Lecture 4

Harmonic Functions (調和関数)

Exponential and Logarithmic Functions

(指数関数 と 対数関数)

What you will learn in Lecture 4

4.1 Harmonic Functions (調和関数)

4.2 Elementary Functions (初等関数) 1:

Exponential and Logarithmic Functions (指数関数と対数関数)

4.1 Harmonic Functions (調和関数)

4.1 Harmonic Functions (調和関数)

Laplace's Equation (ラプラス方程式)

The **second-order partial differential equation (2階偏微分方程式)**

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{3.3.1}$$

is called **Laplace's equation (ラプラス方程式)** in two independent variables x and y.

(The sum of the two second partial derivatives in (3.3.1) is denoted by $\nabla^2\phi$ and is called the Laplacian of ϕ. Laplace's equation is then abbreviated as $\nabla^2\phi = 0$.)

Definition 3.3 Harmonic Functions (調和関数)

A real-valued function ϕ of two real variables x and y that has **continuous (連続) first and second-order partial derivatives (1と2階偏微分) in a domain D** and satisfies **Laplace's equation** is said to be **harmonic** in D.

Harmonic Functions (調和関数)

Theorem 3.7 Analyticity (解析性) and Harmonic Functions (調和関数)

Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D.

then $u(x,y)$ is harmonic in D

and $v(x,y)$ is harmonic in D.

Proof: The Page 160 of Textbook

Harmonic Functions (調和関数)

EXAMPLE (例題) 3.3.1 Harmonic Functions

Show that the real and imaginary parts of function $f(z) = z^2$, where $z = x + iy$, are harmonic in $\mathbb{C}$.

Solution (解答):

The function $f(z) = z^2 = x^2 - y^2 + 2xyi$ is **entire** (**i.e. 整函数**).

Then the function $f(z) = z^2 = x^2 - y^2 + 2xyi$ is **analytic at every point** z in the complex plane.

According to Theorem 3.7,
The functions $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ are necessarily **harmonic** in the complex plane, i.e. in $\mathbb{C}$.

4.1 Harmonic Functions (調和関数)

Harmonic Conjugate Functions (共役調和関数)

Now suppose $u(x, y)$ is a given real function that is **harmonic** in D;

⇩

find another real harmonic function $v(x, y)$ so that u and v satisfy the **Cauchy-Riemann equations** throughout the domain D;

⇩

then this function $v(x, y)$ is called a **harmonic conjugate function (共役調和関数)** of $u(x, y)$.

⇩

By combining the functions as $u(x, y) + iv(x, y) = f(z)$, we obtain a function $f(z)$ that is **analytic** in D.

Harmonic Conjugate Functions (共役調和関数)

EXAMPLE (例題) 3.3.2 Harmonic Conjugate Function

(a) Verify that the function $u(x, y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane.

(b) Find the harmonic conjugate function of $u(x, y)$.

Solution (解答):

(**a**) From the first and second-order partial derivatives

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = -6xy - 5 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -6x$$

we see that u **satisfies Laplace's equation**: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$

Therefore, according to the Definition 3.3, $u(x, y)$ is **harmonic** in **C**.

Solution (解答)(cont.):

(b) Since the **conjugate harmonic function** v must **satisfy the Cauchy-Riemann equations** $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ then we must have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(-6xy - 5) = 6xy + 5 \qquad (3.3.3)$$

Compute the partial integration of $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2$ **with respect to** y gives $v(x, y) = 3x^2y - y^3 + h(x)$.

From this $v(x, y)$, we compute the partial derivative with respect to x as $\frac{\partial v}{\partial x} = 6xy + h'(x)$.

Compare this $\frac{\partial v}{\partial x}$ with the second equation in (3.3.3), we can obtain $h'(x) = 5$, and so $h(x) = 5x + c$, where c is a real constant.

Therefore, the harmonic conjugate function of $u(x, y)$ is $v(x, y) = 3x^2y - y^3 + 5x + c$.

4.2 Elementary Functions (初等関数) 1:

Exponential and Logarithmic Functions

(指数関数と対数関数)

Complex Exponential Function (複素指数函数)

Suppose we know the fact that $e^{\alpha+\beta} = e^{\alpha}e^{\beta}$, where α and β are complex numbers.

Definition 4.1 Complex Exponential Function (複素指数函数)

The function e^z (where $z = x + iy$) defined by

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

i.e. $$e^z = e^x \cos y + ie^x \sin y \qquad (4.1.1)$$

is called the **complex exponential function**.

