CHURCH NUMERALS

The ***Church numeral*** for the number n, abbreviated $\ulcorner n \urcorner$, is:

$$\lambda f x.\, \underbrace{f(\cdots(f}_{n\text{-times}}\, x)\cdots)$$

DEFINABILITY

A function $f : \mathbb{N} \times \cdots \times \mathbb{N} \to \mathbb{N}$ on k-tuples of natural numbers is said to be ***λ-definable*** just if there exists a λ-term F that satisfies the equation:

$$F \ulcorner n_1 \urcorner \cdots \ulcorner n_k \urcorner =_\beta \ulcorner f(n_1, \ldots, n_k) \urcorner$$

ADDITION

$$\mathbf{Add} := \lambda yz.\,\lambda fx.\,yf(zfx) \qquad \mathbf{Add}\,\ulcorner m \urcorner \ulcorner n \urcorner =_\beta \ulcorner m+n \urcorner$$

PREDECESSOR

$$\mathbf{Pred} := \lambda z.\, \lambda fx.\, z(\lambda gh.\, h(gf))(\lambda u.\, x)(\lambda u.\, u)$$

$$\mathbf{Pred}\,\ulcorner 0 \urcorner =_\beta \ulcorner 0 \urcorner$$

$$\mathbf{Pred}\,\ulcorner n+1 \urcorner =_\beta \ulcorner n \urcorner$$

SUBTRACTION

$$\mathbf{Sub} := \lambda mn.\, n\ \mathbf{Pred}\ m$$

$$\begin{aligned}
\mathbf{Sub}\ulcorner m \urcorner \ulcorner n \urcorner &=_\beta \ulcorner 0 \urcorner && \text{if } m \leq n \\
\mathbf{Sub}\ulcorner m \urcorner \ulcorner n \urcorner &=_\beta \ulcorner m - n \urcorner && \text{otherwise}
\end{aligned}$$

TEST FOR ZERO

$$\mathbf{IfZero} := \lambda xyz.\, x(\mathbf{K}\, z)y$$

$$\mathbf{IfZero} \ulcorner 0 \urcorner \ulcorner p \urcorner \ulcorner q \urcorner =_\beta \ulcorner p \urcorner$$
$$\mathbf{IfZero} \ulcorner n+1 \urcorner \ulcorner p \urcorner \ulcorner q \urcorner =_\beta \ulcorner q \urcorner$$