Lecture 9

(Complex) Sequences and Series

(数列と級数)

What you will learn in Lecture 9

9.1 (Complex) Sequences and Series (数列と級数)

9.2 Testing Series

Cauchy's integral formula for derivatives indicates that if a function f is analytic at a point z_0, **then it possesses derivatives of all orders at that point**.

As a consequence of this result we shall see that f can always be expanded in **a power series** centered at that point.

On the other hand, if f fails to be analytic at z_0, we may still be able to expand it in a different kind of series known as a **Laurent series**.

9.1 (Complex) Sequences and Series

(数列と級数)

Sequences (数列)

A **sequence** $\{z_n\}$, where $n = 1, 2, 3, \ldots$, is a function whose domain is the set of positive integers and whose range is a subset of the complex numbers $\mathbf{C}$.

For example, the sequence $\{1 + i^n\}$ is

$$\begin{array}{cccccc} 1+i, & 0, & 1-i, & 2, & 1+i, & \ldots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ n=1, & n=2, & n=3, & n=4, & n=5, & \end{array} \qquad (6.1.1)$$

9.1 (Complex) Sequences and Series (数列と級数)

Sequences (数列)

If $\lim_{n\to\infty} z_n = L$, we say the sequence $\{z_n\}$ is **convergent (収束)**.

Sequence that is not convergent is said to be **divergent (発散)**.

$\{z_n\}$ converges to the number L, if for each positive real number ε, an N can be found such that $|z_n - L| < \varepsilon$ whenever $n > N$. Since $|z_n - L|$ is distance, the terms z_n of a sequence that converges to L can be made arbitrarily close to L.

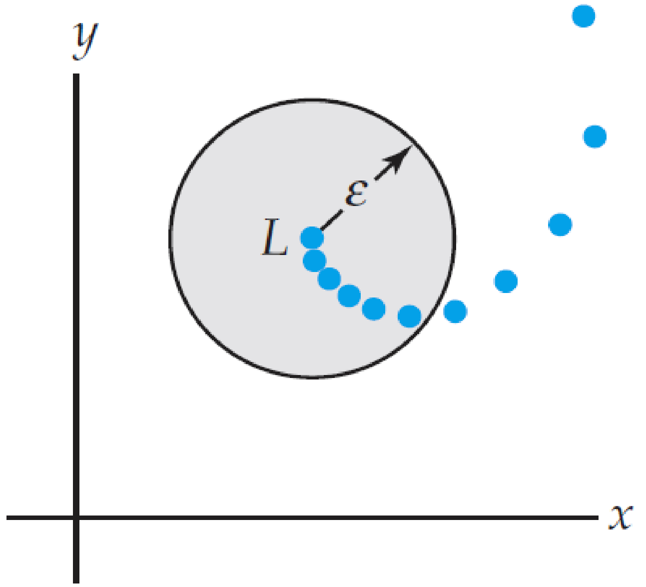

Figure 6.1 If $\{z_n\}$ converges to L, all but a finite number of terms are in every ε–neighborhood of L.

Sequences (数列)

For example, the sequence $\{1 + i^n\}$

$$
\begin{array}{cccccc}
1+i, & 0, & 1-i, & 2, & 1+i, & \ldots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
n=1, & n=2, & n=3, & n=4, & n=5, &
\end{array}
$$

The sequence $\{1 + i^n\}$ **is divergent** because **the general term** $\mathrm{z_n} = 1 + i^n$ **does not approach a fixed complex number** as $n \to \infty$.

Sequences (数列)

EXAMPLE (例題) 6.1.1 A Convergent Sequence

The sequence $\left\{\frac{i^{n+1}}{n}\right\}$ **converges or not.**

Solution (解答):

The sequence $\left\{\frac{i^{n+1}}{n}\right\}$ **converges** since $\lim_{n\to\infty}\frac{i^{n+1}}{n}=0$. As we see from

$$-1,-\frac{i}{2},\frac{1}{3},-\frac{i}{4},-\frac{1}{5},\cdots,$$

and Figure 6.2, the terms of the sequence, marked by colored dots in the figure, spiral in toward the point $z = 0$ as n increases.

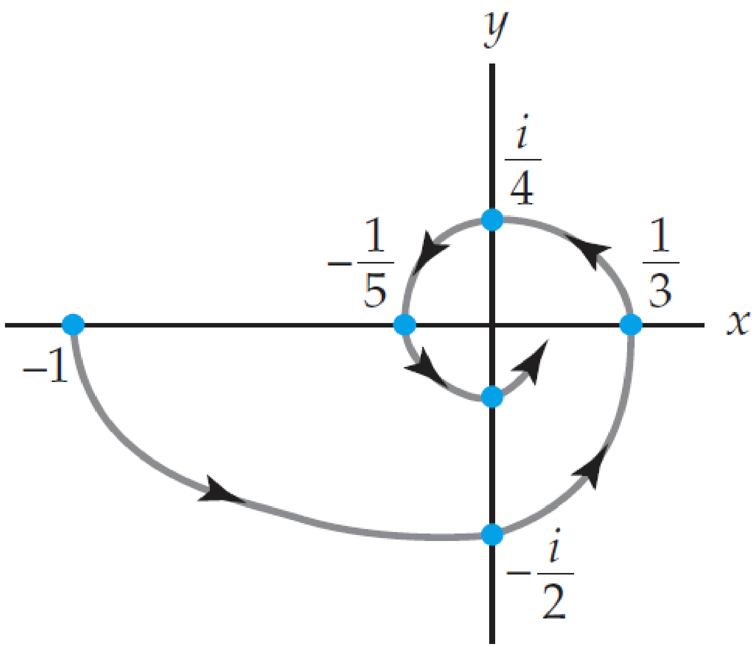

Figure 6.2 The terms of the sequence $\left\{\frac{i^{n+1}}{n}\right\}$ spiral in toward 0.

Sequences (数列)

Theorem 6.1 Criterion (基準) for Sequence Convergence

Suppose that $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ and $L = x + iy$. Then

$$\lim_{n\to\infty} z_n = L$$

if and only if

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y$$

This theorem for sequences is the analogue of Theorem 2.1 in Lecture 2.

Theorem 2.1 Real and Imaginary Parts (実部と虚部) of a Limit

Suppose that $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$, and $L = u_0 + iv_0$. Then $\lim_{z\to z_0} f(z) = L$ **if and only if**

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$$

Additional EXAMPLE (例題) Using Theorem 6.1

The sequence $\left\{\frac{1}{n^3} + i\right\}$ converges or not.

Solution (解答):

(1)The sequence $z_n = \frac{1}{n^3} + i \quad (n = 1,2, \ldots)$ converges to i since

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\left(\frac{1}{n^3} + i\right) = \lim_{n\to\infty}\frac{1}{n^3} + i\lim_{n\to\infty} 1 = 0 + i \cdot 1 = i$$

(2)The definition in Page 6 of this slides can also be used to obtain this result.

For each positive number ε

$$|z_n - i| = \frac{1}{n^3} < \varepsilon \quad \text{whenever } n > \frac{1}{\sqrt[3]{\varepsilon}}$$

EXAMPLE (例題) 6.1.2 Using Theorem 6.1

The sequence $\left\{\frac{3+ni}{n+2ni}\right\}$ converges or not.

Solution (解答):

From $$z_n = \frac{3+ni}{n+2ni} = \frac{(3+ni)(n-2ni)}{n^2+4n^2} = \frac{2n^2+3n}{5n^2} + i\,\frac{n^2-6n}{5n^2}$$

we see that when $n \to \infty$

$$\mathrm{Re}(z_n) = \frac{2n^2+3n}{5n^2} = \frac{2}{5} + \frac{3}{5n} \to \frac{2}{5}$$

$$\mathrm{Im}(z_n) = \frac{n^2-6n}{5n^2} = \frac{1}{5} - \frac{6}{5n} \to \frac{1}{5}$$

From Theorem 6.1, the results are sufficient to conclude that the given sequence converges to $a + ib = \frac{2}{5} + \frac{1}{5}i$.

Series (級数)

An infinite series or series of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + \cdots + z_n + \cdots$$

is convergent if the sequence of **partial sums** $\{S_n\}$, where

$$S_n = z_1 + z_2 + z_3 + \cdots + z_n$$

converges.

If $S_n \to L$ as $n \to \infty$, we say that **the series converges to** L or that **the sum of the series** is L.

Additional Theorem: Criterion (基準) for Series Convergence

Suppose that $z_k = x_k + iy_k$ $(k = 1, 2, \ldots)$ and $S = X + iY$. Then

$$\sum_{k=1}^{\infty} z_k = S$$

if and only if

$$\sum_{k=1}^{\infty} x_k = X \quad \text{and} \quad \sum_{k=1}^{\infty} y_k = Y$$

Additional EXAMPLE (例題) Using the Additional Theorem

Show that if $\sum_{k=1}^{\infty} z_k = S$, then $\sum_{k=1}^{\infty} \overline{z_k} = \overline{S}$.

Solution (解答):

We write $z_k = x_k + iy_k$ $(k = 1, 2, \ldots)$ and $S = X + iY$.

First of all, we note that

$$\sum_{k=1}^{\infty} x_k = X \text{ and } \sum_{k=1}^{\infty} y_k = Y$$

Then since $\sum_{k=1}^{\infty}(-y_k) = -Y$, it follows that

$$\sum_{k=1}^{\infty} \overline{z_k} = \sum_{k=1}^{\infty} (x_k - iy_k) = \sum_{k=1}^{\infty} [x_k + i(-y_k)] = X - iY = \overline{S}$$

Geometric Series (幾何級数)

A **geometric series** is any series of the form

$$\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \cdots + az^{n-1} + \cdots \tag{6.1.2}$$

For (6.1.2), the nth term of the sequence of partial sums is

$$S_n = a + az + az^2 + \cdots + az^{n-1} \tag{6.1.3}$$

Geometric Series (幾何級数)

When an infinite series is a geometric series, it is always possible to find a formula for S_n.

Why? We can multiply S_n in (6.1.3) by z,

$$zS_n = az + az^2 + az^3 + \cdots + az^n$$

and subtract this result from S_n, then we have

$$S_n - zS_n = (a + az + az^2 + \cdots + az^{n-1}) - (az + az^2 + az^3 + \cdots + az^n)$$

$$= a - az^n$$

$$\Rightarrow \qquad S_n = \frac{a(1-z^n)}{1-z} \tag{6.1.4}$$

Now $z^n \to 0$ as $n \to \infty$ whenever $|z| < 1$, and so $S_n \to \frac{a}{1-z}$.

In other words, for $|z| < 1$ the sum of a geometric series (6.1.2) is $\frac{a}{1-z}$:

$$\frac{a}{1-z} = a + az + az^2 + \cdots + az^{n-1} + \cdots \tag{6.1.5}$$

A geometric series (6.1.2) diverges when $|z| \geq 1$.

Special Geometric Series

If we set $a = 1$, the equality in (6.1.5) is

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \tag{6.1.6}$$

If we then replace the symbol z by $-z$ in (6.1.6), we get a similar result

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots \tag{6.1.7}$$

Like (6.1.5), the equality in (6.1.7) is valid for $|z| < 1$ since $|-z| = |z|$.
Now with $a = 1$, (6.1.4) gives us the sum of the first n terms of the series in (6.1.6):

$$\frac{1-z^n}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^{n-1}$$

EXAMPLE (例題) 6.1.3 Convergent Geometric Series

The series $\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k}$ is convergent or divergent?

Solution (解答):

The infinite series $\displaystyle\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{1+2i}{5} + \frac{(1+2i)^2}{5^2} + \frac{(1+2i)^3}{5^3} + \cdots$

is a geometric series. It has the form given in (6.1.2) with $a = \frac{1}{5}(1+2i)$ and $z = \frac{1}{5}(1+2i)$. Since $|z| = \frac{\sqrt{5}}{5} < 1$, the series is convergent and its sum is given by (6.1.5):

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{\frac{1+2i}{5}}{1 - \frac{1+2i}{5}} = \frac{1+2i}{4-2i} = \frac{i}{2}$$

Theorem 6.2 A Necessary Condition for Convergence

If $\sum_{k=1}^{\infty} z_k$ converges, then $\lim\limits_{n\to\infty} z_n = 0.$

Proof

Let L denote the sum of the series. Then $S_n \to L$ and $S_{n-1} \to L$ as $n \to \infty$.

By taking the limit of both sides of $S_n - S_{n-1} = z_n$ as $n \to \infty$, we have

$$\lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} z_n$$

$$L - L = \lim_{n\to\infty} z_n$$

$$0 = \lim_{n\to\infty} z_n$$

we obtain the desired conclusion. ■

9.2 Testing Series

A Test for Divergence

Theorem 6.3 The nth Term Test for Divergence

If $\lim_{n\to\infty} z_n \neq 0$, then $\sum_{k=1}^{\infty} z_k$ diverges.

For example,

the series $\sum_{k=1}^{\infty} \frac{ik+5}{k}$ **diverges** since $z_n = \frac{in+5}{n} \to i \neq 0$ as $n \to \infty$.

The geometric series (6.1.2) **diverges** if $|z| \geq 1$ because even in the case when $\lim_{n\to\infty} |z^n|$ exists, **the limit is not zero**.

Definition 6.1 Absolute and Conditional Convergence (絶対収束と条件収束)

An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **absolutely convergent** if $\sum_{k=1}^{\infty}|z_k|$ converges. An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **conditionally convergent** if it converges but $\sum_{k=1}^{\infty}|z_k|$ diverges.

p-series

In elementary calculus a real series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a p-series and converges for $p > 1$ and diverges for $p \leq 1$.

EXAMPLE (例題) 6.1.4 Absolute Convergence

The series $\sum_{k=1}^{\infty} \frac{i^k}{k^2}$ is absolute convergent or not.

Solution (解答):

The series $\sum_{k=1}^{\infty} \frac{i^k}{k^2}$ is absolutely convergent since the series $\sum_{k=1}^{\infty} \left|\frac{i^k}{k^2}\right|$ is the same as the real convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

Here we identify $p = 2 > 1$.

As in Real-value calculus:

Absolute convergence implies convergence.

We can therefore conclude that the series in Example 6.1.4,

$$\sum_{k=1}^{\infty} \frac{i^k}{k^2} = i - \frac{1}{2^2} - \frac{i}{3^2} + \frac{1}{4^2} + \cdots$$

converges because it is absolutely convergent.

Tests for Convergence

Theorem 6.4 Ratio Test

Suppose is a series of nonzero complex terms such that

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L \tag{6.1.9}$$

(i) If $L < 1$, then the series converges absolutely.

(ii) If $L > 1$ or $L = \infty$, then the series diverges.

(iii) If $L = 1$, the test is inconclusive.

Tests for Convergence

Theorem 6.5 Root Test

Suppose is a series of complex terms such that

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L \qquad (6.1.10)$$

(i) If $L < 1$, then **the series converges absolutely**.

(ii) If $L > 1$ or $L = \infty$, then **the series diverges**.

(iii) If $L = 1$, **the test is inconclusive**.

Review for Lecture 9

- (Complex) Sequences and Series
- Convergence and Divergence
- Geometric Series
- p-series
- Absolute and Conditional Convergence
- Testing Series

Assignment

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: Section 6.1, Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia