



# Lecture 2

## Fourier Series of Functions with Arbitrary Periods

# What you will learn in Lecture 2

**I. Fourier Analysis: Examples**

**II. Fourier Series of Functions with Arbitrary Periods**

# **2.1 Fourier Series: Examples**

## 2.1 Fourier Series: Examples

### Euler Formulas for the Fourier Coefficients

Suppose that  $f$  has the Fourier series representation

$$(1) \quad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Then the coefficients  $a_0$ ,  $a_n$ , and  $b_n$  are called the Fourier coefficients of  $f$  and are given by the following Euler formulas:

$$(2) \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$(3) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad (n = 1, 2, \dots)$$

$$(4) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad (n = 1, 2, \dots)$$

## 2.1 Fourier Series: Examples

### Euler Formulas for the Fourier Coefficients

#### Alternative Euler Formulas

$$(5) \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx$$

$$(6) \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx \quad (n = 1, 2, \dots)$$

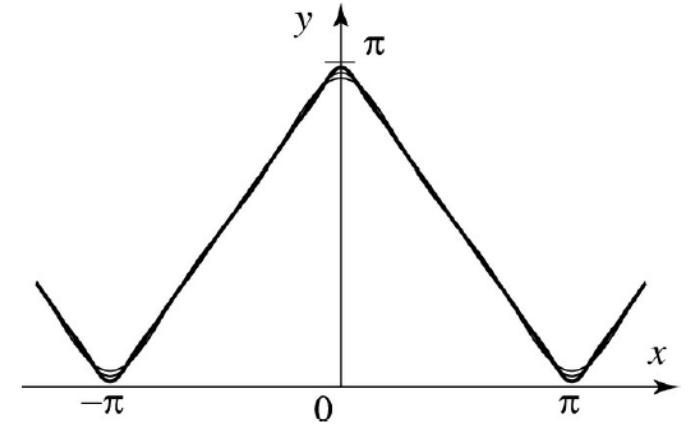
$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \quad (n = 1, 2, \dots)$$

## 2.1 Fourier Series: Examples

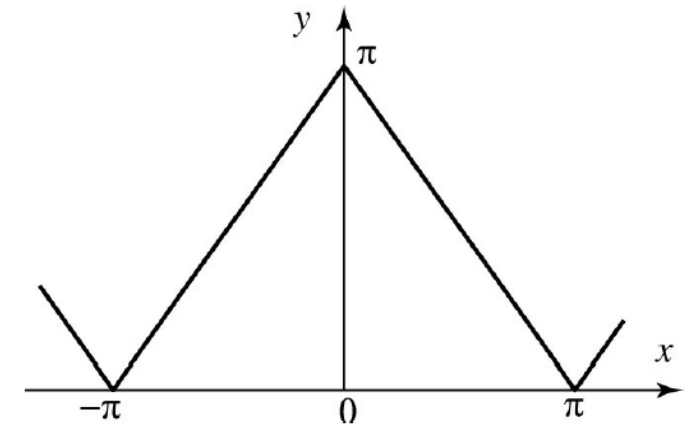
For a positive integer  $N$ , we denote the  $N$ th partial sum of the Fourier series of  $f$  by  $s_N(x)$ . Thus

$$s_N(x) = a_0 + \sum_{n=1}^N (a_n \cos nx + b_n \sin nx)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$



**Figure 7** Partial sums of the Fourier series



**Figure 8** The Fourier series

## 2.1 Fourier Series: Examples

### EXAMPLE 1 Fourier series of the sawtooth function

The sawtooth function, shown in Figure 1, is determined by the formulas

$$f(x) = \begin{cases} \frac{1}{2}(\pi - x) & \text{if } 0 < x \leq 2\pi \\ f(x + 2\pi) & \text{otherwise} \end{cases}$$

- (a) Find its Fourier series.
- (b) With the help of a computer, plot the partial sums  $s_1(x)$ ,  $s_7(x)$ , and  $s_{20}(x)$ , and determine the graph of the Fourier series.

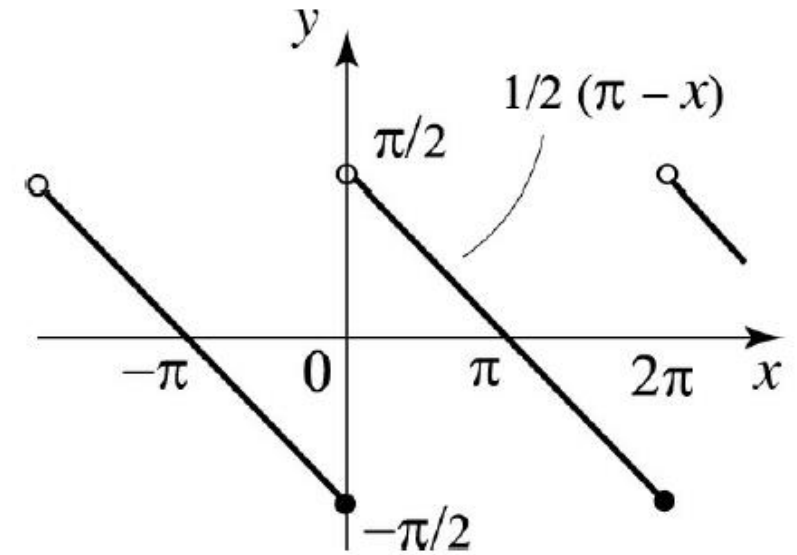


Figure 1 Sawtooth function.

## 2.1 Fourier Series: Examples

### **Solution**

Compute  $a_0, a_n, b_n$

Find complete solution in page 28-30 of the textbook.



# 2.1 Fourier Series: Examples

## Integration by parts

### Product of two functions [\[ edit \]](#)

The theorem can be derived as follows. Suppose  $u(x)$  and  $v(x)$  are two [continuously differentiable functions](#). The [product rule](#) states (in [Leibniz's notation](#)):

$$\frac{d}{dx} (u(x)v(x)) = v(x) \frac{d}{dx} (u(x)) + u(x) \frac{d}{dx} (v(x)).$$

Integrating both sides with respect to  $x$ ,

$$\int \frac{d}{dx} (u(x)v(x)) dx = \int u'(x)v(x) dx + \int u(x)v'(x) dx$$

then applying the definition of [indefinite integral](#),

$$u(x)v(x) = \int u'(x)v(x) dx + \int u(x)v'(x) dx$$

$$\int u(x)v'(x) dx = u(x)v(x) - \int u'(x)v(x) dx$$

gives the formula for **integration by parts**.

Since  $du$  and  $dv$  are [differentials of a function](#) of one variable  $x$ ,

$$du = u'(x)dx \quad dv = v'(x)dx$$

$$\int u(x) dv = u(x)v(x) - \int v(x) du$$

The original integral  $\int uv' dx$  contains  $v'$  ([derivative](#) of  $v$ ); in order to apply the theorem,  $v$  ([antiderivative](#) of  $v'$ ) must be found, and then the resulting integral  $\int vu' dx$  must be evaluated.

### Polynomials and trigonometric functions [\[ edit \]](#)

In order to calculate

$$I = \int x \cos(x) dx ,$$

let:

$$u = x \Rightarrow du = dx$$

$$dv = \cos(x) dx \Rightarrow v = \int \cos(x) dx = \sin(x)$$

then:

$$\begin{aligned} \int x \cos(x) dx &= \int u dv \\ &= u \cdot v - \int v du \\ &= x \sin(x) - \int \sin(x) dx \\ &= x \sin(x) + \cos(x) + C, \end{aligned}$$

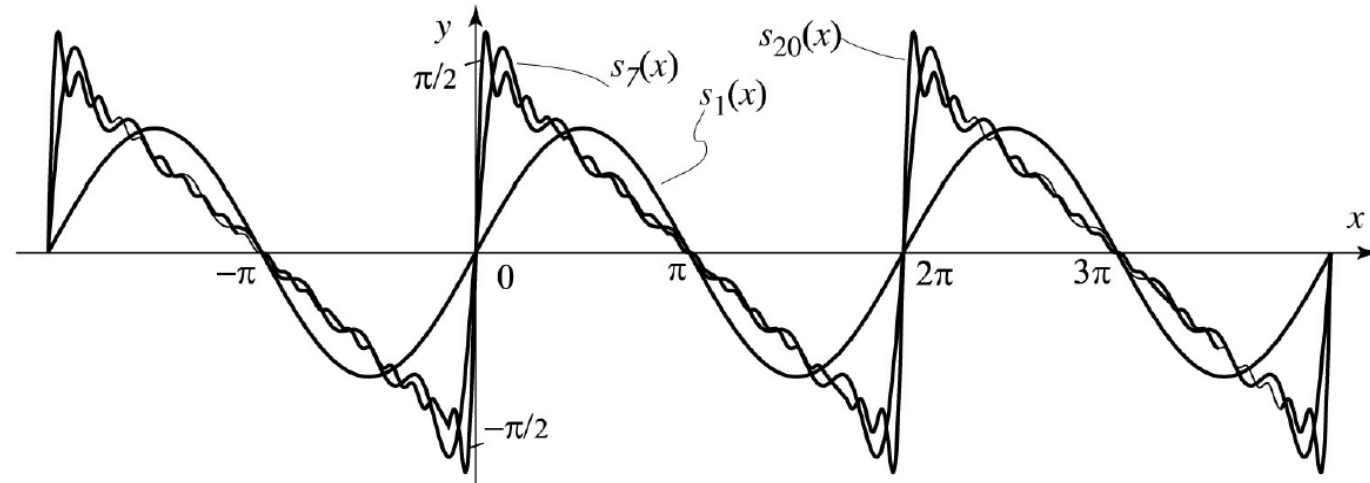
where  $C$  is a [constant of integration](#).

For higher powers of  $x$  in the form

$$\int x^n e^x dx, \quad \int x^n \sin(x) dx, \quad \int x^n \cos(x) dx ,$$

repeatedly using integration by parts can evaluate integrals such as these; each application of the theorem lowers the power of  $x$  by one.

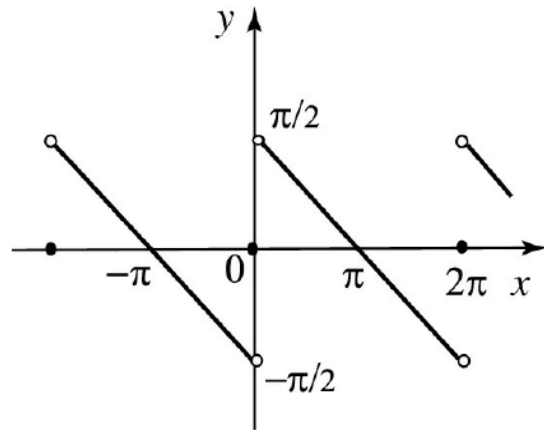
## 2.1 Fourier Series: Examples



**Figure 2** Here the  $n$ th partial sum of the Fourier series is  $s_n(x) = \sum_{k=1}^n \frac{\sin kx}{k}$ . To distinguish the graphs, note that as  $n$  increases, the frequencies of the sine terms increase. This causes the graphs of the higher partial sums to be more wiggly.

**Note 1:** At the points of discontinuity ( $x = 2k\pi$ ) in Example 1, the Fourier series converges to 0 which is the average value of the function from the left and the right at these points.

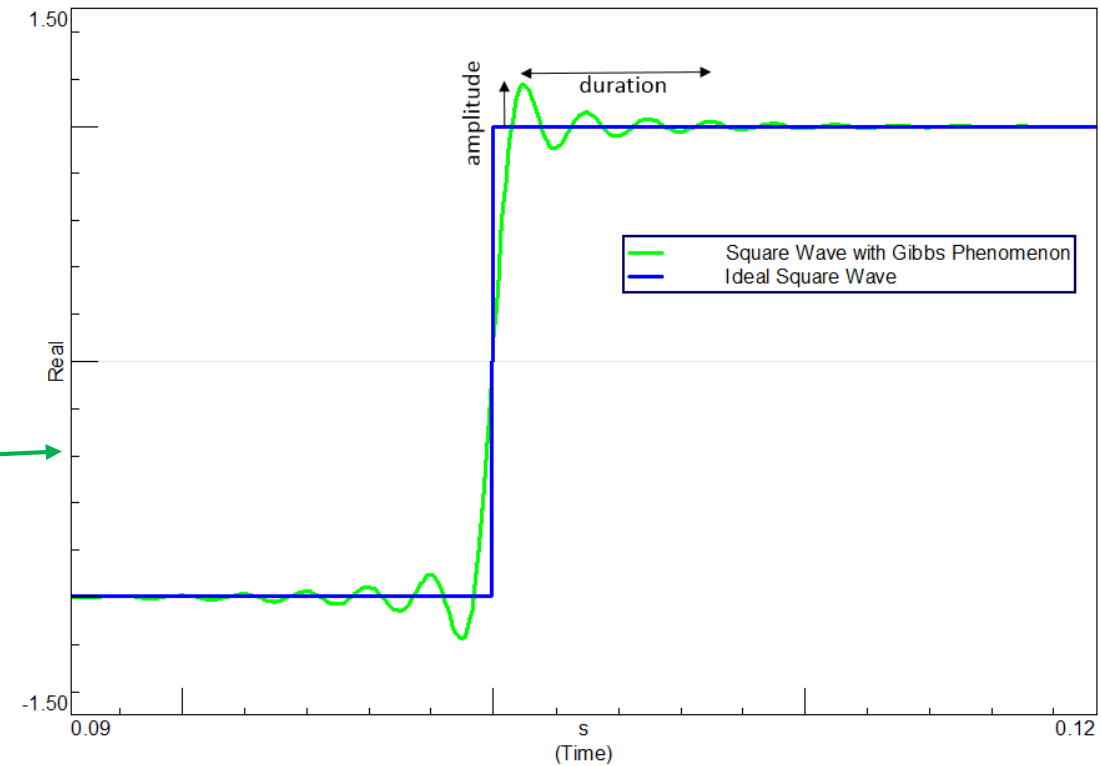
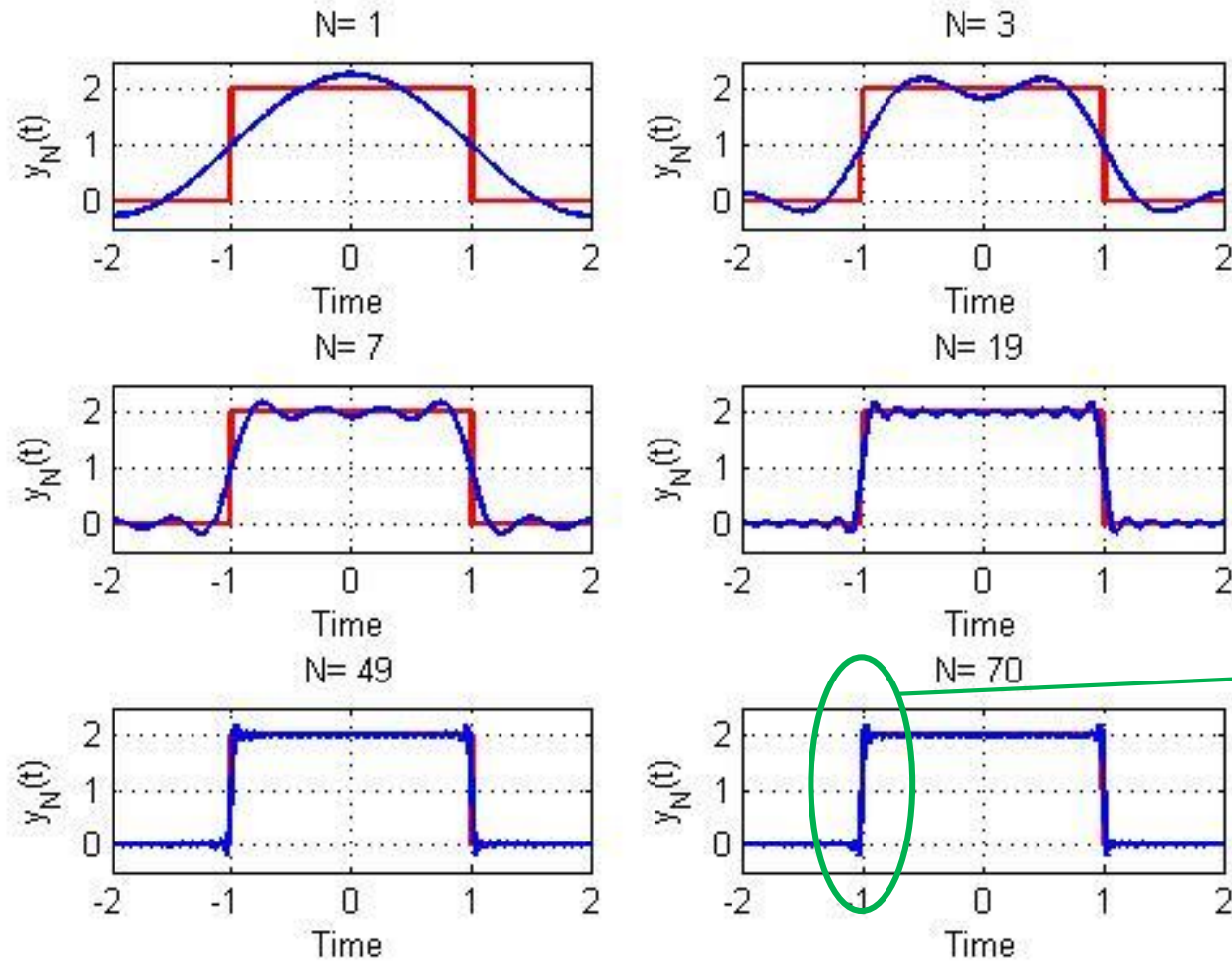
**Note 2:** Near the points of discontinuity, the Fourier series overshoots its limiting values, known as the Gibbs phenomenon.



**Figure 3** The graph of the Fourier series coincides with the graph of the function, except at the points of the discontinuity.  $\sum_{k=1}^{\infty} \frac{\sin kx}{k}$

# 2.1 Fourier Series: Examples

## Gibbs phenomenon



## 2.1 Fourier Series: Examples

### Arithmetic Average

Recall that  $f$  is piecewise smooth if  $f$  and  $f'$  are piecewise continuous.

If  $f$  is piecewise continuous, the average (or arithmetic average) of  $f$  at  $c$  is

$$\frac{f(c-) + f(c+)}{2}$$

where  $f(c-) = \lim_{x \rightarrow c-} f(x)$  and  $f(c+) = \lim_{x \rightarrow c+} f(x)$ .

If  $f$  is continuous at  $c$ , then  $f(c+) = f(c-) = f(c)$  and so the average of  $f$  at  $c$  is  $f(c)$ .

Average will be of interest only at points of discontinuity.

**Example** The function in Figure 4 has a discontinuity at  $x = 1$ . Its average there is  $\frac{1 + \frac{1}{2}}{2} = \frac{3}{4}$ .

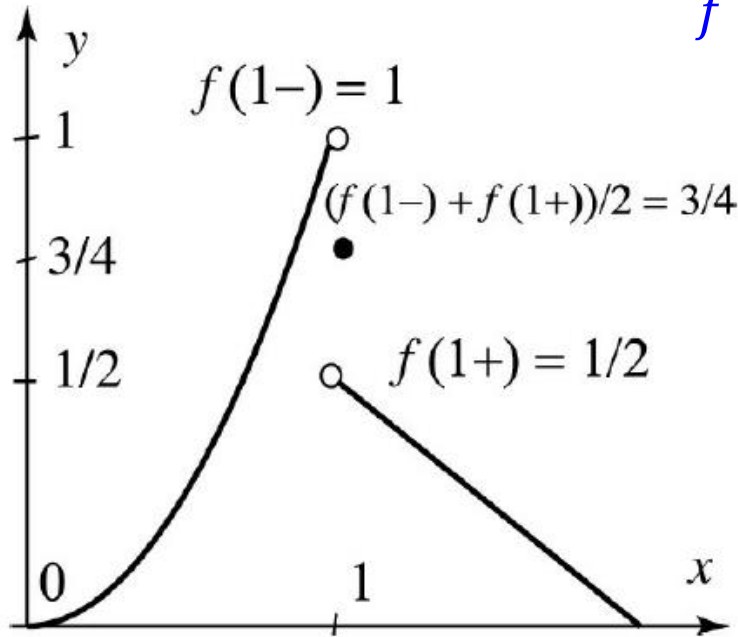


Figure 4 Average of  $f(x)$   $x = 1$ .

