Lecture 9

- *Sampling Theorem
- Discrete Fourier Analysis

What you will learn in Lecture 9

*9.1 Sampling Theorem

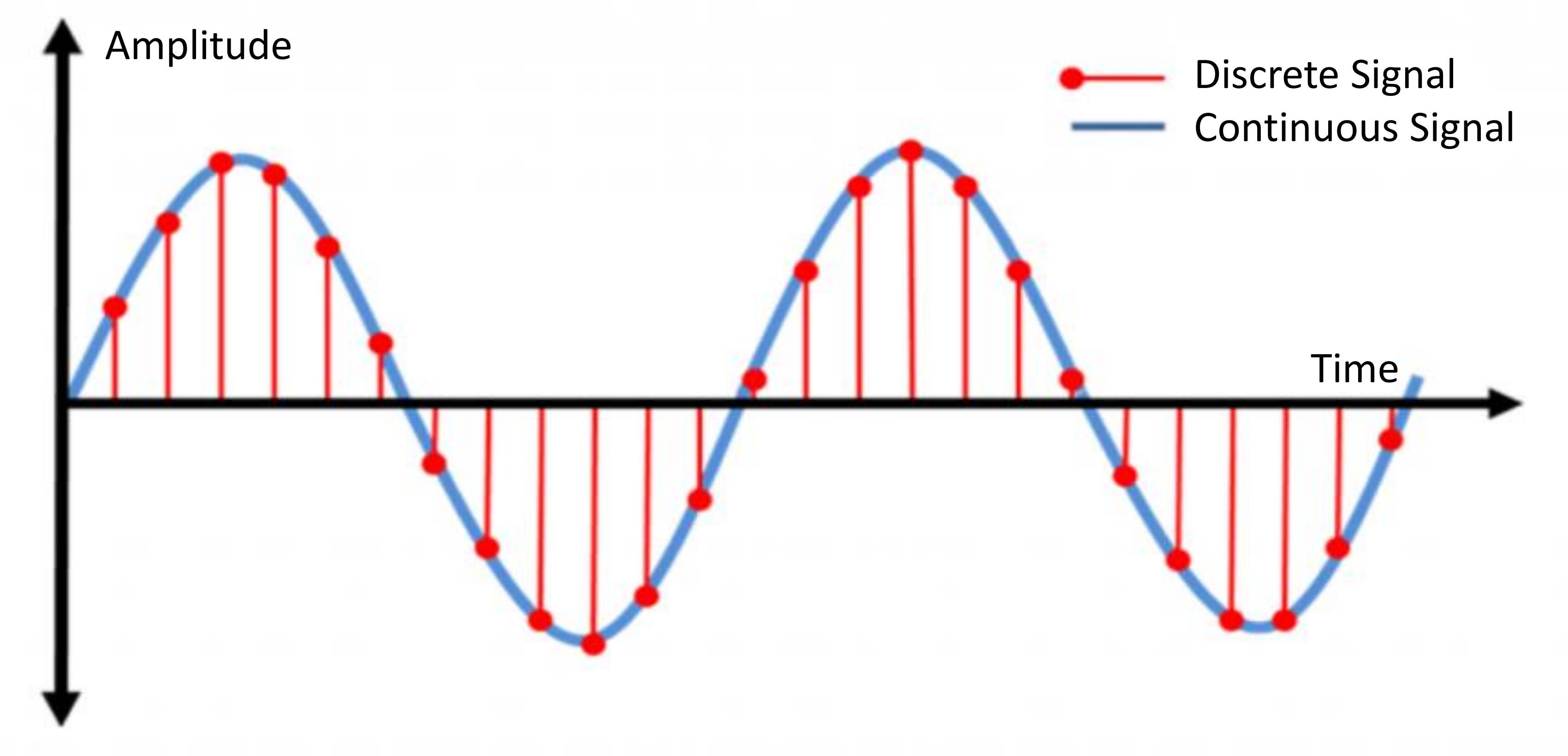
9.2 Discrete Fourier Analysis

Notice: * mark indicates the content that is not in the syllabus, but helpful as the preparation knowledge.

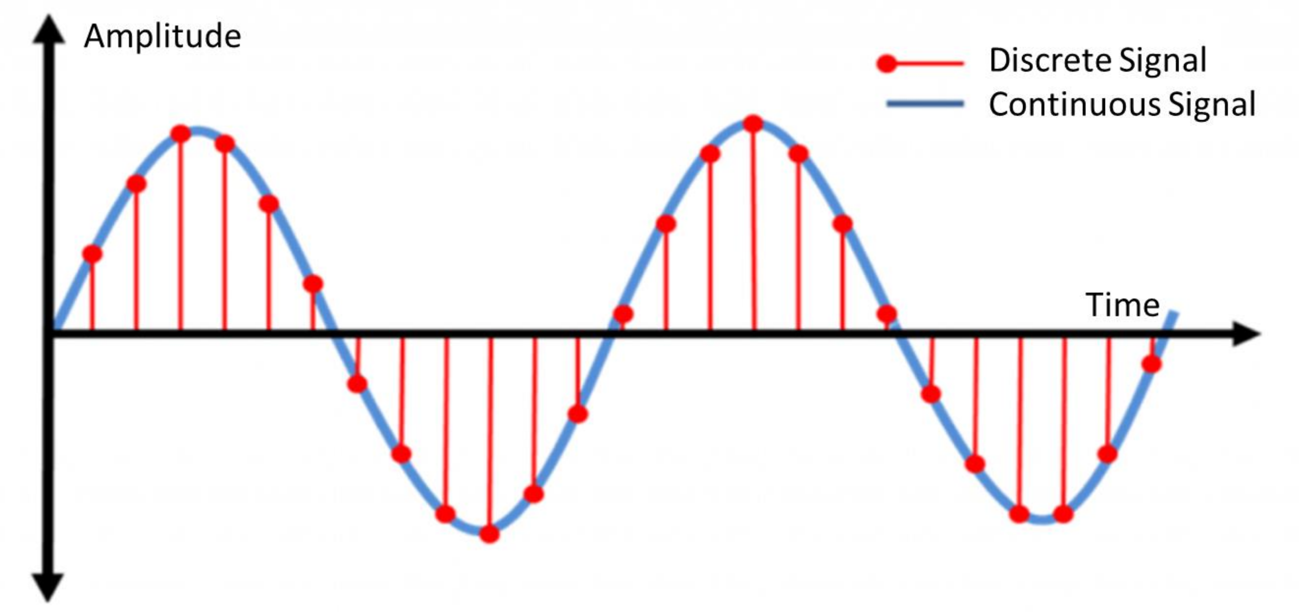
*9.1 Sampling Theorem

9.1 Sampling Theorem

The **sampling theorem** is a striking result that states that **certain functions can be reconstructed completely** from a discrete set of measurements or samples taken at equal intervals.



9.1 Sampling Theorem

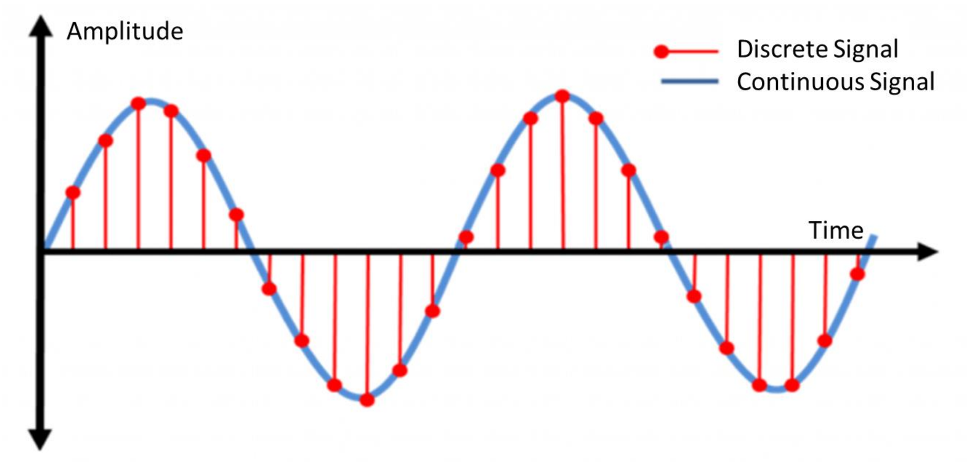


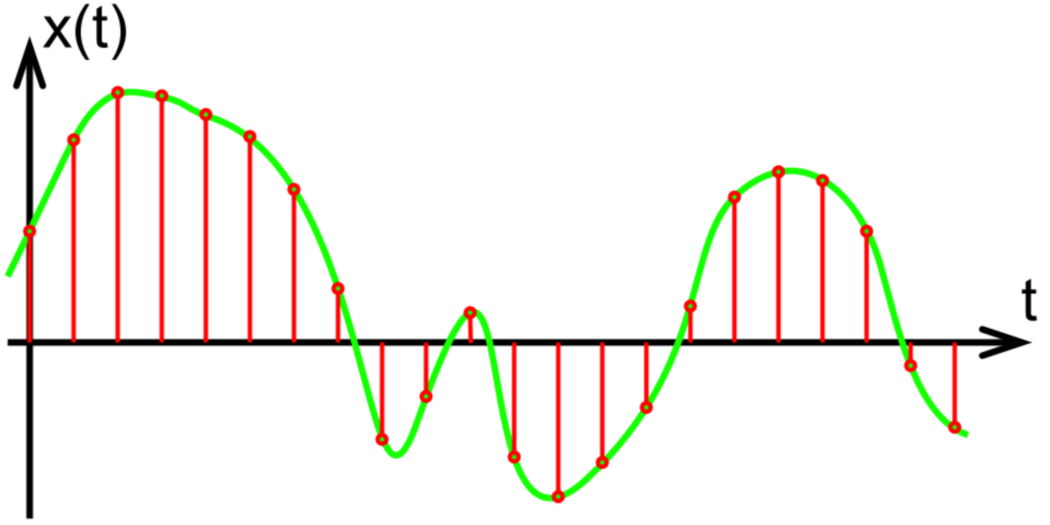
Let continuous function $f(x)$ be periodic, for simplicity of period 2π. We assume that N measurements of $f(x)$ are taken over the interval $0 \leq x \leq 2\pi$ at regularly spaced points

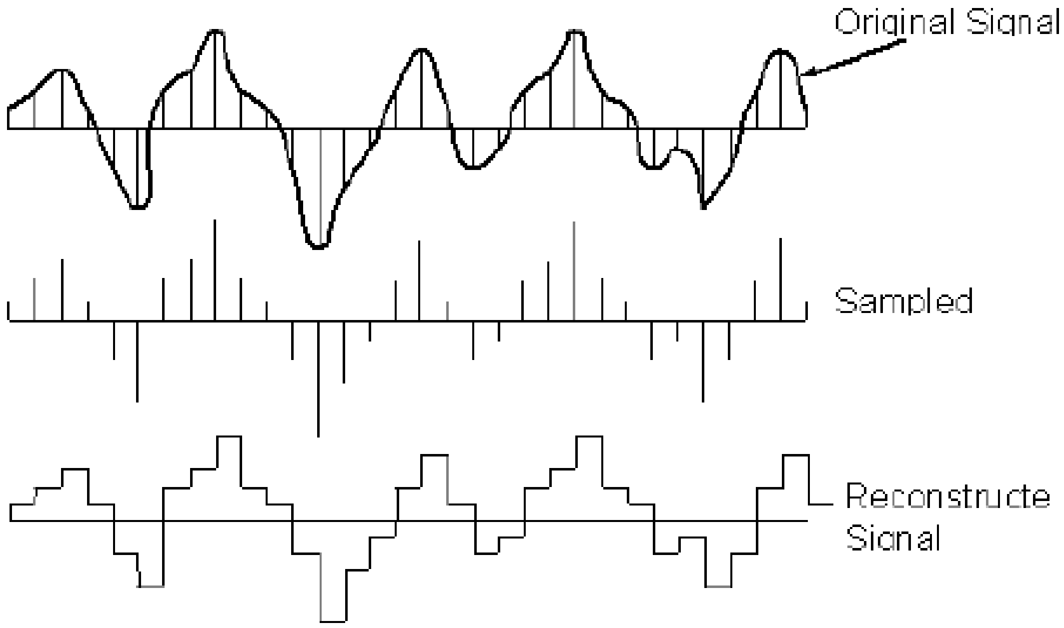
$$x_n = \frac{2\pi n}{N} \qquad n = 0,1,\cdots,N-1$$

We also say that $f(x)$ is being **sampled** at these points.

9.1 Sampling Theorem







9.1 Sampling Theorem

BAND LIMITED FUNCTIONS

A function $f(x)$ is called **band limited** if its **Fourier transform $\hat{f}(\omega)$ vanishes outside a finite interval**. In this case, there is a positive number W such that $\hat{f}(\omega) = 0$ for all $|\omega| > W$. Any such number W is called a **band width** of f.

- It is important to note that only $\hat{f}$ is required to vanish outside a finite interval and not f.
- Indeed, it can be shown that f and $\hat{f}$ cannot both vanish off a finite interval.
- Usually, the "most" functions are not band limited. We can show that at least all the functions that we have dealt with in this text can be approximated as closely as we want by band limited functions.

9.1 Sampling Theorem

EXAMPLE 1 A band limited function

(a) The function $s(x) = \frac{\sin x}{x}$ is shown in Figure 1. From the table of Fourier transforms, we have

$$\hat{s}(\omega)\begin{cases} \sqrt{\frac{\pi}{2}} & if\ |\omega| < 1 \\ \frac{1}{2}\sqrt{\frac{\pi}{2}} & if\ \omega = \pm 1 \\ 0 & if\ |\omega| > 1 \end{cases}$$

Since $\hat{s}$ vanishes for all $|\omega| > 1$, we conclude that s is band limited with band width 1.

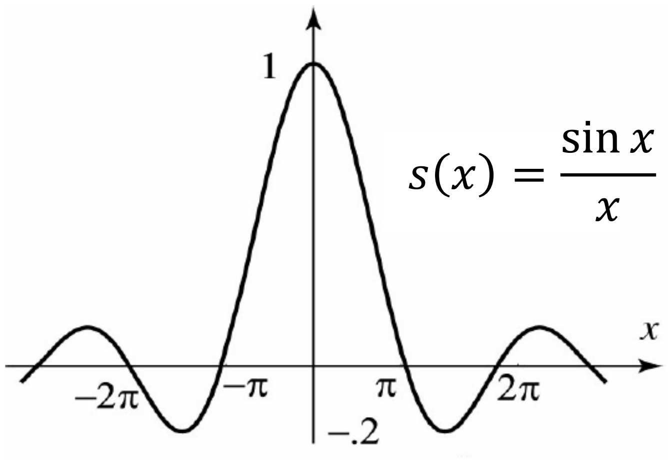


Figure 1 The function $s(x) = \frac{\sin x}{x}$ is band limited with band width 1.

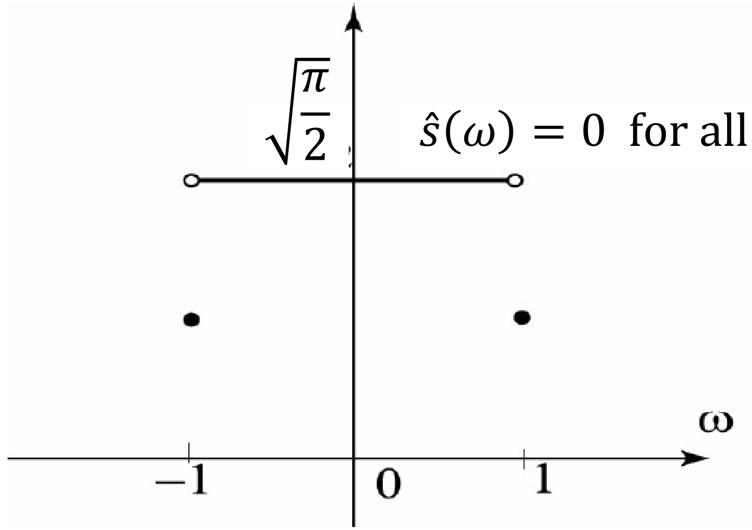


Figure 2 The Fourier transform of $s(x) = \frac{\sin x}{x}$ vanishes for all $|\omega| > 1$

EXAMPLE 1 A band limited function

(b) The function $s_W(x) = \dfrac{\sin Wx}{Wx}$ is shown in Figure 3. From the table of Fourier transforms, we have

$$\widehat{s_W}(\omega) \begin{cases} \dfrac{1}{W}\sqrt{\dfrac{\pi}{2}} & if\ |\omega| < W \\ \dfrac{1}{2W}\sqrt{\dfrac{\pi}{2}} & if\ \omega = \pm W \\ 0 & if\ |\omega| > W \end{cases}$$

Since $\hat{s}$ vanishes for all $|\omega| > W$, we conclude that s is band limited with band width W.

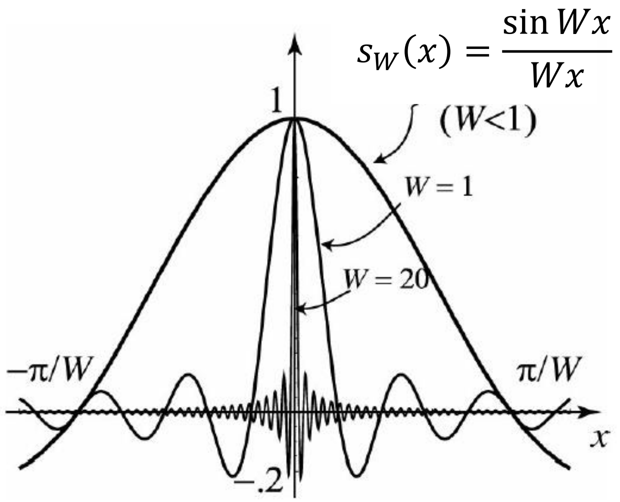


Figure 3

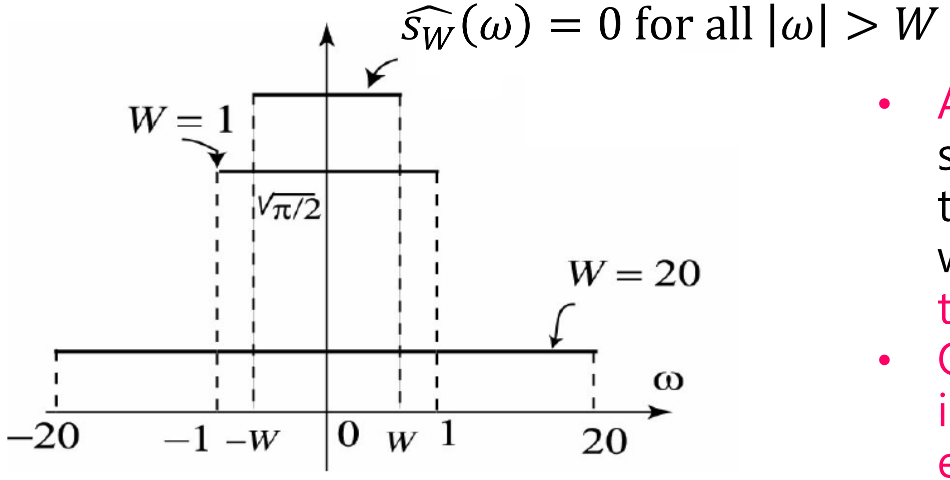


Figure 4

- As W increases, the frequency of $\sin Wx$ increases, and this causes the graph of $s_W(x)$ to be more wavy and more squished toward the origin (Figure 3).
- On the Fourier transform side, increasing W has the opposite effect of spreading or stretching the graph.

9.1 Sampling Theorem

THEOREM 1 Properties of Band Limited Functions

Suppose that a, b are constants and $f(x), g(x)$ are functions.

(a) If $f(x)$ and $g(x)$ are band limited with band widths W_1 and W_2, respectively, then $af + bg$ is band limited with band width W smaller than or equal to the larger of W_1 and W_2.

(b) If $f(x)$ is band limited with band width W, then the translate of $f(x)$ by a, $f(x - a)$, is also band limited with band width W.

(c) If either $f(x)$ or $g(x)$ is band limited with band width W, then the convolution $f * g$ is also band limited with band width W.

9.1 Sampling Theorem

THEOREM 2 Sampling Theory by Band Limited Functions

Suppose that f is band limited with band width W. Then for all x we have

$$(3) \qquad f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{W}\right) \frac{\sin(Wx - nx)}{(Wx - nx)}$$

Thus f can be constructed completely from its sample values $f\left(\frac{n\pi}{W}\right)$, $n = 0, \pm 1, \pm 2, \ldots$

If W and W' are two band widths of f with $W' > W$, then the series corresponding to W requires more sample points per unit length than the one for W, since $\frac{\pi}{W} > \frac{\pi}{W'}$. The least number of sample points per unit length that is required for (3) to hold is called the Nyquist sampling rate and corresponds to the least band width of f.

9.1 Sampling Theorem

THEOREM 3 Sampling Theorem for Time Limited Functions

Suppose that $f(t) = 0$ for all $|t| > T$. Then for all ω we have

$$\text{(4)} \qquad \hat{f}(\omega) = \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{n\pi}{T}\right) \frac{\sin(T\omega - n\pi)}{(T\omega - n\pi)}$$

Thus $\hat{f}$ is completely determined by sampling at the points $\frac{n\pi}{T}$, $n = 0, \pm 1, \pm 2, \cdots$

9.1 Sampling Theorem

EXAMPLE 2 Sampling of a band limited function

The initial temperature distribution of a bar, $f(x)$ $(-\infty < x < \infty)$, has band width $W = 2$. Some of its values are shown in Table 1.

x	0	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$
$f(x)$	0.001765	0.382839	0.792567	0.266685	0.001474	0.033190

Using the sampling theorem, approximate the initial heat distribution at the points 0.1, 0.2, 0.8.

Solution

$$f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{W}\right)\frac{\sin(Wx - nx)}{(Wx - nx)}$$

$$f^*(x) \approx \sum_{n=0}^{5} f\left(\frac{n\pi}{2}\right)\frac{\sin(2x - nx)}{(2x - nx)}$$

Sampling at point $x = \frac{n\pi}{2}$

$$= 0.001765\frac{\sin(2x - 0\cdot x)}{(2x - 0\cdot x)} + 0.382895\frac{\sin(2x - 1\cdot x)}{(2x - 1\cdot x)} + 0.792567\frac{\sin(2x - 2\cdot x)}{(2x - 2\cdot x)} + \cdots + 0.033190\frac{\sin(2x - 5\cdot x)}{(2x - 5\cdot x)}$$

Therefore, by substituting, we can obtain that $f(0.1) = 0.0079$, $f(0.1) = 0.0159$, $f(0.1) = 0.1165$

9.1 Sampling Theorem

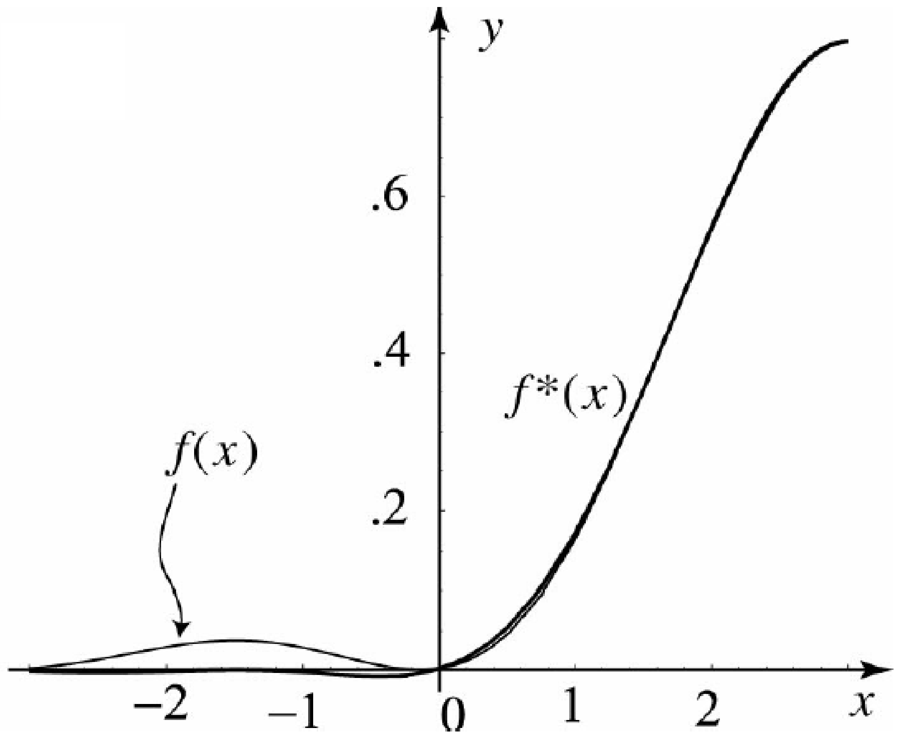


Figure 5 It shows that the approximation of f by its partial sum is much better on the interval $x > 0$ compared to the interval $x < 0$. This is to be expected, since all the sample points were chosen from the interval $x \geq 0$. (In real-life applications f is unknown, except for its sampled values.)

9.2 Discrete Fourier Analysis

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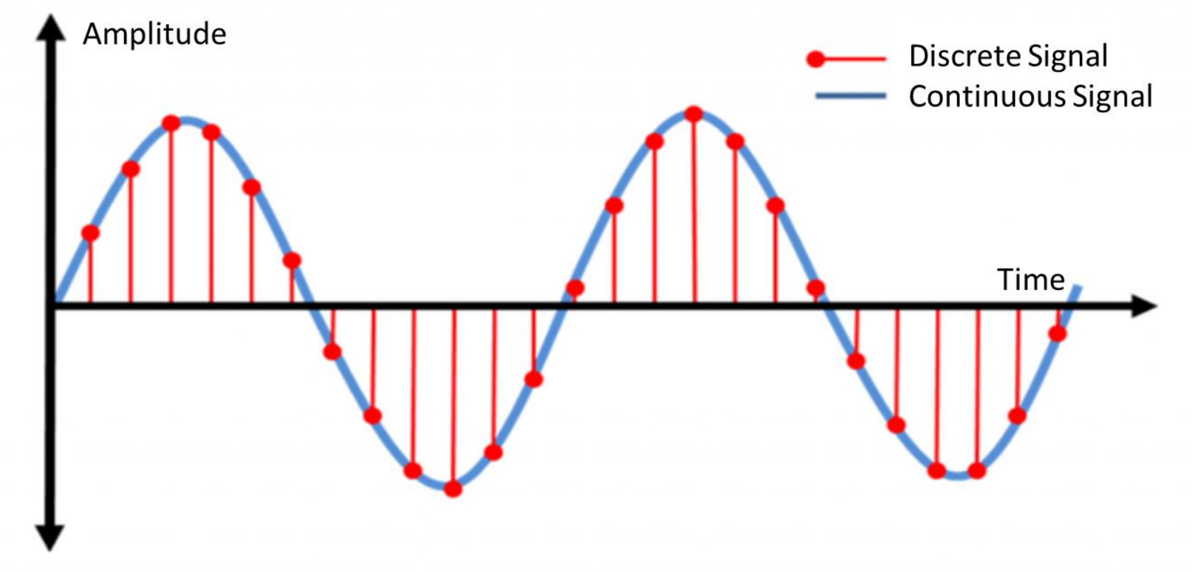
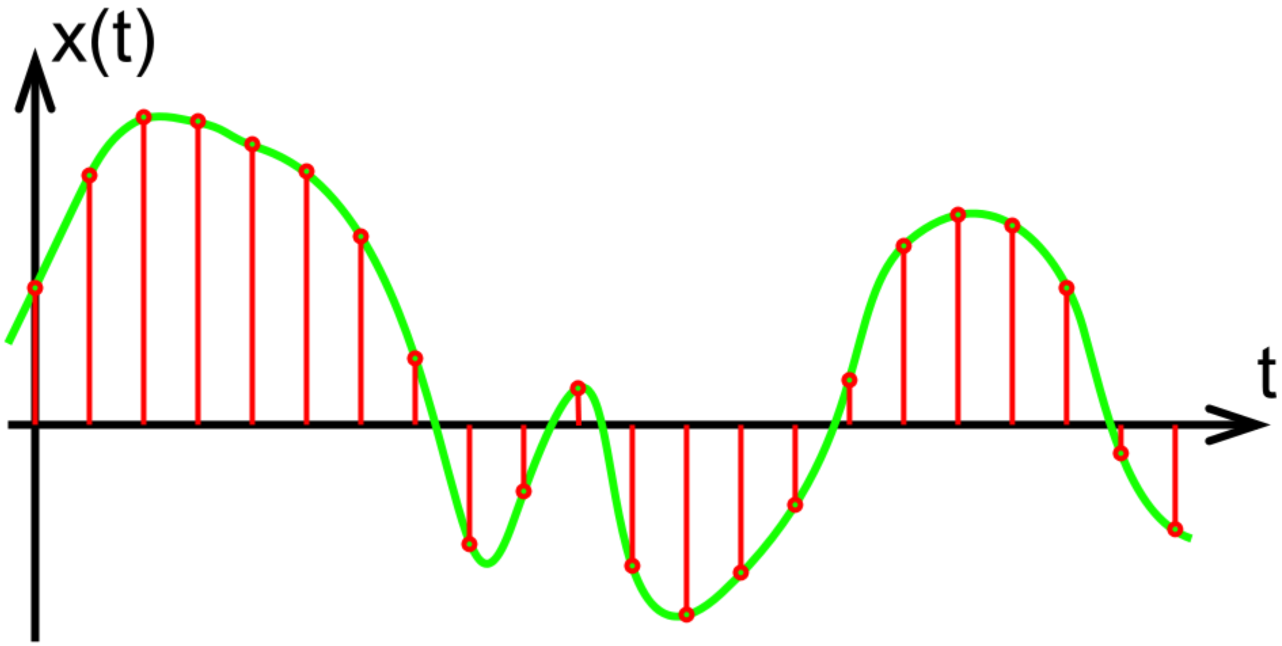
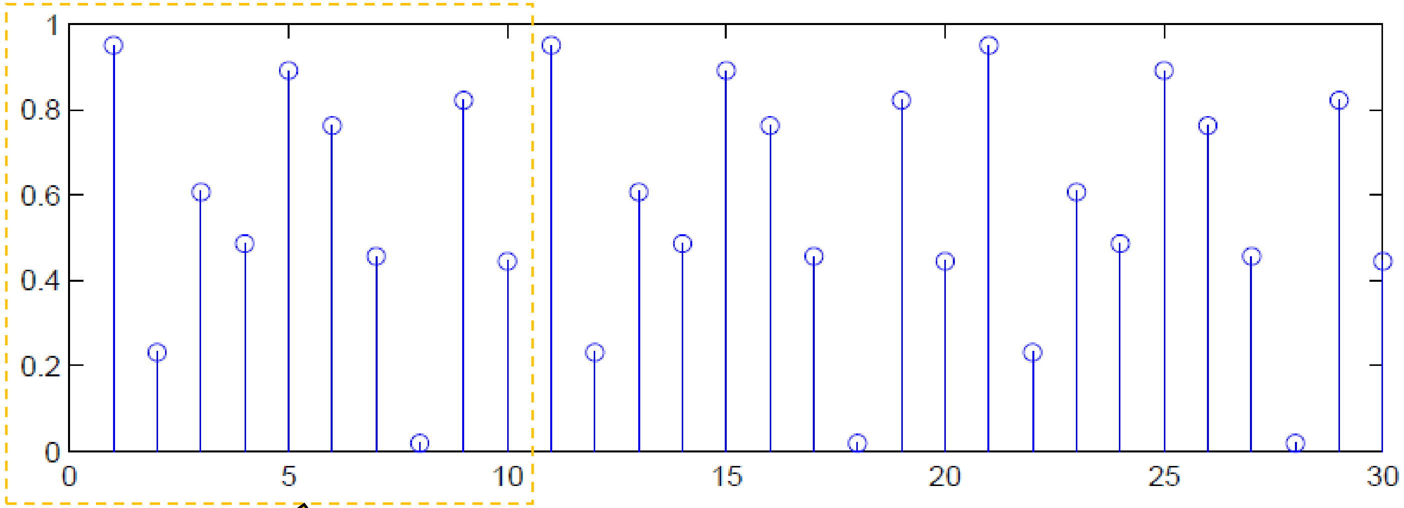




9.2 Discrete Fourier Analysis

Hence the sequence shown below in Figure is considered to be one period of the periodic sequence in plot (b).

(a)



(b)

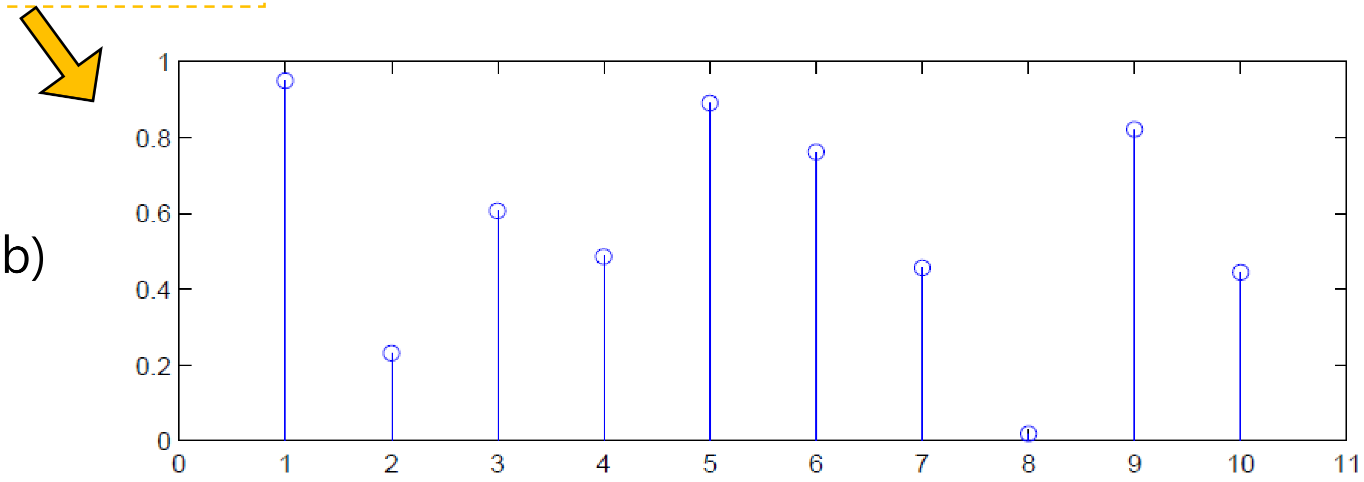
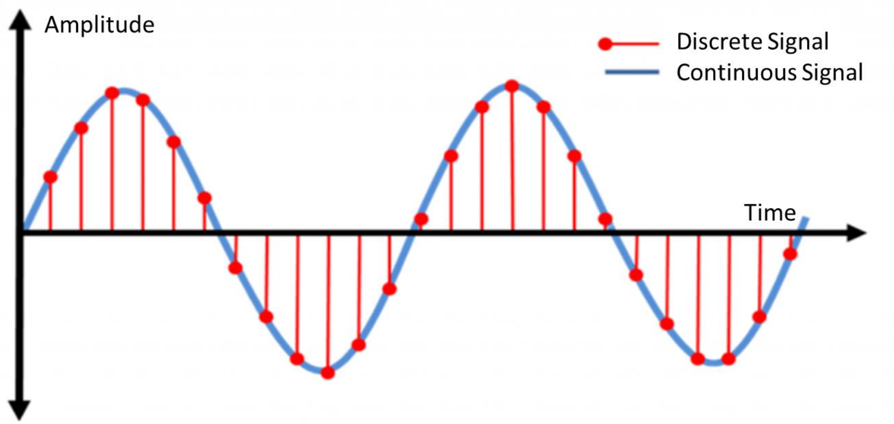


Figure: (a) Sequence of $N = 10$ samples. (b) implicit periodicity in DFT.

9.2 Discrete Fourier Analysis



Now very often a function is given only in terms of values at finitely many points, and one is interested in extending Fourier analysis to this case.

The main application of such a "discrete Fourier analysis" concerns large amounts of **equally spaced data**, as they occur in telecommunication, time series analysis, and various simulation problems.

In these situations, dealing with sampled values rather than with functions, we can replace the Fourier transform by the so-called discrete Fourier transform (DFT)

9.2 Discrete Fourier Analysis

Let these N samples be denoted as $f[n] = \{f[0], f[1], f[2], \cdots, f[\mathrm{k}], \cdots, f[\mathrm{N}-1]\}$.

We think of this as a function over the discrete domain $\{0, 1, \ldots, N - 1\}$.

9.2 Discrete Fourier Analysis

Basic Operations on Sequences

If x and y are two sequences, we define their sum $x + y$ and their product xy, by

$$(x + y)_k = x_k + y_k$$

respectively,

$$(xy)_k = x_k y_k$$

Also, if x is a sequence and a is a real number, then by ax_k we mean the sequence whose kth term is ax_k.

9.2 Discrete Fourier Analysis

The **Discrete Fourier Transform (DFT)** is the equivalent of the continuous Fourier Transform for signals known only at N instants separated by sample times T (i.e. a finite sequence of data).

Recall **FOURIER TRANSFORM**

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \qquad (-\infty < \omega < \infty)$$

9.2 Discrete Fourier Analysis

FOURIER TRANSFORM

$$\hat{f}\,(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \qquad (-\infty < \omega < \infty)$$

- Note the similarity with the definition of the Fourier transform. Here we multiply the values of the sequence by a complex exponential and then sum over the whole domain of the sequence.
- With the Fourier transform, we multiply the function by a complex exponential and integrate over the whole domain of the function.
- In fact, you will see that many properties of the Fourier transform will be translated to properties of the DFT by an appropriate conversion involving changing an integral over a continuous domain to a sum over a discrete domain.

9.2 Discrete Fourier Analysis

Discrete FOURIER TRANSFORM

(Notice: The definitions of DFT and IDFT are consistent with wikipedia but not the textbook.)

The discrete Fourier transform (DFT) transforms a sequence of N real or complex numbers $f[n] = \{f[0], f[1], f[2], \cdots, f[k], \cdots, f[N-1]\}$ into another sequence of complex numbers, $\mathcal{F}[k]$ which is defined by

$$\mathcal{F}[k] = \sum_{n=0}^{N-1} f[n]\, e^{-i\frac{2\pi}{N}nk} \qquad k = 0, 1, \cdots, N-1$$

$$= \sum_{n=0}^{N-1} f[n]\left[\cos\frac{2\pi}{N}nk - i\sin\frac{2\pi}{N}nk\right]$$

where the last expression follows from the first one by Euler's formula.

The output is a complex number which encodes **the amplitude and phase** of a sinusoidal wave.

9.2 Discrete Fourier Analysis

The effect of computing the $\mathcal{F}[k]$ is to find the coefficients of an approximation of the signal by a linear combination of such waves.

Since each wave has an integer number of cycles per N time units, the approximation will be periodic with period N. This approximation is given by the **inverse Fourier transform.**

Like other transforms that we have encountered previously, the DFT has an inverse, known as the **inverse discrete Fourier transform (IDFT),** given by

Inverse Discrete Fourier Transform

$$\boxed{f[n] = \frac{1}{N}\sum_{k=0}^{N-1} \mathcal{F}[k]\, e^{i\frac{2\pi}{N}nk}}$$

9.2 Discrete Fourier Analysis

Thus the IDFT of an N-sequence is another N-sequence. Note the similarity between the two definitions (1) and (2). As you would expect, the effect of the inverse transform is to give you back your original sequence.

Proof

Substitute the formula for $\mathcal{F}[k]$ into the formula for $f[n]$

$$\frac{1}{N}\sum_{k=0}^{N-1}\mathcal{F}[k]\,e^{i\frac{2\pi}{N}nk} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} f[m]\,e^{-i\frac{2\pi}{N}mk}\,e^{i\frac{2\pi}{N}nk}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} f[m]\,e^{i\frac{2\pi}{N}k(n-m)}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} f[m]\sum_{m=0}^{N-1} e^{i\frac{2\pi}{N}k(n-m)}$$

When $m \neq n$, the inner sum is 0 by the formula for a geometric series (as in the first example in the previous section). When $m = n$, the inner sum is N.

So the entire sum is $\frac{1}{N}f[n]N = f[n]$, as desired.

9.2 Discrete Fourier Analysis

The DFT is useful in many applications, including the simple signal spectral analysis outlined above. Knowing how a signal can be expressed as a combination of waves allows for manipulation of that signal and comparisons of different signals:

- **Digital files (jpg, mp3, etc.)** can be shrunk by eliminating contributions from the least important waves in the combination.
- **Different sound files** can be compared by comparing the coefficients $\mathcal{F}[k]$ of the DFT.
- **Radio waves** can be filtered to avoid "noise" and listen to the important components of the signal.

Additional Example

Let $f[n] = \{f[0], f[1], f[2], \cdots, f[\mathrm{k}], \cdots, f[\mathrm{N}-1]\} = \{1, 0, 0, \cdots, \ 0, \cdots, 0\}$

Then the DFT of the $f[n]$ is $\mathcal{F}[k] = \sum_{n=0}^{N-1} f[n]\, e^{-i\frac{2\pi}{N}nk} = 1 \cdot e^{-i\frac{2\pi}{N}\cdot 0\cdot k} + 0 \cdot e^{-i\frac{2\pi}{N}\cdot 1\cdot k} + \cdots + 0 \cdot e^{-i\frac{2\pi}{N}\cdot (N-1)\cdot k} = 1$

So it gives an expression of $f[n]$ as

$$f[n] = \frac{1}{N}\sum_{k=0}^{N-1} 1 \cdot e^{i\frac{2\pi}{N}nk} = \frac{1}{N}\sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}nk}$$

Review for Lecture 9

- Sampling Theorem
- Band Limited Functions
- Discrete Fourier Transform

Exercise

Please Check https://github.com/uoaworks/FourierAnalysisAY2018

Reading: Section10.1, 10.3, 10.4, Textbook

References

[1] Nakhlé H. Asmar, *Partial Differential Equations with Fourier Series and Boundary Value Problems 2nd Edition*, 2004

[2] Erwin Kreyszig, *Advanced Engineering Mathematics 9th Edition*, 2005

[3] Discrete Fourier Transform, https://brilliant.org/wiki/discrete-fourier-transform/

[4] Wikipedia