

Programming Languages and Computation

Week 10: Encoding first-order data

- * 1. Construct a bijection between the set $E = \{0, 2, 4, \dots\}$ of all even numbers, and the set $O = \{1, 3, 5, \dots\}$ of all odd numbers, and show that it is one.
- * 2. In the reference material there is a proof that β is a bijection. Verify that $\beta : \mathbb{Z} \xrightarrow{\cong} \mathbb{N}$ is also an isomorphism: show that the function $\beta^{-1} : \mathbb{N} \rightarrow \mathbb{Z}$ defined in the lecture has the property that $\beta^{-1} \circ \beta = id_{\mathbb{Z}}$ and $\beta \circ \beta^{-1} = id_{\mathbb{N}}$.
- ** 3. Argue that there cannot be a bijection $\mathbb{B} \xrightarrow{\cong} \mathbb{N}$.
- ** 4. Construct a bijection $\phi_3 : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \xrightarrow{\cong} \mathbb{N}$, and prove that it is a bijection.
[Hint: use the pairing function twice.]
- ** 5. Prove that if $f : A \xrightarrow{\cong} B$ is a bijection, then so is its inverse $f^{-1} : B \rightarrow A$.
- ** 6. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions.
 - (a) Prove that if f and g are injections, then so is $g \circ f : A \rightarrow C$.
 - (b) Prove that if f and g are surjections, then so is $g \circ f : A \rightarrow C$.
 - (c) Prove that if $f : A \rightarrow B$ and $g : B \rightarrow C$ are bijections then so is $g \circ f : A \rightarrow C$.
- ** 7. Prove that if $f : A \rightarrow B$ is an isomorphism and $g : B \rightarrow C$ is an isomorphism then so is $g \circ f : A \rightarrow C$.
[Hint: construct an inverse. It is possible to show this in a point-free style using the fact function composition is associative, i.e. $h \circ (g \circ f) = (h \circ g) \circ f$, and that the identity function is a unit for it, i.e. $id_B \circ f = f = f \circ id_A$.]
- *** 8. We define the set \mathcal{T} of *binary trees* by the **Backus-Naur form**
$$t \in \mathcal{T} ::= \bullet \mid \text{fork}(t_1, n, t_2)$$
where $n \in \mathbb{N}$ is a natural number. This is an inductive definition: a tree is either empty (\bullet), or is a fork, consisting of a left subtree t_1 , a number $n \in \mathbb{N}$, and a right subtree t_2 .

Construct a bijection $\mathcal{T} \xrightarrow{\cong} \mathbb{N}$.

[Hint: look at the way lists—also an inductively defined set!—are encoded as natural numbers in the [reference material](#). Try to copy that. Also, use ϕ_3 from the previous exercise.]

- *** 9. Given a bijection $f : A \xrightarrow{\cong} \mathbb{N}$ and a bijection $g : B \xrightarrow{\cong} \mathbb{N}$, show how to construct a bijection $A \times B \xrightarrow{\cong} \mathbb{N}$. Prove that it is a bijection.

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10. Prove that bijections and isomorphisms are the same thing.

- (a) (Easier.) Prove that every isomorphism is a bijection.
- (b) (Harder.) Prove that every bijection is an isomorphism. [Hint: consider the [preimage](#) $f^{-1}(\{b\})$ of a bijection $f : A \rightarrow B$ at every possible $b \in B$. What does it look like?]