Types and λ-calculus

Problem Sheet 5

* 1. Give a type derivation/proof tree for the judgements:

(a) $\vdash \lambda xy.\, yxx : a \to (a \to a \to b) \to b$

(b) $x : (b \to b) \to b \to b,\ y : \forall c. c \to c \vdash \lambda z. x\ (y(\lambda z'.z'))\ (yz) : b \to b$

(c) $\vdash \lambda xyz.\, y(xz) : (a \to b) \to (b \to c) \to a \to c$

** 2. Give terms $M \in \Lambda$ that satisfy each of the following (you are *not* required to justify them with a proof tree, but you may wish to so as to check your answer):

(a) $\vdash M : (a \to b) \to a \to b$

(b) $x : (a \to a) \to c \vdash M : c$

(c) $x : \forall ab.\, a \to a \to b \vdash M : a$

** 3. Prove that $\lambda xy.\, xy(yx)$ is untypable (not typable).

** 4. **(Optional)** Prove: $\forall A$, if $c \notin \{a_1, \ldots, a_n\}$ and $\{a_1, \ldots, a_n\} \cap \mathsf{FV}(C) = \emptyset$ then

$$A[B_1/a_1, \ldots, B_n/a_n][C/c] = A[C/c][B_1[C/c]/a_1, \ldots, B_n[C/c]/a_n]$$

You will want to consider several cases depending on how variables coincide.

The following is the induction principle for the type system, viewed as a set of triples or three-place relation $- \vdash - : -$.

Suppose $\Phi(\Gamma, M, A)$ is a property of triples comprising a type environment, a term and a monotype. Then if the following can all be proven:

- For all type environments Δ, variables x, type schemes $\forall\overline{a}.\,C$ and sequences of monotypes $\overline{B}$ (of the appropriate length), if $x : \forall\overline{a}.\,C \in \Delta$ then $\Phi(\Delta, x, C[\overline{B}/\overline{a}])$.
- For all type environments Δ, terms P and Q and monotypes B and C, if $\Phi(\Delta, P, B \to C)$ and $\Phi(\Delta, Q, B)$ then $\Phi(\Delta, PQ, C)$.
- For all type environments Δ, terms P, variables x and monotypes B and C, if $x \notin \mathsf{dom}(\Gamma)$ and $\Phi(\Delta \cup \{x : B\}, P, C)$ then $\Phi(\Delta, \lambda x.\,P, B \to C)$.

It follows that $\forall(\Gamma, M, A) \in \vdash.\ \Phi(\Gamma, M, A)$.

** 5. Prove, by induction on $\Gamma \vdash M : A$, that:

If $\Gamma \vdash M : A$ then $\Gamma[C/c] \vdash M : A[C/c]$.

Note: by the variable convention, you may assume $(\forall\overline{a}.\,A)[C/c] = \forall\overline{a}.\,A[C/c]$. Also, you will need to use the result of the previous question.

** 6. **(Optional)** Prove the Subject Reduction Theorem by induction on $M \to_\beta N$:

if $M \to_\beta N$ and $\Gamma \vdash M : A$ then $\Gamma \vdash N : A$.

Three tips:

- You will want to have an induction hypothesis of the form $\forall\Gamma.\ \Phi(M, N)$, which will be useful in the (Abs) case, so set up your goal accordingly.
- You will need to use the substitution lemma from the notes.
- You will need to use the Inversion theorem from the notes.