Programming Languages and Computation

Week 10: Encoding first-order data

* 1. Construct a bijection between the set $E = \{0, 2, 4, \ldots\}$ of all even numbers, and the set $O = \{1, 3, 5, \ldots\}$ of all odd numbers, and show that it is one.

* 2. In the reference material there is a proof that β is a bijection. Verify that $\beta : \mathbb{Z} \xrightarrow{\cong} \mathbb{N}$ is also an isomorphism: show that the function $\beta^{-1} : \mathbb{N} \to \mathbb{Z}$ defined in the lecture has the property that $\beta^{-1} \circ \beta = id_{\mathbb{Z}}$ and $\beta \circ \beta^{-1} = \mathrm{id}_{\mathbb{N}}$.

** 3. Argue that there cannot be a bijection $\mathbb{B} \xrightarrow{\cong} \mathbb{N}$.

** 4. Construct a bijection $\phi_3 : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \xrightarrow{\cong} \mathbb{N}$, and prove that it is a bijection.
[Hint: use the pairing function twice.]

** 5. Prove that if $f : A \xrightarrow{\cong} B$ is a bijection, then so is its inverse $f^{-1} : B \to A$.

** 6. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Prove that if f and g are injections, then so is $g \circ f : A \to C$.

(b) Prove that if f and g are surjections, then so is $g \circ f : A \to C$.

(c) Prove that if $f : A \to B$ and $g : B \to C$ are bijections then so is $g \circ f : A \to C$.

** 7. Prove that if $f : A \to B$ is an isomorphism and $g : B \to C$ is an isomorphism then so is $g \circ f : A \to C$.
[Hint: construct an inverse. It is possible to show this in a point-free style using the fact function composition is associative, i.e. $h \circ (g \circ f) = (h \circ g) \circ f$, and that the identity function is a unit for it, i.e. $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.]

*** 8. We define the set $\mathcal{T}$ of *binary trees* by the Backus-Naur form

$$t \in \mathcal{T} ::= \bullet \mid \mathsf{fork}(t_1, n, t_2)$$

where $n \in \mathbb{N}$ is a natural number. This is an inductive definition: a tree is either empty ($\bullet$), or is a fork, consisting of a left subtree t_1, a number $n \in \mathbb{N}$, and a right subtree t_2.

Construct a bijection $\mathscr{T} \xrightarrow{\cong} \mathbb{N}$.
[Hint: look at the way lists—also an inductively defined set!—are encoded as natural numbers in the reference material. Try to copy that. Also, use ϕ_3 from the previous exercise.]

*** 9. Given a bijection $f : A \xrightarrow{\cong} \mathbb{N}$ and a bijection $g : B \xrightarrow{\cong} \mathbb{N}$, show how to construct a bijection $A \times B \xrightarrow{\cong} \mathbb{N}$. Prove that it is a bijection.

**** 10. Prove that bijections and isomorphisms are the same thing.

(a) (Easier.) Prove that every isomorphism is a bijection.

(b) (Harder.) Prove that every bijection is an isomorphism. [Hint: consider the preimage $f^{-1}(\{b\})$ of a bijection $f : A \to B$ at every possible $b \in B$. What does it look like?]

** 11. Is the predicate

$$\mathsf{LUCKY}_{127} = \{ \ulcorner S \urcorner \mid \text{ running } S \text{ on input } 1 \text{ runs for at least } 127 \text{ computational steps } \}$$

decidable? [Hint: if it is, describe a program that decides it. Think simply, write informally, and do not let the syntactic poverty of While confine you.]

** 12. Prove that the set

$$\mathsf{Zero} = \{ \ulcorner S \urcorner \mid [\![S]\!]_{\mathsf{x}} (0) \downarrow \}$$

is semi-decidable. [Hint: As above, think simply, write informally, and do not let the syntactic poverty of While confine you.]

*** 13. Prove that if the predicates U and V are semi-decidable, then so is $U \cup V$. [Hint: use simulations.]

*** 14. Suppose we have a way of encoding every DFA M as a natural number $\delta(M) \in \mathbb{N}$.
Is the predicate

$$\mathsf{EMPTY} = \{\delta(M) \mid L(M) = \emptyset\}$$

decidable? [Hint: if it is, describe a program that decides it. Think simply, write informally, and do not let the syntactic poverty of While confine you.]