Lecture 13

Residues（留数）&

Residue Theorem （定理）Part 2

What you will learn in Lecture 13

13.1 Residues (留数) & Residue Theorem (留数定理) Part 2

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Part 2

Theorem 6.14 Residue at a Simple Pole

If f has a simple pole at $z = z0$, then

$$\operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z) \tag{6.5.1}$$

Theorem 6.15 Residue at a Pole of Order n

If f has a pole of order n at $z = z_0$, then

$$\operatorname{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \tag{6.5.2}$$

13.1 Residues (留数) & Residue Theorem (留数定理) Part 2

When f **is not a rational function**, calculating residues by means of (6.5.1) or (6.5.2) in Lecture 12 can sometimes be tedious.

It is possible to devise **alternative residue formulas**.

In particular, suppose a function f can be written **as a quotient** $f(z) = g(z)/h(z)$, where g and h **are analytic at** $z = z_0$.

If $g(z_0) = 0$ and if the function h **has a zero of order** 1 **at** z_0, then f **has a simple pole at** $z = z_0$ and

$$\mathrm{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)} \tag{6.5.4}$$

13.1 Residues (留数) & Residue Theorem (留数定理) Part 2

To derive this result we shall use **the definition of a zero of order 1**, **the definition of a derivative**, and then (6.5.1).

First, since the function h **has a zero of order 1 at** z_0, we must have $h(z_0) = 0$ and $h(z_0) = 0$.

Second, by definition of the derivative given in (3.1.12) of Lecture 3 (slide 15),

$$h'(z_0) = \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{h(z)}{z - z_0}$$

We then combine the preceding two facts in the following manner in (6.5.1):

$$\mathrm{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)} = \lim_{z \to z_0} \frac{g(z)}{\dfrac{h(z)}{z - z_0}} = \frac{g(z_0)}{h'(z_0)}$$

Recall in Lecture 5

Roots of a Complex Number

Consider to find z in $z^k = w$

where z and w are complex numbers,

k is real, i.e. NOT a complex number.

then

$$z = \sqrt[k]{|w|}\left[\cos\left(\frac{\arg(w) + 2n\pi}{k}\right) + i\sin\left(\frac{\arg(w) + 2n\pi}{k}\right)\right] \qquad (1.4.4)$$

where $n = 0, 1, 2, \ldots, k-1$

EXAMPLE (例題) 6.5.3 **Using (6.5.4) to Compute Residues**

The polynomial z^4+1 can be factored as $(z-z_1)(z-z_2)(z-z_3)(z-z_4)$, where z_1, z_2, z_3, and z_4 are the four distinct roots of the equation $z_4+1=0$ (or equivalently, the four fourth roots of -1). It follows from Theorem 6.13 in Lecture 12 that the function

$$f(z)=\frac{1}{z^4+1}$$

has four simple poles. By using (6.5.4), find its residues.

Solution (解答):

Now from (1.4.4) of Lecture 5, for $z^4=-1$, we have $|-1|=1$, $\arg(-1)=\pi$.

Thus for $n=0,1,2,3$, we obtain $z_1=\cos(\pi i/4)+i\sin(\pi i/4)=e^{\pi i/4}$, $z_2=e^{3\pi i/4}$, $z_3=e^{5\pi i/4}$, and $z_4=e^{7\pi i/4}$.

To compute the residues, we use (6.5.4) of this Lecture by identifying $g(z)=1$, $h(z)=z^4+1$, along with Euler's formula (1.6.6) $e^{\theta}=\cos\theta+i\sin\theta$

Solution (解答)(cont.):

$$\mathrm{Res}(f(z), z_1) = \frac{g(z_1)}{h'(z_1)} = \frac{1}{4z_1^3} = \frac{1}{4}e^{-3\pi i/4} = -\frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

$$\mathrm{Res}(f(z), z_2) = \frac{g(z_2)}{h'(z_2)} = \frac{1}{4z_2^3} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

$$\mathrm{Res}(f(z), z_3) = \frac{g(z_3)}{h'(z_3)} = \frac{1}{4z_3^3} = \frac{1}{4}e^{-15\pi i/4} = \frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}i$$

$$\mathrm{Res}(f(z), z_4) = \frac{g(z_4)}{h'(z_4)} = \frac{1}{4z_4^3} = \frac{1}{4}e^{-21\pi i/4} = -\frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}i$$

Theorem 6.16 Cauchy's Residue Theorem

Let D be **a simply connected domain** and C **a simple closed contour lying entirely within** D. If a function f **is analytic on and within** C, **except at a finite number of isolated singular points** $z_1, z_2, \ldots, z_n$ **within** C, then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k) \qquad (6.5.5)$$

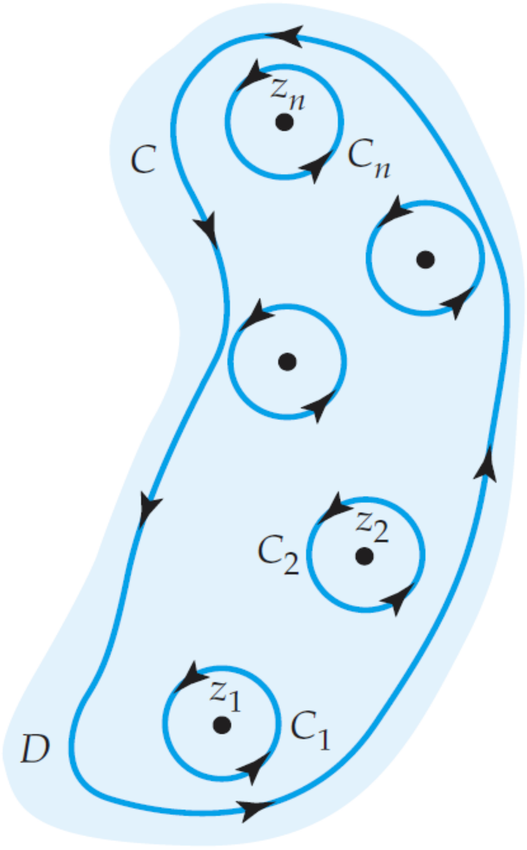

Figure 6.10 n **singular points within contour** C

Proof

Suppose $C_1, C_2, \ldots, C_n$ are circles centered at $z_1, z_2, \ldots, z_n$, respectively. Suppose further that each circle C_k has a radius r_k small enough so that $C_1, C_2, \ldots, C_n$ are mutually disjoint and are interior to the simple closed curve C.

See Figure 6.10. Now in (6.3.20) of Section 6.3 we saw that $\oint_{C_k} f(z)dz = 2\pi i \operatorname{Res}(f(z), z_k)$, and so by Theorem 5.5 we have

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k)$$

13.1 Residues (留数) & Residue Theorem (留数定理) Part 2

Replacing the complex variable s with the usual symbol z, we see that when $k - 1$, formula (6.3.8) in Lecture 11 (slide 13) for the Laurent series coefficients yields $a_{-1} = \frac{1}{2\pi i}\oint_C \; f(z)dz$, or more important,

$$\oint_C \; f(z)dz = 2\pi i \, a_{-1} \tag{6.3.20}$$

EXAMPLE (例題) 6.5.4 Evaluation by the Residue Theorem

Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)} dz$, where

(a) the contour C is the rectangle defined by $x = 0, x = 4, y = -1, y = 1$,

(b) and the contour C is the circle $|z| = 2$.

Solution (解答):

(a) Since both $z = 1$ and $z = 3$ are poles within the rectangle we have from (6.5.5) that

$$\oint_C \frac{1}{(z-1)^2(z-3)} dz = 2\pi i[\mathrm{Res}(f(z), 1\,) + \mathrm{Res}(f(z), 3\,)]$$

We found these residues in Example 6.5.2. Therefore,

$$\oint_C \frac{1}{(z-1)^2(z-3)} dz = 2\pi i\left[\left(-\frac{1}{4}\right) + \frac{1}{4}\right] = 0$$

(b) Since only the pole $z = 1$ lies within the circle $|z| = 2$, we have from (6.5.5)

$$\oint_C \frac{1}{(z-1)^2(z-3)} dz = 2\pi i\ \mathrm{Res}(f(z), 1\,) = 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi}{2} i$$

EXAMPLE (例題) 6.5.5 Evaluation by the Residue Theorem

Evaluate $\oint_C \frac{2z+6}{z^2+4}\,dz$, where the contour C is the circle $|z-i|=2$.

Solution (解答):

By factoring the denominator as $z^2+4=(z-2i)(z+2i)$ we see that the integrand has simple poles at $-2i$ and $2i$. Because only $2i$ lies within the contour C, it follows from (6.5.5) that

$$\oint_C \frac{2z+6}{z^2+4}\,dz = 2\pi i \operatorname{Res}(f(z), 2i\,)$$

But

$$\operatorname{Res}(f(z), 2i) = \lim_{z\to 2i}(z-2i)\frac{2z+6}{(z-2i)(z+2i)} = \frac{3+2i}{2i}$$

Hence,

$$\oint_C \frac{2z+6}{z^2+4}\,dz = 2\pi i\left(\frac{3+2i}{2i}\right) = \pi(3+2i)$$

EXAMPLE (例題) 6.5.6 Evaluation by the Residue Theorem

Evaluate $\oint_C \frac{e^z}{z^4+5z^3}dz$, where the contour C is the circle $|z|=2$.

Solution (解答):

Writing the denominator as $z^4+5z^3=z^3(z+5)$ reveals that the integrand $f(z)$ has a pole of order 3 at $z=0$ and a simple pole at $z=-5$.

But only the pole $z=0$ lies within the given contour and so from (6.5.5) and (6.5.2) we have,

$$\oint_C \frac{e^z}{z^4+5z^3}dz = 2\pi i\,\mathrm{Res}(f(z),0) = 2\pi i \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}z^3\frac{e^z}{z^3(z+5)}$$

$$= \pi i \lim_{z\to 0}\frac{(z^2+8z+17)e^z}{(z+5)^3} = \frac{17\pi}{125}i$$

EXAMPLE (例題) 6.5.7 Evaluation by the Residue Theorem

Evaluate $\oint_C \tan z \, dz$, where the contour C is the circle $|z| = 2$.

Solution (解答):

The integrand $f(z) = \tan z = \sin z / \cos z$ has simple poles at the points where $\cos z = 0$. We saw in the slide 32 in Lecture 5 that the only zeros $\cos z$ are the real numbers $z = (2n+1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$. Since only $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are within the circle $|z| = 2$, we have

$$\oint_C \tan z \, dz = 2\pi i \left[\operatorname{Res}\left(f(z), -\frac{\pi}{2}\right) + \operatorname{Res}\left(f(z), \frac{\pi}{2}\right) \right]$$

With the identifications $g(z) = \sin z$, $h(z) = \cos z$, and $h'(z) = -\sin z$, we see from (6.5.4) that

$$\operatorname{Res}\left(f(z), -\frac{\pi}{2}\right) = \frac{\sin\left(-\frac{\pi}{2}\right)}{-\sin\left(-\frac{\pi}{2}\right)} = -1 \qquad \text{and} \qquad \operatorname{Res}\left(f(z), \frac{\pi}{2}\right) = \frac{\sin\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{\pi}{2}\right)} = -1$$

Therefore, $\oint_C \tan z \, dz = 2\pi i \, [-1 - 1] = -4\pi i$

EXAMPLE (例題) 6.5.8 Evaluation by the Residue Theorem

Evaluate $\oint_C e^{\frac{3}{z}}\,dz$, where the contour C is the circle $|z| = 1$.

Solution (解答):

As we have seen, $z = 0$ is an essential singularity of the integrand $f(z) = e^{\frac{3}{z}}$ and so neither formulas (6.5.1) and (6.5.2) are applicable to find the residue of f at that point.

We saw in Example 6.5.1 that the Laurent series of f at $z = 0$ gives $\mathrm{Res}(f(z), 0) = 3.$

Hence from (6.5.5) we have

$$\oint_C e^{\frac{3}{z}}\,dz = 2\pi i\,\mathrm{Res}(f(z), 0) = 2\pi i \cdot 3 = 6\pi i$$

Review for Lecture 13

- Residues (留数)
- Residue Theorem (留数定理)

Assignment

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: Section 6.5, Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia