

Programming Languages and Computation

Week 9: Computable functions and predicates, data representation

Computable functions and predicates

- * 1. Show that the identity function $\text{id}_{\mathbb{N}} : \mathbb{N} \rightarrow \mathbb{N}$ is computable.
- * 2. Show that the function $\lfloor \sqrt{\cdot} \rfloor : \mathbb{N} \rightarrow \mathbb{N}$ which computes the **integer square root** of a natural number is computable.
- * 3. Argue that the variable name does not matter in the definition of computability. That is, if a program S computes $f : \mathbb{N} \rightarrow \mathbb{N}$ with respect to x , then there is a program S' that also computes f , but with respect to any variable of our choice.
- ** 4. Show that if $f : \mathbb{N} \rightarrow \mathbb{N}$ and $g : \mathbb{N} \rightarrow \mathbb{N}$ are computable, then so is their composition $g \circ f : \mathbb{N} \rightarrow \mathbb{N}$.
- * 5. Show that the predicate
$$P = \{n \in \mathbb{N} \mid n \text{ is a prime number}\}$$
is decidable. (NB 1 is *not* a prime number.)
- ** 6. Show that if the predicates P and Q are decidable, then so is $P \cap Q$. Is the same true if P and Q are semi-decidable?
- **** 7. (Trick question.) Is it true that if P and Q are semi-decidable, then so is $P \cup Q$?

Bijections, sections, and retractions

- * 8. Construct a bijection between the set $E = \{0, 2, 4, \dots\}$ of all even numbers, and the set $O = \{1, 3, 5, \dots\}$ of all odd numbers.
- *** 9. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions.

- (a) Prove that if f and g are injections, then so is $g \circ f : A \rightarrow C$.
- (b) Prove that if f and g are surjections, then so is $g \circ f : A \rightarrow C$.
- (c) Prove that if $f : A \rightarrow B$ and $g : B \rightarrow C$ are bijections then so is $g \circ f : A \rightarrow C$.

*** 10. Prove that if $s : A \rightarrow B$ and $r : B \rightarrow A$ are a section-retraction pair, then

- (a) s is injective, and
- (b) r is surjective.

11.

Prove that predicates $U \subseteq \mathbb{N}$ and functions $f : \mathbb{N} \rightarrow \mathbb{B}$ to the set $\mathbb{B} = \{\top, \perp\}$ of Boolean values are in a perfect correspondence. That is, prove that there is a bijection between the powerset $\mathcal{P}(\mathbb{N}) = \{U \mid U \subseteq \mathbb{N}\}$ and the function space $\mathbb{B}^{\mathbb{N}} = \{f \mid f : \mathbb{N} \rightarrow \mathbb{B}\}$.

- (a) Given a predicate $U \subseteq \mathbb{N}$, construct a function $f_U : \mathbb{N} \rightarrow \mathbb{B}$ that corresponds to it.
- (b) Given a function $f : \mathbb{N} \rightarrow \mathbb{B}$, construct a predicate $U_f \subseteq \mathbb{N}$ that corresponds to it.
- (c) Prove that the constructions $U \mapsto f_U$ and $f \mapsto U_f$ are inverses. That is, for a predicate $S \subseteq \mathbb{N}$ prove that $U_{f_S} = S$; and that for a function $h : \mathbb{N} \rightarrow \mathbb{B}$ prove that $f_{U_h} = h : \mathbb{N} \rightarrow \mathbb{B}$. (What does it mean for two functions to be equal?)