



Lecture 10

Power Series (冪級数)

&

Taylor Series (テイラー級数)

What you will learn in Lecture 10

10.1 Power Series (冪級数)

10.2 Taylor Series (テイラー級数)

10.1 Power Series (冪級数)

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Power Series

The notion of a power series is important in the study of **analytic functions**.

An infinite series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots \quad (6.1.11)$$

where the coefficients (係数) a_k are complex constants, is called a **power series** in $z - z_0$.

The power series (6.1.11) is said to be **centered at z_0** ;
the complex point z_0 is referred to as the **center of the series**.

10.1 Power Series (冪級数)

Radius of Convergence (収束半径)

Every complex power series (6.1.11) has a radius of convergence, which is the circle centered at z_0 of largest radius $R > 0$ for which (6.1.11) converges at every point within the circle $|z - z_0| = R$.

Power series: (i) converges absolutely at all points z within its circle of convergence, i.e. for all z satisfying $|z - z_0| < R$; and (ii) diverges at all points z exterior (外側の) to the circle, i.e. for all z satisfying $|z - z_0| > R$.

10.1 Power Series (冪級数)

Radius of Convergence

The radius of convergence can be:

- (i) $R = 0$ (in which case (6.1.11) converges only at its center $z = z_0$),
- (ii) R is a **finite positive number** (in which case (6.1.11) **converges at all interior points of the circle $|z - z_0| = R$**), or
- (iii) $R = \infty$ (in which case (6.1.11) **converges for all z**).

10.1 Power Series (冪級数)

EXAMPLE (例題) 6.1.5 Circle of Convergence

Evaluate the convergence condition of the power series $\sum_{k=1}^{\infty} \frac{z^{k+1}}{k}$.

Solution (解答):

Hint:

- Ratio Test (6.1.9) in Lecture 9
- Harmonic Series

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

10.1 Power Series (冪級数)

Notice

It should be clear from Theorem 6.4 in Lecture 9 and Example 6.1.5 that for a power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$, the limit (6.1.9) depends only on the coefficients a_k . Thus, if

(i) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \neq 0$, the radius of convergence is $R = \frac{1}{L}$. (6.1.12)

(ii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$, then the series diverges $R = \infty$. (6.1.13)

(iii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, the test is inconclusive $R = 0$. (6.1.14)

10.1 Power Series (冪級数)

Notice

Similar conclusions can be made for the root test (6.1.10) by using

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \quad (6.1.15)$$

For example, if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L \neq 0$, then $R = \frac{1}{L}$

10.1 Power Series (冪級数)

EXAMPLE (例題) 6.1.6 Radius of Convergence by Ratio Test

Evaluate the convergence condition of the power series

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} (z - 1 - i)^k$ and find its radius of convergence.

Solution (解答):

Hint:

- Equation (6.1.13)

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10.1 Power Series (冪級数)

EXAMPLE (例題) 6.1.7 Radius of Convergence by Root Test

Evaluate the convergence condition of the power series

$\sum_{k=1}^{\infty} \left(\frac{6k+1}{2k+5} \right)^k (z - 2i)^k$ and find its radius of convergence.

Solution (解答):

Hint:

- Root Test (6.1.10) in Lecture 9
- Equation similar to (6.1.12)

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10.1 Power Series (冪級数)

Sums, products and ratios of power series

Let $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ and $\sum_{k=0}^{\infty} b_k(z - z_0)^k$ be two convergent power series whose limits are S and T respectively at a given point z . Then the following properties hold:

- (i) $cS = \sum_{k=0}^{\infty} ca_k(z - z_0)^k, \forall c \in \mathbb{C};$
- (ii) $S + T = \sum_{k=0}^{\infty} (a_k + b_k)(z - z_0)^k, \forall c \in \mathbb{C};$
- (iii) If S or T converges absolutely, then

$$ST = \sum_{k=0}^{\infty} c_k(z - z_0)^k$$

where $c_n = \sum_{k=0}^n a_k b_{n-k}$; furthermore, if the two series S and T converge absolutely, the series converges absolutely;

- (iv) If (a) $b_0 \neq 0$,
 - (b) the series $H = \sum_{k=0}^{\infty} d_k(z - z_0)^k$, where the coefficients d_k are obtained by solving the equations $\sum_{k=0}^n a_k b_{n-k} = a_n, n = 0, 1, \dots$, converges,
 - (c) T or H converges absolutely,
 - (d) $T \neq 0$,

then $\frac{S}{T} = \sum_{k=0}^{\infty} d_k(z - z_0)^k$

10.2 Taylor Series (テイラー級数)

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Notice: A power series *defines* or *represents* a function f .

Differentiation and Integration of Power Series

Theorem 6.6 Continuity

A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ represents a continuous function f within its circle of convergence $|z - z_0| = R$.

10.2 Taylor Series (テイラー級数)

Differentiation and Integration of Power Series

Theorem 6.7 Term-by-Term (項別に) Differentiation

A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ can be differentiated term-by-term within its circle of convergence $|z - z_0| = R$.

Differentiating a power series term-by-term gives,

$$\frac{d}{dz} \sum_{k=0}^{\infty} a_k (z - z_0)^k = \sum_{k=0}^{\infty} a_k \frac{d}{dz} (z - z_0)^k = \sum_{k=1}^{\infty} a_k k (z - z_0)^{k-1}$$

Note that the summation index in the last series starts with $k = 1$ because the term corresponding to $k = 0$ is zero.

Differentiation and Integration of Power Series

Theorem 6.8 Term-by-Term Integration

A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ **can be integrated term-by-term within its circle of convergence $|z - z_0| = R$** , for every contour C lying entirely within the circle of convergence.

This theorem gives that

$$\int_C \sum_{k=0}^{\infty} a_k (z - z_0)^k dz = \sum_{k=0}^{\infty} a_k \int_C (z - z_0)^k dz$$

whenever C lies in the interior of $|z - z_0| = R$.

10.2 Taylor Series (テイラー級数)

Taylor Series

Suppose a power series represents a function f within $|z - z_0| = R$, that is,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad (6.2.1)$$

It follows from Theorem 6.7 that the derivatives of f are the series

$$f'(z) = \sum_{k=0}^{\infty} a_k k (z - z_0)^{k-1} = 0 + a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots \quad (6.2.2)$$

$$f''(z) = \sum_{k=0}^{\infty} a_k k(k-1)(z - z_0)^{k-2} = 0 + 0 + 2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3(z - z_0) + \cdots \quad (6.2.3)$$

$$f'''(z) = \sum_{k=0}^{\infty} a_k k(k-1)(k-2)(z - z_0)^{k-3} = 0 + 0 + 0 + 3 \cdot 2 \cdot 1 \cdot a_3 + 4 \cdot 3 \cdot 2 \cdot a_4(z - z_0) + \cdots \quad (6.2.4)$$

and so on.

10.2 Taylor Series (テイラー級数)

Taylor Series

Since the power series (6.2.1) represents a differentiable function f within its circle of convergence $|z - z_0| = R$, where R is either a positive number or infinity (無限大), we conclude that *a power series represents an analytic function* within its circle of convergence.

There is a relationship between the coefficients a_k in (6.2.1) and the derivatives of f . Evaluating (6.2.1), (6.2.2), (6.2.3), and (6.2.4) at $z = z_0$ we have

$$f(z_0) = a_0, \quad f'(z_0) = 1! a_1, \quad f''(z_0) = 2! a_2, \quad f'''(z_0) = 3! a_3,$$

In general, $f^{(n)}(z_0) = n! a_n$, or

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n \geq 0 \tag{6.2.5}$$

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Taylor Series

When $n = 0$ in (6.2.5), we interpret the **zero-order derivative** as $f(z_0)$ and $0! = 1$ so that the formula gives $a_0 = f(z_0)$.

Substituting (代入する) (6.2.5) into (6.2.1), we have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \quad (6.2.6)$$

This series is called **the Taylor series** for f centered at z_0 .

10.2 Taylor Series (テイラー級数)

Maclaurin Series (マクローリン展開)

A Taylor series with center $z_0 = 0$,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \quad (6.2.7)$$

is referred to as **a Maclaurin series**.

10.2 Taylor Series (テイラー級数)

Question

If we are given a function f that is analytic in some domain D ,
can we represent it by a power series of the form (6.2.6) or (6.2.7)?

Check the answer in Theorem 6.9.

Theorem 6.9 Taylor's Theorem

Let f be analytic within a domain D and let z_0 be a point in D .

Then f has the series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \quad (6.2.8)$$

which is valid for the largest circle C with center at z_0 and radius R that lies entirely within D .

10.2 Taylor Series (テイラー級数)

Some Important Maclaurin Series

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad (6.2.12)$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \quad (6.2.13)$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \quad (6.2.14)$$

10.2 Taylor Series (テイラー級数)

EXAMPLE (例題) 6.2.1 Radius of Convergence

Suppose the function $f(z) = \frac{3-i}{1-i+z}$ is expanded in a Taylor series with center $z_0 = 4 - 2i$. What is its radius of convergence R ?

Solution (解答):

Hint:

- The distance between two complex numbers in complex plane.

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10.2 Taylor Series (テイラー級数)

EXAMPLE (例題) 6.2.2 Maclaurin Series

Find the Maclaurin expansion of $f(z) = \frac{1}{(1-z)^2}$.

Solution (解答):

Hint:

- Equation (6.2.8)
- Geometric Series, Equation (6.1.6)
- Theorem 6.7

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10.2 Taylor Series (テイラー級数)

EXAMPLE (例題) 6.2.3 Taylor Series

Expand $f(z) = \frac{1}{1-z}$ in a Taylor series with center $z_0 = 2i$.

Solution (解答):

Hint:

- Geometric Series, Equation (6.1.6)

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Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

Review for Lecture 10

- (Complex) Power Series
- (Complex) Taylor Series
- (Complex) Maclaurin Series

Assignment

Please Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

Reading Materials: Section 6.1, Section 6.2, Textbook

References

- [1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003
- [2] Wikipedia