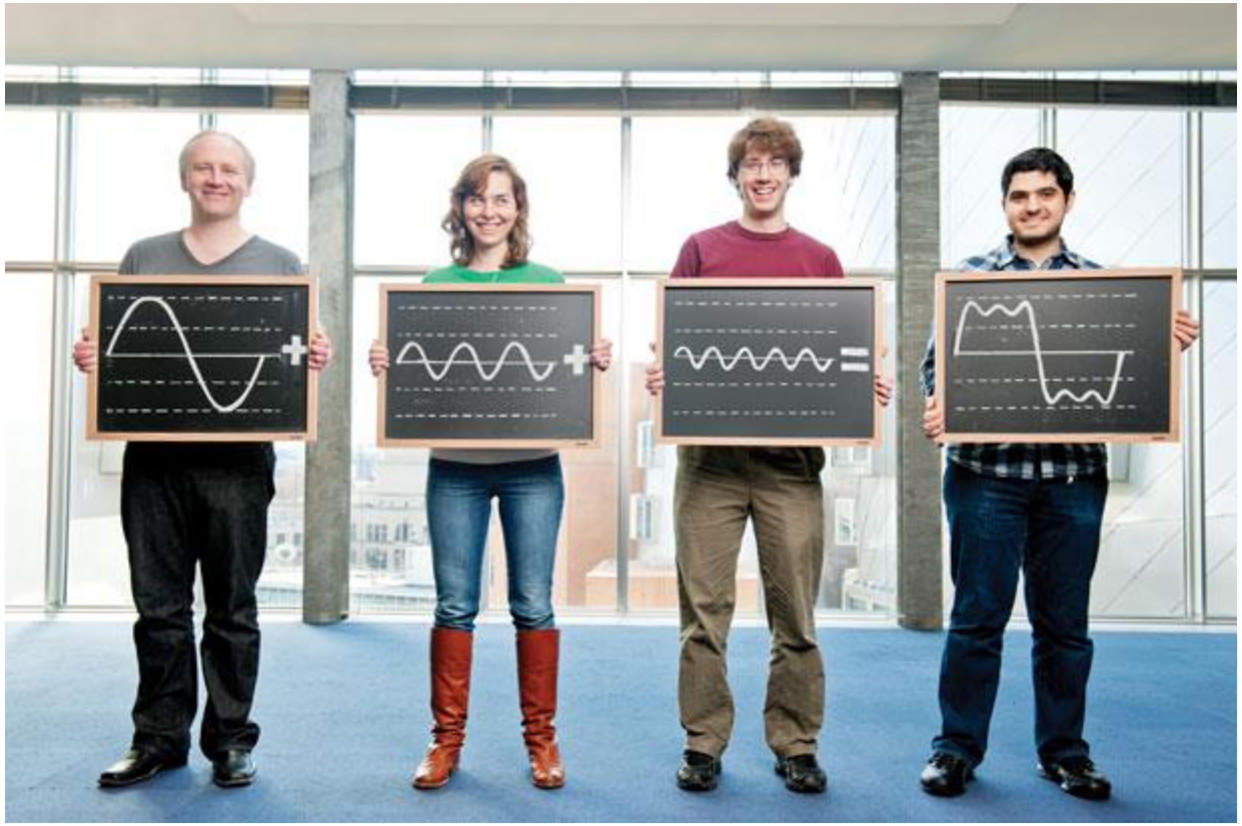
# Fourier Analysis

Instructor: Xiang Li
Teaching Assistant: Lingjun Zhao



# Class Information

**Lecture** (14): Monday (月曜日), Thursday (木曜日)

**Grades**: 10% Attendance (>2/3)

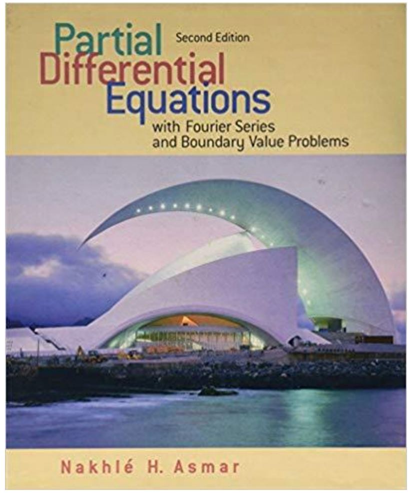
20% Assignment

70% Examination

**Office hours**: Afternoon, Monday and Thursday (Office: #247C )

**Textbook**: **Partial Differential Equations with Fourier Series and Boundary Value Problems** $2^{nd}/3^{rd}$ ,

**Nakhlé H. Asmar, University of Missouri, USA**



# Prerequisites

M-3 **Calculus I** or M-4 **Calculus II**,

M-1 **Linear algebra** or M-2 **Linear algebra II**

# Important related courses:

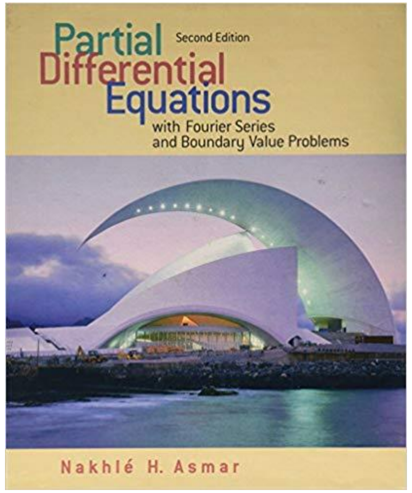
M-6 **Complex analysis**,

A-3 **Image processing**,

A-8 **Digital signal processing**

# What we will cover

## Full syllabus on course website




Chapter 2

1: Part1. Fourier series expansion (Orthogonal system of the function space)
2: Part1. Fourier series expansion (Fourier series of trigonometric functions)
3: Part1. Fourier series expansion (Exercise)
4: Part2. Properties of Fourier series (Convergence condition of Fourier series)
5: Part2. Properties of Fourier series (Parseval's theorem, Weierstrass' theorem)
6: Part2. Properties of Fourier series (Exercise)

Chapter 7

7: Part3. Fourier integral (Introduction from Fourier series, Fourier transform)
8: Part3. Fourier integral (Parseval's theorem, convolution)
9: Part3. Fourier integral (Exercise)

Chapter 8

10: Part4. Laplace transform (Introduction from Fourier transform)
11: Part4. Laplace transform (Ordinary differential equations of constant coefficients)
12: Part4. Laplace transform (Exercise)

Chapter 10

13: Part5. Discrete Fourier transform (Introduction from Fourier series)
14: Part5. Discrete Fourier transform (FFT(Fast Fourier Transform))

Fourier Series Expansion

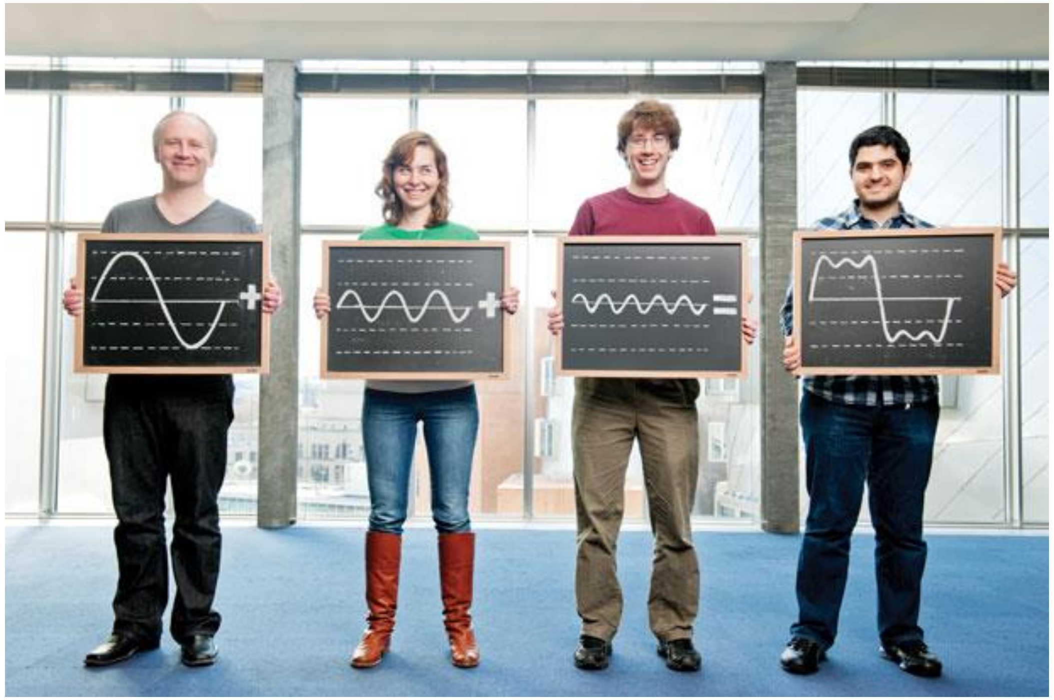
Properties of Fourier Series

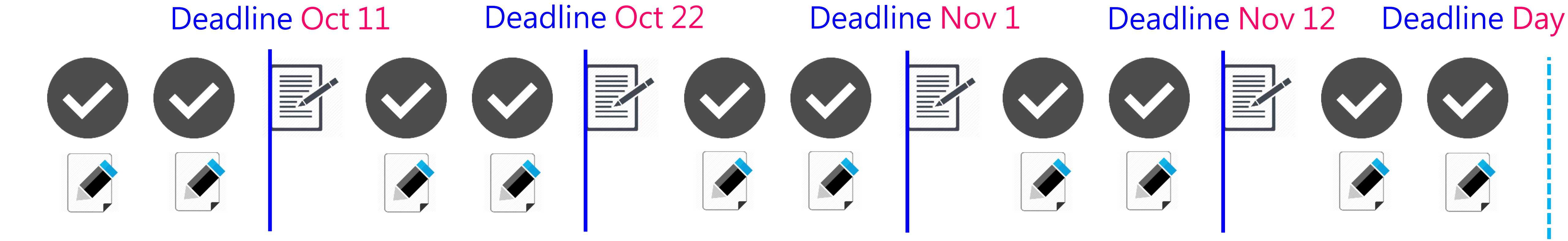
Fourier Integral

Laplace Transform

Discrete Fourier Transform

# Fourier Analysis






# Assignment (10 )

**NOTICE**: The Deadline is **BEFORE** the EXERCISE CLASS beginning.
Day 3, 6, 9, 12 and an additional day of this course.

2018年10月

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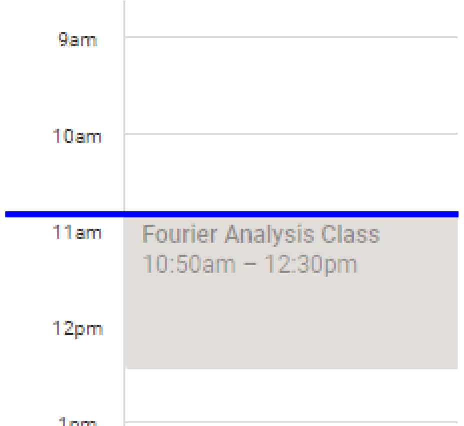
Deadline Hour




2018年11月

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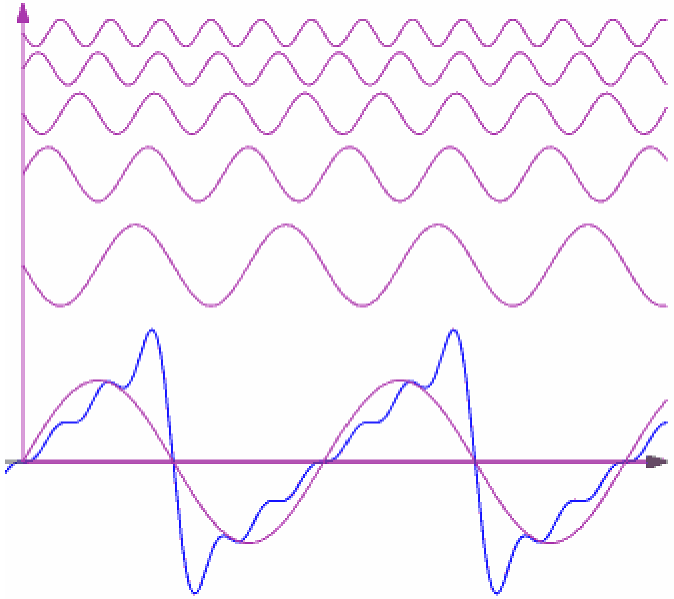
# Fourier Analysis

# Lecture 1

# Orthogonal System and Fourier Series



# What you will learn in Lecture 1

I. Introduction of Fourier Analysis

II. Periodic Functions

III. Piecewise Continuous and Piecewise Smooth Functions

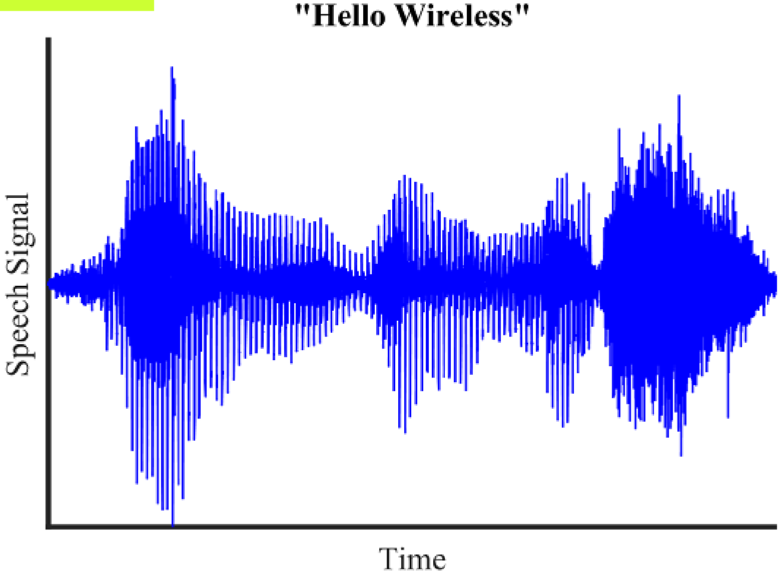
IV. The Trigonometric System and Orthogonality

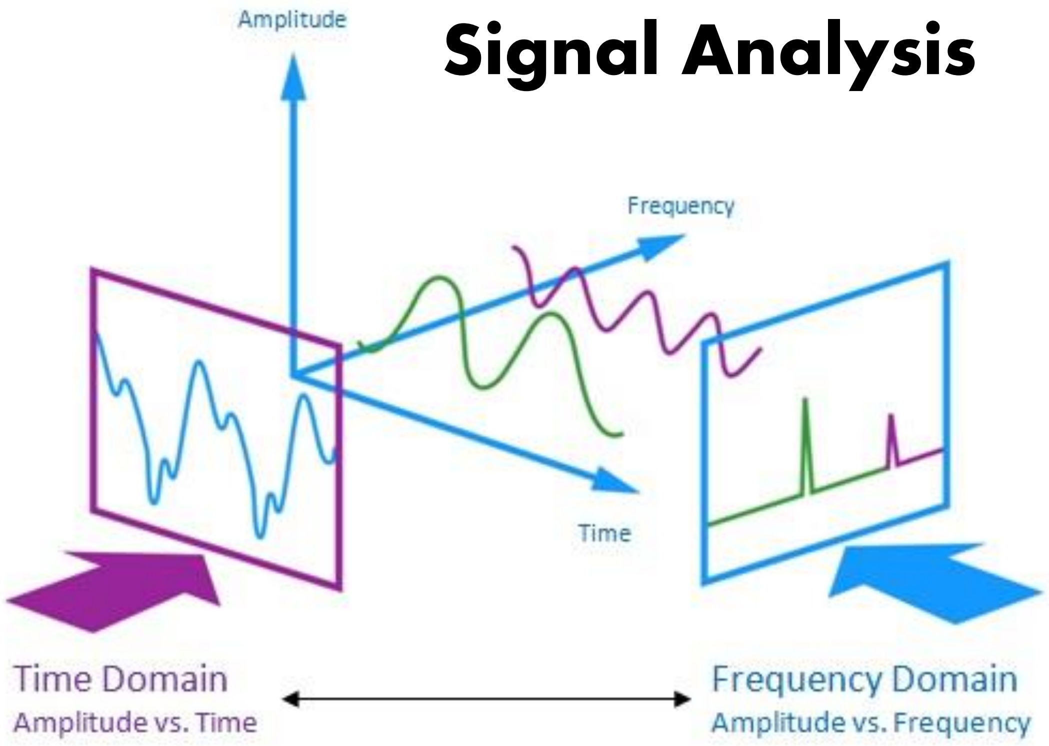
V. Fourier Series

# 1.1 Introduction of Fourier Analysis

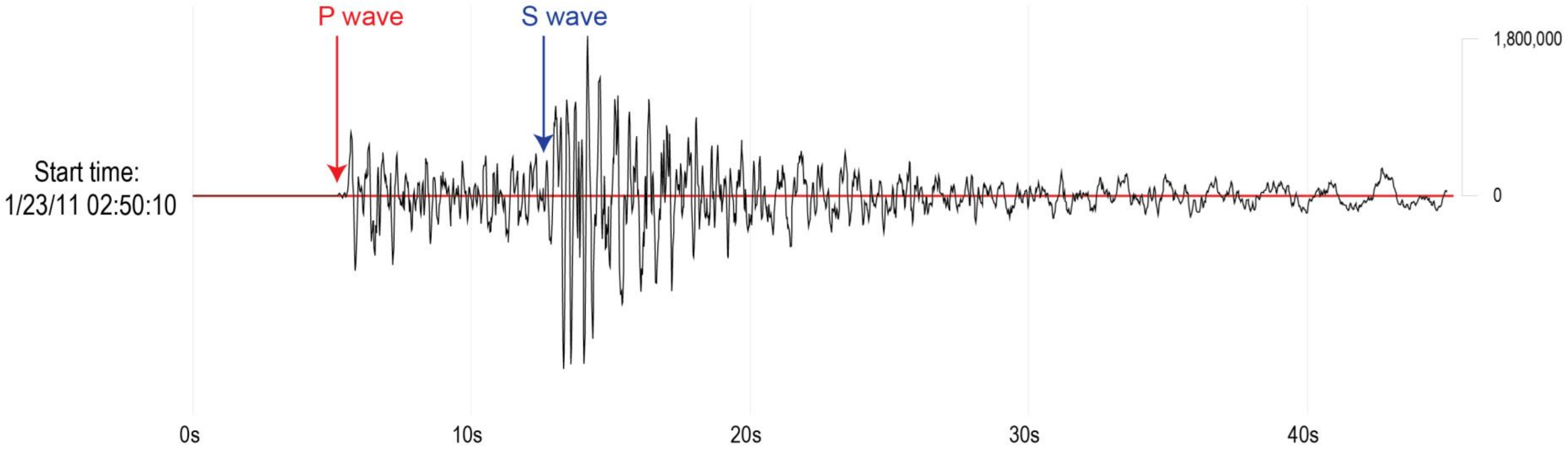
# 1.1 Introduction of Fourier Analysis

## Signals in real-world







# What is Signal?


"A signal is a source of information, generally a physical quantity, which varies with respect to time, space, temperature or like any other independent variable."

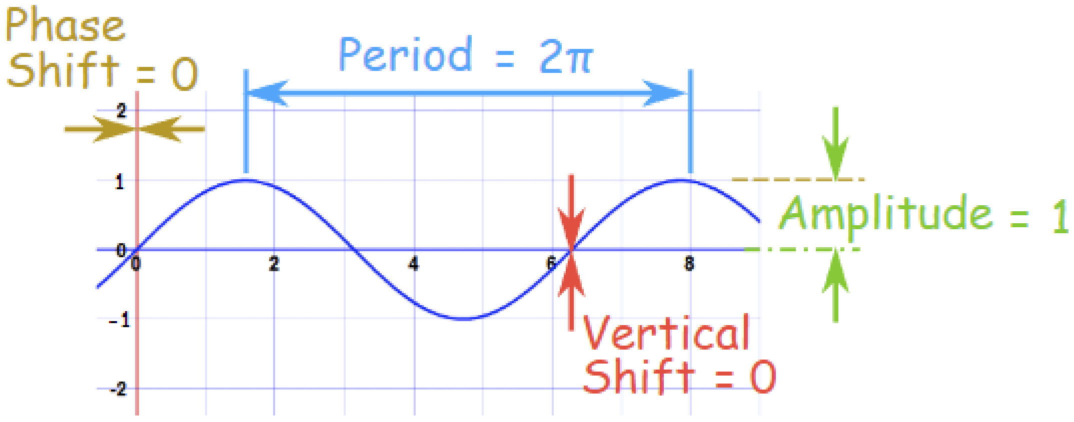
# 1.1 Introduction of Fourier Analysis

signal $y = A\sin(Bx + C) + D$

- amplitude is $\boldsymbol{A}$
- period is $\boldsymbol{2\pi/B}$
- phase shift is $\boldsymbol{-C/B}$
- vertical shift is $\boldsymbol{D}$

Example I: $\sin(x)$

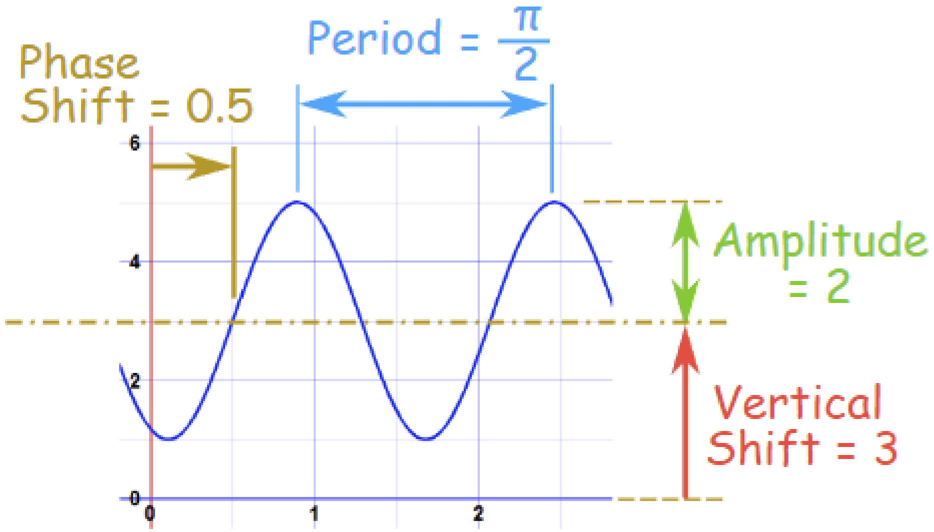
- amplitude $\boldsymbol{A = 1}$
- period $\boldsymbol{2\pi/B = 2\pi}$
- phase shift $\boldsymbol{-C/B = -(0)/1 = 0}$
- vertical shift $\boldsymbol{D = 0}$




https://www.mathsisfun.com/algebra/amplitude-period-frequency-phase-shift.html

Example II: $2\sin(4x - 2) + 3$

- amplitude $\boldsymbol{A = 2}$
- period $\boldsymbol{2\pi/B = 2\pi/4 = \pi/2}$
- phase shift $\boldsymbol{-C/B = -(-2)/4 = 1/2 = 0.5}$
- vertical shift $\boldsymbol{D = 3}$



## How to Represent Signals?

- Taylor series represents any function using polynomials.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

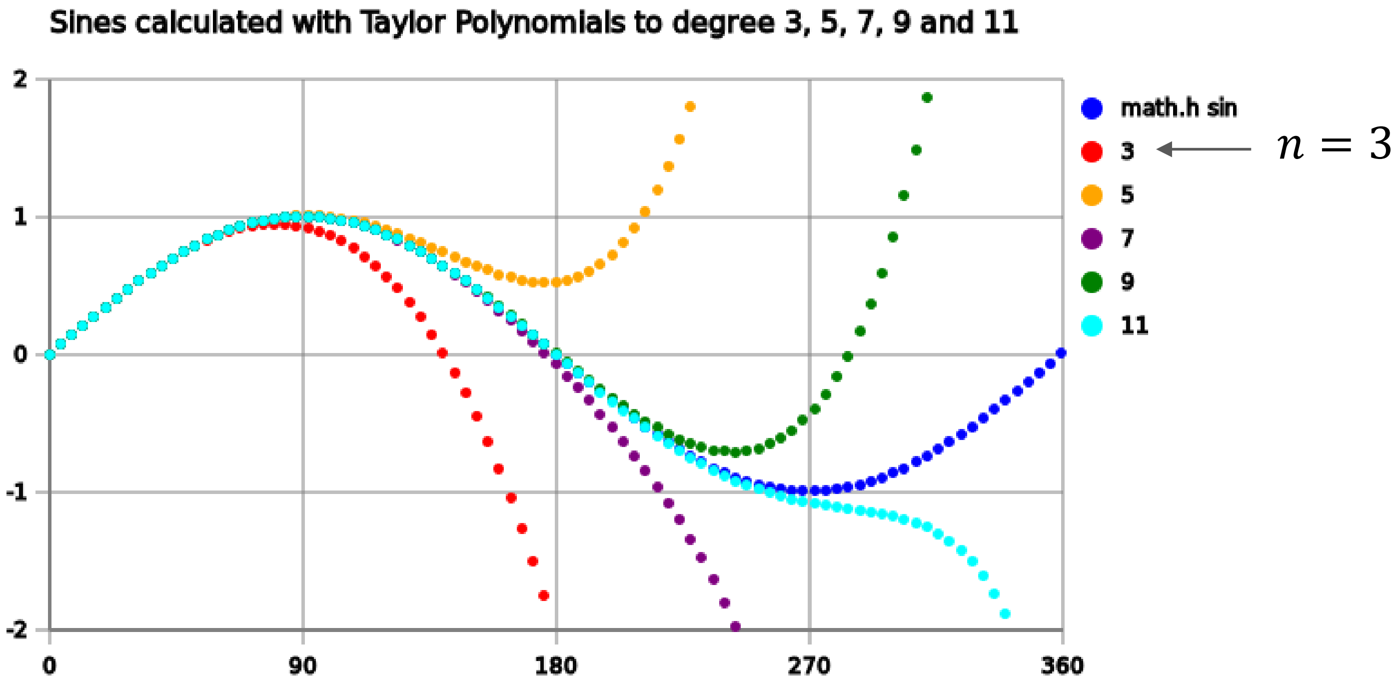
The series is called the **Taylor series of the function** $f$ at $a$ (or **about** $a$ or **centered at** $a$)

James Stewart, Calculus, 6th Edition, 2007

Polynomials are not the best - unstable and not very physically meaningful.

# 1.1 Introduction of Fourier Analysis

## How to Represent Signals?




**Taylor series of the function $f$ at $a$**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

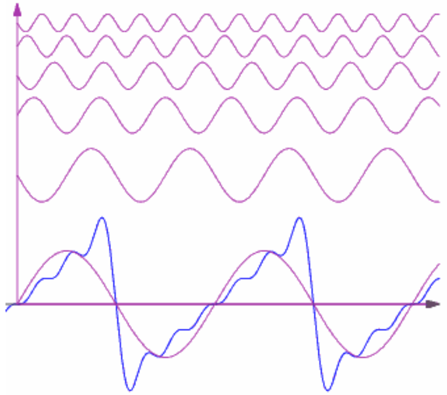
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

# 1.1 Introduction of Fourier Analysis

## How to Represent Signals?

**Joseph Fourier** had an amazing idea (1807):

"**Any** periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies. "

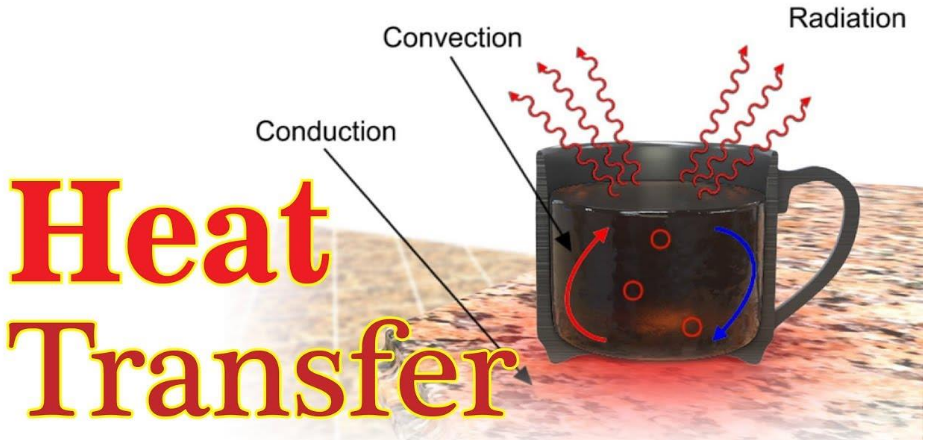


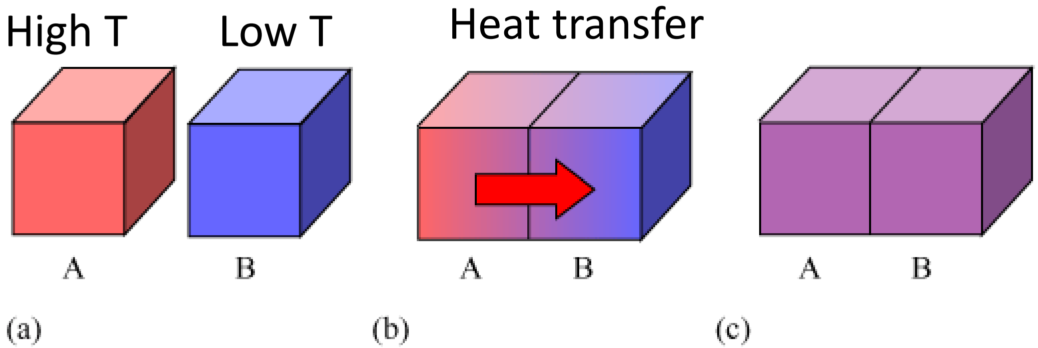

Joseph Fourier (1768-1830)

# 1.1 Introduction of Fourier Analysis

## How to Represent Signals?

Joseph Fourier (1768-1830)




**Partial Differential Equation**: the **Heat Equation**

$$\text{Solve } u(x,t): \quad k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0$$

$$\text{subject to}: \quad u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \qquad \text{Boundary condition}$$

$$u(x,0) = f(x), \quad 0 < x < L \qquad \text{Initial condition}$$

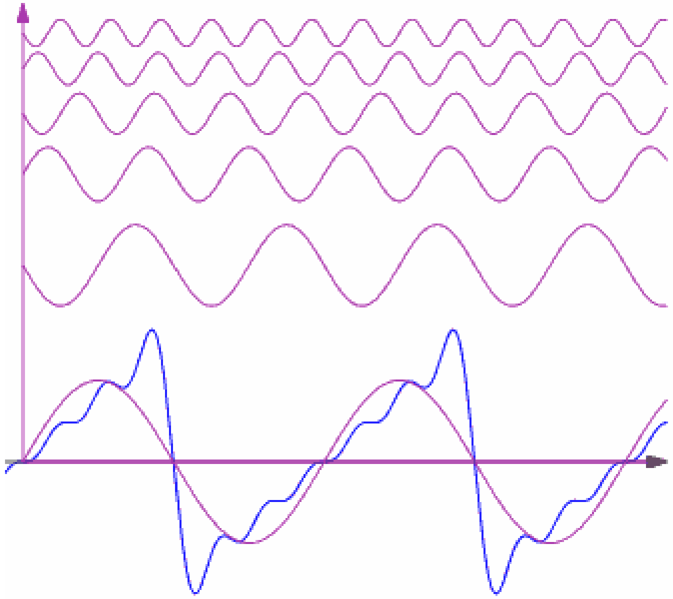
Read more:

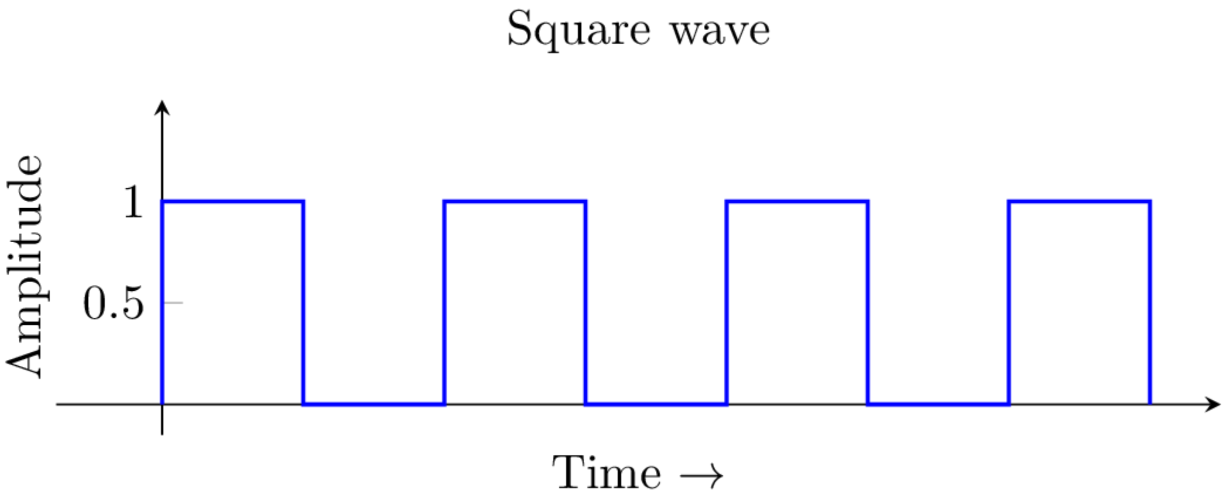
1. I-HSIANG WANG, http://homepage.ntu.edu.tw/~ihwang/Teaching/Fall13/Slides/DE_Lecture_13_handout_v3.pdf
2. Chapter 1, Textbook

# 1.1 Introduction of Fourier Analysis

## How to Represent Signals?

Fourier concludes that
an arbitrary wave can be represented as a sum of an infinite number of weighted sinusoids, i.e., sine and cosine waves.



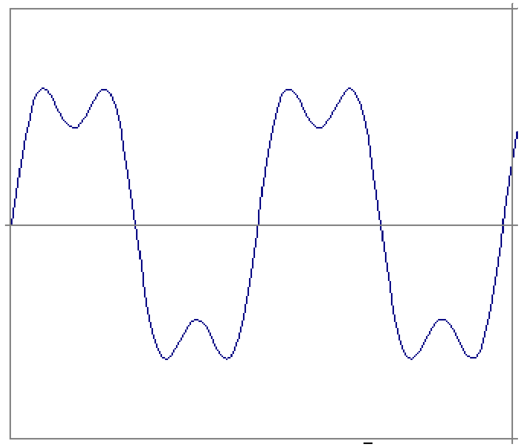


# 1.1 Introduction of Fourier Analysis

## Time and Frequency

Example : $$g(t) = \sin(2\pi f t) + \frac{1}{3}\sin(2\pi(3f)t)$$

signal



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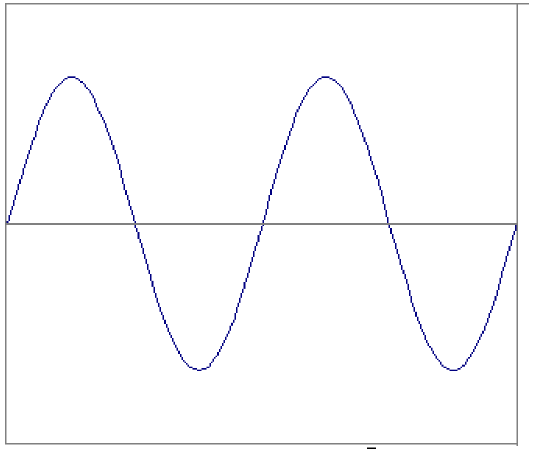

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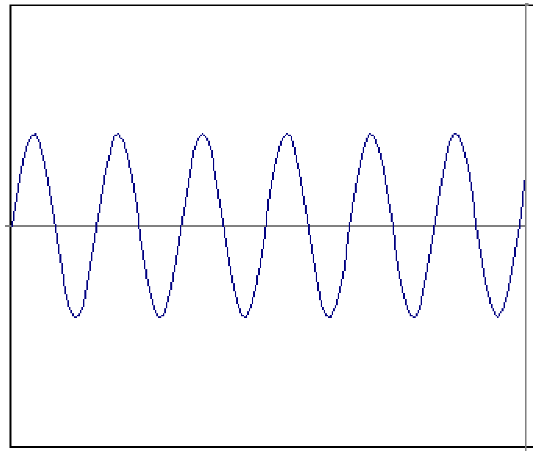
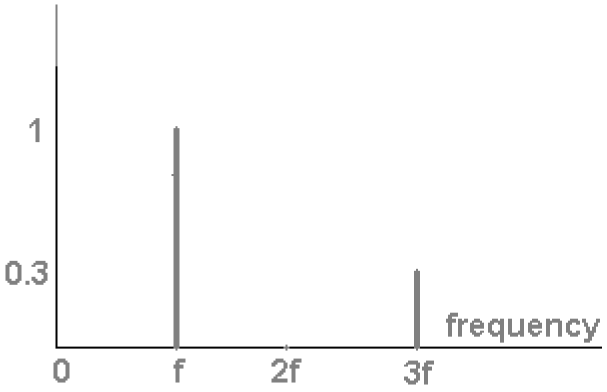
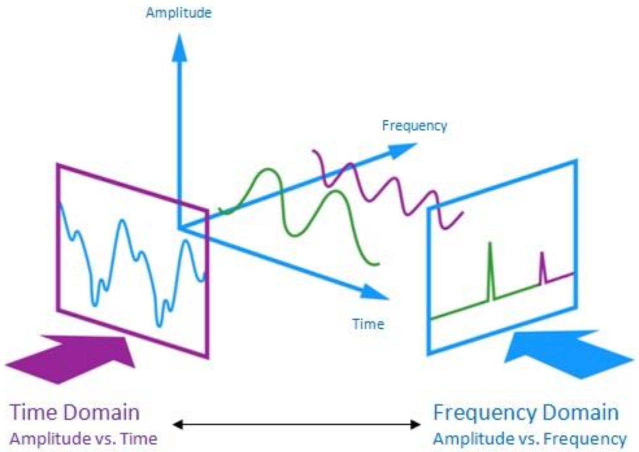


Frequency Spectra

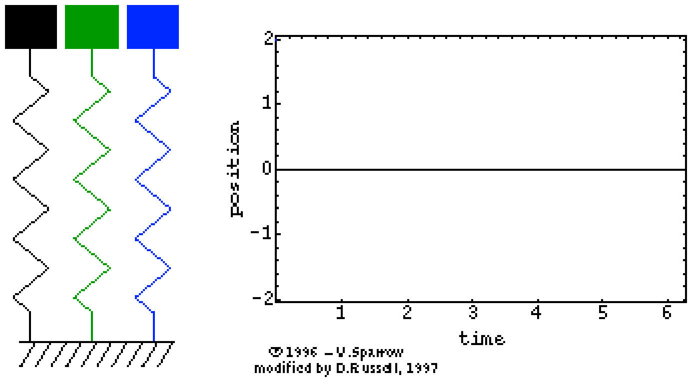


# 1.1 Introduction of Fourier Analysis

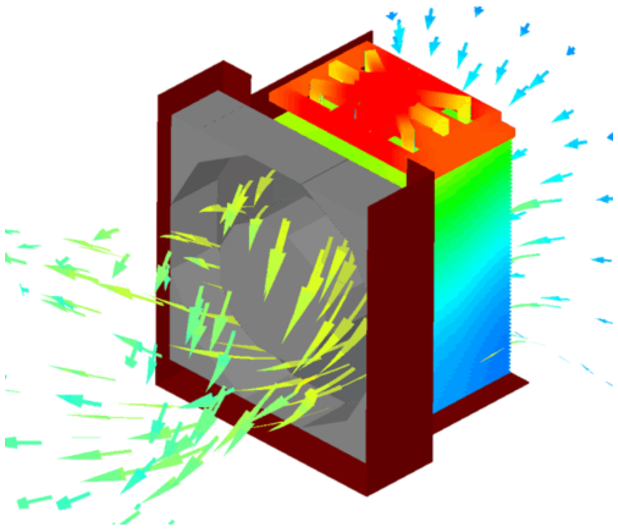
**Fourier series** are indeed the most suitable expansions for solving certain classical problems in **applied mathematics**.

They are fundamental to the **important physical phenomena**, such as

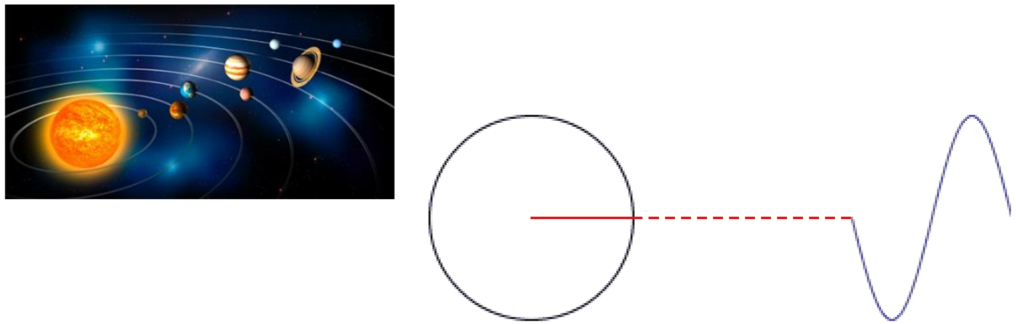



mechanical vibrations

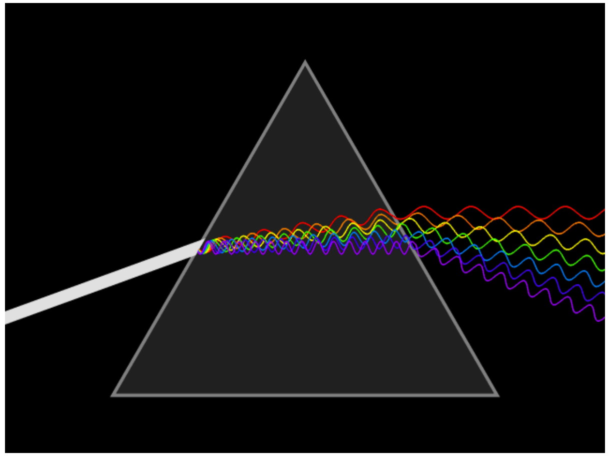
acoustic vibrations



heat transfer



planetary motion



optics

# 1.1 Introduction of Fourier Analysis

Fourier **Analysis** Fourier **Synthesis**

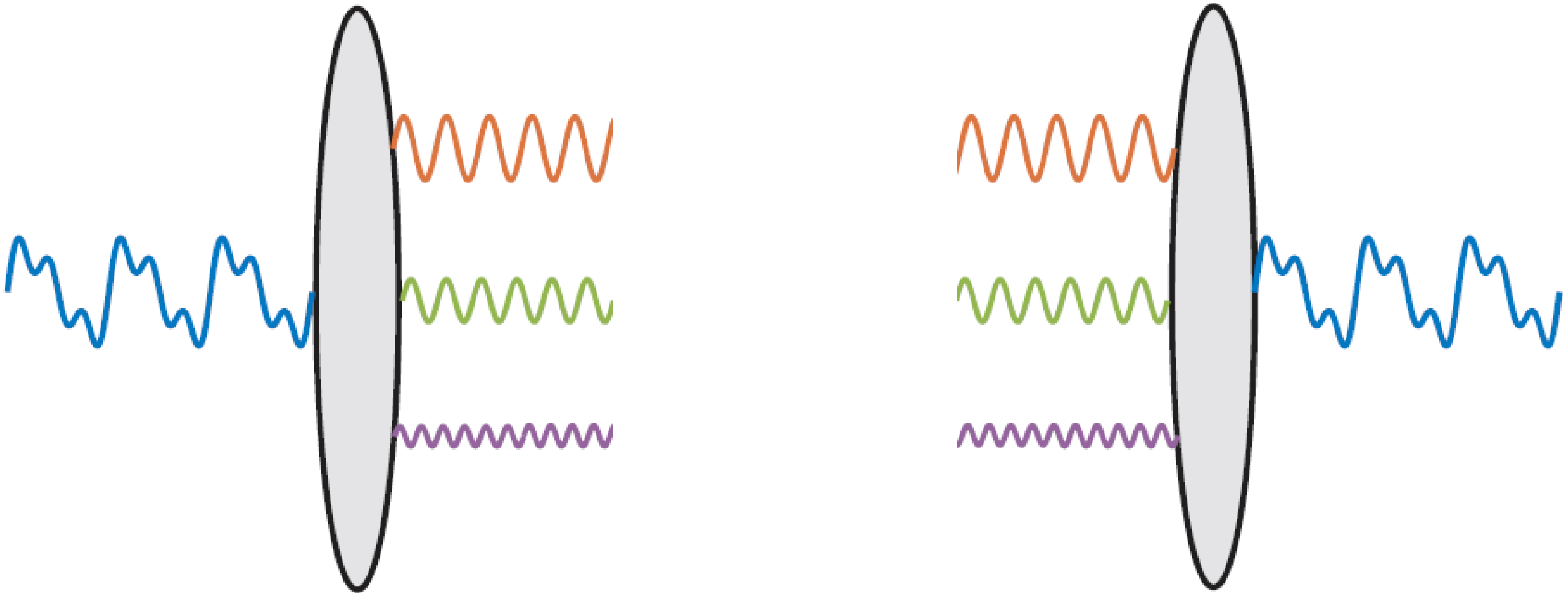


**Figure**: Fourier analysis is used to understand composite waves.

(a) Analysis: breaking a given signal into sine and cosine components

(b) Synthesis: adding certain sine and cosine to create a desired signal.

Charan Langton, Victor Levin, The Intuitive Guide to Fourier Analysis and Spectral Estimation, 2016

# 1.2 Periodic Functions

# 1.2 Periodic Functions

## Periodic Functions

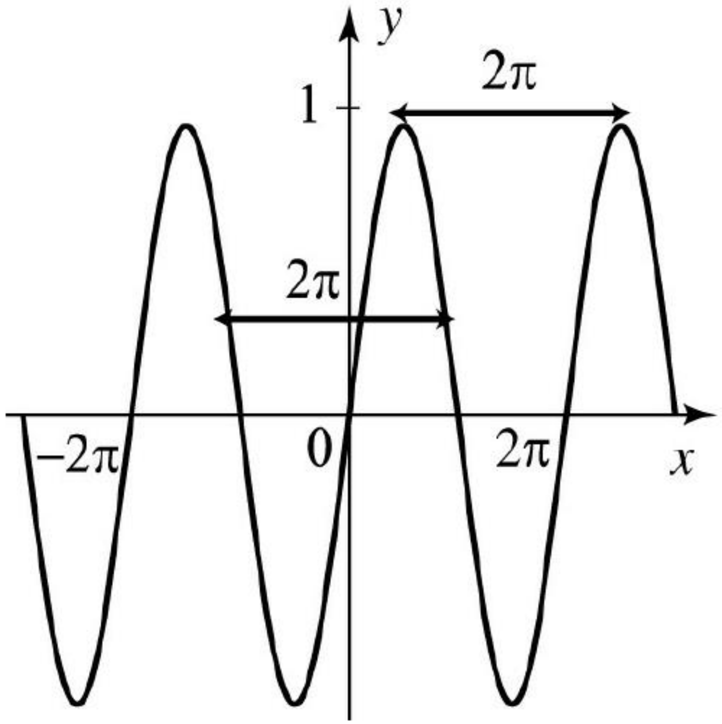



**Figure 1** Graph of $\sin x$

$\sin x$ **repeat** every $2\pi$ units,
its graph is obtained **by repeating** the portion over any interval of length $2\pi$.

This *periodicity* is expressed by the identity

$$\sin x = \sin(x + 2\pi) \quad \text{for all } x$$

# 1.2 Periodic Functions

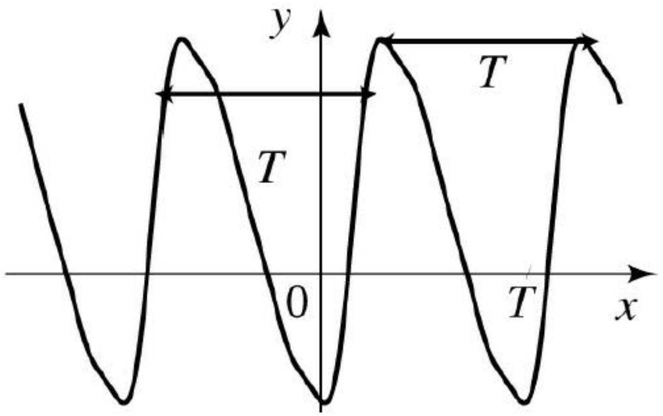


**Figure 2** A $T$-periodic function

In general, a function $f$ satisfying the identity

(1) $$f(x) = f(x+T) \quad \text{for all } x$$

where $T > 0$, is called **periodic**, or more specifically, $T$-**periodic**

The number $\boldsymbol{T}$ is called a **period** of $f$.

If $f$ is nonconstant, we define the **fundamental period**, or simply, the period of $f$ to be the smallest positive number $\boldsymbol{T}$ for which (1) holds.

# 1.2 Periodic Functions

$$f(x) = f(x+T) = f(x+2T) = \cdots = f(x+nT)$$

Hence **if** $T$ **is a period**, **then** $nT$ **is also a period for any integer** $n > 0$. In the case of the sine function, this amounts to saying that $2\pi, 4\pi, 6\pi, \ldots$ are all periods of $\sin x$, but only $2\pi$ is the fundamental period

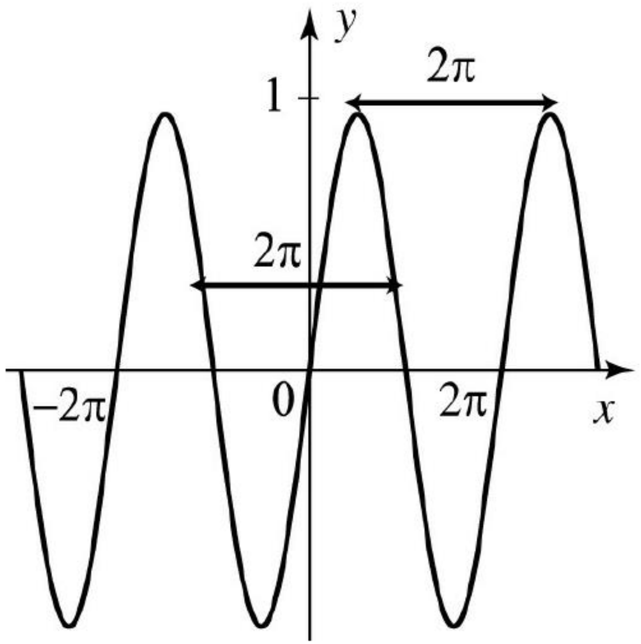



**Figure 1** Graph of $\sin x$

# **EXAMPLE 1** Describing a periodic function

Describe the $2$-periodic function $f$ in Figure 3 in two different ways:

(a) by considering its values on the interval $0 \le x < 2$;
(b) by considering its values on the interval $-1 \le x < 1$.

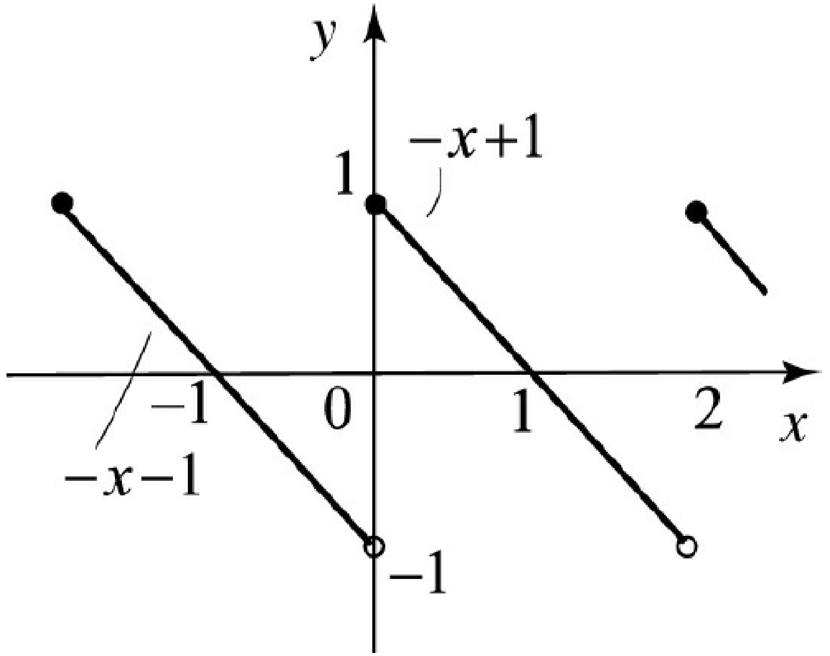



**Figure 3** A 2-periodic function

# 1.2 Periodic Functions

**Solution**

(a) On the interval $0 \le x < 2$ the graph is a portion of the straight line $y = -x + 1$. Thus

$$f(x) = -x + 1 \quad \text{if } 0 \le x < 2$$

Now the relation $f(x + 2) = f(x)$ describes $f$ for all other values of $x$.

(b) On the interval $-1 \le x < 1$, the graph consists of two straight lines (Figure 3). We have

$$f(x) = \begin{cases} -x - 1 & if \ -1 \le x < 0 \\ -x + 1 & if \ 0 \le x < 1 \end{cases}$$

As in part (a), the relation $f(x + 2) = f(x)$ describes $f$ for all values of $x$ outside the interval $[-1, 1)$.

Although the formulas in Example 1(a) and (b) are different, they describe the same periodic function.

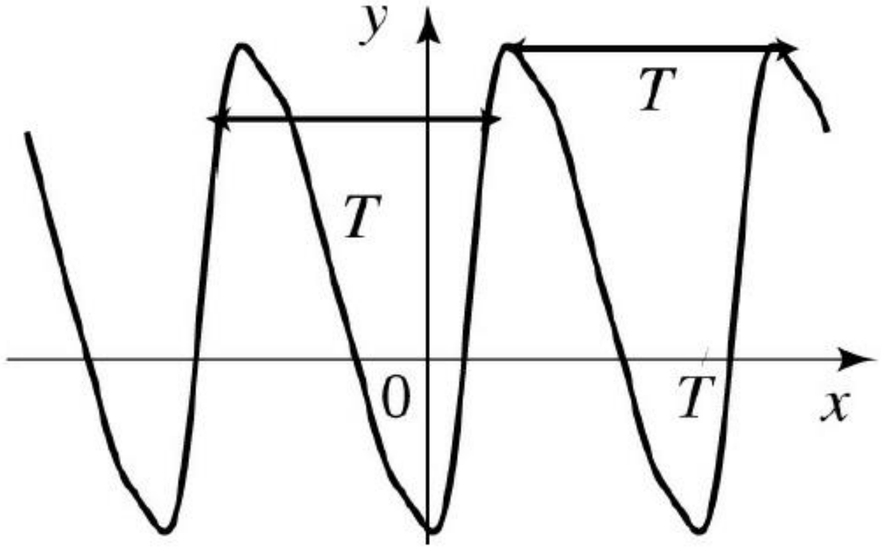



**Figure 2** A $T$-periodic function

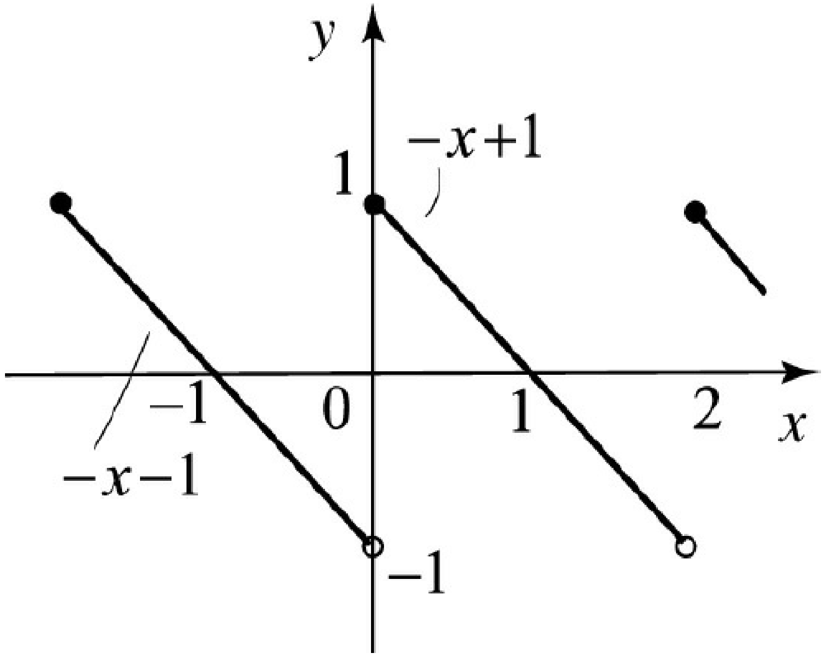



**Figure 3** A 2-periodic function

continuous

**Discontinuous at some certain points**

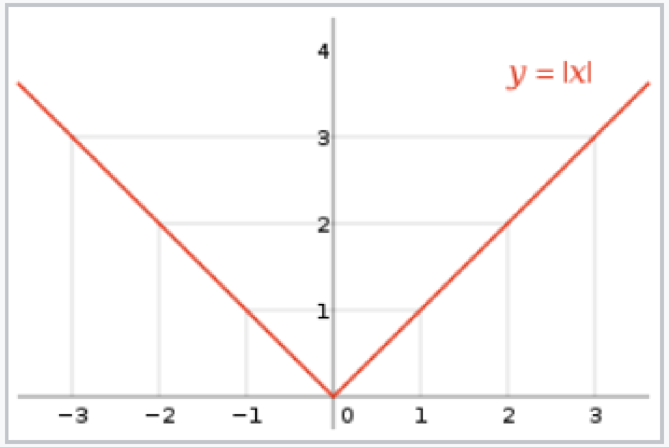
# Piecewise Continuous and Piecewise Smooth Functions

## 1.2 Periodic Functions

A **piecewise-defined function** is defined by **multiple sub-functions**, **each sub-function** applying to **a certain interval** of the main function's domain, a sub-domain

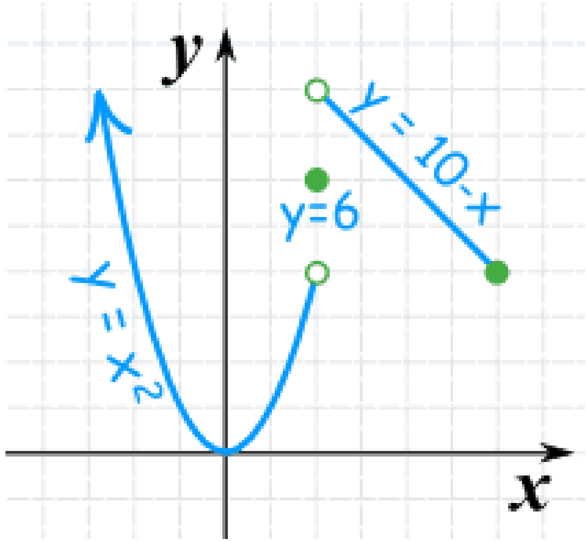
$$|x| = \begin{cases} -x & if\ x < 0 \\ x & if\ x \geq 0 \end{cases}$$

$$f(x) = \begin{cases} x^2 & if\ x < 2 \\ 6 & if\ x = 2 \\ 10 - x & if\ x > 2\ and\ x \leq 6 \end{cases}$$




Graph of the absolute value function, $y = |x|$.

From wikipedia




https://www.mathsisfun.com/sets/functions-piecewise.html

# Discontinuity

Consider the function $f(x)$ in Figure 3. This function is not continuous at $x = 0, \pm 2, \pm 4, \dots$.
Take a point of discontinuity, say $x = 0$. The limit of the function from the left is $-1$, while the limit from the right is $1$.
Symbolically, this is denoted by

$$f(0-) = \lim_{x \to 0^-} f(x) = -1 \quad \text{and} \quad f(0+) = \lim_{x \to 0^+} f(x) = 1$$

In general, we write

$$f(c-) = \lim_{x \to c^-} f(x)$$

$$f(c+) = \lim_{x \to c^+} f(x)$$

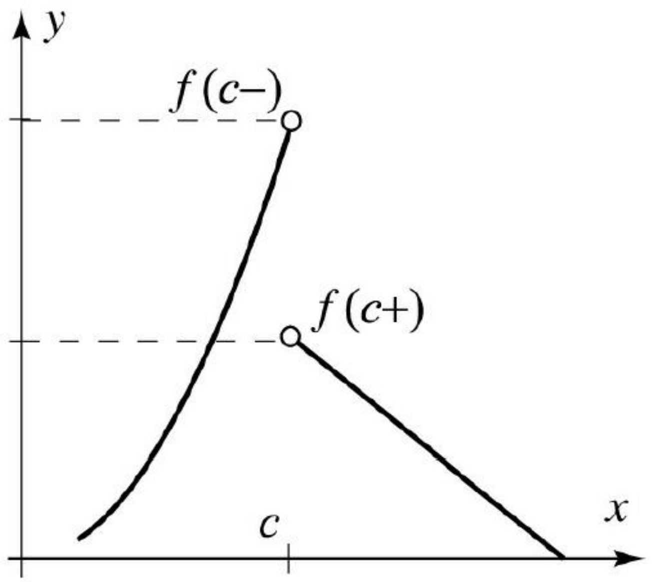



**Figure 4** Left and right limit

# Piecewise Continuous Functions

A function $f$ is said to be **piecewise continuous** on the interval $[a, b]$ if $f(a+)$ and $f(b-)$ exist, and $f$ is defined and continuous on $(a, b)$ except at a finite number of points in $(a, b)$ where the left and right limits exist.

# Piecewise Continuous Functions

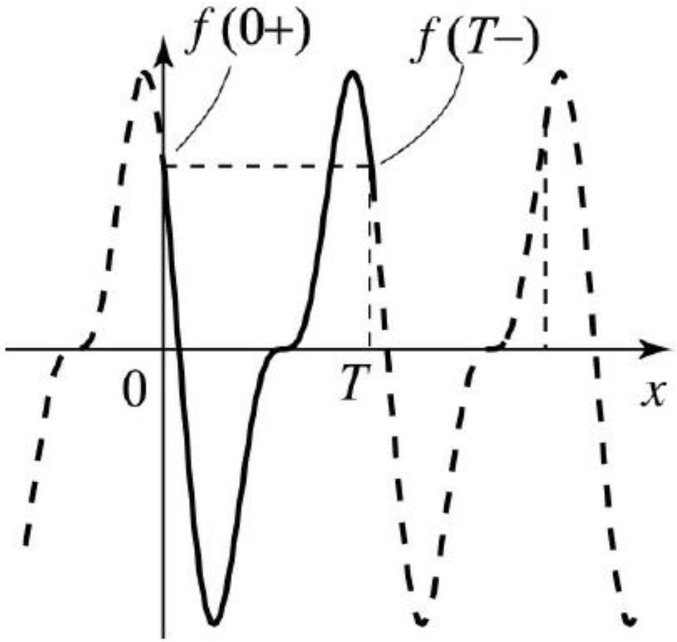



**Figure 5** A continuous $T$-periodic function

At endpoints of the periodic function,

if $f$ is $T$-periodic and continuous, then necessarily $f(0+) = f(T-)$

# Piecewise Smooth Functions

A function $f$, defined on the interval $[a, b]$, is said to be **piecewise smooth** if $f$ and $f'$ are **piecewise continuous** on $[a, b]$.

Thus $f$ is **piecewise smooth** if $f$ is piecewise continuous on $[a, b]$, $f'$ exists and is continuous in $(a, b)$ except possibly at finitely many points $c$ where the one-sided limits $\lim_{x \to c^-} f'(x)$ and $\lim_{x \to c^+} f'(x)$ exist.

Furthermore, $\lim_{x \to a^+} f'(x)$ and $\lim_{x \to b^-} f'(x)$ exist.

**THEOREM 1**: Integral over one period

Suppose that $f$ is piecewise continuous and $T$-periodic. Then, for any real number $a$, we have

$$\int_0^T f(x)dx = \int_a^{a+T} f(x)dx$$

## **THEOREM 1**: Integral over one period

**Proof**

(1) We have

$$\begin{aligned}\int_{nT}^{(n+1)T} f(x)\,dx &= \int_0^T f(s+nT)\,ds \quad (\text{let } x = s+nT,\ dx = ds)\\ &= \int_0^T f(s)\,ds \quad (\text{because } f \text{ is } T\text{-periodic})\\ &= \int_0^T f(x)\,dx.\end{aligned}$$

(2)

$$\begin{aligned}\int_{(n+1)T}^{a+T} f(x)\,dx &= \int_{nT}^{a} f(s+T)\,ds \quad (\text{let } x = s+T,\ dx = ds)\\ &= \int_{nT}^{a} f(s)\,ds \quad (\text{because } f \text{ is } T\text{-periodic})\\ &= \int_{nT}^{a} f(x)\,dx.\end{aligned}$$

(3)

$$\begin{aligned}\int_a^{a+T} f(x)\,dx &= \int_a^{(n+1)T} f(x)\,dx + \int_{(n+1)T}^{a+T} f(x)\,dx\\ &= \int_a^{(n+1)T} f(x)\,dx + \int_{nT}^{a} f(x)\,dx \quad (\text{by (b)})\\ &= \int_{nT}^{(n+1)T} f(x)\,dx = \int_0^T f(x)\,dx \quad (\text{by (a)}).\end{aligned}$$

■

**EXAMPLE 2** Integrating periodic functions

Let $f$ be the 2-periodic function in Example 1. Use **Theorem 1** to compute

(a) $\int_{-1}^{1} f^2(x)\,dx$

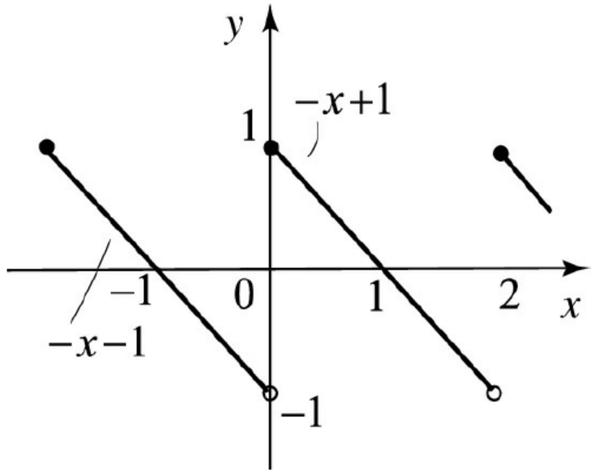



**Figure 3** A 2-periodic function

**Solution**

(a) Observe that $f2(x)$ is also 2-periodic. Thus, by Theorem 1, to compute the integral in (a) we may choose any interval of length 2. Since on the interval $(0, 2)$ the function $f(x)$ is given by a single formula, we choose to work on this interval, and, using the formula from Example 1(a), we find

$$\int_{-1}^{1} f^2(x)\,dx = \int_{0}^{2} f^2(x)\,dx = \int_{0}^{2} (-x+1)^2\,dx = -\frac{1}{3}(-x+1)^3\Big|_0^2 = \frac{2}{3}$$

# 1.2 Periodic Functions

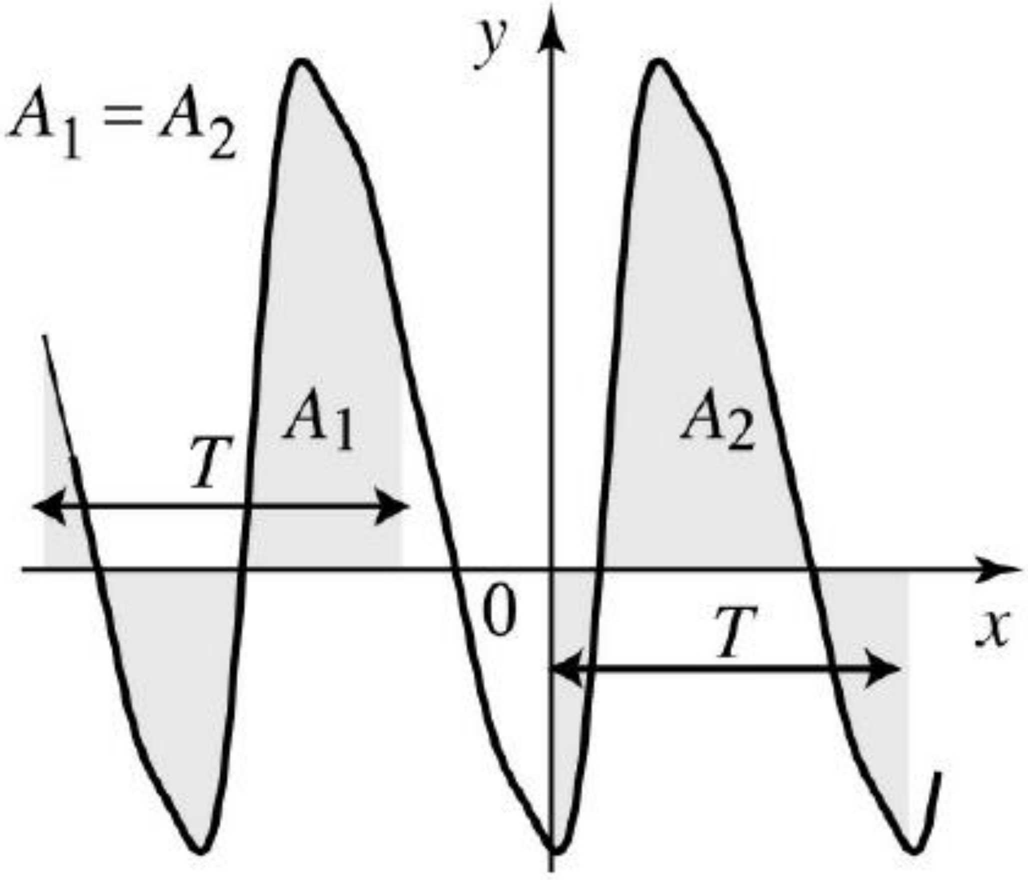



**Figure 6** Areas over one period

# 1.3 The Trigonometric System and Orthogonality

# 1.3 The Trigonometric System and Orthogonality

The most important periodic functions are those in the ($2\pi$-periodic) **trigonometric system**

$$1, \cos x\,, \cos 2x\,, \cos 3x\,, \cdots, \cos mx\,, \cdots,$$
$$\sin x\,, \sin 2x\,, \sin 3x\,, \cdots \sin nx\,, \cdots$$

useful property: **orthogonality**

We say that **two functions *f* and *g* are orthogonal** over the interval $[a, b]$ if

$$\int_a^b f(x)g(x)dx = 0$$

# 1.3 The Trigonometric System and Orthogonality

Dot Product | Scalar product | Inner Product

If $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and b $= \langle b_1, b_2, b_3 \rangle$, then the dot product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is the number $\boldsymbol{a} \cdot \boldsymbol{b}$ given by

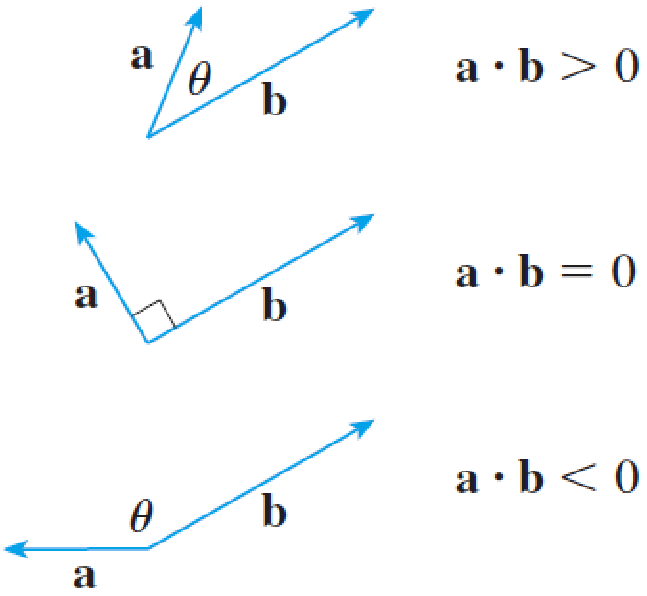
$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

If $\theta$ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

Two vectors and are orthogonal if and only if

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{0}$$




**Note**: Vector Length $||\boldsymbol{a}||$ can be expressed via inner product:

$$||\boldsymbol{a}||^2 = a_1^2 + a_2^2 + \cdots + a_n^2 = \langle \boldsymbol{a}, \boldsymbol{a} \rangle, \quad \text{so} \quad ||\boldsymbol{a}||^2 = \sqrt{\langle \boldsymbol{a}, \boldsymbol{a} \rangle}$$

James Stewart, Calculus, 6th Edition, 2007

A. Kissinger, Matrix Calculations: Inner Products & Orthogonality, Radboud University Nijmegen

# 1.3 The Trigonometric System and Orthogonality

**Definition (Inner Product of Functions)**

The inner product of $f_1(x)$ and $f_2(x)$ on an interval $[a, b]$ is defined as

$$\langle f_1, f_2 \rangle := \int_a^b f_1(x) f_2(x)\, dx$$

Once inner product is defined, we can accordingly define **norm**.

**Definition (Norm of a Function)**

The norm of a function $f(x)$ on an interval $[a, b]$ is

$$||f(x)|| := \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b (f(x))^2\ dx}$$

I-Hsiang Wang, Chapter 11: Fourier Series, Differential Equations, National Taiwan University, 2013

# 1.3 The Trigonometric System and Orthogonality

**Definition (Orthogonal Functions)**

$f_1(x)$ and $f_2(x)$ are **orthogonal** on an interval $[a, b]$ if $\langle f_1, f_2 \rangle = 0$.

**Definition (Orthogonal Set)**

$\{\phi_0(x), \phi_1(x), \cdots\}$ are **orthogonal** on an interval $[a, b]$ if

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)\, dx = 0, \quad m \neq n.$$

I-Hsiang Wang, Chapter 11: Fourier Series, Differential Equations, National Taiwan University, 2013

# 1.3 The Trigonometric System and Orthogonality

**Example (Orthogonal or Not Depends on the Inverval)**

The functions $f_1(x) = x$ and $f_2(x) = x^2$ are orthogonal on the interval $[a, b]$, $a < b$, only if $a = -b$.

**Proof**: When $a < b$,

$$\langle x, x^2 \rangle = \int_a^b x^3 \, dx = \left[ \frac{1}{4} x^4 \right]_a^b = \frac{1}{4} \left( a^4 - b^4 \right) = 0 \iff a + b = 0$$

I-Hsiang Wang, Chapter 11: Fourier Series, Differential Equations, National Taiwan University, 2013

# 1.3 The Trigonometric System and Orthogonality

**Orthogonality properties** of the **trigonometric system** are expressed by

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad m \neq n$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0 \quad \text{for all } m \text{ and } n$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0 \quad m \neq n$$

For the case $n = m$

$$\int_{-\pi}^{\pi} \cos^2 mx \, dx = \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi \quad \text{for all } m \neq 0$$

# 1.3 The Trigonometric System and Orthogonality

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad m \neq n$$

**Proof**

Use a trigonometric identity and write

$$\cos mx \cos nx \, dx = \frac{1}{2}(\cos(m+n)\,x + \cos(m-n)\,x)$$

Since $m \pm n \neq 0$, we get

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \int_{-\pi}^{\pi} \frac{1}{2}(\cos(m+n)\,x + \cos(m-n)\,x)\, dx$$

$$= \frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)\,x + \frac{1}{m-n}\sin(m-n)\,x\right]_{-\pi}^{\pi} = 0 \quad \blacksquare$$

# 1.4 Fourier Analysis

# 1.4 Fourier Analysis

**Fourier series** are special expansions of $2\pi$-periodic functions of the form

$$(1) \qquad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Q: If a function has a Fourier series, how do we compute the coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$?

# Euler Formulas for the Fourier Coefficients

We proceed as Fourier himself did.

We integrate both sides of (1) over the interval $[-\pi, \pi]$

assuming term-by-term integration is justified, and get

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos nx + b_n \sin nx)dx$$

But because

$$\int_{-\pi}^{\pi} \cos nx \, dx = \int_{-\pi}^{\pi} \sin nx \, dx = 0 \quad \text{for } n = 1, 2, \cdots$$

it follows that

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx + 0 = 2\pi a_0 \quad \Rightarrow \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

Note: $a_0$ is the average of $f$ on the interval $[-\pi, \pi]$.

Starting with (1), we **multiply both sides by** $\cos mx\ (m \geq 1)$, integrate term-by-term, use the **orthogonality**

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \overbrace{\int_{-\pi}^{\pi} a_0 \cos mx\,dx}^{=0} + \overbrace{\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n \cos nx \cos mx\,dx}^{=0\ \text{for}\ m \neq n}$$

$$+\overbrace{\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n \sin nx \cos mx\,dx}^{=0}$$

$$= a_m \overbrace{\int_{-\pi}^{\pi} \cos^2 mx\,dx}^{=\pi\ \text{for}\ m=n} = \pi a_m$$

Hence $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad (n = 1, 2, \ldots)$

# 1.4 Fourier Analysis

By a similar procedure, starting with (1), we **multiply both sides by** $\sin mx$ $(m \geq 1)$, integrate term-by-term, use the **orthogonality**

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad (n = 1, 2, \dots)$$

# Euler Formulas for the Fourier Coefficients

Suppose that $f$ has the Fourier series representation

(1) $$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Then the coefficients $a_0$, $a_n$, and $b_n$ are called the Fourier coefficients of $f$ and are given by the following Euler formulas:

(2) $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

(3) $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \quad (n = 1, 2, \ldots)$$

(4) $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \quad (n = 1, 2, \ldots)$$

# Euler Formulas for the Fourier Coefficients

## Alternative Euler Formulas

$$(5) \qquad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$(6) \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \quad (n = 1, 2, \dots)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \quad (n = 1, 2, \dots)$$

# Review for Lecture 1

- Periodic Functions
- Piecewise Continuous Functions
- Piecewise Smooth Functions
- The Trigonometric System and Orthogonality
- Fourier Series

# Exercises

Please Check https://github.com/uoaworks/FourierAnalysisAY2018

Reading: Section 2.1, Textbook

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