COMS30035, Machine learning:

From regression to classification and neural networks:

Sequential Bayesian regression

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Textbooks

We will follow parts of the Chapter 3 of the Bishop book closely:

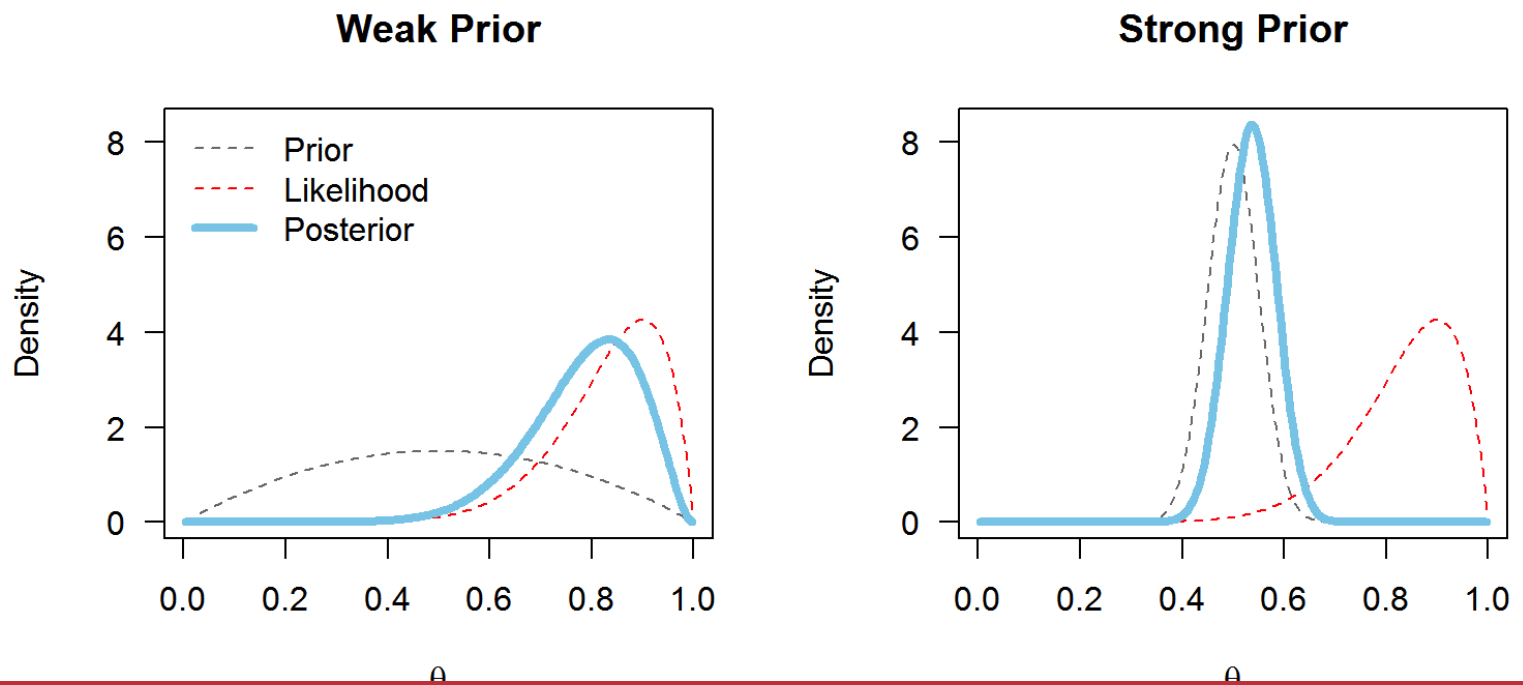
- Bishop, C. M., Pattern recognition and machine learning (2006). Available for free here.

Agenda

- Revising linear and nonlinear regression [see old SPS slides; Chapter 3, Bishop]
  - Linear regression
  - Nonlinear regression
  - Probabilistic models
  - Maximum likelihood estimation
- **Sequential Bayesian regression** [Chapter 3, Bishop]
  - Bayesian formulation
  - Conjugate priors
  - Example
- Classification and neural networks [Chapter 5, Bishop]
  - Architectures (Parametric model)
  - The supervised case
  - Optimising nnets using backprop
  - Highly flexible model $\rightarrow$ overfitting: early stopping/drop-out.

Why Bayesian?

- In the previous lecture we considered likelihood methods
- But these ignore any prior knowledge we often have about θ
- We should use Bayesian inference, which combines prior and likelihood probabilities as
- $\boxed{p(\theta|D) = p(D|\theta)p(\theta)/p(D)}$, where $p(\theta|D)$ is the posterior, $p(D|\theta)$ the likelihood of the data, $p(\theta)$ the prior over the parameters and $p(D)$ is the normalization term also called evidence.



Example: Sequential Bayesian Linear Regression

We are going to go through an example of a Bayesian model that nicely illustrates its advantages.

- Goal: We want a model that iteratively adjusts as new data comes in
- Model: The posterior is given by

$$\boxed{p(\theta|D) = p(D|\theta)p(\theta)/Z}$$

where the likelihood is similar the previous lecture [1]:

$$p(D|\theta) = \prod_{i=1}^{N} p(y_i|x_i, \theta)$$

and we use a conjugate prior $p(\theta) = \mathcal{N}(\theta|m_0, S_0)$, where m_0 and S_0 are the mean and precision (inverse variance), respectively.

[1] To be consistent with Bishop, instead of using σ directly we use the precision parameter $\beta = 1/\sigma^2$ which we assume to be given.

Why conjugate priors?

A conjugate prior is one such that the posterior is in the probability distribution family. For example, given a Gaussian likelihood if we choose a Gaussian prior, then the posterior is guaranteed to be a Gaussian – this makes this prior a conjugate prior for the posterior.

Why are they useful? The Bayes theorem has a nasty normalising term $p(D)$, which is given by $\int p(D|\theta)p(\theta)d\theta$. For models with more than a few θ it is intractable to compute this integral. Conjugate priors save us from this, as they lead to exact posterior for which we do not need to compute integrals.

Example: Sequential Bayesian Linear Regression

Given the conjugate prior $p(\theta) = \mathcal{N}(\theta|m_0, S_0)$, then our posterior is

$$\boxed{p(\theta|D) = \mathcal{N}(\theta|m_N, S_N)}$$

where [2]

$$m_N = S_N(S_0^{-1} m_0 + \beta \Phi^T D)$$
$$S_N^{-1} = S_0^{-1} + \beta \Phi^T \Phi$$

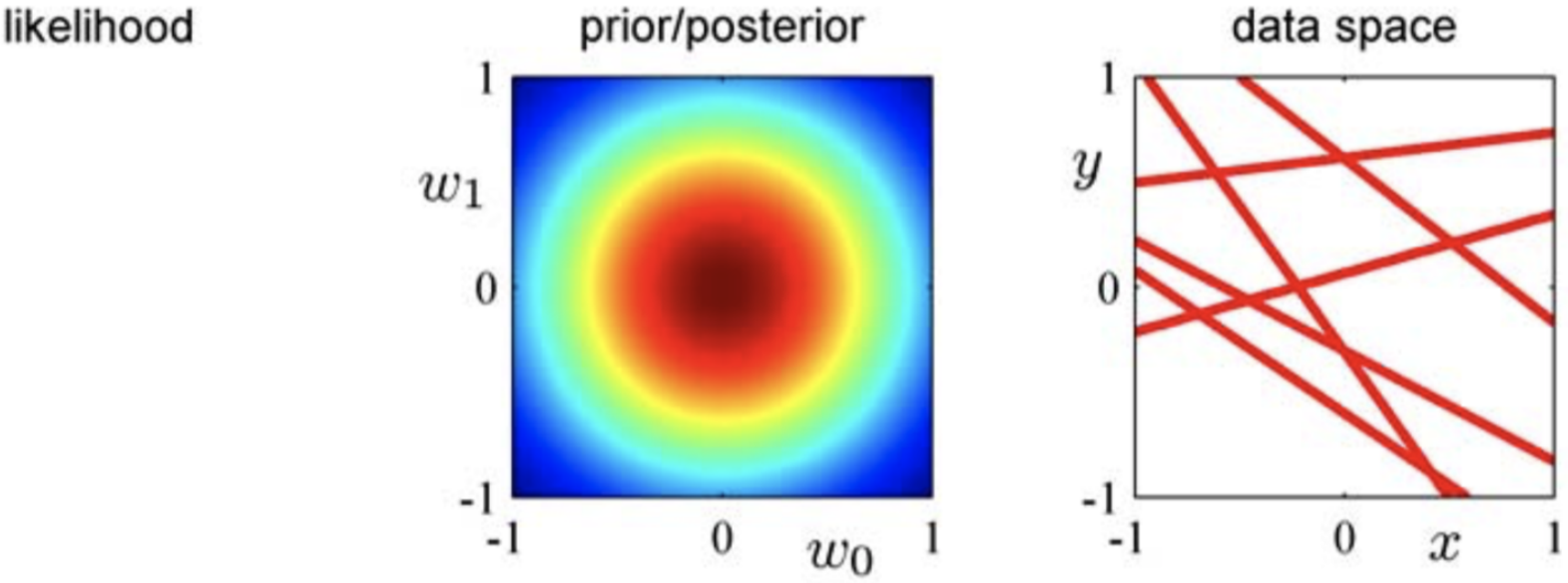
m_0 and m_N are the mean of the prior and posterior, respectively.
S_0 and S_N are the precision of the prior and posterior (i.e. $1/\sigma^2$), respectively.
β and D are the noise precision parameter [3] (which we assume to be given) and the data, respectively.
In our example $\Phi = X$ as we do not use any basis functions (see Bishop).
Note: The posterior becomes the prior (i.e. $S_0 = S_N$ and $m_0 = m_N$) for new data (see example next).

[2] More details on how to derive this equations in Bishop p.153 and here (w/ Python code).
[3] As in the previous lecture $y = a_1 x + \mathcal{N}(0, \beta = 1/\sigma^2)$

Example: Sequential Bayesian Linear Regression

Lets run this Bayesian model using the linear model $y = w_0 + w_1 x$

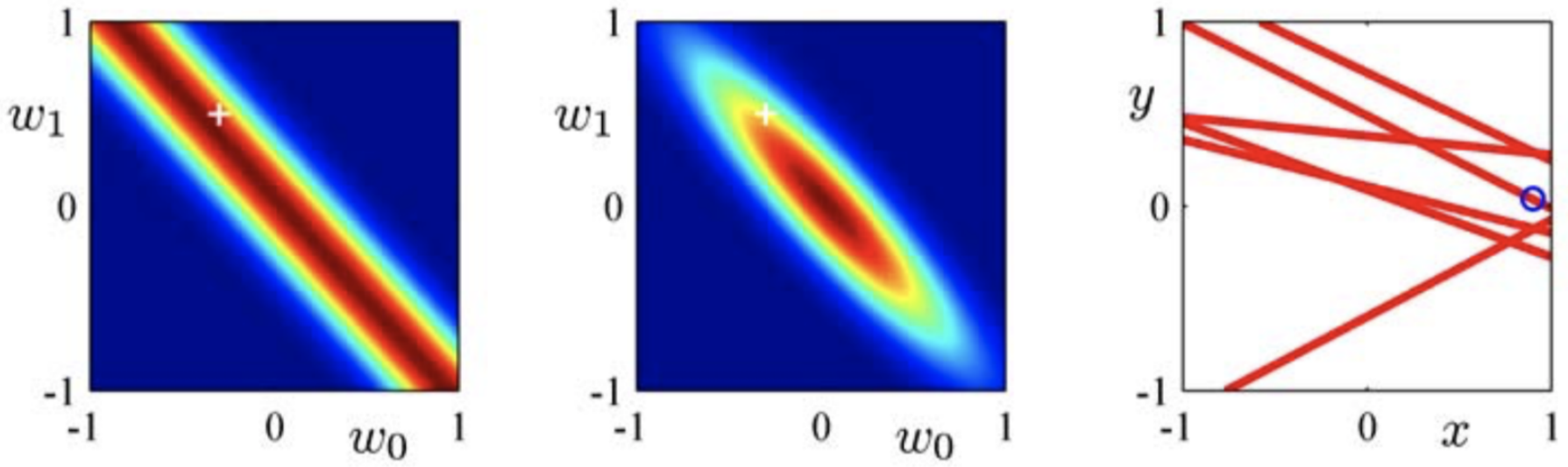
Before data arrives:



Notes: No data yet = no likelihood; prior is broad consequently so is the posterior; the model generates 'totally' random samples (red lines).

Example: Sequential Bayesian Linear Regression

After 1 datapoint:



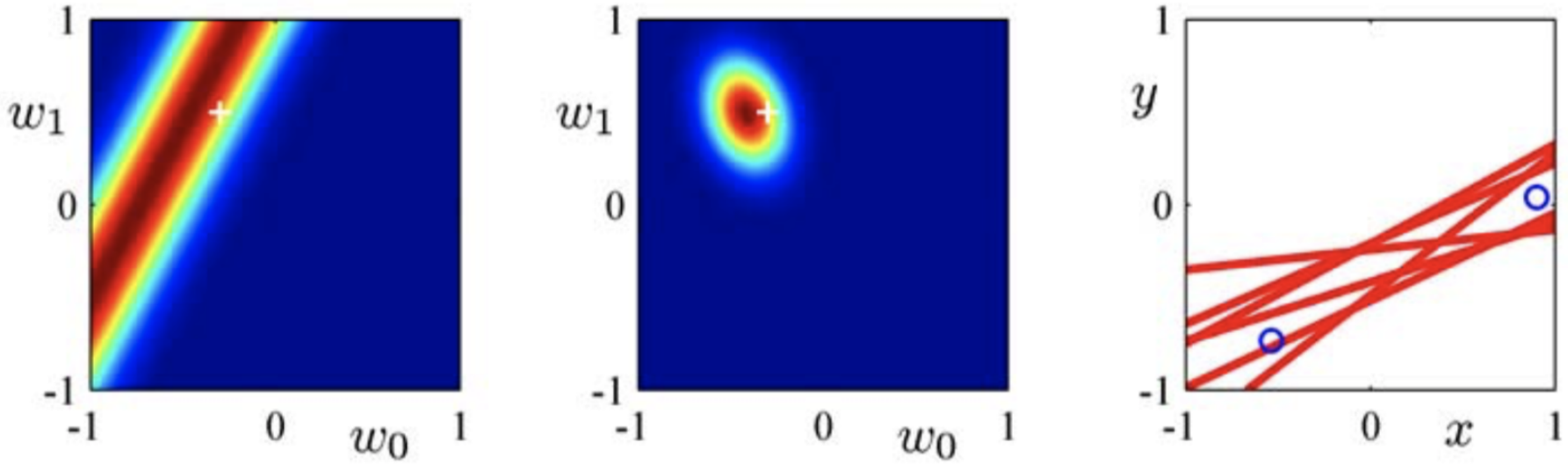
Notes: Data [4] = we have a likelihood; posterior is more precise [5], and is used as prior for next iteration. This is the most important point of this lecture, **a Bayesian framework enables you to automatically consider previous estimates as priors for future model fittings!**

[4] Data samples are represented by blue scatter plot on the right.

[5] Parameter set used to generate the data is indicated by the white cross.

Example: Sequential Bayesian Linear Regression

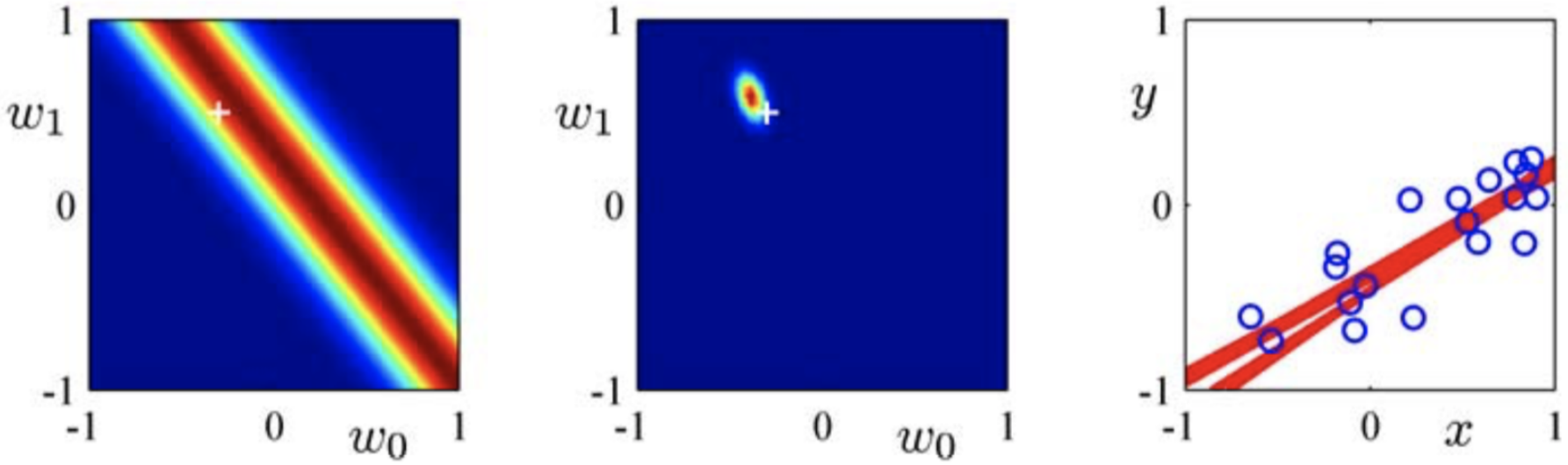
After 2 datapoints:



Notes: The posterior is even more precise, and the lines sampled from the model are better defined.

Example: Sequential Bayesian Linear Regression

After 20 datapoints:



Notes: Posterior is very close to the original parameter set (white cross).

Quiz time!

Go to Blackboard unit page » Quizzes » Lecture 3.2

[Should take you less than 5 minutes]