



Lecture **14**

Application to Integral

What you will learn in Lecture 14

14.1 Laplace Transform (ラプラス変換)

14.2 Inverse Laplace Transform (逆ラプラス変換)

14.1 Laplace Transform (ラプラス変換)

14.1 Laplace Transform (ラプラス変換)

Integral Transforms

Suppose $f(x, y)$ is a real-valued function of two real variables. Then a definite integral of f with respect to one of the variables leads to a function of the other variable.

For example, if we hold y constant, integration with respect to the real variable x gives $\int_1^2 4xy^2 dx = 6y^2$.

Thus a definite integral such as $F(\alpha) = \int_a^b f(x)K(\alpha, x)dx$ transforms a function f of the variable x into a function F of the variable α . We say that

$$F(\alpha) = \int_a^b f(x)K(\alpha, x)dx \quad (6.7.2)$$

is an integral transform of the function f .

Integral transforms usually appear in transform pairs. This means that the original function f can be recovered by another integral transform

$$f(x) = \int_c^d F(\alpha)H(\alpha, x)d\alpha \quad (6.7.3)$$

called the inverse transform. The function $K(\alpha, x)$ in (6.7.2) and the function $H(\alpha, x)$ in (6.7.3) are called the kernels of their respective transforms.

We note that if α represents a complex variable, then the definite integral (6.7.3) is replaced by a contour integral.

14.1 Laplace Transform (ラプラス変換)

The Laplace Transform

Suppose now in (6.7.2) that the symbol α is replaced by the symbol s , and that f represents a real function (On occasion $f(t)$ could be a complex-valued function of a real variable t) that is defined on the unbounded interval $[0, \infty)$. Then (6.7.2) is an improper integral and is defined as

$$\int_0^{\infty} K(s, t) f(t) dt = \lim_{b \rightarrow \infty} \int_0^b K(s, t) f(t) dt \quad (6.7.4)$$

If the limit in (6.7.4) exists, we say that the integral exists or is convergent; if the limit does not exist, then the integral does not exist and is said to be divergent.

Laplace transform of a real function $f(t)$ defined for $t \geq 0$:

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \quad (6.7.1)$$

Here the choice of the kernel in (6.7.4) is $K(s, t) = e^{-st}$, where s is a complex variable.

The integral that defines the Laplace transform may not converge for certain kinds of functions f . For example, neither $\mathcal{L}\{e^{t^2}\}$ nor $\mathcal{L}\{1/t\}$ exist. Also, the limit in (6.7.4) will exist for only certain values of the variable s .

14.1 Laplace Transform (ラプラス変換)

EXAMPLE (例題) 6.7.1 Existence of a Laplace Transform

Evaluate the Laplace transform of $f(t) = 1$, for $t \geq 0$.

Solution (解答):

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

1.4 Application to Integral

Theorem 6.22 Sufficient Conditions for Existence of Laplace Transform

Suppose f is piecewise continuous on $[0, \infty)$ and of exponential order c for $t > T$. Then $\{f(t)\}$ exists for $\operatorname{Re}(s) > c$.

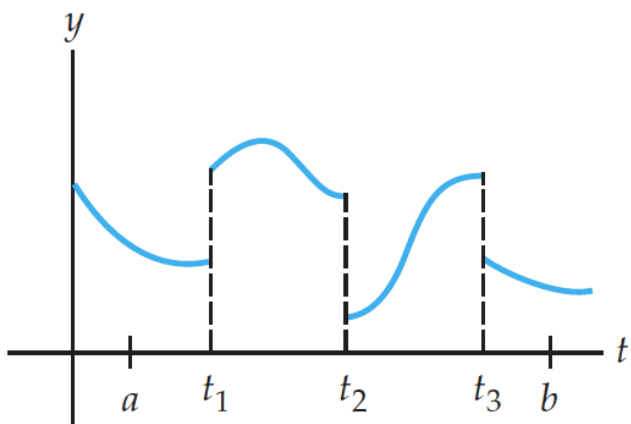


Figure 6.20 Piecewise continuity on $[0, \infty)$

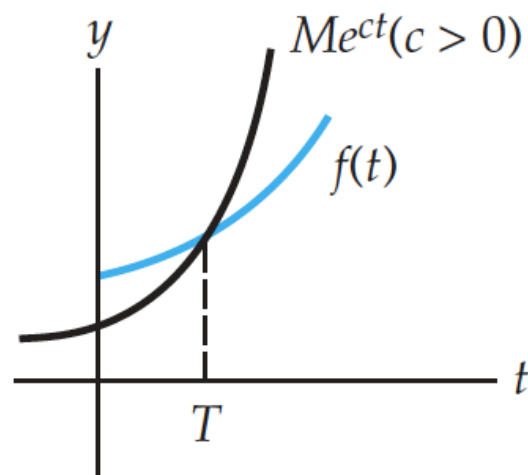


Figure 6.21 Exponential order

A function f is said to be **exponential order** c if there exist constants $c, M > 0$, and $T > 0$ so that $|f(t)| \leq Me^{ct}$, for $t > T$.

The condition $|f(t)| \leq Me^{ct}$ for $t > T$ states that the graph of f on the interval (T, ∞) does not grow faster than the graph of the exponential function Me^{ct} .

14.1 Laplace Transform (ラプラス変換)

Theorem 6.23 Analyticity of the Laplace Transform

Suppose f is piecewise continuous on $[0, \infty)$ and of exponential order c for $t \geq 0$. Then the **Laplace transform** of f ,

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

is an analytic function in the right half-plane defined by $\operatorname{Re}(s) > c$.

14.2 Inverse Laplace Transform (逆ラプラス変換)

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The Inverse Laplace Transform

Although Theorem 6.23 indicates that the complex function $F(s)$ is analytic to the right of the line $x = c$ in the complex plane, $F(s)$ will have singularities to the left of that line in general.

Theorem 6.24 Inverse Laplace Transform

If f and f' are piecewise continuous on $[0, \infty)$ and f is of exponential order c for $t \geq 0$, and $F(s)$ is a Laplace transform, then the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ is

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\gamma - iR}^{\gamma + iR} e^{st} F(s) dt \quad (6.7.7)$$

where $\gamma > c$.

14.2 Inverse Laplace Transform (逆ラプラス変換)

The limit in (6.7.7), which defines a principal value of the integral, is usually written as

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\gamma - iR}^{\gamma + iR} e^{st} F(s) dt \quad (6.7.8)$$

where the limits of integration indicate that the integration is along the infinitely long vertical-line contour $\operatorname{Re}(s) = x = \gamma$.

Here γ is a positive real constant greater than c and greater than all the real parts of the singularities in the left half-plane.

The integral in (6.7.8) is called a **Bromwich contour integral**.

Relating (6.7.8) back to (6.7.3), we see that the kernel of the inverse transform is $H(s, t) = \frac{e^{st}}{2\pi i}$.

14.2 Inverse Laplace Transform (逆ラプラス変換)

Theorem 6.25 Inverse Laplace Transform

Suppose $F(s)$ is a Laplace transform that has a finite number of poles s_1, s_2, \dots, s_n to the left of the vertical line $\operatorname{Re}(s) = \gamma$ and that C is the contour illustrated in Figure 6.23. If $sF(s)$ is bounded as $R \rightarrow \infty$, then

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^n \text{Res}(e^{st} F(s), s_k) \quad (6.7.9)$$

Residue (留数)

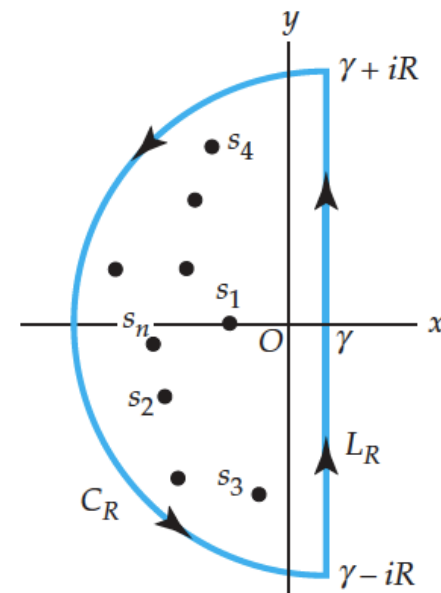


Figure 6.23 Possible contour that could be used to evaluate (6.7.7)

14.2 Inverse Laplace Transform (逆ラプラス変換)

EXAMPLE (例題) 6.7.2 Inverse Laplace Transform

Evaluate $\mathcal{L}^{-1} \left\{ \frac{1}{s^3} \right\}$, $\operatorname{Re}(s) > 0$.

Solution (解答):

Hint:

- Equation (6.7.9)
- Equation (6.5.2) in Lecture 12

The Lecture Slides with complete solution will be uploaded with Assignment sheet after the class.

Check <https://github.com/uoaworks/ComplexAnalysisAY2018>

Review for Lecture 14

- Laplace Transform
- Inverse Laplace Transform
- Application of Residue to Evaluate Inverse Laplace Transform

Assignment

No homework for Lecture 14

Reading Materials: Section 6.7, Textbook

References

- [1] A first course in Complex Analysis with application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003
- [2] Wikipedia

Review

Review for MA06 class.