Lecture 8

Cauchy's Integral Formulas (コーシーの積分公式)

and

Their Consequences (関連事項)

What you will learn in Lecture 8

8.1 Cauchy's Two Integral Formulas

8.2 Some Consequences of Cauchy's Integral Formulas

In this lecture 8, we are going to examine several more consequences of the Cauchy-Goursat theorem.

Unquestionably, the most significant of these is the following result:

The value of a analytic function f at any point z_0 in a simply connected domain can be represented by a contour integral.

After establishing this proposition we shall use it to further show that:

An analytic function f in a simply connected domain possesses derivatives of all orders.

8.1 Cauchy's Two Integral Formulas

The First Formula

If f is **analytic in a simply connected domain** D and z_0 is **any point in** D, the quotient $f(z)/(z-z_0)$ **is not defined at** z_0 and **hence** is **NOT analytic in** D.

Therefore, we **CANNOT conclude** that the **integral of $f(z)/(z-z_0)$ around a simple closed contour C** that **contains z_0 is zero by the Cauchy-Goursat theorem**. We introduce that

Theorem 5.9 Cauchy's Integral Formula (コーシーの積分公式)

Suppose that f is **analytic** in a **simply connected domain** D and C **is any simple closed contour lying entirely within** D. Then **for any point z_0 within** C,

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz \qquad (5.5.1)$$

Therefore, we can see that the integral of $f(z)/(z-z_0)$ around C has the value $2\pi i \cdot f(z_0)$.

The First Formula

Because the symbol z represents a point on the contour C, (5.5.1) indicates that

the values of an analytic function f at points z_0 inside a simple closed contour C are determined by the values of f on the contour C.

We can rewrite the Theorem 5.9 as a more practical manner:

If f **is analytic at all points within and on a simple closed contour** C, and z_0 **is any point interior to** C, then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

EXAMPLE (例題) 5.5.1 Using Cauchy's Integral Formula

Evaluate $\oint_C \frac{z^2-4z+4}{z+i}dz$, where the contour C is the circle $|z|=2$.

Solution (解答):

First, we identify $f(z)=z^2-4z+4$ and $z_0=-i$ as a point within the circle C.

Next, we observe that f is analytic at all points within and on the contour C.

Thus, by the Cauchy integral formula (5.5.1) we obtain

$$\oint_C \frac{z^2-4z+4}{z+i}dz=\oint_C \frac{z^2-4z+4}{z-(-i)}dz=2\pi i\cdot f(-i)$$

$$=2\pi i\big((-i)^2-4(-i)+4\big)=2\pi i(-1+4i+4)=2\pi i(3+4i)=-8\pi+6\pi i$$

EXAMPLE (例題) 5.5.2 **Using Cauchy's Integral Formula**

Evaluate $\oint_C \frac{z}{z^2+9}\,dz$, where the contour C is the circle $|z-2i|=4$.

Solution (解答):

By factoring the denominator as $z^2+9=(z-3i)(z+3i)$ we see that $3i$ is the only point within the closed contour C at which the integrand fails to be analytic. See Figure 5.44. Then by rewriting the integrand as

$$\frac{z}{z^2+9}=\frac{z}{(z-3i)(z+3i)}=\frac{\frac{z}{z+3i}}{z-3i} \quad \}f(z)$$

we can identify $f(z)=z/(z+3i)$. The function f is analytic at all points within and on the contour C. Hence, from Cauchy's integral formula (5.5.1) we have

$$\oint_C \frac{z}{z^2+9}\,dz=\oint_C \frac{\frac{z}{z+3i}}{z-3i}\,dz=2\pi i\cdot f(3i)=2\pi i\frac{3i}{3i+3i}=2\pi i\frac{3i}{6i}=\pi i$$

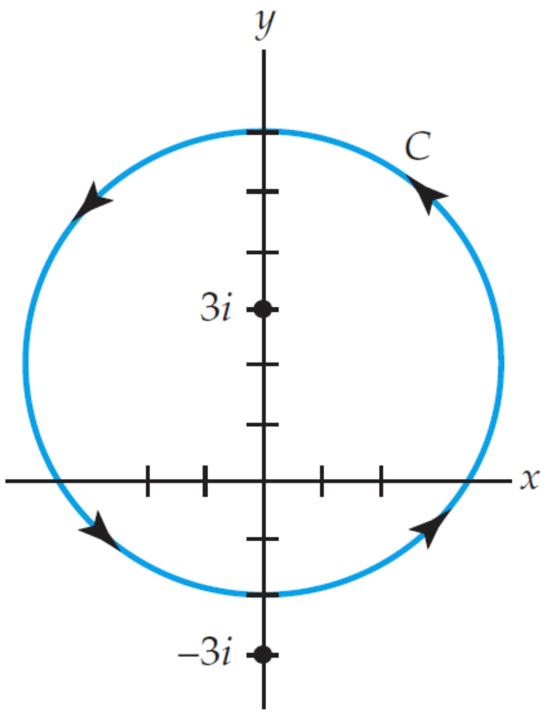

Figure 5.44 Contour for Example 5.5.2

Theorem 5.10 Cauchy's Integral Formula for Derivatives

Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then for any point z_0 within C,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad (5.5.6)$$

Like (5.5.1), formula (5.5.6) can be used to evaluate integrals. See the examples as following.

EXAMPLE (例題) 5.5.3 Using Cauchy's Integral Formula for Derivatives

Evaluate $\oint_C \frac{z+1}{z^4+2iz^3}dz$, where the contour C is the circle $|z|=1$.

Solution (解答):

Inspection of the integrand shows that it is not analytic at $z=0$ and $z=-2i$, but only $z=0$ lies within the closed contour. By writing the integrand as

$$\frac{z+1}{z^4+2iz^3}=\frac{z+1}{(z+2i)z^3}=\frac{\frac{z+1}{z+2i}}{z^3}$$

we can identify, $z_0=0$, $n=2$, and $f(z)=(z+1)/\,(z+2i)$. The quotient rule gives $f''(z)=(2-4i)/(z+2i)^3$ and so $f''(0)=(2i-1)/4i$. Hence from (5.5.6) we find

$$\oint_C \frac{z+1}{z^4+2iz^3}dz=\frac{2\pi i}{2!}f''(0)=-\frac{\pi}{4}+\frac{\pi}{2}i$$

8.2 Some Consequences (関連事項) of Cauchy Integral Formulas

8.2 Some Consequences (關連事項) of Cauchy Integral Formulas

An immediate and important corollary to Theorem 5.10 is summarized next.

Theorem 5.11 Derivative of an Analytic Function Is Analytic

Suppose that f **is analytic in a simply connected domain** D. Then f **possesses derivatives of all orders at every point** z **in** D. The **derivatives** $f', f'', f'''...$ **are analytic functions in** D.

8.2 Some Consequences (關連事項) of Cauchy Integral Formulas

If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a simply connected domain D, we have just seen its derivatives of all orders exist at any point z in D and so $f', f'', f''' \ldots$ are continuous. From

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} - i\frac{\partial^2 u}{\partial y \partial x}$$

$$\vdots$$

we can also conclude that the real functions u and v have continuous partial derivatives of all orders at a point of analyticity.

An inequality (不等式) derived from the Cauchy integral formula for derivatives.

Theorem 5.12 Cauchy's Inequality (コーシーの評価式)

Suppose that f is **analytic in a simply connected domain** D and C is **a circle defined by** $|z - z_0| = r$ **that lies entirely in** D. If $|f(z)| \leq M$ for all points z on C, then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!\,M}{r^n} \tag{5.5.7}$$

The number M in Theorem 5.12 depends on the circle $|z - z_0| = r$. But notice in (5.5.7) that if $n = 0$, then $M \geq |f(z_0)|$ for any circle C centered at z_0 as long as C lies within D. In other words, an upper bound M of $|f(z)|$ on C cannot be smaller than $|f(z_0)|$.

8.2 Some Consequences (関連事項) of Cauchy Integral Formulas

Theorem 5.13 Liouville's Theorem (リウヴィルの定理)

The only bounded entire functions are constants (定数).

Although it bears the name "Liouville's Theorem", it probably was first proved by Cauchy.

Proof:
Suppose f is an entire function and is bounded, that is, $|f(z)| \leq M$ for all z. Then for any point z_0, (5.5.7) gives $|f'(z_0)| \leq M/r$. By making r arbitrarily large we can make $|f'(z_0)|$ as small as we wish. This means $f'(z_0) = 0$ for all points z_0 in the complex plane. Hence, by Theorem 3.6(ii), f must be a constant . ■

Theorem 3.6 Constant Functions

Suppose the function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D.
(i) If $|f(z)|$ is constant in D, then so is $f(z)$.
(ii) If $f'(z) = 0$ in D, then $f(z) = c$ in D, where c is a constant.

8.2 Some Consequences (関連事項) of Cauchy Integral Formulas

Theorem 5.13 enables us to establish a result usually learned—but never proved—in elementary algebra.

Theorem 5.14 Fundamental Theorem of Algebra (代数学の基本定理)

If $p(z)$ is a **nonconstant (非定数) polynomial (多項式)**, then the equation $p(z) = 0$ has at least one root (根).

Using Theorem 5.14, that if $p(z)$ is a nonconstant polynomial of degree n, then $p(z) = 0$ has exactly n roots (counting multiple roots).

8.2 Some Consequences (関連事項) of Cauchy Integral Formulas

The converse of the Cauchy-Goursat theorem:

Theorem 5.15 Morera's Theorem (モレラの定理)

If f is **continuous in a simply connected domain** D and if

$\oint_C f(z)dz = 0$ **for every closed contour** C in D, then f **is analytic in** D.

Theorem 5.16 Maximum Modulus Theorem (最大絶対値の原理あるいは最大値の原理)

Suppose that f **is analytic and nonconstant on a closed region** R **bounded by a simple closed curve** C. Then **the modulus** $|f(z)|$ **attains its maximum on** C.

EXAMPLE (例題) 5.5.5 Maximum Modulus Theorem
Find the maximum modulus of $f(z) = 2z + 5i$ on the closed circular region defined by $|z| \leq 2$.

Solution (解答):

From Equation (1.2.2) $|z|^2 = \bar{z}z$ and by replacing the symbol z by $2z + 5i$ we have

$$|2z + 5i|^2 = (2z + 5i)\overline{(2z + 5i)} = (2z + 5i)(2\bar{z} + (-5i)) = 4\bar{z}z - 10i(z - \bar{z}) + 25. \qquad (5.5.8)$$

But from Equation (1.1.6) of Section 1.1, $z - \bar{z} = 2i\,\mathrm{Im}(z)$, and so (5.5.8) is

$$|2z + 5i|^2 = 4|z|^2 + 20\mathrm{Im}\,(z) + 25. \qquad (5.5.9)$$

Because f is a polynomial, it is analytic on the region defined by $|z| \leq 2$. By Theorem 5.16, $\max_{|z|\leq 2} |\,2z + 5i|$ occurs on the boundary $|z| = 2$. Therefore, on $|z| = 2$, (5.5.9) yields

$$|2z + 5i| = \sqrt{4 \cdot 2^2 + 25 + 20\mathrm{Im}(z)} = \sqrt{41 + 20\mathrm{Im}(z)}$$

This expression attains its maximum when $\mathrm{Im}(z)$ attains its maximum on $|z| = 2$, namely, at the point $z = 2i$. Thus, $\max_{|z|\leq 2}|\,2z + 5i| = \sqrt{41 + 20 \cdot 2} = \sqrt{81} = 9$.

Review for Lecture 8

- Cauchy’s Integral Formula
- Cauchy’s Integral Formula for Derivatives
- Derivative of an Analytic Function Is Analytic
- Cauchy’s Inequality
- Liouville’s Theorem
- Fundamental Theorem of Algebra
- Morera’s Theorem
- Maximum Modulus Theorem

Assignment

Please Check https://github.com/uoaworks/ComplexAnalysisAY2018

Reading Materials: Section 5.5, Textbook

References

[1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia