# Lecture 6

# Convolution

# What you will learn in Lecture 6

**I**. The Fourier Transform: Operational Properties

**II**. Convolution

# 6.1 The Fourier Transform:

# Operational Properties

# 6.1 The Fourier Transform: Operational Properties

## FOURIER TRANSFORM

(1) $$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \qquad (-\infty < \omega < \infty)$$

## INVERSE FOURIER TRANSFORM

(2) $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \qquad (-\infty < x < \infty)$$

# 6.1 The Fourier Transform: Operational Properties

Putting $\omega = 0$ in (1), we find that

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, dx$$

Thus the value of the Fourier transform at $\omega = 0$ is equal to the signed area between the graph of $f(x)$ and the $x$-axis, multiplied by a factor of $\frac{1}{\sqrt{2\pi}}$

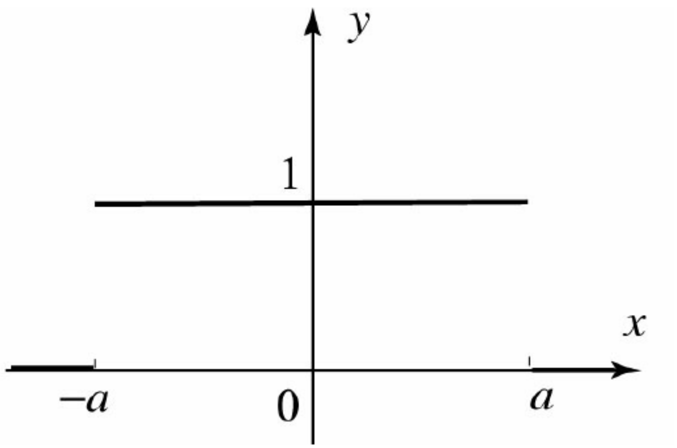
**EXAMPLE 1** A Fourier transform

(a) Find the Fourier transform of the function in Figure 1, given by

$$f(x) = \begin{cases} 1 & if\ |x| < a \\ 0 & if\ |x| > a \end{cases}$$

What is $\hat{f}(0)$?

(b) Express $f$ as an inverse Fourier transform.




**Solution**

Tips:

- Use (1) for $\hat{f}\ (\omega)$
- Introduce L'Hôpital's rule $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$.
- Discuss $\omega = 0$

Find the complete solution in page 399-400 of the textbook.

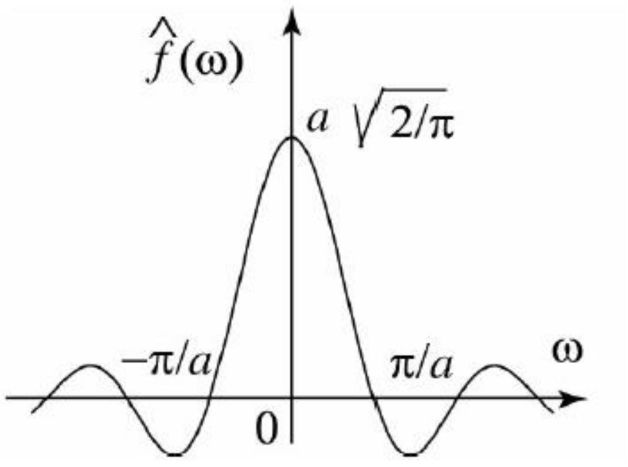



Figure 2 Graph of $\hat{f}$

# 6.1 The Fourier Transform: Operational Properties

## Fourier Transform Symbols

$\mathcal{F}(f)(\omega)$ denotes **Fourier Transform** of $f$

$\mathcal{F}^{-1}(f)(x)$ denotes **Inverse Fourier Transform** of $f$

Other symbols: $FT(f)$, $IFT(f)$

## Operational Properties

We shall investigate the behavior of the Fourier transform in connection with the common operations on functions: **linear combination, translation and convolution etc.**

# Operational Properties

## THEOREM 1 LINEARITY

The Fourier transform is a linear operation; that is, for any integrable functions $f$ and $g$ and any real numbers $a$ and $b$,

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

**Proof**

$$\mathcal{F}[af(x) + bg(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af(x) + bg(x)]e^{-i\omega x}dx$$

$$= a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx + b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x)e^{-i\omega x}dx$$

$$= a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)]$$

## Operational Properties

### THEOREM 2 Fourier Transforms of Derivatives

(i) Suppose $f(x)$ is piecewise smooth, $f(x)$ and $f'(x)$ are integrable, and $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$, then

$$\mathcal{F}(f') = i\omega\mathcal{F}(f)$$

(ii) If in addition $f''(x)$ is integrable, and $f'(x)$ is piecewise smooth and tend to $0$ as $|x| \rightarrow \infty$, then

$$\mathcal{F}(f'') = i\omega\mathcal{F}(f') = -\omega^2\mathcal{F}(f)$$

(iii) In general, if $f$ and $f^{(k)}(x)$ $(k = 1, 2, \ldots, n - 1)$ are piecewise smooth and tend to $0$ as $|x| \rightarrow \infty$, and $f$ and its derivatives of order up to $n$ are integrable, then

$$\mathcal{F}(f^{(n)}) = (i\omega)^n\mathcal{F}(f)$$

**Proof** Find the complete proof in page 402 of the textbook.

## Appendix B. Table of Fourier Transform (Page 1069 of the textbook)

| $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{ix\omega}\, d\omega$ | | $\widehat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$ |
|---|---|---|
| **1.** $\begin{cases} 1 & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases}$ | | $\sqrt{\frac{2}{\pi}} \frac{\sin a\omega}{\omega}$ |
| **2.** $\begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$ | | $\frac{i\left(e^{-ib\omega} - e^{ia\omega}\right)}{\sqrt{2\pi}\omega}$ |
| **3.** $\begin{cases} 1 - \frac{\lvert x\rvert}{a} & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases}$ | $a > 0$ | $2\sqrt{\frac{2}{\pi}} \frac{\sin^2(\frac{a\omega}{2})}{a\omega^2}$ |
| **4.** $\begin{cases} x & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases}$ | $a > 0$ | $i\sqrt{\frac{2}{\pi}} \frac{a\omega\cos(a\omega) - \sin(a\omega)}{\omega^2}$ |
| **5.** $\begin{cases} \sin x & \text{if } \lvert x\rvert < \pi \\ 0 & \text{if } \lvert x\rvert > \pi \end{cases}$ | | $i\sqrt{\frac{2}{\pi}} \frac{\sin(\pi\omega)}{\omega^2 - 1}$ |

⋮

# 6.1 The Fourier Transform: Operational Properties

## THEOREM 3 Derivatives of Fourier Series

(i) Suppose $f(x)$ and $xf(x)$ are integrable; then

$$\mathcal{F}\big(xf(x)\big)(\omega) = i\big[\hat{f}\big]'(\omega) = i\frac{d}{d\omega}\mathcal{F}(f)(\omega)$$

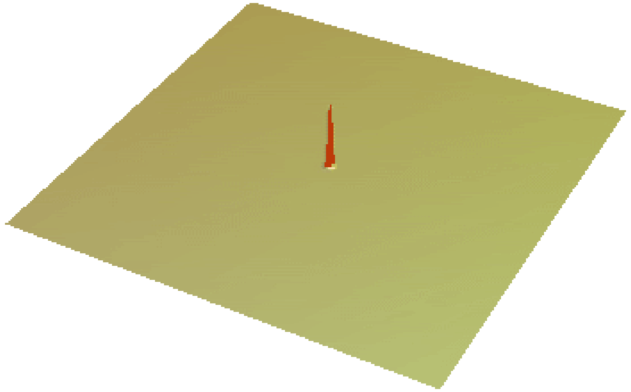
(ii) In general, if $f(x)$ and $x^n f(x)$ are integrable, then

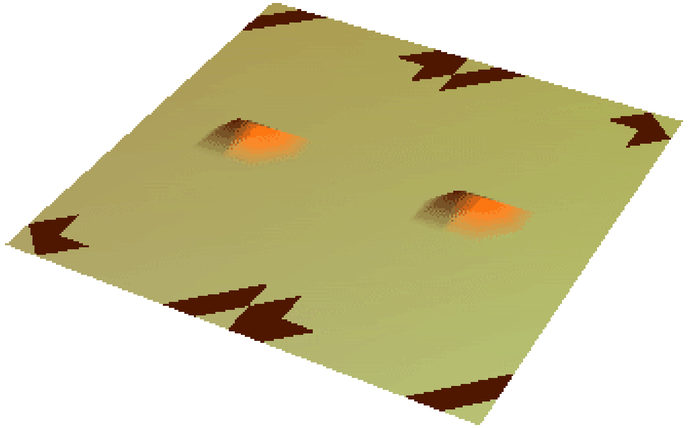
$$\mathcal{F}\big(x^n f(x)\big) = i^n\big[\hat{f}\big]^{(n)}(\omega)$$

**Proof**

Find the complete proof in page 402 of the textbook.

# 6.2 Convolution





# 6.2 Convolution

## Convolution of Functions

Introducing the convolution of two functions $f$ and $g$ by

### CONVOLUTION

$$\boxed{(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt}$$

In this lecture

Convolution Theorem $\mathcal{F}[f(x) * g(x)] = \sqrt{2\pi}\mathcal{F}[f(\omega)]\mathcal{F}[g(\omega)]$

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

In textbook

Convolution Theorem $\mathcal{F}[f(x) * g(x)] = \mathcal{F}[f(\omega)]\mathcal{F}[g(\omega)]$

# 6.2 Convolution

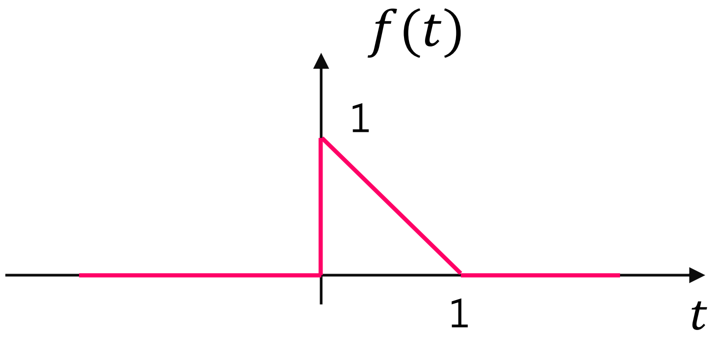
$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

