Lecture 10

Power Series (冪級数)

&

Taylor Series (テイラー級数)

What you will learn in Lecture 10

10.1 Power Series (冪級数)

10.2 Taylor Series (テイラー級数)

10.1 Power Series (冪級数)

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Power Series

The notion of **a power series** is important in the study of **analytic functions**.

An infinite series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad (6.1.11)$$

where **the coefficients (係数)** a_k **are complex constants**, is called **a power series** in $z - z_0$.

The power series (6.1.11) **is said to be centered at** z_0;

the complex point z_0 is referred to as the center of the series.

Radius of Convergence (収束半径)

Every complex power series (6.1.11) has a radius of convergence, which is the circle centered at z_0 of largest radius $R > 0$ for which (6.1.11) converges at every point within the circle $|z - z_0| = R$.

Power series: (i) **converges absolutely at all points z within its circle of convergence**, i.e. for all z satisfying $|z - z_0| < R$;

and (ii) **diverges at all points z exterior (外側の) to the circle**, i.e. for all z satisfying $|z - z_0| > R$.

Radius of Convergence

The radius of convergence can be:

(i) $R = 0$ (in which case (6.1.11) converges only at its center $z = z_0$),

(ii) R is a finite positive number (in which case (6.1.11) **converges at all interior points of the circle** $|z - z_0| = R$), or

(iii) $R = \infty$ (**in which case** (6.1.11) **converges for all** z).

EXAMPLE (例題) 6.1.5 **Circle of Convergence**

Evaluate the convergence condition of the power series $\sum_{k=1}^{\infty} \frac{z^{k+1}}{k}$.

Solution (解答):

By the ratio test (6.1.9) $$\lim_{n\to\infty} \left| \frac{\frac{z^{n+2}}{n+1}}{\frac{z^{n+1}}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1}|z| = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}|z| = |z|$$

Thus the series converges absolutely for $|z| < 1$. The circle of convergence is $|z| = 1$ and the radius of convergence is $R = 1$. Note that on the circle of convergence $|z| = 1$, the series does not converge absolutely since $\sum_{k=1}^{\infty} \frac{1}{k}$ is the **well-known divergent harmonic series**.

Bear in mind this *does not say* that the series diverges on the circle of convergence. In fact, at $z = -1$, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is **the convergent alternating harmonic series**. Indeed, it can be shown that **the series converges at all points on the circle** $|z| = 1$ **except at** $z = 1$.

Notice

It should be clear from **Theorem 6.4** in Lecture 9 and **Example 6.1.5** that **for a power series** $\sum_{k=0}^{\infty} a_k(z-z_0)^k$, **the limit** (6.1.9) depends only on the coefficients a_k. Thus, if

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \neq 0$, **the radius of convergence** is $R = \frac{1}{L}$. (6.1.12)

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$, then **the series diverges** $R = \infty$. (6.1.13)

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, **the test is inconclusive** $R = 0$. (6.1.14)

Notice

Similar conclusions can be made for **the root test (6.1.10)** by using

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} \tag{6.1.15}$$

For example, if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \neq 0$, then $R = \frac{1}{L}$

EXAMPLE (例題) 6.1.6 Radius of Convergence by Ratio Test

Evaluate the convergence condition of the power series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}(z-1-i)^k$ and find its radius of convergence.

Solution (解答):

We note that $a_n = \frac{(-1)^{n+1}}{n!}$ then

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}}{(n+1)!}}{\frac{(-1)^{n+1}}{n!}}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0$$

Hence by (6.1.13) the radius of convergence is ∞;
the power series with center $z_0 = 1+i$ converges absolutely for all z, that is, for $|z-1-i| < \infty$.

EXAMPLE (例題) 6.1.7 Radius of Convergence by Root Test

Evaluate the convergence condition of the power series

$\sum_{k=1}^{\infty}\left(\frac{6k+1}{2k+5}\right)^{k}(z-2i)^{k}$ and find its radius of convergence.

Solution (解答):

We note that $a_n=\left(\frac{6n+1}{2n+5}\right)^{n}$ then

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{6n+1}{2n+5}=3$$

By reasoning similar to that leading to (6.1.12), we conclude that the radius of convergence of the series is $R=\frac{1}{3}$. The circle of convergence is $|z-2i|=\frac{1}{3}$; the power series converges absolutely for $|z-2i|<\frac{1}{3}$.

Sums, products and ratios of power series

Let $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ and $\sum_{k=0}^{\infty} b_k(z-z_0)^k$ be two convergent power series whose limits are S and T respectively at a given point z. Then the following properties hold:

(i) $cS = \sum_{k=0}^{\infty} ca_k(z-z_0)^k$, $\forall c \in \mathbf{C}$;

(ii) $S + T = \sum_{k=0}^{\infty}(a_k + b_k)(z-z_0)^k$, $\forall c \in \mathbf{C}$;

(iii) If S or T converges absolutely, then

$$ST = \sum_{k=0}^{\infty} c_k(z-z_0)^k$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$; furthermore, if the two series S and T converge absolutely, the series converges absolutely;

(iv) If (a) $b_0 \neq 0$,

(b) the series $H = \sum_{k=0}^{\infty} d_k(z-z_0)^k$, where the coefficients d_k are obtained by solving the equations $\sum_{k=0}^{n} a_k b_{n-k} = a_n$, $n = 0, 1, \dots$, converges,

(c) T or H converges absolutely,

(d) $T \neq 0$,

then $\frac{S}{T} = \sum_{k=0}^{\infty} d_k(z-z_0)^k$

10.2 Taylor Series (テイラー級数)

Notice: A power series *defines* or *represents* a function f.

Differentiation and Integration of Power Series

Theorem 6.6 Continuity

A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ represents **a continuous function** f within its circle of convergence $|z - z_0| = R$.

Differentiation and Integration of Power Series

Theorem 6.7 Term-by-Term (項別に) Differentiation

A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ can be differentiated term-by-term within its circle of convergence $|z-z_0|=R$.

Differentiating a power series term-by-term gives,

$$\frac{d}{dz}\sum_{k=0}^{\infty} a_k(z-z_0)^k = \sum_{k=0}^{\infty} a_k \frac{d}{dz}(z-z_0)^k = \sum_{k=1}^{\infty} a_k k(z-z_0)^{k-1}$$

Note that the summation index in the last series starts with $k=1$ because the term corresponding to $k=0$ is zero.

Differentiation and Integration of Power Series

Theorem 6.8 Term-by-Term Integration

A power series $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ **can be integrated term-by-term within its circle of convergence** $|z-z_0|=R$, for every contour C lying entirely within the circle of convergence.

This theorem gives that

$$\int_C \sum_{k=0}^{\infty} a_k (z-z_0)^k \, dz = \sum_{k=0}^{\infty} a_k \int_C (z-z_0)^k \, dz$$

whenever C lies in the interior of $|z-z_0|=R$.

Taylor Series

Suppose a power series represents a function f within $|z - z_0| = R$, that is,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad (6.2.1)$$

It follows from Theorem 6.7 that the derivatives of f are the series

$$f'(z) = \sum_{k=1}^{\infty} a_k k (z - z_0)^{k-1} = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots \qquad (6.2.2)$$

$$f''(z) = \sum_{k=2}^{\infty} a_k k(k-1)(z - z_0)^{k-2} = 2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3(z - z_0) + \cdots \qquad (6.2.3)$$

$$f'''(z) = \sum_{k=3}^{\infty} a_k k(k-1)(k-2)(z - z_0)^{k-3} = 3 \cdot 2 \cdot 1 \cdot a_3 + 4 \cdot 3 \cdot 2 \cdot a_4(z - z_0) + \cdots \qquad (6.2.4)$$

and so on.

Taylor Series

Since the power series (6.2.1) represents **a differentiable function** f **within its circle of convergence** $|z - z_0| = R$, where R **is either a positive number or infinity (無限大)**, we conclude that *a power series represents an analytic function* **within its circle of convergence**.

There is a relationship between the coefficients a_k in (6.2.1) and the **derivatives of** f. Evaluating (6.2.1), (6.2.2), (6.2.3), and (6.2.4) at $z = z_0$ we have

$$f(z_0) = a_0, \qquad f'(z_0) = 1!\,a_1, \qquad f''(z_0) = 2!\,a_2, \qquad f'''(z_0) = 3!\,a_3,$$

In general, $f^{(n)}(z_0) = n!\,a_n,$ or

$$a_n = \frac{f^{(n)}(z_0)}{n!},\ n \geq 0 \tag{6.2.5}$$

Taylor Series

When $n = 0$ in (6.2.5), we interpret **the zero-order derivative as** $f(z_0)$ and $0! = 1$ so that the formula gives $a_0 = f(z_0)$.