Notice: this definition agrees with the real exponential function, i.e.

if z is real number, then $z = x + 0i$, and Definition 4.1 gives:

$$e^{x+i0} = e^x(\cos 0 + i\sin 0) = e^x(1 + i0) = e^x$$

Complex Exponential Function (複素指数函数)

Theorem 4.1 Analyticity (解析性) of e^z

The **exponential function** e^z is **entire** and **its derivative is given by:**

$$\frac{d}{dz}e^z = e^z \tag{4.1.3}$$

Proof: The Page 177 of Textbook

Complex Exponential Function (複素指数函数)

EXAMPLE (例題) 4.1.1 Derivatives of Exponential Functions

Find the derivative of each of the following functions:

(a) $iz^4(z^2 - e^z)$ and (b) $e^{z^2-(1+i)z+3}$

Solution (解答):

(a) Using Equation (4.1.3) and the product rule (積の微分法則) (3.1.4) in Lecture 3:

Product Rule (積の法則): $$\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z) \tag{3.1.4}$$

$$\frac{d}{dz}[iz^4(z^2 - e^z)] = iz^4(2z - e^z) + i4z^3(z^2 - e^z)$$
$$= i6z^5 - iz^4e^z - i4z^3e^z$$

(b) Using Equation (4.1.3) and the chain rule (連鎖律) (3.1.6) in Lecture 3:

Chain Rule (連鎖律): $$\frac{d}{dz}f(g(z)) = f'\big(g(z)\big)g'(z) \tag{3.1.6}$$

$$\frac{d}{dz}\left[e^{z^2-(1+i)z+3}\right] = e^{z^2-(1+i)z+3} \cdot \big(2z - (1+i)\big) = e^{z^2-(1+i)z+3} \cdot (2z - 1 - i)$$

4.2 Exponential and Logarithmic Functions (指数関数と対数関数)

Complex Exponential Function (複素指数函数)

Theorem 4.2 Properties (性質) of e^z

If z_1 and z_2 are complex numbers, then

(i) $e^0 = 1$

(ii) $e^{z_1} e^{z_2} = e^{z_1 + z_2}$

(iii) $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

(iv) $(e^{z_1})^n = e^{n z_1}, n = 0, \pm 1, \pm 2, \ldots$

(v) $|e^z| = e^{\mathrm{Re}(z)},\ \arg(e^z) = \mathrm{Im}(z)$

(vi) $\overline{e^z} = e^{\bar{z}}$

(vii) $e^z \neq 0$, for all $z \in \mathbf{C}$

4.2 Exponential and Logarithmic Functions (指数関数と対数関数)

Complex Exponential Function (複素指数函数)

Modulus (複素数の絶対値) and Argument (偏角)

We have the complex number $w = f(z) = e^z$ in polar form $re^{i\theta}$:

$$w = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = r(\cos\theta + i\sin\theta)$$

then we see that the modulus $r = e^x$ and the argument $\theta = y + 2n\pi$, for $n = 0, \pm 1, \pm 2, \ldots$

Modulus $$|e^z| = r = e^x = e^{\mathrm{Re}(z)} \tag{4.1.4}$$

Argument $$\arg(e^z) = \theta = y + 2n\pi = \mathrm{Im}(z) \quad \text{for } n = 0, \pm 1, \pm 2, \ldots \tag{4.1.5}$$

Conjugate (複素共役)

Because $\cos(-y) = \cos y \quad \sin(-y) = -\sin y$

$$\overline{e^z} = e^x \cos y - ie^x \sin y = e^x \cos(-y) + ie^x \sin(-y) = e^{x-iy} = e^{\bar{z}} \tag{4.1.6}$$

Nonzero (非ゼロ)

From (4.1.4), we know $|e^z| > 0$ because $e^x > 0$ for all $x \in \mathbf{R}$. Then it implies $e^z \neq 0$, for all $z \in \mathbf{C}$.

Complex Exponential Function (複素指数函数)

Periodicity (周期性)

$$\boxed{e^{z+2\pi i} = e^z}$$

The complex exponential function e^z is periodic with a pure imaginary period (純虚数周期) $2\pi i$.

This is because, by (4.1.1) and Theorem 4.2(ii) ,

we have $e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos 2\pi + i \sin 2\pi) = e^z$

Notice that $e^{z+4\pi i} = e^{(z+2\pi i)+2\pi i} = e^{z+2\pi i} = e^z$

By repeating this process we find that

$e^{z+2n\pi i} = e^z$ for $n = 0, \pm 1, \pm 2, \dots$

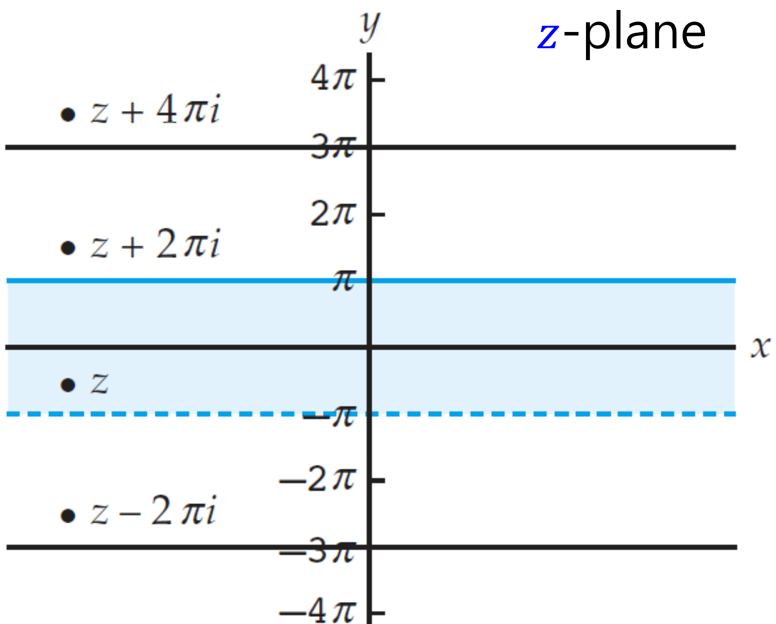


Figure 4.1 The fundamental region of e^z

4.2 Exponential and Logarithmic Functions (指数関数と対数関数)

Complex Exponential Function (複素指数函数)