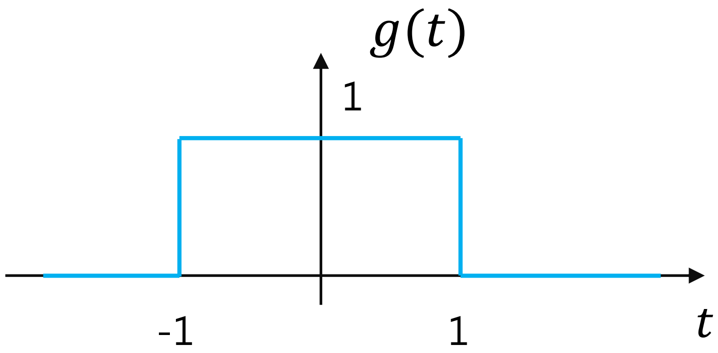
**What does Convolution mean?**

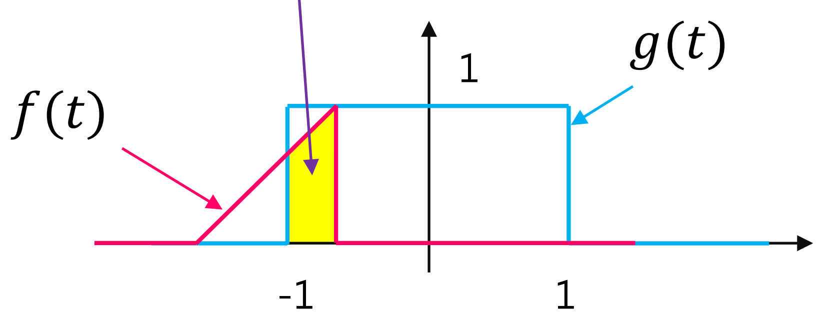
# 6.2 Convolution

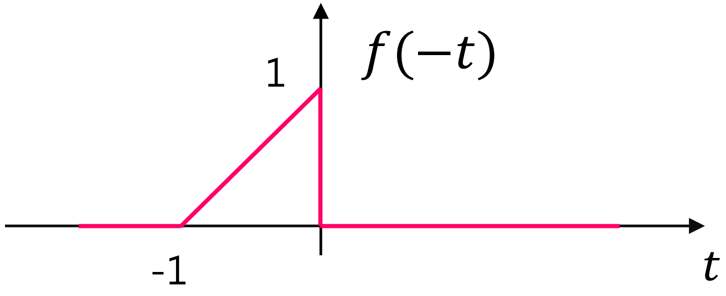
**Case 1**

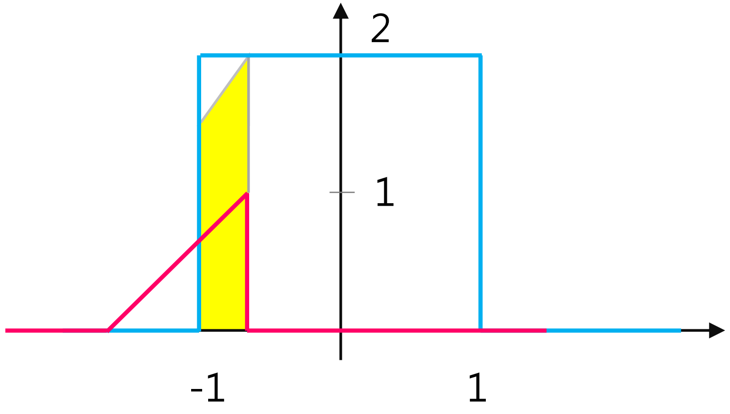
$$h = (f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

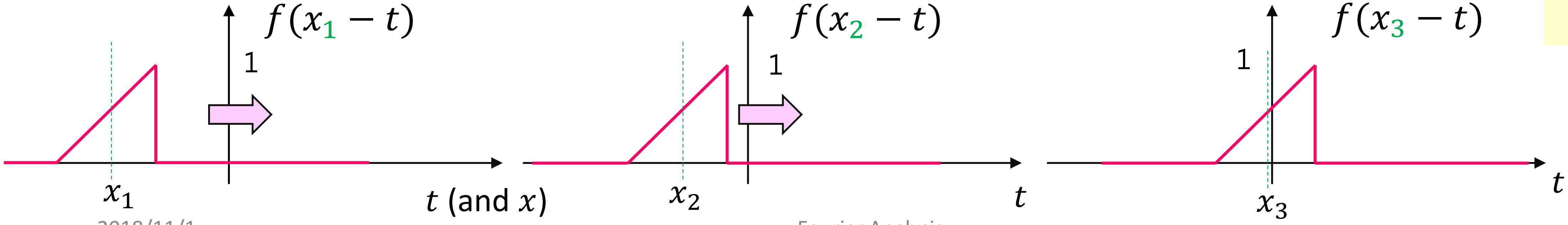


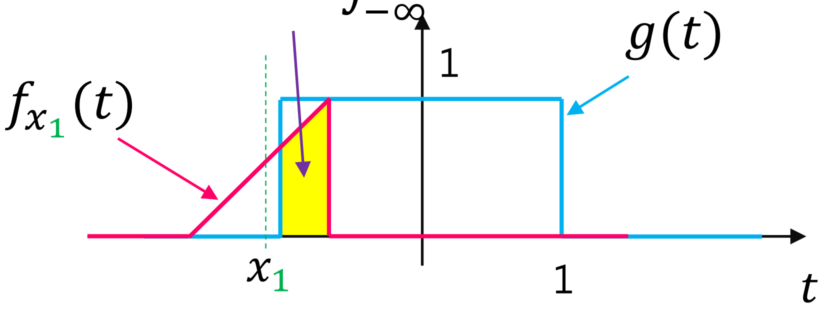




$x_1 < x_2 < x_3$

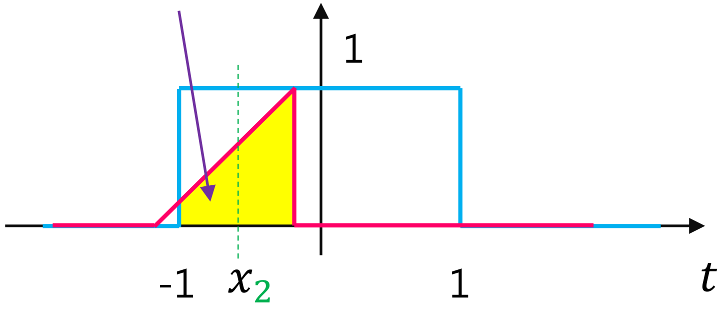
# 6.2 Convolution

Case 1

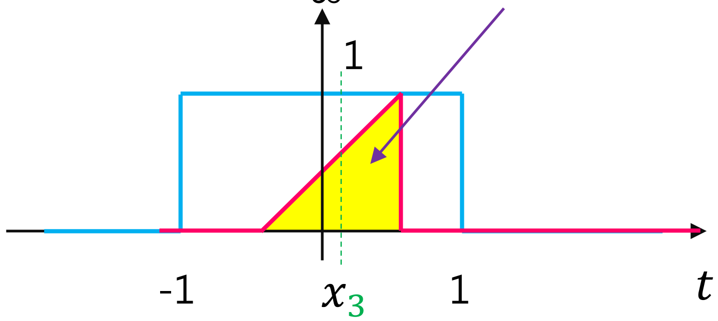
$$f_{x_1} * g = \int_{-\infty}^{\infty} f(x_1 - t)g(t)dt$$



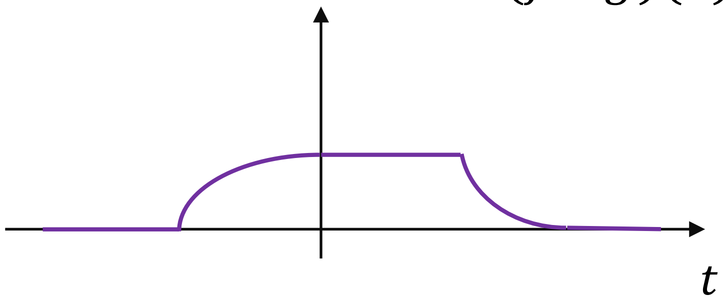

$$f_{x_2} * g = \int_{-\infty}^{\infty} f(x_2 - t)g(t)dt$$




$$f_{x_3} * g = \int_{-\infty}^{\infty} f(x_3 - t)g(t)dt$$



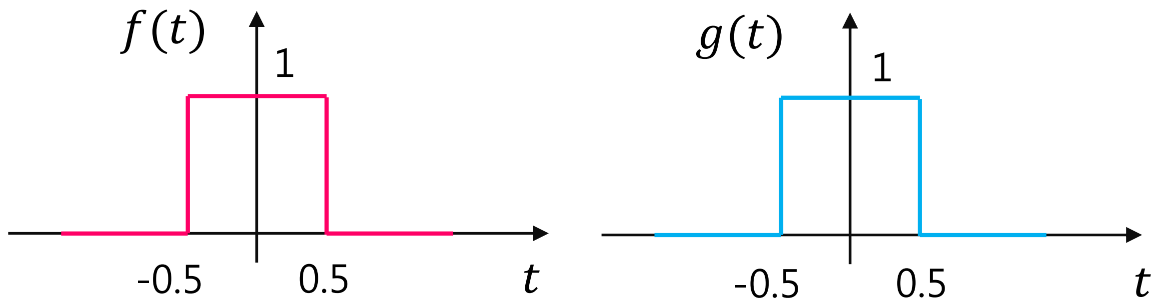

$$h = (f * g)(x) = \int_{-\infty}^{\infty} f(x - t)g(t)dt$$



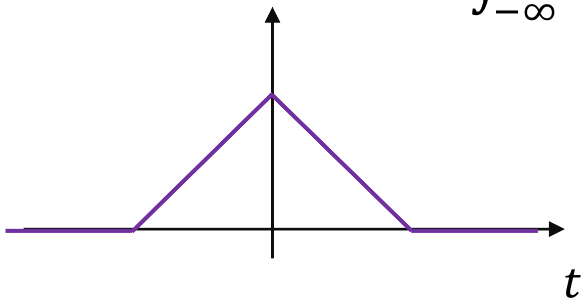
# 6.2 Convolution

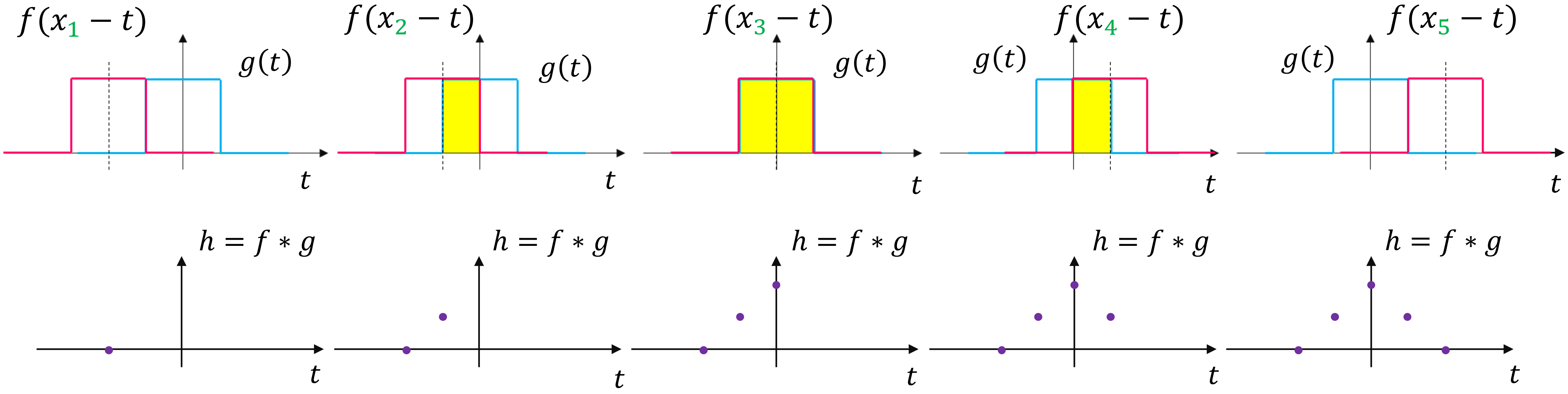
Case 2

$$f(x) = g(x) = \begin{cases} 1 & if\ |x| \le 0.5 \\ 0 & otherwise \end{cases}$$




$$h = f * g = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

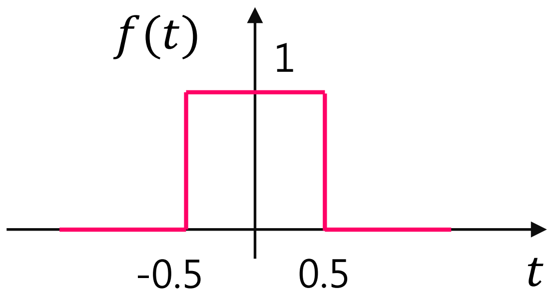


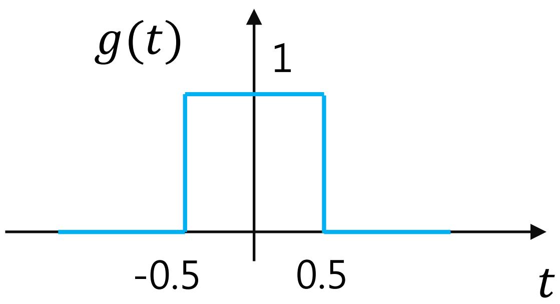



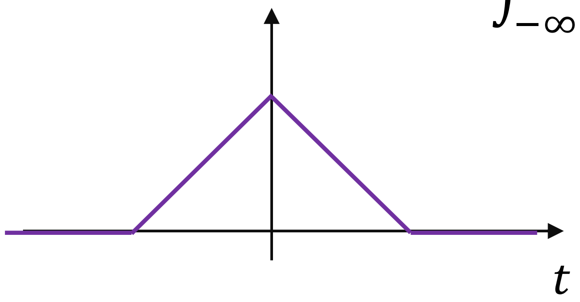
# 6.2 Convolution

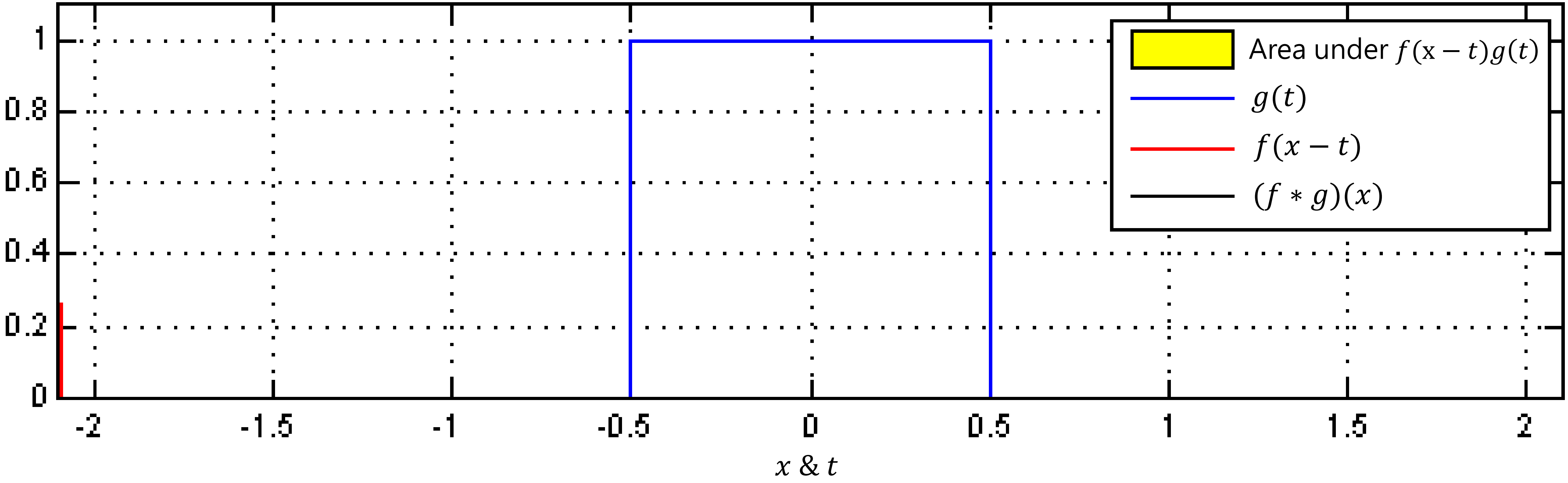
Case 2

$$h = f * g = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

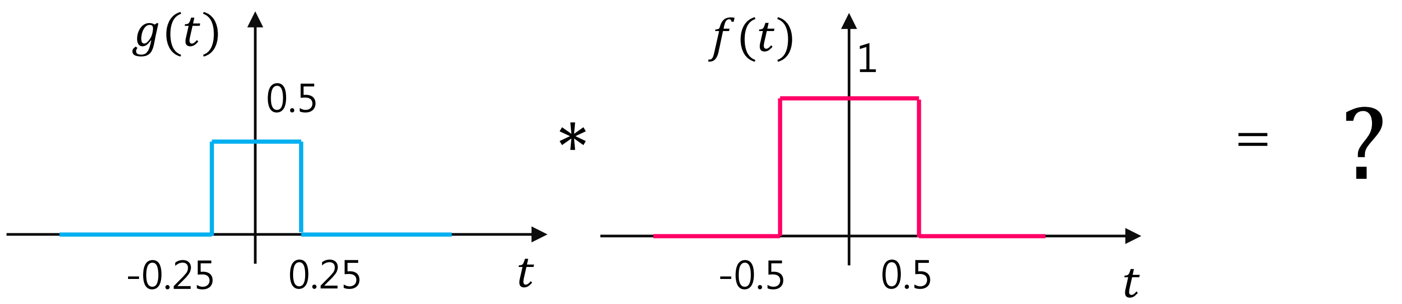


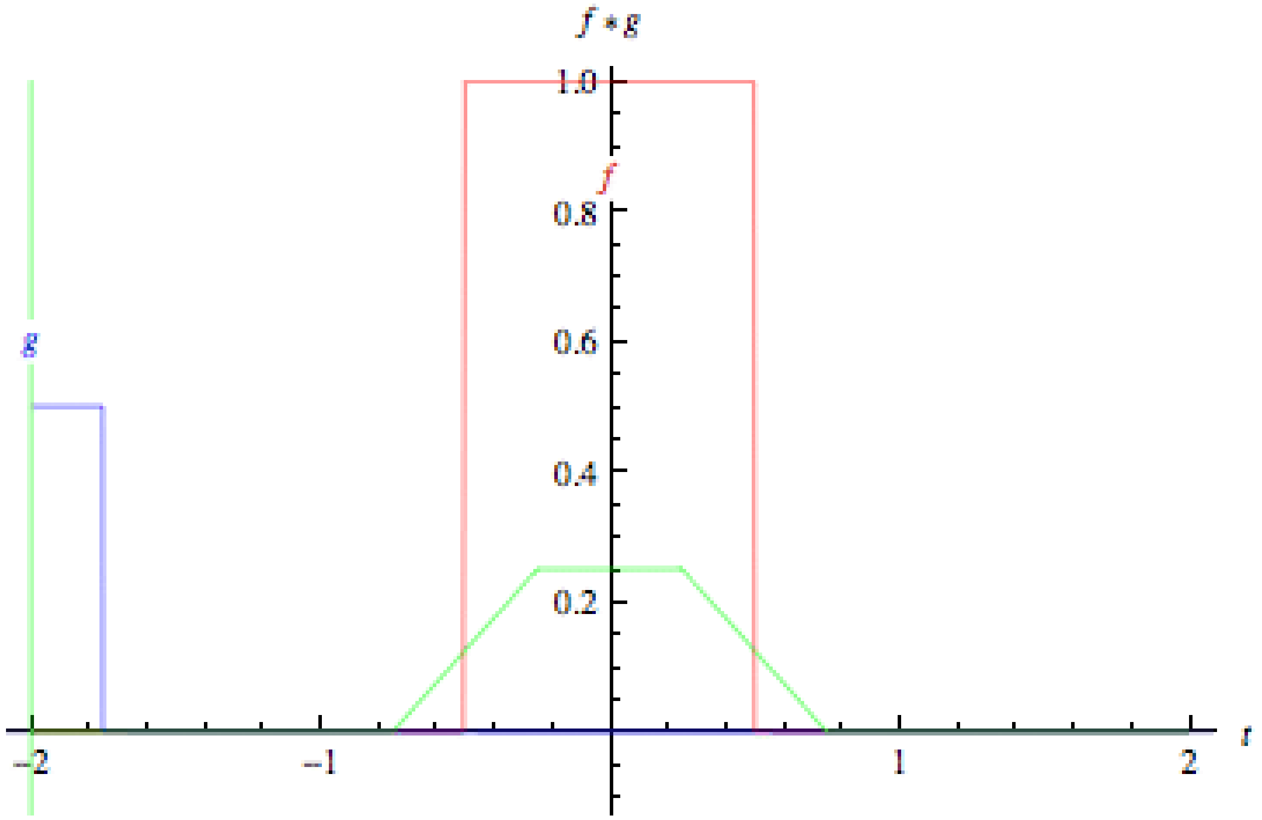



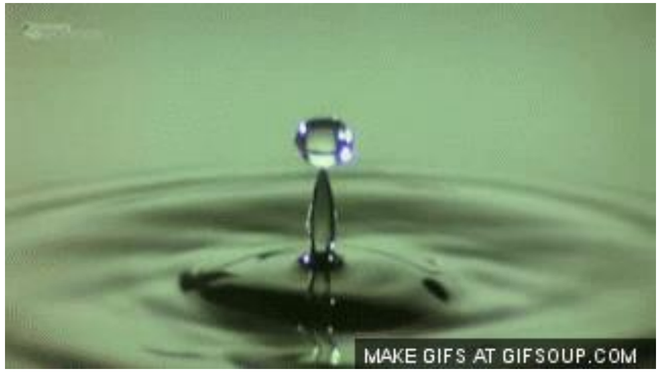
Case 3

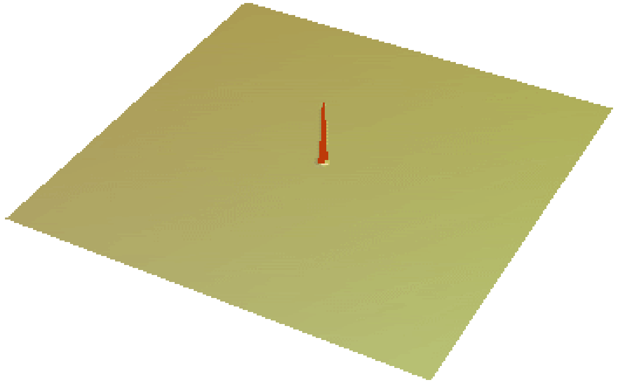


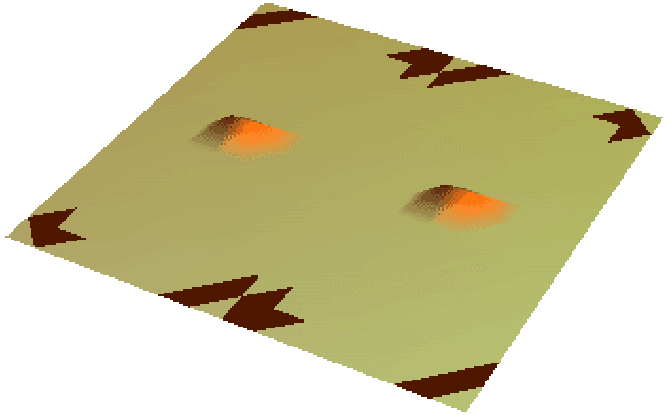
# 6.2 Convolution

Case 4



one stone wave



two stones wave

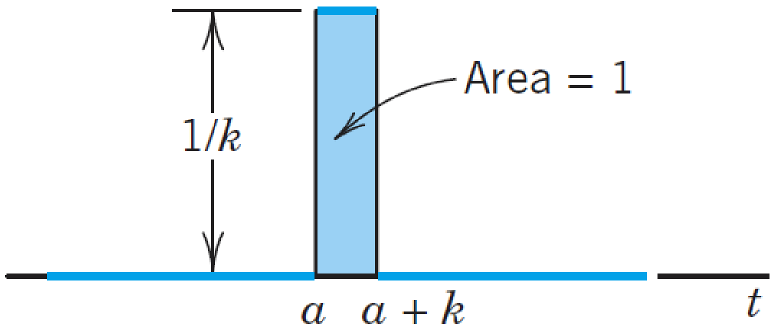
# 6.2 Convolution

Case 4

## Dirac Delta Function

Generalized Functions

$$f_k(t-a)=\begin{cases}\dfrac{1}{k} & \text{if } a \le t \le a+k \\ 0 & otherwise\end{cases}$$

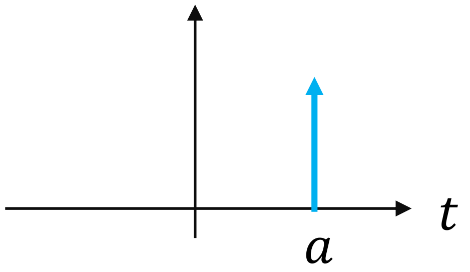



The function $f_k(t-a)$

$$I_k=\int_{-\infty}^{\infty} f_k(t-a)dt=\int_{a}^{a+k}\frac{1}{k}dt=1$$

When $k \to 0$

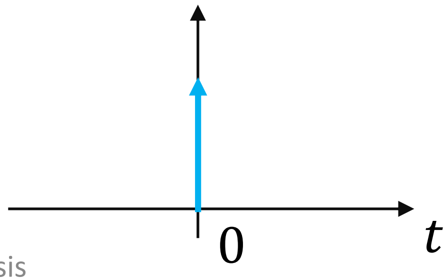
$$\delta(t-a)=\lim_{k\to 0} f_k(t-a)$$




$$\delta(t-a)=\begin{cases}\infty & if \quad t=a \\ 0 & otherwise\end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty}\delta(t-a)dt=1$$

When $a=0$

$$\delta(t)=\begin{cases}\infty & \text{if} \quad t=0 \\ 0 & otherwise\end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty}\delta(t)dt=1$$




Theorem 1 in Section 7.8 ($a=0$)

$$\int_{-\infty}^{\infty} g(t)\delta(t)dt=g(0)$$

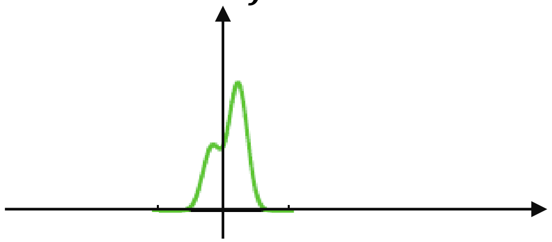
General form ($a \neq 0$)

$$\int_{-\infty}^{\infty} g(t)\delta(t-a)dt=g(a)$$

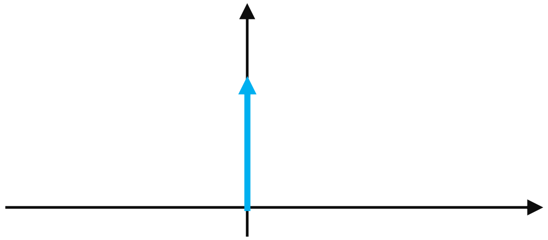
# 6.2 Convolution

**Case 4**

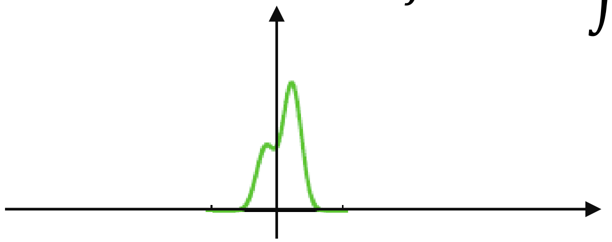
$f$

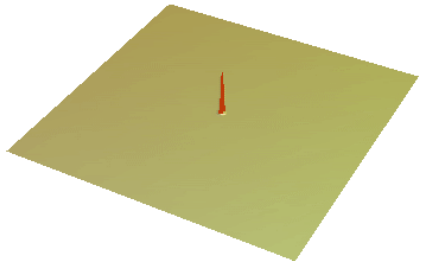


$\delta$

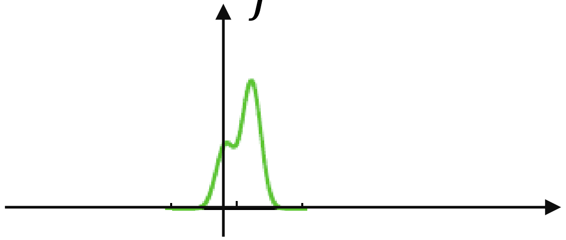


$$h = f * \delta = \int_{-\infty}^{\infty} f(x-t)\delta(t)dt$$

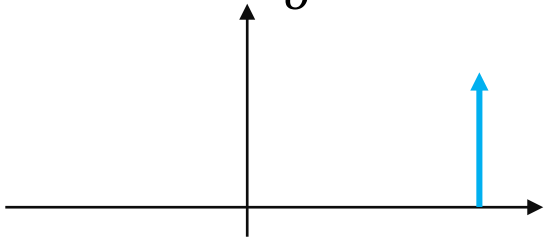




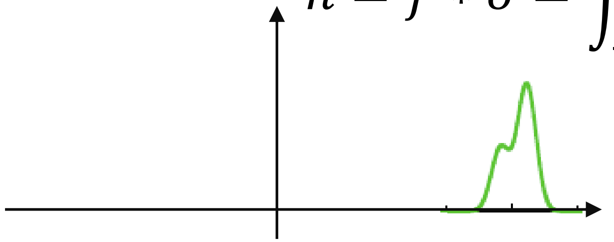
$f$



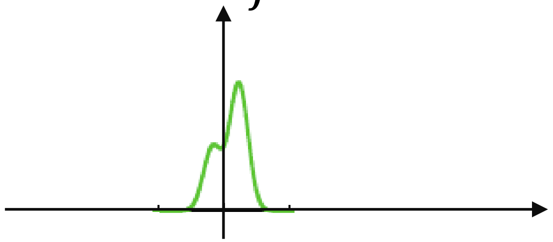
$\delta$



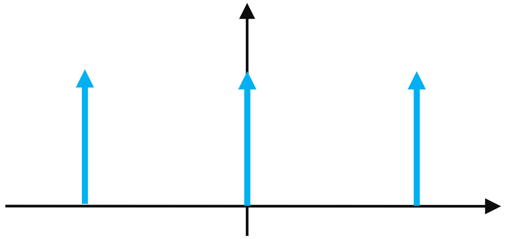
$$h = f * \delta = \int_{-\infty}^{\infty} f(x-t)\delta(t)dt$$



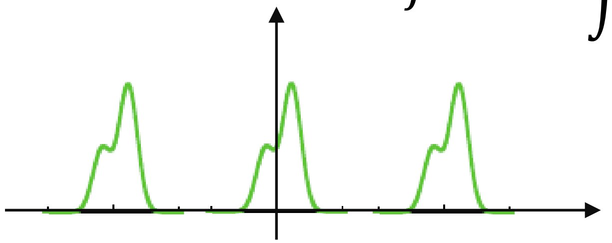
$f$



$\delta$



$$h = f * \delta = \int_{-\infty}^{\infty} f(x-t)\delta(t)dt$$



# 6.2 Convolution

**EXAMPLE 3** **Convolution with the cosine**

Suppose that $f$ is integrable and even ($f(-x) = f(x)$ for all $x$) and let $g(x) = \cos ax$.

Show that, for all real numbers $a$: $(f * g)(x) = \cos(ax)\hat{f}(a)$.

## Solution

Tips:

- Use definition of $f * g$
- $f * g = g * f$
- $f$ is even, then $f \sin at$ is odd
- Trigonometric identity $\cos(a - b) = \cos a \cos b + \sin a \sin b$
- $\int_{-\infty}^{\infty} \cos at\, dt = \int_{-\infty}^{\infty} [\cos at - i\sin(at)] dt$ because $\int_{-\infty}^{\infty} -i\sin(at) dt = 0$ (odd function)

Find the complete solution in page 403 of the textbook.

# 6.2 Convolution

**THEOREM 4 Fourier Transforms of Convolutions (Convolution Theorem)**

Suppose that $f$ and $g$ are integrable; then

$$\mathcal{F}[f(x) * g(x)] = \sqrt{2\pi}\mathcal{F}[f(\omega)]\mathcal{F}[g(\omega)]$$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

In this lecture

Theorem 4 is expressed by saying that the Fourier transform takes convolutions into products.

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-t)g(t)dt$$

$$\mathcal{F}[f(x) * g(x)] = \mathcal{F}[f(\omega)]\mathcal{F}[g(\omega)]$$

In textbook

# 6.2 Convolution

**EXAMPLE 5 Fourier transform of a convolution**

Consider the function $f(x) = 1$ if $|x| < 1$ and $0$ otherwise. The graph of this function is shown in Figure 5. From Example 1, we have

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin a\omega}{\omega}$$

Find the Fourier Transform of $f * f$.

Find $f * f$.

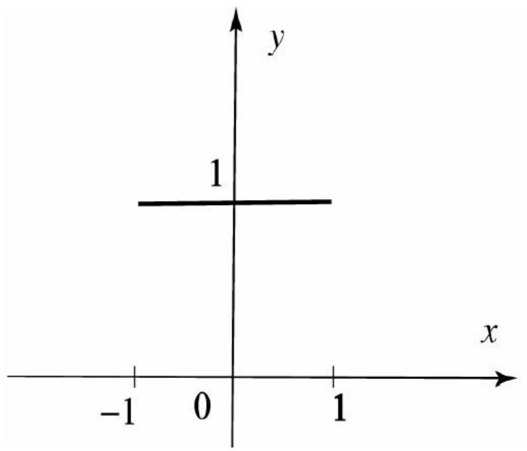



**Figure 5** Graph of $f$.

**Solution:**

Tips:

Instead of $\mathcal{F}[f(x) * f(x)]$, we compute $\mathcal{F}[f(\omega)]\mathcal{F}[f(\omega)] = \hat{f}(\omega)\hat{f}(\omega)$

Find the complete solution in page 405 of the textbook.

# Appendix B. Table of Fourier Transform (Page 1069 of the textbook)

| $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{ix\omega}\, d\omega$ | | $\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$ |
|---|---|---|
| **1.** | $\begin{cases} 1 & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases}$ | $\sqrt{\frac{2}{\pi}} \frac{\sin a\omega}{\omega}$ |
| **2.** | $\begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$ | $\frac{i\left(e^{-ib\omega} - e^{ia\omega}\right)}{\sqrt{2\pi}\omega}$ |
| **3.** | $\begin{cases} 1 - \frac{\lvert x\rvert}{a} & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases} \quad a > 0$ | $\boxed{2\sqrt{\frac{2}{\pi}} \frac{\sin^2\left(\frac{a\omega}{2}\right)}{a\omega^2}}$ |
| **4.** | $\begin{cases} x & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases} \quad a > 0$ | $i\sqrt{\frac{2}{\pi}} \frac{a\omega\cos(a\omega) - \sin(a\omega)}{\omega^2}$ |

⋮

# Review for Lecture 6

- The Fourier Transform: Operational Properties
- Convolution
- Dirac Delta Function

# Exercise

Please Check https://github.com/uoaworks/FourierAnalysisAY2018

Reading: Section 7.2, 7.8, Textbook

# References

[1] Nakhlé  H.  Asmar, *Partial Differential Equations with Fourier Series and Boundary Value Problems 2nd Edition*, 2004

[2] Erwin Kreyszig, *Advanced Engineering Mathematics, 9th Edition*, 2006

[3] Convolution, https://sites.google.com/site/butwhymath/m/convolution

[4] Wikipedia