UNIVERSITY OF BRISTOL

January 2019 Examination Period

FACULTY OF ENGINEERING

Third Year Examination for the Degrees
of
Bachelor of Science
Master of Engineering

COMS30009J
Types and Lambda Calculus

TIME ALLOWED:
2 Hours

Answers to COMS30009J: Types and Lambda Calculus

Intended Learning Outcomes:

Q1. (a) For each of the following reduction steps, give the redex that is contracted:

i. $\underline{\mathsf{id}}\ (\mathbf{pred}\ \underline{2}) \ \triangleright\ \underline{\mathsf{id}}\ \underline{1}$

ii. $\underline{\mathsf{id}}\ (\mathbf{pred}\ \underline{2}) \ \triangleright\ (\mathbf{pred}\ \underline{2})$

iii. $\lambda f x. (\mathsf{S}\ \mathsf{S}\ (\underline{\mathsf{id}}\ x)) \ \triangleright\ \lambda f x. (\mathsf{S}\ \mathsf{S}\ x)$

Solution:

i. $\mathbf{pred}\underline{2}$

ii. $\underline{\mathsf{id}}\ (\mathbf{pred}\ \underline{2})$

iii. $\underline{\mathsf{id}}\ x$

[3 marks]

(b) For each of the following state whether it is true or false (no justification is necessary).

i. $M = N$ implies $M\ \triangleright^*\ N$

ii. $M\ \triangleright\ N$ implies $M\ \triangleright^*\ N$

iii. $M\ \approx\ N$ implies $M\ \triangleright^*\ N$

iv. $M\ \triangleright^*\ N$ implies $M\ \approx\ N$

[4 marks]

Solution:

i. true

ii. true

iii. false

iv. true

(c) For each of the following, give an example of a *closed* term M with that property.

i. M is in normal form.

ii. M is normalising but *not* strongly normalising.

iii. $M\ \triangleright\ M$

iv. $M\ \triangleright^*\ MM$

[4 marks]

Solution:

i. $\underline{\mathsf{id}}$

ii. $\underline{\mathsf{const}}\ \underline{\mathsf{id}}\ \underline{\mathsf{div}}$

iii. $(\lambda x. xx)(\lambda x. xx)$

iv. $\mathsf{fix}\,(\lambda x. xx)$

(d) Prove $N \ \rhd^* \ N'$ implies $M[N/x] \ \rhd^* \ M[N'/x]$ by induction on M.

[6 marks]

Solution: The proof is by induction on M.

- When M is a variable y, assume $N \ \rhd^* \ N'$. Then we distinguish two possible cases:
  - If $x = y$, then, by definition of substitution, $M[N/x] = N$ and $M[N'/x] = N'$ and the goal is therefore $N \ \rhd^* \ N'$ which is just one of our assumptions.
  - If $x \neq y$, then, by definition of substitution, $M[N/x] = y = M[N'/y]$ and the goal follows by reflexivity of $\rhd^*$.
- When M is a constant c, assume $N \ \rhd^* \ N'$. By the definition of substitution, $M[N/x] = c = M[N'/x]$, and so the goal follows by reflexivity of $\rhd^*$.
- When M is an application PQ, we assume the induction hypotheses:

  (IH1) $N \ \rhd^* \ N'$ implies $P[N/x] \ \rhd^* \ P[N'/x]$

  (IH2) $N \ \rhd^* \ N'$ implies $Q[N/x] \ \rhd^* \ Q[N'/x]$

  Assume $N \ \rhd^* \ N'$ (hence, we already are able to use the two IH). By definition of substitution, $(PQ)[N/x] = (P[N/x])(Q[N/x])$ and $(PQ)[N'/x] = (P[N'/x])(Q[N'/x])$. Hence, the goal can be written:

  $$(P[N/x])(Q[N/x]) \ \rhd^* \ (P[N'/x])(Q[N'/x])$$

  By (IH1) and the compatibility of reduction, $(P[N/x])(Q[N/x]) \ \rhd^* \ (P[N'/x])(Q[N/x])$ and by (IH2), $(P[N'/x])(Q[N/x]) \ \rhd^* \ (P[N'/x])(Q[N'/x])$, as required.
- When M is an abstraction $\lambda y.\, P$, we may assume, by the variable convention, that y does not occur outside of P. We assume the induction hypothesis:

  (IH) $N \ \rhd^* \ N'$ implies $P[N/x] \ \rhd^* \ P[N'/x]$

  Suppose $N \ \rhd^* \ N'$. Our goal is to show that $(\lambda y.\, P)[N/x] \ \rhd^* \ (\lambda y.\, P)[N'/x]$. By the definition of substitution (taking into account our assumption about the bound variable name y), $(\lambda y.\, P)[N/x] = \lambda y.\, P[N/x]$ and $(\lambda y.\, P)[N'/x] = \lambda y.\, P[N'/x]$. By the compatibility of reduction and the induction hypothesis $\lambda y.\, P[N/x] \ \rhd^* \ \lambda y.\, P[N'/x]$. Hence, we have proven the goal.

(e) Prove that there cannot be a term M with the property that:

$$M\,(\lambda z.\, z\,(\underline{\mathsf{const}}\,\underline{\mathsf{id}}\,\underline{\mathsf{div}})\,\underline{\mathsf{div}}) \ \approx \ \underline{0} \qquad \text{and} \qquad M\,(\lambda z.\, z\,\underline{\mathsf{id}}\,(\underline{\mathsf{const}}\,\underline{\mathsf{div}}\,\underline{\mathsf{id}})) \ \approx \ \underline{1}$$

[3 marks]

(cont.)

Solution: Suppose for the purposes of obtaining a contradiction that such a term M exists. We have:

$$\lambda z.\, z\,(\underline{\mathsf{const}}\,\underline{\mathsf{id}}\,\underline{\mathsf{div}})\,\underline{\mathsf{div}} \;\approx\; \lambda z.\, z\,\underline{\mathsf{id}}\,(\underline{\mathsf{const}}\,\underline{\mathsf{div}}\,\underline{\mathsf{id}})$$

since both reduce to a common term $\lambda z.\, z\,\underline{\mathsf{id}}\,\underline{\mathsf{div}}$. Call the first of these P and the second Q for short. Then it follows that $\underline{0} \approx M\,P \approx M\,Q \approx \underline{1}$. However, it follows from the Church-Rosser theorem that $\underline{0} \not\approx \underline{1}$.

(f) Let M be a *pure* term. Suppose that the equation $MN \approx NMN$ is true for all terms N. Prove that M cannot have a normal form, i.e. if $M \rhd^* P$ then P is not in normal form.

[5 marks]

Solution: Suppose for contradiction that M satisfies this equation and yet has a normal form P. Then, one instance of the equation is $Mx \approx xMx$. Since $M \rhd^* P$, also $Px \approx xPx$. The term xPx is a β-normal form so, by confluence, it must be that $Px \rhd^* xPx$ (*). We distinguish two cases for P, either P is an abstraction $\lambda y.\, Q$ or it is not. In the first case, $Px \rhd Q[x/y]$ and the latter term must be a normal form. However, $Q[x/y] \neq x(\lambda y.\, Q)x$ because $Q[x/y]$ and Q are strings of the same length. In the second case, Px is already a normal form and, again $Px \neq xPx$. Therefore, it cannot be that $Px \rhd^* xPx$, contradicting (*).

Q2. (a) State the rules of the type system (the rule names are not important).

[3 marks]

Solution:

$$x : \forall \overline{a}.\ A \in \Gamma \ \frac{}{\Gamma \vdash x : A[\overline{B}/\overline{a}]}\ (\mathrm{TVar})$$

$$\frac{\Gamma \vdash M : B \to A \quad \Gamma \vdash N : B}{\Gamma \vdash MN : A}\ (\mathrm{TApp})$$

$$x \notin \mathsf{dom}\ \Gamma\ \frac{\Gamma \cup \{x : B\} \vdash M : A}{\Gamma \vdash \lambda x.\, M : B \to A}\ (\mathrm{TAbs})$$

(b) Give an example of a *closed* term *in normal form* that is not typable.

[1 mark]

Solution: $\lambda x.\, xx$

(c) For each of the following terms M, give a type environment Γ and a type A such that $\Gamma \vdash M : A$ (you need not prove it).

i. $(\lambda x.\, yxz)(\lambda z.\, z)$

ii. $(\lambda xy.\, yx)\, x\, z$

[3 marks]

Solution:

i. $y : (a \to a) \to b \to c,\ z : b \vdash (\lambda x.\, yxz)(\lambda z.\, z) : c$

ii. $x : a,\ z : a \to b \vdash (\lambda xy.\, yx)\, x\, z : b$

(d) Prove the following by induction on M.
For all Γ, B and C: if $\Gamma, x : B \vdash M : C$ and $\Gamma \vdash N : B$ then $\Gamma \vdash M[N/x] : C$

[7 marks]

Solution: The proof is by induction on M.

- In case (Var), M is a variable y. Let Γ be an environment, BA and C be types. Assume $\Gamma, x : B \vdash y : C$ and $\Gamma \vdash N : B$. There are two subcases:
  - If $x = y$ then, by Inversion, $B = C$. By definition, $y[N/x] = N$ and it follows from the second assumption that $\Gamma \vdash N : B$.
  - If $x \neq y$ then, $y[N/x] = y$. It follows from the first assumption, by inversion, that $y : C \in \Gamma$. Therefore, by (Var), $\Gamma \vdash y : C$.
- In case (App), M is an application PQ. Let Γ, B and C be arbitrary and assume $\Gamma, x : B \vdash PQ : C$ and $\Gamma \vdash N : B$. Assume the induction hypotheses:

(IH1) forall Γ', B', C': if $\Gamma', x : B' \vdash P : C'$ and $\Gamma' \vdash N : B'$ then $\Gamma' \vdash P[N/x] : C'$

(IH2) forall Γ', B', C': if $\Gamma', x : B' \vdash Q : C'$ and $\Gamma' \vdash N : B'$ then $\Gamma' \vdash Q[N/x] : C'$

By definition $(PQ)[N/x] = P[N/x][Q/x]$. By inversion on the first assumption, there is a type D such that $\Gamma, x : B \vdash P : D \to C$ and $\Gamma, x : B \vdash Q : D$. Therefore, by (IH1) and the second assumption, $\Gamma \vdash P[N/x] : D \to C$. By (IH2) and the second assumption, $\Gamma \vdash Q[N/x] : D$. Therefore, by (App), $\Gamma \vdash P[N/x]Q[N/x] : C$, and $P[N/x]Q[N/x] = (PQ)[N/x]$ by definition.

- In case (Abs), M is an abstraction $\lambda y.\, P$ and C is an arrow $D \to E$. We can assume by the variable convention that $x \neq y$ and $y \notin \mathsf{FV}(Q)$ and $y \notin \mathbf{dom}(\Gamma)$. Assume the induction hypothesis:

  - forall Γ', B', C': if $\Gamma, x : B' \vdash P : C'$ and $\Gamma \vdash N : C'$ then $\Gamma \vdash P[N/x] : C'$.

  Let Γ, B and C be arbitrary. Assume $\Gamma, x : B \vdash \lambda y.\, P : D \to E$ and $\Gamma \vdash N : B$. It follows by inversion from the first assumption that $\Gamma, x : B, y : D \vdash P : E$. Therefore, it follows from the induction hypothesis, with $\Gamma' := \Gamma \cup \{y : D\}$ and $B' = B$ and $C' = E$, that $\Gamma, y : D \vdash P[N/x] : E$. Therefore, it follows from (Abs) that $\Gamma \vdash \lambda y.\, P[N/x] : D \to E$. By the assumptions on y and definition, $\lambda y.\, P[N/x] = (\lambda y.\, P)[N/x]$.

(e) Prove that $a \to (a \to b) \to b$ is the principal type of $\lambda xy.\, yx$, i.e. that:

- $\vdash \lambda xy.\, yx : a \to (a \to b) \to b$
- and, every type C such that $\vdash \lambda xy.\, yx : C$ has shape $A \to (A \to B) \to B$ for some types A and B.

[5 marks]

Solution: First, observe that $a \to (a \to b) \to b$ is a type of $\lambda xy.\, yx$ because:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : a, y : a \to b \vdash y : a \to b} \quad \dfrac{}{x : a, y : a \to b \vdash x : a}}{x : a, y : a \to b \vdash yx : b}}{x : a \vdash \lambda y.\, yx : (a \to b) \to b}}{\vdash \lambda xy.\, yx : a \to (a \to b) \to b}$$

Next, suppose that A is another type of $\lambda xy.\, yx$. By Inversion, A must have shape $B \to C$ with $x : B \vdash \lambda y.\, yx : C$. By inversion on this judgement, C must have shape $D \to E$ with $x : B, y : D \vdash yx : E$. By inversion on this judgment, there is a type F such that $x : B, y : D \vdash y : F \to E$ and $x : B, y : D \vdash x : F$. By inversion on these final two judgements, we have $D = F \to E$ and $B = F$. Therefore,

$\vdash \lambda xy.\, yx : F \to (F \to E) \to E$. We have $(a \to (a \to b) \to b)[F/a, E/b] = F \to (F \to E) \to E$, as required.

(f) Suppose $M \approx \lambda x.\, xx$. Prove that M is *not* typable.

[3 marks]

Solution: Suppose for the purpose of obtaining a contradiction that M is typable, i.e. there is a type A such that $\vdash M : A$. Observe that, since $\lambda x.\, xx$ is a normal form, it follows from the definition of $\approx$ that $M \rhd^* \lambda x.\, xx$. By Subject-Reduction, it follows that $\vdash \lambda x.\, xx : A$. However, we know that $\lambda x.\, xx$ is not typable.

(g) Give two terms M and N and a type A such that $M \rhd N$ and, additionally, both of the following are true:

- There are no proof trees for $\vdash M : A$
- There are infinitely many proof trees for $\vdash N : A$

[3 marks]

Solution: Take $N = \underline{\mathsf{const}}\,\underline{\mathsf{id}}\,\underline{\mathsf{id}}$ and $M = \underline{\mathsf{const}}\,N\,\underline{\mathsf{div}}$ and $A = a \to a$. Then, clearly $M \rhd N$. M is untypable because it contains $\underline{\mathsf{div}}$ as a subterm. On the other hand, there are infinitely many proof trees for $\vdash \underline{\mathsf{const}}\,\underline{\mathsf{id}}\,\underline{\mathsf{id}} : a \to a$ because the following is a proof tree for all types B:

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \underline{\mathsf{const}} : (a \to a) \to (B \to B) \to a \to a} \quad \dfrac{\vdots}{\vdash \underline{\mathsf{id}} : a \to a}}{\vdash \underline{\mathsf{const}}\,\underline{\mathsf{id}} : (B \to B) \to a \to a} \quad \dfrac{\vdots}{\vdash \underline{\mathsf{id}} : B \to B}}{\vdash \underline{\mathsf{const}}\,\underline{\mathsf{id}}\,\underline{\mathsf{id}} : a \to a}$$

END OF PAPER