Programming Languages and Computation

Week 9: Encoding data

* 1. Construct a bijection between the set $E = \{0, 2, 4, \dots\}$ of all even numbers, and the set $O = \{1, 3, 5, \dots\}$ of all odd numbers.

* 2. Argue that there cannot be a bijection $\mathbb{B} \xrightarrow{\cong} \mathbb{N}$.

** 3. Construct a bijection $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \xrightarrow{\cong} \mathbb{N}$. [Hint: use the pairing function.]

** 4. Prove that if $f : A \xrightarrow{\cong} B$ is a bijection, then so is its inverse $f^{-1} : B \to A$.

*** 5. We define the set $\mathscr{T}$ of *binary trees* by the Backus-Naur form

$$t \in \mathscr{T} ::= \bullet \mid \mathsf{fork}(t_1, n, t_2)$$

where $n \in \mathbb{N}$ is a natural number. This is an inductive definition: a tree is either empty ($\bullet$), or is a fork, consisting of a left subtree t_1, a number $n \in \mathbb{N}$, and a right subtree t_2.
Construct a bijection $\mathscr{T} \xrightarrow{\cong} \mathbb{N}$.
[Hint: look at the way lists—also an inductively defined set!—are encoded as natural numbers in the reference material. Try to copy that. Also, use ϕ_3 from the previous exercise.]

*** 6. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Prove that if f and g are injections, then so is $g \circ f : A \to C$.

(b) Prove that if f and g are surjections, then so is $g \circ f : A \to C$.

(c) Prove that if $f : A \to B$ and $g : B \to C$ are bijections then so is $g \circ f : A \to C$.

*** 7. Prove that if $s : A \to B$ and $r : B \to A$ are a section-retraction pair, then

(a) s is injective, and

(b) r is surjective.

**** 8. Prove that bijections and isomorphisms are the same thing.

> (a) (Easier.) Prove that every isomorphism is a bijection.
>
> (b) (Harder.) Prove that every isomorphism is a bijection. [Hint: consider the preimage $f^{-1}(\{b\})$ of a bijection $f : A \to B$ at every possible $b \in B$. What does it look like?]