

# Problem Sheet 9: Computable functions and predicates, data representation

## Computable functions and predicates

- \* 1. Show that the identity function  $\text{id}_{\mathbb{N}} : \mathbb{N} \rightarrow \mathbb{N}$  is computable.
- \* 2. Show that the function  $\lfloor \sqrt{-} \rfloor : \mathbb{N} \rightarrow \mathbb{N}$  which computes the **integer square root** of a natural number is computable.
- \* 3. Argue that the variable name does not matter in the definition of computability. That is, if a program  $S$  computes  $f : \mathbb{N} \rightarrow \mathbb{N}$  with respect to  $x$ , then there is a program  $S'$  that also computes  $f$ , but with respect to any variable of our choice.
- \*\* 4. Show that if  $f : \mathbb{N} \rightarrow \mathbb{N}$  and  $g : \mathbb{N} \rightarrow \mathbb{N}$  are computable, then so is their composition  $g \circ f : \mathbb{N} \rightarrow \mathbb{N}$ .
- \* 5. Show that the predicate
$$P = \{n \in \mathbb{N} \mid n \text{ is a prime number}\}$$
is decidable. (NB 1 is *not* a prime number.)
- \*\* 6. Show that if the predicates  $P$  and  $Q$  are decidable, then so is  $P \cap Q$ . Is the same true if  $P$  and  $Q$  are semi-decidable?
- \*\*\*\* 7. (Trick question.) Is it true that if  $P$  and  $Q$  are semi-decidable, then so is  $P \cup Q$ ?

## Bijections, sections, and retractions

- \* 8. Construct a bijection between the set  $E = \{0, 2, 4, \dots\}$  of all even numbers, and the set  $O = \{1, 3, 5, \dots\}$  of all odd numbers.

- \*\*\* 9. Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be functions.

- (a) Prove that if  $f$  and  $g$  are injections, then so is  $g \circ f : A \rightarrow C$ .
- (b) Prove that if  $f$  and  $g$  are surjections, then so is  $g \circ f : A \rightarrow C$ .
- (c) Prove that if  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are bijections then so is  $g \circ f : A \rightarrow C$ .

- \*\*\* 10. Prove that if  $s : A \rightarrow B$  and  $r : B \rightarrow A$  are a section-retraction pair, then

- (a)  $s$  is injective, and
- (b)  $r$  is surjective.

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11.

Prove that predicates  $U \subseteq \mathbb{N}$  and functions  $f : \mathbb{N} \rightarrow \mathbb{B}$  to the set  $\mathbb{B} = \{\top, \perp\}$  of Boolean values are in a perfect correspondence. That is, prove that there is a bijection between the powerset  $\mathcal{P}(\mathbb{N}) = \{U \mid U \subseteq \mathbb{N}\}$  and the function space  $\mathbb{B}^{\mathbb{N}} = \{f \mid f : \mathbb{N} \rightarrow \mathbb{B}\}$ .

- (a) Given a predicate  $U \subseteq \mathbb{N}$ , construct a function  $f_U : \mathbb{N} \rightarrow \mathbb{B}$  that corresponds to it.
- (b) Given a function  $f : \mathbb{N} \rightarrow \mathbb{B}$ , construct a predicate  $U_f \subseteq \mathbb{N}$  that corresponds to it.
- (c) Prove that the constructions  $U \mapsto f_U$  and  $f \mapsto U_f$  are inverses. That is, for a predicate  $S \subseteq \mathbb{N}$  prove that  $U_{f_S} = S$ ; and that for a function  $h : \mathbb{N} \rightarrow \mathbb{B}$  prove that  $f_{U_h} = h : \mathbb{N} \rightarrow \mathbb{B}$ . (What does it mean for two functions to be equal?)