Fundamental Region of the complex exponential function

- Because $e^{z+2n\pi i} = e^z$ for $n = 0, \pm 1, \pm 2, \ldots$

thus there are many points in the z-plane, for example, $z - 2\pi i, z + 4\pi i, z + 6\pi i$, ...will correspond to the same single point $w = e^z$ in the w-plane,

i.e. the complex exponential function $w = f(z) = e^z$ is not one-to-one (一対一) mapping from z-plane to w-plane.

- **We divide the complex plane into horizontal strips.**
- The **infinite horizontal (水平な) strip** defined by:

$$-\infty < x < \infty, -\pi < y \le \pi$$

is called the **fundamental region (基本領域)** of the complex exponential function e^z.

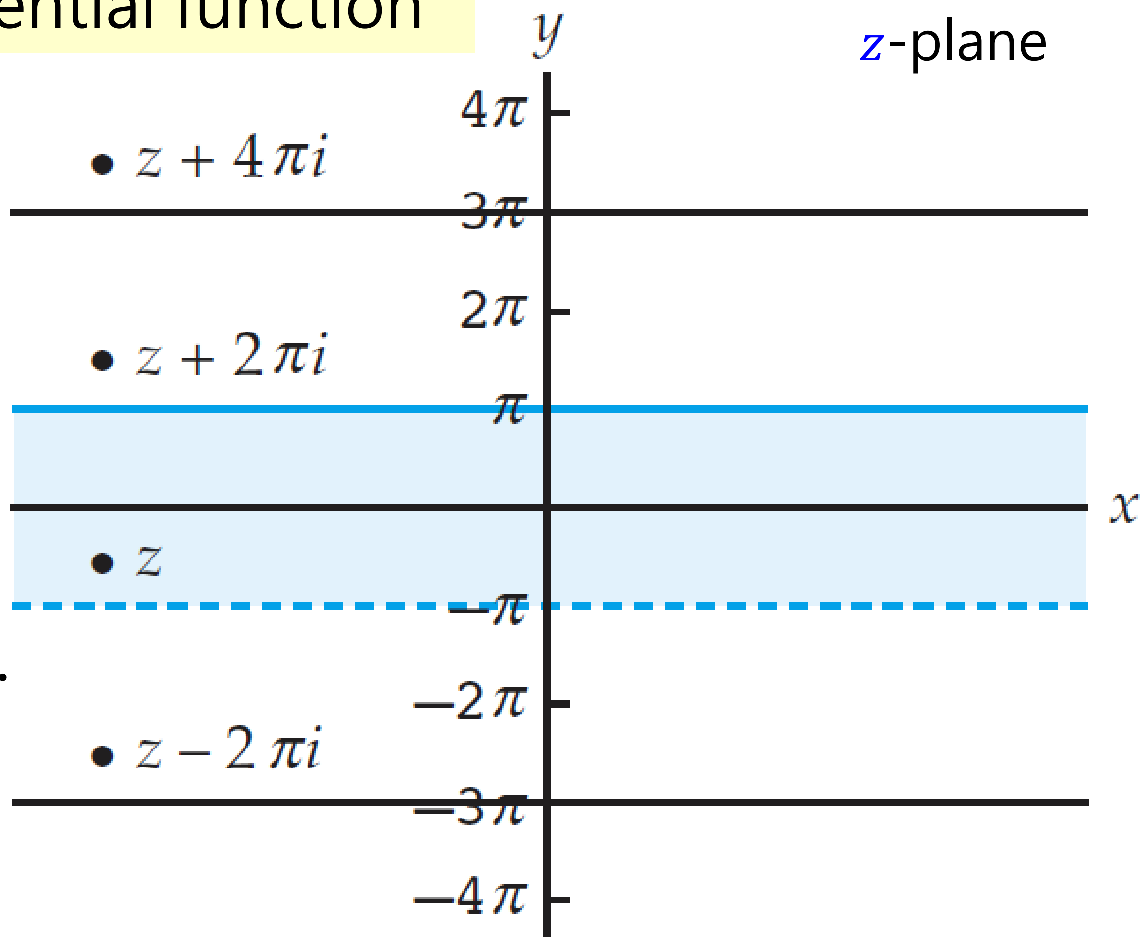
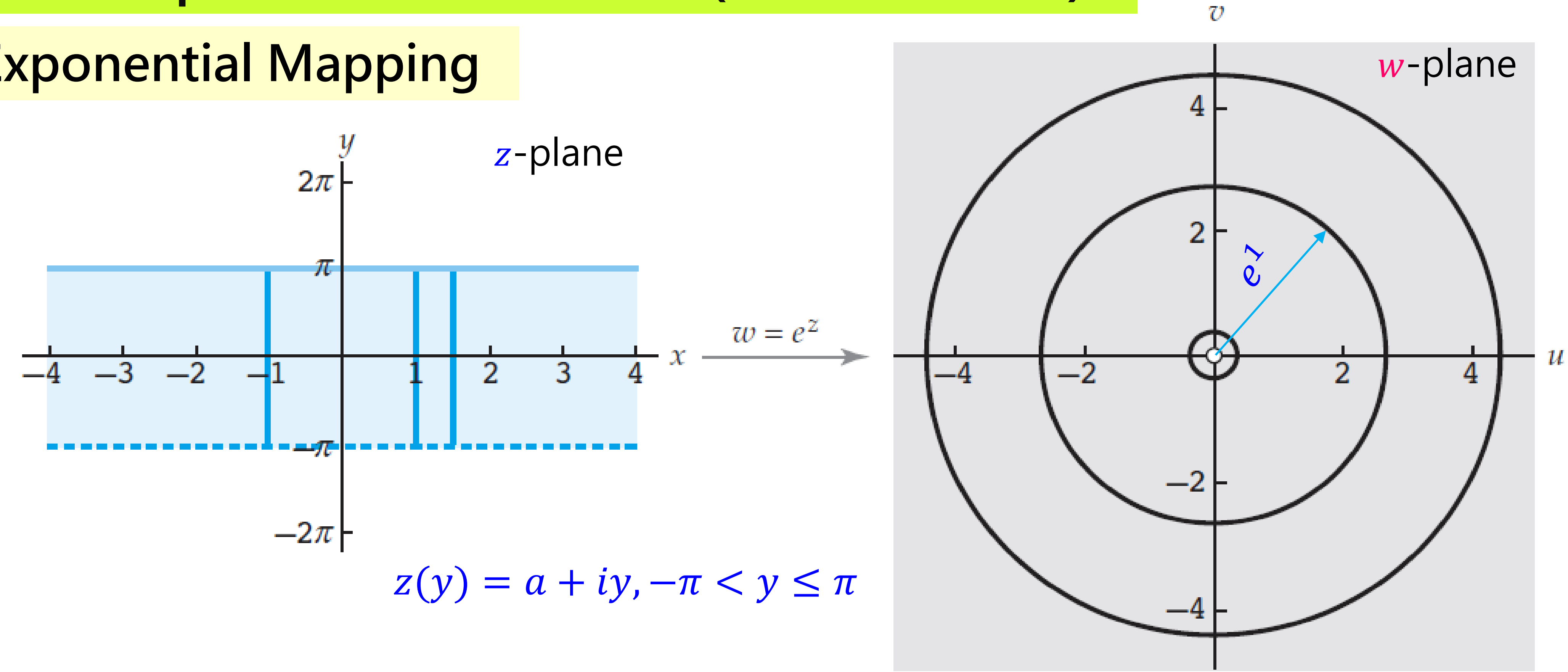


Figure 4.1 The fundamental region of e^z

4.2 Exponential and Logarithmic Functions (指数関数と対数関数)

Complex Exponential Function (複素指数函数)

*The Exponential Mapping



(a) Vertical line segments in the fundamental region (b) Images of the line segments in (a) are circles (円).

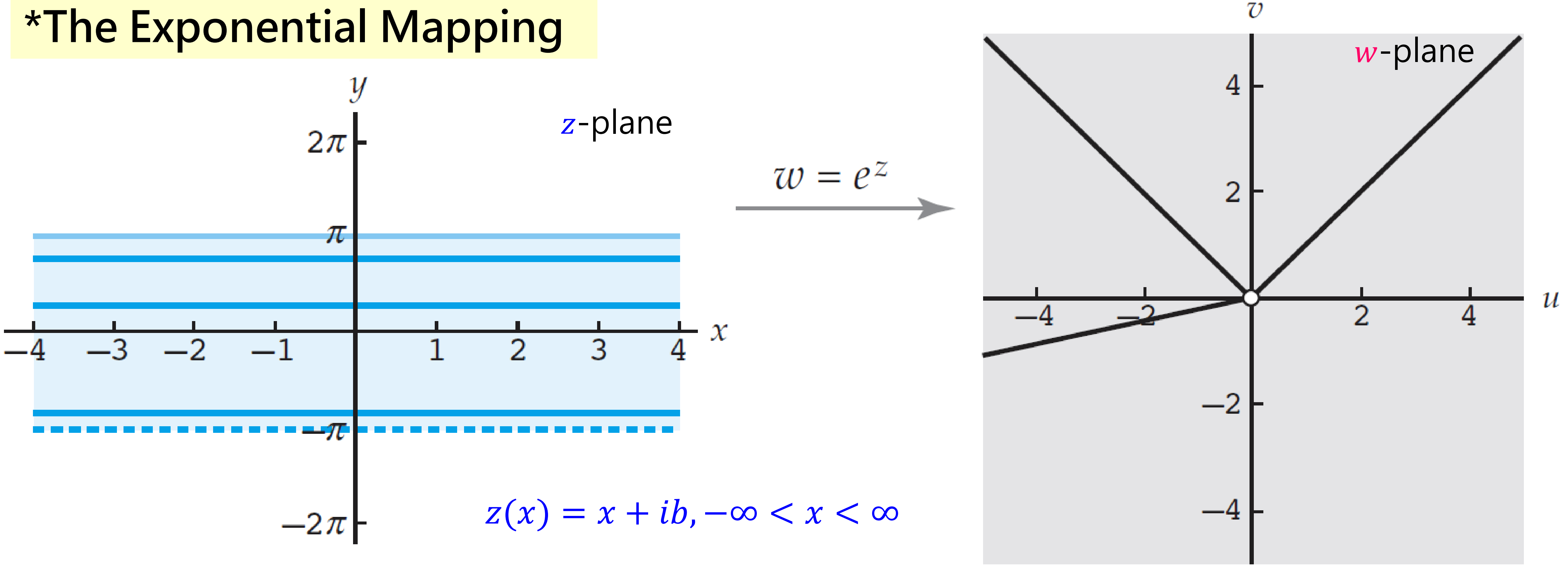
Figure 4.2 The image (i.e. mapping result) of the fundamental region under $w = f(z) = e^z$ for the vertical line segments

Notice: In all lectures, the contents marked with * are not in the scope of the final examination.

4.2 Exponential and Logarithmic Functions (指数関数と対数関数)

Complex Exponential Function (複素指数函数)

*The Exponential Mapping



(a) Horizontal lines in the fundamental region

(b) Images of the lines in (a) are rays (半直線)

Figure 4.3 The mapping $w = f(z) = e^z$ for the horizontal lines

4.2 Exponential and Logarithmic Functions (指数関数と対数関数)

Complex Exponential Function (複素指数函数)

*Exponential Mapping Properties

(i) $w = e^z$ maps the **fundamental region** (基本領域) $-\infty < x < \infty$, $-\pi < y \leq \pi$, onto the set $|w| > 0$, i.e. the points that satisfies $|w| \neq 0$.

(Recall Theorem 4.2(vii), Nonzero property.)

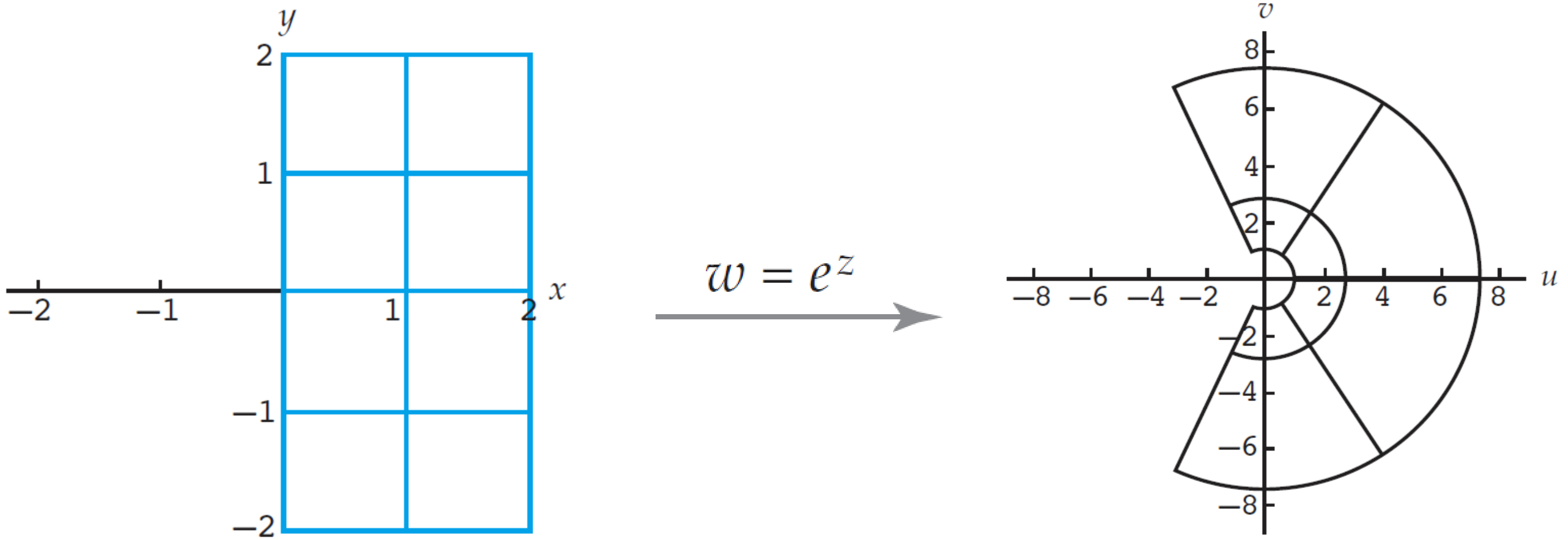
(ii) $w = e^z$ **maps** the **vertical line segment** (垂直の線分) $x = a$, $-\pi < y \leq \pi$, **onto** the **circle** (円) $|w| = e^a$, where a is a real number.

(iii) $w = e^z$ **maps** the **horizontal line** (水平線) $y = b$, $-\infty < x < \infty$, **onto** the **ray** (半直線) $\arg(w) = b$.

Complex Exponential Function (複素指数函数)

***EXAMPLE (例題) 4.1.2 Exponential Mapping of a Grid**

Find the image of the grid shown in Figure 4.4(a) under $w = f(z) = e^z$.



(a) The grid (i.e. Line segments) in z-plane

(b) Image of the grid in w-plane

Figure 4.4 The mapping $w = f(z) = e^z$

Read more in the Page 181 of Textbook.

Complex Logarithmic Functions (複素対数関数)

In **real** domain, the **natural logarithm function** $\ln x$ is often defined as an **inverse function (逆関数)** of **the real exponential function** e^x.

From now on, we will use the alternative notation $\log_e x$ to represent the real natural logarithm function.

- The **real exponential function** e^x is **one-to-one (一対一) on its domain** $\mathbf{R}$,
- **But the complex exponential function** e^z is **NOT a one-to-one function on its domain** $\mathbf{C}$, because there are **infinitely (無限に) many arguments (偏角)** of z.

Complex Logarithmic Functions (複素対数関数)

If $e^w = z$, then $w = \log_e |z| + i \arg(z)$ (4.1.10)

Because of the Periodicity (周期性), there are infinitely (無限に) many arguments (偏角) of z, thus **(4.1.10) gives infinitely many solutions** w to the equation $e^w = z$.

The set of solutions given by (4.1.10) defines a multiple-valued function as:

Definition 4.2 Complex Logarithm Function (複素対数関数)

The multi-valued function $\ln z$ (where $z = x + iy$) defined by

$$\ln z = \log_e |z| + i \arg(z) \tag{4.1.11}$$

is called the complex logarithm.

Notice: We use the lowercase (小文字) first letter for symbol $\ln z$.

Complex Logarithmic Functions (複素対数関数)

EXAMPLE (例題) 4.1.3 Solving Exponential Equations

Find all complex solutions to each of the following equations.

(a) $e^w = i$ (b) $e^w = 1+i$ (c) $e^w = -2$

Solution (解答):

Because $\lim_{a\to\infty} \arctan(a) = \frac{\pi}{2}$

(a) For $e^w = z = i$, we have $|i| = 1$ and $\arg(i) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}) = \frac{\pi}{2} + 2n\pi$.

Thus, from (4.1.11) we obtain:

$$w = \ln i = \log_e |i| + i \arg(i)$$

$$= \log_e 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = 0 + i\left(\frac{\pi}{2} + 2n\pi\right) = \frac{(4n+1)\pi}{2} i \quad n = 0, \pm 1, \pm 2, \ldots$$

Therefore, each of the values $w = \cdots, -\frac{3\pi}{2}i, \frac{\pi}{2}i, \frac{5\pi}{2}i, \ldots$ satisfies the equation $e^w = i$.

Complex Logarithmic Functions (複素対数関数)

Solution (解答)(cont.):

(b) For $z = 1 + i$, we have $|1 + i| = \sqrt{2}$ and $\arg(1 + i) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}) = \frac{\pi}{4} + 2n\pi$.

Thus, from (4.1.11) we obtain:

$$w = \ln(1 + i) = \log_e |1 + i| + i \arg(1 + i)$$
$$= \log_e \sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right) i$$
$$= \frac{1}{2}\log_e 2 + \frac{(8n + 1)\pi}{4} i \qquad n = 0, \pm 1, \pm 2, \ldots$$

(c) For $z = -2$, we have $|-2| = 2$ and $\arg(-2) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}) = \pi + 2n\pi$.

Thus, from (4.1.11) we obtain:

$$w = \ln(-2) = \log_e |-2| + i \arg(-2)$$
$$= \log_e 2 + i(\pi + 2n\pi) i$$
$$= \log_e 2 + (2n + 1)\pi i \qquad n = 0, \pm 1, \pm 2, \ldots$$

Complex Logarithmic Functions (複素対数関数)

Theorem 4.3 Algebraic Properties of $\ln z$

If z_1 and z_2 are nonzero complex numbers and n is an integer, then

(i) $$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

(ii) $$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$$

(iii) $$\ln z_1^n = n \ln z_1$$

Complex Logarithmic Functions (複素対数関数)

Definition 4.3 Principal Value (主値) of the Complex Logarithm

The multi-valued function $\mathrm{Ln}\,z$ (where $z = x + iy$) defined by

$$\mathrm{Ln}\,z = \log_e |z| + i\,\mathrm{Arg}(z),\ -\pi < \mathrm{Arg}(z) \le \pi \qquad (4.1.14 \text{ and } 4.1.15)$$

is called **the principal value (主値) of the complex logarithm**.

Notice: We use the uppercase (大文字) first letter for $\mathrm{Ln}\,z$ here!

