TYPES AND λ-CALCULUS

Problem Sheet 4

* 1. Put in all the implicit parentheses required by the official syntax of types for the following examples:

(a) $a \to b \to a$

(b) $\forall bc.\, (b \to c) \to c$

(c) $\forall ab.\, (c \to c \to d) \to a \to b$

(d) $\forall a.\, (a \to b) \to a \to b$

Solution

(a) $(a \to (b \to a))$

(b) $\forall bc.\, ((b \to c) \to c)$

(c) $\forall ab.\, ((c \to (c \to d)) \to (a \to b))$

(d) $\forall a.\, ((a \to b) \to (a \to b))$

The following is the induction principle for the set $\mathbb{T}$ of monotypes.

> Suppose Φ is a property of monotypes. Then if the following can both be proven:
>
> - For all type variables a, $\Phi(a)$.
> - For all monotypes B and C, if $\Phi(B)$ and $\Phi(C)$ then $\Phi(B \to C)$.
>
> It follows that $\forall A \in \mathbb{T}.\, \Phi(A)$.

** 2. Prove, by induction on A, that $A(\sigma_1\sigma_2) = (A\sigma_1)\sigma_2$.

Solution

The proof is by induction on A:

(TyVar) In this case, A is a type variable a. The goal is to show $a(\sigma_1\sigma_2) = (a\sigma_1)\sigma_2$. By definition $a(\sigma_1\sigma_2) = (\sigma_1(a))\sigma_2$. Also by definition, $\sigma_1(a) = a\sigma_1$, so we have $(\sigma_1(a))\sigma_2 = (a\sigma_1)\sigma_2$ as required.

(Arrow) In this case, A is an arrow type $B \to C$. Assume the induction hypotheses:

(IH1) $B(\sigma_1\sigma_2) = (B\sigma_1)\sigma_2$

(IH2) $C(\sigma_1\sigma_2) = (C\sigma_1)\sigma_2$

Then the goal is to show $(B \to C)(\sigma_1\sigma_2) = ((B \to C)\sigma_1)\sigma_2$. We reason equationally:

$$\begin{aligned}
&(B \to C)(\sigma_1\sigma_2) && (1)\\
&= B(\sigma_1\sigma_2) \to C(\sigma_1\sigma_2) && (2)\\
&= (B\sigma_1)\sigma_2 \to (C\sigma_1)\sigma_2 && (3)\\
&= (B\sigma_1 \to C\sigma_1)\sigma_2 && (4)\\
&= ((B \to C)\sigma_1)\sigma_2 && (5)
\end{aligned}$$

Where lines (2), (4) and (5) follow from the definition of type substitution application and line (3) follows from the two induction hypotheses.

** 3. Prove, by induction on M, that:

if $x \in \mathsf{FV}(M)$ then $\mathsf{FV}(M[N/x]) = (\mathsf{FV}(M) \setminus \{x\}) \cup \mathsf{FV}(N)$.

Hint: You will want to use Lemma 6.1 of the notes.
Hint: In the application case, consider splitting on whether x is free in the operator only, the operand only, or both.

Solution

By induction on M.

(Var) In this case, M is a variable y. Assume $x \in \mathsf{FV}(M)$. The goal is to show $\mathsf{FV}(y[N/x]) = (\mathsf{FV}(y) \setminus \{x\}) \cup \mathsf{FV}(N)$. By the assumption, it must be that $x = y$ so, by definition, the left-hand side of the goal is $\mathsf{FV}(N)$ and the right-hand side is $\emptyset \cup \mathsf{FV}(N)$. So they are equal.

(App) In this case, M is an application PQ. Assume the induction hypotheses:

(IH1) if $x \in \mathsf{FV}(P)$ then $\mathsf{FV}(P[N/x]) = (\mathsf{FV}(P) \setminus \{x\}) \cup \mathsf{FV}(N)$

(IH2) if $x \in \mathsf{FV}(Q)$ then $\mathsf{FV}(Q[N/x]) = (\mathsf{FV}(Q) \setminus \{x\}) \cup \mathsf{FV}(N)$

The goal is to show: if $xin\mathsf{FV}(PQ)$ then $\mathsf{FV}((PQ)[N/x]) = (\mathsf{FV}(PQ) \setminus \{x\}) \cup \mathsf{FV}(N)$, so assume $x \in \mathsf{FV}(PQ)$. By definition of substitution and FV, $\mathsf{FV}((PQ)[N/x]) = \mathsf{FV}(P[N/x]) \cup \mathsf{FV}(Q[N/x])$ and by definition of FV, $\mathsf{FV}(PQ) = \mathsf{FV}(P) \cup \mathsf{FV}(Q)$ (*). Therefore, the goal is to prove:

$$\mathsf{FV}(P[N/x]) \cup \mathsf{FV}(Q[N/x]) = ((\mathsf{FV}(P) \cup \mathsf{FV}(Q)) \setminus \{x\}) \cup \mathsf{FV}(N)$$

Now, from (*), we know that either $x \in \mathsf{FV}(P)$ or $x \in \mathsf{FV}(Q)$. We split by three cases:

- If $x \in \mathsf{FV}(P)$ and $x \in \mathsf{FV}(Q)$, then (IH1) and (IH2) are both applicable, and we obtain $\mathsf{FV}(P[N/x]) = (\mathsf{FV}(P) \setminus \{x\}) \cup \mathsf{FV}(N)$ and $\mathsf{FV}(Q[N/x]) = (\mathsf{FV}(Q) \setminus \{x\}) \cup \mathsf{FV}(N)$. We note that, by simple set theory, $(\mathsf{FV}(P) \cup \mathsf{FV}(Q)) \setminus \{x\}$ in the goal is the same thing as $(\mathsf{FV}(P) \setminus \{x\}) \cup (\mathsf{FV}(Q) \setminus \{x\})$. So, in summary, we have:

$$\begin{array}{ll} & \mathsf{FV}(P[N/x]) \cup \mathsf{FV}(Q[N/x]) \\ = & ((\mathsf{FV}(P) \setminus \{x\}) \cup \mathsf{FV}(N)) \cup ((\mathsf{FV}(Q) \setminus \{x\}) \cup \mathsf{FV}(N)) \\ = & (\mathsf{FV}(P) \setminus \{x\}) \cup (\mathsf{FV}(Q) \setminus \{x\}) \cup \mathsf{FV}(N) \\ = & (\mathsf{FV}(P) \cup \mathsf{FV}(Q)) \setminus \{x\} \end{array}$$

  as required, with the penultimate line following by simple set theory.

- If $x \in \mathsf{FV}(P)$ but $x \notin \mathsf{FV}(Q)$, then (IH1) is applicable and we obtain $\mathsf{FV}(P[N/x]) = (\mathsf{FV}(P) \setminus \{x\}) \cup \mathsf{FV}(N)$. Since $x \notin \mathsf{FV}(Q)$, by Lemma 6.1, $\mathsf{FV}(Q[N/x]) = \mathsf{FV}(Q)$. Thus we have:

$$\begin{array}{ll} & \mathsf{FV}(P[N/x]) \cup \mathsf{FV}(Q[N/x]) \\ = & (\mathsf{FV}(P) \setminus \{x\}) \cup \mathsf{FV}(N) \cup \mathsf{FV}(Q) \\ = & ((\mathsf{FV}(P) \cup \mathsf{FV}(Q)) \setminus \{x\}) \cup \mathsf{FV}(N) \end{array}$$

  following by simple set theory.

- Otherwise $x \notin \mathsf{FV}(P)$ but $x \in \mathsf{FV}(Q)$, and the proof is analogous to the above.

(Abs) In this case, M is an abstraction $\lambda y.\,P$ and we may assume by the variable convention that y does not occur anywhere else (and in particular is not x and does not occur in N). Assume the induction hypothesis:

(IH) if $x \in \mathsf{FV}(P)$ then $\mathsf{FV}(P[N/x]) = (\mathsf{FV}(P) \setminus \{x\}) \cup \mathsf{FV}(N)$

Then our goal is to show: if $x \in \mathsf{FV}(\lambda y.\,P)$ then $\mathsf{FV}(((\lambda y.\,P)[N/x]) = (\mathsf{FV}(\lambda y.\,P) \setminus \{x\}) \cup \mathsf{FV}(N)$, so assume $x \in \mathsf{FV}(\lambda y.\,P)$. By definition, we therefore have $x \in \mathsf{FV}(P)$. Hence, the induction hypothesis is applicable and we obtain $\mathsf{FV}(P[N/x]) = (\mathsf{FV}(P) \setminus \{x\}) \cup \mathsf{FV}(N)$ (*). