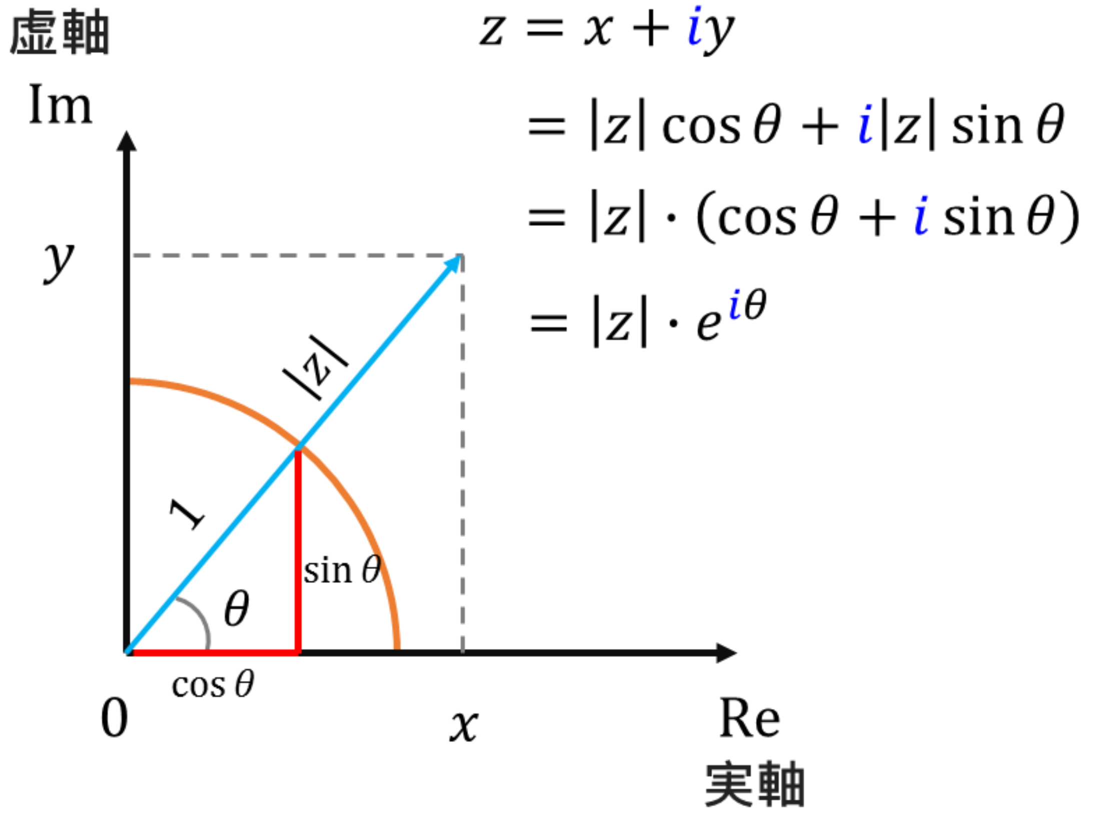
MA06 AY2018 Q4

Complex Analysis

複素関数論



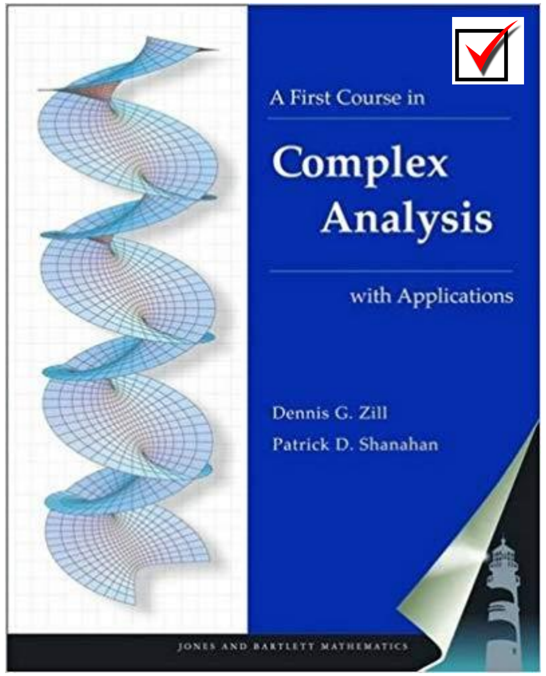
Class Information

Lectures: Tuesday (火曜日), Friday (金曜日)

Grades: 20% Assignment (Attendance >2/3)

80% Examination

Office hours: Period 5 and 6, Tuesday and Friday; 研究棟#247C



Textbook: (教科書) [Eng] **A first course in Complex Analysis with application,** Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

参考書 [Jap] **工学基礎 複素関数論,** 矢嶋 徹, 及川 正行, サイエンス社, 2007

About Final Examination

90% $\left\{ \begin{array}{l} \textbf{Lecture Slides (Example, Definition, Theorem)} \\ \textbf{Assignments} \end{array} \right.$

10% Ability to solve certain problems

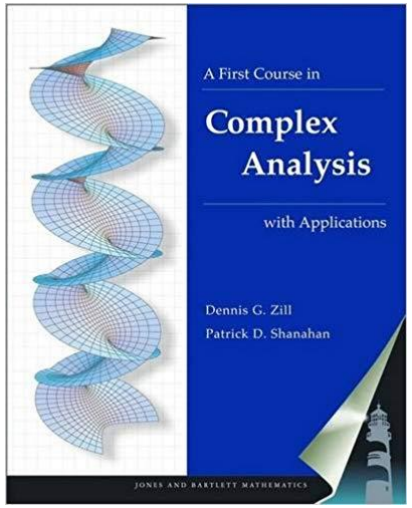
About Deadline of Assignments

- Submit past week assignments before 11:59 AM at Tuesday to Office#247C.

For example: submit Assignment 1 (20181204) and Assignment 2 (20181207) before 11:59 AM at Tuesday, 20181211

What we will cover

Full syllabus on course website



Chapter 1	1. complex plane, point at infinity
Chapter 2, 3	2. holomorphic functions, Cauchy-Riemann equations
Chapter 3	3. harmonic functions
Chapter 4	4. exponent functions, trigonometric functions, logarithm functions, roots, complex pow complex numbers
Chapter 5	5. complex integrals
	6. Cauchy's integral theorem, integrals of holomorphic functions
	7. Cauchy's integral formula, Liouville's theorem, maximum modulus principle
Chapter 6	8. complex sequence and series
	9. sequence and series of functions, uniform convergence
	10. power series and its convergence domain
	11. Taylor series expansion
	12. Laurent series expansion, zero points, singularities
	13. residue theorem
	14. application to several (real) definite integrals (Details depend on each class.)

Prerequisites

MA03 **Calculus I**

MA04 **Calculus II**

Important related courses:

MA05 **Fourier analysis**

NS02 **Electromagnetism**

You should know

This number means **the equation** is corresponding to (**Section 1.1, Equation (3)**) in **the textbook**.

The sum (和) and product (積) of a complex number z with its conjugate (複素共役) $\bar{z}$ is a real number:

$$z + \bar{z} = (a + ib) + (a - ib) = 2a \qquad \boxed{(1.1.3)}$$

$$z\bar{z} = (a + ib)(a - ib) = a^2 - i^2b^2 = a^2 + b^2 \qquad (1.1.4)$$

This number means **the example** is corresponding to (**Section 1.2, Example 1**) in **the textbook**.

EXAMPLE (例題) 1.2.1 **Find the Modulus of a Complex Number**

(a) $z = 2 - 3i$ (b) $z = -9i$.

Lecture 1

Complex Number（複素数）

Complex Plane （複素平面）

What you will learn in Lecture 1

1.1 Why Complex Number (複素数) ?

1.2 Complex Number (複素数) and Their Properties (性質)

1.3 Complex Plane (複素平面)

1.4 Polar form (極形式) of Complex Plane (複素平面)

1.1 Why Complex Number (複素数) ?

1.1 Why **Complex Number (複素数)** ?

Let's consider a problem that find solutions of equations.

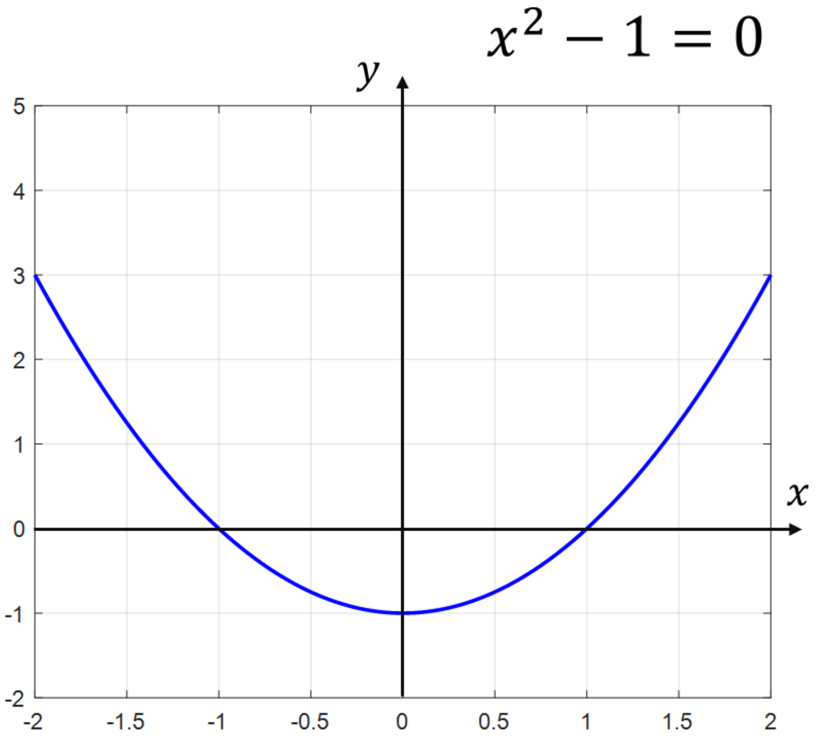
Equation 1

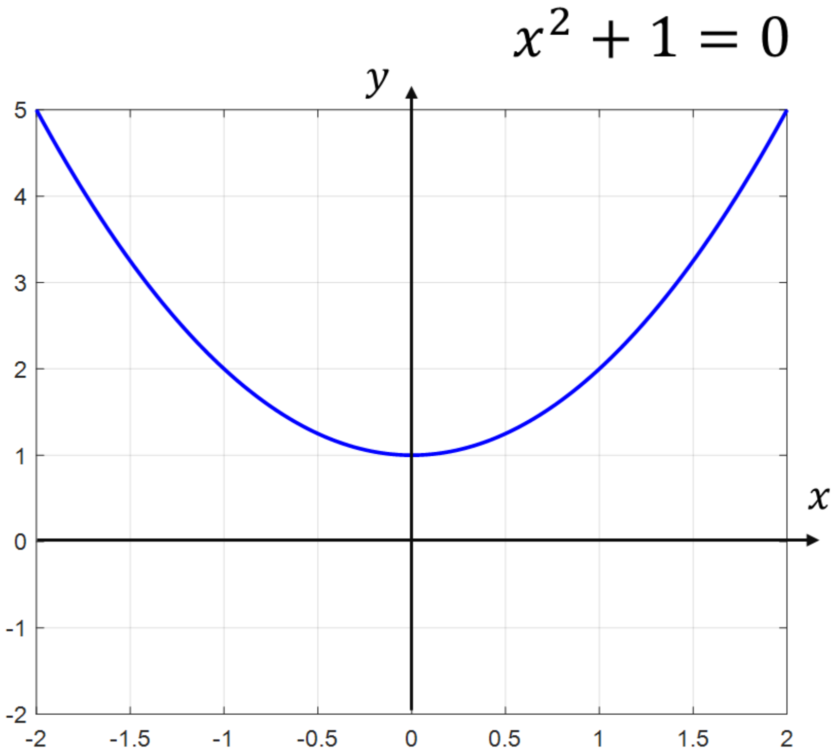
$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm\sqrt{1} = \pm 1$$

Equation 2

$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$x = ?$$

The Equation 2 has no solutions in real number domain, we must create the definition for $\sqrt{-1}$.





Imaginary Unit (虚数単位)

Definition (定義)　Imaginary Unit (虚数単位)

The imaginary unit i is defined by $i = \sqrt{-1}$.

The definition of i tells us that $\boxed{i^2 = -1}$

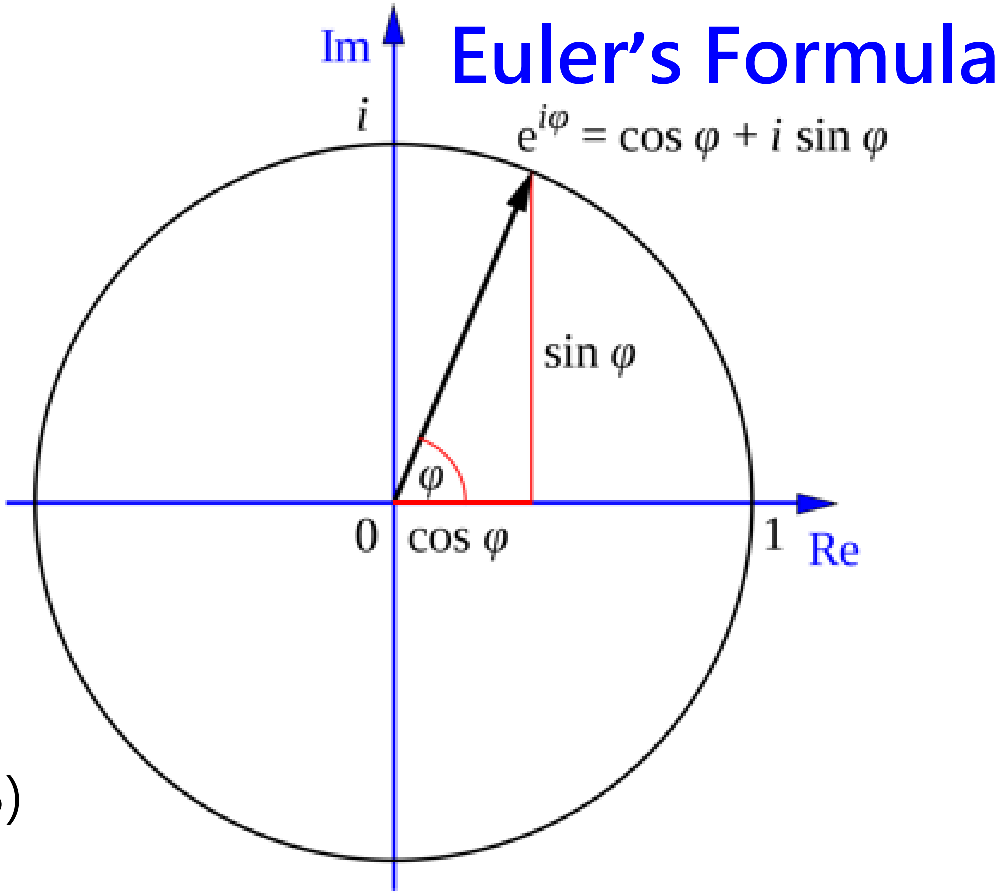
We can use this fact to find other powers of i.

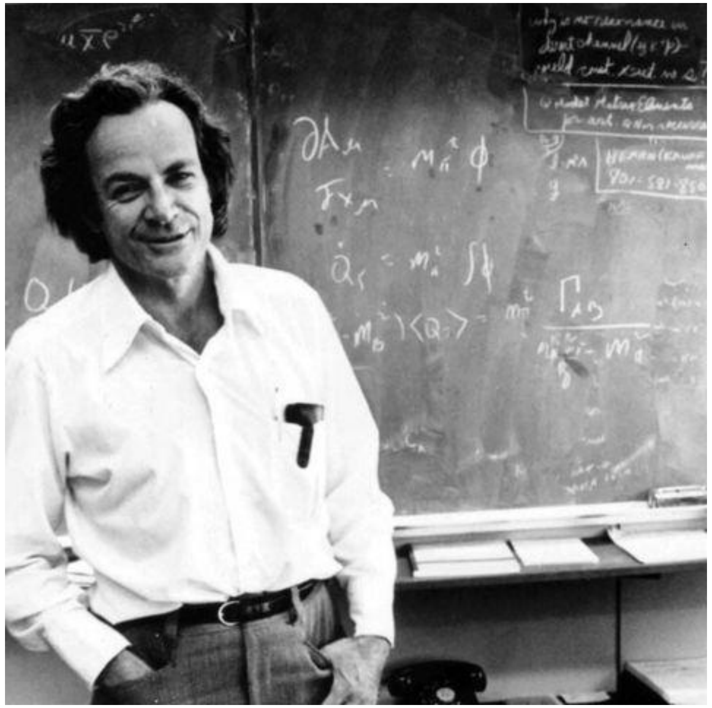
Example

$$i^3 = i^2 \cdot i = (-1) \cdot i = -i \qquad i^4 = i^2 \cdot i^2 = (-1) \cdot (-1) = 1$$

1.1 Why **Complex Number (複素数)** ?

Leonhard Euler
レオンハルト・オイラー
(Switzerland) (1707~1783)





Richard Feynman
リチャード・ファインマン
(U.S.) (1918~1988)

The physicist **Richard Feynman** called the **Euler's Formula (オイラーの公式)** "our jewel" and "the most remarkable formula in mathematics".

[1] https://en.wikipedia.org/wiki/Euler%27s_formula

1.2 Complex Number (複素数)

and Their Properties (性質)

1.2 Complex Number (複素数) and Their Properties (性質)

Complex Number (複素数)

Imaginary Unit (虚数単位)

$$i = \sqrt{-1} \quad \Rightarrow \quad \boxed{i^2 = -1}$$

Pure Imaginary Number (純虚数)

Define **pure imaginary number (純虚数) as** $z = bi$,

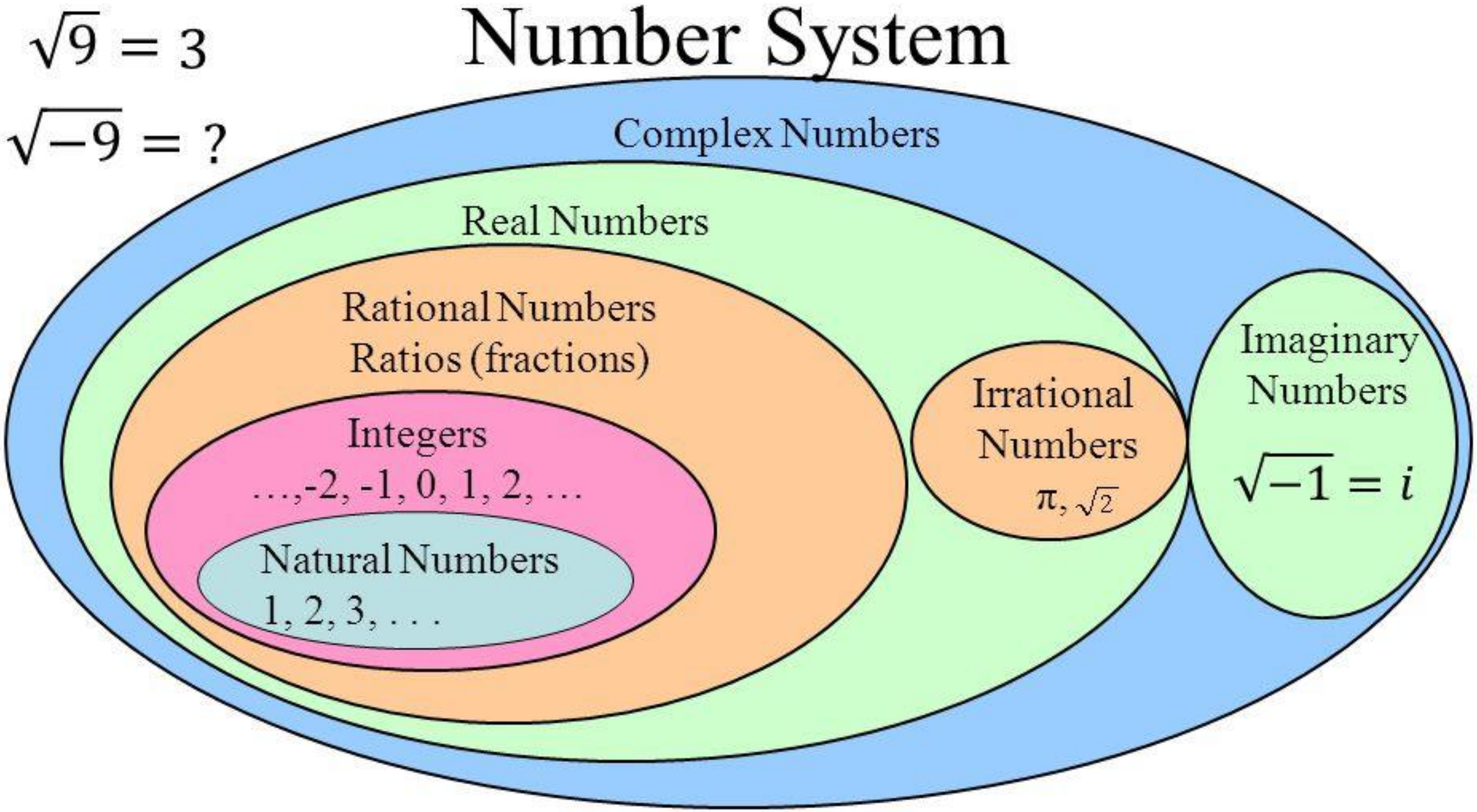
where b is a **real number (実数)** and i is the **imaginary unit (虚数単位)**.

For example, $z = 6i$ or $z = -2i$ is a pure imaginary number (純虚数) .

Definition (定義) 1.1 Complex Number (複素数)

A **complex number** is defined as $z = a + ib$, where a and b are **real numbers** (実数) and i is **the imaginary unit(虚数単位)**.

1.2 Complex Number (複素数) and Their Properties (性質)



Source: https://www.slideserve.com/ankti/complex-numbers

1.2 Complex Number (複素数) and Their Properties (性質)

Real Part (実部) and Imaginary Part (虚部) of Complex Number

In $z = a + ib$,

the real number a is called **the Real part (実部)** of z, i.e. $\mathrm{Re}(z)$;

the real number b is called the **Imaginary part (虚部)** of z, i.e. $\mathrm{Im}(z)$.

For example:

if $z = 4 - 9i$, then $\mathrm{Re}(z) = 4$ and $\mathrm{Im}(z) = -9$.

1.2 Complex Number (複素数) and Their Properties (性質)

Definition 1.2 Equality (相等関係)

If real numbers (実数) $a_1 = a_2$ and $b_1 = b_2$, then Complex numbers (複素数) $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal, i.e. $z_1 = z_2$.

(Two complex numbers are equal if their corresponding real and imaginary parts are equal.)

If we use the symbols $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$,

Definition 1.2 states that $z_1 = z_2$ if $\underset{\underset{a_1}{\shortparallel}}{\mathrm{Re}(z_1)} = \underset{\underset{a_2}{\shortparallel}}{\mathrm{Re}(z_2)}$ and $\underset{\underset{b_1}{\shortparallel}}{\mathrm{Im}(z_1)} = \underset{\underset{b_2}{\shortparallel}}{\mathrm{Im}(z_2)}$.

1.2 Complex Number (複素数) and Their Properties (性質)

The set of Complex numbers (複素数の集合)

The set of Complex numbers (複素数全体の集合, i.e. 複素数体) is usually denoted by the symbol $\mathbf{C}$ or $\mathbb{C}$.

i.e.

$$a + ib \in \mathbf{C}, \quad ib \in \mathbf{C}, \quad a + i0 \in \mathbf{C}$$

Notice: Because any real number a can be written as $z = a + i0 = a$, we see that the set $\mathbf{R}$ of real numbers (実数の集合) is a subset (部分集合) of $\mathbf{C}$.

1.2 Complex Number (複素数) and Their Properties (性質)

Arithmetic Operations (四則演算)

If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, we have the operations as follows.

加法 Addition

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

減法 Subtraction

$$z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2)$$

乗法 Multiplication

$$z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2) = a_1a_2 + i^2 b_1b_2 + ib_1a_2 + ia_1b_2$$

$$= a_1a_2 - b_1b_2 + i(b_1a_2 + a_1b_2)$$

除法 Division

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \qquad a_2 \neq 0 \text{ or } b_2 \neq 0$$

$$= \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)} = \frac{a_1a_2 + b_1b_2 + i(a_2b_1 - a_1b_2)}{a_2^2 - i^2 b_2^2 + i(a_2b_2 - a_2b_2)}$$

$$= \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\,\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2}$$

1.2 Complex Number (複素数) and Their Properties (性質)

The familiar commutative, associative, and distributive laws hold for complex numbers :

交換法則 Commutative laws

$$\begin{cases} z_1 + z_2 = z_2 + z_1 \\ z_1 z_2 = z_2 z_1 \end{cases}$$

結合法則 Associative laws

$$\begin{cases} z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \\ z_1(z_2 z_3) = (z_1 z_2) z_3 \end{cases}$$

分配法則 Distributive laws

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

1.2 Complex Number (複素数) and Their Properties (性質)

Notice (注意):

For Addition (加法), Subtraction (減法), and Multiplication (乗法)

(1) To add (subtract) two complex numbers, simply add (subtract) the corresponding real and imaginary parts (対応する実部と虚部を加算（減算）するだけです).

(2) To multiply (乗算) two complex numbers, use the distributive law (分配法則) and $i^2 = -1$.

*We will discuss the division (除法) later.

EXAMPLE (例題) 1.1.1 Addition (加法) and Multiplication (乗法)

If $z_1 = 2 + 4i$ and $z_2 = -3 + 8i$, find (a) $z_1 + z_2$ and (b) $z_1 z_2$.

Solution (解答):

(a) By adding real and imaginary parts, the sum of the two complex numbers z_1 and z_2 is

$$z_1 + z_2 = (2 + 4i) + (-3 + 8i) = (2 - 3) + i(4 + 8) = -1 + 12i$$

(b) By the distributive law and $i^2 = -1$, the product of z_1 and z_2 is

$$\begin{aligned} z_1 z_2 = (2 + 4i)(-3 + 8i) &= (2 + 4i)(-3) + (2 + 4i)(8i) \\ &= -6 - 12i + 16i + 32i^2 \\ &= (-6 - 32) + i(16 - 12) = -38 + 4i \end{aligned}$$

Zero (ゼロ)

The zero in the **complex number (複素数)** system is the number $0 + 0i$, i.e. 0.

The zero satisfies the **additive identity (加法単位元)** in the complex number system that, for any complex number $z = a + ib$, we have

$z + 0 = z.$

$$z + 0 = (a + ib) + (0 + 0i) = a + 0 + i(b + 0) = a + ib = z.$$

1.2 Complex Number (複素数) and Their Properties (性質)

Conjugate (複素共役、複素共軛)

If z is a **complex number**, then **the complex number** $\bar{z}$ **obtained by** *changing the sign of its imaginary part* (*虚部の符号を変える*) is called **the complex conjugate** (**複素共役**), or simply **conjugate**.

In other words (換言すれば),

if $z = a + ib$, then its **conjugate** is $\bar{z} = a - ib$.

Example: if $z = 6 + 3i$, then $\bar{z} = 6 - 3i$;
if $z = -5 - i$, then $\bar{z} = -5 + i$.

If z is a real number, e.g. $z = 7 + 0i = 7$, then $\bar{z} = 7 - 0i = 7$.

1.2 Complex Number (複素数) and Their Properties (性質)

The sum (和) and product (積) of a complex number z with its conjugate (複素共役) $\bar{z}$ is a real number:

$$z + \bar{z} = (a + ib) + (a - ib) = 2a \tag{1.1.3}$$

$$z\bar{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2 \tag{1.1.4}$$

The difference (差) of a complex number z with its conjugate $\bar{z}$ is a pure imaginary number (純虚数):

$$z - \bar{z} = (a + ib) - (a - ib) = 2bi \tag{1.1.5}$$

Because $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$, (3) and (5) give two useful formulas:

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i} \tag{1.1.6}$$

1.2 Complex Number (複素数) and Their Properties (性質)

Division (除法)

To compute $\frac{z_1}{z_2}$, multiply the numerator (分子) and denominator (分母)

of $\frac{z_1}{z_2}$ by the conjugate (複素共役) of z_2. That is,

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)} \qquad (1.1.7)$$

and then use Equation (1.1.4)

$$z\bar{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2 \qquad (1.1.4)$$

Thus $$\frac{z_1}{z_2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\,\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$$

EXAMPLE (例題) 1.1.2 Division (除法)

If $z_1 = 2 - 3i$ and $z_2 = 4 + 6i$, find z_1/z_2.

Solution (解答):

By using Equation (7): $\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)}$

$$\frac{z_1}{z_2} = \frac{(2 - 3i)(4 - 6i)}{(4 + 6i)(4 - 6i)} = \frac{2 \cdot 4 + i^2 \cdot (-3) \cdot (-6)}{4^2 + 6^2} + i\frac{(-3) \cdot 4 + 2 \cdot (-6)}{4^2 + 6^2}$$

$$= \frac{8 - 18}{16 + 36} + i\frac{-12 - 12}{4^2 + 6^2}$$

$$= -\frac{10}{52} - i\frac{24}{52} \quad = -\frac{5}{26} - i\frac{6}{13}$$

EXAMPLE (例題) 1.1.3 Reciprocal (逆数)

Find the reciprocal of $z = 2 - 3i$.

Solution (解答):

$$\frac{1}{z} = \frac{1}{2-3i} = \frac{1}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{2+3i}{4+9} = \frac{2+3i}{13}$$

Thus $\frac{1}{z} = z^{-1} = \frac{2}{13} + i\frac{3}{13}$

You can verify that $zz^{-1} = (2 - 3i)\left(\frac{2}{13} + i\frac{3}{13}\right) = 1$

Remarks

- We cannot compare two complex numbers $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ by means of inequality (不等式), which means that $z_1 < z_2$ or $z_1 \geq z_2$ have no meaning in $\mathbf{C}$ (複素数全体の集合) except $b_1 = b_2 = 0$ i.e. z_1 and z_2 are both real numbers.

Notice: We can compare **the modulus of complex numbers** by means of inequality, i.e. $|z_1| < |z_2|$ or $|z_1| \leq |z_2|$.

1.3 Complex Plane (複素平面)

1.3 Complex Plane (複素平面)

A complex number (複素数) $z = x + iy$ can be plotted on complex plane **(複素平面)** by a pair of real numbers $(x\,, y)$.

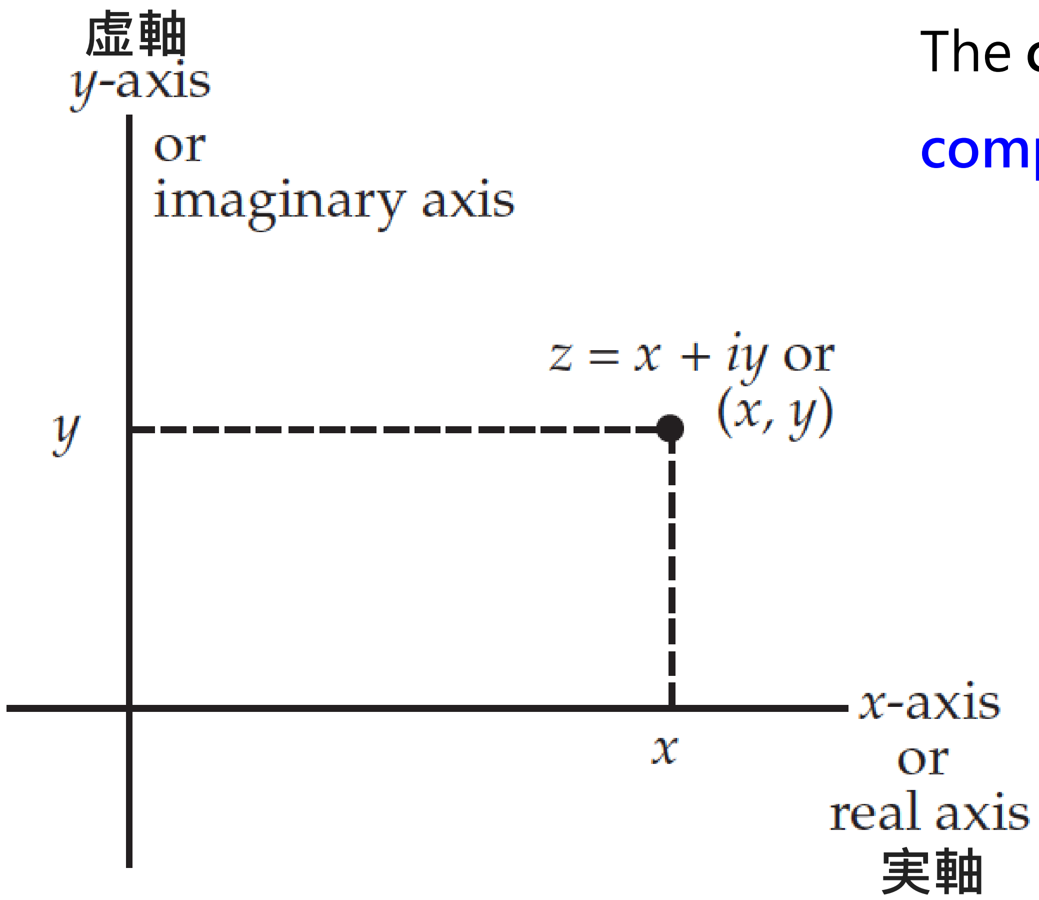


Figure 1.1 z-plane

The **coordinate plane** (座標平面) in Figure 1.1 is called the **complex plane (複素平面)** or simply the **z-plane** (z -平面).

- The x-axis (**横軸**) is called the **Real axis** (実軸) because each point on that axis is a **real number**.
- The y-axis (**縦軸**) is called the **Imaginary axis** (虚軸) because each point on that axis is a **pure imaginary number (純虚数)**.

1.3 Complex Plane (複素平面)

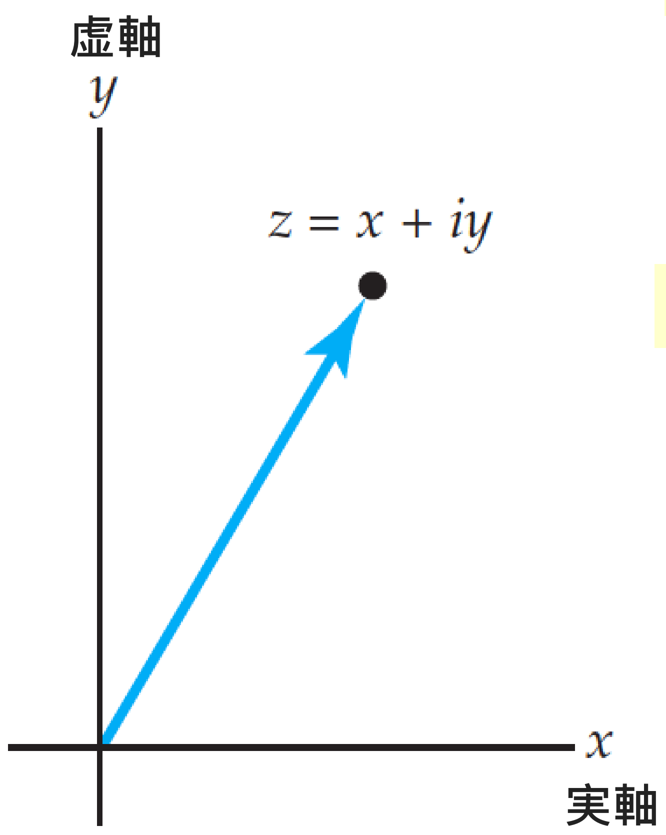


Figure 1.2 A vector z

A **complex number** $z = x + iy$ can be viewed as a **vector (ベクトル)**.

Its initial point (**始点**) is the origin (**原点**) and terminal point (**終点**) is the point (x, y).

The length of vector z (ベクトルz の大きさ) has a special name: Modulus.

Definition 1.3 Modulus (複素数の絶対値)

The modulus (**複素数の絶対値**) of a Complex numbers (**複素数**) $z = x + yi$, is the real number (**実数**)

$$|z| = \sqrt{x^2 + y^2} \tag{1.2.1}$$

Notice: Modulus of z can also be called as absolute value of z or magnitude of z.

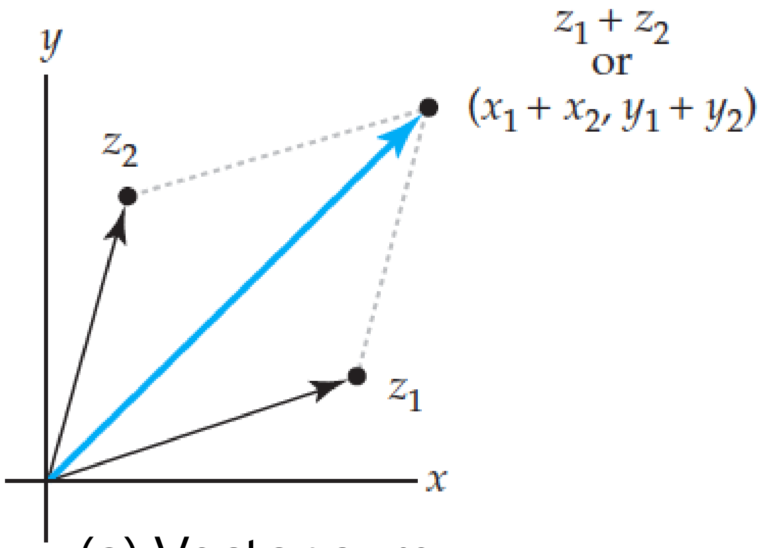
EXAMPLE (例題) 1.2.1 **Find the Modulus (複素数の絶対値) of a Complex Number** (a) $z = 2 - 3i$ (b) $z = -9i$.

Solution (解答):

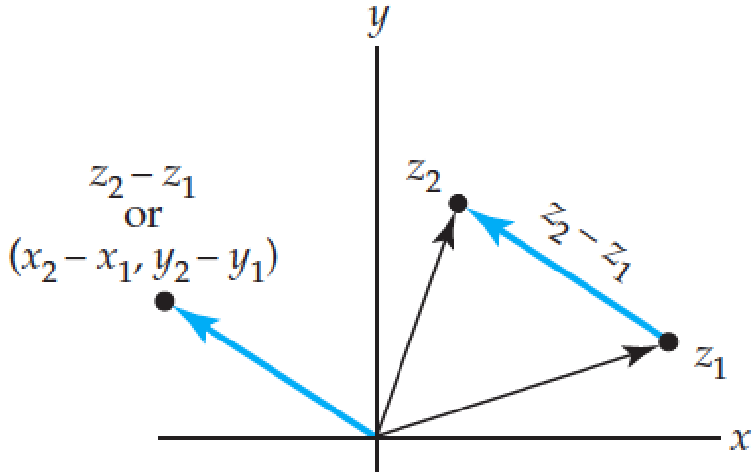
(a) $|z| = \sqrt{x^2 + y^2} = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

(b) $|z| = \sqrt{x^2 + y^2} = \sqrt{0^2 + (-9)^2} = 9$

1.3 Complex Plane (複素平面)



(a) Vector sum



(b) Vector difference

Figure 1.3 Sum and difference of vectors

Sum (和) and Difference (差) of complex numbers by using vectors

$$z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2$$

$z_1 + z_1$ is the vector from the origin to the point

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

Distance (距離) between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$|z_2 - z_1| = |(x_2 - x_1) + i(y_2 - y_1)| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Notice:

When $z_1 = 0$, we see $|z_2 - z_1| = |z_2|$, represents the distance between the origin and the point z_2

Inequalities (不等式)

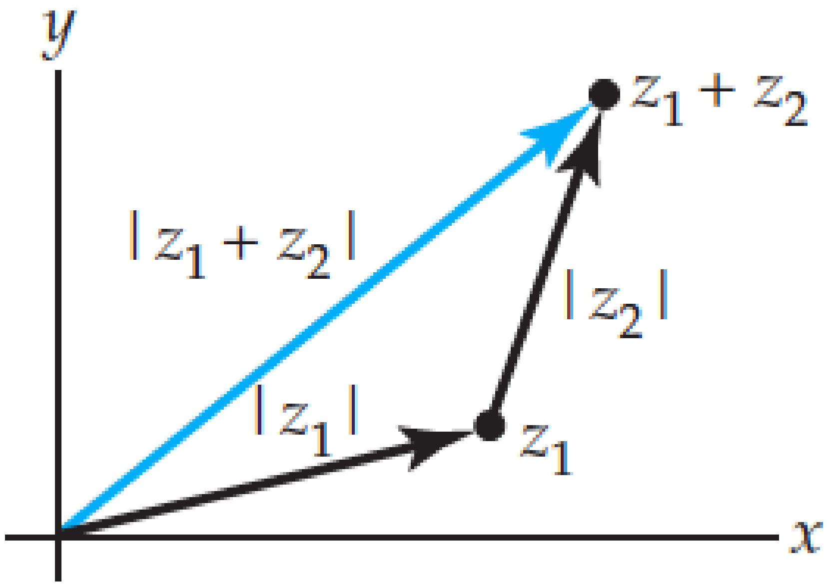


Figure 1.5 Triangle with vector sides

$$|z_1 + z_2| \leq |z_1| + |z_2| \tag{1.2.6}$$

This (1.2.6) is known as **triangle inequality (三角不等式)**.

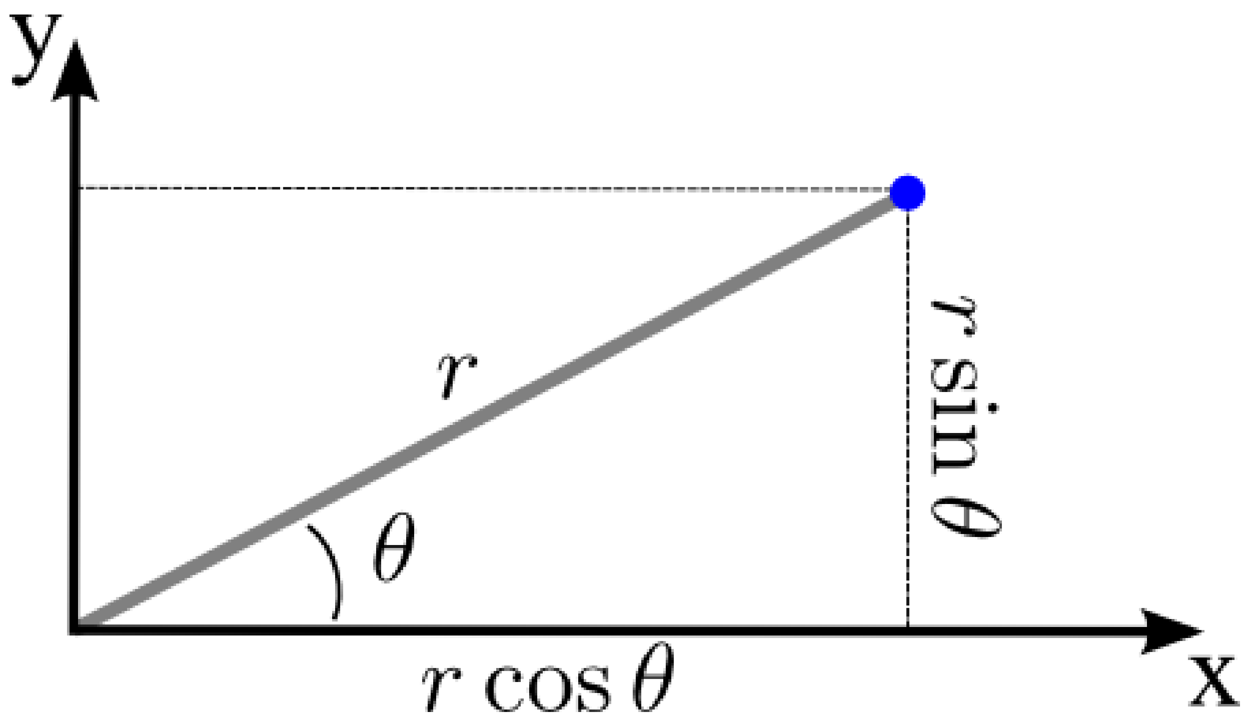
$$|z_1| = |z_1 + z_2 + (-z_2)| \leq |z_1 + z_2| + |-z_2|$$
$$= |z_1 + z_2| + |z_2|$$

$$\Rightarrow \quad |z_1 + z_2| \geq |z_1| - |z_2| \tag{1.2.7}$$

1.4 Polar form (極形式) of

Complex Plane (複素平面)

1.4 Polar form (極形式) of Complex Plane (複素平面)



- Cartesian coordinate system (デカルト座標系、直交座標系): (x, y)
- Polar Coordinate System (極座標系): (r, θ)

1.4 Polar form (極形式) of Complex Plane (複素平面)

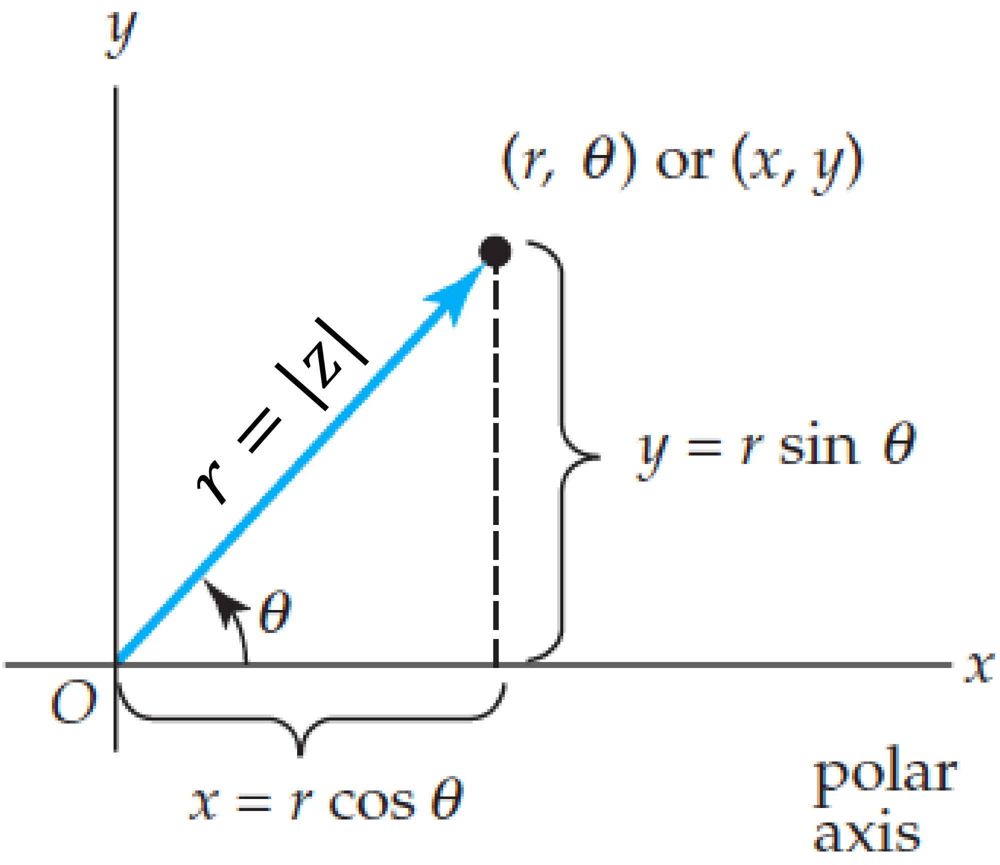


Figure 1.7 **Polar coordinates (極座標系)** in the complex plane

$$z = x + iy$$
$$= (r\cos\theta) + i(r\sin\theta)$$
$$= r(\cos\theta + i\sin\theta) \quad (1.3.1)$$

$$z = re^{i\theta}$$

By using Euler's Formula
$e^{i\theta} = \cos\theta + i\sin\theta$

We say that (1.3.1) is the **polar form (極形式)** of the complex number z.

We call $r = |z| = \sqrt{x^2 + y^2}$ as **modulus** or **magnitude (複素数の絶対値) of** z,

$\theta = \arg(z) = \arctan\frac{y}{x}$ as **argument (偏角) of** z. (Notice quadrant)

EXAMPLE (例題) 1.3.1 Find the polar form of the complex number

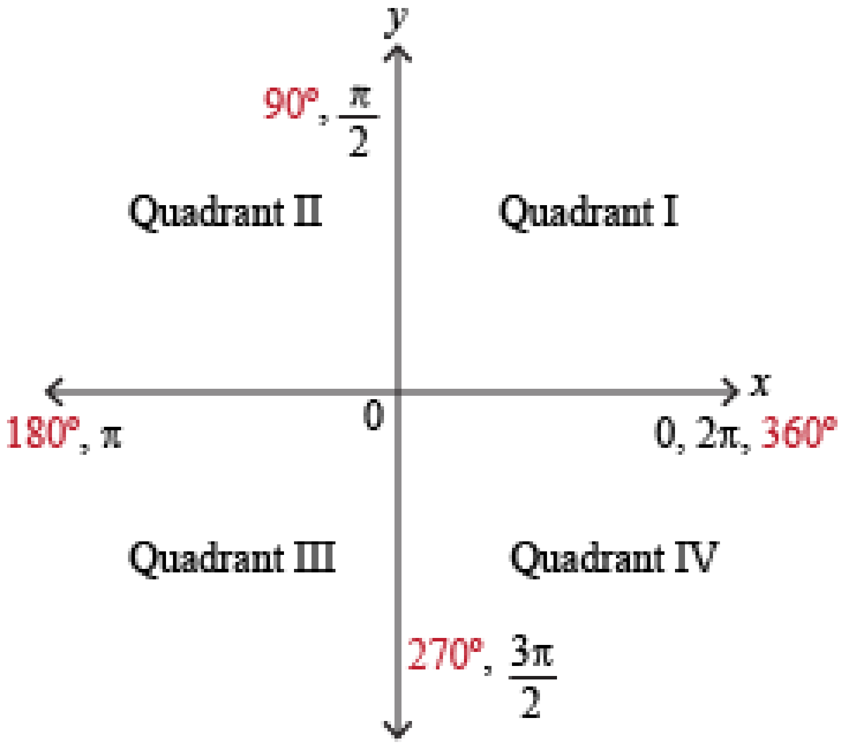
$$z = -\sqrt{3} - i$$

Solution (解答):

$$r = |z| = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2}$$
$$= 2$$
$$\arctan\frac{y}{x} = \arctan\frac{-1}{-\sqrt{3}} = \frac{\pi}{6}$$

Because the point $(-\sqrt{3}, -1)$ is in the **third quadrant (第三象限)** and $\tan\theta$ is π-periodic

$$\theta = \arg(z) = \frac{\pi}{6} + \pi = \frac{7\pi}{6}$$



By Equation (1.3.1) $z = 2(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6})$

Review for Lecture 1

- Imaginary Unit
- Complex Number
- Arithmetic Operations
- Conjugate
- Complex Plane
- Polar form of the Complex Plane

Assignment

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: Section 1.1~1.3, Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Complex Number, BLOSSOMS Fractals Lesson, https://blossoms.mit.edu/sites/default/files/video/download/zager-math-tutorial.pdf

[3] Wikipedia

Appendix (付録)

1.1 Why **Complex Number (複素数)** ?

Recall that in *Calculus II*, we have **Taylor Series**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

Appendix (付録)

1.1 Why **Complex Number (複素数)** ?

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$$= 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots$$

because $i^2 = -1$

$$= 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= \cos x + i \sin x$$

Appendix (付録)

1.2 Complex Number (複素数) and Their Properties (性質)

Additional (追加の) properties

- The conjugate (複素共役) of a sum (和) of two complex numbers **(複素数)** is the sum (和) of the conjugates (複素共役)

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \qquad (1.1.1a)$$

- **Similarly (同様に), for the** difference (差) ,

$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2 \qquad (1.1.1b)$$

And more

- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \qquad (1.1.2)$
- $\bar{\bar{z}} = z$