Lecture 12

- Zeros（零点）& Poles（極）
- Residues (留数) & Residue Theorem (留数定理) Part 1

What you will learn in Lecture 12

12.1 Zeros (零点) & Poles (極)

12.2 Residues (留数) & Residue Theorem (留数定理) Part 1

12.1 Zeros (零点) & Poles (極)

Recall Laurent Series（ローラン級数）

Nevertheless, f can expanded in a series of the form given in (6.3.1) that is valid for all z near 1:

$$f(z) = \cdots + \frac{0}{(z-1)^2} + \frac{1}{(z-1)} + 0 + 0 \cdot (z-1) + 0 \cdot (z-1)^2 + \cdots \tag{6.3.2}$$

The series representation in (6.3.2) is valid for $0 < |z-1| < \infty$.

Using summation notation, we can write (6.3.1) as the sum of two series

$$f(z) = \boxed{\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}} + \boxed{\sum_{k=0}^{\infty} a_k (z-z_0)^k} \tag{6.3.3}$$

The two series on the right-hand side in (6.3.3) are given special names.

(1) The part with negative powers of $z - z_0$, that is,

$$\boxed{\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}} = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k} \tag{6.3.4}$$

is called **the principal part of the series (6.3.1)** and **will converge for** $|1/(z-z_0)| < r^*$ or equivalently for $|z-z_0| > 1/r^* = r$.

12.1 Zeros (零点) & Poles (極)

The discussion that follows we will assign different names to the isolated singularity $z = z_0$ according to the number of terms in the principal part.

Classification of Isolated Singular Points

This classification depends on whether the principal part (6.3.4) of its Laurent expansion (6.3.3) in Lecture 11 contains zero, a finite number, or an infinite number of terms.

(i) If the principal part is zero, that is, all the coefficients a_{-k} in (6.3.4) are zero, then $z = z_0$ is called a removable singularity.

(ii) If the principal part contains a finite number of nonzero terms, then $z = z_0$ is called a pole. If, in this case, the last nonzero coefficient in (6.3.4) is a_{-n}, $n \geq 1$, then we say that $z = z_0$ is a pole of order n. If $z = z_0$ is pole of order 1, then the principal part (6.3.4) contains exactly one term with coefficient a_{-1}. A pole of order 1 is commonly called a simple pole.

(iii) If the principal part (6.3.4) contains an infinitely many nonzero terms, then $z = z_0$ is called an essential singularity.

12.1 Zeros (零点) & Poles (極)

Table 6.1 summarizes the form of a Laurent series for a function f when $z = z_0$ is one of the above types of isolated singularities. Of course, R in the table could be ∞.

Table 6.1 Forms of Laurent series

$z = z_0$	Laurent Series for $0 < \lvert z - z_0 \rvert < R$
Removable singularity	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Pole of order n	$\dfrac{a_{-n}}{(z-z_0)^n} + \dfrac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$
Simple pole	$\dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$
Essential singularity	$\cdots + \dfrac{a_{-2}}{(z-z_0)^2} + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$

EXAMPLE (例題) 6.4.1 Removable singularity

Classify the isolated singularity for the given function $f(z) = \dfrac{\sin z}{z}$.

Solution (解答):

Following the Example 6.3.1 of Lecture 11 by dividing the Maclaurin series for $\sin z$ by z, we see from

$$f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \tag{6.4.3}$$

that all the coefficients in the principal part of the Laurent series are zero. Hence $z = 0$ is a removable singularity of the function $f(z) = (\sin z)/z$.

12.1 Zeros (零点) & Poles (極)

If a function f has a removable singularity at the point $z = z_0$, then we can always supply an appropriate definition for the value of $f(z_0)$ so that f **becomes analytic** at $z = z_0$.

For instance, since the right-hand side of (6.4.3) is 1 when we set $z = 0$, it makes sense to define $f(0) = 1$.

Hence the function $f(z) = (\sin z)/z$, as given in (6.4.3), is now defined and continuous at every complex number z.

Indeed, f is also analytic at $z = 0$ because it is represented by the Taylor series $1 - z^2/3! + z^4/5! - \cdots$ centered at 0 (a Maclaurin series).

EXAMPLE (例題) 6.4.2 Poles and Essential Singularity

Classify the isolated singularity for the given function

(a) $f(z) = \dfrac{\sin z}{z^2}$ valid for $0 < |z| < \infty$ (b) $f(z) = 1/(z-1)^2(z-3)$ valid for $0 < |z-1| < 2$ (c) $f(z) = e^3/z$ valid for $0 < |z| < \infty$

Solution (解答): (a) Dividing the Maclaurin series for $\sin z$ by z^2

$$f(z) = \frac{\sin z}{z^2} = \underbrace{\frac{1}{z}}_{\text{principal part}} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots$$

for $0 < |z| < \infty$.

From this series we see that $a_{-1} \neq 0$ and so $z = 0$ is **a simple pole** of the function $f(z) = (\sin z)/z^2$.

We see that $z = 0$ is **a pole of order 3** of the function $f(z) = (\sin z)/z^4$ considered in Example 6.3.1 of Lecture 11.

Solution (解答)(cont.):

(b) In Example 6.3.3 of Lecture 11 we showed that the Laurent expansion of $f(z) = 1/(z-1)^2(z-3)$ valid for $0 < |z-1| < 2$ was

$$f(z) = \underbrace{-\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)}}_{\text{principal part}} - \frac{1}{8} - \frac{1}{16}(z-1) - \cdots$$

Since $a_{-2} = -\frac{1}{2} \neq 0$, we conclude that $z = 1$ is a pole of order 2.

(c) In Example 6.3.6 of Lecture 11 we see from (6.3.19) that the principal part of the Laurent expansion of the function $f(z) = e^{3/z}$ valid for $0 < |z| < \infty$ contains an infinite number of nonzero terms. This shows that $z = 0$ is an essential singularity of f.

12.1 Zeros (零点) & Poles (極)

Zeros (零点)

Recall, a number z_0 is zero of a function f if $f(z_0) = 0$. We say that an analytic function f **has a zero of order** n at $z = z_0$ if

$$\underbrace{f(z_0) = 0,\ f'(z_0) = 0,\ f''(z_0) = 0,\ \dots,\ f^{(n-1)}(z_0) = 0}_{z_0 \text{ is a zero of } f \text{ and of its first } n-1 \text{ derivatives}},\ \text{but } f^{(n)}(z_0) \neq 0 \qquad (6.4.4)$$

A zero of order n is also referred to as a zero of multiplicity n.

For example, for $f(z) = (z - 5)^3$ we see that $f(5) = 0$, $f'(5) = 0$, $f''(5) = 0$, but $f'''(5) = 6 \neq 0$. Thus f has a zero of order (or multiplicity) 3 at $z_0 = 5$.

A zero of order 1 is called a **simple zero**.

12.1 Zeros (零点) & Poles (極)

Theorem 6.11 Zero of Order n

A function f that is analytic in some disk $|z - z_0| < R$ has a **zero of order** n at $z = z_0$ if and only if f can be written

$$f(z) = (z - z_0)^n \phi(z) \tag{6.4.5}$$

where ϕ is **analytic at** $z = z_0$ and $\phi(z_0) \neq 0$.

EXAMPLE (例題) 6.4.3 Order of a Zero

Determine the order of the zero for the given function

$f(z) = z \sin z^2.$

Solution (解答):

The analytic function $f(z) = z \sin z^2$ has a zero at $z = 0$. If we replace z by z^2 in Maclaurin Series of $\sin z$ of (6.2.13) of Lecture 10, we obtain the Maclaurin expansion

$$\sin z^2 = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \cdots$$

Then by factoring z^2 out of the foregoing series we can rewrite f as

$$f(z) = z \sin z^2 = z^3 \phi(z) \quad \text{where} \quad \phi(z) = 1 - \frac{z^4}{3!} + \frac{z^8}{5!} - \cdots \tag{6.4.6}$$

and $\phi(0) = 1$. When compared to (6.4.5), the result in (6.4.6) shows that $z = 0$ is a zero of order 3 of f.

12.1 Zeros (零点) & Poles (極)

Poles (極)

We can characterize a pole of order n in a manner analogous to (6.4.5).

Theorem 6.12 Pole of Order n

A function f analytic in a punctured disk $0 < |z - z_0| < R$ has a pole of order n at $z = z_0$ if and only if f can be written

$$f(z) = \frac{1}{(z - z_0)^n}\phi(z) \tag{6.4.7}$$

where ϕ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$.

Zeros Again

A zero $z = z_0$ **of an analytic function** f is *isolated* in the sense that there exists **some neighborhood of** z_0 for which $f(z) = 0$ **at every point** z **in that neighborhood except** at $z = z_0$.

As a consequence, if z_0 is **a zero of a nontrivial analytic function** f, then the function $1/f(z)$ has **an isolated singularity at the point** $z = z_0$.

The following result enables us, in some circumstances, to determine **the poles of a function** by inspection.

Theorem 6.13 Pole of Order n

If the functions g and h are analytic at $z = z_0$ and h has a zero of order n at $z = z_0$ and $g(z_0) = 0$, then the function $f(z) = g(z)/h(z)$ has a pole of order n at $z = z_0$.

Proof

Because the function h has zero of order n, (6.4.5) gives $h(z) = (z - z_0)^n \phi(z)$, where ϕ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$. Thus f can be written

$$f(z) = \frac{g(z)/\phi(z)}{(z - z_0)^n} \tag{6.4.10}$$

Proof (Cont.)

Since g and ϕ are analytic at $z = z_0$ and $\phi(z_0) \neq 0$, it follows that the function g/ϕ is analytic at z_0. Moreover, $g(z_0) \neq 0$ implies $g(z_0)/\phi(z_0) \neq 0$. We conclude from Theorem 6.12 that the function f has a pole of order n at z_0. ■

When $n = 1$ in (6.4.10), we see that a zero of order 1, or a simple zero, in the denominator h of $f(z) = g(z)/h(z)$ corresponds to a simple pole of f.

