Lecture 2

Fourier Series of Functions with Arbitrary Periods

What you will learn in Lecture 2

I. Fourier Analysis: Examples

II. Fourier Series of Functions with Arbitrary Periods

2.1 Fourier Series: Examples

2.1 Fourier Series: Examples

Euler Formulas for the Fourier Coefficients

Suppose that f has the Fourier series representation

(1) $$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Then the coefficients a_0, a_n, and b_n are called the Fourier coefficients of f and are given by the following Euler formulas:

(2) $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

(3) $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \quad (n = 1, 2, \dots)$$

(4) $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \quad (n = 1, 2, \dots)$$

Euler Formulas for the Fourier Coefficients

Alternative Euler Formulas

(5) $$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

(6) $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \quad (n = 1, 2, \dots)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \quad (n = 1, 2, \dots)$$

2.1 Fourier Series: Examples

For a positive integer N, we denote the Nth partial sum of the Fourier series of f by $s_N(x)$. Thus

$$s_N(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

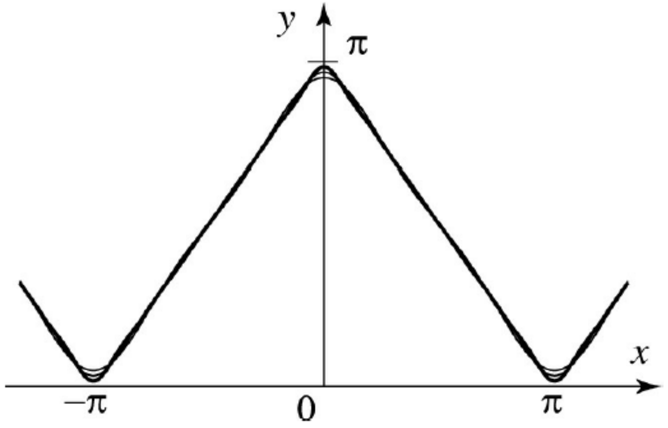


Figure 7 Partial sums of the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

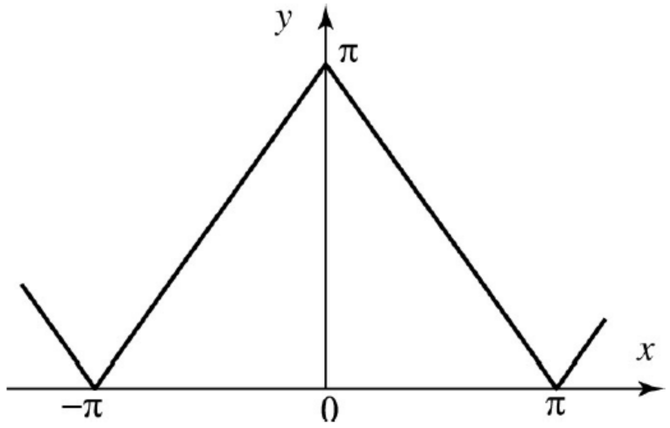


Figure 8 The Fourier series

EXAMPLE 1 Fourier series of the sawtooth function

The sawtooth function, shown in Figure 1, is determined by the formulas

$$f(x) = \begin{cases} \frac{1}{2}(\pi - x) & if\ 0 < x \le 2\pi \\ f(x + 2\pi) & otherwise \end{cases}$$

(a) Find its Fourier series.

(b) With the help of a computer, plot the partial sums $s_1(x), s_7(x)$, and $s_{20}(x)$, and determine the graph of the Courier series.

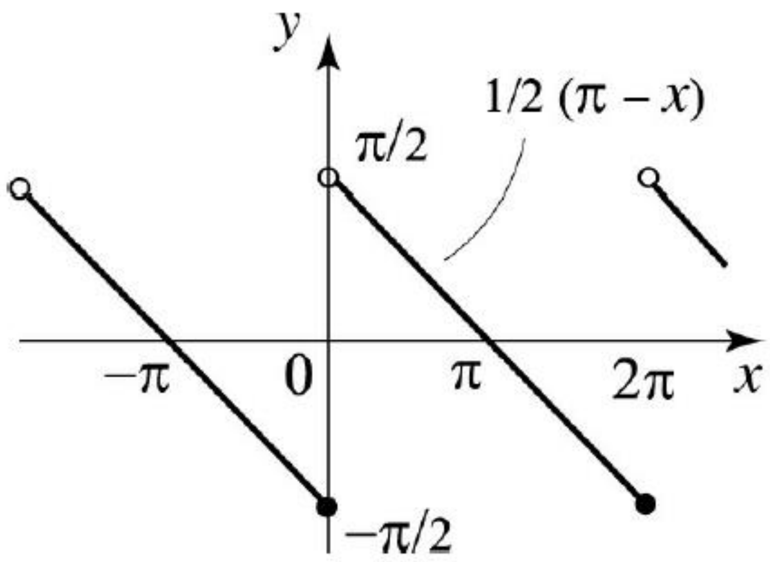


Figure 1 Sawtooth function.

2.1 Fourier Series: Examples

Solution

Compute a_0, a_n, b_n

Find complete solution in page 28-30 of the textbook.

2.1 Fourier Series: Examples

Integration by parts

Product of two functions [edit]

The theorem can be derived as follows. Suppose $u(x)$ and $v(x)$ are two continuously differentiable functions. The product rule states (in Leibniz's notation):

$$\frac{d}{dx}\Big(u(x)v(x)\Big) = v(x)\frac{d}{dx}\left(u(x)\right) + u(x)\frac{d}{dx}\left(v(x)\right).$$

Integrating both sides with respect to x,

$$\int \frac{d}{dx}\left(u(x)v(x)\right)\,dx = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx$$

then applying the definition of indefinite integral,

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx$$

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

gives the formula for **integration by parts**.

Since du and dv are differentials of a function of one variable x,

$$du = u'(x)dx \quad dv = v'(x)dx$$

$$\int u(x)\,dv = u(x)v(x) - \int v(x)\,du$$

The original integral $\int uv'\,dx$ contains v' (derivative of v); in order to apply the theorem, v (antiderivative of v') must be found, and then the resulting integral $\int vu'\,dx$ must be evaluated.

Polynomials and trigonometric functions [edit]

In order to calculate

$$I = \int x\cos(x)\,dx\,,$$

let:

$$u = x \ \Rightarrow\ du = dx$$

$$dv = \cos(x)\,dx \ \Rightarrow\ v = \int \cos(x)\,dx = \sin(x)$$

then:

$$\begin{aligned}\int x\cos(x)\,dx &= \int u\,dv\\ &= u\cdot v - \int v\,du\\ &= x\sin(x) - \int \sin(x)\,dx\\ &= x\sin(x) + \cos(x) + C,\end{aligned}$$

where C is a constant of integration.

For higher powers of x in the form

$$\int x^n e^x\,dx,\ \int x^n \sin(x)\,dx,\ \int x^n\cos(x)\,dx\,,$$

repeatedly using integration by parts can evaluate integrals such as these; each application of the theorem lowers the power of x by one.

https://en.wikipedia.org/wiki/Integration_by_parts

2.1 Fourier Series: Examples

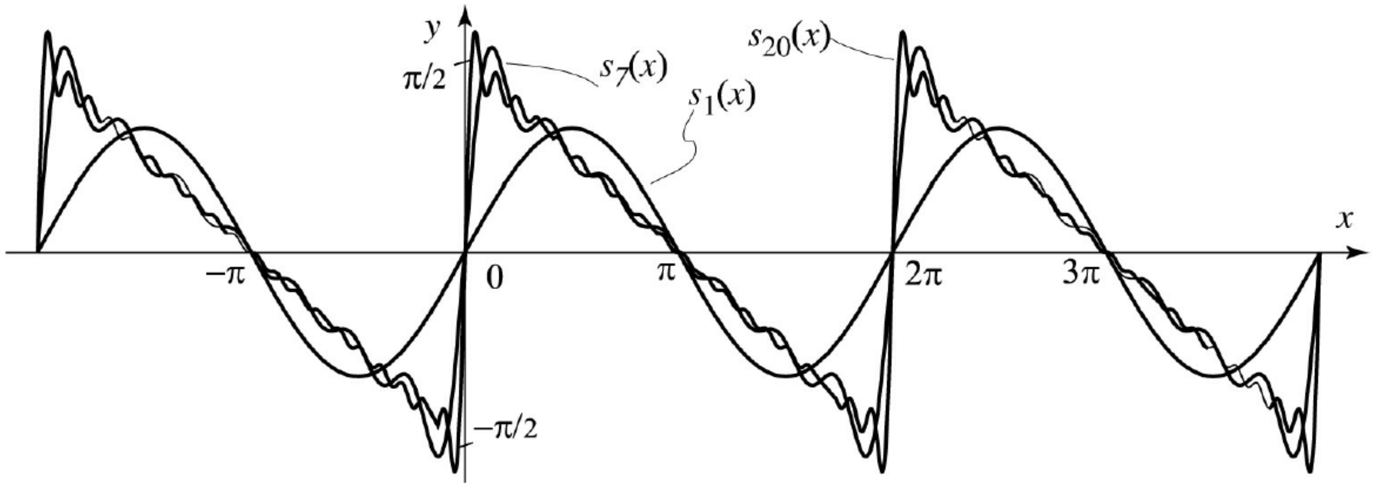


Figure 2 Here the nth partial sum of the Fourier series is $s_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k}$. distinguish the graphs, note that as n increases, the frequencies of the sine terms increase. This causes the graphs of the higher partial sums to be more wiggly.

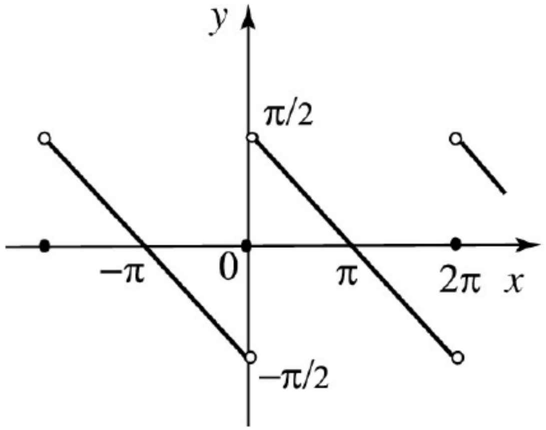


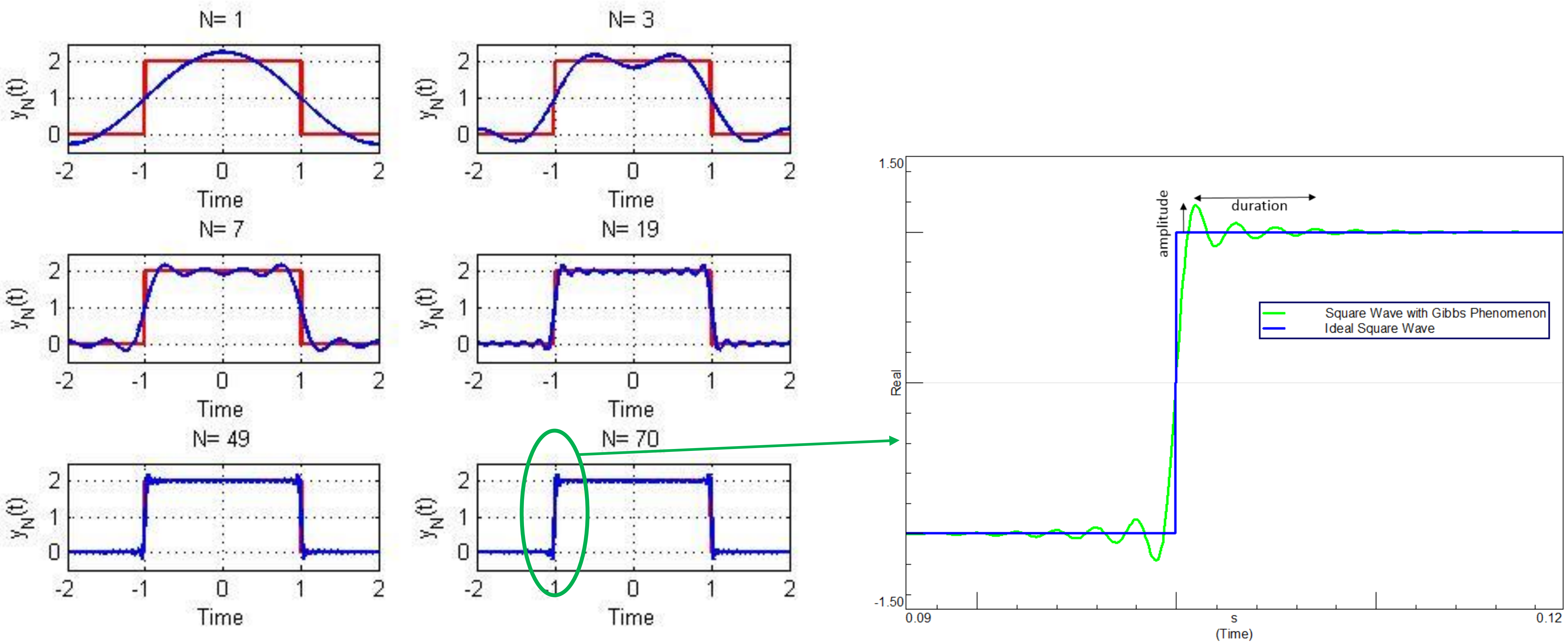
Figure 3 The graph of the Fourier series coincides with the graph of $\sum_{k=1}^{\infty} \frac{\sin kx}{k}$ the function, except at the points of the discontinuity.

Note 1: **At the points of discontinuity** $(x = 2k\pi)$ in Example 1, **the Fourier series converges to** 0 which is **the average value of the function** from the left and the right at these points.

Note 2: **Near the points of discontinuity**, **the Fourier series overshoots its limiting values**, known as the Gibbs phenomenon.

2.1 Fourier Series: Examples

Gibbs phenomenon



Arithmetic Average

Recall that f is piecewise smooth if f and f' are piecewise continuous.

If f is piecewise continuous, the **average** (or **arithmetic average**) of f at c is

$$\frac{f(c-)+f(c+)}{2}$$

where $f(c-) = \lim_{x\to c^-} f(x)$ and $f(c+) = \lim_{x\to c^+} f(x)$.

If f is continuous at c, then $f(c+) = f(c-) = f(c)$ and so the **average** of f at c is $f(c)$.

Average will be of interest only at points of discontinuity.

Example The function in Figure 4 has a discontinuity at $x = 1$. Its **average** there is $\frac{1+\frac{1}{2}}{2} = \frac{3}{4}$.

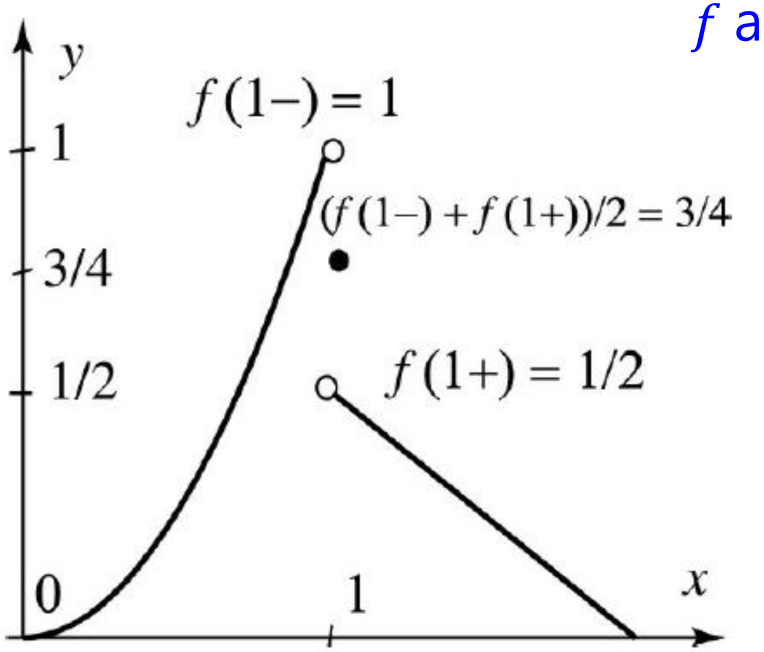


Figure 4 Average of $f(x)$ $x = 1$.

2.1 Fourier Series: Examples

THEOREM 1 Fourier Series Representation

Suppose that f is a 2π-**periodic piecewise smooth function**. Then for all x we have

$$\frac{f(x+)+f(x-)}{2}=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

where the Fourier coefficients a_0, a_n, b_n are given by (2)–(4). In particular, if f is **piecewise smooth and continuous at** x, then

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

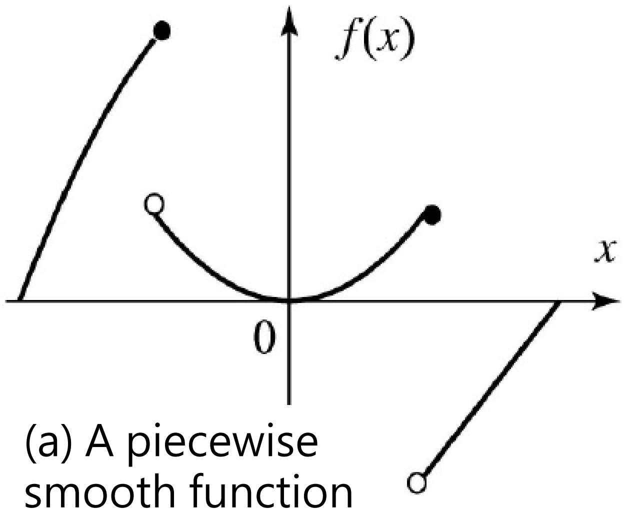
Find the complete proof in Section 2.8 of the textbook

1. At a point of continuity of **a piecewise smooth function** the **Fourier series converges to the value of the function**.

2. **At a point of discontinuity, the Fourier series** does its best to converge, and having no reason to favor one side over the other, **it converges to the average of the left and right limits** (see Figure 5).

2.1 Fourier Series: Examples

THEOREM 1 Fourier Series Representation



(a) A piecewise smooth function

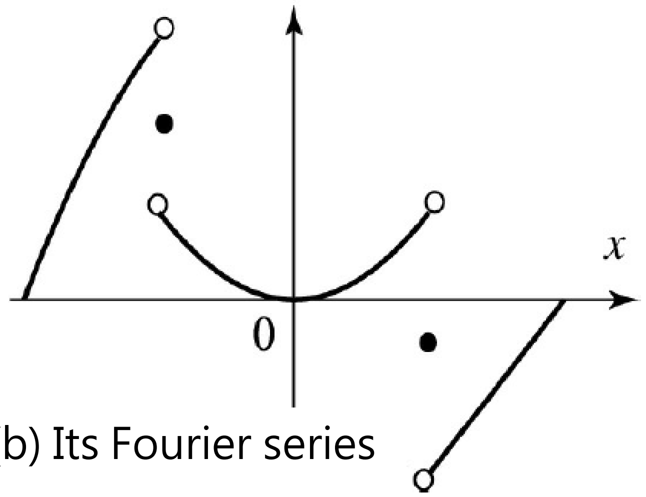


Figure 5 (b) Its Fourier series

2.1 Fourier Series: Examples

THEOREM 1 Fourier Series Representation

For this reason, **we may define (or redefine) f at isolated points without affecting its Fourier series.**

If we redefine the function at points of discontinuity to be $\frac{f(c-)+f(c+)}{2}$ we then have the equality

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

holding at all x.

2.1 Fourier Series: Examples

EXAMPLE 2 Triangular wave

The 2π-periodic triangular wave is given on the interval $[-\pi, \pi]$ by

$$g(x) = \begin{cases} \pi + x & if \ -\pi \le x \le 0 \\ \pi - x & if \ 0 \le x \le \pi \end{cases}$$

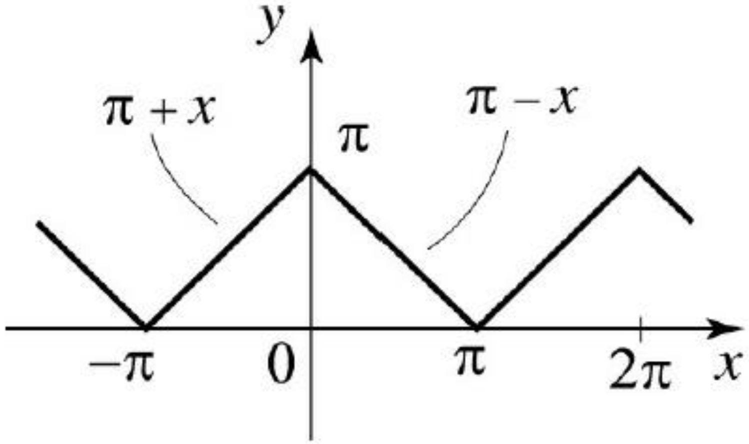


Figure 6 Triangular wave

(a) Find its Fourier series.

(b) Plot some partial sums and the Fourier series.

Even and Odd Functions

A function f is **even** if $f(-x) = f(x)$ for all x.

A function f is **odd** if $f(-x) = -f(x)$ for all x.

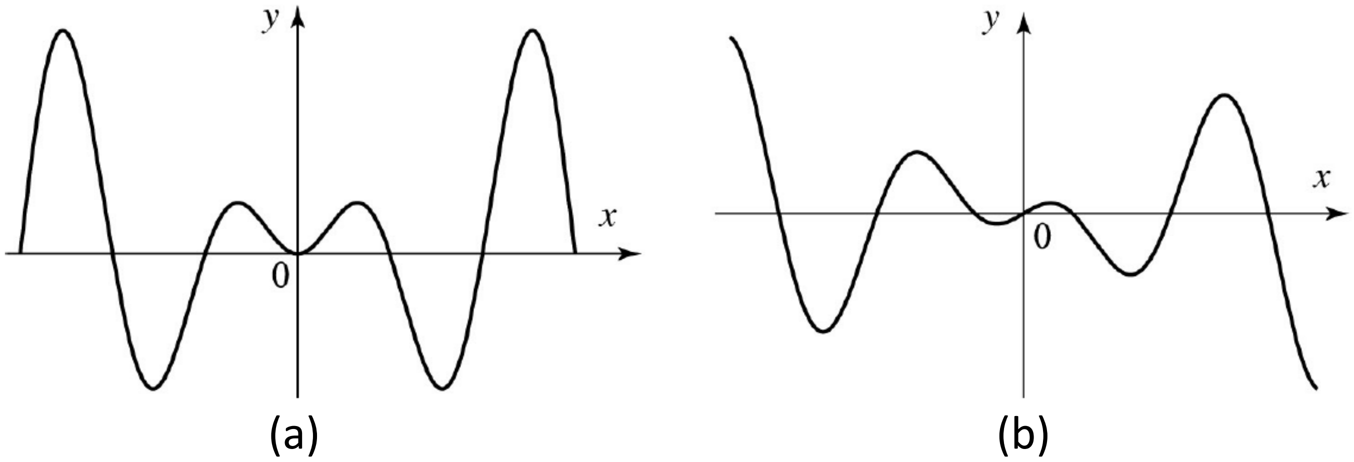


Figure 6 in Section 2.3 of the textbook
(a) Even function: The graph is symmetric with respect to the y-axis.
(b) Odd function: The graph is symmetric with respect to the origin.

If f is even (Figure 6(a)), then

$$\int_{-p}^{p} f(x)dx = 2\int_{0}^{p} f(x)dx$$

and if f is odd (Figure 6(b)), then

$$\int_{-p}^{p} f(x)dx = 0$$

Even and Odd Functions

The following **useful properties** concerning **the products of these functions** are easily verified.

$$(\text{Even})\cdot(\text{Even})=\text{Even}$$

$$(\text{Even})\cdot(\text{Odd})=\text{Odd}$$

$$(\text{Odd})\cdot(\text{Odd})=\text{Even}$$

2.1 Fourier Series: Examples

Solution Find the solution in page 31-32 of the textbook.

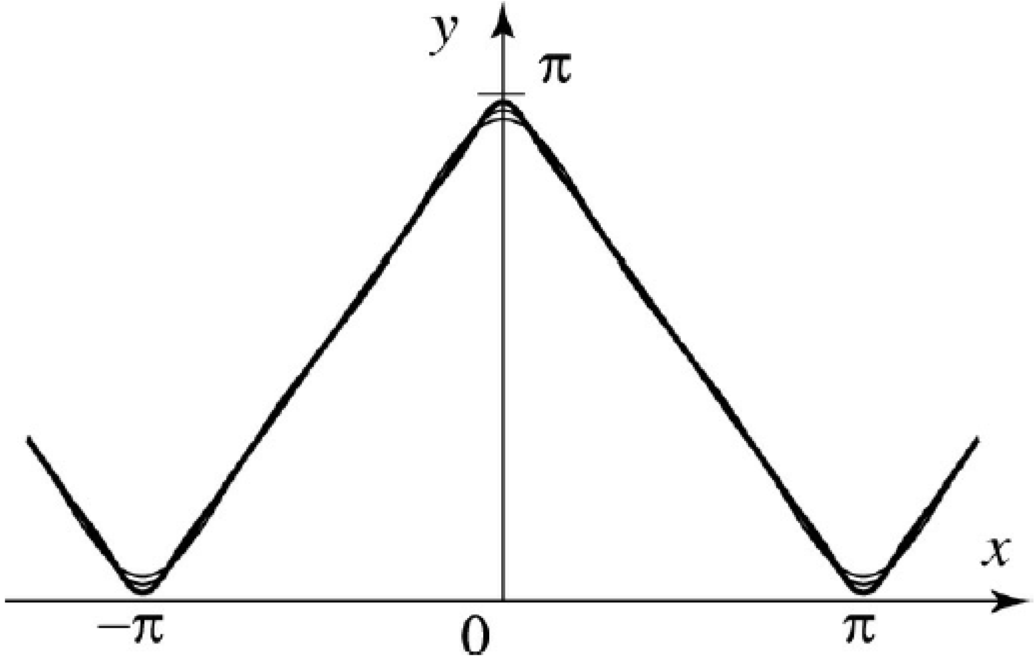


Figure 7 Partial sums of the Fourier series

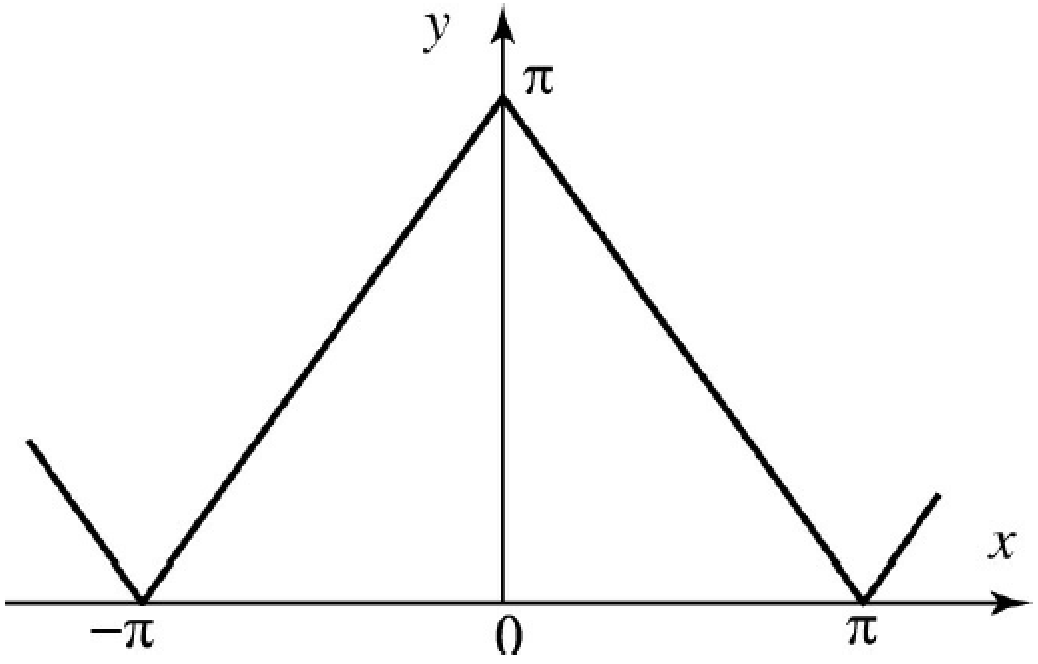


Figure 8 The Fourier series

Notice: Figure 7 is converging very fast, much faster than those in Example 1.

This is due to **the magnitudes of the Fourier coefficients**.

In Example 1 the coefficients are of the order $1/n$,

while in Example 2 the coefficients are of the order $1/n^2$.

2.1 Fourier Series: Examples

Operations on Fourier Series

The **Fourier series** in Examples 1 is an **odd function** and **contains only sine terms**, while the Examples 2 is an **even function** and **contains only cosine terms**.

We will **derive the Fourier series containing both sine and cosine terms** without computing Fourier coefficients **but by applying operations such as multiplying a Fourier series by a constant, adding two Fourier series, changing variables (x to $-x$), and translating.**

Operations on Fourier Series

EXAMPLE 4 Linear combinations of Fourier series

The 2π-periodic function

$$h(x) = \begin{cases} \pi - x & if\ 0 < x \leq \pi \\ 0 & if\ \pi < x \leq 2\pi \end{cases}$$

is related to the functions in Examples 1 and 2 by

$$h(x) = f(x) + \frac{1}{2} g(x)$$

2.1 Fourier Series: Examples

Solution Find the solution in page 33 of the textbook.

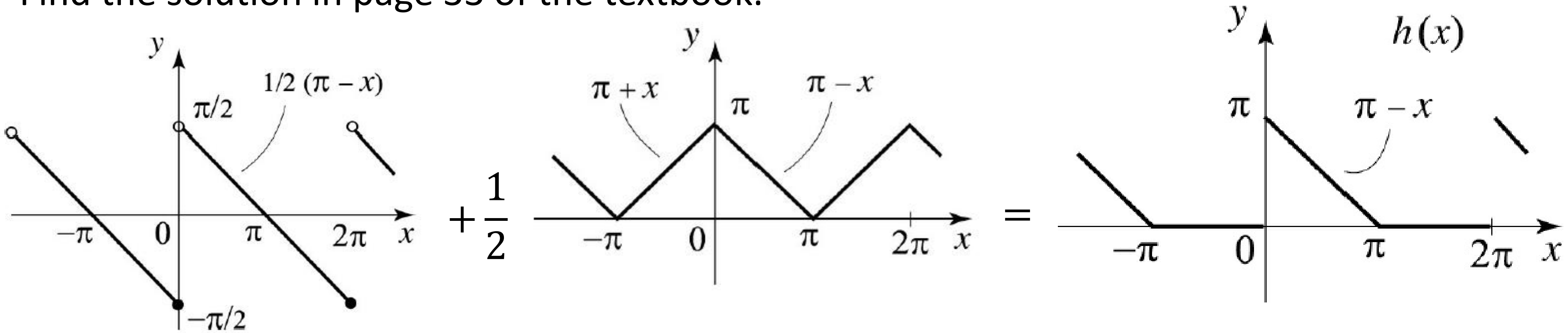
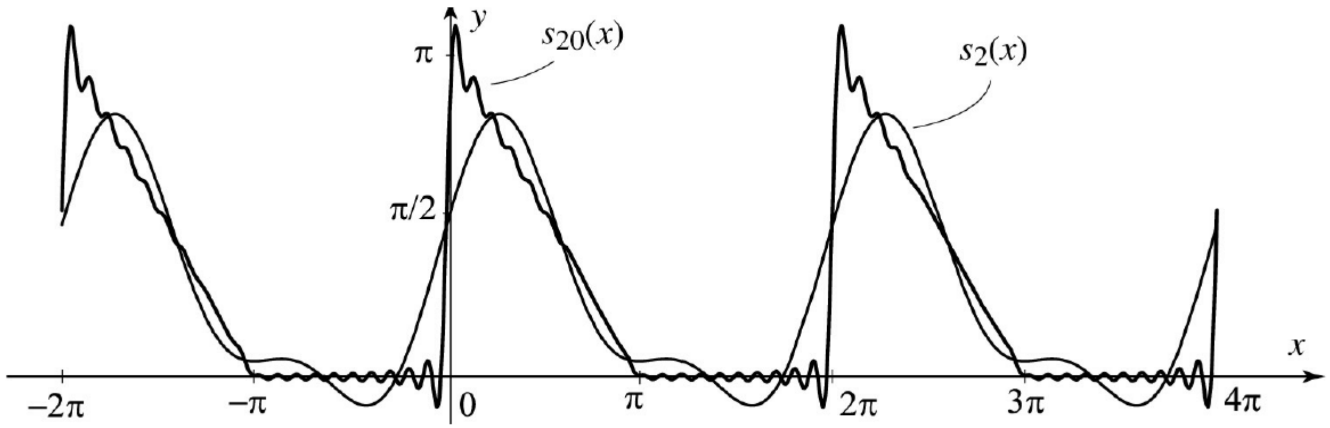


Figure 9 The function of Example 4



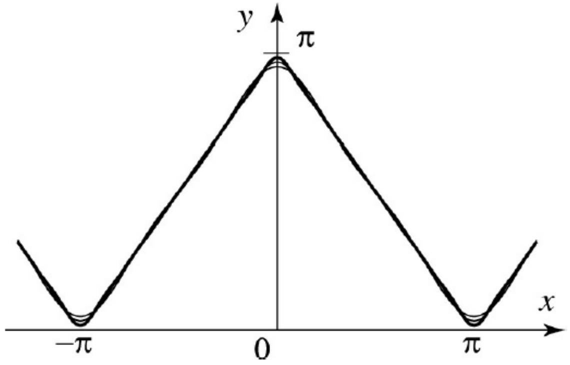


Figure 7 Partial sums of the Fourier series

Figure 10 Note the Gibbs phenomenon at the points of discontinuity ($x = 2k\pi$).
This is due to the fact that the **Fourier series consists of a cosine part that is converging very fast** (Figure 7) and a sine part that overshoots at the points of discontinuity.

2.1 Fourier Series: Examples

EXAMPLE 5 Changing variables and translating

The graph of the function $k(x)$ in Figure 11 is obtained by reflecting through the y-axis the graph in Figure 9 and then translating by π units to the left or right. Thus $k(x) = h(-x-\pi)$ and the Fourier series representation of $k(x)$ is

$$k(x) = h(-x-\pi) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1}{\pi}\left(\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right)\cos n\,(-x-\pi) + \frac{\sin n(-x-\pi)}{n}\right\}$$

But $\cos n(-x-\pi) = (-1)^n \cos nx$ and $\sin n(-x-\pi) = (-1)^{n+1}\sin nx$. So

$$k(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1}{\pi}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)\cos n\,x + (-1)^{n+1}\frac{\sin nx}{n}\right\}$$

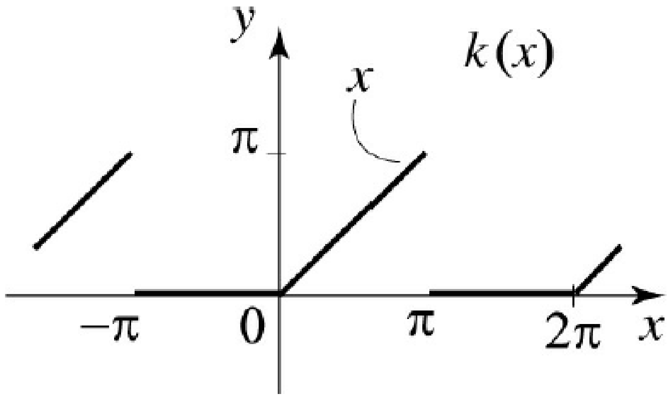


Figure 11 The function in Example 5(a).

2.2 Fourier Series of Functions with Arbitrary Periods

2.2 Fourier Series of Functions with Arbitrary Periods

In the preceding section we worked with functions of period 2π.

The choice of this period was merely for convenience.

We show how to extend our results to functions with arbitrary period (Figure 1)

by using a simple change of variables.

Suppose that f is a function with period $T = 2p > 0$, and let

$$g(x) = f\left(\frac{p}{\pi}x\right) \tag{1}$$

Since f is $2p$-periodic, we have

$$g(x + 2\pi) = f\left(\frac{p}{\pi}(x + 2\pi)\right) = f\left(\frac{p}{\pi}x + 2p\right) = f\left(\frac{p}{\pi}x\right) = g(x)$$

Hence g is 2π-periodic.

2.2 Fourier Series of Functions with Arbitrary Periods

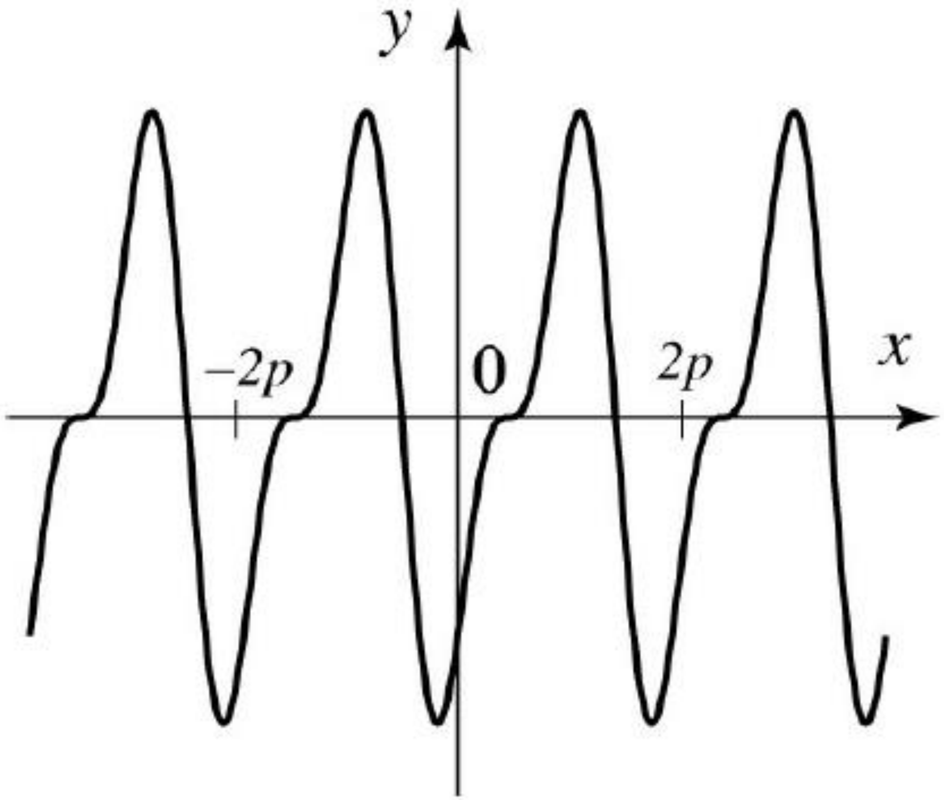


Figure 1 A $2p$-periodic function

2.2 Fourier Series of Functions with Arbitrary Periods

THEOREM 1 Fourier Series Representation: Arbitrary Period

Suppose that f is a $2p$-periodic piecewise smooth function. The Fourier series of f is given by

$$a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right) \tag{2}$$

where

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f(x)\,dx \tag{3}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx \quad (n = 1, 2, \dots) \tag{4}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx \quad (n = 1, 2, \dots) \tag{5}$$

The Fourier series converges to $f(x)$ if f is continuous at x and to $\frac{f(x-)+f(x+)}{2}$ otherwise.

By Theorem 1, Section 2.1, all the integrals can be replaced $\int_{-p}^{p}$ by $\int_{0}^{2p}$ without changing the values of the coefficients.

2.2 Fourier Series of Functions with Arbitrary Periods

Proof

Find complete proof in page 39 of the textbook.

EXAMPLE 1 **A Fourier series with arbitrary period**

Find the Fourier series of the $2p$-periodic function given by $f(x) = |x|$ $if\ -p \le x \le p$ (Figure 2).

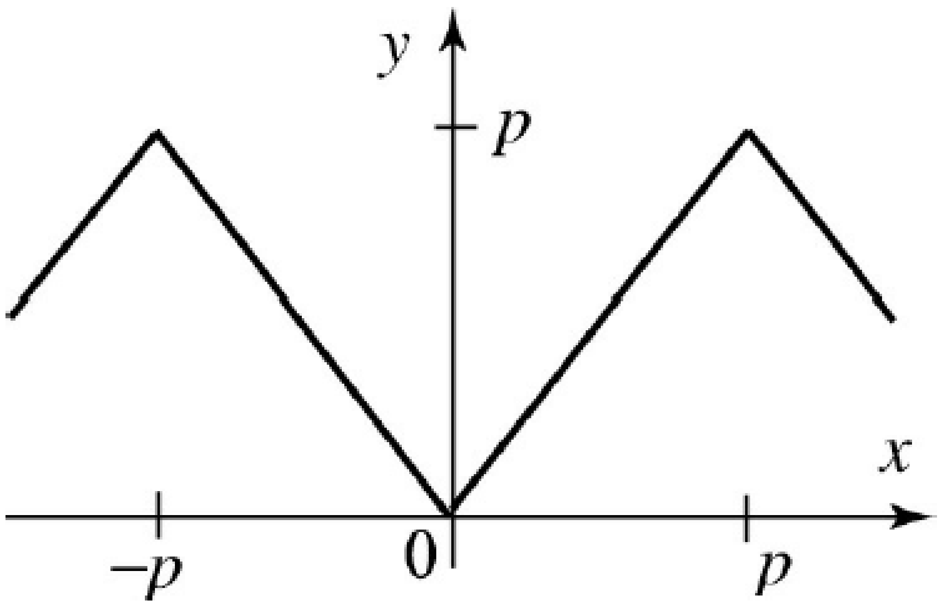


Figure 2 Triangular wave with period $2p$.

2.2 Fourier Series of Functions with Arbitrary Periods

Solution Find the solution in page 40 of the textbook.

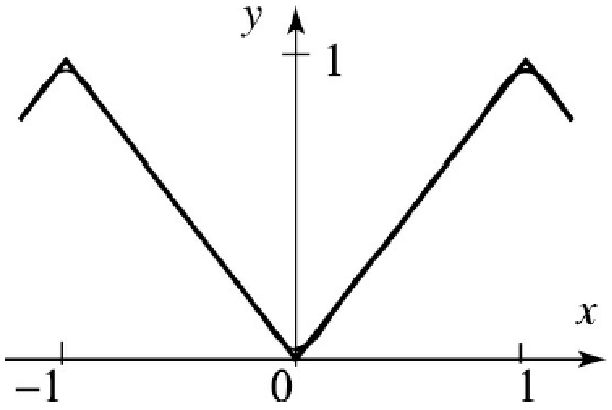


Figure 3 Partial sums of the Fourier series ($p = 1$), in Example 1.

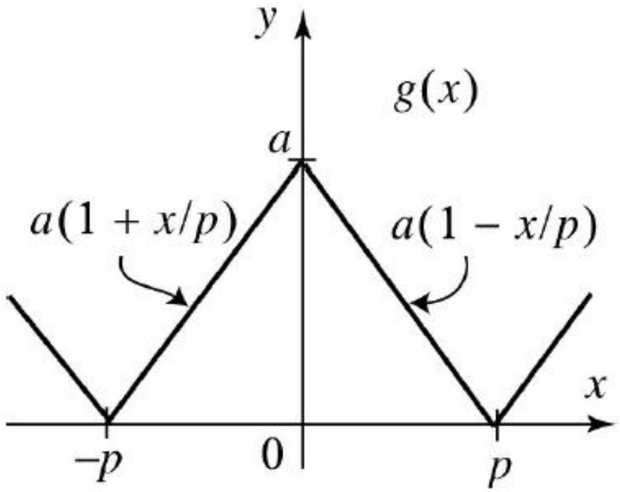
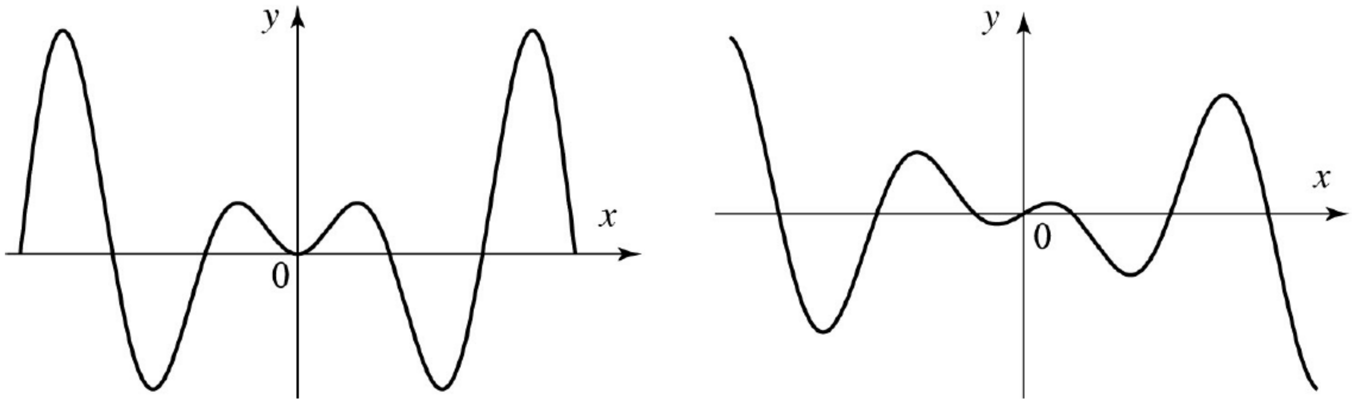


Figure 4 A $2p$-periodic triangular wave.

Even and Odd Functions

A function f is **even** if $f(-x) = f(x)$ for all x.

A function f is **odd** if $f(-x) = -f(x)$ for all x.



(a) (b)

Figure 6
(a) Even function: The graph is symmetric with respect to the y-axis.
(b) Odd function: The graph is symmetric with respect to the origin.

If f is even (Figure 6(a)), then

$$\int_{-p}^{p} f(x)dx = 2\int_{0}^{p} f(x)dx$$

and if f is odd (Figure 6(b)), then

$$\int_{-p}^{p} f(x)dx = 0$$

Even and Odd Functions

The following **useful properties** concerning **the products of these functions** are easily verified.

$$(\text{Even})\cdot(\text{Even}) = \text{Even}$$

$$(\text{Even})\cdot(\text{Odd}) = \text{Odd}$$

$$(\text{Odd})\cdot(\text{Odd}) = \text{Even}$$

Even and Odd Functions

THEOREM 2 Fourier Series of Even and Odd Functions

Suppose that f is $2p$-periodic and has the Fourier series representation (2). Then

(i) f is even if and only if $b_n = 0$ for all n. In this case

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x$$

where

$$a_0 = \frac{1}{p} \int_0^p f(x)\, dx \quad \text{and} \quad a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x\, dx \quad (n = 1, 2, \dots)$$

(ii) f is odd if and only if $a_n = 0$ for all n. In this case

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

where

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x\, dx \quad (n = 1, 2, \dots)$$

Even and Odd Functions

THEOREM 2 Fourier Series of Even and Odd Functions

Proof

Find complete proof in page 43 of the textbook.

EXAMPLE 4 Fourier series of an even function

Find the Fourier series of the 2-periodic function $f(x) = 1 - x^2$ if $-1 < x < 1$.

Solution

Compute a_0, a_n, b_n by Theorem 2.

Find complete solution in page 43, 44 of the textbook.

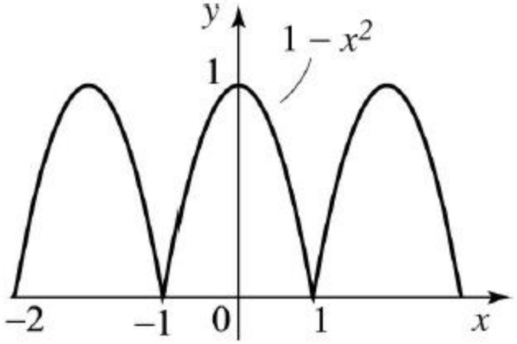


Figure 7 An even function.

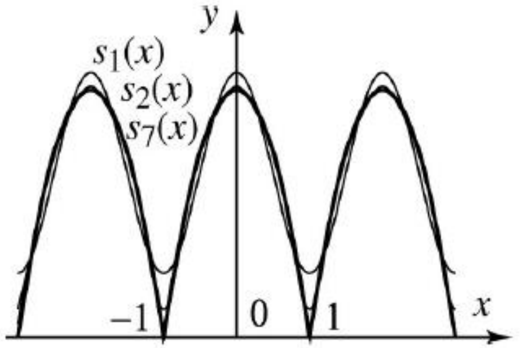


Figure 8 Partial sums of the Fourier series in Example 4.

Review for Lecture 2

- Fourier Series
- Fourier Series of Functions with Arbitrary Periods

Exercise

Please Check https://github.com/uoaworks/FourierAnalysisAY2018

Reading: Section 2.2, 2.3, Textbook

References

[1] Nakhlé H. Asmar, *Partial Differential Equations with Fourier Series and Boundary Value Problems 2nd Edition*, 2004

[2] Wikipedia