



# Lecture 2

## Complex Functions (複素関数)

# What you will learn in Lecture 2

## 2.1 Complex Functions (複素関数)

## 2.2 Complex Function as Mappings (写像、変換)

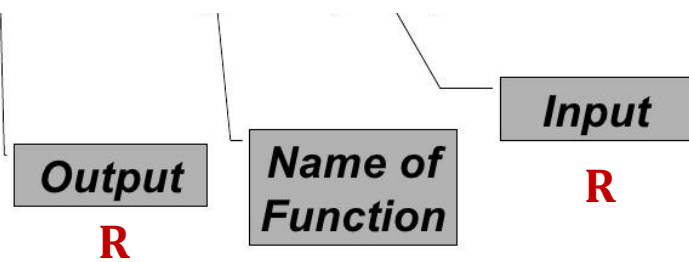
## 2.3 Limits (極限) and Continuity (連続性)

# 2.1 Complex Functions

## (複素関数)

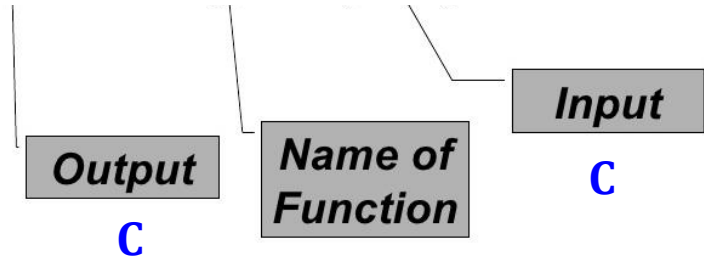
## 2.1 Complex Functions (複素関数)

$$y = f(x)$$



real-valued functions (実数値関数)  
of a real variable (実変数)

$$w = f(z)$$



complex-valued function of a  
complex variable (複素変数)

### Definition (定義) 2.1 Complex Function (複素関数)

A **complex function (複素関数)** is a function  $f$  whose **domain (定義域)** and **range (値域)** are subsets of the set  $\mathbf{C}$  of complex numbers.

We denote the domain and range of a function  $f$  by  $\text{Dom}(f)$  and  $\text{Range}(f)$ , respectively.

## 2.1 Complex Functions (複素関数)

### EXAMPLE (例題) 2.1.1 Complex Function (複素関数)

(a) Evaluate  $f(z) = z^2 - (2 + i)z$  when (1)  $z = i$  and (2)  $z = 1 + i$

(b) Evaluate  $g(z) = z + 2\operatorname{Re}(z)$  when (1)  $z = i$  and (2)  $z = 2 - 3i$

#### **Solution (解答):**

$$(a) \quad (1) \quad f(i) = i^2 - (2 + i)(i) = -1 - 2i + 1 = -2i.$$

$$\begin{aligned} (2) \quad f(1 + i) &= (1 + i)^2 - (2 + i)(1 + i) \\ &= (1 + 2i - 1) - (2 + 2i + i - 1) = -1 - i. \end{aligned}$$

$$(b) \quad (1) \quad g(i) = i + 2\operatorname{Re}(i) = i + 2 \cdot (0) = i$$

$$(2) \quad g(2 - 3i) = 2 - 3i + 2\operatorname{Re}(2 - 3i) = 2 - 3i + 2 \cdot (2) = 6 - 3i$$

Notice: When the domain (定義域) of a complex function (複素関数) is not explicitly stated, we assume the domain (定義域) to be the set of all complex numbers  $z$  for which  $f(z)$  is defined.

## 2.1 Complex Functions (複素関数)

### Real and Imaginary Parts of a Complex Function

If  $w = f(z)$  is a complex function (複素関数), then the image (値域) of a complex number  $z = x + iy$  under  $f$  is a complex number  $w = u + iv$ .

For example, suppose we have the complex function  $w = f(z) = z^2$ , then

$$\begin{aligned} w = z^2 &= (x + iy)^2 = (x^2 - y^2) + 2xyi \\ &= u + iv \end{aligned} \tag{2.1.1}$$

It shows that, if  $w = u + iv = f(z) = f(x + iy)$  is a complex function, then both  $u$  and  $v$  are real functions of the two real variables  $x$  and  $y$ , i.e.

$$w = f(z) = u(x, y) + iv(x, y) \tag{2.1.2}$$

The functions (i.e. 実2変数関数)  $u(x, y)$  and  $v(x, y)$  in (2.1.2) are called the real and imaginary parts of  $f$ , respectively.

## 2.1 Complex Functions (複素関数)

### EXAMPLE (例題) 2.1.2

If  $z = x + iy$ , find the real and imaginary parts (実部と虚部) of the functions (a)  $f(z) = z^2 - (2 + i)z$  (b)  $g(z) = z + 2\operatorname{Re}(z)$

**Solution (解答):**

$$\begin{aligned} \text{(a) } f(z) &= z^2 - (2 + i)z = (x + iy)^2 - (2 + i)(x + iy) \\ &= x^2 + 2xyi - y^2 - (2x + 2yi + ix - y) \\ &= x^2 - 2x + y - y^2 + (2xy - x - 2y)i \end{aligned}$$

$$\text{Therefore } u(x, y) = x^2 - 2x + y - y^2 \quad v(x, y) = 2xy - x - 2y$$

$$\text{(b) } g(z) = z + 2\operatorname{Re}(z) = x + iy + 2\operatorname{Re}(x + iy) = x + iy + 2x = 3x + iy$$