EXAMPLE (例題) 6.4.4 **Order of Poles**

Determine the order of the poles for the given function.

(a) $f(z) = \frac{2z+5}{(z-1)(z+5)(z-2)^4}$ (b) $f(z) = 1/(z \sin z^2)$

Solution (解答):

(a) Inspection of the rational function

$$f(z) = \frac{2z+5}{(z-1)(z+5)(z-2)^4}$$

shows that **the denominator** has zeros of order 1 at $z = 1$ and $z = -5$, and a zero of order 4 at $z = 2$. **Since the numerator is not zero at any of these points**, it follows from Theorem 6.13 and (6.4.10) that f has simple poles at $z = 1$ and $z = -5$, and a pole of order 4 at $z = 2$.

(b) In Example 6.4.3 we saw that $z = 0$ is a zero of order 3 of $z \sin z^2$. From Theorem 6.13 and (6.4.10) we conclude that the reciprocal function $f(z) = 1/(z \sin z^2)$ has a pole of order 3 at $z = 0$.

12.2 Residues (留数) &

Residue Theorem (留数定理)

Part 1

Residues (留数)

The coefficient a_{-1} of $1/(z - z_0)$ in the Laurent series given above is called **the residue of the function** f at the isolated singularity z_0. We shall use the notation

$$a_{-1} = \operatorname{Res}(f(z), z_0)$$

to denote the residue of f at z_0.

Recall,

if the principal part of the Laurent series valid for $0 < |z - z_0| < R$ contains a finite number of terms with a_{-n} the last nonzero coefficient, then z_0 is a pole of order n;

if the principal part of the series contains an infinite number of terms with nonzero coefficients, then z_0 is an essential singularity.

EXAMPLE (例題) 6.5.1 Residues
Find the residues for (a) The part (b) of Example 6.4.2; (b) The Example 6.3.6 of Lecture 11.

Solution (解答):

(a) In part (b) of Example 6.4.2 in this Lecture, we saw that $z = 1$ is a pole of order two of the function $f(z) = \frac{1}{(z-1)^2(z-3)}$. From the Laurent series obtained in that example valid for the deleted neighborhood of $z = 1$ defined by $0 < |z-1| < 2$,

$$f(z) = \frac{-\frac{1}{2}}{(z-1)^2} + \overbrace{\frac{-\frac{1}{4}}{(z-1)}}^{a_{-1}} - \frac{1}{8} - \frac{1}{16}(z-1) - \cdots$$

we see that the coefficient of $1/(z-1)$ is $a_{-1} = \mathrm{Res}(f(z), 1) = -\frac{1}{4}$.

Solution (解答)(cont.):

(b) In Example 6.3.6 of Lecture 11 we saw that $z = 0$ is an essential singularity of $f(z) = e^{\frac{3}{z}}$. Inspection of the Laurent series obtained in that example,

$$e^{\frac{3}{z}} = 1 + \overbrace{\frac{3}{z}}^{a_{-1}} + \frac{3^2}{2!\,z^2} + \frac{3^3}{3!\,z^3} + \cdots$$

$0 < |z| < \infty$, shows that the coefficient of $1/z$ is $a_{-1} = \text{Res}(f(z), 0) = 3$.

12.2 Residues (留数) & Residue Theorem (留数定理) Part 1

We will see why the coefficient a_{-1} is so important later on in this section.

In the meantime we are going to examine ways of obtaining this complex number when z_0 is a pole of a function f without the necessity of expanding f in a Laurent series at z_0.

We begin with the residue at a simple pole.

Theorem 6.14 Residue at a Simple Pole

If f has a simple pole at $z = z0$, then

$$\text{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z) \qquad (6.5.1)$$

Theorem 6.15 Residue at a Pole of Order n

If f has a pole of order n at $z = z_0$, then

$$\operatorname{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \tag{6.5.2}$$

EXAMPLE (例題) 6.5.2 **Residue at a Pole**

The function $f(z) = \frac{1}{(z-1)^2(z-3)}$ has a simple pole at $z = 3$ and a pole of order 2 at $z = 1$. Use Theorems 6.14 and 6.15 to find the residues.

Solution (解答):

Since $z = 3$ is a simple pole, we use (6.5.1):

$$\mathrm{Res}(f(z), 3) = \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3}(z-3)\frac{1}{(z-1)^2(z-3)} = \lim_{z\to 3}\frac{1}{(z-1)^2} = \frac{1}{4}$$

Now at the pole of order 2, the result in (6.5.2) gives

$$\mathrm{Res}(f(z), 1) = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}(z-1)^2 f(z) = \lim_{z\to 1}\frac{d}{dz}\frac{1}{z-3} = \lim_{z\to 1}\frac{-1}{(z-3)^2} = -\frac{1}{4}$$

Review for Lecture 12

- Classification of Isolated Singular Points
- Zeros (零点)
- Poles (極)
- Residues (留数)

Assignment

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: Section 6.4, 6.5, Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia