Lecture 6

Convolution

What you will learn in Lecture 6

I. The Fourier Transform: Operational Properties

II. Convolution

6.1 The Fourier Transform:

Operational Properties

6.1 The Fourier Transform: Operational Properties

FOURIER TRANSFORM

(1) $$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \qquad (-\infty < \omega < \infty)$$

INVERSE FOURIER TRANSFORM

(2) $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \qquad (-\infty < x < \infty)$$

6.1 The Fourier Transform: Operational Properties

Putting $\omega = 0$ in (1), we find that

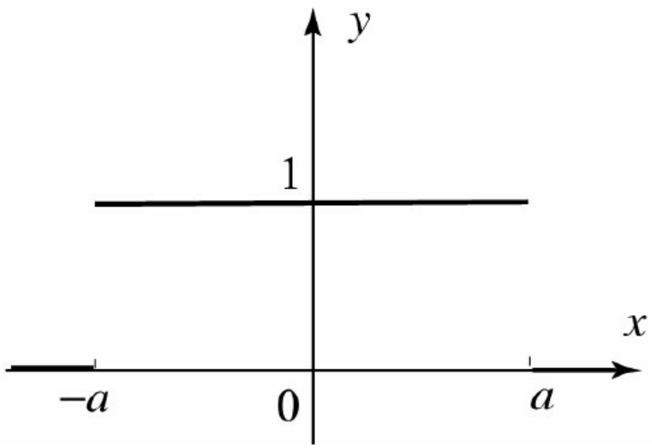
$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, dx$$

Thus the value of the Fourier transform at $\omega = 0$ is equal to the signed area between the graph of $f(x)$ and the x-axis, multiplied by a factor of $\frac{1}{\sqrt{2\pi}}$

EXAMPLE 1 A Fourier transform

(a) Find the Fourier transform of the function in Figure 1, given by

$$f(x) = \begin{cases} 1 & if\ |x| < a \\ 0 & if\ |x| > a \end{cases}$$



What is $\hat{f}(0)$?

(b) Express f as an inverse Fourier transform.

Solution

Tips:

- Use (1) for $\hat{f}\ (\omega)$
- Introduce L'Hôpital's rule $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$
- Discuss $\omega = 0$

Find the complete solution in page 399-400 of the textbook.

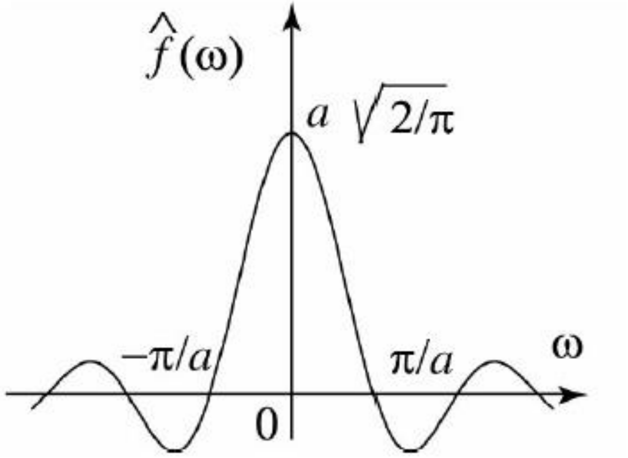


Figure 2 Graph of $\hat{f}$

6.1 The Fourier Transform: Operational Properties

Fourier Transform Symbols

$\mathcal{F}(f)(\omega)$ denotes **Fourier Transform** of f

$\mathcal{F}^{-1}(f)(x)$ denotes **Inverse Fourier Transform** of f

Other symbols: $FT(f)$, $IFT(f)$

Operational Properties

We shall investigate the behavior of the Fourier transform in connection with the common operations on functions: **linear combination, translation and convolution etc.**

Operational Properties

THEOREM 1 LINEARITY

The Fourier transform is a linear operation; that is, for any integrable functions f and g and any real numbers a and b,

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

Proof

$$\mathcal{F}[af(x) + bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{-i\omega x} dx$$

$$= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx$$

$$= a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)]$$

6.1 The Fourier Transform: Operational Properties

Operational Properties

THEOREM 2 Fourier Transforms of Derivatives

(i) Suppose $f(x)$ is piecewise smooth, $f(x)$ and $f'(x)$ are integrable, and $f(x) \to 0$ as $|x| \to \infty$, then

$$\mathcal{F}(f') = i\omega\mathcal{F}(f)$$

(ii) If in addition $f''(x)$ is integrable, and $f'(x)$ is piecewise smooth and tend to 0 as $|x| \to \infty$, then

$$\mathcal{F}(f'') = i\omega\mathcal{F}(f') = -\omega^2\mathcal{F}(f)$$

(iii) In general, if f and $f^{(k)}(x)$ $(k = 1, 2, \ldots, n-1)$ are piecewise smooth and tend to 0 as $|x| \to \infty$, and f and its derivatives of order up to n are integrable, then

$$\mathcal{F}(f^{(n)}) = (i\omega)^n\mathcal{F}(f)$$

Proof Find the complete proof in page 402 of the textbook.

6.1 The Fourier Transform: Operational Properties

Appendix B. Table of Fourier Transform (Page 1069 of the textbook)

$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(\omega)e^{ix\omega}\,d\omega$	$\widehat{f}(\omega)=\mathcal{F}(f)(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}\,dx$
1. $\begin{cases} 1 & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega}$
2. $\begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$	$\frac{i\left(e^{-ib\omega}-e^{ia\omega}\right)}{\sqrt{2\pi}\omega}$
3. $\begin{cases} 1-\frac{\lvert x\rvert}{a} & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases}$ $a>0$	$2\sqrt{\frac{2}{\pi}}\frac{\sin^2(\frac{a\omega}{2})}{a\omega^2}$
4. $\begin{cases} x & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases}$ $a>0$	$i\sqrt{\frac{2}{\pi}}\frac{a\omega\cos(a\omega)-\sin(a\omega)}{\omega^2}$
5. $\begin{cases} \sin x & \text{if } \lvert x\rvert < \pi \\ 0 & \text{if } \lvert x\rvert > \pi \end{cases}$	$i\sqrt{\frac{2}{\pi}}\frac{\sin(\pi\omega)}{\omega^2-1}$

⋮

THEOREM 3 Derivatives of Fourier Series

(i) Suppose $f(x)$ and $xf(x)$ are integrable; then

$$\mathcal{F}\big(xf(x)\big)(\omega) = i\big[\hat{f}\big]'(\omega) = i\frac{d}{d\omega}\mathcal{F}(f)(\omega)$$

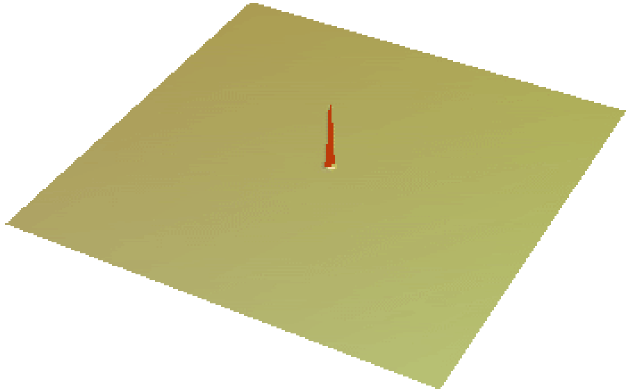
(ii) In general, if $f(x)$ and $x^n f(x)$ are integrable, then

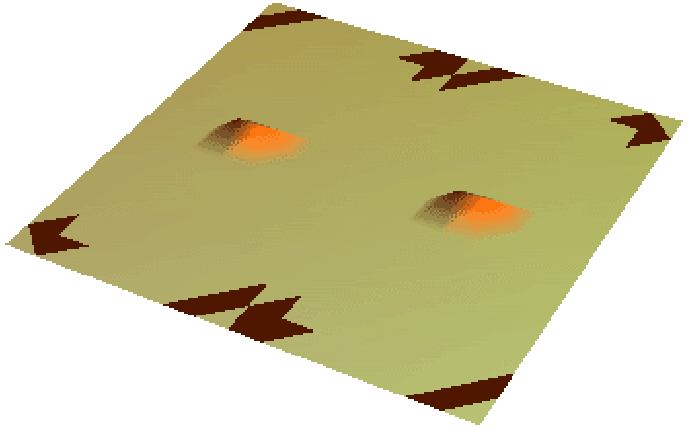
$$\mathcal{F}\big(x^n f(x)\big) = i^n\big[\hat{f}\big]^{(n)}(\omega)$$

Proof

Find the complete proof in page 402 of the textbook.

6.2 Convolution





6.2 Convolution

Convolution of Functions

Introducing the convolution of two functions f and g by

CONVOLUTION

$$\boxed{(f * \mathrm{g})(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt}$$

In this lecture

Convolution Theorem $\mathcal{F}[f(x) * \mathrm{g}(x)] = \sqrt{2\pi}\mathcal{F}[f(\omega)]\mathcal{F}[g(\omega)]$

$$(f * \mathrm{g})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

In textbook

Convolution Theorem $\mathcal{F}[f(x) * \mathrm{g}(x)] = \mathcal{F}[f(\omega)]\mathcal{F}[g(\omega)]$

6.2 Convolution

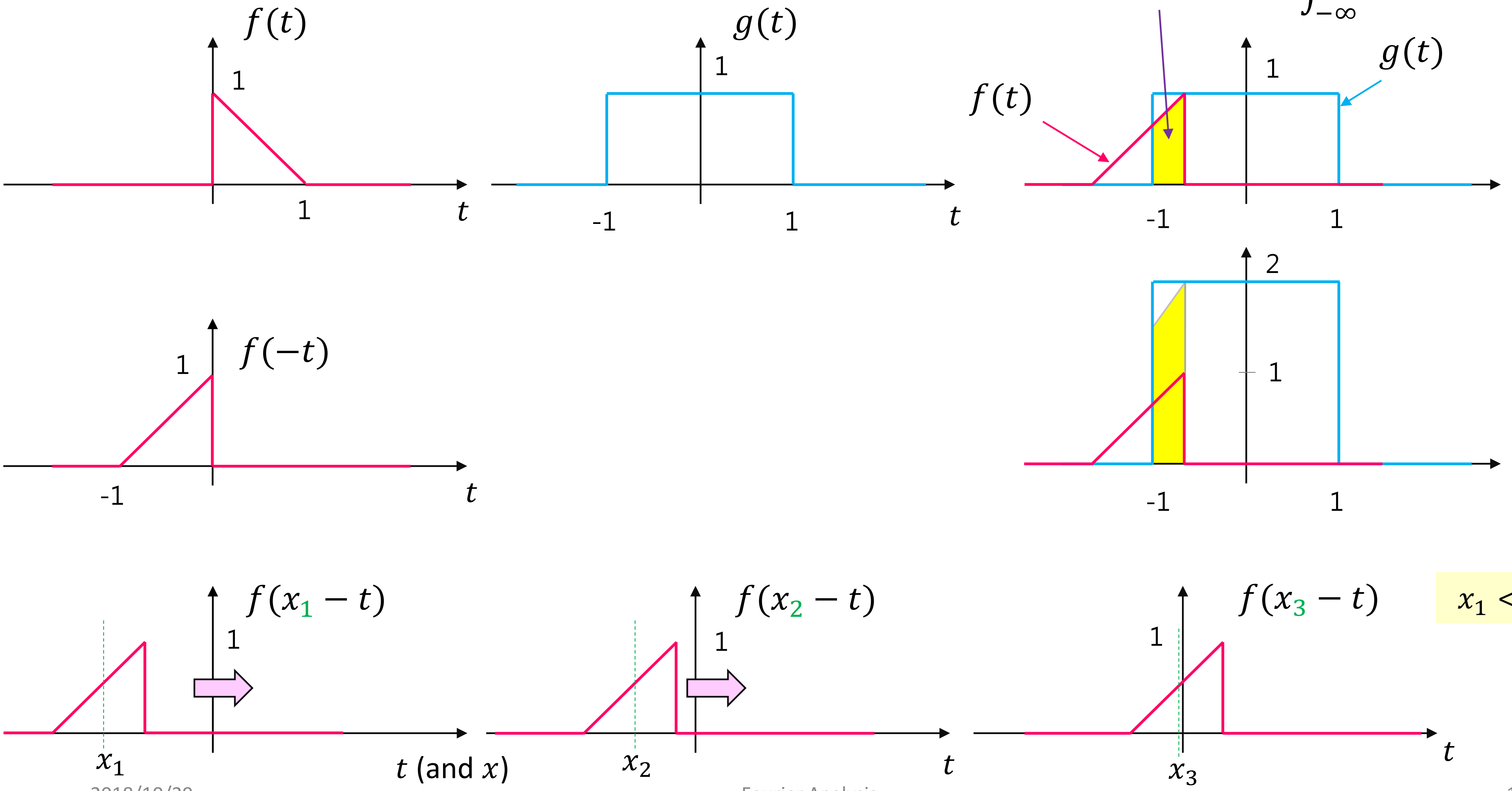
$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

What does Convolution mean?

6.2 Convolution

Case 1

$$h = (f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$



$x_1 < x_2 < x_3$

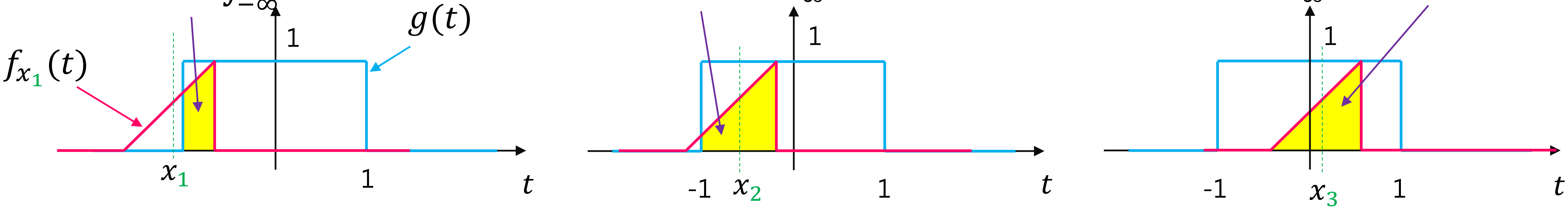
6.2 Convolution

Case 1

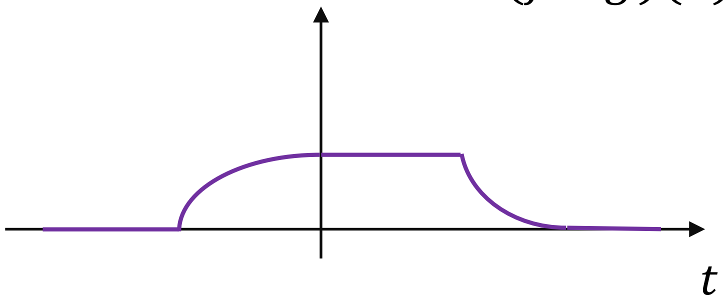
$$f_{x_1} * g = \int_{-\infty}^{\infty} f(x_1 - t)g(t)dt$$

$$f_{x_2} * g = \int_{-\infty}^{\infty} f(x_2 - t)g(t)dt$$

$$f_{x_3} * g = \int_{-\infty}^{\infty} f(x_3 - t)g(t)dt$$



$$h = (f * g)(x) = \int_{-\infty}^{\infty} f(x - t)g(t)dt$$

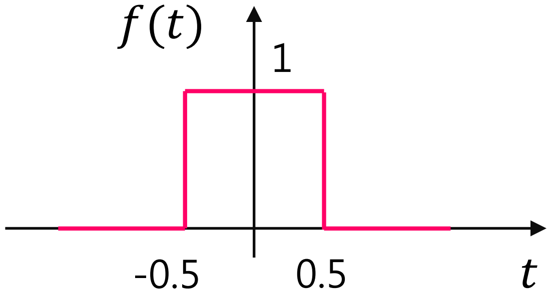


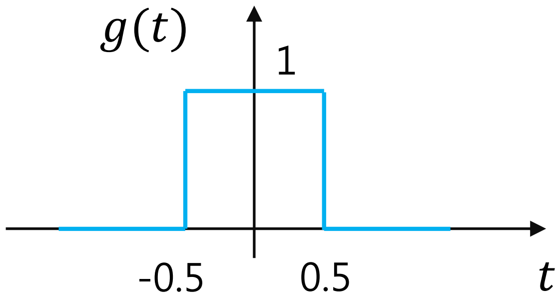
6.2 Convolution

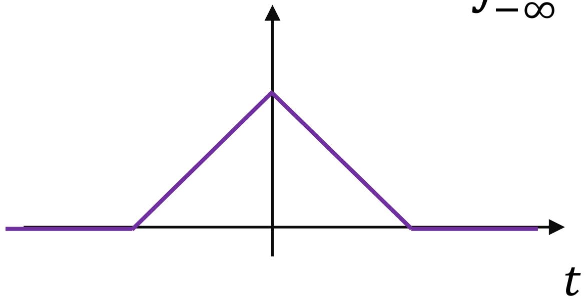
Case 2

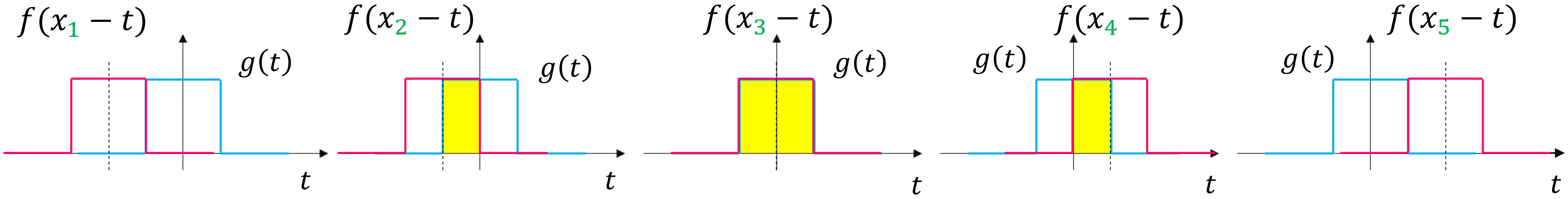
$$f(x) = g(x) = \begin{cases} 1 & if\ |x| \le 0.5 \\ 0 & otherwise \end{cases}$$

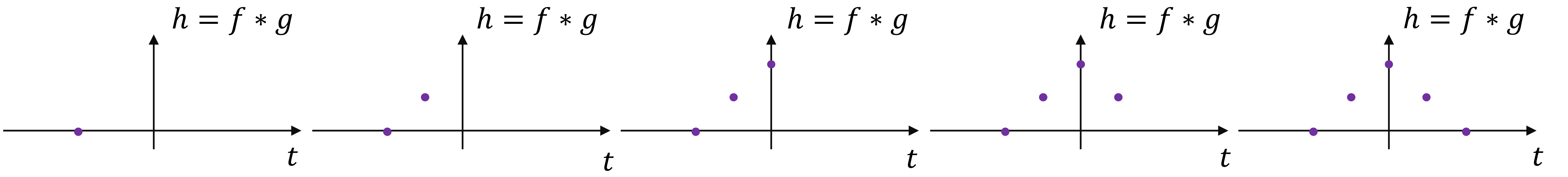
$$h = f * g = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$







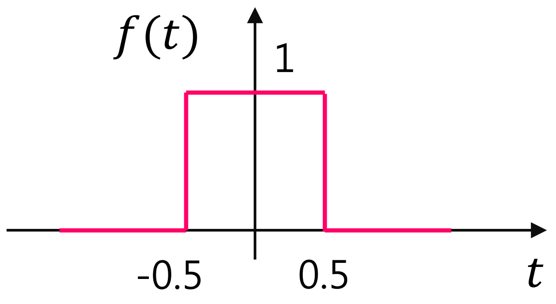


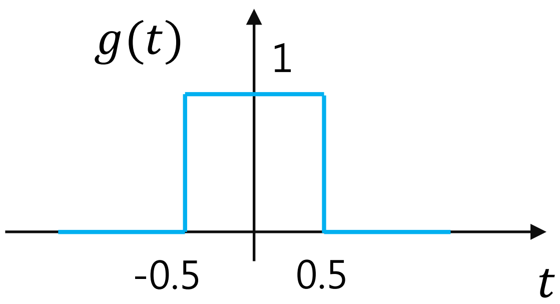


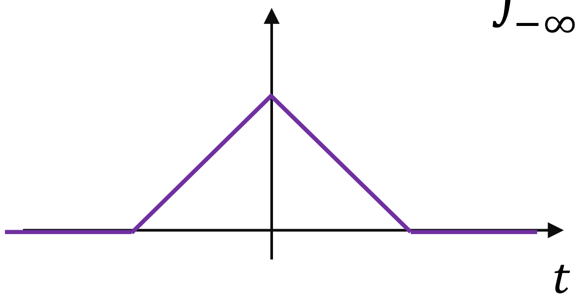
6.2 Convolution

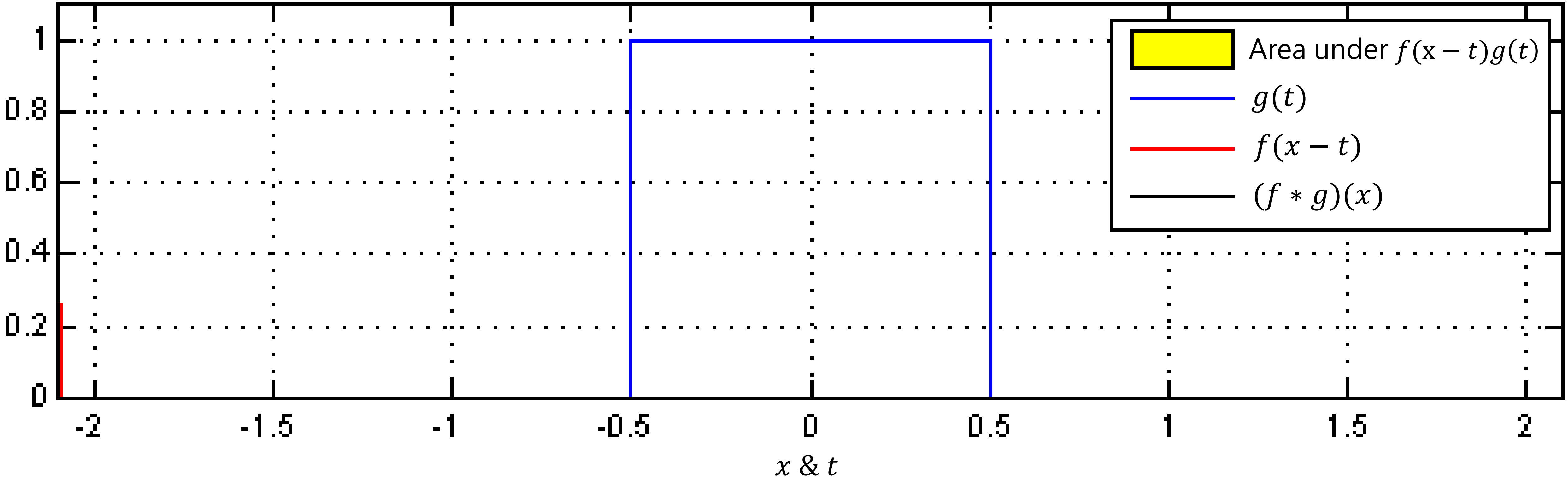
Case 2

$$h = f * g = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$



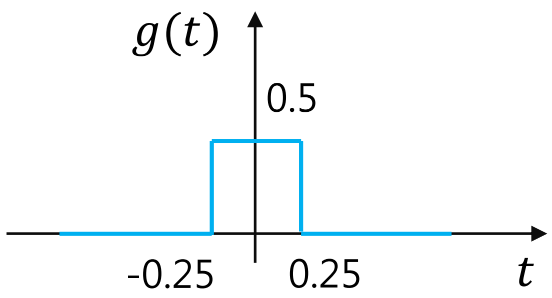






6.2 Convolution

Case 3



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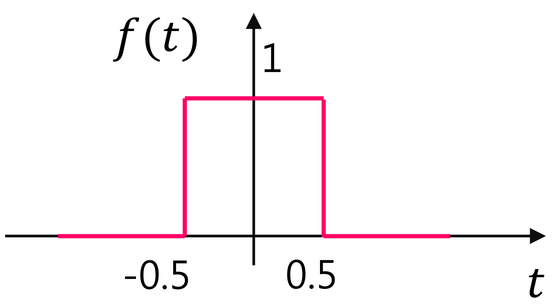

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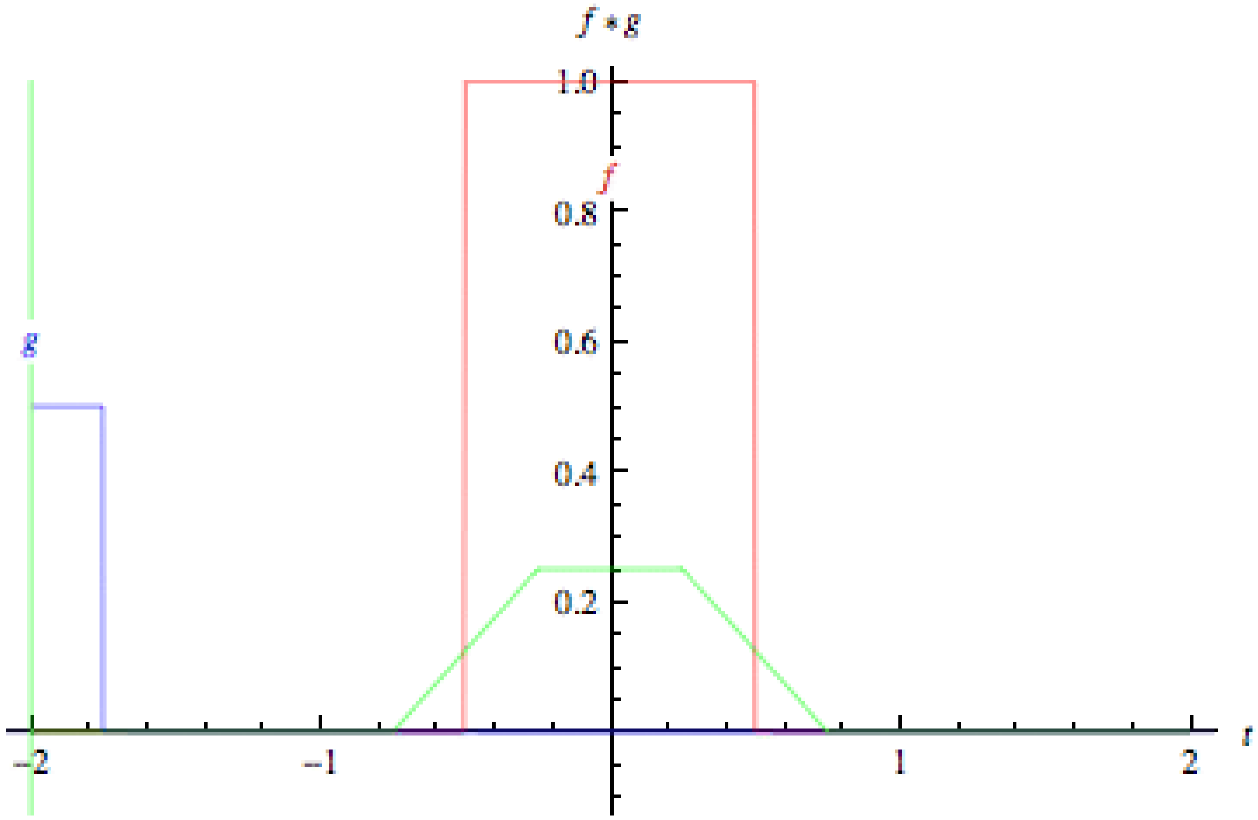
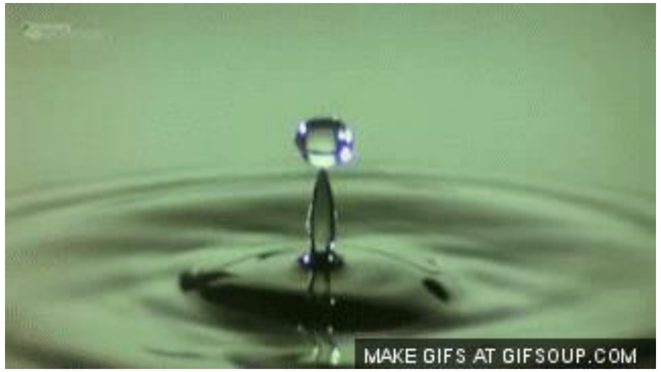
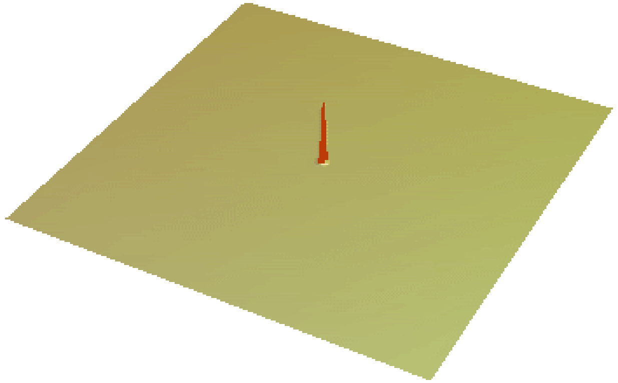


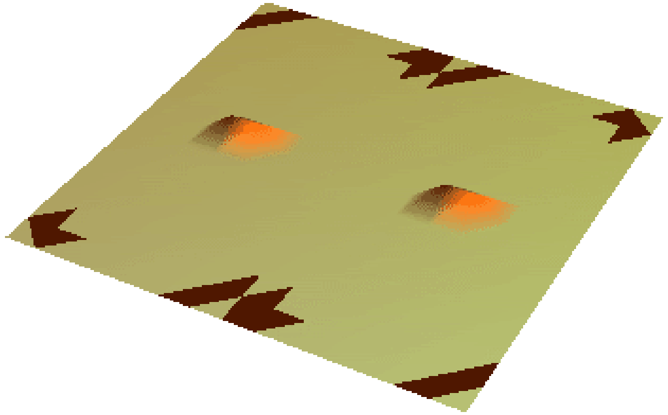
6.2 Convolution

Case 4





one stone wave



two stones wave

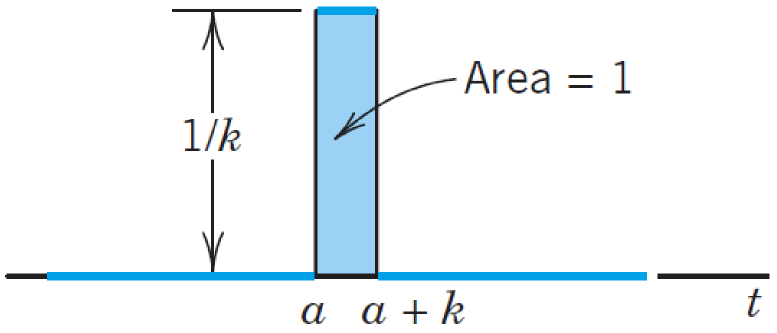
6.2 Convolution

Case 4

Dirac Delta Function

Generalized Functions

$$f_k(t-a) = \begin{cases} \dfrac{1}{k} & \text{if } a \le t \le a+k \\ 0 & otherwise \end{cases}$$

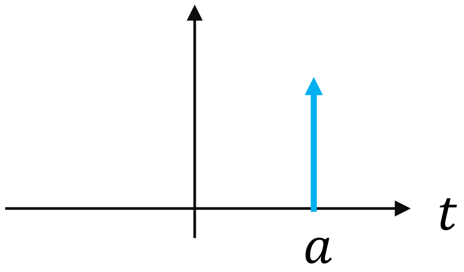


The function $f_k(t-a)$

$$I_k = \int_{-\infty}^{\infty} f_k(t-a)dt = \int_a^{a+k} \frac{1}{k} dt = 1$$

When $k \to 0$

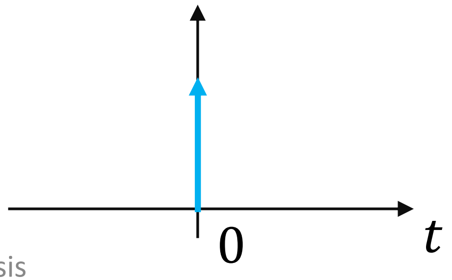
$$\delta(t-a) = \lim_{k\to 0} f_k(t-a)$$



$$\delta(t-a) = \begin{cases} \infty & if \quad t = a \\ 0 & otherwise \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t-a)dt = 1$$

When $a = 0$

$$\delta(t) = \begin{cases} \infty & \text{if} \quad t = 0 \\ 0 & otherwise \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)dt = 1$$



Theorem 1 in Section 7.8 ($a = 0$)

$$\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0)$$

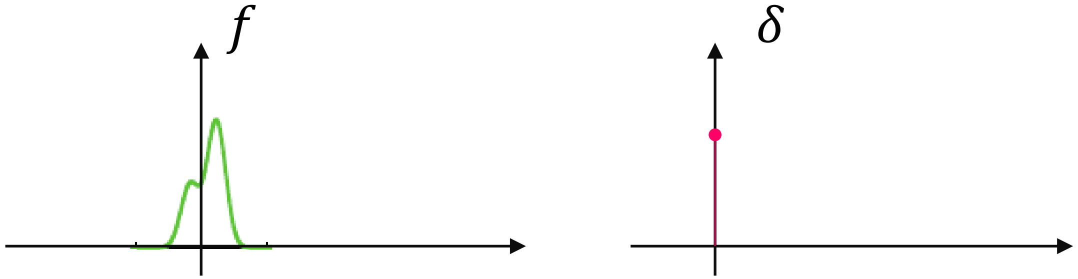
General form ($a \neq 0$)

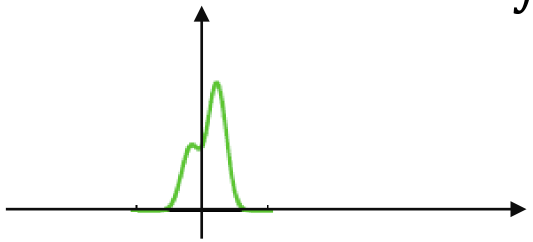
$$\int_{-\infty}^{\infty} g(t)\delta(t-a)dt = g(a)$$

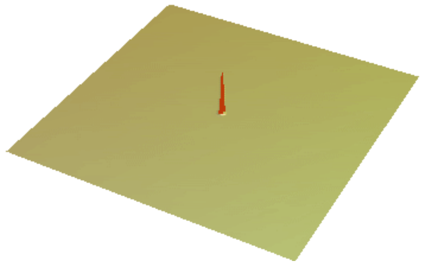
6.2 Convolution

Case 4

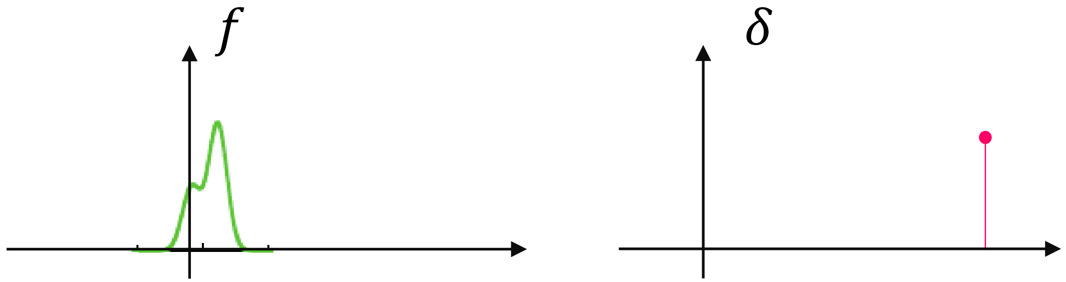
$$h = f * \delta = \int_{-\infty}^{\infty} f(x-t)\delta(t)dt$$

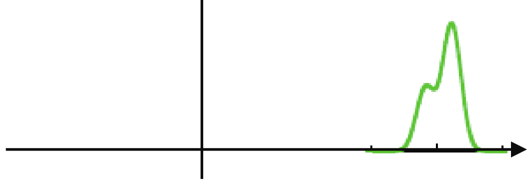




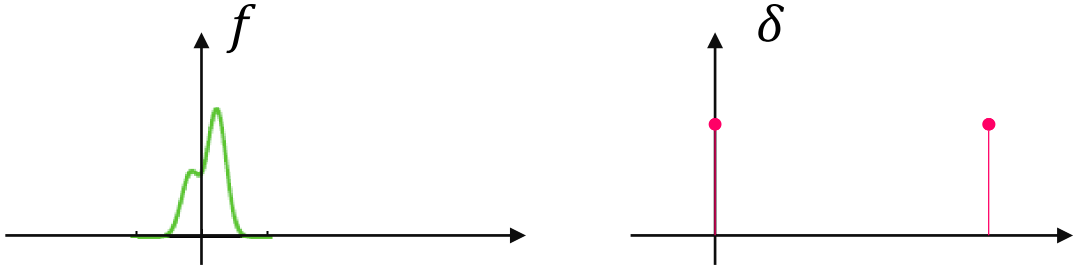


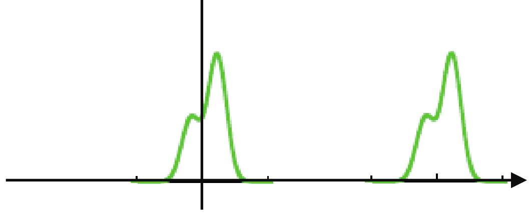
$$h = f * \delta = \int_{-\infty}^{\infty} f(x-t)\delta(t)dt$$

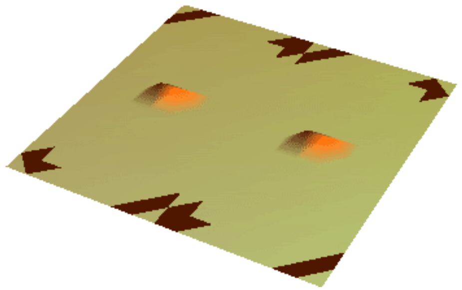




$$h = f * \delta = \int_{-\infty}^{\infty} f(x-t)\delta(t)dt$$







6.2 Convolution

EXAMPLE 3 Convolution with the cosine

Suppose that f is integrable and even ($f(-x) = f(x)$ for all x) and let $g(x) = \cos ax$.

Show that, for all real numbers a: $(f * g)(x) = \cos(ax)\hat{f}(a)$.

Solution

Tips:

- Use definition of $f * g$
- $f * g = g * f$
- f is even, then $f \sin at$ is odd
- Trigonometric identity $\cos(a-b) = \cos a \cos b + \sin a \sin b$
- $\int_{-\infty}^{\infty} -i\sin(at)dx = 0$

Find the complete solution in page 403 of the textbook.

6.2 Convolution

THEOREM 4 Fourier Transforms of Convolutions (Convolution Theorem)

Suppose that f and g are integrable; then

$$\mathcal{F}[f(x) * g(x)] = \sqrt{2\pi}\mathcal{F}[f(\omega)]\mathcal{F}[g(\omega)]$$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

In this lecture

Theorem 4 is expressed by saying that the Fourier transform takes convolutions into products.

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-t)g(t)dt$$

$$\mathcal{F}[f(x) * g(x)] = \mathcal{F}[f(\omega)]\mathcal{F}[g(\omega)]$$

In textbook

6.2 Convolution

EXAMPLE 5 **Fourier transform of a convolution**

Consider the function $f(x) = 1$ if $|x| < 1$ and 0 otherwise. The graph of this function is shown in Figure 5. From Example 1, we have

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin a\omega}{\omega}$$

Find the Fourier Transform of $f * f$.

Find $f * f$.

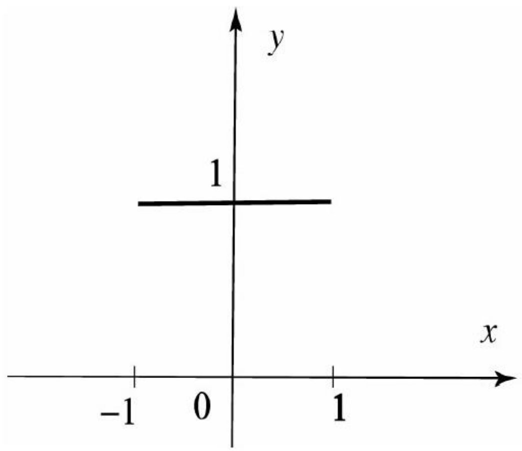


Figure 5 Graph of f.

Solution:

Tips:

Instead of $\mathcal{F}[f(x) * f(x)]$, we compute $\mathcal{F}[f(\omega)]\mathcal{F}[f(\omega)] = \hat{f}(\omega)\hat{f}(\omega)$

Find the complete solution in page 405 of the textbook.

Appendix B. Table of Fourier Transform (Page 1069 of the textbook)

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{ix\omega}\, d\omega$		$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$
1.	$\begin{cases} 1 & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin a\omega}{\omega}$
2.	$\begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$	$\frac{i\left(e^{-ib\omega} - e^{ia\omega}\right)}{\sqrt{2\pi}\omega}$
3.	$\begin{cases} 1 - \frac{\lvert x\rvert}{a} & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases} \quad a > 0$	$2\sqrt{\frac{2}{\pi}} \frac{\sin^2\left(\frac{a\omega}{2}\right)}{a\omega^2}$
4.	$\begin{cases} x & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases} \quad a > 0$	$i\sqrt{\frac{2}{\pi}} \frac{a\omega\cos(a\omega) - \sin(a\omega)}{\omega^2}$

⋮

Review for Lecture 6

- The Fourier Transform: Operational Properties
- Convolution
- Dirac Delta Function

Exercise

Please Check https://github.com/uoaworks/FourierAnalysisAY2018

Reading: Section 7.2, 7.8, Textbook

References

[1] Nakhlé H. Asmar, *Partial Differential Equations with Fourier Series and Boundary Value Problems 2nd Edition*, 2004

[2] Erwin Kreyszig, *Advanced Engineering Mathematics, 9th Edition*, 2006

[3] Convolution, https://sites.google.com/site/butwhymath/m/convolution

[4] Wikipedia