Substituting (代入する) (6.2.5) into (6.2.1), we have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \tag{6.2.6}$$

This series is called **the Taylor series** for f **centered at** z_0.

10.2 Taylor Series (テイラー級数)

Maclaurin Series (マクローリン展開)

A Taylor series with center $z_0 = 0$,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \qquad (6.2.7)$$

is referred to as **a Maclaurin series**.

10.2 Taylor Series (テイラー級数)

Question

If we are **given a function** f that **is analytic in some domain** D, can we **represent it by a power series of the form** (6.2.6) or (6.2.7)?

Check the answer in Theorem 6.9.

Theorem 6.9 Taylor's Theorem

Let f **be analytic within a domain** D and let z_0 be a point in D.

Then f has the series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \qquad (6.2.8)$$

which is valid for the largest circle C **with center at** z_0 **and radius** R **that lies entirely within** D.

Some Important Maclaurin Series

$$e^{z} = 1 + \frac{z}{1!} + \frac{z^{2}}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^{k}}{k!} \tag{6.2.12}$$

$$\sin z = z - \frac{z^{3}}{3!} + \frac{z^{5}}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^{k} \frac{z^{2k+1}}{(2k+1)!} \tag{6.2.13}$$

$$\cos z = 1 - \frac{z^{2}}{2!} + \frac{z^{4}}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^{k} \frac{z^{2k}}{(2k)!} \tag{6.2.14}$$

EXAMPLE (例題) 6.2.1 Radius of Convergence

Suppose the function $f(z) = \frac{3-i}{1-i+z}$ is expanded in a Taylor series with center $z_0 = 4 - 2i$. What is its radius of convergence R?

Solution (解答):

Observe that the function **is analytic at every point except at** $z = -1 + i$, **which is an isolated singularity of** f. The distance from $z = -1 + i$ to $z_0 = 4 - 2i$ is

$$|z - z_0| = \sqrt{(-1-4)^2 + \left(1 - (-2)\right)^2} = \sqrt{34}$$

This last number **is the radius of convergence** R for the Taylor series centered at $4 - 2i$.

EXAMPLE (例題) 6.2.2 **Maclaurin Series**

Find the Maclaurin expansion of $f(z) = \frac{1}{(1-z)^2}$.

Solution (解答):

We could, of course, begin by computing the coefficients using (6.2.8). However, recall from (6.1.6) in Lecture 9 that for $|z| < 1$,

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \tag{6.1.6}$$

If we differentiate both sides of the last result with respect to z, then

$$\frac{d}{dz}\frac{1}{1-z} = \frac{d}{dz}1 + \frac{d}{dz}z + \frac{d}{dz}z^2 + \frac{d}{dz}z^3 + \cdots$$

or

$$\frac{1}{(1-z)^2} = 0 + 1 + 2z + 3z^2 + \cdots = \sum_{k=1}^{\infty} kz^{k-1}$$

Since we are using Theorem 6.7, the radius of convergence of the last power series is the same as the original series, $R = 1$.

EXAMPLE (例題) 6.2.3 Taylor Series

Expand $f(z) = \frac{1}{1-z}$ in a Taylor series with center $z_0 = 2i$.

Solution (解答):

We again use the geometric series (6.1.6) in Lecture 9. By adding and subtracting $2i$ in the denominator of $1/(1-z)$, we can write

$$\frac{1}{1-z} = \frac{1}{1-z+2i-2i} = \frac{1}{1-2i-(z-2i)} = \frac{1}{1-2i} \frac{1}{1-\frac{z-2i}{1-2i}}$$

We now write $\frac{1}{1-\frac{z-2i}{1-2i}}$ as a power series by using (6.1.6) with that z replaced by the expression $\frac{z-2i}{1-2i}$

$$\frac{1}{1-z} = \frac{1}{1-2i}\left[1 + \frac{z-2i}{1-2i} + \left(\frac{z-2i}{1-2i}\right)^2 + \left(\frac{z-2i}{1-2i}\right)^3 + \cdots\right] \tag{6.2.17}$$

Because the distance from the center $z_0 = 2i$ to the nearest singularity $z = 1$ is $\sqrt{5}$, we conclude that the circle of convergence for (6.2.17) is $|z_0 - 2i| = \sqrt{5}$. This can be verified by the ratio test of the Lecture 9.

Review for Lecture 10

- (Complex) Power Series
- (Complex) Taylor Series
- (Complex) Maclaurin Series

Assignment

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: Section 6.1, Section 6.2, Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia