Types and λ-calculus

Problem Sheet 6

* 1. Give a type derivation/proof tree for the judgements:

(a) $\vdash (\lambda x.\, x)\,\underline{2} : \mathsf{Nat}$

(b) $x : \mathsf{Nat},\, y : \mathsf{Nat} \vdash \mathsf{ifz}\ y\ x\ (\mathsf{pred}\ x) : \mathsf{Nat}$

(c) $\vdash \lambda xy.\, yxx : a \to (a \to a \to b) \to b$

(d) $\vdash \lambda xyz.\, y(xz) : (a \to b) \to (b \to c) \to a \to c$

** 2. Give terms M *in normal form* that satisfy each of the following (you are *not* required to justify them with a proof tree, but you may wish to so as to check your answer):

(a) $\vdash M : (a \to b) \to a \to b$

(b) $x : (a \to a) \to c \vdash M : c$

(c) $\vdash M : a \to b \to \mathsf{Nat}$

** 3. Use inversion to prove that the following terms are not typable:

(a) $\underline{1}\ (\lambda x.\, x)$

(b) $\mathsf{pred}\ (\lambda x.\, x)$

(c) $\lambda xy.\, xy(yx)$

** 4. The following property is called *Weakening*:

For all Γ, Γ' and A: if $\Gamma \vdash M : A$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash M : A$.

We can prove Weakening by induction on M.

Proof. The proof is by induction on M.

- When M is a variable x ... (a)

- When M is a constant c, let A be a type, Γ and Γ' be type environments such that $\Gamma \subseteq \Gamma'$ and suppose $\Gamma \vdash c : A$. By inversion, it follows that $c{:}A \in \mathbb{C}$. Therefore, the side condition is fulfilled to use (TCst) to also justify $\Gamma' \vdash c : A$ (this rule does not place any requirements on the environment).

- When M is an application PQ, assume the induction hypotheses:

  (IH1) For all Γ'' and Γ''' and A', if $\Gamma'' \subseteq \Gamma'''$ and $\Gamma'' \vdash P : A'$ then $\Gamma''' \vdash P : A'$.

  (IH2) For all Γ'' and Γ''' and A', if $\Gamma'' \subseteq \Gamma'''$ and $\Gamma'' \vdash Q : A'$ then $\Gamma''' \vdash Q : A'$.

  Let A be a type, Γ and Γ' be environments such that $\Gamma \subseteq \Gamma'$. Then suppose $\Gamma \vdash PQ : A$. By inversion, there must be a type B such that $\Gamma \vdash P : B \to A$ and $\Gamma \vdash Q : B$. It follows from (IH1) with $\Gamma'' := \Gamma$ and $\Gamma''' := \Gamma'$ and $A' := B \to A$ that $\Gamma' \vdash P : B \to A$. It follows from (IH2) with $\Gamma'' := \Gamma$, $\Gamma''' := \Gamma'$ and $A' := B$ that $\Gamma' \vdash Q : B$. Therefore, by (TApp), $\Gamma' \vdash PQ : A$.

- When M is an abstraction $\lambda x. P$ ... (b)

□

Complete the remaining two cases.

*** 5. Find terms M and N such that:

(i) M is not typable

(ii) N is typable

(iii) $M \rhd N$

** 6.

(a) Prove the following result by induction on M:

> Let B be a type and N a term. For all terms M, types A and environments Γ: if $\Gamma, x{:}B \vdash M : A$ and $\Gamma \vdash N : B$ then $\Gamma \vdash M[N/x] : A$.

(b) Prove the missing case in the proof of Lemma 12.2 from the notes: if $\Gamma \vdash (\lambda x. M)N : A$ then $\Gamma \vdash M[N/x] : A$.