Lecture 4

- Complex Form of Fourier Series
- Uniform Convergence and Fourier Series

What you will learn in Lecture 4

I. Complex Form of Fourier Series

II. Uniform Convergence and Fourier Series

4.1 Complex Form of Fourier Series

4.1 Complex Form of Fourier Series

Recall that in *Calculus II*, we have **Taylor Series**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

4.1 Complex Form of Fourier Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$$= 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots$$

because $i^2 = -1$

$$= 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= \cos x + i \sin x$$

4.1 Complex Form of Fourier Series

Leonhard Euler (1707-1783)

① $e^{ix} = \cos x + i \sin x$ **Euler's formula**

② $e^{-ix} = \cos(-x) + i\sin(-x)$

$= \cos x - i \sin x$

① + ② $e^{ix} + e^{-ix} = 2\cos x \Rightarrow \cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

① - ② $e^{ix} - e^{-ix} = i2\sin x \Rightarrow \sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

4.1 Complex Form of Fourier Series

We know the trigonometric identity

$$\cos x \cdot \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

Because $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ then we have

$$\begin{aligned}
\cos x \cdot \cos y &= \frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^{iy} + e^{-iy}}{2} \\
&= \frac{1}{2} \cdot \frac{e^{i(x+y)} + e^{i(x-y)} + e^{i(-x+y)} + e^{i(-x-y)}}{2} \\
&= \frac{1}{2}\left(\underbrace{\frac{e^{i(x+y)} + e^{-i(x+y)}}{2}}_{\cos(x+y)} + \underbrace{\frac{e^{i(x-y)} + e^{-i(x-y)}}{2}}_{\cos(x-y)}\right).
\end{aligned}$$

4.1 Complex Form of Fourier Series

For complex form of Fourier series , let us start with the two identities

(1) $$\cos u = \frac{e^{iu} + e^{-iu}}{2} \quad \text{and} \quad \sin u = \frac{e^{iu} - e^{-iu}}{2i}$$

We will use these identities to find a complex form for the Fourier series expansion of a $2p$-periodic function

(2) $$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

4.1 Complex Form of Fourier Series

THEOREM 1 Complex Form of Fourier Analysis

Let f be a $2p$-periodic piecewise smooth function. The **complex form of the Fourier series** of f is

(3) $$\sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{p}x}$$

where **the Fourier coefficients** c_n are given by

(4) $$c_n = \frac{1}{2p}\int_{-p}^{p} f(t)e^{-i\frac{n\pi}{p}t}\,dt \quad (n = 0, \pm 1, \pm 2, \cdots)$$

For all x, the Fourier series converges to $f(x)$ if f is continuous at x, and to $\dfrac{f(x+)+f(x-)}{2}$ otherwise.

4.1 Complex Form of Fourier Series

Proof

Find the complete proof in page 62, Section 2.6 of the textbook.

4.1 Complex Form of Fourier Series

The Nth **partial sum** of (3) is by definition the **symmetric sum**

$$s_N(x) = \sum_{n=-N}^{N} c_n e^{i\frac{n\pi}{p}x}$$

We will see in a moment that is the same as the usual partial sum of the Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right)$$

4.1 Complex Form of Fourier Series

If $h(t)$ is **complex-valued**, write $h(t) = u(t) + iv(t)$, where u and v are the real and imaginary parts of h. We define

$$\int h(t)dt = \int (u(t) + iv(t))dt \equiv \int u(t)dt + i \int v(t)dt$$

For example, if $h(t) = f(t)e^{ict}$, where $f(t)$ is real-valued and c is a real number. Then $h(t) = f(t)e^{ict} = f(t)(\cos ct + i \sin ct)$. So

$$\int f(t)e^{ict}dt = \int f(t)\cos ct\, dt + i \int f(t)\sin ct\, dt$$

where now both integrals on the right are integrals or antiderivatives of real-valued functions.

4.1 Complex Form of Fourier Series

It is straightforward to show that the **integral is linear**:

If h and g are complex-valued and α and β are complex numbers, then

$$\int (\alpha h(t) + \beta g(t))dt = \alpha \int h(t)dt + \beta \int g(t)\, dt$$

4.1 Complex Form of Fourier Series

We now highlight some interesting identities that relate the **complex Fourier coefficients** to **the Fourier cosine and sine coefficients.**

(5) $$c_0 = a_0$$

(6) $$c_n = \frac{1}{2}(a_n - ib_n) \qquad c_{-n} = \frac{1}{2}(a_n + ib_n) \quad (n > 0)$$

(7) $$a_n = c_n + c_{-n} \qquad b_n = i(c_n - c_{-n}) \quad (n > 0)$$

(8) $$S_N(x) = s_N(x)$$

(6) shows that c_{-n} is the complex conjugate of c_n. In symbols,

(9) $$c_{-n} = \bar{c}_n$$

EXAMPLE 1 A complex Fourier series

Find the complex form of the Fourier series of the 2π-periodic function $f(x) = e^{ax}$ for $-\pi < x < \pi$, where $a \neq 0, \pm i, \pm 2i, \pm 3i, \ldots$. Determine the values of the Fourier series at $x = \pm\pi$.

Solution

Find the complete solution in page 63 and 64, Section 2.6 of the textbook.

4.1 Complex Form of Fourier Series

EXAMPLE 1 A complex Fourier series

- Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} = \frac{1 - e^{-2x}}{2e^{-x}}.$$

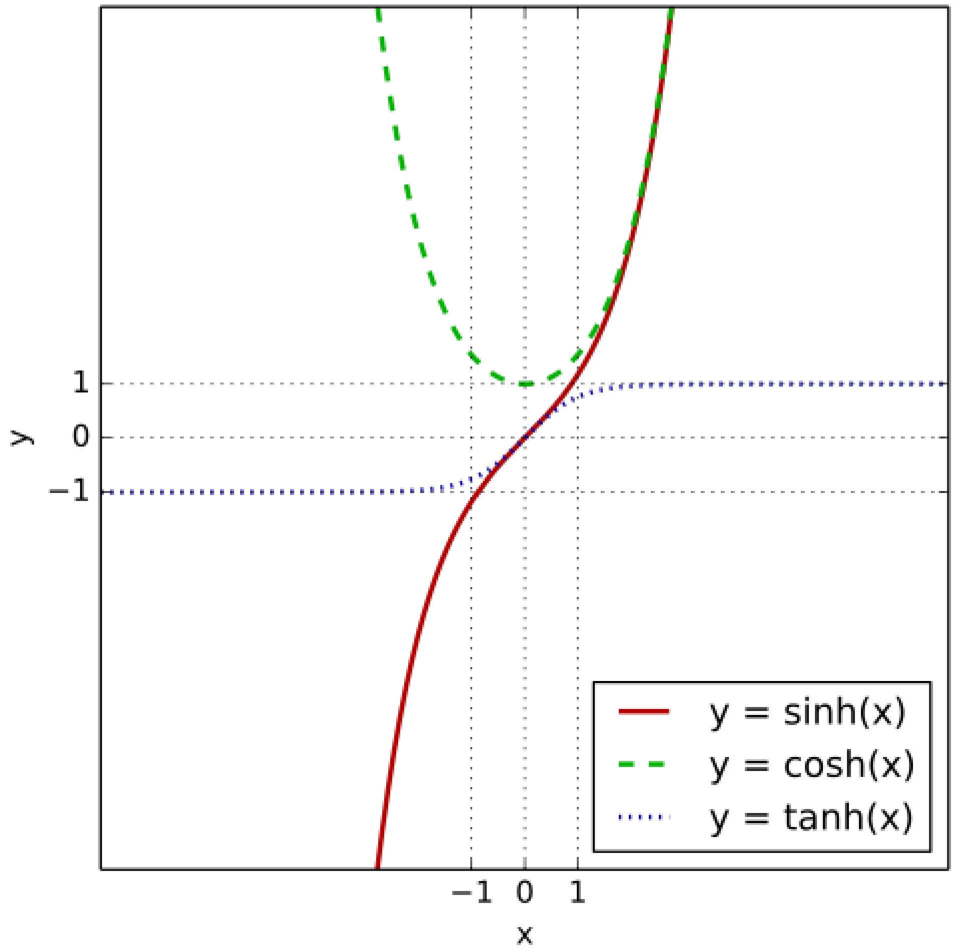
- Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x} = \frac{1 + e^{-2x}}{2e^{-x}}.$$

- Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} =$$

$$= \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}.$$



https://en.wikipedia.org/wiki/Hyperbolic_function

4.1 Complex Form of Fourier Series

Similarly, the complex form of the Fourier coefficients can be obtained by appealing to **the orthogonality** of the **complex exponential system**

$$1, e^{i\frac{\pi}{p}x}, e^{-i\frac{\pi}{p}x}, e^{i\frac{2\pi}{p}x}, e^{-i\frac{2\pi}{p}x}, \cdots, e^{i\frac{n\pi}{p}x}, e^{-i\frac{n\pi}{p}x}, \cdots$$

The orthogonality of this system is expressed by

(11) $$\frac{1}{2p}\int_{-p}^{p} e^{i\frac{m\pi}{p}x} e^{-i\frac{n\pi}{p}x}\,dx = \begin{cases} 0 & if\ m \neq n \\ 1 & if\ m = n \end{cases}$$

THEOREM 2 Complex Form of Parseval's Identity

Let f be a real-valued square integrable function on $[-p, p]$ with Fourier coefficients c_n given by (4). Then

$$\frac{1}{2p}\int_{-p}^{p} f(x)^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Proof

Find the complete proof in page 65, Section 2.6 of the textbook.

4.2 Uniform Convergence and Fourier Series

4.2 Uniform Convergence and Fourier Series

Recall that in Lecture 2

EXAMPLE 1 **Fourier series of the sawtooth function**

The sawtooth function, shown in Figure 1, is determined by the formulas

$$f(x) = \begin{cases} \frac{1}{2}(\pi - x) & if\ 0 < x \leq 2\pi \\ f(x+2\pi) & otherwise \end{cases}$$

(a) Find its Fourier series.

(b) Plot some **partial sums** and the Fourier series.

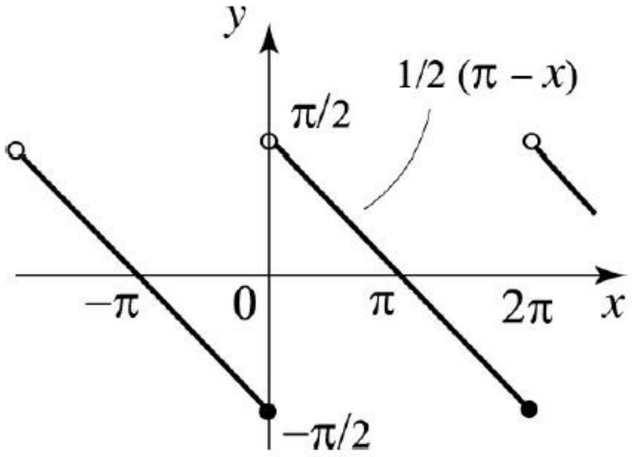


Figure 1 Sawtooth function.

Pointwise convergence.

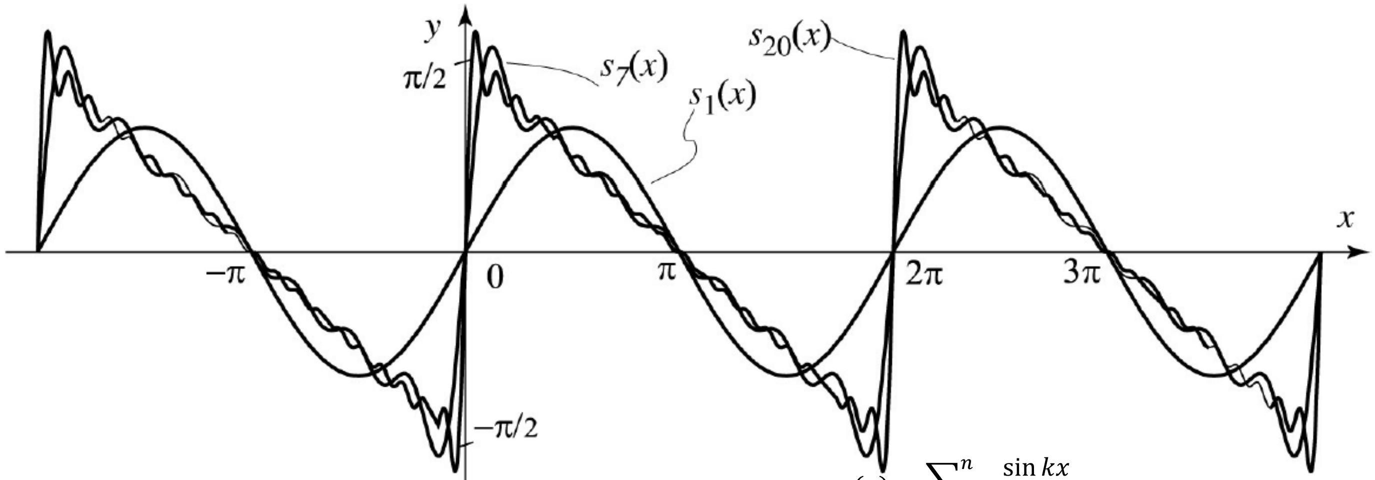


Figure 2 Here the nth partial sum of the Fourier series is $s_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k}$. distinguish the graphs, note that as n increases, the frequencies of the sine terms increase. This causes the graphs of the higher partial sums to be more wiggly.

4.2 Uniform Convergence and Fourier Series

Recall that in Lecture 2

EXAMPLE 2 Triangular wave

The 2π-periodic triangular wave is given on the interval $[-\pi, \pi]$ by

$$g(x) = \begin{cases} \pi + x & if \ -\pi \le x \le 0 \\ \pi - x & if \ 0 \le x \le \pi \end{cases}$$

(a) Find its Fourier series.

(b) Plot some **partial sums** and the Fourier series.

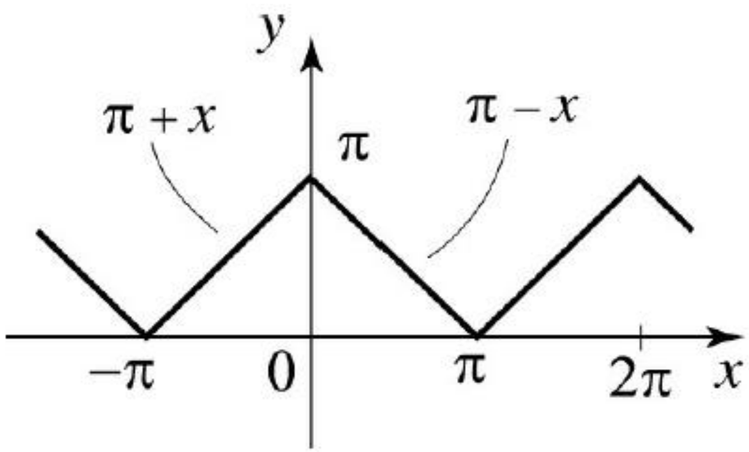


Figure 6 Triangular wave

Uniform convergence.

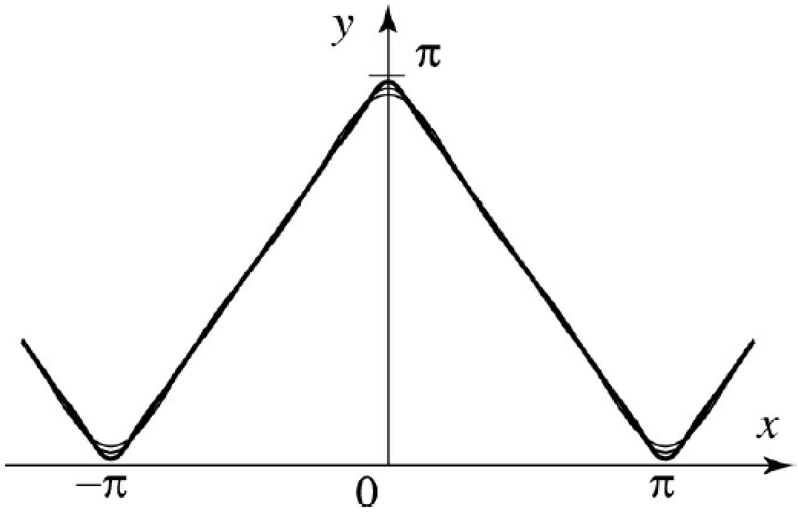


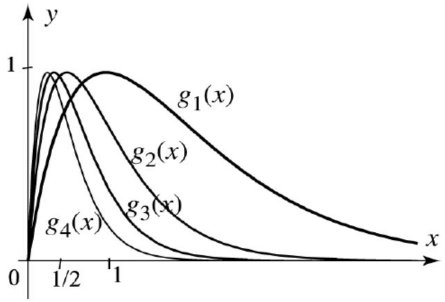
Figure 7 Partial sums of the Fourier series

4.2 Uniform Convergence and Fourier Series

Pointwise Convergence and Uniform Convergence

- A sequence of functions $(f_n(x))_{k=0}^{\infty}$ with common domain X is said to be **pointwise convergent** to a limit function $f(x)$ if for all $x \in X$ and for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then:

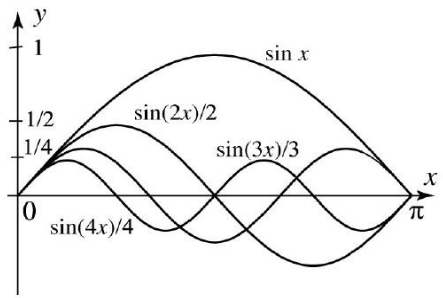
$$|f_n(x) - f(x)| < \epsilon$$



Pointwise Convergence

- A sequence of functions $(f_n(x))_{k=0}^{\infty}$ with common domain X is said to be **uniformly convergent** to a limit function $f(x)$ if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ we have that for all $x \in X$ that then:

$$|f_n(x) - f(x)| < \epsilon$$



Uniform Convergence

4.2 Uniform Convergence and Fourier Series

Pointwise Convergence and Uniform Convergence

The best way to **understand the difference** between these two definitions is this think of the definitions as **a "game" process** of sorts.

For pointwise convergence, think of it this way:

- You select an $x \in X$, and a friend gives you an $\epsilon > 0$.
- You then find an $N \in \mathbb{N}$ such that if $n \geq N$ we have that:

$$|f_n(x) - f(x)| < \epsilon$$

- If you succeed for each $\epsilon > 0$, then your friend asks you to choose another $x \in X$. You do, and once again, you show that for any $\epsilon > 0$ you can find such an N to satisfy the inequality above.
- For every $x \in X$ and for any $\epsilon > 0$ you can find such an $N \in \mathbb{N}$ then you will have shown that the sequence of functions $(f_n(x))_{k=0}^{\infty}$ converges pointwise to the limit function $f(x)$.

In essence, the sequence of functions $(f_n(x))_{k=0}^{\infty}$ converges pointwise to the limit function $f(x)$ if the sequence of numbers $(f_n(x))_{k=0}^{\infty}$ converges to $f(x_0)$ for each $x_0 \in X$.

4.2 Uniform Convergence and Fourier Series

Pointwise Convergence and Uniform Convergence

Now, for **uniform convergence**, think of it this way:

- Your friend gives you an $\epsilon > 0$.
- You must then find an $N \in \mathbb{N}$ such that if $n \geq N$ then for EVERY $x \in X$ we have that:

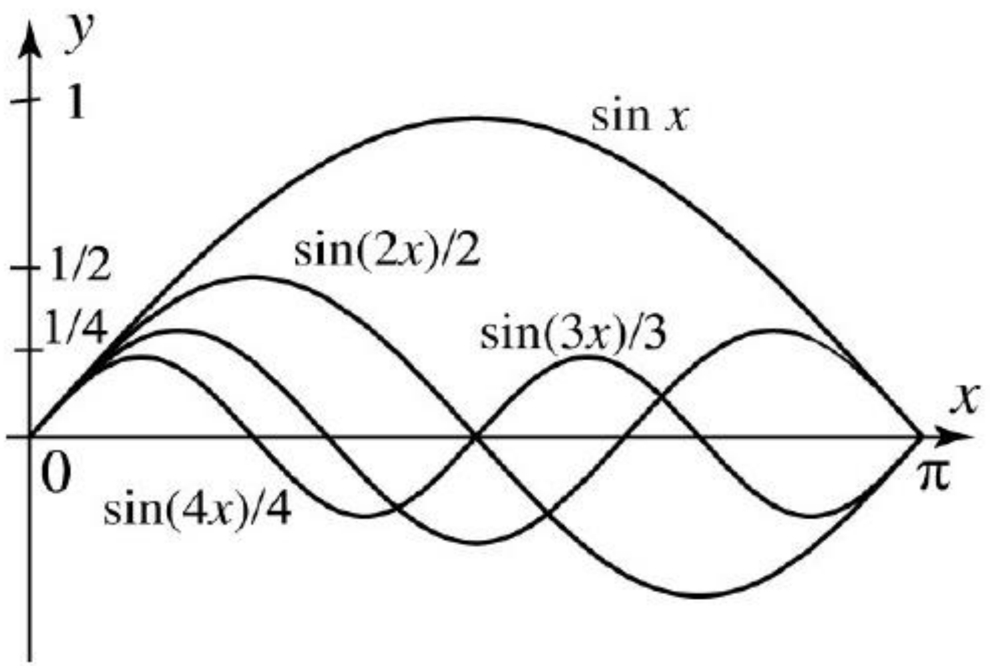
$$|f_n(x) - f(x)| < \epsilon$$

- If you succeed for each $\epsilon > 0$ then you will have shown that the sequence of functions $(f_n(x))_{k=0}^{\infty}$ converges uniformly to the limit function $f(x)$.

In essence, the sequence of functions $(f_n(x))_{k=0}^{\infty}$ converges uniformly to the limit function $f(x)$ if the sequence of numbers $(f_n(x))_{k=0}^{\infty}$ converge to $f(x_0)$ for each $x_0 \in X$ at a somewhat similar/uniform rate.

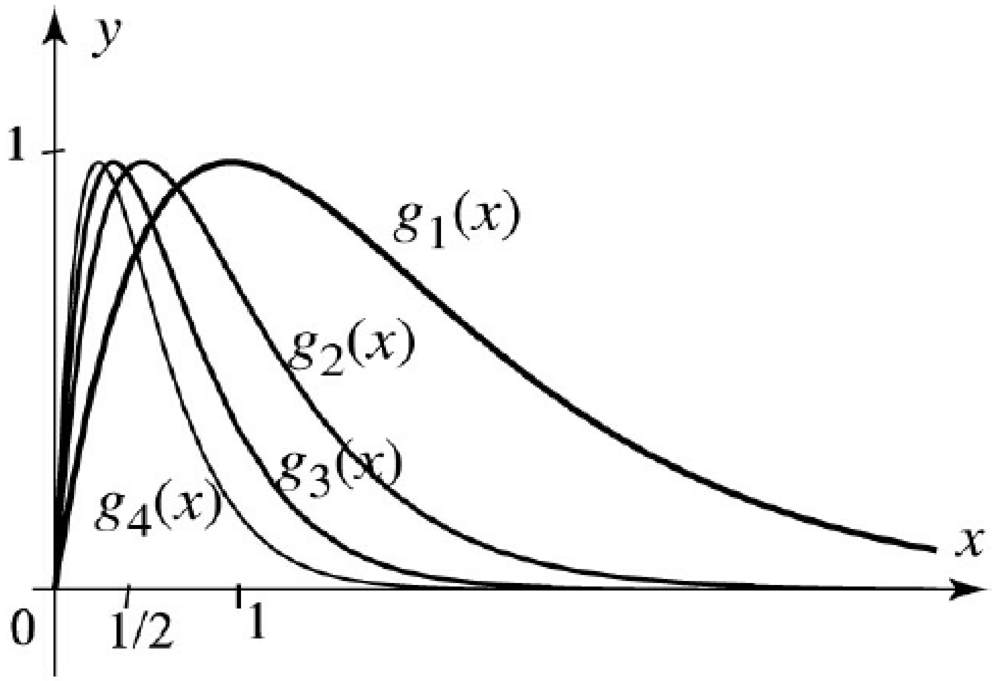
4.2 Uniform Convergence and Fourier Series

Pointwise Convergence and Uniform Convergence



(a) Uniform convergence.

$f_n(x) \to 0$ as $n \to \infty$



(b) Pointwise convergence.

$g_n(x) \to 0$ as $n \to \infty$

THEOREM 1 Weierstrass M-Test

Let $(u_k)_{k=0}^{\infty}$ be a sequence of real- or complex-valued functions on E.
Suppose that there is a sequence $(M_k)_{k=0}^{\infty}$ of nonnegative real numbers such that the following two conditions hold:

(1) $$|u_k(x)| \leq M_k \quad \text{for all } x \text{ in } E.$$

and

(2) $$\sum_{k=0}^{\infty} M_k < \infty$$

Then $\sum_{k=0}^{\infty} u_k(x)$ converges uniformly on E.

4.2 Uniform Convergence and Fourier Series

EXAMPLE 1 Weierstrass *M*-test

(a) The Fourier series of the function $g(x)$ in Example 2, Section 2.2 (Figure 1)

$$\frac{\pi}{2}+\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos[(2k+1)x]}{(2k+1)^2}$$

converges uniformly on the entire real line.

Solution

For all x in E

Because $|\cos[(2k+1)x]| \leq 1$

$$\left|\frac{\cos[(2k+1)x]}{(2k+1)^2}\right| \leq \frac{1}{(2k+1)^2}$$

$$\sum \frac{1}{(2k+1)^2} < \infty$$ Hint: p-series

i.e. $\sum M_k < \infty$

According to Wierstrass M-test, we conclude that the series converges uniformly on E.

EXAMPLE 1 Weierstrass *M*-test

(b) Let $E = [1, \infty)$, and consider the series

$$\sum_{k=0}^{\infty} e^{-kx} \sin kx$$

Solution

For all x in E

Because $|\sin kx| \leq 1$ and $E = [1, \infty)$

$$\left|e^{-kx} \sin kx\right| \leq e^{-k}$$

$$\sum_{k=0}^{\infty} e^{-k} = \frac{1}{1 - e^{-1}} = \frac{e}{e - 1} < \infty$$

Hint: geometric series

i.e. $\sum M_k < \infty$

According to Wierstrass M-test, we conclude that the series converges uniformly on E.

Review for Lecture 4

- Complex Form of Fourier Series
- Uniform Convergence and Fourier Series

Exercise

Please Check https://github.com/uoaworks/FourierAnalysisAY2018

Reading: Section 2.6, 2.9, Textbook

References

[1] Nakhlé H. Asmar, *Partial Differential Equations with Fourier Series and Boundary Value Problems 2nd Edition*, 2004

[2] Wikipedia