



# Lecture 3

**Differentiability (微分可能性) & Analyticity (解析性)**

**Cauchy-Riemann Equations (コーシー・リーマンの方程式)**

# What you will learn in Lecture 3

**3.1** Differentiability (微分可能性) & Analyticity (解析性)

**3.2** Cauchy-Riemann Equations (コーシー・リーマンの方程式)

# **3.1 Differentiability (微分可能性) & Analyticity (解析性)**

## 3.1 Differentiability (微分可能性) & Analyticity (解析性)

### The Derivative (微分係数)

#### Definition (定義) 3.1 Derivative of Complex Function (複素関数)

Suppose (仮定する) the complex function  $f$  is defined in a neighborhood (近傍) of a point  $z_0$ . The derivative (微分係数) of  $f$  at  $z_0$ , denoted by  $f'(z_0)$ , is

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad (3.1.1)$$

when this limit exists.

- If the limit in (3.1.1) exists, then the function  $f$  is said to be **differentiable (微分可能)** at  $z_0$ .
- Besides  $f'(z_0)$ , we have two other symbols denoting (表示する) the derivative of  $w = f(z)$ , which are  $w'$  and  $\left. \frac{dw}{dz} \right|_{z=z_0}$ .

## 3.1 Differentiability (微分可能性) & Analyticity (解析性)

### EXAMPLE (例題) 3.1.1

Use Definition 3.1 to find the derivative of  $f(z) = z^2 - 5z$ .

#### Solution (解答):

We replace  $z_0$  in (3.1.1) by the symbol  $z$ . **First**, compute the complex function

$$\begin{aligned} f(z + \Delta z) &= (z + \Delta z)^2 - 5(z + \Delta z) \\ &= z^2 + 2z\Delta z + (\Delta z)^2 - 5z - 5\Delta z. \end{aligned}$$

**Second**,

$$\begin{aligned} f(z + \Delta z) - f(z) &= z^2 + 2z\Delta z + (\Delta z)^2 - 5z - 5\Delta z - (z^2 - 5z) \\ &= 2z\Delta z + (\Delta z)^2 - 5\Delta z. \end{aligned}$$

**Finally**, (3.1.1) gives

$$\begin{aligned} f'(z) &= \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{2z\Delta z + (\Delta z)^2 - 5\Delta z}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{\Delta z(2z + \Delta z - 5)}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} (2z + \Delta z - 5) \quad \Rightarrow \text{The limit is } f'(z) = 2z - 5 \end{aligned}$$

## 3.1 Differentiability (微分可能性) & Analyticity (解析性)

### Rules of Differentiation (微分法則)

If  $f$  and  $g$  are **differentiable** at a point  $z$ , and  $c$  is a **complex constant**, then (3.1.1) can be used to show:

Constant Rules (定数の法則):  $\frac{d}{dz} c = 0$  and  $\frac{d}{dz} cf(z) = cf'(z)$  (3.1.2)

Sum Rule (和の法則):  $\frac{d}{dz} [f(z) \pm g(z)] = f'(z) \pm g'(z)$  (3.1.3)

Product Rule (積の法則):  $\frac{d}{dz} [f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$  (3.1.4)

Quotient Rule (商の法則):  $\frac{d}{dz} \left[ \frac{f(z)}{g(z)} \right] = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$  (3.1.5)

Chain Rule (連鎖律):  $\frac{d}{dz} f(g(z)) = f'(g(z))g'(z)$  (3.1.6)

Power Rule (冪乗の法則):  $\frac{d}{dz} z^n = nz^{n-1}$ , where  $n$  is an integer. (3.1.7)

Combine (6) and (7),  $\frac{d}{dz} [f(z)]^n = n[f(z)]^{n-1}f'(z)$ ,  $n$  is an integer. (3.1.8)

### 3.1 Differentiability (微分可能性) & Analyticity (解析性)

#### EXAMPLE (例題) 3.1.2 Using the Rules of Differentiation.

Differentiate:

$$(a) f(z) = 3z^4 - 5z^3 + 2z \quad (b) f(z) = \frac{z^2}{4z+1} \quad (c) f(z) = (iz^2 + 3z)^5$$

**Solution (解答):**

(a) Using the power rule (3.1.7), the sum rule (3.1.3), along with (3.1.2), we obtain

$$f'(z) = 3 \cdot 4z^3 - 5 \cdot 3z^2 + 2 \cdot 1 = 12z^3 - 15z^2 + 2$$

(b) From the quotient rule (3.1.5),

$$f'(z) = \frac{(4z+1) \cdot 2z - z^2 \cdot 4}{(4z+1)^2} = \frac{4z^2 + 2z}{(4z+1)^2}$$

(c) In the power rule for functions (3.1.8) we identify  $n = 5$ ,  $f(z) = iz^2 + 3z$ , and  $f'(z) = 2iz + 3$ , so that

$$f'(z) = 5(iz^2 + 3z)^4(2iz + 3)$$

### 3.1 Differentiability (微分可能性) & Analyticity (解析性)

**EXAMPLE (例題) 3.1.3** A Function That Is Nowhere Differentiable. Show that the function  $f(z) = x + 4yi$  is not differentiable at any point  $z$ .

**Solution (解答):**

Let  $z$  be any point in the complex plane. With  $\Delta z = \Delta x + i\Delta y$ ,  
 $f(z + \Delta z) - f(z) = (x + \Delta x) + 4(y + \Delta y)i - (x + 4yi) = \Delta x + 4\Delta yi$

and so 
$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta x + 4\Delta yi}{\Delta x + i\Delta y} \quad (3.1.9)$$

Now, as shown in Figure 3.1(a), if we let  $\Delta z \rightarrow 0$  along a line parallel to the  $x$ -axis, then  $\Delta y = 0$  and  $\Delta z = \Delta x$  and

$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta x}{\Delta x} = 1 \quad (3.1.10)$$

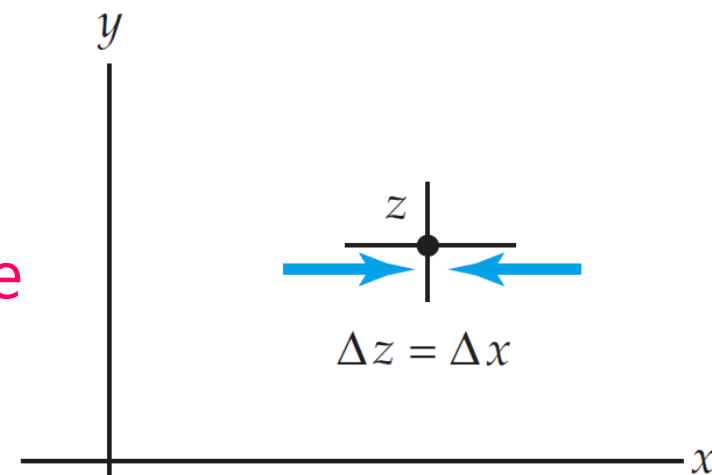


Figure 3.1(a)  $\Delta z \rightarrow 0$  along a line parallel to  $x$ -axis



### 3.1 Differentiability (微分可能性) & Analyticity (解析性)

On the other hand, if we let  $\Delta z \rightarrow 0$  along a line parallel to the  $y$ -axis as shown in Figure 3.1(b), then  $\Delta x = 0$  and  $\Delta z = i\Delta y$  so that

$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{4\Delta y i}{i\Delta y} = 4 \quad (3.1.11)$$

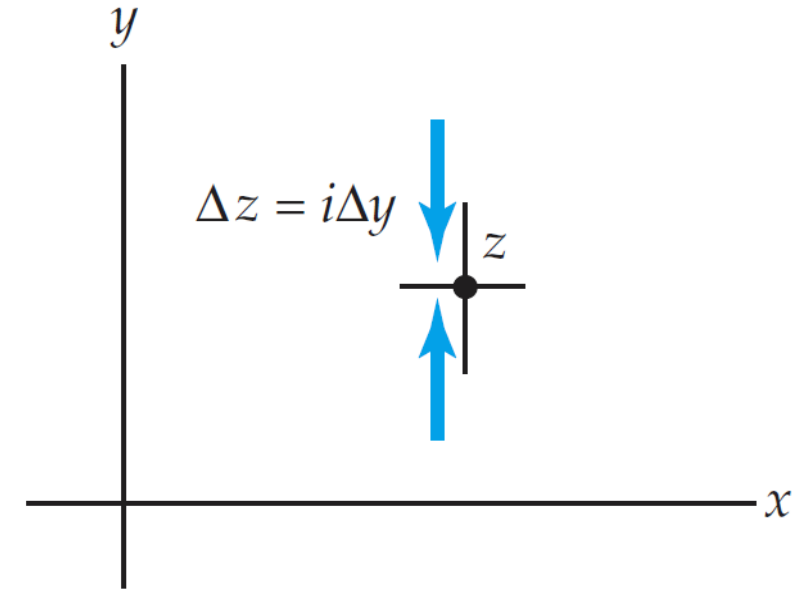


Figure 3.1(b)  $\Delta z \rightarrow 0$  along a line parallel to  $y$ -axis

In view of the obvious fact that the values in (3.1.10) and (3.1.11) are different, we conclude that  $f(z) = x + 4yi$  is nowhere differentiable; that is,  $f$  is not differentiable at any point  $z$ .

## 3.1 Differentiability (微分可能性) & Analyticity (解析性)

### Analytic Functions (解析関数)

#### Definition (定義) 3.2 Analyticity (解析性) at a Point

A **complex function** (複素関数)  $w = f(z)$  is said to be **analytic (解析的)** at a point  $z_0$  if  $f$  is differentiable at  $z_0$  and at every point in some neighborhood of  $z_0$ .

- A function  $f$  is analytic in a domain  $D$  if it is analytic at every point in  $D$ .

Notice: A function  $f$  that is **analytic throughout a domain  $D$**  is also called **holomorphic function (正則関数)** or regular function.

- Analyticity is a neighborhood property that is defined over an open set (開集合) (i.e. not only for a single point).

### 3.1 Differentiability (微分可能性) & Analyticity (解析性)

If the functions  $f$  and  $g$  are analytic in a domain  $D$ , then

#### Analyticity of Sum (和), Product (積), and Quotient (商)

The sum  $f(z) + g(z)$ , difference (差)  $f(z) - g(z)$ , and product  $f(z)g(z)$  are analytic. The quotient  $f(z)/g(z)$  is analytic provided  $g(z) \neq 0$  in  $D$ .

## 3.1 Differentiability (微分可能性) & Analyticity (解析性)

### Entire Functions (整函数)

A function that is analytic at every point  $z$  in the complex plane is said to be an entire function (整函数).

### Theorem 3.1 Analyticity of Polynomial and Rational Functions

(i) A polynomial function (多項式関数)  $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ , where  $n$  is a nonnegative integer, is an entire function (整函数).

(ii) A rational function (有理関数)  $f(z) = \frac{p(z)}{q(z)}$ , where  $p$  and  $q$  are polynomial functions (多項式関数), is analytic in any domain  $D$  that contains NO point  $z_0$  for which  $q(z_0) = 0$ .

## 3.1 Differentiability (微分可能性) & Analyticity (解析性)

### Singular Points (特異点)

In general, a point  $z$  at which a complex function  $w = f(z)$  fails (失敗する) to be analytic is called a singular point (特異点) of  $f$ .

## 3.1 Differentiability (微分可能性) & Analyticity (解析性)

### Theorem 3.2 Differentiability (微分可能性) Implies (ならば) Continuity (連続性)

If  $f$  is **differentiable (微分可能)** at a point  $z_0$  in a domain  $D$ , then  $f$  is **continuous (連続)** at  $z_0$ .

### 3.1 Differentiability (微分可能性) & Analyticity (解析性)

#### An Alternative (代わりの) Definition of the Derivative (微分係数) $f'(z)$

We know

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad (3.1.1)$$

Since  $\Delta z = z - z_0$ , then  $z = z_0 + \Delta z$ , and so (3.1.1) can be written as

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad (3.1.12)$$

## 3.1 Differentiability (微分可能性) & Analyticity (解析性)

### Theorem 3.3 L'Hôpital's Rule (ロピタルの定理)

Suppose  $f$  and  $g$  are functions that are analytic at a point  $z_0$  and  $f(z_0) = 0, g(z_0) = 0$ , but  $g'(z_0) \neq 0$ . Then

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \quad (3.1.13)$$



## 3.1 Differentiability (微分可能性) & Analyticity (解析性)

### EXAMPLE (例題) 3.1.4 Using L'Hôpital's Rule (ロピタルの定理)

Compute  $\lim_{z \rightarrow 2+i} \frac{z^2 - 4z + 5}{z^3 - z - 10i}$ .

#### Solution (解答):

We know  $z_0 = 2 + i$

If we identify  $f(z) = z^2 - 4z + 5$  and  $g(z) = z^3 - z - 10i$ , you should verify that

$$f(z_0) = f(2 + i) = (2 + i)^2 - 4(2 + i) + 5 = 4 + 4i + i^2 - 8 - 4i + 5 = 0$$

$$\begin{aligned} g(z_0) &= g(2 + i) = (2 + i)^3 - (2 + i) - 10i = (4 + 4i + i^2)(2 + i) - 2 - i - 10i \\ &= (3 + 4i)(2 + i) - 2 - 11i = 6 + 8i + 3i + 4i^2 - 2 - 11i = 0 \end{aligned}$$

The given limit has the indeterminate form  $\frac{0}{0}$ .

Now since  $f$  and  $g$  are polynomial functions, both functions are necessarily analytic at  $z_0 = 2 + i$ .

$$f'(z) = \frac{d(z^2 - 4z + 5)}{dz} = 2z - 4, \quad g'(z) = \frac{d(z^3 - z - 10i)}{dz} = 3z^2 - 1,$$

$$\text{then } f'(2 + i) = 2(2 + i) - 4 = 2i, \quad g'(2 + i) = 3(2 + i)^2 - 1 = 8 + 12i$$

$$\text{we see that (3.1.13) gives } \lim_{z \rightarrow 2+i} \frac{z^2 - 4z + 5}{z^3 - z - 10i} = \frac{f'(2 + i)}{g'(2 + i)} = \frac{2i}{8 + 12i} = \frac{2i(8 - 12i)}{(8 + 12i)(8 - 12i)} = \frac{3}{26} + \frac{1}{13}i$$

## 3.2 Cauchy-Riemann Equations

(コーシー・リーマンの方程式)

## 3.2 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

### A Necessary Condition (必要条件) for Analyticity (解析性)

#### Theorem 3.4 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

Suppose  $f(z) = u(x, y) + iv(x, y)$  is differentiable at a point  $z = x + iy$ . Then at  $z$  the first-order partial derivatives of  $u(x, y)$  and  $v(x, y)$  exist and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (3.2.1)$$

Proof: Check P152 of Textbook

## 3.2 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

**EXAMPLE (例題) 3.2.1** Verifying Cauchy-Riemann Equations for the polynomial function  $f(z) = z^2 + z$

**Solution (解答):**

The polynomial function  $f(z) = z^2 + z$  is analytic for all  $z$  and can be written as  $f(z) = x^2 - y^2 + x + i(2xy + y)$ . Thus,  $u(x, y) = x^2 - y^2 + x$  and  $v(x, y) = 2xy + y$ .

For any point  $(x, y)$  in the complex plane, we can see that the Cauchy-Riemann equations are satisfied:

$$\left. \begin{array}{l} \frac{\partial u}{\partial x} = 2x - 0 + 1 \\ \frac{\partial v}{\partial y} = 2x + 1 \end{array} \right\} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \left. \begin{array}{l} \frac{\partial u}{\partial y} = 0 - 2y + 0 \\ \frac{\partial v}{\partial x} = 2y + 0 \end{array} \right\} \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

## 3.2 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

### Criterion (基準) for Non-analyticity

If the Cauchy-Riemann equations are not satisfied at every point  $z$  in a domain  $D$ ,  
then the function  $f(z) = u(x, y) + iv(x, y)$  CANNOT be analytic in  $D$ .

## 3.2 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

**EXAMPLE (例題) 3.2.2** Using the Cauchy-Riemann Equations  
Show that the complex function  $f(z) = 2x^2 + y + i(y^2 - x)$  is not analytic at any point.

**Solution (解答):**

We identify  $u(x, y) = 2x^2 + y$  and  $v(x, y) = y^2 - x$ . From

$$\left. \begin{array}{l} \frac{\partial u}{\partial x} = 4x + 0 \\ \frac{\partial v}{\partial y} = 2y - 0 \end{array} \right\} \begin{array}{l} \text{If } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \text{then we have} \\ y = 2x \end{array} \quad \text{and} \quad \left. \begin{array}{l} \frac{\partial u}{\partial y} = 0 + 1 \\ \frac{\partial v}{\partial x} = 0 - 1 \end{array} \right\} \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

we see that  $\partial u/\partial y = -\partial v/\partial x$  but that the equality  $\partial u/\partial x = \partial v/\partial y$  is satisfied only on the line  $y = 2x$ .

However, for any point  $z$  on the line, there is no neighborhood or open disk about  $z$  in which  $f$  is differentiable at every point. We conclude that  $f$  is nowhere analytic.

## 3.2 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

### A Sufficient Condition (十分条件) for Analyticity (解析性)

#### Theorem 3.5 Criterion (基準) for Analyticity

Suppose the real functions  $u(x, y)$  and  $v(x, y)$  are continuous and have continuous first-order partial derivatives  $\partial u/\partial x$ ,  $\partial u/\partial y$ ,  $\partial v/\partial x$ , and  $\partial v/\partial y$ , in a domain  $D$ .

If  $u$  and  $v$  satisfy the Cauchy-Riemann equations (3.2.1) at all points of  $D$ , then the complex function  $f(z) = u(x, y) + iv(x, y)$  is analytic in  $D$ .

## 3.2 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

**EXAMPLE (例題) 3.2.3** Using Theorem 3.5 to evaluate the analyticity of the function  $f(z) = \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$

**Solution (解答):**

For the function  $f(z) = \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$ , the real functions  $u(x, y) = \frac{x}{x^2+y^2}$  and  $v(x, y) = -\frac{y}{x^2+y^2}$  are continuous except at the point where  $x^2 + y^2 = 0$ , that is, at  $z = 0 - i0 = 0$ .

Moreover, we can verify that the first four first-order partial derivatives

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{x^2 + y^2},$$

$$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2},$$

$$\frac{\partial v}{\partial y} = \frac{2xy}{(x^2 + y^2)^2}, \quad \text{and}$$

$$\frac{\partial v}{\partial x} = \frac{y^2 - x^2}{x^2 + y^2}$$

are continuous except at  $z = 0$ .



## 3.2 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

### **Solution (解答)(cont.):**

Finally, we see from

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{x^2 + y^2} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2} = \frac{\partial v}{\partial x}$$

that the Cauchy-Riemann equations are satisfied except at  $z = 0$ .

Thus we conclude from Theorem 3.5 that  $f$  is analytic in any domain  $D$  that does not contain the point  $z = 0$ .

We call this  $z = 0$  a singular point (特異点).

## 3.2 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

### Sufficient Conditions (十分条件) for Differentiability (微分可能性)

If **the real functions**  $u(x, y)$  and  $v(x, y)$  are **continuous** and **also have continuous first-order partial derivatives** in some neighborhood of a point  $z$ , and if  $u$  and  $v$  satisfy the Cauchy-Riemann equations (3.2.1) at  $z$ , then the complex function  $f(z) = u(x, y) + iv(x, y)$  is differentiable at  $z$  and  $f'(z)$  is given by (3.2.9).

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad (3.2.9)$$

Namely, **we have a formula for computing the derivative  $f'(z)$ .**

## 3.2 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

**Statement: Analyticity implies Differentiability but not conversely.**

Analyticity  $\Rightarrow$  Differentiability  
 $\nLeftarrow$

### EXAMPLE (例題) 3.2.4 A Function Differentiable on a Line

#### **Solution (解答):**

In Example 3.2.2 we saw that the complex function  $f(z) = 2x + y + i(y - x)$  was nowhere analytic, but yet the Cauchy-Riemann equations were satisfied on the line  $y = 2x$ .

Since the functions  $u(x, y) = 2x + y$ ,  $\partial u / \partial x = 2$ ,  $\partial u / \partial y = 1$ ,  $v(x, y) = y - x$ ,  $\partial v / \partial x = -1$  and  $\partial v / \partial y = 1$  are continuous at every point, it follows that  $f$  is differentiable on the line  $y = 2x$ .

Moreover, from (3.2.9) we see that the derivative of  $f$  at points on this line is given by  $f'(z) = 2 - i = 1 - i$ .

# Review for Lecture 3

- Differentiability (微分可能性)
- Analyticity (解析性)
- Holomorphic function (正則関数)
- Singular Point (特異点)
- L'Hôpital's Rule (ロピタルの定理)
- Cauchy-Riemann Equations (コーシー・リーマンの方程式)
- Criterion (基準) for Analyticity

# Exercise

Please Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

Reading Materials: Section 3.1, 3.2, Textbook

# References

- [1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003
- [2] Wikipedia

# Appendix (付録)

In logic, **necessity** (必要性) and **sufficiency** (十分性) are terms used to describe a conditional or implicational relationship between **statements** (命題).

"**If  $P$  then  $Q$** ", we say that " **$Q$  is necessary (必要な) for  $P$** " because  **$P$  cannot be true unless  $Q$  is true.**

$$P \Leftarrow Q$$

Similarly, " **$P$  is sufficient (十分な) for  $Q$** " because  **$P$  being true always implies that  $Q$  is true**, but  **$P$  not being true does not always imply that  $Q$  is not true.**

$$P \Rightarrow Q$$

"**necessary and sufficient**" condition of another means that the former statement is true *if and only if* (iif, 同値) the latter is true. That is, the two statements must be either simultaneously true or simultaneously false.

$$P \Leftrightarrow Q$$

Read more: [https://en.wikipedia.org/wiki/Necessity\\_and\\_sufficiency](https://en.wikipedia.org/wiki/Necessity_and_sufficiency)