

COMS30035, Machine learning:

From regression to classification
and neural networks:

Revising regression

(based on slides by Dima Damen)

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Textbooks

Chapter 3 of the Bishop book is directly relevant:

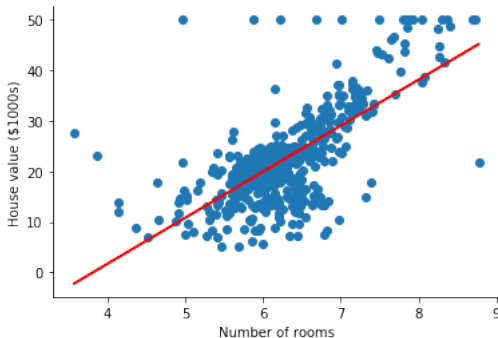
- ▶ Bishop, C. M., Pattern recognition and machine learning (2006). Available for free [here](#).
- ▶ **Note:** this first part is a revision of what we covered in SPS last year; for the full lecture notes see [here](#).

Agenda

- ▶ **Revising linear and nonlinear regression** [[see SPS slides](#); Chapter 3, Bishop]
 - ▶ Linear regression
 - ▶ Nonlinear regression
 - ▶ Probabilistic models
 - ▶ Maximum likelihood estimation
- ▶ **Sequential Bayesian regression** [Chapter 3, Bishop]
 - ▶ Bayesian formulation
 - ▶ Maximum a posteriori
 - ▶ Example
- ▶ **Classification and neural networks** [Chapter 5, Bishop]
 - ▶ Architectures (Parametric model)
 - ▶ The supervised case
 - ▶ Optimising nnets using backprop
 - ▶ Highly flexible model → overfitting: early stopping/drop-out.

Revisiting regression

- ▶ Goal: Finding a relationship between two variables (e.g. regress *house value* against *number of rooms*)
- ▶ Model: Linear relationship between *house value* and *number of rooms*?



Revisiting regression – deterministic model

Data: a set of data points $D = \{(x_1, y_1), (x_2, y_2), \dots, (x_N, y_N)\}$ where x_i is the house value i and y_i is the number of rooms i .

Task: build a model that can predict the house value from the number of rooms

Model Type: parametric; assumes a polynomial relationship between house value and number of rooms

Model Complexity: assume the relationship is linear
house value = $a_0 + a_1 * \text{rooms}$

$$y_i = a_0 + a_1 x_i \quad (1)$$

Model Parameters: model has two parameters a_0 and a_1 which should be estimated.

- ▶ a is the y-intercept
- ▶ b is the slope of the line

Least Squares Solution - matrix form

- ▶ To find a solution to the parameters $\theta = \{a_0, a_1\}$ solve least squares problem which **in matrix form**, means to find \mathbf{a}_{LS} ; ¹

$$\|\mathbf{y} - \mathbf{X} \mathbf{a}_{LS}\|^2 = 0 \quad (2)$$

$$\mathbf{a}_{LS} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} \quad (3)$$

- ▶ Matrix formulation also allows least squares method to be extended to **polynomial fitting**
- ▶ For a polynomial of degree $p + 1$ we use (note: $p > 1$ gives nonlinear regression)

$$y_i = a_0 + a_1 x_i + a_2 x_i^2 + \cdots + a_p x_i^p$$

¹ $\|\mathbf{A}\|^2 = \sqrt{\sum \sum |a_{ij}|^2}$ denotes the Frobenius norm, defined as the square root of the sum of the absolute squares of its elements. For a detailed derivation see [this derivation - p8](#)

Least Squares Solution

Example

Find the best least squares fit by a linear function to the data using $p = 1$

x	-1	0	1	2
y	0	1	3	9

$$\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{X}^T \mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$$

$$\mathbf{a}_{LS} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} = \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1.8 \\ 2.9 \end{bmatrix}$$

$$y = 1.8 + 2.9x$$

Regression with probabilistic models

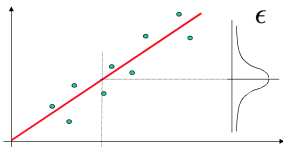
Probabilistic models are a core part of ML, as they allow us to also capture the uncertainty the model has about the data, which is critical for real world applications. For simplicity, let's drop a_0 from the previous model and add a random variable ϵ that captures the uncertainty

$$\text{house price} = a_1 \times \text{number of rooms} + \epsilon$$

We can assume, for example, that ϵ is given by $\mathcal{N}(\mu = 0, \sigma^2)$ which gives the likelihood

$$p(D|\theta) = \prod_{i=1}^N p(\text{price}_i | \text{rooms}_i, \theta) = \prod_{i=1}^N \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \frac{(\text{price}_i - a_1 \text{rooms}_i)^2}{\sigma^2}}$$

This model has **two** parameters: the slope a_1 and variance σ^2



²Note that here $\mu = a_0$ which, for simplicity, we assume to be zero.

Maximum Likelihood Estimation

- ▶ Similar to building deterministic models, probabilistic model parameters need to be tuned/trained
- ▶ **Maximum-likelihood estimation (MLE)** is a method of estimating the parameters of a probabilistic model.
- ▶ Assume θ is a vector of all parameters of the probabilistic model. (e.g. $\theta = \{a_1, \sigma\}$).
- ▶ **MLE** is an extremum estimator³ obtained by maximising an objective function of θ

³"Extremum estimators are a wide class of estimators for parametric models that are calculated through maximization (or minimization) of a certain objective function, which depends on the data." wikipedia.org

Maximum Likelihood Estimation

Definition

Assume $f(\theta)$ is an objective function to be optimised (e.g. maximised), the *arg max* corresponds to the value of θ that attains the maximum value of the objective function f

$$\hat{\theta} = \arg \max_{\theta} f(\theta)$$

- Tuning the parameter is then equal to finding the maximum argument *arg max*

Maximum Likelihood Estimation - General

- ▶ Maximum Likelihood Estimation (MLE) is a common method for solving such problems

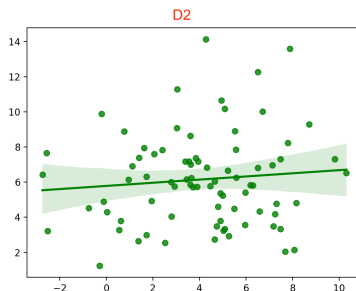
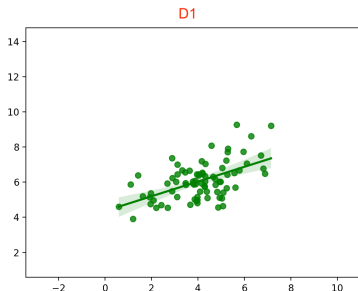
$$\begin{aligned}\theta_{MLE} &= \arg \max_{\theta} p(D|\theta) \\ &= \arg \max_{\theta} \ln p(D|\theta) \\ &= \arg \min_{\theta} -\ln p(D|\theta)\end{aligned}$$

MLE Recipe

1. Determine θ , D and expression for likelihood $p(D|\theta)$
2. Take the natural logarithm of the likelihood
3. Take the derivative of $\ln p(D|\theta)$ w.r.t. θ . If θ is a multi-dimensional vector, take partial derivatives
4. Set derivative(s) to 0 and solve for θ

Data Modelling - Deterministic vs Probabilistic

- ▶ **Probabilistic Models** can tell us **more**
- ▶ We could use the same MLE recipe to find σ_{ML} . This would tell us how uncertain our model is about the data D .
- ▶ For example: if we apply this method to two datasets (D_1 and D_2) what would the parameters $\theta = \{a_1, \sigma\}$ be?



$$a_{1_{ML}}^{D_1} > a_{1_{ML}}^{D_2} \text{ [slope]} \text{ and } \sigma_{ML}^{D_1} < \sigma_{ML}^{D_2} \text{ [uncertainty]}^4$$

⁴The uncertainty (σ) is represented by the light green bar in the plots. Test it yourself.

Quiz time!



Go to Blackboard unit page » Quizzes » Lecture 3.1

[Should take you less than 5 minutes]