Lecture 3

- Half-Range Expansions: The Cosine and Sine Series
- Approximation and Parseval' Theorem
- Complex Form of Fourier Series

What you will learn in Lecture 3

I. Half-Range Expansions: The Cosine and Sine Series

II. Mean-Square Approximation and Parseval' Theorem

III. Complex Form of Fourier Series

3.1 Half-Range Expansions:

The Cosine and Sine Series

3.1 Half-Range Expansions: The Cosine and Sine Series

- In many applications we are interested in representing by a Fourier series a function $f(x)$ that is defined only in a finite interval, say $0 < x < p$.

- **Since f is clearly not periodic**, the results of the previous sections are not readily applicable.

- **Our goal** in this section is to show how we can represent f by a Fourier series, after extending it to a periodic function.

3.1 Half-Range Expansions: The Cosine and Sine Series

THEOREM 1 Half-Range Expansions

Suppose that $f(x)$ is a piecewise smooth function defined on an interval $0 < x < p$.

Then f has a **cosine series expansion**

(1)
$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x \qquad (0 < x < p)$$

where

(2)
$$a_0 = \frac{1}{p} \int_0^p f(x)\, dx \qquad \text{and} \qquad a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x\, dx \quad (n \geq 1)$$

Also, f has a **sine series expansion**

(3)
$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x \qquad (0 < x < p)$$

where

(4)
$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x\, dx \quad (n \geq 1)$$

On the interval $0 < x < p$, the series (1) and (3) converge to $f(x)$ if f is continuous at x and to $\frac{f(x-) + f(x+)}{2}$ otherwise.

3.1 Half-Range Expansions: The Cosine and Sine Series

THEOREM 1 Half-Range Expansions

- The series (1) and (3) are commonly referred to as the **half-range expansions** of f.

- They are **two different series representations of the same function** on the interval $0 < x < p$.

3.1 Half-Range Expansions: The Cosine and Sine Series

- Define the **even periodic extension** of f by $f_1(x) = f(x)$ if $0 < x < p$, $f_1(x) = f(-x)$ if $-p < x < 0$, and $f_1(x) = f_1(x + 2p)$ otherwise.
- Define the **odd periodic extension** of f by $f_2(x) = f(x)$ if $0 < x < p$, $f_2(x) = -f(-x)$ if $-p < x < 0$, and $f_2(x) = f_2(x + 2p)$ otherwise.

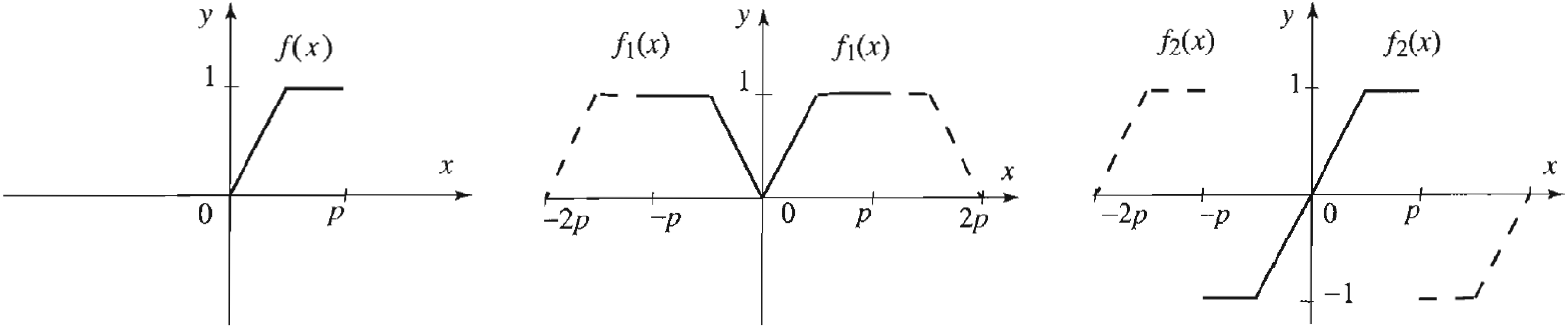


Figure 1 (a) $f(x), 0 < x < p$ (b) Even $2p$-periodic extension, f_1. (c) Odd $2p$-periodic extension, f_2.

EXAMPLE 2 Half-range expansions

Consider the function $f(x) = \sin x, 0 \le x \le \pi.$ If we take its odd extension, we get the usual sine function, $f_2(x) = \sin x$ for all $x.$ Thus, the sine series expansion is just sin x.

Solution

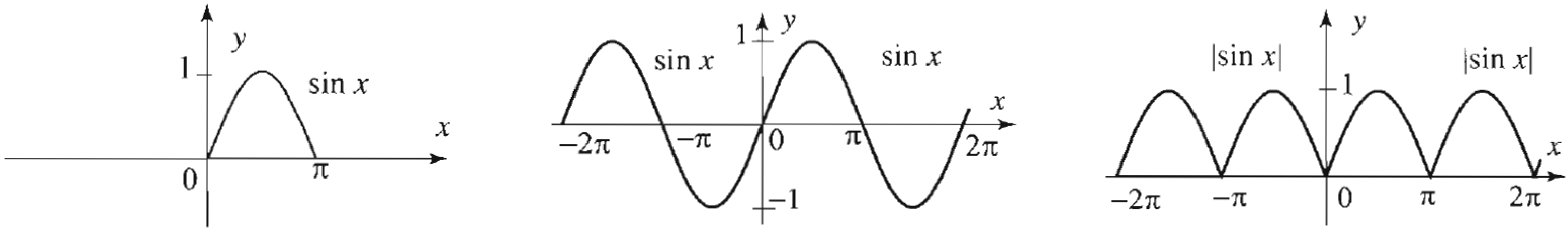


Figure 3 (a) $f(x) = \sin x\,, 0 \le x \le p$ (b) Odd extension of f, $\sin x$. (c) Even extension of f, $|\sin x|$.

Solution

Consider the even extension of f, we get the function $|\sin x|$, notice that it is 2π-periodic function.

Using the Theorem 2 (Fourier Series of Even and Odd Functions) in Section 2.3 of the textbook, we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\,dx = \frac{2}{2\pi}\int_{0}^{\pi} \sin x\,\,dx = -\frac{1}{\pi}\cos x\,\Big|_0^{\pi} = \frac{2}{\pi}$$

For $n \geq 1$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} \sin x\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} \frac{1}{2}(\sin(1-n)x + \sin(1+n)x)dx$$

because $\sin a\cos b = \frac{1}{2}(\sin(a-b) + \sin(a+b))$

$$= \frac{1}{\pi}\left[-\frac{1}{1-n}\cos(1-n)x - \frac{1}{1+n}\cos(1+n)x\right]\Big|_0^{\pi} \qquad (\text{if } n \neq 1)$$

$$= \frac{1}{\pi}\left[\frac{-1}{1-n}(-1)^{1-n} - \frac{1}{1+n}(-1)^{1+n} + \frac{1}{1-n} + \frac{1}{1+n}\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{4}{\pi(1-n^2)} & \text{if } n \text{ is even} \end{cases}$$

Solution

For $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin(2x)\, dx = 0$$

Thus, the Fourier series is:

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k)^2 - 1} \cos 2kx$$

3.2 Mean-Square Approximation and Parseval’ Theorem

3.2 Mean-Square Approximation and Parseval' Theorem

When a $2p$-periodic function is represented by its **Fourier series**,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

it is important to know **how well** the Nth **partial sums** approximate f.

$$s_N(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

Q: How well the partial sums of the Fourier series approximate the function

$$\boxed{E_n = \frac{1}{2p} \int_{-p}^{p} \left(f(x) - s_N(x) \right)^2 dx}$$

known as the **mean** (or **total**) **square error** of the partial sum S_N relative to f.

We also say that S_N **approximates** f **in the mean with error** E_N.

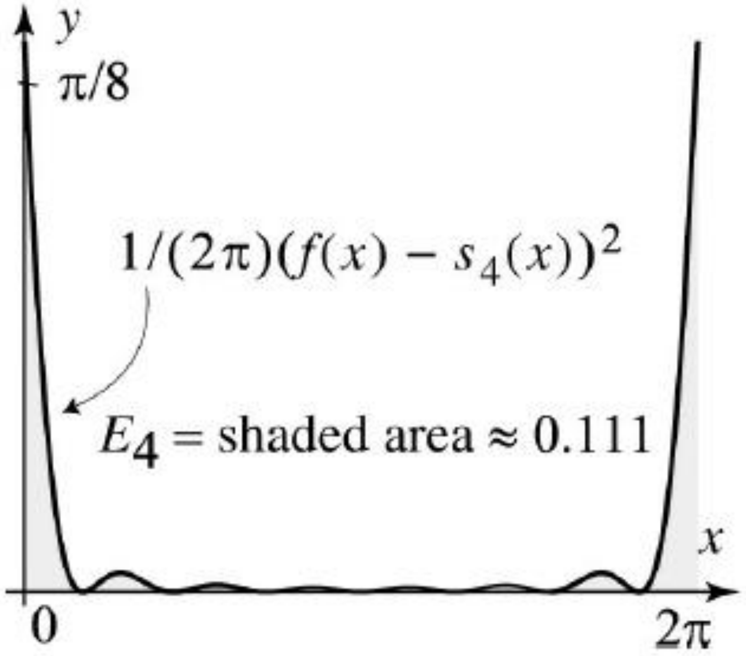
3.2 Mean-Square Approximation and Parseval' Theorem

EXAMPLE 1 Approximation in the mean by Fourier series

Compute E_N for $N = 1, 2, \dots, 10$, in the case of the 2π-periodic sawtooth function $f(x) = \frac{1}{2}(\pi - x)$, $0 < x < 2\pi$.

Solution

Find the complete solution in page 54, Section 2.5 of the textbook.



3.2 Mean-Square Approximation and Parseval' Theorem

Recall in Lecture 2, we have an example

EXAMPLE 1 Fourier series of the sawtooth function

The sawtooth function, shown in Figure 1, is determined by the formulas

$$f(x) = \begin{cases} \frac{1}{2}(\pi - x) & if\ 0 < x \le 2\pi \\ f(x+2\pi) & otherwise \end{cases}$$

(a) Find its Fourier series.

(b) With the help of a computer, plot the partial sums $s_1(x), s_7(x)$, and $s_{20}(x)$, and determine the graph of the Courier series.

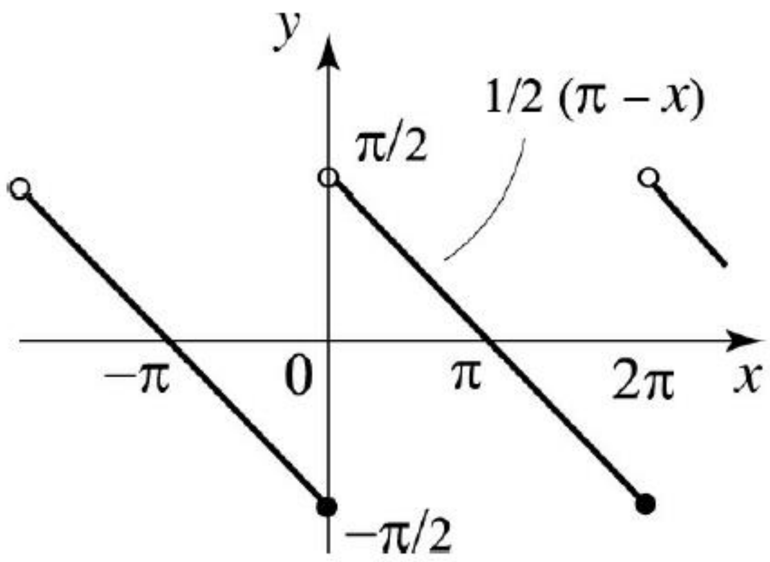


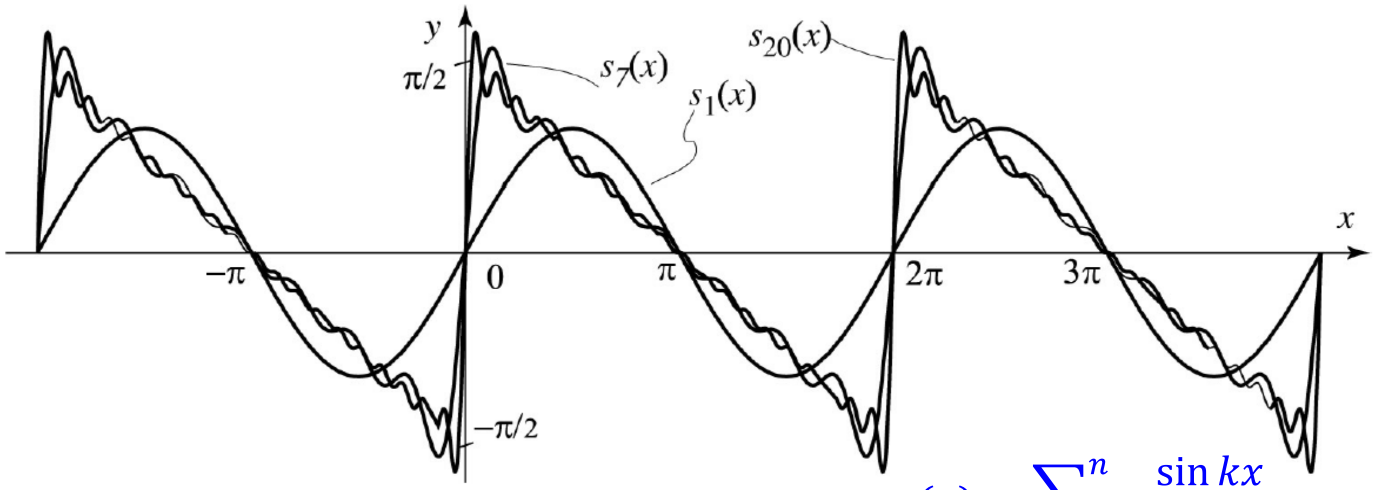
Figure 1 Sawtooth function.

3.2 Mean-Square Approximation and Parseval' Theorem

Recall in Lecture 2, we have an example

Solution

Find the complete solution in page 28 and 29, Section 2.2 of the textbook.



$$s_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k}$$

Figure 2 Here the nth partial sum of the Fourier series is distinguish the graphs, note that as n increases, the frequencies of the sine terms increase. This causes the graphs of the higher partial sums to be more wiggly.

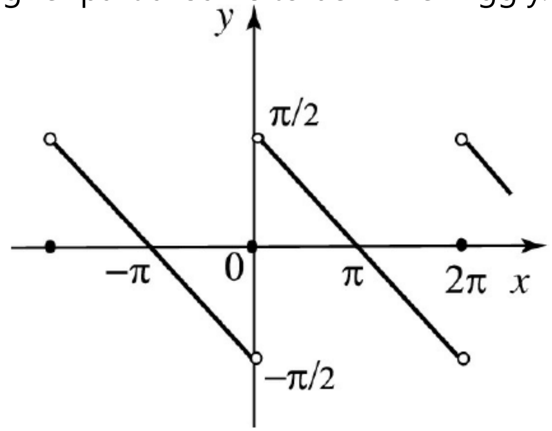


Figure 3 The graph of the Fourier series coincides with the graph of $\sum_{k=1}^{\infty} \frac{\sin kx}{k}$ the function, except at the points of the discontinuity.

3.2 Mean-Square Approximation and Parseval’ Theorem

We introduce the class of **square integrable functions** on $[a, b]$, which consists of functions f defined on $[a, b]$ and such that

$$\int_{-p}^{p} f(x)^2 dx < \infty$$

THEOREM 1 Approximation in the Mean by Fourier Series

Suppose that f is **square integrable** on $[-p, p]$. Then S_N, the Nth partial sum of the Fourier series of f, approximates (or converges to) f in the mean with an error E_N that decreases to zero as $N \to \infty$. In symbols, we have

(3) $$\lim_{N\to\infty} E_n = \lim_{N\to\infty} \frac{1}{2p}\int_{-p}^{p} (f(x) - s_N(x))^2 dx = 0$$

$$\lim_{N\to\infty} \frac{1}{2p}\int_{-p}^{p} (f(x) - s_N(x))^2 dx = \frac{1}{2p}\int_{-p}^{p} \lim_{N\to\infty} (f(x) - s_N(x))^2 dx$$

(4) $$= \frac{1}{2p}\int_{-p}^{p} 0 dx = 0$$

THEOREM 2 Mean Square Error

Suppose that f is square integrable on $[-p, p]$. Then

$$(5) \qquad E_n = \frac{1}{2p}\int_{-p}^{p} f(x)^2\,dx - a_0^2 - \frac{1}{2}\sum_{n=1}^{N}(a_n^2 + b_n^2)$$

3.2 Mean-Square Approximation and Parseval’ Theorem

Proof

Find the complete proof in page 55, 56, Section 2.5 of the textbook.

3.2 Mean-Square Approximation and Parseval' Theorem

One useful consequence of (5) is the following **inequality**:

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leq \frac{1}{2p}\int_{-p}^{p} f(x)^2 dx$$

known as **Bessel' s inequality**.

To **prove** it, note that $E_N \geq 0$, from (2). Hence (5) implies that

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{N}(a_n^2 + b_n^2) \leq \frac{1}{2p}\int_{-p}^{p} f(x)^2 dx$$

The desired inequality follows by letting $N \to \infty$.

3.2 Mean-Square Approximation and Parseval' Theorem

$$0 = \lim_{N\to\infty} E_n = \frac{1}{2p}\int_{-p}^{p} f(x)^2 dx - a_0^2 - \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

PARSEVAL' S IDENTITY

$$\frac{1}{2p}\int_{-p}^{p} f(x)^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{6}$$

3.2 Mean-Square Approximation and Parseval' Theorem

It is valid for all **square integrable functions** on $[-p, p]$ and has many interesting applications.

Parseval's identity, or **Bessel's inequality**, imply that the Fourier coefficients of a square integrable function are **square summable.** That is, we have

$$a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) < \infty$$

EXAMPLE 2 Evaluating series with Parseval's identity

Evaluate the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

Solution For the Example 1 in Section 2.2 of the textbook, we notice that

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{1}{n}$$

Therefore it may correspond to the series we have now

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

(Find the complete solution in page 57, Section 2.5 of the textbook.)

3.3 Complex Form of Fourier Series

3.3 Complex Form of Fourier Series

Let us start with the two identities

(1)
$$\cos u = \frac{e^{iu} + e^{-iu}}{2} \quad \text{and} \quad \sin u = \frac{e^{iu} - e^{-iu}}{2i}$$

We will use these identities to find a complex form for the Fourier series expansion of a $2p$-periodic function

(2)
$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

3.3 Complex Form of Fourier Series

THEOREM 1 Complex Form of Fourier Analysis

Let f be a $2p$-periodic piecewise smooth function. The **complex form of the Fourier series** of f is

$$\sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{p}x}$$

where **the Fourier coefficients** c_n are given by

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(t)e^{-i\frac{n\pi}{p}t}dt \quad (n = 0, \pm 1, \pm 2, \cdots)$$

For all x, the Fourier series converges to $f(x)$ if f is continuous at x, and to $\frac{f(x+)+f(x-)}{2}$ otherwise.

3.3 Complex Form of Fourier Series

The Nth **partial sum** of (3) is by definition the **symmetric sum**

$$s_N(x) = \sum_{n=-N}^{N} c_n e^{i\frac{n\pi}{p}x}$$

We will see in a moment that is the same as the usual partial sum of the Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

Review for Lecture 3

- Half-Range Expansions: The Cosine and Sine Series
- Mean-Square Approximation and Parseval' Theorem
- Complex Form of Fourier Series

Exercise

Please Check https://github.com/uoaworks/FourierAnalysisAY2018

Reading: Section 2.4, 2.5, 2.6, Textbook

References

[1] Nakhlé H. Asmar, *Partial Differential Equations with Fourier Series and Boundary Value Problems 2nd Edition*, 2004