

## Programming Languages and Computation

# Week 10: Encoding first-order data

- \* 1. Construct a bijection between the set  $E = \{0, 2, 4, \dots\}$  of all even numbers, and the set  $O = \{1, 3, 5, \dots\}$  of all odd numbers, and show that it is one.
- \* 2. In the reference material there is a proof that  $\beta$  is a bijection. Verify that  $\beta : \mathbb{Z} \xrightarrow{\cong} \mathbb{N}$  is also an isomorphism: show that the function  $\beta^{-1} : \mathbb{N} \rightarrow \mathbb{Z}$  defined in the lecture has the property that  $\beta^{-1} \circ \beta = id_{\mathbb{Z}}$  and  $\beta \circ \beta^{-1} = id_{\mathbb{N}}$ .
- \*\* 3. Argue that there cannot be a bijection  $\mathbb{B} \xrightarrow{\cong} \mathbb{N}$ .
- \*\* 4. Construct a bijection  $\phi_3 : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \xrightarrow{\cong} \mathbb{N}$ , and prove that it is a bijection.  
[Hint: use the pairing function twice.]
- \*\* 5. Prove that if  $f : A \xrightarrow{\cong} B$  is a bijection, then so is its inverse  $f^{-1} : B \rightarrow A$ .
- \*\* 6. Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be functions.
  - (a) Prove that if  $f$  and  $g$  are injections, then so is  $g \circ f : A \rightarrow C$ .
  - (b) Prove that if  $f$  and  $g$  are surjections, then so is  $g \circ f : A \rightarrow C$ .
  - (c) Prove that if  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are bijections then so is  $g \circ f : A \rightarrow C$ .
- \*\* 7. Prove that if  $f : A \rightarrow B$  is an isomorphism and  $g : B \rightarrow C$  is an isomorphism then so is  $g \circ f : A \rightarrow C$ .  
[Hint: construct an inverse. It is possible to show this in a point-free style using the fact function composition is associative, i.e.  $h \circ (g \circ f) = (h \circ g) \circ f$ , and that the identity function is a unit for it, i.e.  $id_B \circ f = f = f \circ id_A$ .]
- \*\*\* 8. We define the set  $\mathcal{T}$  of *binary trees* by the **Backus-Naur form**
$$t \in \mathcal{T} ::= \bullet \mid \text{fork}(t_1, n, t_2)$$
where  $n \in \mathbb{N}$  is a natural number. This is an inductive definition: a tree is either empty ( $\bullet$ ), or is a fork, consisting of a left subtree  $t_1$ , a number  $n \in \mathbb{N}$ , and a right subtree  $t_2$ .

Construct a bijection  $\mathcal{T} \xrightarrow{\cong} \mathbb{N}$ .

[Hint: look at the way lists—also an inductively defined set!—are encoded as natural numbers in the [reference material](#). Try to copy that. Also, use  $\phi_3$  from the previous exercise.]

- \*\*\* 9. Given a bijection  $f : A \xrightarrow{\cong} \mathbb{N}$  and a bijection  $g : B \xrightarrow{\cong} \mathbb{N}$ , show how to construct a bijection  $A \times B \xrightarrow{\cong} \mathbb{N}$ . Prove that it is a bijection.

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10. Prove that bijections and isomorphisms are the same thing.

- (a) (Easier.) Prove that every isomorphism is a bijection.
- (b) (Harder.) Prove that every bijection is an isomorphism. [Hint: consider the [preimage](#)  $f^{-1}(\{b\})$  of a bijection  $f : A \rightarrow B$  at every possible  $b \in B$ . What does it look like?]

- \*\* 11. Is the predicate

$$\text{LUCKY}_{127} = \{ \ulcorner S \urcorner \mid \text{running } S \text{ on input } 1 \text{ runs for at least } 127 \text{ computational steps} \}$$

decidable? [Hint: if it is, describe a program that decides it. Think simply, write informally, and do not let the syntactic poverty of While confine you.]

- \*\* 12. Prove that the set

$$\text{Zero} = \{ \ulcorner S \urcorner \mid \llbracket S \rrbracket_x(0) \downarrow \}$$

is semi-decidable. [Hint: As above, think simply, write informally, and do not let the syntactic poverty of While confine you.]

- \*\*\* 13. Prove that if the predicates  $U$  and  $V$  are semi-decidable, then so is  $U \cup V$ . [Hint: use simulations.]