Programming Languages and Computation

Week 10: Encoding data

* 1. Construct a bijection between the set $E = \{0, 2, 4, \dots\}$ of all even numbers, and the set $O = \{1, 3, 5, \dots\}$ of all odd numbers.

* 2. In the reference material there is a proof that β is a bijection. Verify also that $\beta : \mathbb{Z} \xrightarrow{\cong} \mathbb{N}$ is an isomorphism: show that the function $\beta^{-1} : \mathbb{N} \to \mathbb{Z}$ defined in the lecture has the property that $\beta^{-1} \circ \beta = id_{\mathbb{Z}}$ and $\beta \circ \beta^{-1} = \mathrm{id}_{\mathbb{N}}$.

* 3. Verify that $\phi_* : \mathbb{N}^* \xrightarrow{\cong} \mathbb{N}$ is an isomorphism.

** 4. Argue that there cannot be a bijection $\mathbb{B} \xrightarrow{\cong} \mathbb{N}$.

** 5. Construct a bijection $\phi_3 : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \xrightarrow{\cong} \mathbb{N}$. [Hint: use the pairing function twice.]

** 6. Prove that if $f : A \xrightarrow{\cong} B$ is a bijection, then so is its inverse $f^{-1} : B \to A$.

** 7. Let $f : A \to B$ and $g : B \to C$ be functions.

   (a) Prove that if f and g are injections, then so is $g \circ f : A \to C$.

   (b) Prove that if f and g are surjections, then so is $g \circ f : A \to C$.

   (c) Prove that if $f : A \to B$ and $g : B \to C$ are bijections then so is $g \circ f : A \to C$.

*** 8. We define the set $\mathscr{T}$ of *binary trees* by the Backus-Naur form

$$t \in \mathscr{T} ::= \bullet \mid \mathsf{fork}(t_1, n, t_2)$$

where $n \in \mathbb{N}$ is a natural number. This is an inductive definition: a tree is either empty ($\bullet$), or is a fork, consisting of a left subtree t_1, a number $n \in \mathbb{N}$, and a right subtree t_2.
Construct a bijection $\mathscr{T} \xrightarrow{\cong} \mathbb{N}$.
[Hint: look at the way lists—also an inductively defined set!—are encoded as natural numbers in the reference material. Try to copy that. Also, use ϕ_3 from the previous exercise.]

**** 9. Prove that bijections and isomorphisms are the same thing.

> (a) (Easier.) Prove that every isomorphism is a bijection.
>
> (b) (Harder.) Prove that every bijection is an isomorphism. [Hint: consider the preimage $f^{-1}(\{b\})$ of a bijection $f : A \to B$ at every possible $b \in B$. What does it look like?]

*** 10. Prove that if $s : A \to B$ and $r : B \to A$ are a section-retraction pair, then

> (a) s is injective, and
>
> (b) r is surjective.