## 2.1 Fourier Series: Examples

### THEOREM 1 Fourier Series Representation

Suppose that  $f$  is a  $2\pi$ -periodic piecewise smooth function. Then for all  $x$  we have

$$\frac{f(x+) + f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where the Fourier coefficients  $a_0, a_n, b_n$  are given by (2)–(4). In particular, if  $f$  is piecewise smooth and continuous at  $x$ , then

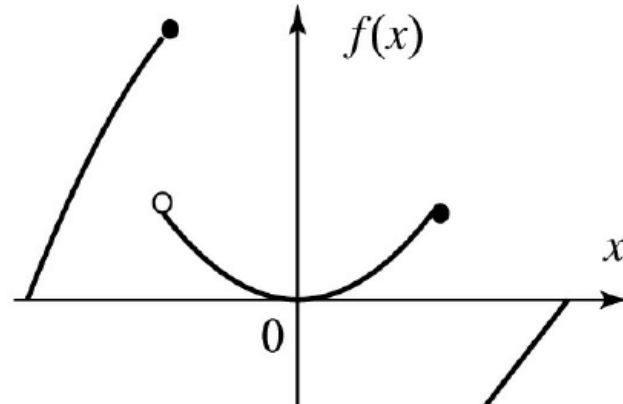
$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Find the complete proof in Section 2.8 of the textbook

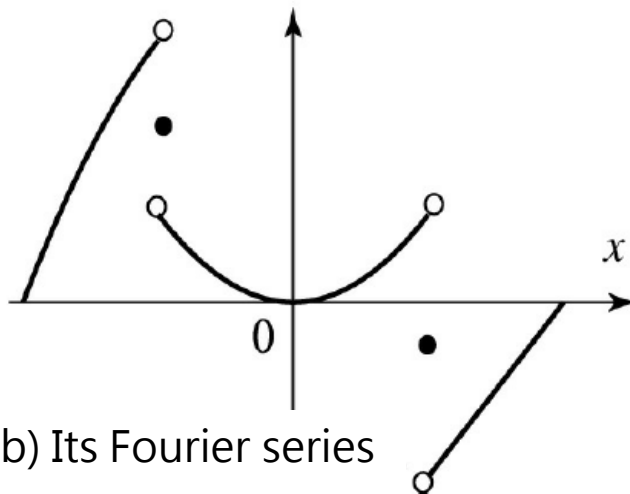
1. At a point of continuity of a piecewise smooth function the Fourier series converges to the value of the function.
2. At a point of discontinuity, the Fourier series does its best to converge, and having no reason to favor one side over the other, it converges to the average of the left and right limits (see Figure 5).

## 2.1 Fourier Series: Examples

### THEOREM 1 Fourier Series Representation



(a) A piecewise smooth function



(b) Its Fourier series

Figure 5

## 2.1 Fourier Series: Examples

### **THEOREM 1** Fourier Series Representation

For this reason, we may define (or redefine)  $f$  at isolated points without affecting its Fourier series.

If we **redefine the function at points of discontinuity to be  $\frac{f(c-) + f(c+)}{2}$**  we then have the equality

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

holding at all  $x$ .

## 2.1 Fourier Series: Examples

### EXAMPLE 2 Triangular wave

The  $2\pi$ -periodic triangular wave is given on the interval  $[-\pi, \pi]$  by

$$g(x) = \begin{cases} \pi + x & \text{if } -\pi \leq x \leq 0 \\ \pi - x & \text{if } 0 \leq x \leq \pi \end{cases}$$

(a) Find its Fourier series.

(b) Plot some partial sums and the Fourier series.

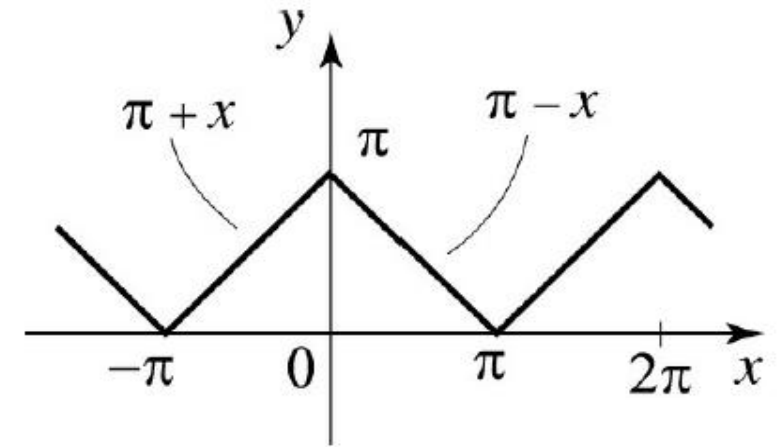


Figure 6 Triangular wave



## 2.1 Fourier Series: Examples

### Even and Odd Functions

A function  $f$  is **even** if  $f(-x) = f(x)$  for all  $x$ .

A function  $f$  is **odd** if  $f(-x) = -f(x)$  for all  $x$ .

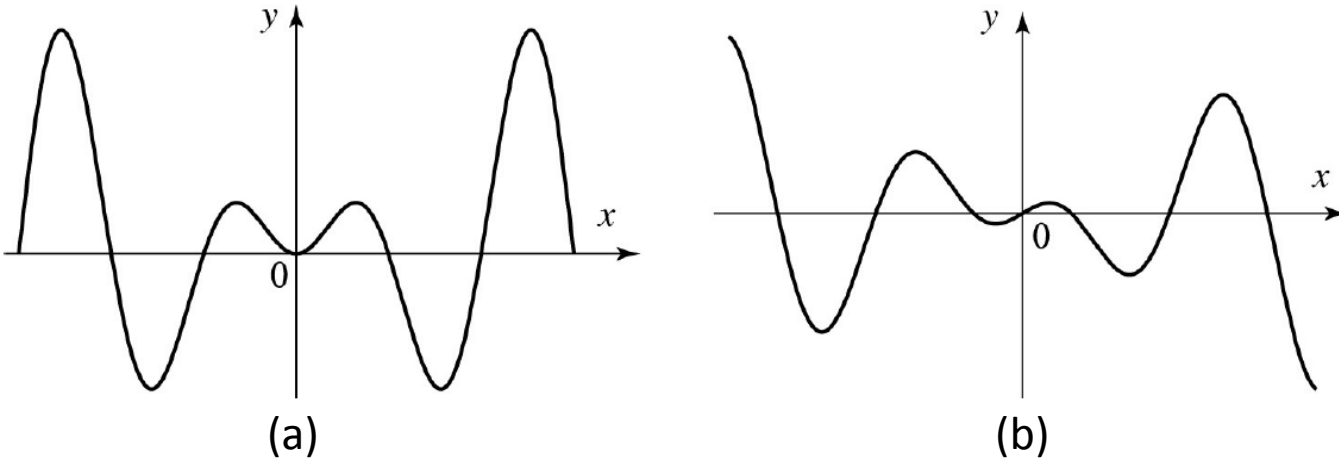


Figure 6 in Section 2.3 of the textbook

(a) Even function: The graph is symmetric with respect to the  $y$ -axis.

(b) Odd function: The graph is symmetric with respect to the origin.

If  $f$  is even (Figure 6(a)), then

$$\int_{-p}^p f(x) dx = 2 \int_0^p f(x) dx$$

and if  $f$  is odd (Figure 6(b)), then

$$\int_{-p}^p f(x) dx = 0$$

## 2.1 Fourier Series: Examples

### Even and Odd Functions

The following **useful properties concerning the products of these functions** are easily verified.

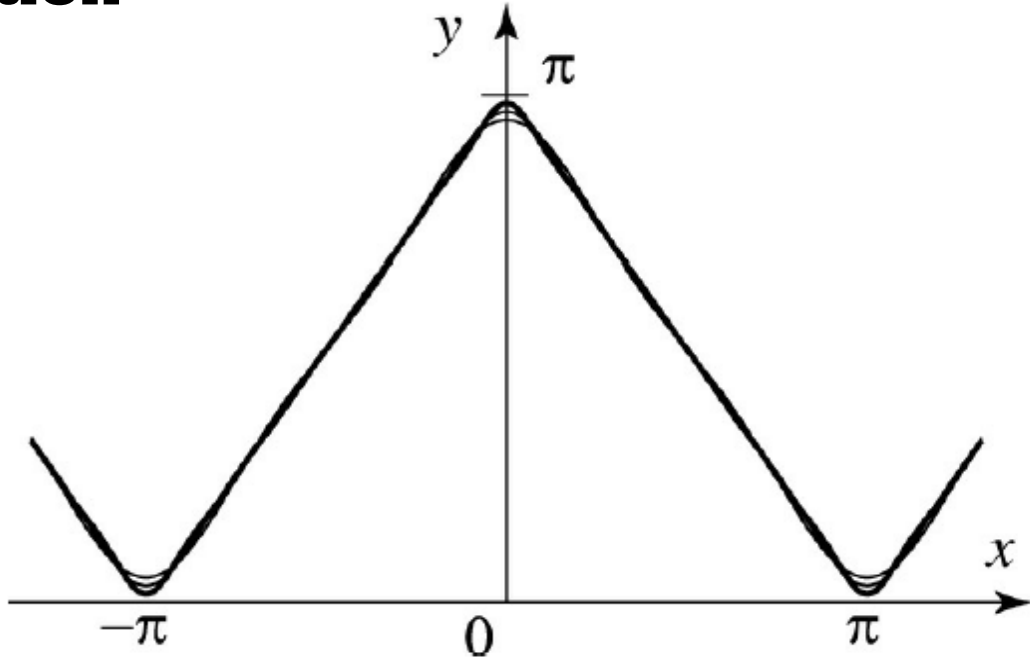
$$(\text{Even}) \cdot (\text{Even}) = \text{Even}$$

$$(\text{Even}) \cdot (\text{Odd}) = \text{Odd}$$

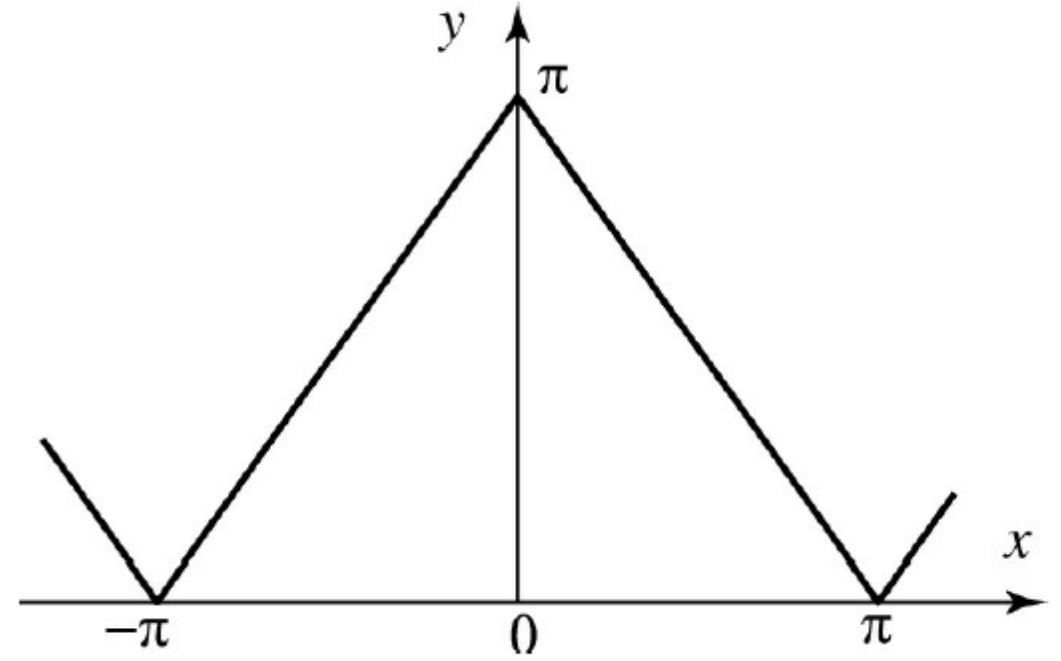
$$(\text{Odd}) \cdot (\text{Odd}) = \text{Even}$$

## 2.1 Fourier Series: Examples

### Solution



**Figure 7** Partial sums of the Fourier series



**Figure 8** The Fourier series

**Notice:** Figure 7 is converging very fast, much faster than those in Example 1.

This is due to the magnitudes of the Fourier coefficients.

In Example 1 the coefficients are of the order  $1/n$ ,

while in Example 2 the coefficients are of the order  $1/n^2$ .

## 2.1 Fourier Series: Examples

### Operations on Fourier Series

The Fourier series in [Examples 1](#) is an odd function and contains only sine terms, while the [Examples 2](#) is an even function and contains only cosine terms.

We will derive the Fourier series containing both sine and cosine terms **without computing Fourier coefficients but by applying operations such as multiplying a Fourier series by a constant, adding two Fourier series, changing variables ( $x$  to  $-x$ ), and translating.**

## 2.1 Fourier Series: Examples

### Operations on Fourier Series

#### **EXAMPLE 4** Linear combinations of Fourier series

The  $2\pi$ -periodic function

$$h(x) = \begin{cases} \pi - x & \text{if } 0 < x \leq \pi \\ 0 & \text{if } \pi < x \leq 2\pi \end{cases}$$

is related to the functions in [Examples 1](#) and [2](#) by

$$h(x) = f(x) + \frac{1}{2}g(x)$$

## 2.1 Fourier Series: Examples

**Solution** Find the solution in page 33 of the textbook.

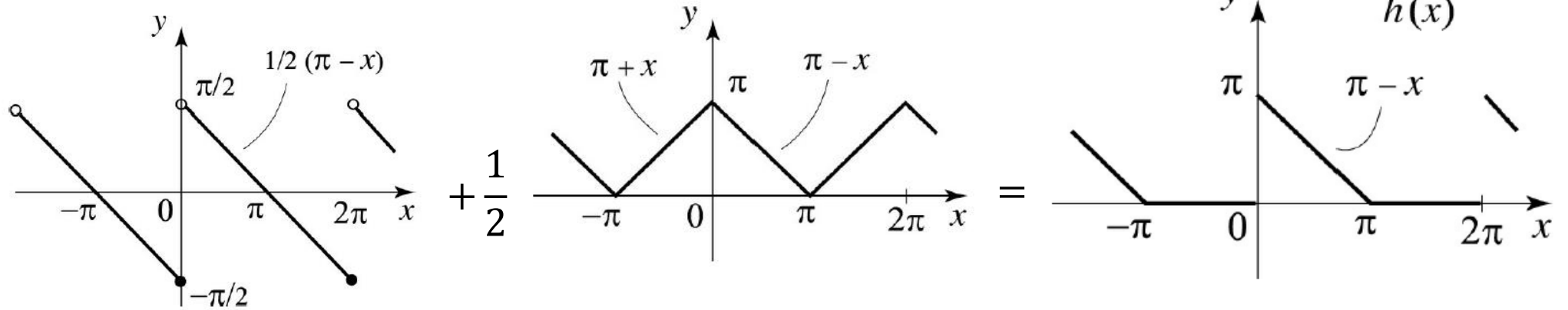


Figure 9 The function of Example 4

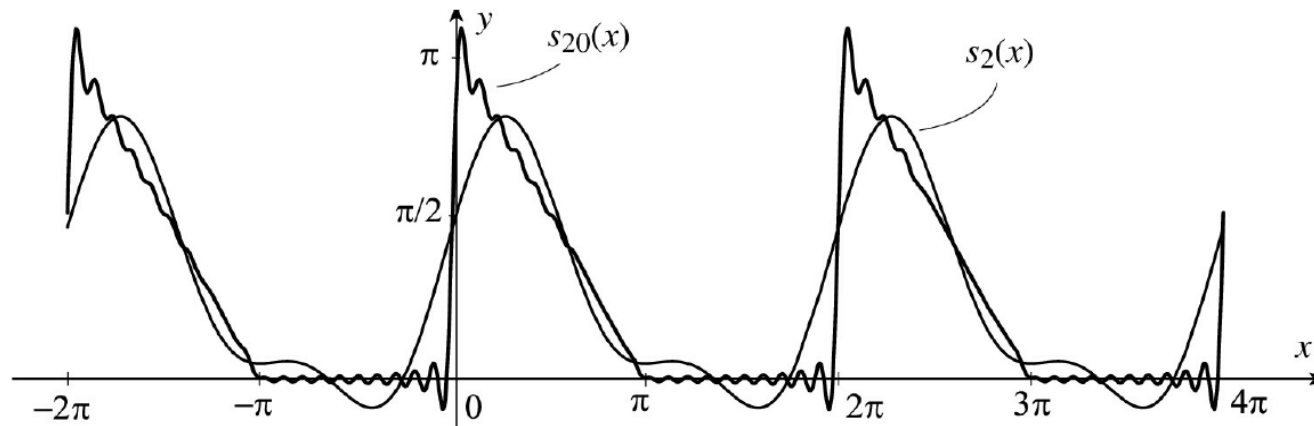


Figure 10 Note the Gibbs phenomenon **at the points of discontinuity** ( $x = 2k\pi$ ).

This is due to the fact that the Fourier series consists of a cosine part that is converging very fast (Figure 7) and a sine part that overshoots at the points of discontinuity.

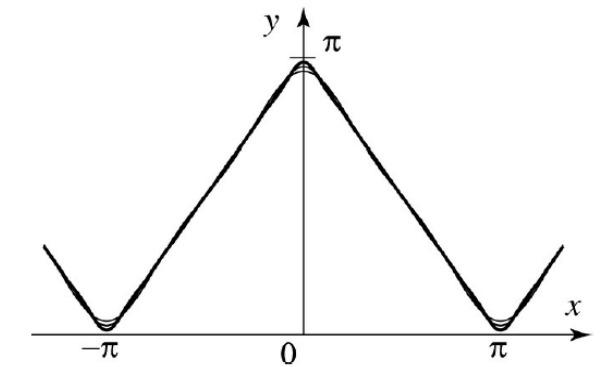


Figure 7 Partial sums of the Fourier series

## 2.1 Fourier Series: Examples

### EXAMPLE 5 Changing variables and translating

The graph of the function  $k(x)$  in Figure 11 is obtained by reflecting through the  $y$ -axis the graph in Figure 9 and then translating by  $\pi$  units to the left or right. Thus  $k(x) = h(-x - \pi)$  and the Fourier series representation of  $k(x)$  is

$$k(x) = h(-x - \pi) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{1}{\pi} \left( \frac{1}{n^2} - \frac{(-1)^n}{n^2} \right) \cos n(-x - \pi) + \frac{\sin n(-x - \pi)}{n} \right\}$$

But  $\cos n(-x - \pi) = (-1)^n \cos nx$  and  $\sin n(-x - \pi) = (-1)^{n+1} \sin nx$ . So

$$k(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{1}{\pi} \left( \frac{(-1)^n}{n^2} - \frac{1}{n^2} \right) \cos nx + (-1)^{n+1} \frac{\sin nx}{n} \right\}$$

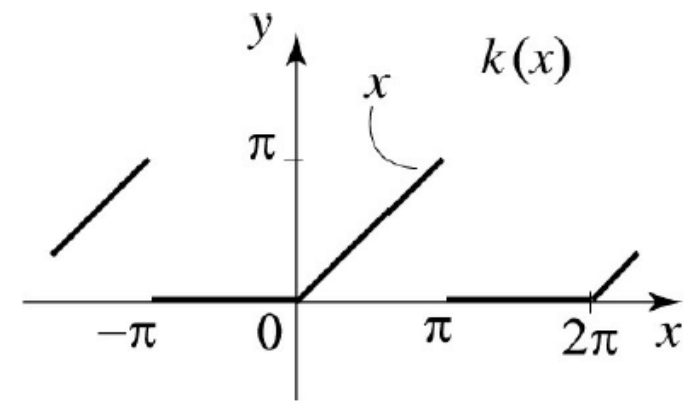


Figure 11 The function in Example 5(a).

## **2.2 Fourier Series of Functions with Arbitrary Periods**



## 2.2 Fourier Series of Functions with Arbitrary Periods

In the preceding section we worked with functions of period  $2\pi$ .

The choice of this period was merely for convenience.

We show how to extend our results to functions with arbitrary period (Figure 1) by using a simple change of variables.

Suppose that  $f$  is a function with period  $T = 2p > 0$ , and let

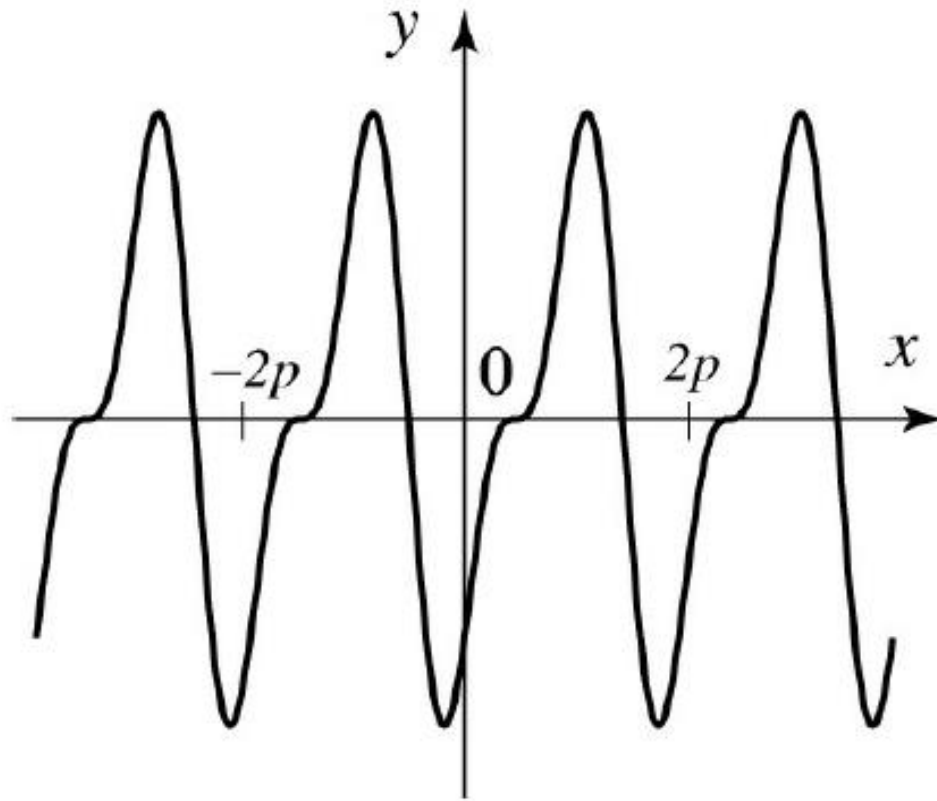
$$(1) \qquad g(x) = f\left(\frac{p}{\pi}x\right)$$

Since  $f$  is  $2p$ -periodic, we have

$$g(x + 2\pi) = f\left(\frac{p}{\pi}(x + 2\pi)\right) = f\left(\frac{p}{\pi}x + 2p\right) = f\left(\frac{p}{\pi}x\right) = g(x)$$

Hence  $g$  is  $2\pi$ -periodic.

## 2.2 Fourier Series of Functions with Arbitrary Periods



**Figure 1** A  $2p$ -periodic function

## 2.2 Fourier Series of Functions with Arbitrary Periods

### **THEOREM 1** Fourier Series Representation: Arbitrary Period

Suppose that  $f$  is a  $2p$ -periodic piecewise smooth function. The Fourier series of  $f$  is given by

$$(2) \quad a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

where

$$(3) \quad a_0 = \frac{1}{2p} \int_{-p}^p f(x) dx$$

$$(4) \quad a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx \quad (n = 1, 2, \dots)$$

$$(5) \quad b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi}{p} x dx \quad (n = 1, 2, \dots)$$

The Fourier series converges to  $f(x)$  if  $f$  is continuous at  $x$  and to  $\frac{f(x-) + f(x+)}{2}$  otherwise.

By [Theorem 1, Section 2.1](#), all the integrals can be replaced  $\int_{-p}^p$  by  $\int_0^{2p}$  without changing the values of the coefficients.

## 2.2 Fourier Series of Functions with Arbitrary Periods

### **Proof**

Find complete proof in page 39 of the textbook.

## 2.2 Fourier Series of Functions with Arbitrary Periods

### **EXAMPLE 1** A Fourier series with arbitrary period

Find the Fourier series of the  $2p$ -periodic function given by  $f(x) = |x|$  if  $-p \leq x \leq p$  (Figure 2).

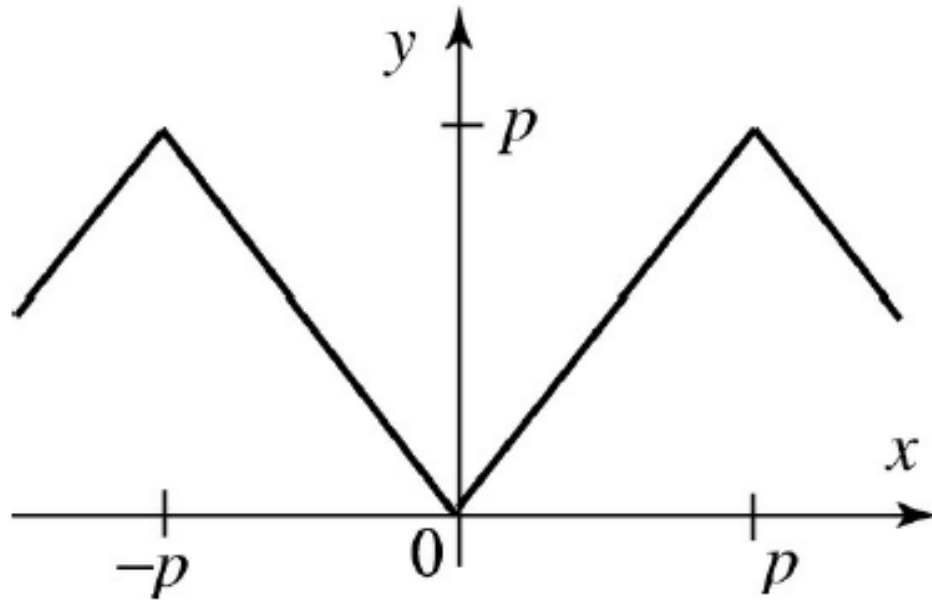


Figure 2 Triangular wave with period  $2p$ .

## 2.2 Fourier Series of Functions with Arbitrary Periods

### Solution

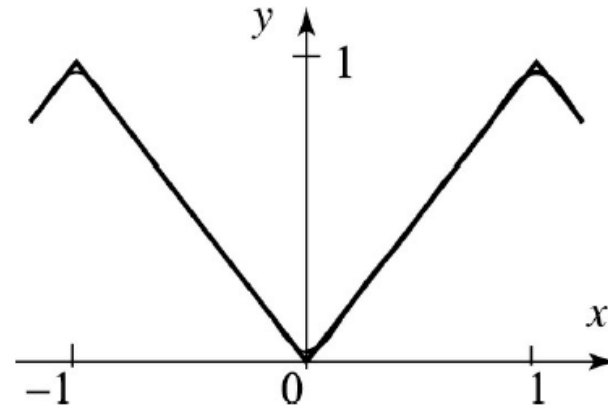


Figure 3 Partial sums of the Fourier series ( $p = 1$ ), in Example 1.

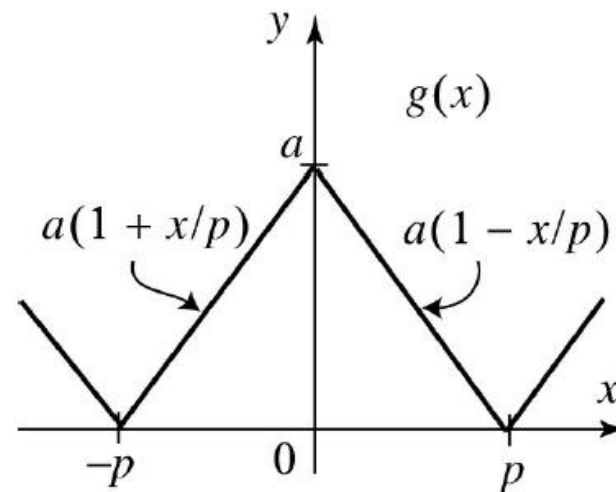


Figure 4 A  $2p$ -periodic triangular wave.

## 2.2 Fourier Series of Functions with Arbitrary Periods

### Even and Odd Functions

A function  $f$  is **even** if  $f(-x) = f(x)$  for all  $x$ .

A function  $f$  is **odd** if  $f(-x) = -f(x)$  for all  $x$ .

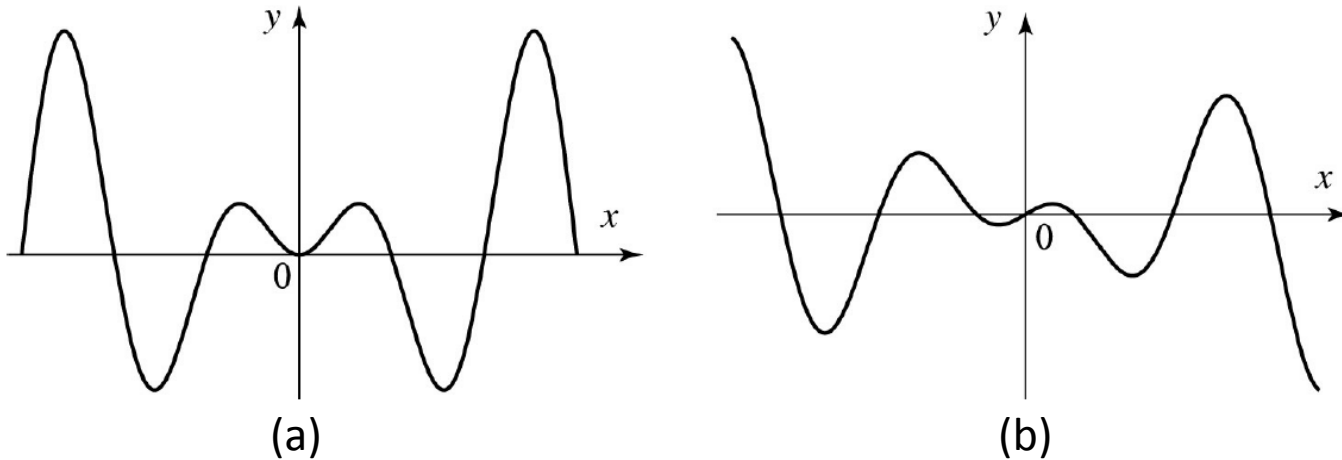


Figure 6

(a) Even function: The graph is symmetric with respect to the  $y$ -axis.

(b) Odd function: The graph is symmetric with respect to the origin.

If  $f$  is even (Figure 6(a)), then

$$\int_{-p}^p f(x) dx = 2 \int_0^p f(x) dx$$

and if  $f$  is odd (Figure 6(b)), then

$$\int_{-p}^p f(x) dx = 0$$

## 2.2 Fourier Series of Functions with Arbitrary Periods

### Even and Odd Functions

The following **useful properties concerning the products of these functions** are easily verified.

$$(\text{Even}) \cdot (\text{Even}) = \text{Even}$$

$$(\text{Even}) \cdot (\text{Odd}) = \text{Odd}$$

$$(\text{Odd}) \cdot (\text{Odd}) = \text{Even}$$



## 2.2 Fourier Series of Functions with Arbitrary Periods

### Even and Odd Functions

#### THEOREM 2 Fourier Series of Even and Odd Functions

Suppose that  $f$  is  $2p$ -periodic and has the Fourier series representation (2). Then

(i)  $f$  is even if and only if  $b_n = 0$  for all  $n$ . In this case

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x$$

where

$$a_0 = \frac{1}{p} \int_0^p f(x) dx \quad \text{and} \quad a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x dx \quad (n = 1, 2, \dots)$$

(ii)  $f$  is odd if and only if  $a_n = 0$  for all  $n$ . In this case

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

where

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx \quad (n = 1, 2, \dots)$$

## 2.2 Fourier Series of Functions with Arbitrary Periods

### Even and Odd Functions

#### **THEOREM 2** Fourier Series of Even and Odd Functions

#### **Proof**

Find complete proof in page 43 of the textbook.

## 2.2 Fourier Series of Functions with Arbitrary Periods

### EXAMPLE 4 Fourier series of an even function

Find the Fourier series of the 2-periodic function  $f(x) = 1 - x^2$  if  $-1 < x < 1$ .

### Solution

Compute  $a_0, a_n, b_n$  by Theorem 2.

Find complete solution in page 43, 44 of the textbook.

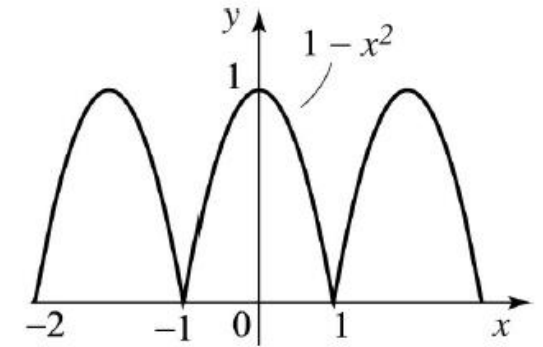


Figure 7 An even function.

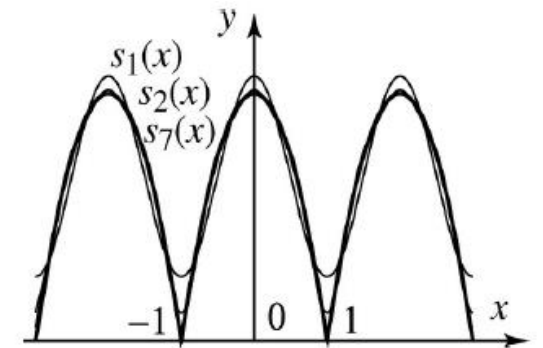


Figure 8 Partial sums of the Fourier series in [Example 4](#).

# Review for Lecture 2

- Fourier Series
- Fourier Series of Functions with Arbitrary Periods

# Exercise

Please Check <https://github.com/uoaworks/FourierAnalysisAY2018>

Reading: Section 2.2, 2.3, Textbook

# References

- [1] Nakhlé H. Asmar, *Partial Differential Equations with Fourier Series and Boundary Value Problems 2<sup>nd</sup> Edition*, 2004
- [2] Wikipedia