Reference for Types and Lambda Calculus

Terms

$$\begin{array}{rrcl} \text{(Terms)} & M, N & ::= & x \mid c \mid (\lambda x.\, M) \mid (MN) \\ \text{(Constants)} & c & ::= & \mathsf{fix} \mid \mathsf{Z} \mid \mathsf{S} \mid \mathsf{pred} \mid \mathsf{ifz} \end{array}$$

A term is said to be *pure* just if it contains no constants. Abbreviations:

$$\begin{array}{rcl} \underline{n} & = & \mathsf{S}^n\ \mathsf{Z} \\ \underline{\mathsf{id}} & = & \lambda x.\, x \\ \underline{\mathsf{const}} & = & \lambda xy.\, x \\ \underline{\mathsf{sub}} & = & \lambda xyz.\, xz(yz) \\ \underline{\mathsf{div}} & = & \mathsf{fix}\ \underline{\mathsf{id}} \end{array}$$

Free Variables

$$\begin{array}{rcl} \mathsf{FV}(x) & = & \{x\} \\ \mathsf{FV}(c) & = & \emptyset \\ \mathsf{FV}(PQ) & = & \mathsf{FV}(P) \cup \mathsf{FV}(Q) \\ \mathsf{FV}(\lambda x.\, N) & = & \mathsf{FV}(N) \setminus \{x\} \end{array}$$

A term M is said to be *closed* just if $\mathsf{FV}(M) = \emptyset$.

Substitution

$$\begin{array}{rcll} c[N/x] & = & c & \\ y[N/x] & = & y & \text{if } x \neq y \\ y[N/x] & = & N & \text{if } x = y \\ (PQ)[N/x] & = & P[N/x]Q[N/x] & \\ (\lambda y.\, P)[N/x] & = & \lambda y.\, P & \text{if } y = x \\ (\lambda y.\, P)[N/x] & = & \lambda y.\, P[N/x] & \text{if } y \neq x \text{ and } y \notin \mathsf{FV}(N) \end{array}$$

Redexes

$$\begin{array}{r} \mathsf{pred}\ \mathsf{Z}\ /\ \mathsf{Z} \\ \mathsf{pred}\ (\mathsf{S}\ N)\ /\ N \\ \mathsf{ifz}\ \mathsf{Z}\ N\ P\ /\ N \\ \mathsf{ifz}\ (\mathsf{S}\ M)\ N\ P\ /\ P \\ (\lambda x.\, M)\ N\ /\ M[N/x] \\ \mathsf{fix}\ M\ /\ M\ (\mathsf{fix}\ M) \end{array}$$

One Step

$$C[] ::= [] \mid M\ C[] \mid C[]\ N \mid \lambda x.\, C[]$$

Define $M \ \triangleright\ N$ just if there is a context $C[]$ and a redex/contraction pair $P\ /\ Q$ such that $M = C[P]$ and $N = C[Q]$.

Reduction and Conversion

- $P \ \triangleright^0 \ Q$ just if $P = Q$.
- $P \ \triangleright^{k+1} \ Q$ just if there is some U such that $P \ \triangleright^k \ U$ and $U \ \triangleright \ Q$.

Define $M \ \triangleright^* \ N$ just if there is some n such that $M \ \triangleright^n \ N$.

We write $M \ \approx \ N$ just if there is a term P such that $M \ \triangleright^* \ P$ and $N \ \triangleright^* \ P$.

- If $M \ \triangleright^* \ N$ then the term N is said to be a ***reduct*** of M.
- If $M \ \triangleright^* \ N$ and $M \neq N$ then the term N is said to be a ***proper reduct*** of M.
- A term M without a proper reduct is a ***normal form***.
- A term M that can reduce to normal form ***has a normal form*** or is ***normalisable***.
- A term M that has no infinite reduction sequences is said to be ***strongly normalisable***.

Computability

A ***Gödel numbering*** is a pair of computable (partial) functions:

- $\# : \Lambda \to \mathbb{N}$
- $\#^{-1} : \mathbb{N} \to \Lambda$

with the property that $\#^{-1}\ (\#\ M) = M$.

Let $\Phi \subseteq \Lambda$ be a property of PCF-terms. We say that Φ is ***decidable*** just if the characteristic function of Φ is computable, i.e. there is an algorithm for computing $\chi : \Lambda \to \mathbb{N}$ satisfying, for all terms $M \in \Lambda$:

$$\chi(M) = \begin{cases} 1 & \text{if } M \in \Phi \\ 0 & \text{if } M \notin \Phi \end{cases}$$

Type Assignment

$$(\mathsf{Types}) \quad A,\, B \quad ::= \mathsf{Nat} \mid a \mid (A \to B)$$

Let $\mathbb{C}$ be the following collection of type assignments:

$$\begin{aligned} &\{\mathsf{Z} : \mathsf{Nat}\} \cup \{\mathsf{S} : \mathsf{Nat} \to \mathsf{Nat}\} \cup \{\mathsf{pred} : \mathsf{Nat} \to \mathsf{Nat}\} \\ &\qquad\qquad \cup \{\mathsf{ifz} : \mathsf{Nat} \to A \to A \to A \mid A \in \mathbb{T}\} \\ &\qquad\qquad \cup \{\mathsf{fix} : (A \to A) \to A \mid A \in \mathbb{T}\} \end{aligned}$$

The typing rules are:

$$\frac{x{:}A \in \Gamma}{\Gamma \vdash x : A}\,(\mathsf{TVar}) \qquad \frac{c{:}A \in \mathbb{C}}{\Gamma \vdash c : A}\,(\mathsf{TCst})$$

$$\frac{\Gamma \vdash M : B \to A \quad \Gamma \vdash N : B}{\Gamma \vdash MN : A}\,(\mathsf{TApp}) \qquad x \notin \mathsf{dom}\;\Gamma\,\frac{\Gamma \cup \{x : B\} \vdash M : A}{\Gamma \vdash \lambda x.\, M : B \to A}\,(\mathsf{TAbs})$$

We say that a closed term M is ***typable*** just if there is some type A such that $\vdash M : A$ is derivable in the type system. If $\vdash M : A$, then M is said to be an ***inhabitant*** of A. The ***pure-term inhabitation problem***, is the problem of, given a type A, determining if there a closed, *pure* term M such that $\vdash M : A$.

Useful Theorems

Theorem (Confluence)**.** *If $M \rhd^* P$ and $M \rhd^* Q$ then there exists a term N such that $P \rhd^* N$ and $Q \rhd^* N$.*

Theorem (Unique Normal Forms)**.** *If $M \rhd^* N$ and $M \rhd^* P$ with N and P normal forms, then $N = P$.*

Theorem (Scott-Curry)**.** *Let $\Phi \subseteq \Lambda$ satisfy both the following properties:*

- $\emptyset \neq \Phi \neq \Lambda$
- *if $M \in \Phi$ and $M \approx N$, then $N \in \Phi$*

Then it follows that Φ is undecidable.

Theorem (Preservation)**.** *If $\Gamma \vdash M : A$ and $M \rhd N$ then $\Gamma \vdash N : A$*

Theorem (Progress)**.** *If $\vdash M : A$ then either M is a value or there is some N such that $M \rhd N$.*