Complex Logarithmic Functions (複素対数関数)

EXAMPLE (例題) 4.1.4 Principal Value of the Complex Logarithm

Compute the principal value of the complex logarithm $\mathrm{Ln}\, z$ for

(a) $z = i$ (b) $z = 1 + i$ (c) $z = -2$

Solution (解答):

(a) For $e^w = z = i$, we have $|i| = 1$ and $\mathrm{Arg}(i) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}) = \frac{\pi}{2} \cancel{+ 2n\pi}$

Thus, from (4.1.11) we obtain:

$$w = \mathrm{Ln}\, i = \log_e |i| + i\, \mathrm{Arg}(i)$$
$$= \log_e 1 + \frac{\pi}{2} i = 0 + \frac{\pi}{2} i = \frac{\pi}{2} i \qquad \cancel{n = 0, \pm 1, \pm 2, \ldots}$$

Complex Logarithmic Functions (複素対数関数)

Solution (解答)(cont.):

(b) For $e^w = z = 1 + i$, we have $|1+i| = \sqrt{2}$ and $\mathrm{Arg}(1+i) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}) = \frac{\pi}{4}$.

Thus, from (4.1.11) we obtain:

$$w = \mathrm{Ln}(1+i) = \log_e|1+i| + i\,\mathrm{Arg}(1+i)$$

$$= \log_e\sqrt{2} + \frac{\pi}{4}i = \frac{1}{2}\log_e 2 + \frac{\pi}{4}i \approx 0.3466 + 0.7854i$$

(c) For $e^w = z = -2$, we have $|-2| = 2$ and $\mathrm{Arg}(-2) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}) = \pi$.

Thus, from (4.1.11) we obtain:

$$w = \mathrm{Ln}(-2) = \log_e|-2| + i\,\mathrm{Arg}(-2)$$

$$= \log_e 2 + \pi i \approx 0.6931 + 3.1416i$$

Complex Logarithmic Functions (複素対数関数)

$\mathrm{Ln}\, z$ as an Inverse Function (逆関数) of e^z

Follows from (4.1.10) that

$$e^{\mathrm{Ln}\, z} = z \text{ for all } z \neq 0. \tag{4.1.16}$$

If the complex exponential function $f(z) = e^z$ is defined on the fundamental region $-\infty < x < \infty, -\pi < y \leq \pi$,

then f is **one-to-one (一対一)** and the **inverse function (逆関数)** of f is the **principal value of the complex logarithm** $f^{-1}(z) = \mathrm{Ln}\, z$.

Complex Logarithmic Functions (複素対数関数)

Recall that

Theorem 2.3 Real and Imaginary Parts of a Continuous Function

Suppose that $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$.
Then the complex function (複素関数) f is continuous at the point z_0 **if and only if** both real functions (実数値関数) u and v are continuous at the point (x_0, y_0).

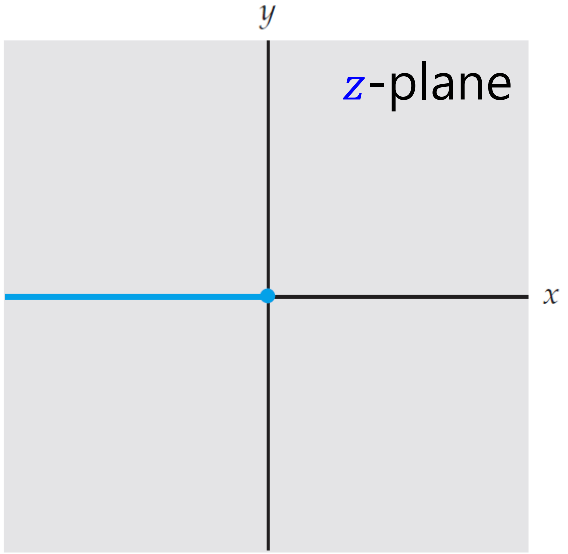


Figure 4.6 $f_1(z)$ defines on domain in gray color

The principal value of the complex logarithm function

$$\mathrm{Ln}\, z = \log_e |z| + i\,\mathrm{Arg}(z), \quad -\pi < \mathrm{Arg}(z) \le \pi$$

Real part $u(x, y) = \log_e |z| = \log_e \sqrt{x^2 + y^2}$ is **continuous at all points** in the plane **except** $(0, 0)$

Imaginary part $v(x, y) = \mathrm{Arg}(z)$ is **continuous on the domain** $|z| > 0$, $-\pi < \mathrm{Arg}(z) < \pi$

Therefore, $\mathrm{Ln}\, z$ is **a continuous function on the domain** $|z| > 0$, $-\pi < \mathrm{Arg}(z) < \pi$

We give this new function a name by "principal branch of the complex logarithm function"

$$f_1(z) = \log_e |z| + i\,\mathrm{Arg}(z), \quad -\pi < \mathrm{Arg}(z) < \pi \qquad (4.1.19)$$

Here, $f_1(z)$ is $\mathrm{Ln}\, z$ except $\mathrm{Arg}(z) = \pi$

Complex Logarithmic Functions (複素対数関数)

Theorem 4.4 Analyticity of the Principal Branch of $\ln z$

The principal branch f_1 of the complex logarithm defined by (4.1.19) is **an analytic function** and **its derivative is given by**:

$$f_1'(z) = \frac{1}{z} \tag{4.1.20}$$

The theorem 4.4 implies that $\mathrm{Ln}\, z$ is differentiable in the domain $|z| > 0$, $-\pi < \mathrm{Arg}(z) < \pi$, and its derivative is given by $f_1'(z)$.

