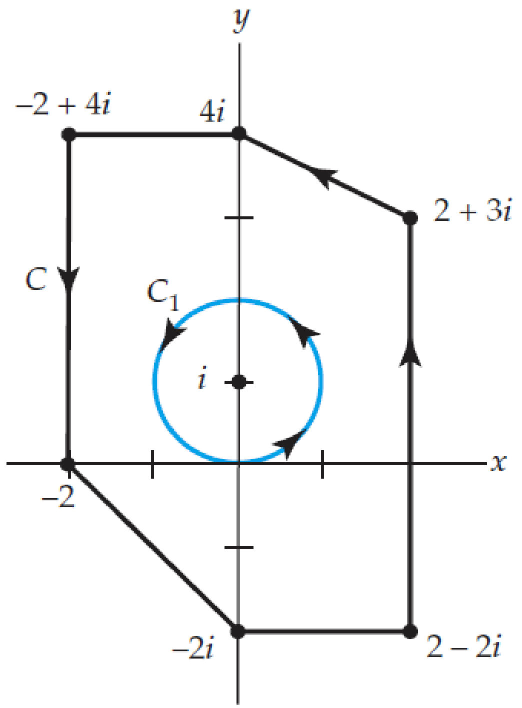


Figure 5.30 We use the simpler contour C_1 in Example 5.3.3.

7.1.3 Cauchy-Goursat Theorem for Multiply Connected Domains

Additional Point: **Common Parametric Curves in the Complex Plane**

Line

A parametrization of the line containing the points z_0 and z_1 is:

$$z(t) = z_0(1-t) + z_1 t, \qquad -\infty \leq t \leq \infty. \qquad (2.2.7)$$

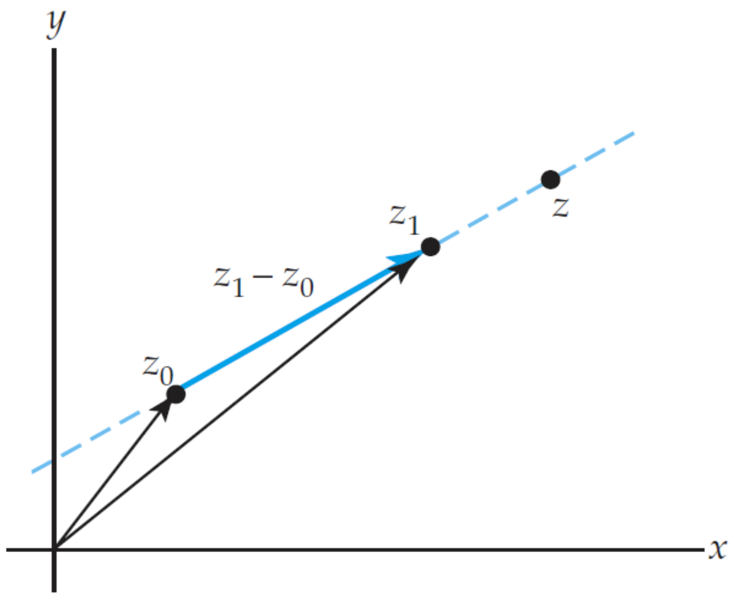


Figure 2.4 Parametrization of a line

Circle

A parametrization of the circle centered at z_0 **with radius** r is:

$$z(t) = z_0 + r(\cos t + i \sin t), \qquad 0 \leq t \leq 2\pi. \quad (2.2.9)$$

In exponential notation, this parametrization is:

$$z(t) = z_0 + re^{it}, \qquad 0 \leq t \leq 2\pi. \qquad (2.2.10)$$

7.1.3 Cauchy-Goursat Theorem for Multiply Connected Domains

The result obtained in Example 5.3.3 can be generalized.

By using the principle of deformation of contours (5.3.5), it can be shown that **if z_0 is any constant complex number interior to any simple closed contour C**, then **for an integer n** we have

$$\oint_C \frac{1}{(z-z_0)^n}\,dz = \begin{cases} 2\pi i, & n = 1 \\ 0, & n \neq 1 \end{cases} \qquad (5.3.6)$$

EXAMPLE (例題) 5.3.4 Applying Formula (5.3.6)

Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}dz$, where the contour C is the circle $|z-2|=2$.

Solution (解答):

Because the denominator factors as $z^2+2z-3=(z-1)(z+3)$ the integrand fails to be analytic at $z=1$ and $z=-3$. Of these two points, only $z=1$ lies within the contour C, which is a circle centered at $z=2$ of radius $r=2$. Now by partial fractions

$$\frac{5z+7}{z^2+2z-3}=\frac{5z+7}{(z-1)(z+3)}=\frac{3(z+3)}{(z-1)(z+3)}+\frac{2(z-1)}{(z-1)(z+3)}=\frac{3}{z-1}+\frac{2}{z+3}$$

$$\oint_C \frac{5z+7}{z^2+2z-3}dz=\oint_C \frac{3}{z-1}dz+\oint_C \frac{2}{z+3}dz \tag{5.3.7}$$

By (5.3.6), the first integral in (5.3.7) has the value $2\pi i$, whereas the value of the second integral is 0 by the Cauchy-Goursat theorem.

Hence, (5.3.7) becomes

$$\oint_C \frac{5z+7}{z^2+2z-3}dz=3\cdot(2\pi i)+2\cdot(0)=6\pi i$$

Theorem 5.5 Cauchy-Goursat Theorem for Multiply Connected Domains

Suppose $C,\ C_1, \ldots, C_n$ are **simple closed curves with a positive orientation** such that $C_1, C_2, \ldots, C_n$ are interior to C but the regions interior to each C_k, $k = 1, 2, \ldots, n$, have no points in common. If f **is analytic on each contour and at each point interior to** C but exterior to all the C_k, $k = 1, 2, \ldots, n$, then

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz \qquad (5.3.8)$$

EXAMPLE (例題) 5.3.5 Applying Theorem 5.5

Evaluate $\oint_C \frac{1}{z^2+1}dz$, where the contour C is the circle $|z| = 3$.

Solution (解答): We know that $\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$,

Consequently, the integrand $1/(z^2+1)$ is not analytic at $z = i$ and at $z = -i$. Both of these points lie within the contour C.

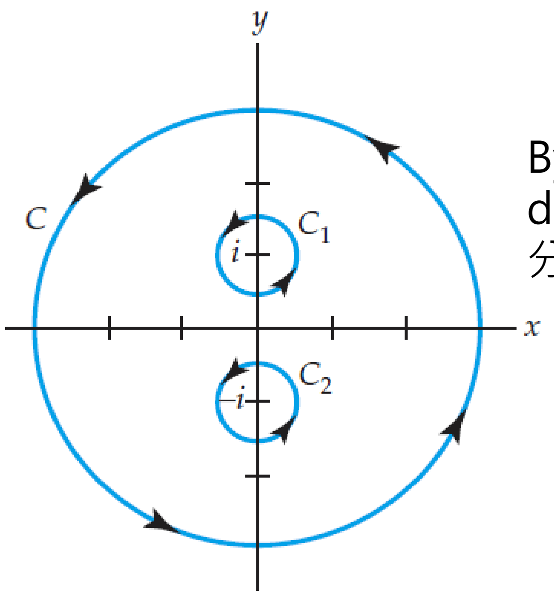


Figure 5.32 Contour for Example 5.3.5

By using partial fraction decomposition (部分分数分解):

$$\frac{1}{(z+i)(z-i)} = \frac{\frac{1}{2i}(z+i)}{(z+i)(z-i)} - \frac{\frac{1}{2i}(z-i)}{(z+i)(z-i)} = \frac{1}{2i}\frac{1}{z-i} - \frac{1}{2i}\frac{1}{z+i}$$

$$\oint_C \frac{1}{z^2+1}dz = \oint_C \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]dz$$

We now choose to surround the points $z = i$ and $z = -i$ by circular contours C_1 and C_2, respectively, that lie entirely within C. Specifically, the choice $|z-i| = \frac{1}{2}$ for C_1 and $|z+i| = \frac{1}{2}$ for C_2 will suffice (十分である). See Figure 5.32. From Theorem 5.5 we can write:

7.1.3 Cauchy-Goursat Theorem for Multiply Connected Domains

Solution (解答)(cont.):

$$\oint_C \frac{1}{z^2+1}dz = \oint_C \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]dz = \oint_{C_1} \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]dz + \oint_{C_2} \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]dz$$

$$= \frac{1}{2i}\oint_{C_1} \frac{1}{z-i}dz - \underbrace{\frac{1}{2i}\oint_{C_1} \frac{1}{z+i}dz}_{0} + \underbrace{\frac{1}{2i}\oint_{C_2} \frac{1}{z-i}dz}_{0} - \frac{1}{2i}\oint_{C_2} \frac{1}{z+i}dz \qquad (5.3.9)$$

Because $1/(z+i)$ is analytic on C_1 and at each point in its interior and because $1/(z-i)$ is analytic on C_2 and at each point in its interior, it follows from (5.3.4) that the second and third integrals in (5.3.9) are zero. Moreover, it follows from (5.3.6), with $n = 1$, that

$$\oint_C \frac{1}{z^2+1}dz = \frac{1}{2i}\oint_{C_1} \frac{1}{z-i}dz - 0 + 0 - \frac{1}{2i}\oint_{C_2} \frac{1}{z+i}dz$$

$$\oint_{C_1} \frac{1}{z-i}dz = 2\pi i \qquad \text{and} \qquad \oint_{C_2} \frac{1}{z+i}dz = 2\pi i$$

$$\oint_C \frac{1}{z^2+1}dz = \frac{1}{2i}\oint_{C_1} \frac{1}{z-i}dz - \frac{1}{2i}\oint_{C_2} \frac{1}{z+i}dz = \frac{2\pi i}{2i} - \frac{2\pi i}{2i} = 0$$

*7.2 Independence (独立) of Path (経路) for Contour Integral

Notice: In all lecture notes, the contents marked with * are not in the scope of the final examination.

*7.2 Independence (独立) of Path (経路) for Contour Integral

There exist **Real line integrals (実·線積分)** $\int_C Pdx + Qdy$ **whose value depends only on the initial point** (始点) A **and terminal point** (終点) B **of the curve** C, and **not on** C **itself.**

In this case we say that **the line integral is independent of the path**.

For example, $\int_C ydx + xdy$ **is independent of the path**.

(1) *Can a contour integral* $\int_C f(z)dz$ ***be independent of the path?***

(2) *Is there* ***a complex version*** *of* ***the fundamental theorem*** *of* ***calculus?***

we will see that the answer to both of these questions is **YES**.

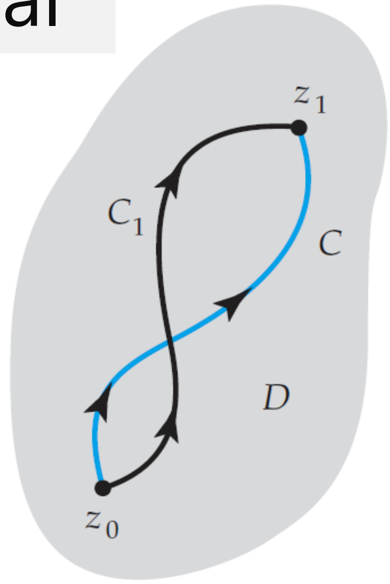
*7.2 Independence (独立) of Path (経路) for Contour Integral

Definition 5.4 Independence of the Path for Contour Integral

Let z_0 and z_1 be points in a domain D. A contour integral $\int_C f(z)dz$ **is said to be independent of the path if its value is the same for all contours C in D with initial point z_0 and terminal point z_1.**

*7.2 Independence (独立) of Path (経路) for Contour Integral

Now suppose, as shown in Figure 5.38, that C and C_1 are two contours lying entirely in a simply connected domain D and both with initial point z_0 and terminal point z_1.



Thus, if f is analytic in D, it follows from the Cauchy-Goursat theorem that

$$\int_C f(z)dz + \int_{-C_1} f(z)dz = 0$$

$$\Rightarrow \quad \int_C f(z)dz = \int_{C_1} f(z)dz$$

Figure 5.38 If f is analytic in D, integrals on C and C_1 are equal.

Theorem 5.6 Analyticity Implies Path Independence

Suppose that a function f is **analytic** in a **simply connected domain** D and C **is any contour in** D. Then $\int_C f(z)dz$ is **independent of the path** C.

EXAMPLE (例題) 5.4.1 **Choosing a Different Path**

Evaluate $\int_C 2z dz$, where the contour C is shown in blue color in Figure 5.39.

Solution (解答):

Because the function $f(z) = 2z$ is entire, by Theorem 5.6, we can replace the piecewise smooth path C by any convenient contour C_1 joining $z_0 = -1$ and $z_1 = -1 + i$. Specifically, if we **choose the contour** C_1 to **be the vertical line segment (線分)** $x = -1, 0 \leq y \leq 1$, shown **in black color** in Figure 5.39, then $z = -1 + iy, dz = idy$. Therefore,

$$\int_C 2z dz = \int_{C_1} 2z dz = \int_0^1 2(-1 + iy) idy = -2\int_0^1 ydy - 2i\int_0^1 dy = -1 - 2i$$

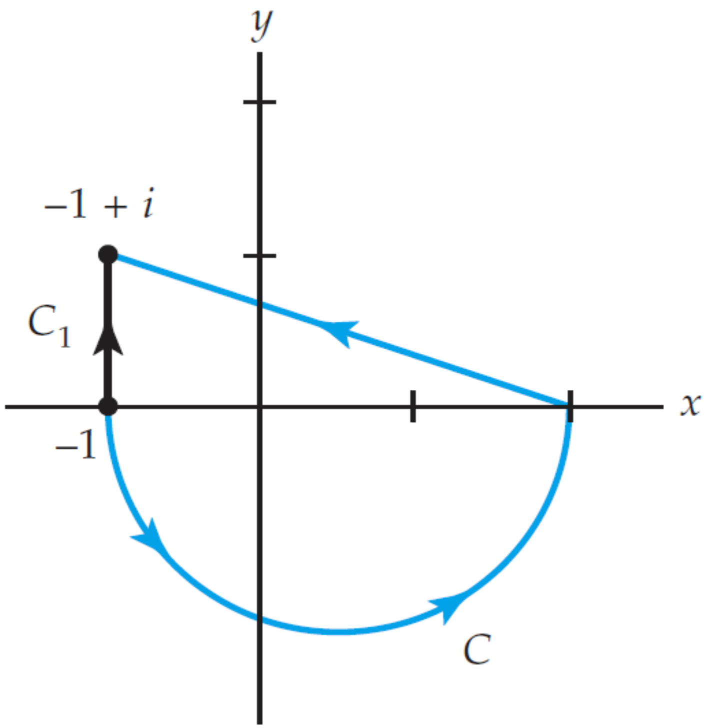


Figure 5.39 Contour for Example 5.4.1

*7.2 Independence (独立) of Path (経路) for Contour Integral

Definition 5.5 Antiderivative

Suppose that a function f is continuous on a domain D. If there exists a function F such that $F'(z) = f(z)$ for each z in D, then F is called an antiderivative of f.

For example, the function $F(z) = -\cos z$ is an antiderivative of $f(z) = \sin z$ because $F'(z) = \sin z$.

*7.2 Independence (独立) of Path (経路) for Contour Integral

Theorem 5.7 Fundamental Theorem for Contour Integrals

Suppose that a function f is continuous on a domain D and F is an antiderivative of f in D. Then for any contour C in D with initial point (始点) z_0 and terminal point (終点) z_1,

$$\int_C f(z)dz = F(z_1) - F(z_0) \tag{5.4.4}$$

EXAMPLE (例題) 5.4.2 Applying Theorem 5.7

Evaluate $\int_C 2z\,dz$, where the contour C is shown in color in Figure 5.39.

Solution (解答):

In Example 5.4.1 we know that $\int_C 2z\,dz$, where C is shown in Figure 5.39, is independent of the path.

Here because the $f(z) = 2z$ is an entire function, it is continuous.

Moreover, $F(z) = z^2$ is an antiderivative of f since $F'(z) = 2z = f(z)$.

Hence, by (5.4.4) of Theorem 5.7 we have

$$\int_{-1}^{-1+i} 2z\,dz = z^2\Big|_{-1}^{-1+i} = (-1+i)^2 - (-1)^2 = -1 - 2i$$

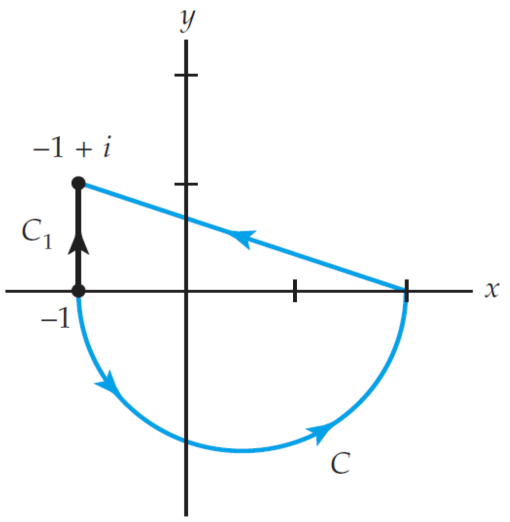


Figure 5.39 Contour for Example 5.4.1

EXAMPLE (例題) 5.4.3 Applying Theorem 5.7

Evaluate $\int_C \cos z\, dz$, where C is any contour with initial point $z_0 = 0$ and terminal point $z_1 = 2 + i$.

Solution (解答):

$F(z) = \sin z$ is an antiderivative of $f(z) = \cos z$ since $F'(z) = \cos z = f(z)$.

Therefore, from (5.4.4) we have

$$\int_C \cos z\, dz = \int_0^{2+i} \cos z\, dz = \sin z \Big|_0^{2+i} = \sin(2+i) - \sin 0 = \sin(2+i)$$

Review for Lecture 7

- Simply and Multiply Connected Domains
- Cauchy-Goursat Theorem
- Cauchy-Goursat Theorem for Multiply Connected Domains
- *Independence of Path for Contour Integral
- *Fundamental Theorem for Contour Integrals

Assignment

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: Section 5.3, 5.4, Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia