Types and λ-calculus

Problem Sheet 3

This week, questions 2, 8 and 9 will be marked.

Recall that a *closed* term has no free variables.

* 1. Perform one step of reduction for each of the following terms:

(a) $\underline{\mathsf{const}}\,\mathsf{pred}\,\mathsf{pred}$

(b) $\underline{\mathsf{sub}}\,\underline{\mathsf{const}}$

(c) $(\lambda x.\,x\,x)(\lambda x.\,x\,x)$

(d) $\underline{\mathsf{const}}\,(\mathsf{pred}\,\mathsf{pred})$

* 2. For each of the following reduction steps $M \rhd N$, identify the redex P, the contraction Q, and the context $C[\,]$ in which the contraction happens, i.e. such that $M = C[P]$ and $N = C[Q]$.

(a) $\lambda x.\,\mathsf{pred}\ (\mathsf{pred}\ \underline{2}) \rhd \lambda x.\,\mathsf{pred}\ \underline{1}$

(b) $\underline{\mathsf{id}}\ (\underline{\mathsf{const}}\ \underline{\mathsf{div}}\ \underline{0}) \rhd \underline{\mathsf{id}}\ (\underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{div}})\ \underline{0})$

(c) $\underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{id}})\ (\mathsf{S}\ x) \rhd (\lambda y.\,\underline{\mathsf{id}}\ \underline{\mathsf{id}})\ (\mathsf{S}\ x)$

* 3. Let us define the Booleans as follows:

$$\underline{\mathsf{false}} = \underline{0}$$
$$\underline{\mathsf{true}} = \underline{1}$$

Define Boolean conjunction as a term $\underline{\mathsf{and}}$, disjunction as a term $\underline{\mathsf{or}}$ and negation as a term $\underline{\mathsf{not}}$.

* 4. Define terms $\underline{\mathsf{curry}}$ and $\underline{\mathsf{uncurry}}$ with the following behaviour:

$$\underline{\mathsf{curry}}\, M\, N\, P \;\rhd^*\; M\,(N, P)$$
$$\underline{\mathsf{uncurry}}\, M\,(N, P) \;\rhd^*\; M\, N\, P$$

* 5. For each of the following specifications, give an example of a *closed* term N *in normal form* that satisfies it (i.e do some reduction):

(a) $\underline{\mathsf{id}}\;\underline{\mathsf{id}} \;\rhd^*\; N$
(b) $\underline{\mathsf{sub}}\;\underline{\mathsf{const}}\;\underline{\mathsf{const}} \;\rhd^*\; N$
(c) $\mathsf{fix}\,(\lambda x y.\, y) \;\rhd^*\; N$
(d) $(\lambda x y.\, y x)\,(\underline{\mathsf{const}}\;\underline{\mathsf{const}})\,(\lambda x.\, x x) \;\rhd^*\; N$

* 6. Draw the reduction graph of the term $(\lambda x y.\, y y)\,((\lambda z.zz)\,(\lambda z.zz))\,(\lambda x.\, x)$. (This graph will have 4 vertices). What I mean by this is to draw a directed graph where:

- The nodes are all N s.t. $(\lambda x y.\, y y)\,((\lambda z.zz)\,(\lambda z.zz))\,(\lambda x.\, x) \;\rhd^*\; N$
- There is an edge from node M to node N iff $M \;\rhd\; N$

** 7. Give an example of a *closed* term M for each of the following properties:

(a) M is in normal form.
(b) M has exactly one reduct.
(c) M contains strictly fewer redexes than one of its reducts (here we mean "fewer in number", the redexes may be quite different).
(d) A reduct of M contains a redex that did not occur anywhere in M.

** 8. Prove the following statement:

For all M, N and $C[\,]$: if $M \;\rhd\; N$ then $C[M] \;\rhd\; C[N]$.

Note that "if $M \;\rhd\; N$ then $C[M] \;\rhd\; C[N]$" is subtly different from the definition of $\rhd$ which says that $C[P] \;\rhd\; C[Q]$ whenever P is a redex and Q the contraction. Here, M and N can be any terms.

You do *not* need to use induction to prove it. You will need to work closely with the definition of $\rhd$: on the one hand you will assume $M \rhd N$ and want to know what you get out of it and, on the other hand, you will want to show $C[M] \rhd C[N]$ and thus need to know what evidence is required to put into it.

Look again at the definition of $\rhd$ using contexts. In the definition, "just if" means the same as "iff", so the definition of $\rhd$ can be seen as a pair of implications: one direction tells you what follows from $M \rhd N$ when you have it as an assumption (forwards reasoning) and the other tells you what you need in order to deduce $M \rhd N$ (backwards reasoning).

** 9.

(a) Complete the following proof by filling in (a):

For all P, $C[\,]$, for all $n \in \mathbb{N}$: for all Q, if $P \rhd^n Q$ then $C[P] \rhd^n C[Q]$.

Proof. Let P be a term and $C[\,]$ a context. We show that, for all $n \in \mathbb{N}$, for all Q, $P \rhd^n Q$ implies $C[P] \rhd^n C[Q]$ by induction on n:

- When $n = 0$, let Q be a term and suppose $P \rhd^0 Q$. Then, by definition, $P = Q$ and hence $C[P] = C[Q]$. By definition, therefore $C[P] \rhd^0 C[Q]$.
- When n is of shape $k+1$, we can assume the induction hypothesis:

  (IH) forall Q, $P \rhd^k Q$ implies $C[P] \rhd^k C[Q]$.

  ... (a) ...

$\square$

(b) Deduce that (i.e. give a short proof of): For all $P, Q, C[\,]$: if $P \rhd^* Q$ then $C[P] \rhd^* C[Q]$.

** 10. Show that there is no term P that satisfies: for all M and N, $P\,(MN) \rhd^* N$. In other words, prove that we cannot write a PCF program that extracts the argument of an application.