That is, if $|z| > 0$, $-\pi < \mathrm{Arg}(z) < \pi$ then

$$\boxed{\frac{d}{dx}\mathrm{Ln}\, z = \frac{1}{z}} \tag{4.1.21}$$

Complex Logarithmic Functions (複素対数関数)

EXAMPLE (例題) 4.1.5 Derivatives of Logarithmic Functions

Find the derivatives of the following functions in an appropriate domain:

(a) $z\,\mathrm{Ln}\,z$ and *(b) $\mathrm{Ln}\,(z+1)$

Solution (解答):

(a) The function $z\mathrm{Ln}\,z$ is differentiable at all points where both of the functions z and $\mathrm{Ln}\,z$ are differentiable.

Because z is entire (整函数) and $\mathrm{Ln}\,z$ is differentiable on the domain given in (4.1.19), as $|z| > 0$, $-\pi < \mathrm{Arg}(z) < \pi$, it follows that $z\mathrm{Ln}\,z$ is differentiable on the domain defined by $|z| > 0$, $-\pi < \mathrm{Arg}(z) < \pi$

In this domain, the derivative is given by the product rule (積の微分法則) (3.1.4) of Lecture 3 and (4.1.21):

$$\frac{d}{dx}[\mathrm{zLn}\,z] = z \cdot \frac{1}{z} + 1 \cdot \mathrm{Ln}\,z = 1 + \mathrm{Ln}\,z$$

Complex Logarithmic Functions (複素対数関数)

Solution (解答)(cont.):

*(b) The function $\mathrm{Ln}\,(z+1)$ is a composition ($f(g(z))$) of the functions $\mathrm{Ln}\,z$ and $z+1$.

Because $z+1$ is entire (整函数) and $\mathrm{Ln}\,z$ is differentiable on the domain given in (4.1.19), as $|z|>0$, $-\pi<\mathrm{Arg}(z)<\pi$, it follows from the chain rule (連鎖律) that $\mathrm{Ln}\,(z+1)$ is differentiable at all points $g=z+1$ such that $|g|>0$, $-\pi<\mathrm{Arg}(g)<\pi$

Recall the domain $|g|>0$, $-\pi<\mathrm{Arg}(g)<\pi$ for $f_1(g)$ in (4.1.19) as figure 4.6, we obtain the domain for $g=z+1$ as figure T1. (Notice: The g here is equivalent to the original "z" in (4.1.19).)

The equation $z=g-1$ defines a linear mapping of the g-plane onto the z-plane given by translation (i.e. shift along x-axis) by -1.

In this domain in z-plane, the derivative is given by the chain rule (連鎖律) (3.1.6) of Lecture 3 and (4.1.21):

$$\frac{d}{dx}[\mathrm{Ln}\,(z+1)]=\frac{1}{z+1}\cdot 1=\frac{1}{z+1}$$

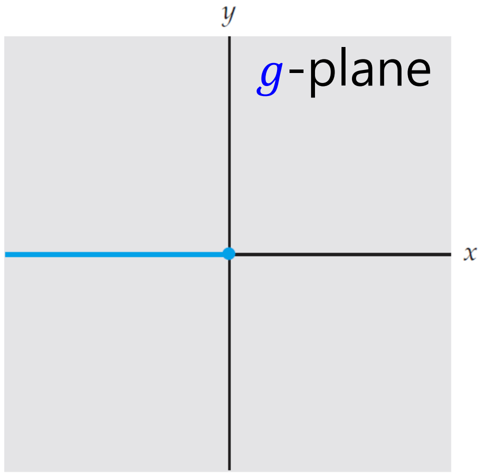


Figure T1. The defined domain in g-plane with gray color

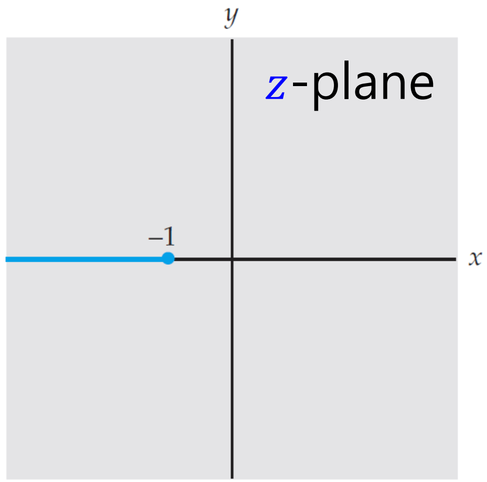


Figure 4.7 $\mathrm{Ln}(z+1)$ is not differentiable on the ray shown in blue color.

Review for Lecture 4

- Harmonic Functions
- Exponential Functions
- Exponential Mapping
- Logarithmic Functions
- The principal value of the Logarithmic Functions
- Analyticity of the Principal Branch of $\ln z$

Exercise

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: Section , Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia

Appendix (付録)

Elementary function:

https://en.wikipedia.org/wiki/Elementary_function

初等関数とは:

http://www.cc.miyazaki-u.ac.jp/yazaki/teaching/di/di-function.pdf

Appendix (付録)

Horizontal strip とは: