COMS30035, Machine learning:

Revisiting regression

Edwin Simpson (based on slides by Rui Ponte Costa and Dima Damen)

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Textbooks

Chapter 3 of the Bishop book is directly relevant:

- Bishop, C. M., Pattern recognition and machine learning (2006). Available for free here.
- **Note:** this first part is a revision of should be covered in Data-driven Computer Science in your 2nd year; more complete (but old!) full lecture notes here.

Agenda

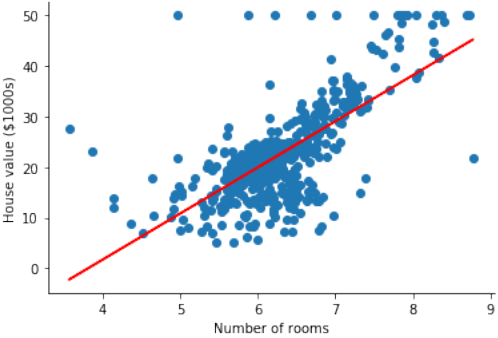
- Linear regression
- Nonlinear regression
- Probabilistic models
- Maximum likelihood estimation

[see old SPS slides; Chapter 3, Bishop]

Revisiting regression

- Goal: Finding a relationship between two variables (e.g. regress *house value* against *number of rooms*)

- Model: Linear relationship between *house value* and *number of rooms*?



Revisiting regression – deterministic model

Data: a set of data points $D = \{(x_1, y_1), (x_2, y_2), \cdots, (x_N, y_N)\}$ where x_i is the house value i and y_i is the number of rooms i.

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Model Parameters: model has two parameters a_0 and a_1 which should be estimated.

- a_0 is the y-intercept
- a_1 is the slope of the line

Least Squares Solution - matrix form

- To find a solution to the parameters $\theta = \{a_0, a_1\}$ solve least squares problem which in matrix form, means to find $\mathbf{a}_{LS}$; [1]

(2)

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- Matrix formulation also allows least squares method to be extended to **polynomial fitting**
- For a polynomial of degree $p+1$ we use (note: $p > 1$ gives nonlinear regression)

$$y_i = a_0 + a_1 x_i + a_2 x_i^2 + \cdots + a_p\, x_i^p$$

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$$y = 1.8 + 2.9x$$

Regression with probabilistic models

Probabilistic models are a core part of ML, as they allow us to also capture the uncertainty the model has about the data, which is critical for real world applications. For simplicity, lets drop a_0 from the previous model and add a random variable ϵ that captures the uncertainty

$$\text{house price} = a_1 \times \text{number of rooms} + \epsilon$$

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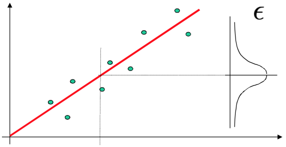
This model has two parameters: the slope a_1 and variance σ [2]



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- Assume $\boldsymbol{\theta}$ is a vector of all parameters of the probabilistic model. (e.g. $\boldsymbol{\theta} = \{a_1, \sigma\}$).
- **MLE** is an extremum estimator[3] obtained by maximising an objective function of $\boldsymbol{\theta}$

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- Tuning the parameter is then equal to finding the maximum argument *arg max*

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4. Set derivative(s) to 0 and solve for θ

Data Modelling - Deterministic vs Probabilistic

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[4] The uncertainty (σ) is represented by the light green bar in the plots. Test it yourself.

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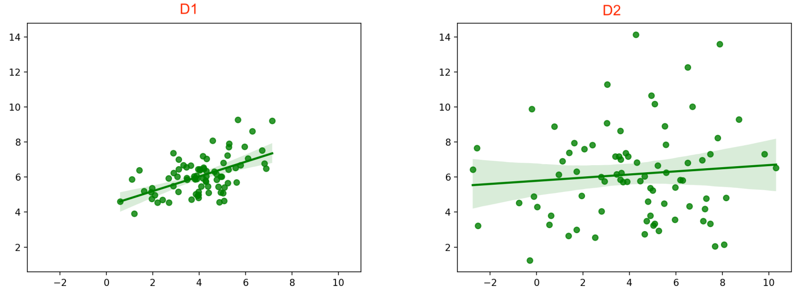

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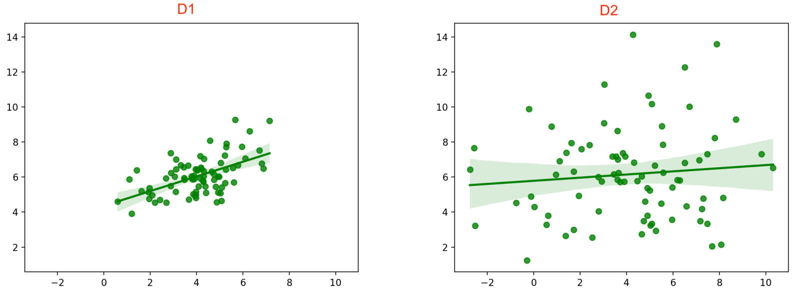

$$a_{1\,ML}^{\;D_1} > a_{1\,ML}^{\;D_2} \text{ [slope] and } \sigma_{ML}^{D_1} < \sigma_{ML}^{D_2} \text{ [uncertainty}^4\text{]}$$

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Quiz time!

Go to Blackboard unit page » Quizzes » Week 1, Revisiting Regression

[Should take you less than 5 minutes]