



# Lecture 4

- **Complex Form of Fourier Series**
- **Uniform Convergence and Fourier Series**

# What you will learn in Lecture 4

**I. Complex Form of Fourier Series**

**II. Uniform Convergence and Fourier Series**

# **4.1 Complex Form of Fourier Series**

## 4.1 Complex Form of Fourier Series

Recall that in *Calculus II*, we have Taylor Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad R = \infty$$

## 4.1 Complex Form of Fourier Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \dots$$

$$= 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \dots$$

because  $i^2 = -1$

$$= 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \dots$$

$$= \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) + i \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)$$

$$= \cos x + i \sin x$$

# 4.1 Complex Form of Fourier Series

$$\textcircled{1} \quad e^{ix} = \cos x + i \sin x \quad \text{Euler's formula}$$

$$\begin{aligned} \textcircled{2} \quad e^{-ix} &= \cos(-x) + i \sin(-x) \\ &= \cos x - i \sin x \end{aligned}$$

$$\textcircled{1} + \textcircled{2} \quad e^{ix} + e^{-ix} = 2 \cos x \quad \rightarrow \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\textcircled{1} - \textcircled{2} \quad e^{ix} - e^{-ix} = i2 \sin x \quad \rightarrow \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$



Leonhard Euler (1707-1783)

## 4.1 Complex Form of Fourier Series

We know the trigonometric identity

$$\cos x \cdot \cos y = \frac{1}{2} [\cos(x + y) + \cos(x - y)]$$

Because  $\cos x = \frac{e^{ix} + e^{-ix}}{2}$   $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$  then we have

$$\begin{aligned}\cos x \cdot \cos y &= \frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^{iy} + e^{-iy}}{2} \\&= \frac{1}{2} \cdot \frac{e^{i(x+y)} + e^{i(x-y)} + e^{i(-x+y)} + e^{i(-x-y)}}{2} \\&= \frac{1}{2} \left( \underbrace{\frac{e^{i(x+y)} + e^{-i(x+y)}}{2}}_{\cos(x+y)} + \underbrace{\frac{e^{i(x-y)} + e^{-i(x-y)}}{2}}_{\cos(x-y)} \right).\end{aligned}$$

## 4.1 Complex Form of Fourier Series

For complex form of Fourier series , let us start with the two identities

$$(1) \quad \cos u = \frac{e^{iu} + e^{-iu}}{2} \quad \text{and} \quad \sin u = \frac{e^{iu} - e^{-iu}}{2i}$$

We will use these identities to find a complex form for the Fourier series expansion of a  $2p$ -periodic function

$$(2) \quad f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$



## 4.1 Complex Form of Fourier Series

### THEOREM 1 Complex Form of Fourier Analysis

Let  $f$  be a  $2p$ -periodic piecewise smooth function. The **complex form of the Fourier series of  $f$**  is

$$(3) \quad \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi}{p} x}$$

where **the Fourier coefficients  $c_n$**  are given by

$$(4) \quad c_n = \frac{1}{2p} \int_{-p}^p f(t) e^{-i \frac{n\pi}{p} t} dt \quad (n = 0, \pm 1, \pm 2, \dots)$$

For all  $x$ , the Fourier series converges to  $f(x)$  **if  $f$  is continuous at  $x$** , and to  $\frac{f(x+) + f(x-)}{2}$  otherwise.

# 4.1 Complex Form of Fourier Series

## **Proof**

Find the complete solution in page 62, Section 2.6 of the textbook.

## 4.1 Complex Form of Fourier Series

The  $N$ th partial sum of (3) is by definition the symmetric sum

$$s_N(x) = \sum_{n=-N}^N c_n e^{i \frac{n\pi}{p} x}$$

We will see in a moment that is the same as the usual partial sum of the Fourier Series

$$f(x) = a_0 + \sum_{n=1}^N \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

## 4.1 Complex Form of Fourier Series

If  $h(t)$  is **complex-valued**, write  $h(t) = u(t) + iv(t)$ , where  $u$  and  $v$  are the real and imaginary parts of  $h$ . We define

$$\int h(t)dt = \int (u(t) + iv(t))dt \equiv \int u(t)dt + i \int v(t)dt$$

For example, if  $h(t) = f(t)e^{ict}$ , where  $f(t)$  is **real-valued** and  $c$  is a real number.

Then  $h(t) = f(t)e^{ict} = f(t)(\cos ct + i \sin ct)$ . So

$$\int f(t)e^{ict}dt = \int f(t) \cos ct dt + i \int f(t) \sin ct dt$$

where now **both integrals on the right are integrals or antiderivatives of real-valued functions.**

## 4.1 Complex Form of Fourier Series

It is straightforward to show that the integral is linear:

If  $h$  and  $g$  are complex-valued and  $\alpha$  and  $\beta$  are complex numbers, then

$$\int (\alpha h(t) + \beta g(t)) dt = \alpha \int h(t) dt + \beta \int g(t) dt$$

## 4.1 Complex Form of Fourier Series

We now highlight some interesting identities that relate the **complex Fourier coefficients** to the Fourier cosine and sine coefficients.

$$(5) \quad c_0 = a_0$$

$$(6) \quad c_n = \frac{1}{2}(a_n - ib_n) \quad c_{-n} = \frac{1}{2}(a_n + ib_n) \quad (n > 0)$$

$$(7) \quad a_n = c_n + c_{-n} \quad b_n = i(c_n - c_{-n}) \quad (n > 0)$$

$$(8) \quad S_N(x) = s_N(x)$$

(6) shows that  $c_{-n}$  is the complex conjugate of  $c_n$ . In symbols,

$$(9) \quad c_{-n} = \bar{c}_n$$

## 4.1 Complex Form of Fourier Series

### **EXAMPLE 1** A complex Fourier series

Find the complex form of the Fourier series of the  $2\pi$ -periodic function  $f(x) = e^{ax}$  for  $-\pi < x < \pi$ , where  $a \neq 0, \pm i, \pm 2i, \pm 3i, \dots$ . Determine the values of the Fourier series at  $x = \pm\pi$ .

### **Solution**

Find the complete solution in page 63 and 64, Section 2.6 of the textbook.

## 4.1 Complex Form of Fourier Series

### EXAMPLE 1 A complex Fourier series

- Hyperbolic sine:

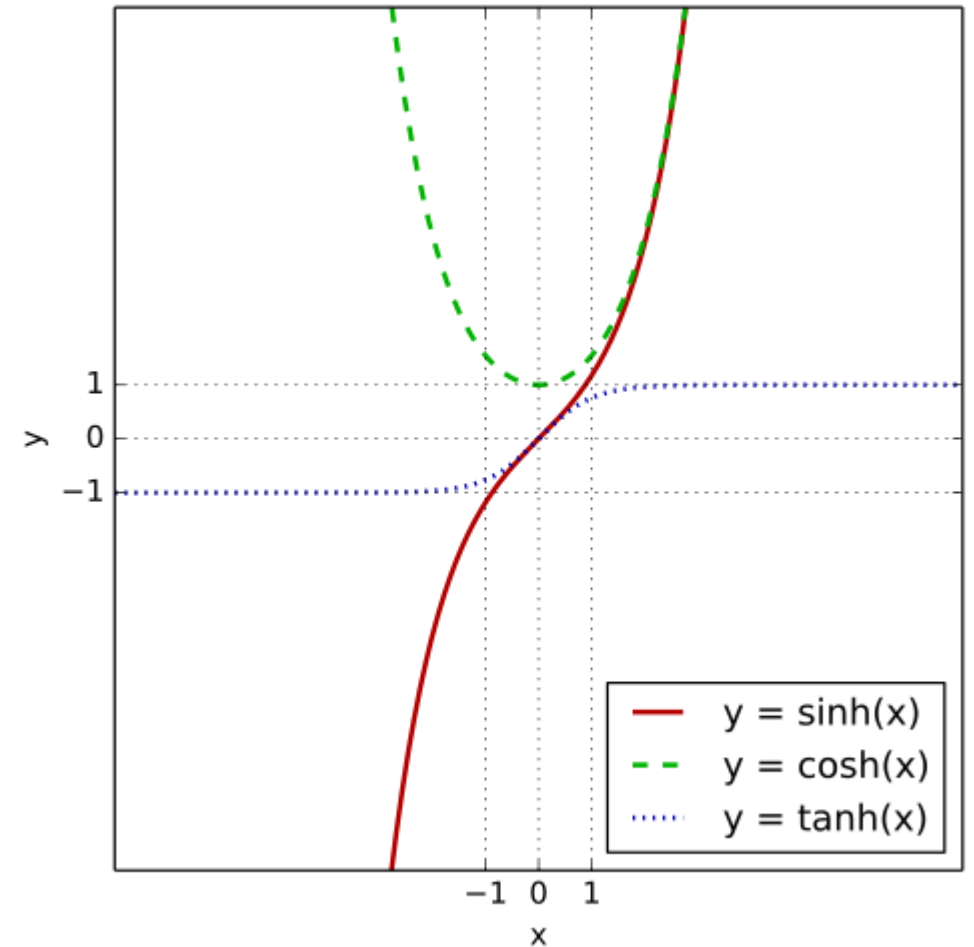
$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} = \frac{1 - e^{-2x}}{2e^{-x}}.$$

- Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x} = \frac{1 + e^{-2x}}{2e^{-x}}.$$

- Hyperbolic tangent:

$$\begin{aligned}\tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \\ &= \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}.\end{aligned}$$



[https://en.wikipedia.org/wiki/Hyperbolic\\_function](https://en.wikipedia.org/wiki/Hyperbolic_function)



## 4.1 Complex Form of Fourier Series

Similarly, the complex form of the Fourier coefficients can be obtained by appealing to **the orthogonality of the complex exponential system**

$$1, e^{i\frac{\pi}{p}x}, e^{-i\frac{\pi}{p}x}, e^{i\frac{2\pi}{p}x}, e^{-i\frac{2\pi}{p}x}, \dots, e^{i\frac{n\pi}{p}x}, e^{-i\frac{n\pi}{p}x}, \dots$$

**The orthogonality of this system** is expressed by

$$(11) \quad \frac{1}{2p} \int_{-p}^p e^{i\frac{m\pi}{p}x} e^{-i\frac{n\pi}{p}x} dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

## 4.1 Complex Form of Fourier Series

### **THEOREM 2** Complex Form of Parseval's Identity

Let  $f$  be a real-valued square integrable function on  $[-p, p]$  with Fourier coefficients  $c_n$  given by (4). Then

$$\frac{1}{2p} \int_{-p}^p f(x)^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

### **Proof**

Find the complete solution in page 65, Section 2.6 of the textbook.

# **4.2 Uniform Convergence and Fourier Series**

## 4.2 Uniform Convergence and Fourier Series

Recall that in Lecture 2

### EXAMPLE 1 Fourier series of the sawtooth function

The sawtooth function, shown in Figure 1, is determined by the formulas

$$f(x) = \begin{cases} \frac{1}{2}(\pi - x) & \text{if } 0 < x \leq 2\pi \\ f(x + 2\pi) & \text{otherwise} \end{cases}$$

- (a) Find its Fourier series.
- (b) Plot some **partial sums** and the Fourier series.

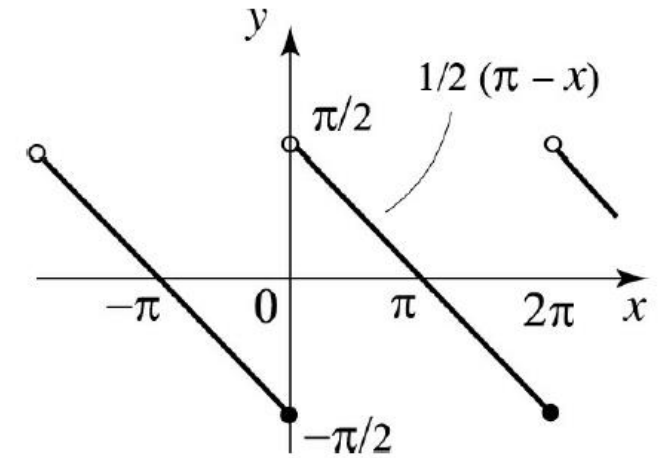


Figure 1 Sawtooth function.

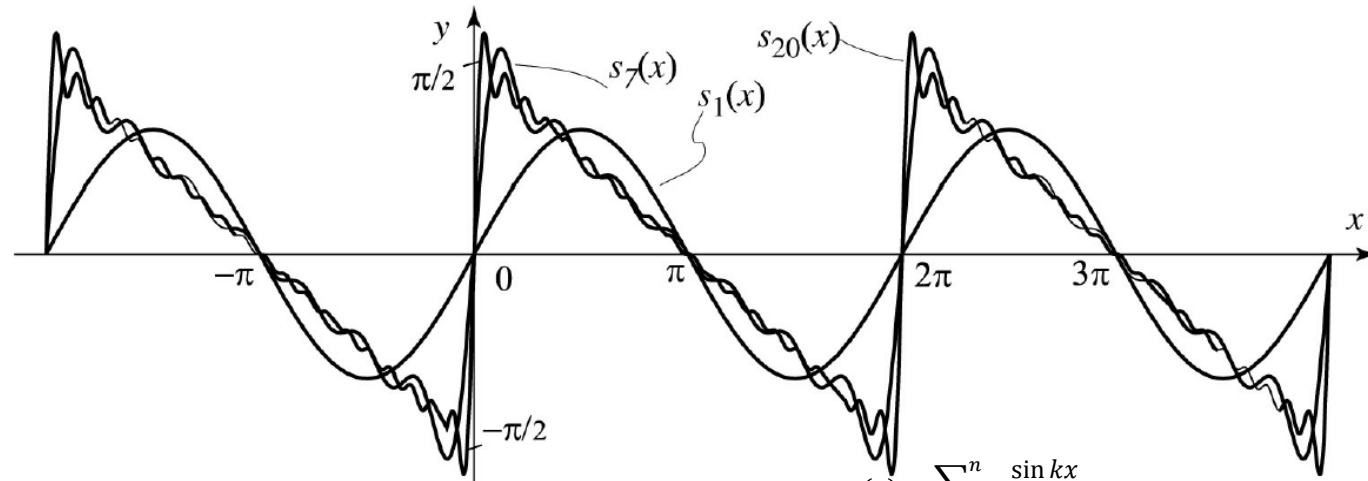


Figure 2 Here the  $n$ th partial sum of the Fourier series is  $s_n(x) = \sum_{k=1}^n \frac{\sin kx}{k}$ . To distinguish the graphs, note that as  $n$  increases, the frequencies of the sine terms increase. This causes the graphs of the higher partial sums to be more wiggly.

Pointwise convergence.

## 4.2 Uniform Convergence and Fourier Series

Recall that in Lecture 2

### EXAMPLE 2 Triangular wave

The  $2\pi$ -periodic triangular wave is given on the interval  $[-\pi, \pi]$  by

$$g(x) = \begin{cases} \pi + x & \text{if } -\pi \leq x \leq 0 \\ \pi - x & \text{if } 0 \leq x \leq \pi \end{cases}$$

(a) Find its Fourier series.

(b) Plot some **partial sums** and the Fourier series.

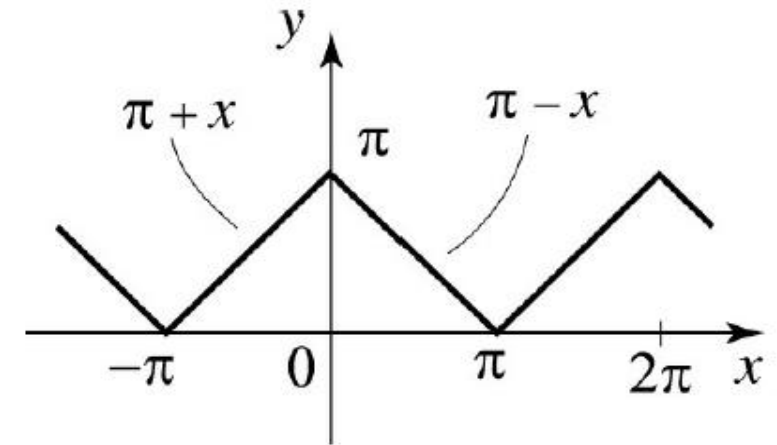


Figure 6 Triangular wave

Uniform convergence.

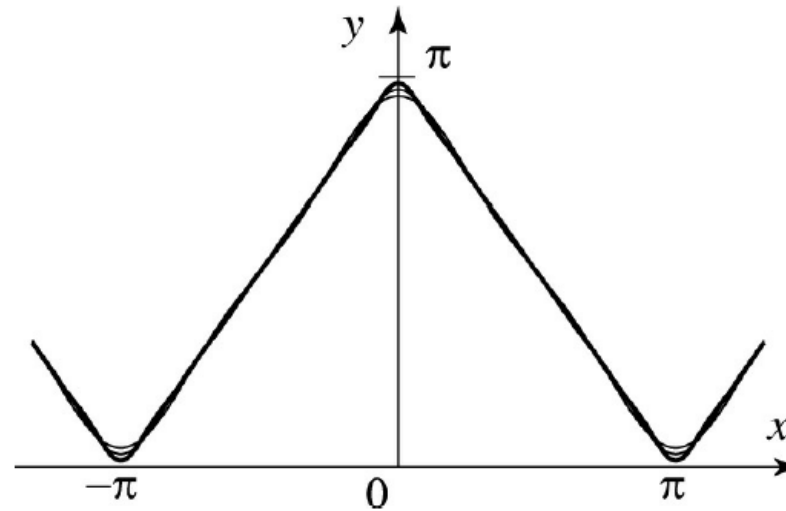


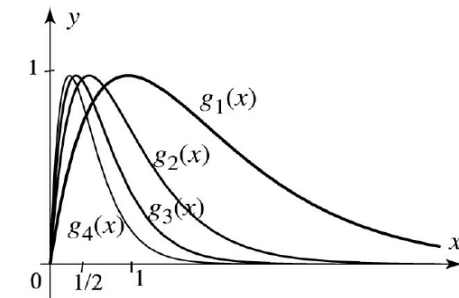
Figure 7 Partial sums of the Fourier series

## 4.2 Uniform Convergence and Fourier Series

### Pointwise Convergence and Uniform Convergence

- A sequence of functions  $(f_n(x))_{k=0}^{\infty}$  with common domain  $X$  is said to be **pointwise convergent** to a limit function  $f(x)$  if for all  $x \in X$  and for all  $\epsilon > 0$  there exists an  $N \in \mathbb{N}$  such that if  $n \geq N$  then:

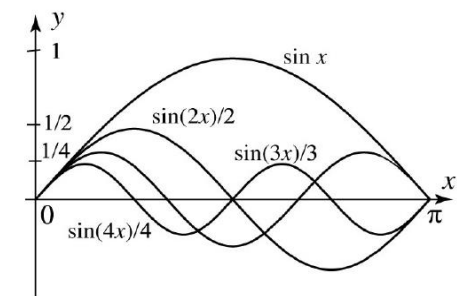
$$|f_n(x) - f(x)| < \epsilon$$



Pointwise Convergence

- A sequence of functions  $(f_n(x))_{k=0}^{\infty}$  with common domain  $X$  is said to be **uniformly convergent** to a limit function  $f(x)$  if for all  $\epsilon > 0$  there exists an  $N \in \mathbb{N}$  such that if  $n \geq N$  we have that for all  $x \in X$  that then:

$$|f_n(x) - f(x)| < \epsilon$$



Uniform Convergence

## 4.2 Uniform Convergence and Fourier Series

### Pointwise Convergence and Uniform Convergence

The best way to **understand the difference** between these two definitions is to think of the definitions as a **"game" process** of sorts.

For **pointwise convergence**, think of it this way:

- You select an  $x \in X$ , and a friend gives you an  $\epsilon > 0$ .
- You then find an  $N \in \mathbb{N}$  such that if  $n \geq N$  we have that:

$$|f_n(x) - f(x)| < \epsilon$$

- If you succeed for each  $\epsilon > 0$ , then your friend asks you to choose another  $x \in X$ . You do, and once again, you show that for any  $\epsilon > 0$  you can find such an  $N$  to satisfy the inequality above.
- For every  $x \in X$  and for any  $\epsilon > 0$  you can find such an  $N \in \mathbb{N}$  then you will have shown that the sequence of functions  $(f_n(x))_{k=0}^{\infty}$  converges pointwise to the limit function  $f(x)$ .

In essence, the sequence of functions  $(f_n(x))_{k=0}^{\infty}$  converges pointwise to the limit function  $f(x)$  if the sequence of numbers  $(f_n(x))_{k=0}^{\infty}$  converges to  $f(x_0)$  for each  $x_0 \in X$ .

## 4.2 Uniform Convergence and Fourier Series

### Pointwise Convergence and Uniform Convergence

Now, for **uniform convergence**, think of it this way:

- Your friend gives you an  $\epsilon > 0$ .
- You must then find an  $N \in \mathbb{N}$  such that if  $n \geq N$  then for EVERY  $x \in X$  we have that:

$$|f_n(x) - f(x)| < \epsilon$$

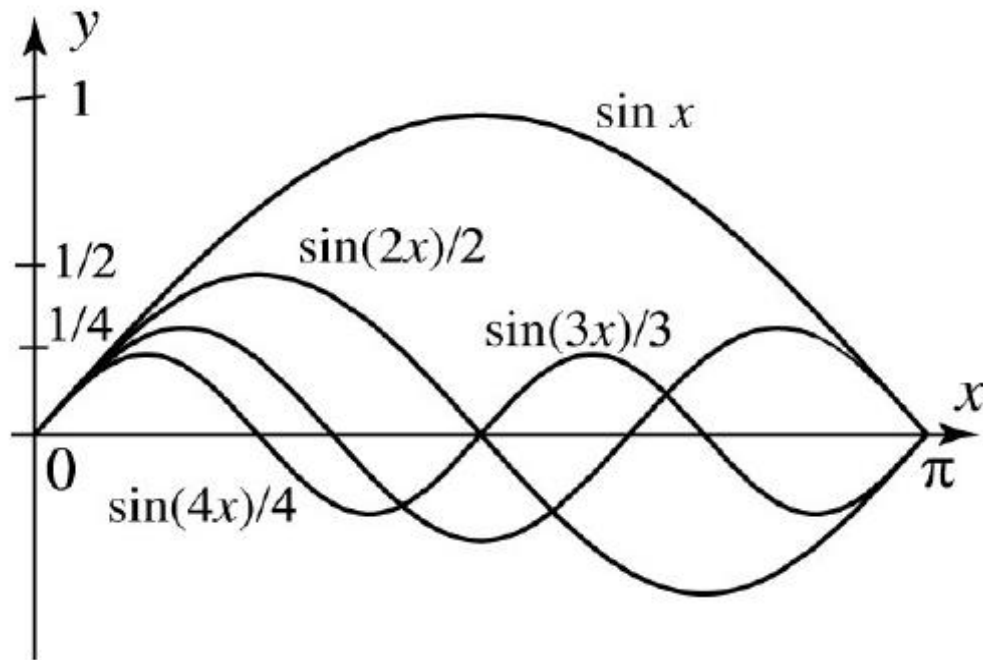
- If you succeed for each  $\epsilon > 0$  then you will have shown that the sequence of functions  $(f_n(x))_{k=0}^{\infty}$  converges uniformly to the limit function  $f(x)$ .

In essence, the sequence of functions  $(f_n(x))_{k=0}^{\infty}$  converges uniformly to the limit function  $f(x)$  if the sequence of numbers  $(f_n(x))_{k=0}^{\infty}$  converge to  $f(x_0)$  for each  $x_0 \in X$  at a somewhat similar/uniform rate.



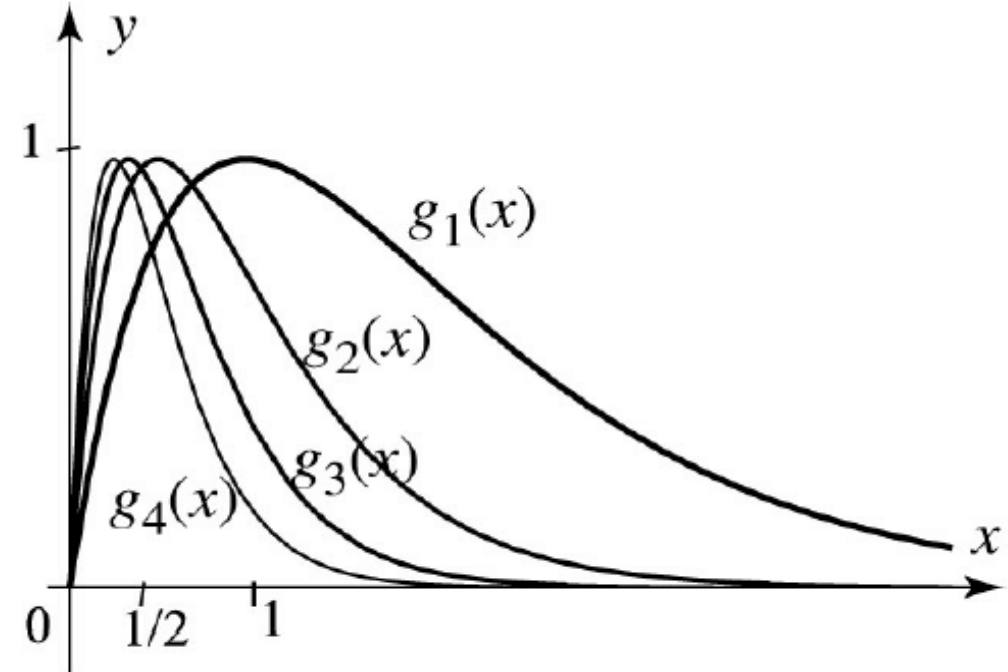
## 4.2 Uniform Convergence and Fourier Series

### Pointwise Convergence and Uniform Convergence



(a) Uniform convergence.

$$f_n(x) \rightarrow 0 \text{ as } n \rightarrow \infty$$



(b) Pointwise convergence.

$$g_n(x) \rightarrow 0 \text{ as } n \rightarrow \infty$$

## 4.2 Uniform Convergence and Fourier Series

### THEOREM 1 Weierstrass $M$ -Test

Let  $(u_k)_{k=0}^{\infty}$  be a sequence of real- or complex-valued functions on  $E$ . Suppose that there is a sequence  $(M_k)_{k=0}^{\infty}$  of nonnegative real numbers such that the following two conditions hold:

$$(1) \quad |u_k(x)| \leq M_k \quad \text{for all } x \text{ in } E.$$

and

$$(2) \quad \sum_{k=0}^{\infty} M_k < \infty$$

Then  $\sum_{k=0}^{\infty} u_k(x)$  converges uniformly on  $E$ .

## 4.2 Uniform Convergence and Fourier Series

### EXAMPLE 1 Weierstrass $M$ -test

(a) The Fourier series of the function  $g(x)$  in [Example 2, Section 2.2](#) ([Figure 1](#))

$$\frac{\pi}{2} + \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos[(2k+1)x]}{(2k+1)^2}$$

converges uniformly on the entire real line.

#### Solution

For all  $x$  in  $E$

Because  $|\cos[(2k+1)x]| \leq 1$

$$\left| \frac{\cos[(2k+1)x]}{(2k+1)^2} \right| \leq \frac{1}{(2k+1)^2}$$

$$\sum \frac{1}{(2k+1)^2} < \infty \quad \text{Hint: } p\text{-series}$$

$$\text{i.e.} \quad \sum M_k < \infty$$

According to Weierstrass  $M$ -test, we conclude that the series converges uniformly on  $E$ .

## 4.2 Uniform Convergence and Fourier Series

### EXAMPLE 1 Weierstrass $M$ -test

(b) Let  $E = [1, \infty)$ , and consider the series

$$\sum_{k=0}^{\infty} e^{-kx} \sin kx$$

#### Solution

For all  $x$  in  $E$

Because  $|\sin kx| \leq 1$  and  $E = [1, \infty)$

$$|e^{-kx} \sin kx| \leq e^{-k}$$

$$\sum_{k=0}^{\infty} e^{-k} = \frac{1}{1 - e^{-1}} = \frac{e}{e - 1} < \infty$$

Hint: geometric series

$$\text{i.e. } \sum M_k < \infty$$

According to Weierstrass  $M$ -test, we conclude that the series converges uniformly on  $E$ .

# Review for Lecture 4

- Complex Form of Fourier Series
- Uniform Convergence and Fourier Series

# Exercise

Please Check <https://github.com/uoaworks/FourierAnalysisAY2018>

Reading: Section 2.6, 2.9, Textbook

# References

- [1] Nakhlé H. Asmar, *Partial Differential Equations with Fourier Series and Boundary Value Problems 2<sup>nd</sup> Edition*, 2004
- [2] Wikipedia