Lecture 5

Fourier Integral and Fourier Transform

What you will learn in Lecture 5

I. The Fourier Integral Representation

II. The Fourier Transform

5.1 The Fourier Integral Representation

5.1 The Fourier Integral Representation

THEOREM Fourier Series Representation: Arbitrary Period

Recall in Lecture 2

Suppose that f_p is a $2p$-periodic piecewise smooth function. The Fourier series of f_p is given by

$$f_p(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right) \qquad (1)$$

where

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f_p(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f_p(x)\cos\frac{n\pi}{p}x\,dx \quad (n = 1, 2, \dots)$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f_p(x)\sin\frac{n\pi}{p}x\,dx \quad (n = 1, 2, \dots)$$

The Fourier series converges to $f_p(x)$ if f_p is continuous at x and to $\dfrac{f_p(x-) + f_p(x+)}{2}$ otherwise.

5.1 The Fourier Integral Representation

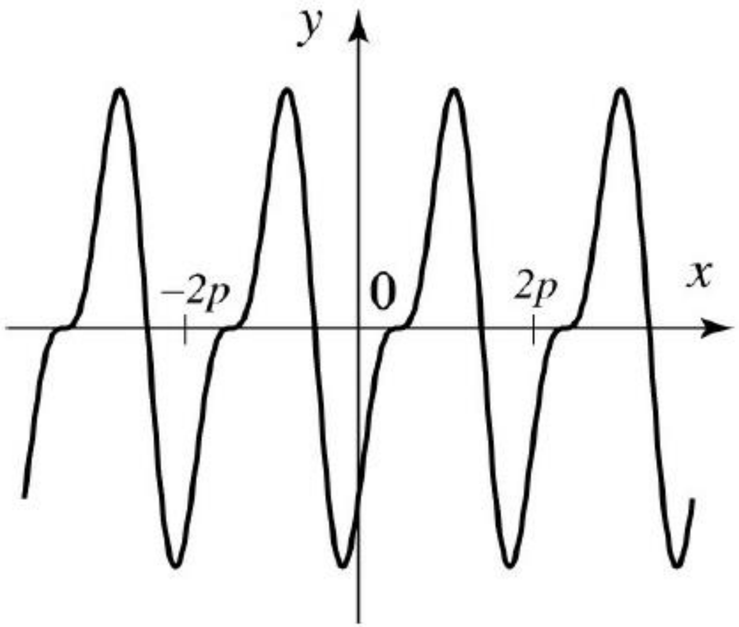
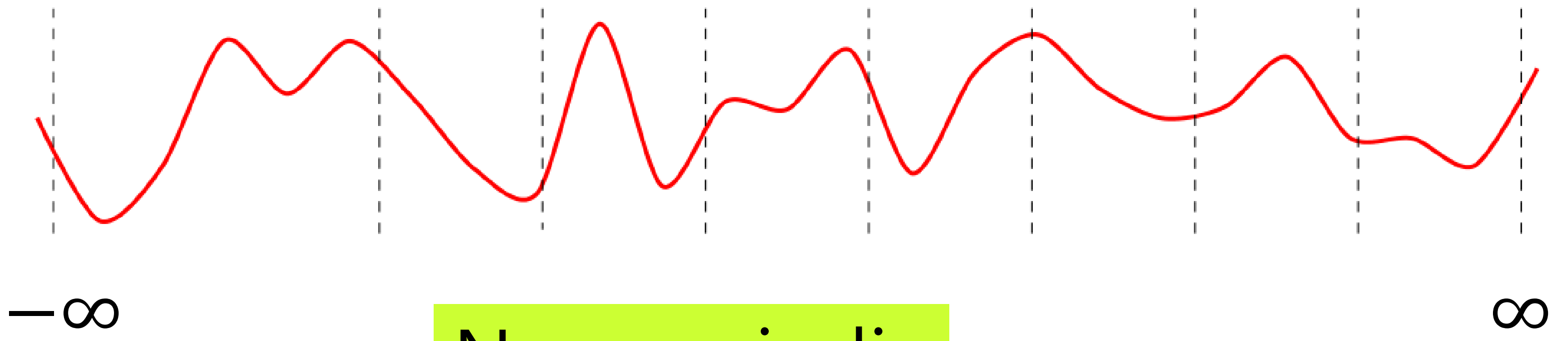
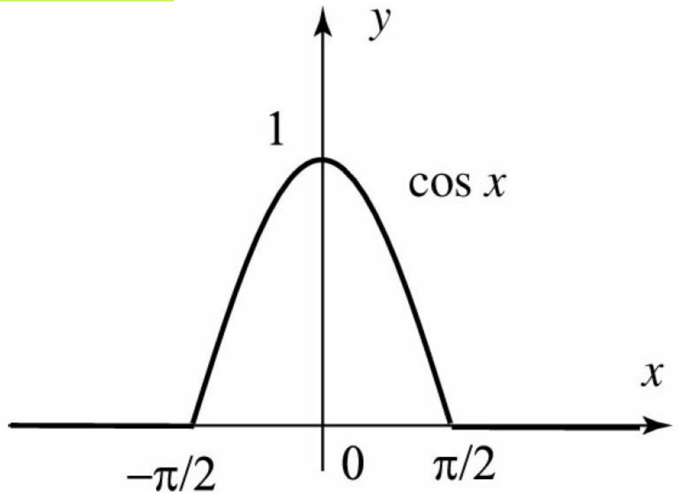


Figure A $2p$-periodic function



Non-periodic



$$\begin{cases} \cos x & \text{if } |x| < \pi/2, \\ 0 & \text{if } |x| > \pi/2. \end{cases}$$

Now **suppose** that f **is defined on the entire real line but is NOT periodic**.

Q: Can we represent f by a Fourier series?

NO. But we can present as **Fourier Integral**.

THEOREM Fourier Series Representation: Arbitrary Period

$$f_p(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f_p(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f_p(x)\cos\frac{n\pi}{p}x\,dx \quad (n = 1, 2, \dots)$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f_p(x)\sin\frac{n\pi}{p}x\,dx \quad (n = 1, 2, \dots)$$

When $p \to \infty$, what will happen to the theorem?

$$\lim_{p\to\infty} a_0 = \lim_{p\to\infty}\frac{1}{2p}\int_{-p}^{p} f_p(x)\,dx = 0$$

For a large p

$$a_n \approx \frac{1}{p}\int_{-\infty}^{\infty} f_p(x)\cos\frac{n\pi}{p}x\,dx \quad (n = 1, 2, \dots)$$

$$= \Delta\omega\frac{1}{\pi}\int_{-\infty}^{\infty} f_p(x)\cos\omega_n x\,dx \qquad \omega_n = \frac{n\pi}{p} \quad \Delta\omega = \frac{\pi}{p}$$

$$= \Delta\omega\cdot A(\omega)$$

Similarly, $b_n \approx \Delta\omega\cdot B(\omega)$

$$\int_{-\infty}^{\infty}|f(x)|dx < \infty \Rightarrow f(x) = \lim_{p\to\infty} f_p(x) \approx 0 + \sum_{n=1}^{\infty}(\Delta\omega\cdot A(\omega)\ \cos\omega_n x + \Delta\omega\cdot B(\omega)\ \sin\omega_n x)$$

$$= \sum_{n=1}^{\infty}(A(\omega)\ \cos\omega_n x + B(\omega)\ \sin\omega_n x)\,\Delta\omega$$

$n \to \infty$ $\omega_n \to \omega$

ω is nonnegative

$$\lim_{\Delta\omega\to 0} f(x) = \int_{0}^{\infty}[A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega \quad (-\infty < x < \infty)$$

5.1 The Fourier Integral Representation

THEOREM 1 Fourier Integral Representation

Suppose that f **is piecewise smooth on every finite interval** and that

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty$$

Then f has the following **Fourier integral representation**

(2) $$f(x) = \int_{0}^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega \quad (-\infty < x < \infty)$$

(3) $$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$$

The integral in (2) converges to $f(x)$ if f is continuous at x and to $\frac{f(x+)+f(x-)}{2}$ otherwise.

5.1 The Fourier Integral Representation

Note the similarity between the Fourier series and Fourier integral

Fourier series

(1) $$f_p(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f_p(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f_p(x)\cos\frac{n\pi}{p}x\,dx \quad (n = 1, 2, \dots)$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f_p(x)\sin\frac{n\pi}{p}x\,dx \quad (n = 1, 2, \dots)$$

Fourier integral

(2) $$f(x) = \int_0^{\infty}[A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega \quad (-\infty < x < \infty)$$

(3) $$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$$

① The **sum** in (1) is replaced by an **integral** in (2)

② The **integrals from** $-p$ **to** p are replaced by **integrals from** $-\infty$ **to** ∞ in (3).

③ In (3), the "Fourier coefficients" are computed **over a continuous range** $\omega \geq 0$, whereas the Fourier coefficients of a periodic function are computed **over a discrete range of values** $n = 0, 1, 2, \dots$.

5.1 The Fourier Integral Representation

EXAMPLE 1 A Fourier integral representation

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & if\ |x| \leq 1 \\ 0 & otherwise \end{cases}$$

Solution

From (3),

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos \omega t\, dt = \frac{1}{\pi}\int_{-1}^{1} 1 \cdot \cos \omega t\, dt = \left[\frac{\sin \omega\, t}{\pi\omega}\right]_{-1}^{1} = \frac{2\sin \omega}{\pi\omega}$$

(here should consider $\omega = 0$, Check Page 391-392 Of the textbook)

$f(x)$ is even

$$B(\omega) = 0$$

For $|x| \neq 1$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin \omega \cos \omega x}{\omega} d\omega$$

For $x = \pm 1$, we have the average

$$f(x) = \frac{1}{2}$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin \omega \cos \omega x}{\omega} d\omega = \begin{cases} 1 & if\ |x| < 1 \\ 1/2 & if\ |x| = 1 \\ 0 & if\ |x| > 1 \end{cases}$$

5.1 The Fourier Integral Representation

Setting $x = 0$ in the integral representation of Example 1 yields the important integral

(5) $$\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega = \frac{\pi}{2}$$

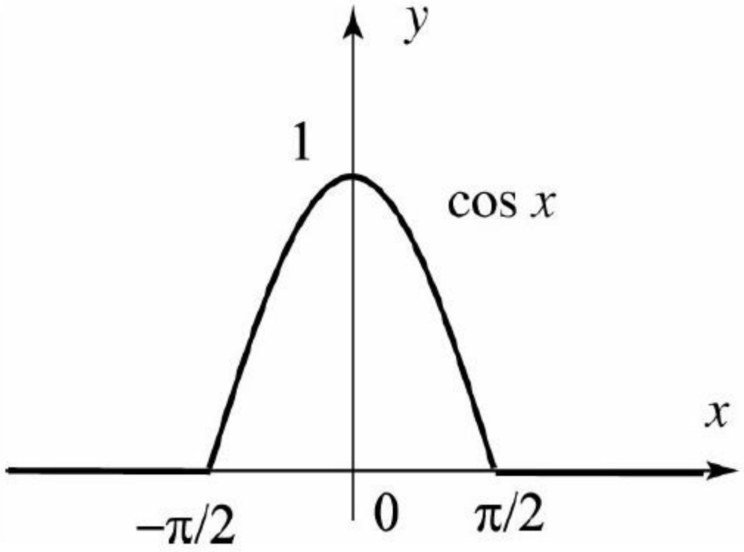
known as the **Dirichlet integral**, after the German mathematician Peter Gustave Lejeune Dirichlet (1805–1859).

5.1 The Fourier Integral Representation

EXAMPLE 2 Computing integrals via the Fourier integral

Show that

$$\frac{2}{\pi}\int_0^\infty \frac{\cos\frac{\pi\omega}{2}}{1-\omega^2}\cos\omega x\, d\omega = \begin{cases} \cos x & if\ |x| < \frac{\pi}{2} \\ 0 & if\ |x| > \frac{\pi}{2} \end{cases}$$



Solution

Tips: Even function, Trigonometric identity

Find the complete solution in page 392-393 of the textbook.

5.1 The Fourier Integral Representation

Partial Fourier Integrals and the Gibbs Phenomenon

In analogy with the partial sums of Fourier series, we define the partial Fourier integral of f by

$$S_\nu(x) = \int_0^\nu [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega \quad (\text{for } \nu > 0)$$

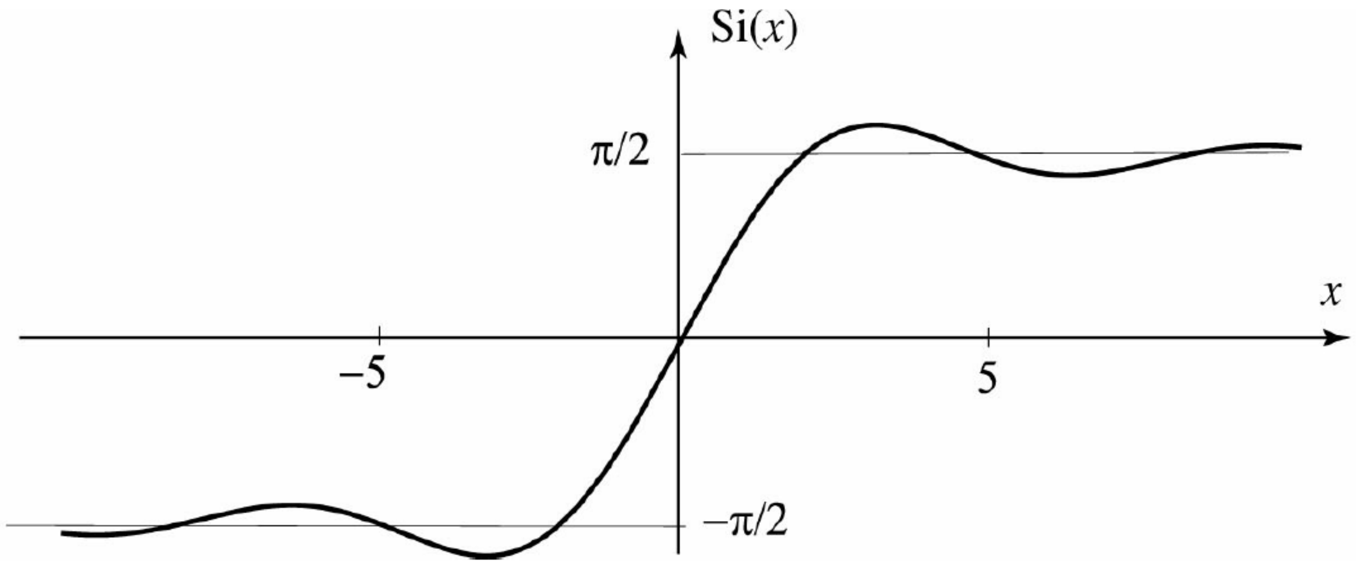
where $A(\omega)$ and $B(\omega)$ are given by (3). With this notation, Theorem 1 states

$$\lim_{\nu\to\infty} S_\nu(x) = \frac{f(x+) + f(x-)}{2}$$

Like Fourier series, near a point of discontinuity the Fourier integral exhibits a **Gibbs phenomenon**. To illustrate this phenomenon, we introduce the **sine integral function**

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt \; (-\infty < x < \infty)$$

5.1 The Fourier Integral Representation



Sine integral function $\mathrm{Si}(x)$

Because of its frequent occurrence, the function $Si(x)$ is tabulated and is available as a standard function in most computer systems. See Figure 3 for its graph. From (5), it follows that

$$\lim_{x\to\infty} \mathrm{Si}(x) = \int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$

EXAMPLE 3 Gibbs phenomenon for partial Fourier integrals

(a) Show that the partial Fourier integral of the function in Example 1 can be written as

$$S_\nu(x) = \frac{1}{\pi}[\mathrm{Si}(\nu(1+x)) + \mathrm{Si}(\nu(1-x))]$$

(b) To illustrate the representation of the function by its Fourier integral, plot several partial Fourier integrals and discuss their behavior near the points $x = \pm 1$.

Solution

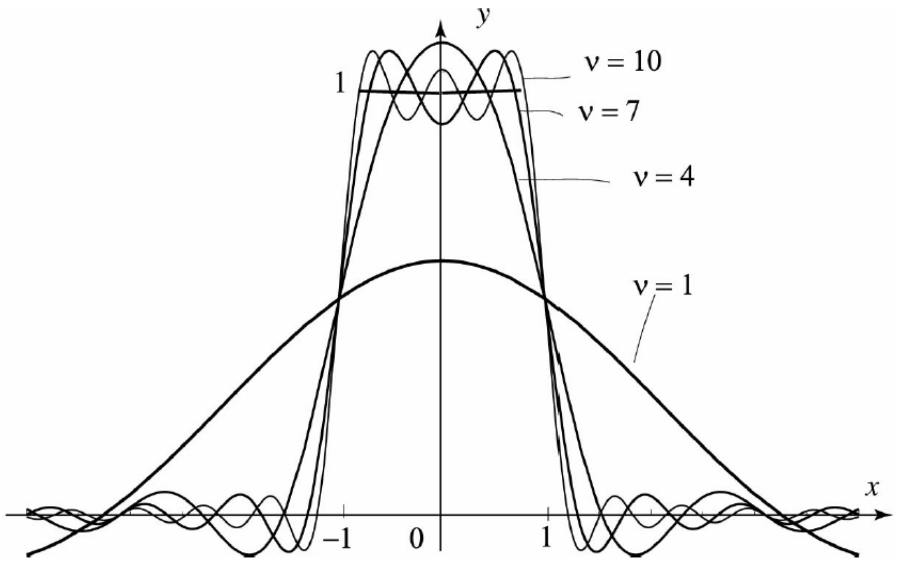


Figure 4 Approximation by partial Fourier integrals and Gibbs phenomenon

Find the complete solution in page 394 of the textbook.

5.2 The Fourier Transform

5.2 The Fourier Transform

Consider **a continuous piecewise smooth integrable function** f. Starting with the **Fourier integral representation**, we have

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$$

$$= \int_0^\infty \left[\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt\cdot\cos\omega x + \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt\cdot\sin\omega x\right]d\omega$$

$$= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(t)(\cos\omega t\cos\omega x + \sin\omega t\sin\omega x)\,dtd\omega$$

$$\cos(a-b) = \cos a\cos b + \sin a\sin b$$

$$= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(t)\cos\omega(x-t)\,dtd\omega$$

$$\cos u = \frac{e^{iu}+e^{-iu}}{2}$$

$$= \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^{\infty} f(t)\left(e^{i\omega(x-t)} + e^{-i\omega(x-t)}\right)dtd\omega$$

$$= \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^{\infty} f(t)e^{i\omega(x-t)}dtd\omega + \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^{\infty} f(t)e^{-i\omega(x-t)}dtd\omega$$

The second term, change ω to $-\omega$, then interval for ω becomes $-\infty$ to 0

5.2 The Fourier Transform

(cont.)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{i\omega(x-t)}dtd\omega$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x}\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt}_{\hat{f}\,(\omega)}d\omega$$

This is the complex form of the Fourier integral representation, which features the following transform pair:

5.2 The Fourier Transform

FOURIER TRANSFORM

(1) $$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \qquad (-\infty < \omega < \infty)$$

INVERSE FOURIER TRANSFORM

(2) $$f(\mathrm{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \qquad (-\infty < x < \infty)$$

5.2 The Fourier Transform

Putting $\omega = 0$ in (1), we find that

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, dx$$

Thus the value of the Fourier transform at $\omega = 0$ is equal to the signed area between the graph of $f(x)$ and the x-axis, multiplied by a factor of $\frac{1}{\sqrt{2\pi}}$

5.2 The Fourier Transform

EXAMPLE 1 **A Fourier transform**

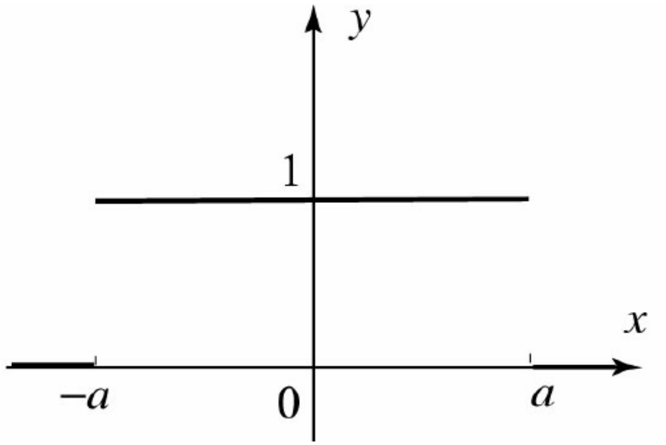
(a) Find the Fourier transform of the function in Figure 1, given by

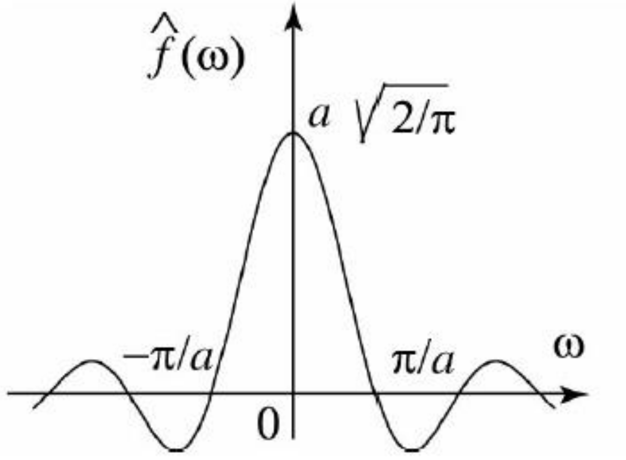
$$f(x) = \begin{cases} 1 & if\ |x| < a \\ 0 & if\ |x| > a \end{cases}$$

What is $\hat{f}(0)$?

(b) Express f as an inverse Fourier transform.

Solution





Find the complete solution in page 399-400 of the textbook.

Review for Lecture 5

- Fourier Integral Representation
- Fourier Transform
- Gibbs Phenomenon

Exercise

Please Check https://github.com/uoaworks/FourierAnalysisAY2018

Reading: Section 7.1, 7.2, Textbook

References

[1] Nakhlé H. Asmar, *Partial Differential Equations with Fourier Series and Boundary Value Problems 2nd Edition*, 2004