Lecture 8

Laplace Transform for Solving ODEs

What you will learn in Lecture 8

I. Laplace Transform for Solving ODEs

II. Further Properties of the Laplace Transforms

III. Convolutions and Laplace Transforms

8.1 Laplace Transform for Solving Ordinary Differential Equations (ODEs)

8.1 Laplace Transform for Solving ODEs

Let us now discuss how the Laplace transform method **solves ODEs** and initial value problems.

We consider an initial value problem.

$$\boxed{y'' + ay' + by = r(t), \quad y(0) = K_0, \;\; y'(0) = K_1}$$

where a and b are constant. Here $r(t)$ is the given **input** applied to the system and $y(t)$ is the **output** (*response to the input*) to be obtained.

8.1 Laplace Transform for Solving ODEs

Recall that in the previous Lecture

THEOREM 3 Laplace transform of derivatives

(i) Suppose that f is continuous on $[0, \infty)$ and of exponential order as in (2).

Suppose further that f' is piecewise continuous on $[0, \infty)$ and of exponential order. Then

(3) $$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

(ii) More generally, if $f, f', \dots, f^{(n-1)}$ are continuous on $[0, \infty)$ and of exponential order as in (2), and $f^{(n)}$ is piecewise continuous on $[0, \infty)$ and of exponential order, then

(4) $$\mathcal{L}\left(f^{(n)}\right) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

When $n = 2$, (4) gives

(5) $$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

Proof

Find the complete proof in the page 483 of the textbook.

8.1 Laplace Transform for Solving ODEs

In **Laplace's method** we do three steps:

Step 1. Setting up the subsidiary equation, namely take the Laplace Transform for both sides of the differential equation and use the Theorem 3

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R$$

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R$$

Step 2. Find the solution of the subsidiary equation by algebra

$$Y = \frac{(s + a)y(0) + y'(0) + R}{s^2 + as + b}$$

Step 3. Find the Inversion of Y to obtain $y = \mathcal{L}^{-1}(Y)$

We may reduce the original function by partial fractions to a sum of terms whose inverse can be found from the table of the Laplace Transforms.

EXAMPLE 8 A second order ordinary differential equation

Solve $y'' + y = 2, y(0) = 0, y'(0) = 1$

Solution

Step 1. Setting up the subsidiary equation, namely take the Laplace Transform for both sides of the differential equation and use the Theorem 3

$$[s^2 Y - sy(0) - y'(0)\,] + bY = \mathcal{L}(2) = 2 \cdot \frac{1}{s} = \frac{2}{s}$$

$$(s^2 + 1)Y = \frac{2}{s} + 1$$

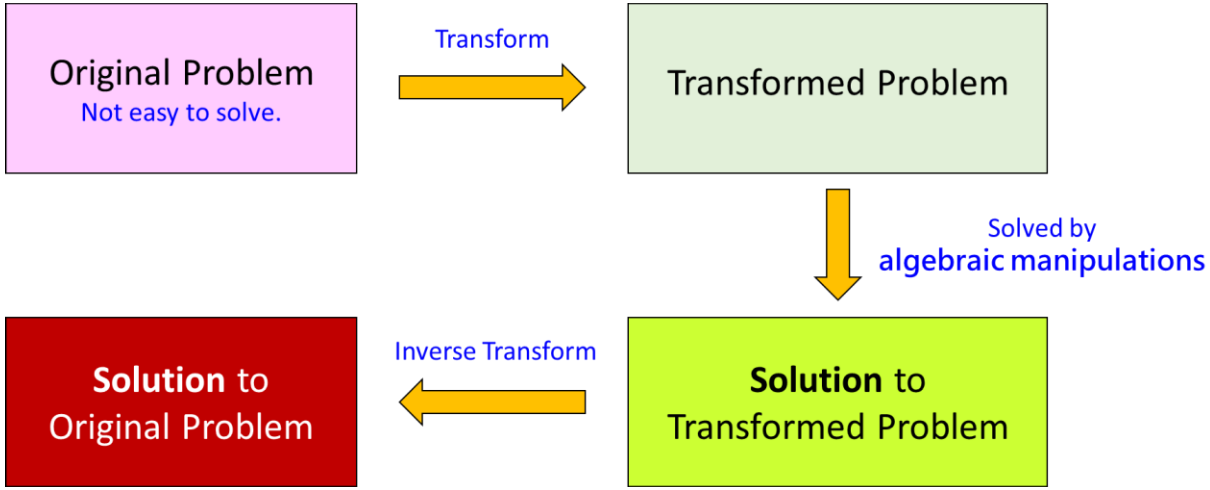
Step 2. Find the solution of the subsidiary equation by algebra

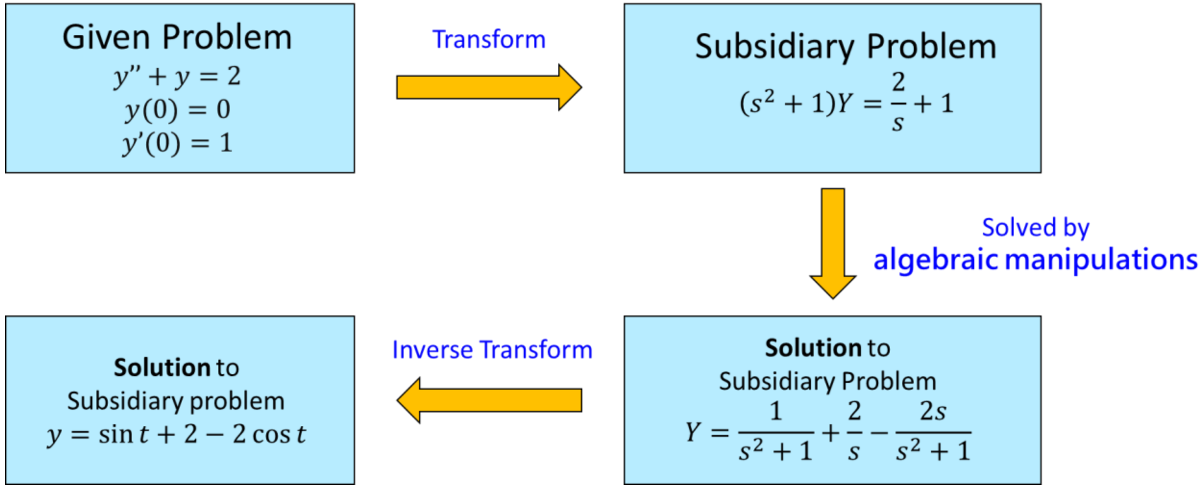
$$Y = \frac{\frac{2}{s} + 1}{s^2 + 1} = \frac{1}{s^2 + 1} + \frac{2}{s(s^2 + 1)} = \frac{1}{s^2 + 1} + \frac{2}{s} - \frac{2s}{s^2 + 1}$$

Step 3. Find the Inversion of Y to obtain

$$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\right) + \mathcal{L}^{-1}\left(\frac{2}{s}\right) - \mathcal{L}^{-1}\left(\frac{2s}{s^2 + 1}\right) = \sin t + 2 - 2\cos t$$

8.1 Laplace Transform for Solving ODEs





EXAMPLE 9 Shifting the time variable

Solve $y'' + 2y' + y = t, y(1) = 0, y'(1) = 0$

Solution

Making the change of variables $t = \tau + 1$, then $\tau = t - 1$

$$y'' + 2y' + y = \tau + 1, y(0) = 0, y'(0) = 0$$

Step 1.

$$s^2Y + 2sY + Y = \mathcal{L}(\tau) + \mathcal{L}(1) = \frac{1}{s^2} + \frac{1}{s}$$

$$(s^2 + 2s + 1)Y = \frac{1}{s^2} + \frac{1}{s}$$

Step 2.

$$Y = \frac{\frac{1}{s^2} + \frac{1}{s}}{s^2 + 2s + 1} = \frac{\frac{s+1}{s^2}}{(s+1)^2} = \frac{1}{s^2(s+1)} = \frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}$$

Step 3.

$$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = e^{-\tau} - 1 + \tau$$

Making the change of variables $\tau = t - 1$

$$y = e^{1-t} + t - 2$$

8.1 Laplace Transform for Solving ODEs

Advantages of the Laplace Method

1. Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE.

2. Initial values are automatically taken care of.

3. Complicated inputs $r(t)$ (right sides of linear ODEs) can be handled very efficiently, as we show in the next sections.

Homogeneous and non-homogeneous equations

Recall that a linear differential equation

$$f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x)y = g(x)$$

was called homogeneous if $g(x) = 0$, and non-homogeneous or inhomogeneous otherwise.

8.2 Further Properties of the Laplace Transforms

8.2 Further Properties of the Laplace Transforms

We introduce an auxiliary function:

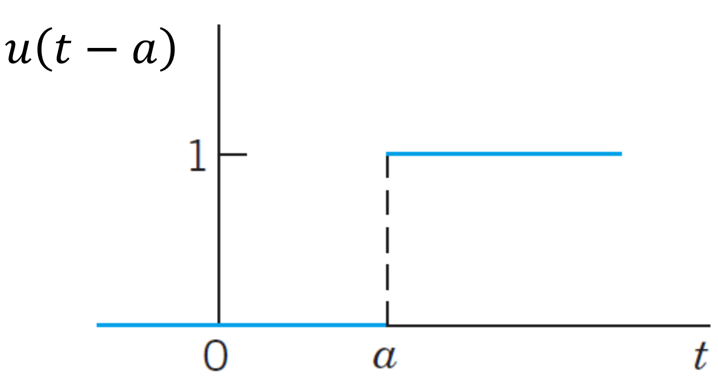
the *unit step function* or *Heaviside function*

These functions are suitable for solving ODEs with complicated right sides of **considerable engineering interest**, such as **single waves, inputs** (driving forces) that **are discontinuous or act for some time only, periodic inputs more general than just cosine and sine**, or **impulsive forces acting for an instant**.

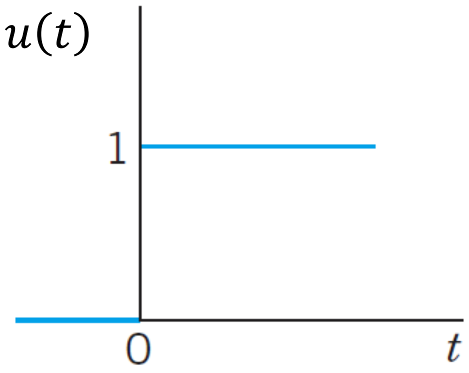
Unit Step Function (Heaviside Function)

The **unit step function** or **Heaviside function** $u(t-a)$ is 0 for $t<a$, has a jump of size 1 at $t=a$ (where we can leave it undefined), and is 1 for in a formula

$$u(t-a)=\begin{cases}0 & if\ t<a\\1 & if\ t\geq a\end{cases}\qquad a\geq 0$$



Unit step function $u(t-a)$



Unit step function $u(t)$, when $a=0$

8.2 Further Properties of the Laplace Transforms

Given a function $f(t)$, consider the product $u(t-a)f(t-a)$. Written explicitly, we have

$$u(t-a)f(t-a) = \begin{cases} 0 & if\ t < a \\ f(t-a) & if\ t \geq a \end{cases}$$

Thus, if $f(t)$ represents, say a signal, then $u(t-a)f(t-a)$ represents the same signal, but delayed by a units of time.

8.2 Further Properties of the Laplace Transforms

THEOREM 1 Shifting on The t-Axis

If a is a positive real number, then

$$\mathcal{L}\big(u(t-a)f(t-a)\big)(s) = e^{-as}F(s)$$

where $F(s) = \mathcal{L}(f(t))(s)$.

Proof

$$\begin{aligned}
\mathcal{L}\big(u(t-a)f(t-a)\big)(s) &= \int_a^{\infty} f(t-a)e^{-st}dt \\
&= \int_0^{\infty} f(T)e^{-s(T+a)}dT \quad (\text{where } t-a=T, dt=dT) \\
&= e^{-as}\int_0^{\infty} f(T)e^{-sT}dT \\
&= e^{-as}F(s)
\end{aligned}$$

8.2 Further Properties of the Laplace Transforms

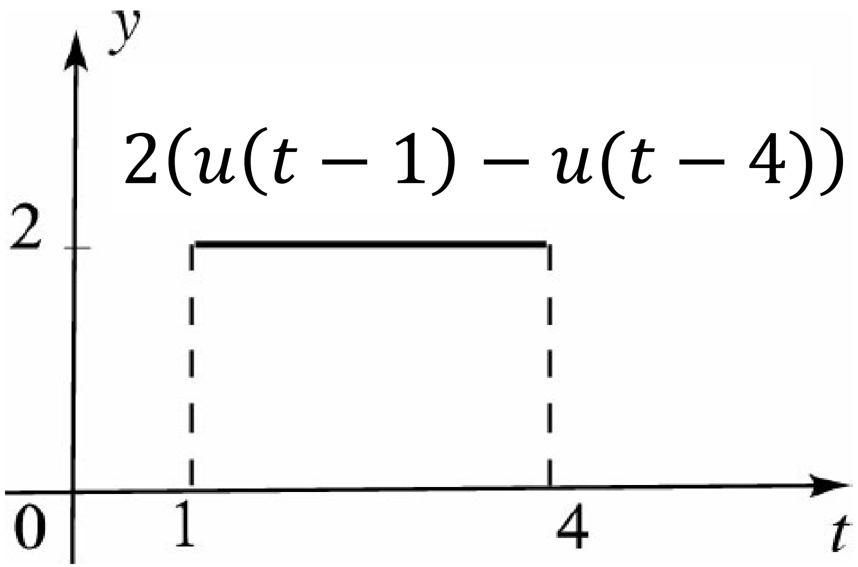
The unit step function is a typical "engineering function" made to measure for engineering applications, which often involve functions (mechanical or electrical driving forces) that are either "off" or "on."

EXAMPLE 1 Transforms involving unit step functions

(a) Evaluate $\mathcal{L}(u(t-a))$

(b) Evaluate $\mathcal{L}(f(t))$ where

$$f(t) = \begin{cases} 2 & if\ 1 \le t < 4 \\ 0 & \text{otherwise} \end{cases}$$



8.2 Further Properties of the Laplace Transforms

Solutions

(a) The transform of follows directly from the defining integral in

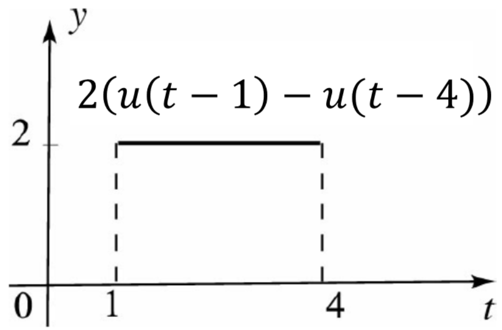
$$\mathcal{L}\big(u(t-a)\big) = \int_0^\infty e^{-st}u(t-a)dt = \int_a^\infty e^{-st}\cdot 1dt = -\frac{e^{-st}}{s}\Big|_{t=a}^{\infty}$$

here the integration begins at $t = a(\geq 0)$ because $u(t-a)$ is 0 for $t < a$. Hence

$$\mathcal{L}\big(u(t-a)\big) = \frac{e^{-as}}{s} \qquad s > 0$$

(b) Verify that $f(t) = 2(u(t-1) - u(t-4))$ Then

$$\mathcal{L}\big(f(t)\big) = 2\left(\mathcal{L}\big(u(t-1)\big) - \mathcal{L}(u(t-4))\right) = \frac{2}{s}(e^{-s} - e^{-4s}),\ s > 0$$



8.2 Further Properties of the Laplace Transforms

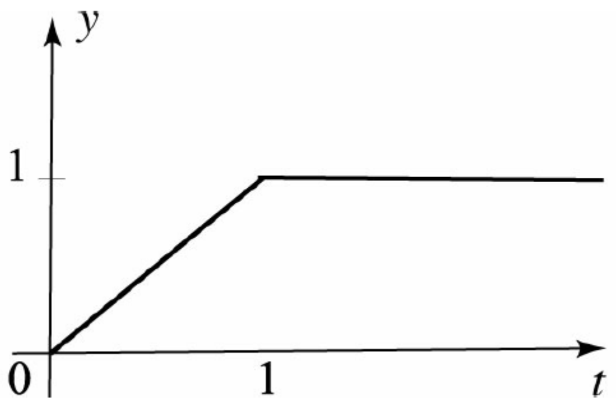


Figure A ramp function

EXAMPLE 2 A ramp function

Evaluate the Laplace transform of the ramp function shown in Figure 3.

Solution

For $t > 1$, we have $f(t) = u(t-1)$, and for $0 < t < 1$, we have $f(t) = t(u(t) - u(t-1))$. Verify that, by combining these two formulas, we have

$$f(t) = t(u(t) - u(t-1)) + u(t-1)$$

Rewrite f(t) as follows:

$$f(t) = -(t-1)u(t-1) + tu(t) = -(t-1)u(t-1) + t$$

Recall that $\mathcal{L}(t) = \frac{1}{s^2}$. And by Theorem 1,

$$\mathcal{L}\big((t-1)u(t-1)\big) = \frac{e^{-s}}{s^2}$$

Hence

$$\mathcal{L}(f(t)) = -\frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

8.2 Further Properties of the Laplace Transforms

EXAMPLE 3 A nonhomogeneous differential equation

Solve $y'' + y = f(t), y(0) = 0, y'(0) = 0$, where $f(t)$ is as in Example 2.

$$f(t) = -(t-1)u(t-1) + tu(t) = -(t-1)u(t-1) + t$$

Solutions

Step 1.

$$s^2Y + Y = -\frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

$$(s^2+1)Y = -\frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

Step 2.

$$Y = \frac{1-e^{-s}}{s^2(s^2+1)} = (1-e^{-s})\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) = \frac{1}{s^2} - \frac{1}{s^2+1} - e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)$$

Step 3. Using Inverse Laplace Transform and Theorem 1, we have

$$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \mathcal{L}^{-1}\left(e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)\right)$$

$$= t - \sin t + u(t-1)[(t-1) - \sin(t-1)]$$

8.3 Convolutions and Laplace Transforms

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Given two functions f and g, defined for all $t > 0$, we define their **convolution** $(f * g)(t)$ by

$$\boxed{(2) \qquad (f * g)(t) = \int_o^t f(t-\tau)g(\tau)d\tau \qquad \text{for all } t \geq 0}$$

8.3 Convolutions and Laplace Transforms

THEOREM 2 Transforms of Convolutions

Suppose that f and g are piecewise continuous and of exponential order;

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(\mathrm{g})$$

Proof

If we extend the functions f and g to be zero for t<0, then the integral in (2) is same as

$$\int_0^\infty f(t-\tau)g(\tau)\,d\tau$$

Thus, throughout this proof, we assume that f and g are extended to the whole line with $f(t) = 0$ and $g(t) = 0$, For all $t < 0$. We have

$$\mathcal{L}(f * g)(s) = \int_0^\infty \left[\int_0^\infty f(t-\tau)g(\tau)\,d\tau\right] e^{-st} dt$$

$$= \int_0^\infty \int_0^\infty f(t-\tau)e^{-st} dt\, g(\tau) d\tau \quad \text{(interchange order of integration)}$$

$$= \int_0^\infty \int_0^\infty f(u)e^{-s(u+\tau)} du\, g(\tau) d\tau \quad (u = t-\tau, du = dt)$$

$$= \int_0^\infty \int_0^\infty f(u)e^{-su} du\, g(\tau)e^{-s\tau} d\tau \;\; = \mathcal{L}(f)\mathcal{L}(\mathrm{g})$$

EXAMPLE 5 Transforms involving convolutions

(a) Evaluate $\mathcal{L}\left(\int_0^t (t-\tau)\sin(\tau)d\tau\right)$

Solution

(a) From Theorem 2, we have

$$\mathcal{L}\left(\int_0^t (t-\tau)\sin(\tau)d\tau\right) = \mathcal{L}(t)\mathcal{L}(\sin t) = \frac{1}{s^2}\frac{1}{s^2+1}$$

8.3 Convolutions and Laplace Transforms

Additional EXAMPLE Transforms involving convolutions

Let $H(s) = \frac{1}{(s-a)s}$. Find $h(t)$.

Solution

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{(s-a)s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-a} \cdot \frac{1}{s}\right) = \mathcal{L}^{-1}(F(s) \cdot G(s))$$

$$F(s) = \frac{1}{s-a} \quad \rightarrow \ \mathcal{L}^{-1}\big(F(s)\big) = e^{as}$$

$$G(s) = \frac{1}{s} \quad \rightarrow \ \mathcal{L}^{-1}\big(G(s)\big) = 1$$

By using Theorem 2.

$$h(t) = \mathcal{L}^{-1}(\mathcal{L}(e^{as})\mathcal{L}(1)) = e^{at} * 1 = \int_0^t e^{a\tau} \cdot 1 \mathrm{d}\tau = \frac{1}{a}(e^{at} - 1)$$

EXAMPLE 6 Solving differential equations with convolutions

Express the solution of the initial value problem.

$$y'' - 2y' + 5y = f(t), \quad y(0) = 0, \; y'(0) = 0$$

as a convolution.

Solutions

Step 1.

$$s^2Y - 2sY + 5Y = F(s)$$

$$(s^2 - 2s + 5)Y = F(s)$$

Step 2.

$$Y = \frac{F(s)}{s^2 - 2s + 5} = F(s)\frac{1}{(s-1)^2+2^2} = F(s)\left[\frac{1}{2}\frac{2}{(s-1)^2+2^2}\right]$$

Step 3. Find the Inverse Laplace Transform in the table and using Theorem 2, we have

$$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(F(s)\left[\frac{1}{2}\frac{2}{(s-1)^2+2^2}\right]\right) = f(t) * \frac{1}{2}e^t \sin 2t$$

$$= \frac{1}{2}\int_0^t e^{t-\tau} \sin 2(t-\tau) f(\tau) d\tau$$

Review for Lecture 8

- Laplace Transform for Solving ODEs
- Further Properties of Laplace Transforms
- Convolution and Laplace Transforms

Exercise

Please Check https://github.com/uoaworks/FourierAnalysisAY2018

Reading: Section 8.2, Textbook

References

[1] Nakhlé H. Asmar, *Partial Differential Equations with Fourier Series and Boundary Value Problems 2nd Edition*, 2004

[2] Erwin Kreyszig, *Advanced Engineering Mathematics 9th Edition*, 2005

[3] Wikipedia