$$\text{Therefore } u(x, y) = 3x \quad v(x, y) = y$$

## 2.2 Complex Function as Mappings

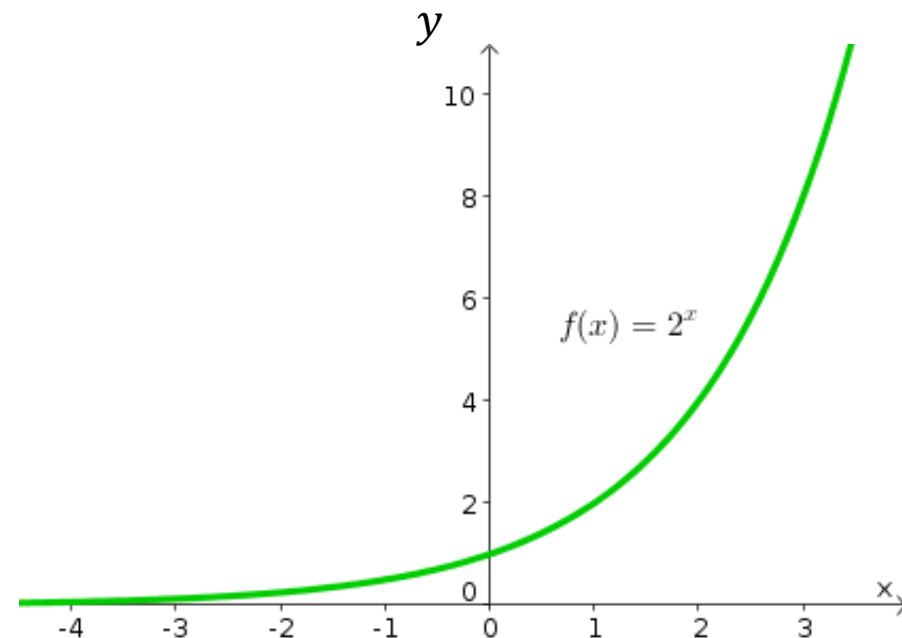
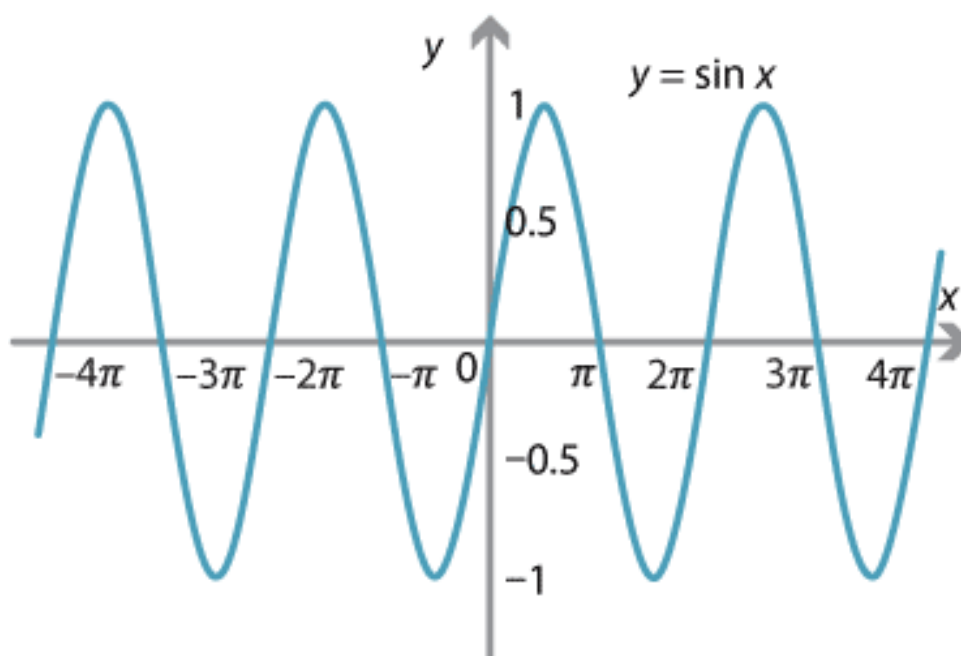
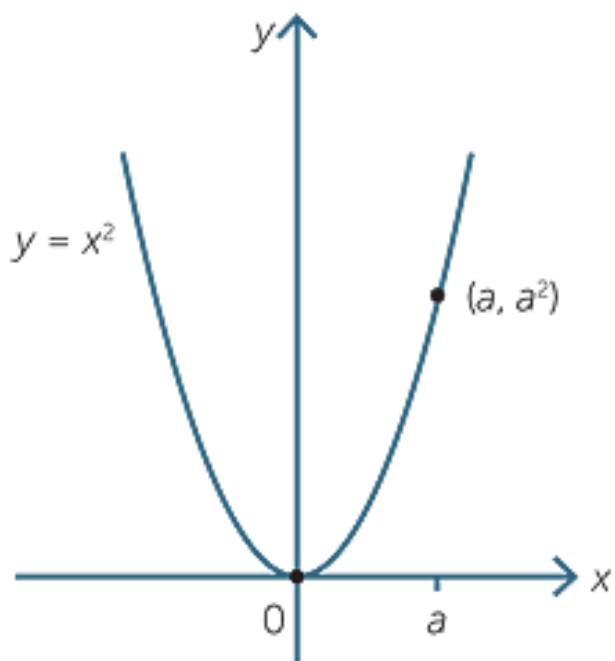
(写像、変換)



## 2.2 Complex Function as Mappings (写像、変換)

We can plot the **graph (グラフ)** of **real-valued function** !

Recall that in *Calculus I*, if  $y = f(x)$  is a real-valued function (実数値関数) of a real variable  $x$ , then **the graph (グラフ)** of  $f$  is defined to be **the set of all points  $(x, f(x))$  (i.e.  $(x, y)$ )** in the **two-dimensional Cartesian plane (i.e. 2次元空間)** (デカルト座標系、直交座標系).



## 2.2 Complex Function as Mappings (写像、変換)

Can we plot **graph** of **complex function**?

However, if  $w = f(z)$  is a complex function, then both  $z$  and  $w$  lie in a complex plane (複素平面).

It follows that **the set** (集合) of all points  $(z, f(z))$  (i.e.  $(z, w)$ ) lies in **four-dimensional space** (4次元空間) (two dimensions from the input  $z$  and two dimensions from the output  $w$ ).

