Lecture 7

Laplace Transform

What you will learn in Lecture 7

I. Laplace Transform

7.1 Laplace Transform

7.1 Laplace Transform

Laplace transforms are invaluable for any engineer's mathematical toolbox as they make solving **linear Ordinary Differential Equations (ODEs)** and related initial value problems

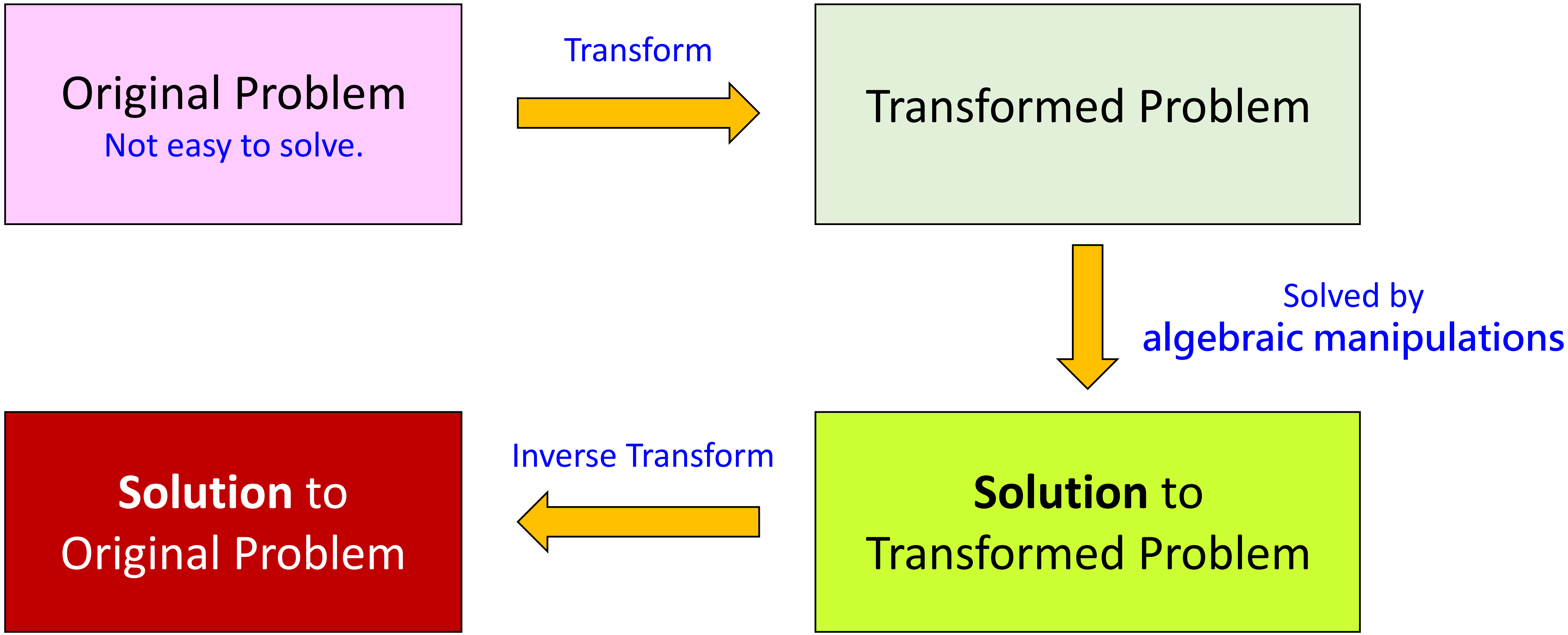
The process of solving an ODE using the **Laplace transform** method consists of three steps:

Step 1. The given **ODE** is **transformed into** an **algebraic equation**, called the **subsidiary equation**.

Step 2. The **subsidiary equation** is **solved by purely algebraic manipulations**.

Step 3. **The solution in Step 2 is transformed back**, resulting in the solution of the given problem.

7.1 Laplace Transform



7.1 Laplace Transform

Suppose that $f(t)$ is defined for all $t \geq 0$. The **Laplace transform** of f is the function

(1) $$F(s) = \mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}dt$$
$$= \int_0^\infty f(t)k(s,t)dt$$

with "**kernel**"

$$k(s,t) = e^{-st}$$

7.1 Laplace Transform

Furthermore, the given function $f(t)$ in (1) is called the **Inverse Laplace Transform** of $F(s)$ and is denoted by $\mathcal{L}^{-1}(\mathcal{L}(f))$; we shall write

(1*) $$f(t) = \mathcal{L}^{-1}\big(\mathcal{L}(f)\big) = \mathcal{L}^{-1}(F)$$

Note that (1) and (1*) together imply that we have a new transform pair

$$\boxed{\begin{aligned} f(t) &= \mathcal{L}^{-1}\big(\mathcal{L}(f)\big) \\ F(s) &= \mathcal{L}\big(\mathcal{L}^{-1}(F)\big) \end{aligned}}$$

Notation

Original functions depend on t and **their transforms on** s,

keep this in mind!

EXAMPLE 1 Find $\mathcal{L}(1), \mathcal{L}(t), \mathcal{L}(e^{at})$

Solutions We compute these transforms using (1). We have

$$\mathcal{L}(1)(s) = \int_0^{\infty} 1 \cdot e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_0^{\infty}$$

$$\int_0^{\infty} 1 \cdot e^{-st} dt = \lim_{T\to\infty} \int_0^{T} e^{-st} dt = \lim_{T\to\infty} \left[-\frac{1}{s} e^{-st}\right]_0^T = \lim_{T\to\infty} \left[-\frac{1}{s} e^{-sT} + \frac{1}{s} e^{0}\right] = \frac{1}{s} \qquad s > 0$$

$$\mathcal{L}(t)(s) = \int_0^{\infty} t \cdot e^{-st} dt = \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st}\right]_0^T = \frac{1}{s^2} \qquad s > 0$$

$$\mathcal{L}(e^{at})(s) = \int_0^{\infty} e^{-(s-a)t} dt = -\frac{1}{s-a} e^{-(s-a)t}\Big|_0^{\infty} = \frac{1}{s-a} \qquad s > a$$

7.1 Laplace Transform

THEOREM 2 Linearity of Laplace Transform

If f and g are functions and α and β are numbers, then

$$\mathcal{L}(\alpha f + \beta g) = \alpha\mathcal{L}(f) + \beta\mathcal{L}(g)$$

Proof

$$\begin{aligned}\mathcal{L}(\alpha f(t) + \beta g(t)) &= \int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]dt \\ &= \alpha\int_0^\infty e^{-st} f(t)dt + \beta\int_0^\infty e^{-st} g(t)dt \\ &= \alpha\mathcal{L}(f(t)) + \beta\mathcal{L}(g(t))\end{aligned}$$

$$\mathcal{L}^{-1}(\alpha F + \beta G) = \alpha\mathcal{L}^{-1}(F) + \beta\mathcal{L}^{-1}(G)$$

Additional EXAMPLE Application of Theorem 1: Hyperbolic Functions

Find the transforms of $\cosh at$ and $\sinh at$

Solutions

$$\mathcal{L}(\cosh at) = \mathcal{L}\left(\frac{e^{at} + e^{-at}}{2}\right) = \frac{1}{2}\left(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})\right) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}$$

$$\mathcal{L}(\sinh at) = \mathcal{L}\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2}\left(\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at})\right) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2}$$

EXAMPLE 3 Application of Theorem 1: Cosine and Sine

Derive the formulas $\mathcal{L}(\cos kt)$ and $\mathcal{L}(\sin kt)$

Solutions

According to Euler's Identity $e^{ikt} = \cos kt + i \sin kt$

$$\mathcal{L}(\cos kt) + i\mathcal{L}(\sin kt) = \int_0^\infty e^{-st}(\cos kt + i \sin kt)dt$$
$$= \int_0^\infty e^{-(s-ik)t}dt$$
$$= -\frac{1}{s-ik}e^{-(s-ik)t}\bigg|_0^\infty$$
$$= \frac{1}{s-ik}$$
$$= \frac{s+ik}{s^2+k^2}$$
$$= \frac{s}{s^2+k^2} + i\frac{k}{s^2+k^2}$$

Equating real and imaginary parts, we have

$$\mathcal{L}(\cos kt) = \frac{s}{s^2+k^2}$$
$$\mathcal{L}(\sin kt) = \frac{k}{s^2+k^2}$$

7.1 Laplace Transform

Table of Laplace Transform. We shall see that from these almost all the others can be obtained by the use of the general properties of the Laplace transform.

Table of Fourier Transforms

$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(\omega)e^{ix\omega}\,d\omega$		$\widehat{f}(\omega)=\mathcal{F}(f)(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}\,dx$
1.	$\begin{cases}1 & \text{if } \lvert x\rvert<a\\ 0 & \text{if } \lvert x\rvert>a\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega}$
2.	$\begin{cases}1 & \text{if } a<x<b\\ 0 & \text{otherwise}\end{cases}$	$\frac{i\,(e^{-i\,b\omega}-e^{i\,a\omega})}{\sqrt{2\,\pi}\omega}$
3.	$\begin{cases}1-\frac{\lvert x\rvert}{a} & \text{if } \lvert x\rvert<a\\ 0 & \text{if } \lvert x\rvert>a\end{cases}\quad a>0$	$2\sqrt{\frac{2}{\pi}}\frac{\sin^2(\frac{a\,\omega}{2})}{a\,\omega^2}$
4.	$\begin{cases}x & \text{if } \lvert x\rvert<a\\ 0 & \text{if } \lvert x\rvert>a\end{cases}\quad a>0$	$i\sqrt{\frac{2}{\pi}}\frac{a\omega\cos(a\omega)-\sin(a\omega)}{\omega^2}$
5.	$\begin{cases}\sin x & \text{if } \lvert x\rvert<\pi\\ 0 & \text{if } \lvert x\rvert>\pi\end{cases}$	$i\sqrt{\frac{2}{\pi}}\frac{\sin(\pi\omega)}{\omega^2-1}$
6.	$\begin{cases}\sin(ax) & \text{if } \lvert x\rvert<b\\ 0 & \text{if } \lvert x\rvert>b\end{cases}\quad a,b>0$	$i\sqrt{\frac{2}{\pi}}\frac{\omega\cos(b\omega)\sin(ab)-a\cos(ab)\sin(b\omega)}{\omega^2-a^2}$
7.	$\frac{1}{a^2+x^2},\ a>0$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert\omega\rvert}}{a}$
8.	$\frac{x}{a^2+x^2},\ a>0$	$-i\sqrt{\frac{\pi}{2}}\operatorname{sgn}\omega\, e^{-a\lvert\omega\rvert}$
9.	$\sqrt{\frac{2}{\pi}}\frac{a}{1+a^2x^2},\ a>0$	$e^{-\frac{\lvert\omega\rvert}{a}}$

⋮

Appendix B, Nakhlé H. Asmar, *Partial Differential Equations with Fourier Series and Boundary Value Problems 2nd Edition*, 2004

EXAMPLE 5 Inverse Laplace transforms

(a) Evaluate $\mathcal{L}^{-1}\left(\frac{2}{4+(s-1)^2}\right)$ (b) Evaluate $\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+3}\right)$

Solutions (a) From the table of Laplace transforms in Appendix B of the textbook, we know

$$\mathcal{L}(e^{at}\sin kt) = \frac{k}{(s-a)^2+k^2}$$

Consider that $a = 1$ and $k = 2$, we have

$$e^t \sin 2t = \mathcal{L}^{-1}\left(\frac{2}{2^2+(s-1)^2}\right)$$

(b) $$\frac{1}{s^2+2s+3} = \frac{1}{(s+1)^2+\sqrt{2}^2} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+\sqrt{2}^2}$$

Consider that $a = -1$ and $k = \sqrt{2}$, we have

$$e^{-t}\sin\sqrt{2}t = \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}^2+(s+1)^2}\right)$$

THEOREM 5 Shifting on the s-axis

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at}f(t)$ has the transform $F(s-a)$, where $s-a>k$. In formulas,

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

or, if we take the inverse on both sides.

$$e^{at}f(t) = \mathcal{L}^{-1}[F(s-a)\ ]$$

Proof

For $s > a + k$

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{at}e^{-st}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a)$$

Additional EXAMPLE s-Shifting: Damped Vibrations. Completing the Square

Known that

$$\mathcal{L}[e^{at}\cos kt] = \frac{s-a}{(s-a)^2+k^2} \qquad \mathcal{L}[e^{at}\sin kt] = \frac{k}{(s-a)^2+k^2}$$

Find the inverse transform of

$$\mathcal{L}[f] = \frac{3s-137}{s^2+2s+401}$$

Solutions

$$f(t) = \mathcal{L}^{-1}\left[\frac{3(s+1) - 140}{(s+1)^2 + 400}\right] = 3\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2 + 20^2}\right] - 7\mathcal{L}^{-1}\left[\frac{20}{(s+1)^2 + 20^2}\right]$$

Taking $a = -1$, $k = 20$, we have

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t)$$

Existence of Laplace Transforms

A piecewise continuous function $f(t)$ has a Laplace transform if it does not grow too fast, say, if for all $t \geq 0$ and some constants M and a it satisfies the "**growth restriction**"

(2) $$|f(t)| \leq Me^{at}$$

We say that f is of **exponential order** if there exist nonnegative numbers a and M.

7.1 Laplace Transform

THEOREM 1 Existence Theorem for Laplace Transforms

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and satisfies (2) for all $t \geq 0$ and some constants M and a, then the Laplace Transform $\mathcal{L}(f)$ exists for all $s > a$.

Proof

Find the complete proof in the page 480 of the textbook.

Uniqueness of Laplace Transform.

If the **Laplace transform of a given function exists**, **it is uniquely determined**.

Conversely, it can be shown that if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length, although they may differ at isolated points.

Hence we may say that the inverse of a given transform is essentially unique.

In particular, if two *continuous* functions have the same transform, they are completely identical.

7.1 Laplace Transform

THEOREM 3 Laplace transform of derivatives

(i) Suppose that f is continuous on $[0, \infty)$ and of exponential order as in (2).

Suppose further that f' is piecewise continuous on $[0, \infty)$ and of exponential order. Then

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0) \tag{3}$$

(ii) More generally, if $f, f', \dots, f^{(n-1)}$ are continuous on $[0, \infty)$ and of exponential order as in (2), and $f^{(n)}$ is piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathcal{L}\left(f^{(n)}\right) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}\mathcal{L}(0) \tag{4}$$

When $n = 2$, (4) gives

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0) \tag{5}$$

Proof

Find the complete proof in the page 483 of the textbook.

THEOREM 4 Derivative of Laplace transform

(i) Suppose $f(t)$ is piecewise continuous and of exponential order. Then

$$\mathcal{L}\big(tf(t)\big)(s) = -\frac{d}{ds}\mathcal{L}(f)(s) \tag{6}$$

(ii) In general, if $f(t)$ is piecewise continuous and of exponential order, then

$$\mathcal{L}\big(t^n f(t)\big)(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}(f)(s) \tag{7}$$

Proof

Find the complete proof in the page 484 of the textbook.

EXAMPLE 4 Derivative of Laplace transform

(a) Evaluate $\mathcal{L}(t \sin 2t)$

(b) Evaluate $\mathcal{L}(t^2 \sin t)$

Solutions

(a) $\mathcal{L}(t \sin 2t) = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right] = \frac{4s}{(s^2+4)^2}$

(b) $\mathcal{L}(t^2 \sin 2t) = -\frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right] = \frac{2(3s^2-1)}{(s^2+1)^3}$

EXAMPLE 7 Inverse Laplace transform of rational functions

Evaluate $\mathcal{L}^{-1}(\frac{1}{s^2+2s-3})$

Solutions

$$\frac{1}{s^2+2s-3} = \frac{1}{(s+1)^2-2^2} = \frac{1}{2}\frac{2}{(s+1)^2-2^2}$$

From the table of Laplace transforms in Appendix B of the textbook, we know

$$\mathcal{L}(e^{at}\sinh kt) = \frac{k}{(s-a)^2-k^2}$$

Taking $a=-1$, $k=2$, we have

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+2s-3}\right) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2-2^2}\right) = \frac{1}{2}e^{-t}\sinh 2t$$

Review for Lecture 7

- Laplace transform

Exercise

Please Check https://github.com/uoaworks/FourierAnalysisAY2018

Reading: Section 8.1, Textbook

References

[1] Nakhlé H. Asmar, *Partial Differential Equations with Fourier Series and Boundary Value Problems 2nd Edition*, 2004

[2] Wikipedia