



Lecture 9

(Complex) Sequences and Series

(数列と級数)

What you will learn in Lecture 9

9.1 (Complex) Sequences and Series (数列と級数)

9.2 Testing Series

Cauchy's integral formula for derivatives indicates that if a function f is analytic at a point z_0 , then it possesses derivatives of all orders at that point.

As a consequence of this result we shall see that f can always be expanded in a power series centered at that point.

On the other hand, if f fails to be analytic at z_0 , we may still be able to expand it in a different kind of series known as a Laurent series.

9.1 (Complex) Sequences and Series

(数列と級数)

A **sequence** $\{z_n\}$, where $n = 1, 2, 3, \dots$, is a function whose domain is the set of positive integers and whose range is a subset of the complex numbers \mathbb{C} .

For example, the sequence $\{1 + i^n\}$ is

$$\begin{array}{ccccccccc} 1 + i, & 0, & 1 - i, & 2, & 1 + i, & \dots & & & (6.1.1) \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & & & \\ n = 1, & n = 2, & n = 3, & n = 4, & n = 5, & & & & \end{array}$$

9.1 (Complex) Sequences and Series (数列と級数)

Sequences (数列)

If $\lim_{n \rightarrow \infty} z_n = L$, we say the sequence $\{z_n\}$ is **convergent** (収束).

Sequence that is not convergent is said to be **divergent** (発散).

$\{z_n\}$ converges to the number L , if for each positive real number ε , an N can be found such that $|z_n - L| < \varepsilon$ whenever $n > N$. Since $|z_n - L|$ is distance, the terms z_n of a sequence that converges to L can be made arbitrarily close to L .

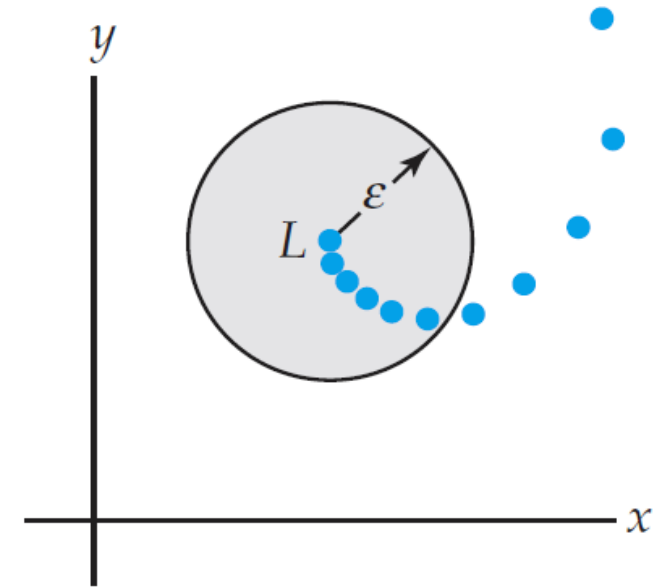


Figure 6.1 If $\{z_n\}$ converges to L , all but a finite number of terms are in every ε -neighborhood of L .

For example, the sequence $\{1 + i^n\}$

$$\begin{array}{ccccccccc} 1 + i, & 0, & 1 - i, & 2, & 1 + i, & \dots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ n = 1, & n = 2, & n = 3, & n = 4, & n = 5, & \end{array}$$

The sequence $\{1 + i^n\}$ is divergent because the general term $z_n = 1 + i^n$ does not approach a fixed complex number as $n \rightarrow \infty$.

EXAMPLE (例題) 6.1.1 A Convergent Sequence

The sequence $\left\{\frac{i^{n+1}}{n}\right\}$ converges or not.

Solution (解答):

Hint:

- Page 6 of this slides

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

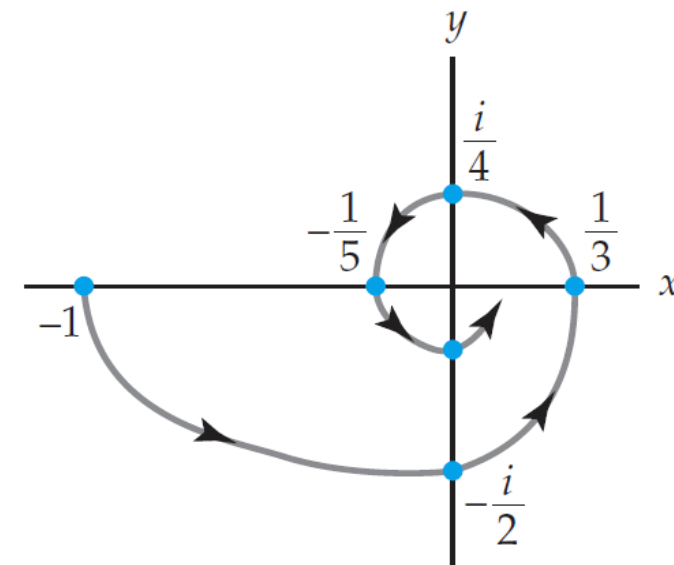


Figure 6.2 The terms of the sequence $\left\{\frac{i^{n+1}}{n}\right\}$ spiral in toward 0.

Theorem 6.1 Criterion (基準) for Sequence Convergence

Suppose that $z_n = x_n + iy_n$ ($n = 1, 2, \dots$) and $L = x + iy$. Then

$$\lim_{n \rightarrow \infty} z_n = L$$

if and only if

$$\lim_{n \rightarrow \infty} x_n = x \quad \text{and} \quad \lim_{n \rightarrow \infty} y_n = y$$

This theorem for sequences is the analogue of Theorem 2.1 in Lecture 2.

Theorem 2.1 Real and Imaginary Parts (実部と虚部) of a Limit

Suppose that $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$, and

$L = u_0 + iv_0$. Then $\lim_{z \rightarrow z_0} f(z) = L$ if and only if

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0$$

Additional EXAMPLE (例題) Using Theorem 6.1

The sequence $\left\{\frac{1}{n^3} + i\right\}$ converges or not.

Solution (解答):

Hint:

- Theorem 6.1

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

EXAMPLE (例題) 6.1.2 Using Theorem 6.1

The sequence $\left\{ \frac{3+ni}{n+2ni} \right\}$ converges or not.

Solution (解答):

Hint:

- Theorem 6.1

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

An infinite series or series of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + \cdots + z_n + \cdots$$

is convergent if the sequence of partial sums $\{S_n\}$, where

$$S_n = z_1 + z_2 + z_3 + \cdots + z_n$$

converges.

If $S_n \rightarrow L$ as $n \rightarrow \infty$, we say that the series converges to L or that the sum of the series is L .

Additional Theorem: Criterion (基準) for Series Convergence

Suppose that $z_k = x_k + iy_k$ ($k = 1, 2, \dots$) and $S = X + iY$. Then

$$\sum_{k=1}^{\infty} z_k = S$$

if and only if

$$\sum_{k=1}^{\infty} x_k = X \quad \text{and} \quad \sum_{k=1}^{\infty} y_k = Y$$

Additional EXAMPLE (例題) Using the Additional Theorem

Show that if $\sum_{k=1}^{\infty} z_k = S$, then $\sum_{k=1}^{\infty} \overline{z_k} = \overline{S}$.

Solution (解答):

Hint:

- Additional Theorem

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

Geometric Series (幾何級数)

A geometric series is any series of the form

$$\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \cdots + az^{n-1} + \cdots \quad (6.1.2)$$

For (6.1.2), the n th term of the sequence of partial sums is

$$S_n = a + az + az^2 + \cdots + az^{n-1} \quad (6.1.3)$$

Geometric Series (幾何級数)

When an infinite series is a geometric series, it is always possible to find a formula for S_n .

Why? We can multiply S_n in (6.1.3) by z ,

$$zS_n = az + az^2 + az^3 + \cdots + az^n$$

and subtract this result from S_n , then we have

$$S_n - zS_n = (a + az + az^2 + \cdots + az^{n-1}) - (az + az^2 + az^3 + \cdots + az^n)$$

$$= a - az^n$$

$$\Rightarrow S_n = \frac{a(1 - z^n)}{1 - z} \tag{6.1.4}$$

Now $z^n \rightarrow 0$ as $n \rightarrow \infty$ whenever $|z| < 1$, and so $S_n \rightarrow \frac{a}{1-z}$.

In other words, for $|z| < 1$ the sum of a geometric series (6.1.2) is $\frac{a}{1-z}$:

$$\frac{a}{1-z} = a + az + az^2 + \cdots + az^{n-1} + \cdots \tag{6.1.5}$$

A geometric series (6.1.2) diverges when $|z| \geq 1$.

Special Geometric Series

If we set $a = 1$, the equality in (6.1.5) is

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots \quad (6.1.6)$$

If we then replace the symbol z by $-z$ in (6.1.6), we get a similar result

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots \quad (6.1.7)$$

Like (6.1.5), the equality in (6.1.7) is valid for $|z| < 1$ since $|-z| = |z|$. Now with $a = 1$, (6.1.4) gives us the sum of the first n terms of the series in (6.1.6):

$$\frac{1 - z^n}{1 - z} = 1 + z + z^2 + z^3 + \dots + z^{n-1}$$

EXAMPLE (例題) 6.1.3 Convergent Geometric Series

The series $\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k}$ is convergent or divergent?

Solution (解答):

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

p -series

In elementary calculus a real series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a p -series and converges for $p > 1$ and diverges for $p \leq 1$.

Theorem 6.2 A Necessary Condition for Convergence

If $\sum_{k=1}^{\infty} z_k$ converges, then $\lim_{n \rightarrow \infty} z_n = 0$.

Proof

The Lecture Slides with complete proof will be uploaded with Assignment sheet after the class.

Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

9.2 Testing Series

9.2 Testing Series

A Test for Divergence

Theorem 6.3 The n th Term Test for Divergence

If $\lim_{n \rightarrow \infty} z_n \neq 0$, then $\sum_{k=1}^{\infty} z_k$ diverges.

For example,

the series $\sum_{k=1}^{\infty} \frac{ik+5}{k}$ **diverges** since $z_n = \frac{in+5}{n} \rightarrow i \neq 0$ as $n \rightarrow \infty$.

The geometric series (6.1.2) **diverges** if $|z| \geq 1$ because even in the case when $\lim_{n \rightarrow \infty} |z^n|$ exists, the limit is not zero.

9.2 Testing Series

Definition 6.1 Absolute and Conditional Convergence (絶対収束と条件収束)

An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **absolutely convergent** if $\sum_{k=1}^{\infty} |z_k|$ converges. An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **conditionally convergent** if it converges but $\sum_{k=1}^{\infty} |z_k|$ diverges.

9.2 Testing Series

EXAMPLE (例題) 6.1.4 Absolute Convergence

The series $\sum_{k=1}^{\infty} \frac{i^k}{k^2}$ is absolute convergent or not.

Solution (解答):

Hint:

- Definition 6.1

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

9.2 Testing Series

As in Real-value calculus:

Absolute convergence implies convergence.

We can therefore conclude that the series in Example 6.1.4,

$$\sum_{k=1}^{\infty} \frac{i^k}{k^2} = i - \frac{1}{2^2} - \frac{i}{3^2} + \frac{1}{4^2} + \dots$$

converges because it is absolutely convergent.

9.2 Testing Series

Tests for Convergence

Theorem 6.4 Ratio Test

Suppose $\sum z_n$ is a series of nonzero complex terms such that

$$\lim_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right| = L \quad (6.1.9)$$

- (i) If $L < 1$, then the series converges absolutely.
- (ii) If $L > 1$ or $L = \infty$, then the series diverges.
- (iii) If $L = 1$, the test is inconclusive.

Tests for Convergence

Theorem 6.5 Root Test

Suppose $\sum z_n$ is a series of complex terms such that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|z_n|} = L \quad (6.1.10)$$

- (i) If $L < 1$, then **the series converges absolutely**.
- (ii) If $L > 1$ or $L = \infty$, then **the series diverges**.
- (iii) If $L = 1$, **the test is inconclusive**.

Review for Lecture 9

- (Complex) Sequences and Series
- Convergence and Divergence
- Geometric Series
- p -series
- Absolute and Conditional Convergence
- Testing Series

Assignment

Please Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

Reading Materials: Section 6.1, Textbook

References

- [1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003
- [2] Wikipedia