Lecture 14

Application to Integral

What you will learn in Lecture 14

14.1 Laplace Transform (ラプラス変換)

14.2 Inverse Laplace Transform (逆ラプラス変換)

14.1 Laplace Transform (ラプラス変換)

14.1 Laplace Transform (ラプラス変換)

Integral Transforms

Suppose $f(x,y)$ is **a real-valued function of two real variables**. Then a definite integral of f with respect to one of the variables leads to a function of the other variable.

For example, if we hold y constant, integration with respect to the real variable x gives $\int_1^2 4xy^2 dx = 6y^2$. Thus a definite integral such as $F(\alpha) = \int_a^b f(x)K(\alpha,x)dx$ transforms a function f of the variable x into a function F of the variable α. We say that

$$F(\alpha) = \int_a^b f(x)K(\alpha,x)dx \qquad (6.7.2)$$

is an **integral transform** of the function f.

Integral transforms usually appear in **transform pairs**. This means that the original function f can be recovered by another integral transform

$$f(x) = \int_c^d F(\alpha)H(\alpha,x)d\alpha \qquad (6.7.3)$$

called **the inverse transform**. The function $K(\alpha,x)$ in (6.7.2) and the function $H(\alpha,x)$ in (6.7.3) are called **the kernels** of their respective transforms.

We note that if α represents a complex variable, then the definite integral (6.7.3) is replaced by **a contour integral**.

14.1 Laplace Transform (ラプラス変換)

The Laplace Transform

Suppose now in (6.7.2) that the symbol α is replaced by the symbol s, and that f represents a represents a real function (On occasion $f(t)$ could be a complex-valued function of a real variable t) that is defined on the unbounded interval $[0, \infty)$. Then (6.7.2) is an improper integral and is defined as

$$\int_0^\infty K(s,t)f(t)dt = \lim_{b\to\infty}\int_0^b K(s,t)f(t)dt \qquad (6.7.4)$$

If the limit in (6.7.4) exists, we say that the integral exists or is convergent; if the limit does not exist, then the integral does not exist and is said to be divergent.

Laplace transform of a real function $f(t)$ defined **for $t \geq 0$**:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt \qquad (6.7.1)$$

Here the choice of the kernel in (6.7.4) is $K(s,t) = e^{-st}$, where s is a complex variable.

The integral that defines the Laplace transform may not converge for certain kinds of functions f. For example, neither $\mathcal{L}\{e^{t^2}\}$ nor $\mathcal{L}\{1/t\}$ exist. Also, the limit in (6.7.4) will exist for only certain values of the variable s.

EXAMPLE (例題) 6.7.1 Existence of a Laplace Transform

Evaluate the Laplace transform of $f(t) = 1$, for $t \geq 0$.

Solution (解答):

The Laplace transform of $f(t) = 1$, t ≥ 0 is

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st}(1)dt = \lim_{b\to\infty} \int_0^{b} e^{-st}(1)dt = \lim_{b\to\infty} \left.\frac{-e^{-st}}{s}\right|_0^b = \lim_{b\to\infty} \frac{1-e^{-bs}}{s} \tag{6.7.5}$$

If s is a complex variable, $s = x + iy$, then recall

$$e^{-bs} = e^{-bx}e^{-iby} = e^{-bx}(\cos by - i\sin by) \tag{6.7.6}$$

From (6.7.6) we see in (6.7.5) that $e^{-bs} \to 0$ as $b \to \infty$ if $x > 0$. In other words, (6.7.5) gives $\mathcal{L}\{1\} = \frac{1}{s}$, provided $\mathrm{Re}(s) = x > 0$.

Theorem 6.22 Sufficient Conditions for Existence of Laplace Transform

Suppose f is piecewise continuous on $[0, \infty)$ and of exponential order c for $t > T$. Then $\mathcal{L}\{f(t)\}$ exists for $\mathrm{Re}(s) > c$.

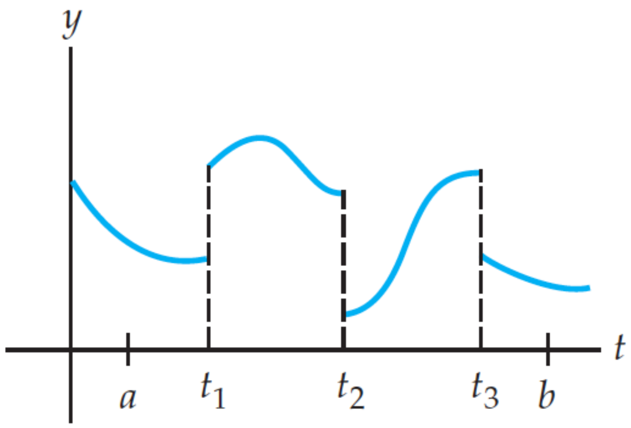

Figure 6.20 Piecewise continuity on $[0, \infty)$

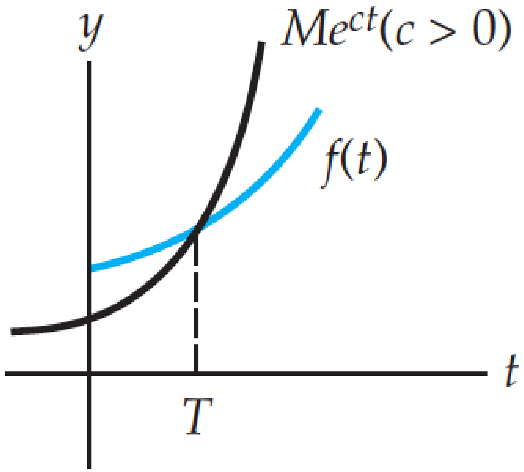

Figure 6.21 Exponential order

A function f is said to be **exponential order** c if there exist constants c, $M > 0$, and $T > 0$ so that $|f(t)| \leq Me^{ct}$, for $t > T$.
The condition $|f(t)| \leq Me^{ct}$ for $t > T$ states that the graph of f on the interval (T, ∞) does not grow faster than the graph of the exponential function Me^{ct}.

Theorem 6.23 Analyticity of the Laplace Transform

Suppose f is piecewise continuous on $[0, \infty)$ and of exponential order c for $t \geq 0$. Then the **Laplace transform** of f,

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

is an analytic function in the right half-plane defined by $\mathrm{Re}(s) > c$.

14.2 Inverse Laplace Transform (逆ラプラス変換)

14.2 Inverse Laplace Transform (逆ラプラス変換)

The Inverse Laplace Transform

Although Theorem 6.23 indicates that the complex function $F(s)$ is analytic to the right of the line $x = c$ in the complex plane, $F(s)$ will have singularities to the left of that line in general.

Theorem 6.24 Inverse Laplace Transform

If f and f' are piecewise continuous on $[0, \infty)$ and f is of exponential order c for $t \geq 0$, and $F(s)$ is a Laplace transform, then the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ is

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\gamma - iR}^{\gamma + iR} e^{st} F(s) dt \qquad (6.7.7)$$

where $\gamma > c$.

14.2 Inverse Laplace Transform (逆ラプラス変換)

The limit in (6.7.7), which defines a principal value of the integral, is usually written as

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)dt \tag{6.7.8}$$

where the limits of integration indicate that the integration is along the infinitely long vertical-line contour $\mathrm{Re}(s) = x = \gamma$.

Here γ is a positive real constant greater than c and greater than all the real parts of the singularities in the left half-plane.

The integral in (6.7.8) is called a **Bromwich contour integral**.

Relating (6.7.8) back to (6.7.3), we see that the kernel of the inverse transform is $H(s,t) = \frac{e^{st}}{2\pi i}$.

14.2 Inverse Laplace Transform (逆ラプラス変換)

Theorem 6.25 Inverse Laplace Transform

Suppose $F(s)$ is a Laplace transform that has a finite number of poles $s_1, s_2, \ldots, s_n$ to the left of the vertical line $\mathrm{Re}(s) = \gamma$ and that C is the contour illustrated in Figure 6.23. If $sF(s)$ is bounded as $R \to \infty$, then

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \mathrm{Res}(e^{st}F(s), s_k) \qquad (6.7.9)$$

Residue (留数)

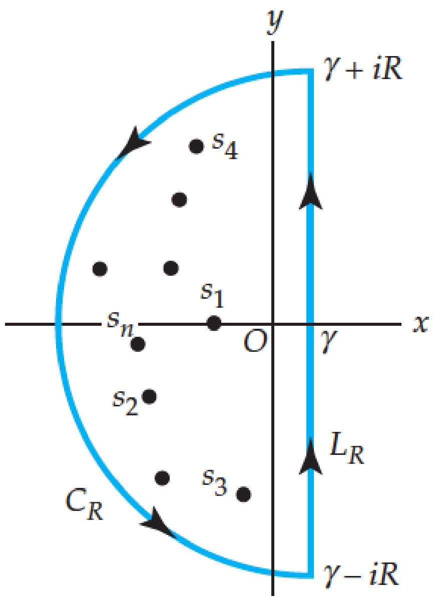

Figure 6.23 Possible contour that could be used to evaluate (6.7.7)

EXAMPLE (例題) 6.7.2 Inverse Laplace Transform

Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$, $\mathrm{Re}(s) > 0$.

Solution (解答):

Considered as a function of a complex variable s, the function $F(s) = \frac{1}{s^3}$ has a pole of order 3 at $s = 0$. Thus by (6.7.9) and (6.5.2) of Lecture 12:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \mathrm{Res}\left(e^{st}\frac{1}{s^3}, 0\right)$$

$$= \frac{1}{2!}\lim_{s\to 0}\frac{d^2}{ds^2}(s-0)^3 e^{st}\frac{1}{s^3}$$

$$= \frac{1}{2}\lim_{s\to 0}\frac{d^2}{ds^2}e^{st}$$

$$= \frac{1}{2}\lim_{s\to 0}t^2 e^{st}$$

$$= \frac{1}{2}t^2$$

Review for Lecture 14

- Laplace Transform
- Inverse Laplace Transform
- Application of Residue to Evaluate Inverse Laplace Transform

Assignment

No homework for Lecture 14

Reading Materials: Section 6.7, Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia

Review

Important Examples for the closed-book Final Exam

Make sure that you can solve the following examples. (Parts of the Assignments will also be covered.)

Lecture 1: Example 1.1.1; 1.1.2; 1.1.3; 1.2.1; 1.3.1
Lecture 2: Example 2.1.1; 2.1.2; 2.6.1; 2.6.4
Lecture 3: Example 3.1.1; 3.1.2; 3.1.3; 3.1.4; 3.2.1; 3.2.2
Lecture 4: Example 3.2.2; 4.1.1; 4.1.3; 4.1.4
Lecture 5: Example 4.2.1; 4.2.2; 4.2.3; 4.3.1; 4.3.2
Lecture 6: Example 5.2.1; 5.2.2; 5.2.3
Lecture 7: Example 5.3.1; 5.3.3; 5.3.4; 5.3.5
Lecture 8: Example 5.5.1; 5.5.2; 5.5.3
Lecture 9: Example 6.1.2; 6.1.3; 6.1.4
Lecture 10: Example 6.1.5; 6.1.6; 6.1.7; 6.2.1; 6.2.2; 6.2.3
Lecture 11: Example 6.3.1 ~ 6.3.4; 6.4.6
Lecture 12: Example 6.4.2 ~ 6.4.4; 6.5.1; 6.5.2
Lecture 13: Example 6.5.4 ~ 6.5.7