A Comparison of Pointwise and Uniform Conv. of Seqs. of Functions

A Comparison of Pointwise and Uniform Convergence of Sequences of Functions

Recall from the Pointwise Convergence of Sequences of Functions page that a sequence of functions $(f_n(x))_{n=1}^{\infty}$ with common domain X is said to be pointwise convergent to a limit function $f(x)$ if for all $x \in X$ and for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then:

$$\mid f_n(x) - f(x) \mid < \epsilon \quad (1)$$

Also recall from the Uniform Convergence of Sequences of Functions page that a sequence of functions $(f_n(x))_{n=1}^{\infty}$ with common domain X is said to be uniformly convergent to a limit function $f(x)$ if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ we have that for all $x \in X$ that then:

$$\mid f_n(x) - f(x) \mid < \epsilon \quad (2)$$

At first glance it may seem as though the definitions of pointwise and uniform convergence of a sequence of functions are the same, however, it is extremely important to note the differences between the definitions and understand them thoroughly.

The best way to understand the difference between these two definitions is this think of the definitions as a "game" process of sorts. For pointwise convergence, think of it this way:

- You select an $x \in X$, and a friend gives you an $\epsilon > 0$.
- You then find an $N \in \mathbb{N}$ such that if $n \geq N$ we have that:

$$\mid f_n(x) - f(x) \mid < \epsilon \quad (3)$$

- If you succeed for each $\epsilon > 0$, then your friend asks you to choose another $x \in X$. You do, and once again, you show that for any $\epsilon > 0$ you can find such an N to satisfy the inequality above.
- If you can convince you friend that for every $x \in X$ and for any $\epsilon > 0$ you can find such an $N \in \mathbb{N}$ then you will have shown that the sequence of functions $(f_n(x))_{n=1}^{\infty}$ converges pointwise to the limit function $f(x)$. In essence, the sequence of functions $(f_n(x))_{n=1}^{\infty}$ converges pointwise to the limit function $f(x)$ if the sequence of numbers $(f_n(x_0))_{n=1}^{\infty}$ converges to $f(x_0)$ for each $x_0 \in X$.

Now, for uniform convergence, think of it this way:

- Your friend gives you an $\epsilon > 0$.
- You must then find an $N \in \mathbb{N}$ such that if $n \geq N$ then for EVERY $x \in X$ we have that:

$$\mid f_n(x) - f(x) \mid < \epsilon \quad (4)$$

- If you succeed for each $\epsilon > 0$ then you will have shown that the sequence of functions $(f_n(x))_{n=1}^{\infty}$ converges uniformly to the limit function $f(x)$. In essence, the sequence of functions $(f_n(x))_{n=1}^{\infty}$ converges uniformly to the limit function $f(x)$ if the sequence of numbers $(f_n(x_0))_{n=1}^{\infty}$ converge to $f(x_0)$ for each $x_0 \in X$ at a somewhat similar/uniform rate.