

Lecture 8

# Laplace Transform for Solving ODEs

# What you will learn in Lecture 8

I. Laplace Transform for Solving ODEs

**II**. Further Properties of the Laplace Transforms

III. Convolutions and Laplace Transforms

**Ordinary Differential Equations** 

(ODEs)

Let us now discuss how the Laplace transform method solves ODEs and initial value problems.

We consider an initial value problem.

$$y'' + ay' + by = r(t), \quad y(0) = K_0, \quad y'(0) = K_1$$

where a and b are constant. Here r(t) is the given input applied to the system and y(t) is the output (response to the input) to be obtained.

### **THEOREM 3 Laplace transform of derivatives**

(i) Suppose that f is continuous on  $[0, \infty)$  and of exponential order as in (2).

Suppose further that f' is piecewise continuous on  $[0, \infty)$  and of exponential order. Then

(3) 
$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

(ii) More generally, if  $f, f', ..., f^{(n-1)}$  are continuous on  $[0, \infty)$  and of exponential order as in (2), and  $f^{(n)}$  is piecewise continuous on  $[0, \infty)$  and of exponential order, then

(4) 
$$\mathcal{L}(f^{(n)}) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

When n = 2, (4) gives

(5) 
$$\mathcal{L}(f'') = s^2 \mathcal{L}(f) - sf(0) - f'(0)$$

#### **Proof**

Find the complete proof in the page 483 of the textbook.

In Laplace's method we do three steps:

**Step 1**. Setting up the subsidiary equation, namely take the Laplace Transform for both sides of the differential equation and use the Theorem 3

$$[s^{2}Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R$$
$$(s^{2} + as + b)Y = (s + a)y(0) + y'(0) + R$$

Step 2. Find the solution of the subsidiary equation by algebra

$$Y = \frac{(s+a)y(0) + y'(0) + R}{s^2 + as + b}$$

**Step 3**. Find the Inversion of Y to obtain  $y = \mathcal{L}^{-1}(Y)$ 

We may reduce the original function by partial fractions to a sum of terms whose inverse can be found from the table of the Laplace Transforms.

### **EXAMPLE 8 A second order ordinary differential equation**

Solve 
$$y'' + y = 2$$
,  $y(0) = 0$ ,  $y'(0) = 1$ 

#### **Solution**

**Step 1**. Setting up the subsidiary equation, namely take the Laplace Transform for both sides of the differential equation and use the Theorem 3

$$[s^{2}Y - sy(0) - y'(0)] + bY = \mathcal{L}(2) = 2 \cdot \frac{1}{s} = \frac{2}{s}$$
$$(s^{2} + 1)Y = \frac{2}{s} + 1$$

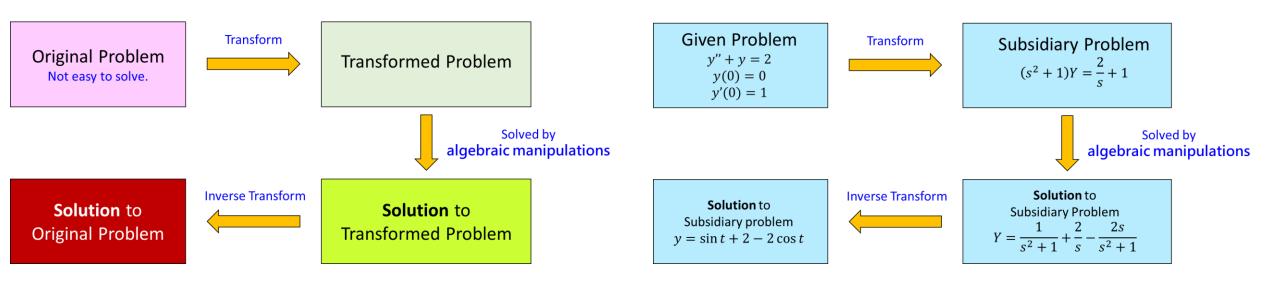
Step 2. Find the solution of the subsidiary equation by algebra

$$Y = \frac{\frac{2}{s} + 1}{s^2 + 1} = \frac{1}{s^2 + 1} + \frac{2}{s(s^2 + 1)} = \frac{1}{s^2 + 1} + \frac{2}{s} - \frac{2s}{s^2 + 1}$$

**Step 3**. Find the Inversion of *Y* to obtain

$$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\right) + \mathcal{L}^{-1}\left(\frac{2}{s}\right) - \mathcal{L}^{-1}\left(\frac{2s}{s^2 + 1}\right) = \sin t + 2 - 2\cos t$$

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### **EXAMPLE 9 Shifting the time variable**

Solve 
$$y'' + 2y' + y = t$$
,  $y(1) = 0$ ,  $y'(1) = 0$ 

#### **Solution**

Making the change of variables  $t = \tau + 1$ , then  $\tau = t - 1$ 

$$y'' + 2y' + y = \tau + 1, y(0) = 0, y'(0) = 0$$

Step 1.

$$s^{2}Y + 2sY + Y = \mathcal{L}(\tau) + \mathcal{L}(1) = \frac{1}{s^{2}} + \frac{1}{s}$$
$$(s^{2} + 2s + 1)Y = \frac{1}{s^{2}} + \frac{1}{s}$$

Step 2.

$$Y = \frac{\frac{1}{s^2} + \frac{1}{s}}{s^2 + 2s + 1} = \frac{\frac{s+1}{s^2}}{(s+1)^2} = \frac{1}{s^2(s+1)} = \frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}$$

Step 3.

$$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = e^{-\tau} - 1 + \tau$$

Making the change of variables  $\tau = t - 1$ 

$$y = e^{1-t} + t - 2$$

#### Advantages of the Laplace Method

- 1. Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE.
- 2. Initial values are automatically taken care of.
- **3.** Complicated inputs r(t) (right sides of linear ODEs) can be handled very efficiently, as we show in the next sections.

#### Homogeneous and non-homogeneous equations

Recall that a linear differential equation

$$f_n(x)y^{(n)} + \dots + f_1(x)y' + f_0(x)y = g(x)$$

was called homogeneous if g(x) = 0, and non-homogeneous or inhomogeneous otherwise.

# 8.2 Further Properties of

the Laplace Transforms

We introduce an auxiliary function:

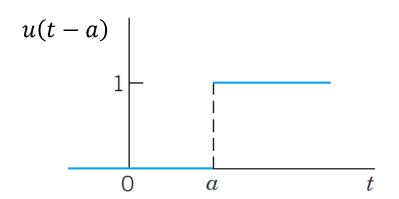
the *unit step function* or *Heaviside function* 

These functions are suitable for solving ODEs with complicated right sides of considerable engineering interest, such as single waves, inputs (driving forces) that are discontinuous or act for some time only, periodic inputs more general than just cosine and sine, or impulsive forces acting for an instant.

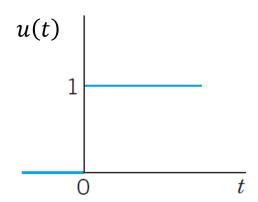
#### **Unit Step Function (Heaviside Function)**

The unit step function or Heaviside function u(t - a) is 0 for t < a, has a jump of size 1 at t = a (where we can leave it undefined), and is 1 for in a formula

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \ge a \end{cases} \qquad a \ge 0$$



Unit step function u(t - a)



Unit step function u(t), when a=0

Given a function f(t), consider the product u(t-a)f(t-a). Written explicitly, we have

$$u(t-a)f(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t \ge a \end{cases}$$

Thus, if f(t) represents, say a signal, then u(t-a)f(t-a) represents the same signal, but delayed by a units of time.

#### THEOREM 1 Shifting on The *t*-Axis

If a is a positive real number, then

$$\mathcal{L}(u(t-a)f(t-a))(s) = e^{-as}F(s)$$

where  $F(s) = \mathcal{L}(f(t))(s)$ .

#### **Proof**

$$\mathcal{L}(u(t-a)f(t-a))(s) = \int_{a}^{\infty} f(t-a)e^{-st}dt$$

$$= \int_{0}^{\infty} f(T)e^{-s(T+a)}dT \text{ (where } t-a=T, dt=dT)$$

$$= e^{-as} \int_{0}^{\infty} f(T)e^{-sT}dT$$

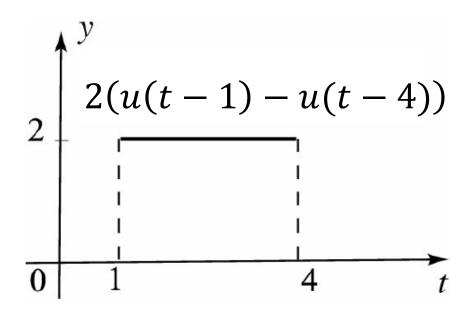
$$= e^{-as}F(s)$$

The unit step function is a typical "engineering function" made to measure for engineering applications, which often involve functions (mechanical or electrical driving forces) that are either "off" or "on."

### **EXAMPLE 1 Transforms involving unit step functions**

- (a) Evaluate  $\mathcal{L}(u(t-a))$  (b) Evaluate  $\mathcal{L}(f(t))$  where

$$f(t) = \begin{cases} 2 & if \ 1 \le t < 4 \\ 0 & \text{otherwise} \end{cases}$$



#### **Solutions**

(a) The transform of follows directly from the defining integral in

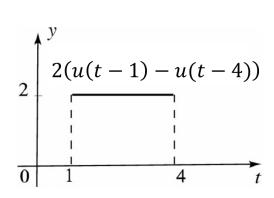
$$\mathcal{L}(u(t-a)) = \int_0^\infty e^{-st} u(t-a) dt = \int_a^\infty e^{-st} \cdot 1 dt = -\frac{e^{-st}}{s} \Big|_{t=a}^\infty$$

here the integration begins at  $t = a \ge 0$  because u(t - a) is 0 for t < a. Hence

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s} \qquad s > 0$$

(b) Verify that f(t) = 2(u(t-1) - u(t-4)) Then

$$\mathcal{L}(f(t)) = 2\left(\mathcal{L}(u(t-1)) - \mathcal{L}(u(t-4))\right) = \frac{2}{s}(e^{-s} - e^{-4s}), s > 0$$



#### **EXAMPLE 2 A ramp function**

Evaluate the Laplace transform of the ramp function shown in Figure 3.

#### **Solution**

For t > 1, we have f(t) = u(t-1), and for 0 < t < 1, we have f(t) = t(u(t) - u(t-1)). Verify that, by combining these two formulas, we have

$$f(t) = t(u(t) - u(t-1)) + u(t-1)$$

Rewrite f(t) as follows:

$$f(t) = -(t-1)u(t-1) + tu(t) = -(t-1)u(t-1) + t$$

Recall that  $\mathcal{L}(t) = \frac{1}{s^2}$ . And by Theorem 1,  $\mathcal{L}((t-1)u(t-1)) = \frac{e^{-s}}{s^2}$ 

$$\mathcal{L}((t-1)u(t-1)) = \frac{e^{-s}}{s^2}$$

Hence

$$\mathcal{L}(f(t)) = -\frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

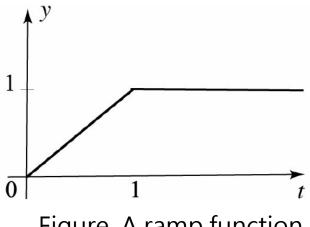


Figure A ramp function

#### **EXAMPLE 3 A nonhomogeneous differential equation**

Solve y'' + y = f(t), y(0) = 0, y'(0) = 0, where f(t) is as in Example 2. f(t) = -(t-1)u(t-1) + tu(t) = -(t-1)u(t-1) + t

#### **Solutions**

Step 1.

$$s^{2}Y + Y = -\frac{e^{-s}}{s^{2}} + \frac{1}{s^{2}}$$
$$(s^{2} + 1)Y = -\frac{e^{-s}}{s^{2}} + \frac{1}{s^{2}}$$

Step 2.

$$Y = \frac{1 - e^{-s}}{s^2(s^2 + 1)} = (1 - e^{-s}) \left( \frac{1}{s^2} - \frac{1}{s^2 + 1} \right) = \frac{1}{s^2} - \frac{1}{s^2 + 1} - e^{-s} \left( \frac{1}{s^2} - \frac{1}{s^2 + 1} \right)$$

Step 3. Using Inverse Laplace Transform and Theorem 1, we have

$$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \mathcal{L}^{-1}\left(e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)\right)$$
$$= t - \sin t + u(t-1)[(t-1) - \sin(t-1)]$$

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### 8.3 Convolutions and

# **Laplace Transforms**

Given two functions f and g, defined for all t > 0, we define their convolution (f \* g)(t) by

(2) 
$$(f * g)(t) = \int_{0}^{t} f(t - \tau)g(\tau)d\tau \quad \text{for all } t \ge 0$$

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#### **THEOREM 2 Transforms of Convolutions**

Suppose that f and g are piecewise continuous and of exponential order;

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$$

#### **Proof**

If we extend the functions f and g to be zero for t<0, then the integral in (2) is same as

$$\int_0^\infty f(t-\tau)g(\tau)\,d\tau$$

Thus, throughout this proof, we assume that f and g are extended to the whole line with f(t) = 0 and g(t) = 0, For all t < 0. We have

$$\mathcal{L}(f*g)(s) = \int_0^\infty \left[ \int_0^\infty f(t-\tau)g(\tau) \, d\tau \right] e^{-st} dt$$

$$= \int_0^\infty \int_0^\infty f(t-\tau)e^{-st} dt \, g(\tau) d\tau \quad \text{(interchange order of integration)}$$

$$= \int_0^\infty \int_0^\infty f(u)e^{-s(u+\tau)} du \, g(\tau) d\tau \quad (u=t-\tau, du=dt)$$

$$= \int_0^\infty \int_0^\infty f(u)e^{-su} du \, g(\tau)e^{-s\tau} d\tau = \mathcal{L}(f)\mathcal{L}(g)$$
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#### **EXAMPLE 5 Transforms involving convolutions**

(a) Evaluate 
$$\mathcal{L}\left(\int_0^t (t-\tau)\sin(\tau)d\tau\right)$$

#### **Solution**

(a) From Theorem 2, we have

$$\mathcal{L}\left(\int_0^t (t-\tau)\sin(\tau)d\tau\right) = \mathcal{L}(t)\mathcal{L}(\sin t) = \frac{1}{s^2}\frac{1}{s^2+1}$$

#### Additional EXAMPLE Transforms involving convolutions

Let 
$$H(s) = \frac{1}{(s-a)s}$$
. Find  $h(t)$ .

#### **Solution**

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{(s-a)s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-a} \cdot \frac{1}{s}\right) = \mathcal{L}^{-1}(F(s) \cdot G(s))$$
$$F(s) = \frac{1}{s-a} \rightarrow \mathcal{L}^{-1}(F(s)) = e^{as}$$
$$G(s) = \frac{1}{s} \rightarrow \mathcal{L}^{-1}(G(s)) = 1$$

By using Theorem 2.

$$h(t) = \mathcal{L}^{-1}(\mathcal{L}(e^{as})\mathcal{L}(1)) = e^{at} * 1 = \int_0^t e^{a\tau} \cdot 1 d\tau = \frac{1}{a}(e^{at} - 1)$$

#### **EXAMPLE 6 Solving differential equations with convolutions**

Express the solution of the initial value problem.

$$y'' - 2y' + 5y = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

as a convolution.

#### **Solutions**

Step 1.

$$s^2Y - 2sY + 5Y = F(s)$$

$$(s^2 - 2s + 5)Y = F(s)$$

Step 2.

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$$Y = \frac{F(s)}{s^2 - 2s + 5} = F(s) \frac{1}{(s - 1)^2 + 2^2} = F(s) \left[ \frac{1}{2} \frac{2}{(s - 1)^2 + 2^2} \right]$$

Step 3. Find the Inverse Laplace Transform in the table and using Theorem 2, we have

$$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(F(s)\left[\frac{1}{2}\frac{2}{(s-1)^2 + 2^2}\right]\right) = f(t) * \frac{1}{2}e^t \sin 2t$$

$$= \frac{1}{2}\int_0^t e^{t-\tau} \sin 2(t-\tau)f(\tau)d\tau$$
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# Review for Lecture 8

- Laplace Transform for Solving ODEs
- Further Properties of Laplace Transforms
- Convolution and Laplace Transforms

# Exercise

Please Check https://github.com/uoaworks/FourierAnalysisAY2018

Reading: Section 8.2, Textbook

## References

[1] Nakhlé H. Asmar, *Partial Differential Equations with Fourier Series and Boundary Value Problems 2<sup>nd</sup> Edition*, 2004

[2] Erwin Kreyszig, Advanced Engineering Mathematics 9th Edition, 2005

[3] Wikipedia