Therefore,

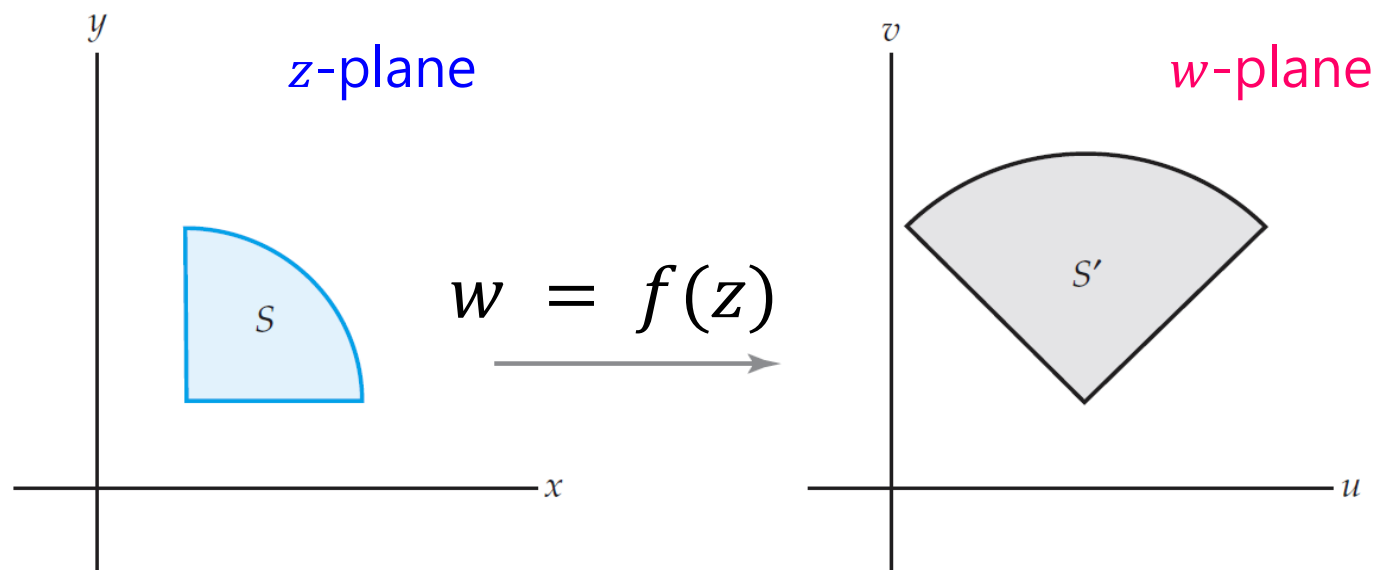
**We cannot draw the graph of a complex function.**

Instead (代わりに), we use the idea of mapping (写像、変換).

## 2.2 Complex Function as Mappings (写像、变换)

### Complex Function as Mappings (写像、变换)

- Define two complex planes.
- The point  $z$  in the  $z$ -plane is associated with the unique point  $w = f(z)$  in the  $w$ -plane.
- Every complex function  $w = f(z)$  describes a correspondence (i.e. mapping) between points in two complex planes.



## 2.2 Complex Function as Mappings (写像、变换)

### Complex Function as Mappings (写像、变换)

If  $w = f(z)$  is a complex mapping and if  $S$  is a set of points in the  $z$ -plane, then we call the image (值域) of  $S$  under  $f$  as the set  $S'$ .

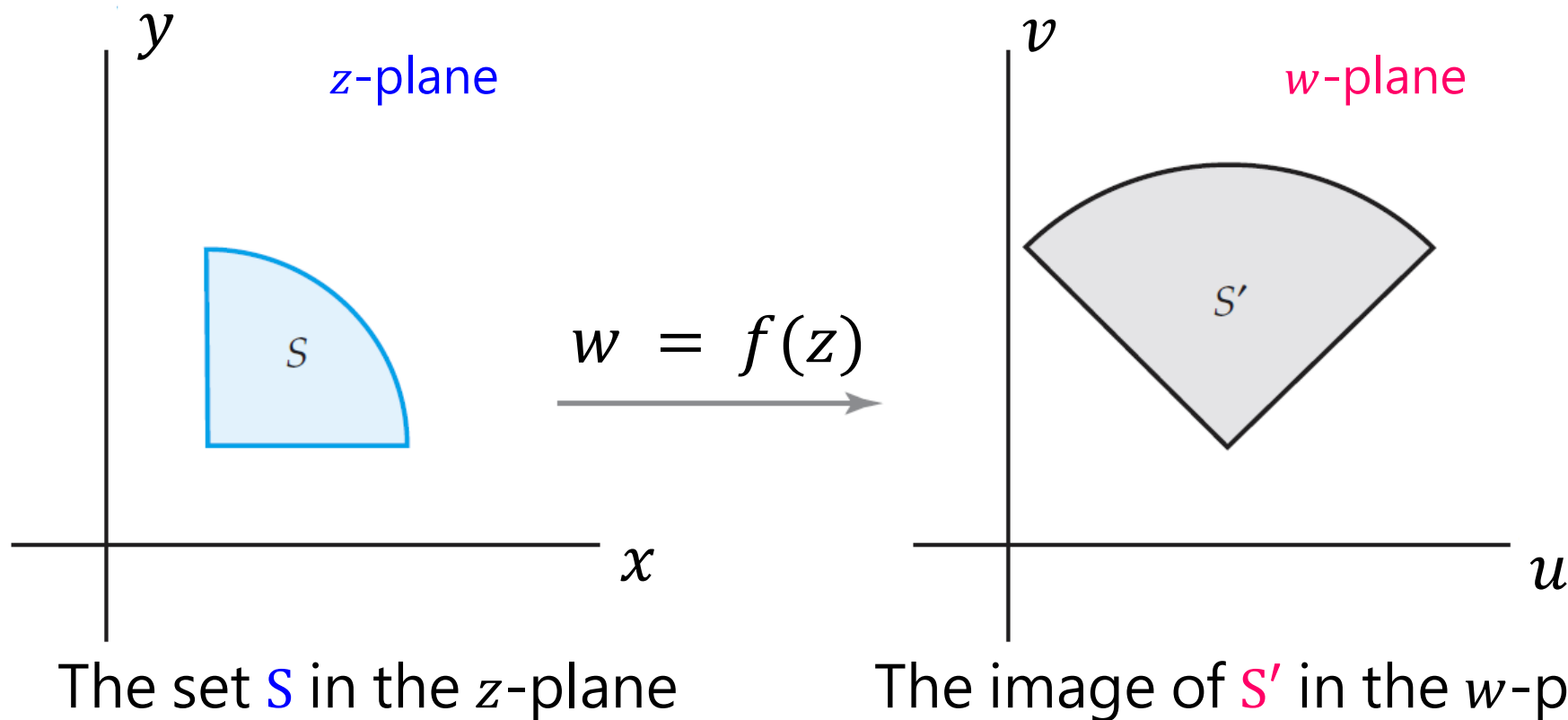


Figure 2.1 The image (值域) of a set  $S$  under a mapping  $w = f(z)$

## 2.2 Complex Function as Mappings (写像、変換)

### EXAMPLE (例題) 2.2.1 Image of a Half-Plane under $w = iz$

Find the image of the half-plane  $\operatorname{Re}(z) \geq 2$  under the complex mapping  $w = f(z) = iz$  and represent the mapping graphically.

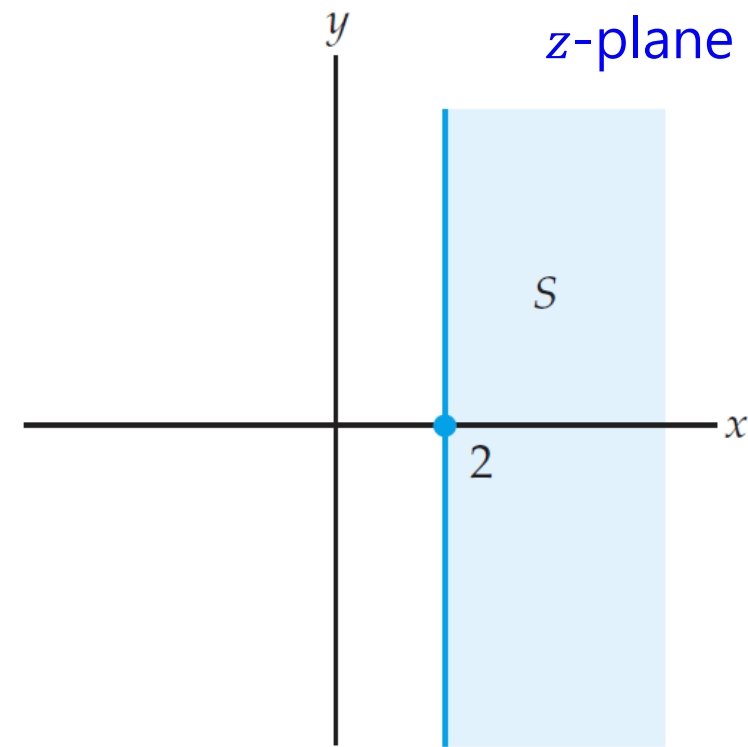
**Solution (解答):**

The vertical (垂直の) boundary line (境界線)

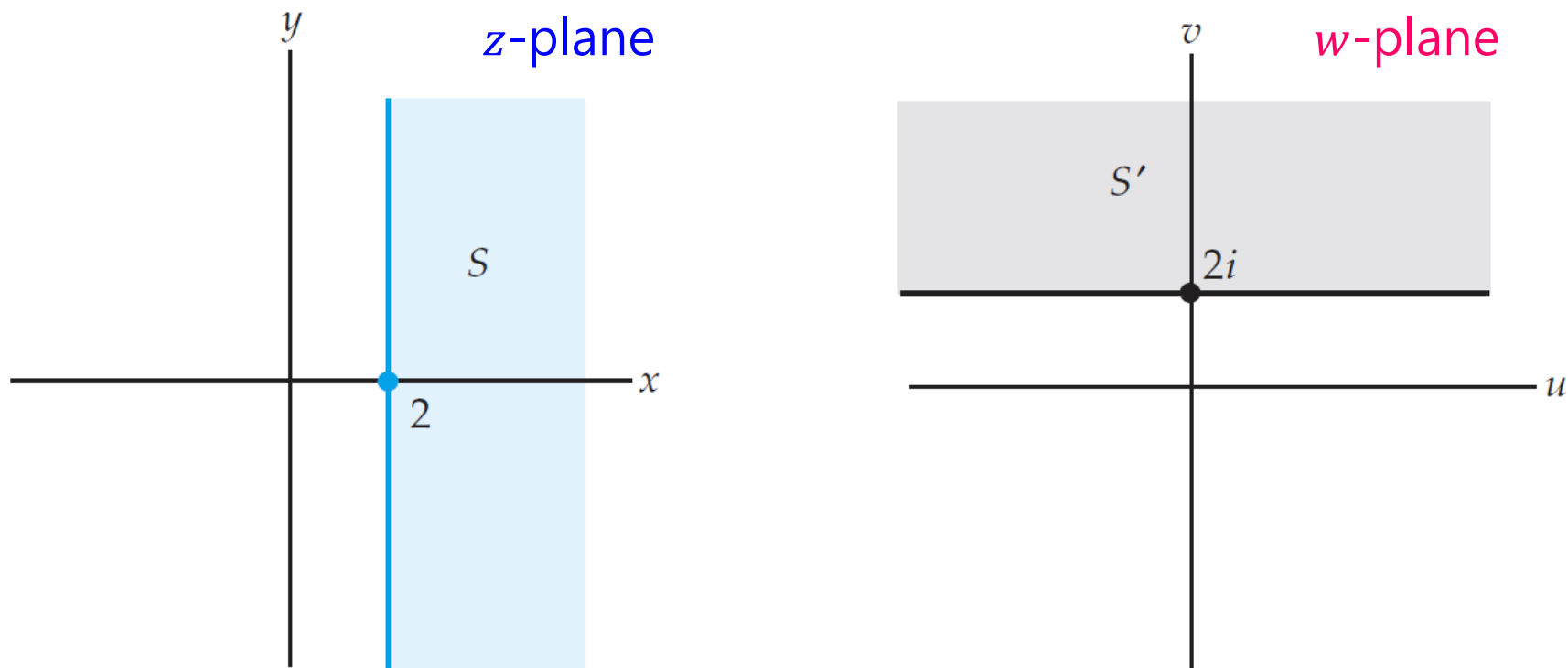
$$\operatorname{Re}(z) = x = 2 \text{ of } S$$

For any point  $z$  on this line we have  $z = 2 + iy$  where  $-\infty < y < \infty$ .

$$w = f(z) = f(2 + iy) = i(2 + iy) = -y + 2i$$



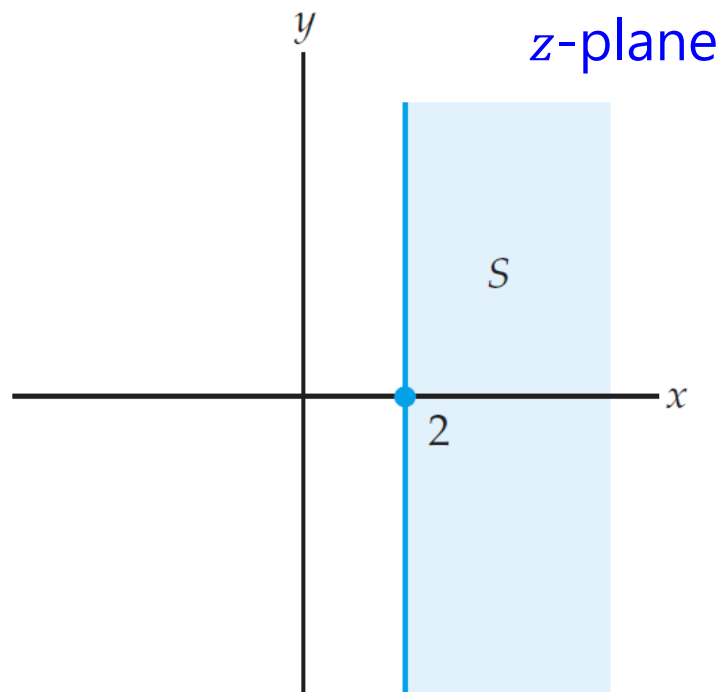
## 2.2 Complex Function as Mappings (写像、变换)



Because the set of points  $w = -y + 2i$ ,  $-\infty < y < \infty$ , is the line  $v = 2$  in the  $w$ -plane,

We conclude that the vertical line (垂直線)  $x = 2$  in the  $z$ -plane is mapped onto the horizontal line (水平線)  $v = 2$  in the  $w$ -plane by the mapping  $w = f(z) = iz$ .

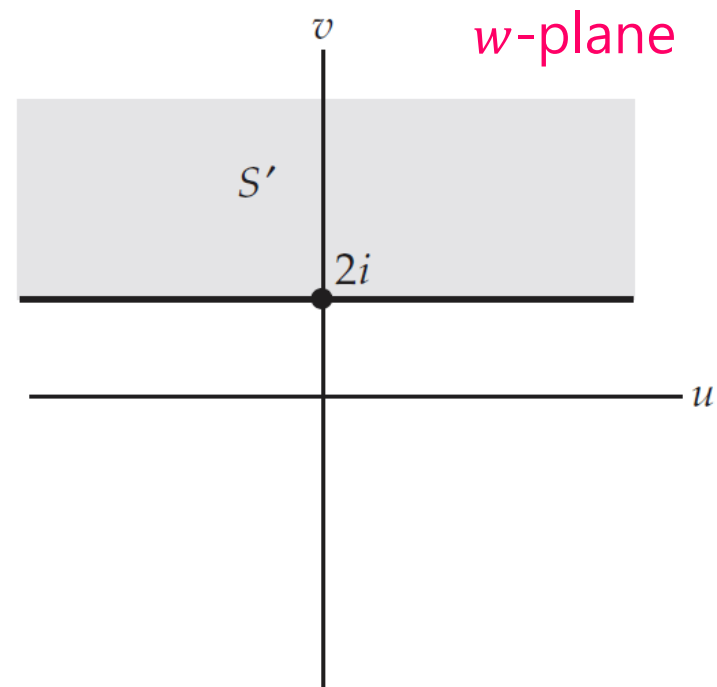
## 2.2 Complex Function as Mappings (写像、变换)



The set  $S$  satisfies inequalities (不等式):

$$x \geq 2 \text{ and } -\infty < y < \infty$$

➡  $\text{Re}(z) \geq 2$



The set  $S'$  satisfies inequalities (不等式):

$$v \geq 2 \text{ and } -\infty < u < \infty.$$

➡  $\text{Im}(w) \geq 2$

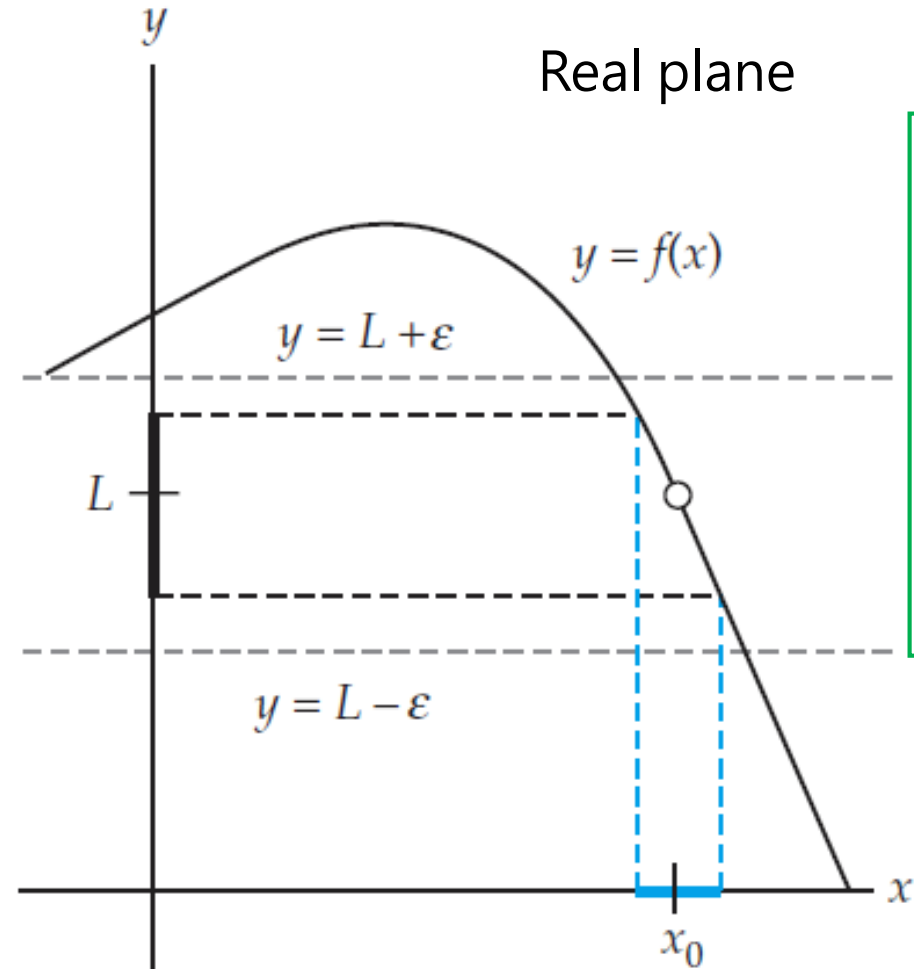
In summary, the half-plane  $\text{Re}(z) \geq 2$  shown in blue color of left figure is mapped onto the half-plane  $\text{Im}(w) \geq 2$  shown in gray color (灰色) in right figure by the complex mapping  $w = f(z) = iz$ .

## 2.3 Limits (極限) and Continuity (連続性)



## 2.3 Limits (極限) and Continuity (連続性)

### Limit of Real Function



The limit of  $f$  as  $x$  tends  $x_0$  exists and is equal to  $L$  if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $|f(x) - L| < \varepsilon$  whenever  $0 < |x - x_0| < \delta$ .

$$\lim_{x \rightarrow x_0} f(x) = L$$

Figure 2.50 Geometric (幾何学の) meaning of a real limit.

## 2.3 Limits (極限) and Continuity (連続性)

### Limit of Complex Function (複素関数の極限)

#### Definition (定義) 2.8

#### Limit of a Complex Function (複素関数の極限)

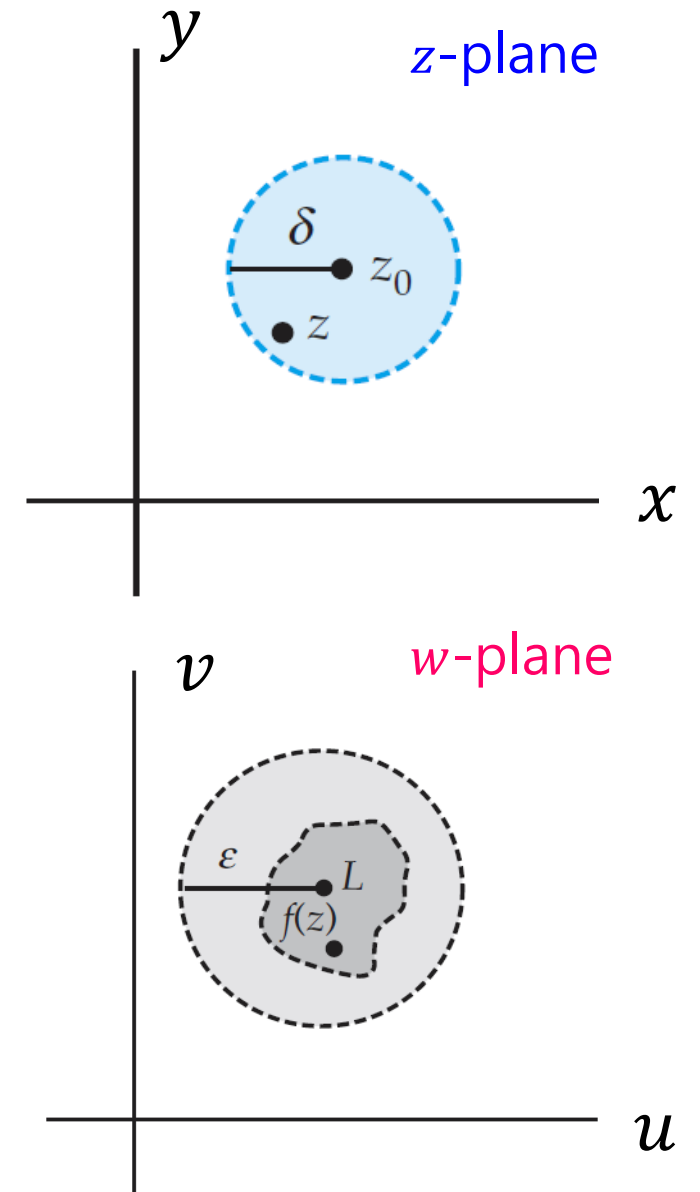
Suppose that a **complex function**  $f$  is defined in a deleted neighborhood of  $z_0$  and suppose that  $L$  is a **complex number**. **The limit of  $f$**  as  $z$  tends to  $z_0$  exists and is equal to  $L$ , written as  $\lim_{z \rightarrow z_0} f(z) = L$ , if for

every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that

$$|f(z) - L| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

in  $w$ -plane

in  $z$ -plane



## 2.3 Limits (極限) and Continuity (連続性)

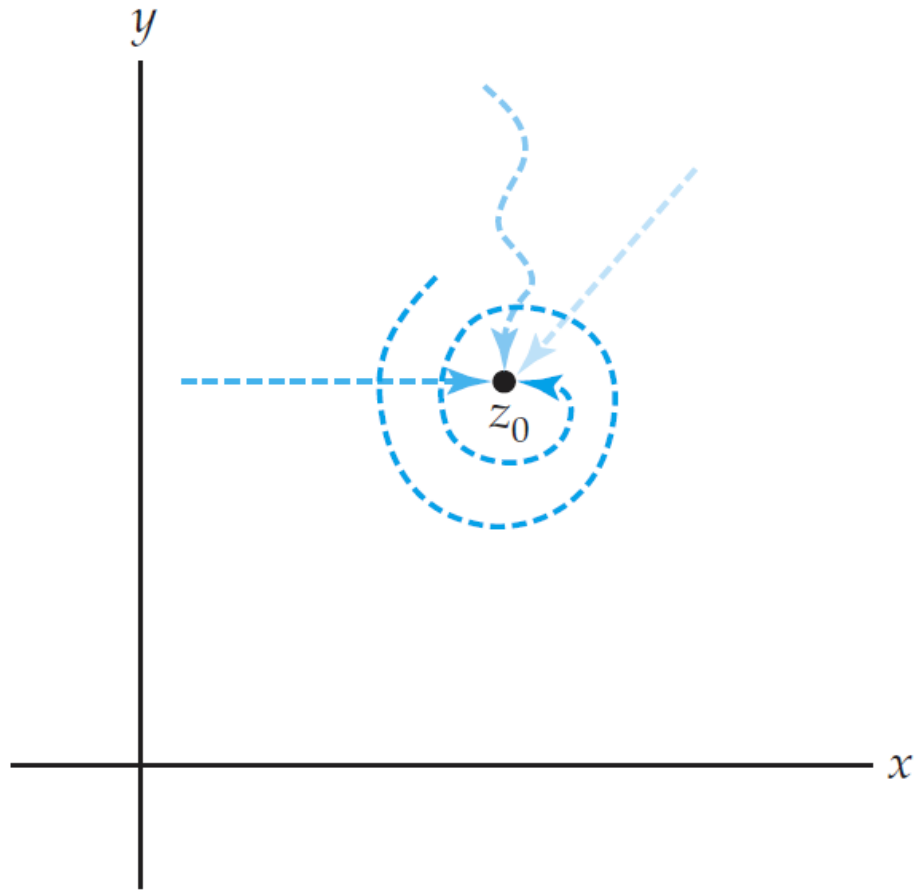


Figure 2.53 Different ways to approach  $z_0$  in a limit.

### Criterion (基準) for the Nonexistence (存在しない) of a Limit

If  $f$  approaches two complex numbers  $L_1 \neq L_2$  for two different curves or paths through  $z_0$ ,

then  $\lim_{z \rightarrow z_0} f(z) = L$  does not exist.

## 2.3 Limits (極限) and Continuity (連續性)

### EXAMPLE (例題) 2.6.1

Show that  $\lim_{z \rightarrow 0} \frac{z}{\bar{z}}$  does not exist.

#### Solution (解答):

First, we let  $z$  approach 0 along the real axis, i.e. we consider complex numbers of the form  $z = x + 0i$  where the real number  $x$  is approaching 0

$$\lim_{z \rightarrow 0} \frac{z}{\bar{z}} = \lim_{x \rightarrow 0} \frac{x + 0i}{x - 0i} = \lim_{x \rightarrow 0} 1 = 1$$

Second, we let  $z$  approach 0 along the imaginary axis, then  $z = 0 + iy$  where the real number  $y$  is approaching 0

$$\lim_{z \rightarrow 0} \frac{z}{\bar{z}} = \lim_{y \rightarrow 0} \frac{0 + iy}{0 - iy} = \lim_{y \rightarrow 0} (-1) = -1$$

The two limits are not same, then conclude that  $\lim_{z \rightarrow 0} \frac{z}{\bar{z}}$  does not exist.

## 2.3 Limits (極限) and Continuity (連続性)

### Theorem 2.1 Real and Imaginary Parts (実部と虚部) of a Limit

Suppose that  $f(z) = u(x, y) + iv(x, y)$  and  $z_0 = x_0 + iy_0$ , and

$L = u_0 + iv_0$ . Then  $\lim_{z \rightarrow z_0} f(z) = L$  if and only if

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0$$

## 2.3 Limits (極限) and Continuity (連續性)

### EXAMPLE (例題) 2.6.3

Use Theorem 2.1 to compute  $\lim_{z \rightarrow 1+i} (z^2 + i)$ , where  $z = x + iy$ .

**Solution (解答):**

Since  $f(z) = z^2 + i = x^2 - y^2 + (2xy + 1)i$ ,

Apply Theorem 2.1 with  $u(x, y) = x^2 - y^2$ ,  $v(x, y) = 2xy + 1$ , and  $z_0 = 1 + i \Rightarrow x_0 = 1, y_0 = 1$

$$u_0 = \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = \lim_{(x,y) \rightarrow (1,1)} (x^2 - y^2) = \lim_{(x,y) \rightarrow (1,1)} (1^2 - 1^2) = 0$$

$$v_0 = \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = \lim_{(x,y) \rightarrow (1,1)} (2xy + 1) = \lim_{(x,y) \rightarrow (1,1)} (2 \cdot 1 \cdot 1 + 1) = 3$$

so  $L = u_0 + iv_0 = 0 + i(3) = 3i$ . Therefore,  $\lim_{z \rightarrow 1+i} (z^2 + i) = 3i$

## 2.3 Limits (極限) and Continuity (連續性)

### Theorem 2.2 Properties (性質) of Complex Limits

Suppose that  $f$  and  $g$  are complex functions. Then  $\lim_{z \rightarrow z_0} f(z) = L$  and  $\lim_{z \rightarrow z_0} g(z) = M$ , then

(i)  $\lim_{z \rightarrow z_0} cf(z) = cL$ , where  $c$  is a complex constant,

(ii)  $\lim_{z \rightarrow z_0} (f(z) \pm g(z)) = L \pm M$ ,

(iii)  $\lim_{z \rightarrow z_0} f(z) \cdot g(z) = L \cdot M$ , and

(iv)  $\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{L}{M}$ , provided  $M \neq 0$ .

## 2.3 Limits (極限) and Continuity (連續性)

We establish two basic complex limits:

- The **complex constant** (定数) function  $f(z) = c$ , where  $c$  is a complex constant (定数)

$$\lim_{z \rightarrow z_0} c = c \quad (2.6.15)$$

- The **complex identity** (恒等) function  $f(z) = z$

$$\lim_{z \rightarrow z_0} z = z_0 \quad (2.6.16)$$



## 2.3 Limits (極限) and Continuity (連續性)

### EXAMPLE (例題) 2.6.4

Use Theorem 2.2 and the basic limits (2.6.15) and (2.6.16) to compute the limits  $\lim_{z \rightarrow i} \frac{(3+i)z^4 - z^2 + 2z}{z+1}$ .

### Solution (解答):

By Theorem 2.2(iii) and (2.6.16),

$$\lim_{z \rightarrow i} z^2 = \lim_{z \rightarrow i} z \cdot z = \left( \lim_{z \rightarrow i} z \right) \cdot \left( \lim_{z \rightarrow i} z \right) = i \cdot i = -1$$

$$\text{Similarly, } \lim_{z \rightarrow i} z^4 = \left( \lim_{z \rightarrow i} z \right) \cdot \left( \lim_{z \rightarrow i} z \right) \cdot \left( \lim_{z \rightarrow i} z \right) \cdot \left( \lim_{z \rightarrow i} z \right) = i^4 = i^2 \cdot i^2 = 1$$

## 2.3 Limits (極限) and Continuity (連續性)

### **Solution (解答) (cont.):**

Using these limits, Theorems 2.2(i), 2.2(ii), and (2.6.16), we obtain:

$$\begin{aligned}\lim_{z \rightarrow i} ((3 + i)z^4 - z^2 + 2z) &= (3 + i) \lim_{z \rightarrow i} z^4 - \lim_{z \rightarrow i} z^2 + 2 \lim_{z \rightarrow i} z \\ &= (3 + i) \cdot (1) - (-1) + 2 \cdot (i) \\ &= 4 + 3i\end{aligned}$$

$$\lim_{z \rightarrow i} z + 1 = 1 + i$$

Therefore, by Theorem 2.2(iv), we have:

$$\lim_{z \rightarrow i} \frac{(3 + i)z^4 - z^2 + 2z}{z + 1} = \frac{\lim_{z \rightarrow i} ((3 + i)z^4 - z^2 + 2z)}{\lim_{z \rightarrow i} z + 1} = \frac{4 + 3i}{1 + i} = \frac{7}{2} - \frac{1}{2}i$$

## 2.3 Limits (極限) and Continuity (連続性)

### Continuity (連続性) of Complex Functions

#### Definition 2.9 Continuity (連続性) of a Complex Function

A complex function  $f$  is continuous at a point  $z_0$  if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

## 2.3 Limits (極限) and Continuity (連続性)

### Continuity (連続性) of Complex Functions

#### Criteria (基準) for Continuity (連続性) at a Point

A complex function  $f$  is continuous at a point  $z_0$  if each of the following three conditions (条件) hold (満たす):

- (i)  $\lim_{z \rightarrow z_0} f(z)$  exists,
- (ii)  $f$  is defined at  $z_0$ , and
- (iii)  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$

## 2.3 Limits (極限) and Continuity (連續性)

### Continuity (連續性) of Complex Functions

#### EXAMPLE (例題) 2.6.5 Checking Continuity at a Point

Consider the function  $f(z) = z^2 - iz + 2$  to determine if  $f$  is continuous at the point  $z_0 = 1 - i$ .

#### Solution (解答):

From Theorem 2.2 and the limits in (2.6.15) and (2.6.16) we obtain:

$$\lim_{z \rightarrow z_0} f(z) = \lim_{z \rightarrow z_0} (z^2 - iz + 2) = (1 - i)^2 - i(1 - i) + 2 = 1 - 3i.$$

Furthermore, for  $z_0 = 1 - i$  we have:

$$f(z_0) = f(1 - i) = (1 - i)^2 - i(1 - i) + 2 = 1 - 3i.$$

Since  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$ , we conclude that  $f$  is continuous at the point  $z_0 = 1 - i$ .

## 2.3 Limits (極限) and Continuity (連続性)

### Continuity (連続性) of Complex Functions

#### Theorem 2.3 Real and Imaginary Parts of a Continuous Function

Suppose that  $f(z) = u(x, y) + iv(x, y)$  and  $z_0 = x_0 + iy_0$ .

Then the complex function (複素関数)  $f$  is continuous at the point  $z_0$  if and only if both real functions (実数値関数)  $u$  and  $v$  are continuous at the point  $(x_0, y_0)$ .

## 2.3 Limits (極限) and Continuity (連続性)

### Continuity (連続性) of Complex Functions

#### EXAMPLE (例題) 2.6.7 Checking Continuity Using Theorem 2.3

Show that the function  $f(z) = \bar{z}$  is continuous on  $\mathbb{C}$ .

#### Solution (解答):

According to Theorem 2.3,  $f(z) = \bar{z} = x + iy = x - iy$  is continuous at  $z_0 = x_0 + iy_0$  if both  $u(x, y) = x$  and  $v(x, y) = -y$  are continuous at  $(x_0, y_0)$ .

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = x_0 \text{ and } \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = -y_0$$

Because  $u$  and  $v$  are two-variable polynomial functions, we have (2.6.13) that

$$\lim_{(x,y) \rightarrow (x_0,y_0)} p(x, y) = p(x_0, y_0) \quad (2.6.13)$$

This implies that  $u$  and  $v$  are continuous at  $(x_0, y_0)$ , and, therefore, that  $f$  is continuous at  $z_0 = x_0 + iy_0$  by Theorem 2.3.

Since  $z_0 = x_0 + iy_0$  was an arbitrary (任意の) point, we conclude that the function  $f(z) = \bar{z}$  is continuous on  $\mathbb{C}$ .

## 2.3 Limits (極限) and Continuity (連續性)

### Continuity (連續性) of Complex Functions

#### Theorem 2.4 Properties (性質) of Continuous Functions

If  $f$  and  $g$  are continuous at the point  $z_0$ , then the following functions are continuous at the point  $z_0$ :

- (i)  $cf$ , where  $c$  is a complex constant,
- (ii)  $f \pm g$ ,
- (iii)  $f \cdot g$ ,
- (iv)  $\frac{f}{g}$ , provided  $g(z_0) \neq 0$ .



## 2.3 Limits (極限) and Continuity (連続性)

### Continuity (連続性) of Complex Functions

#### Theorem 2.5 Continuity of Polynomial Functions (多項式関数)

Polynomial functions (多項式関数) are continuous on the entire complex plane  $\mathbb{C}$ .

# Review for Lecture 2

- Complex Functions
- Complex Functions as Mapping
- Limit of Complex Function
- Continuity of Complex Function

# Exercise

Please Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

Reading Materials: Section 2.1, 2.2, 2.6, Textbook

# References

- [1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003
- [2] Wikipedia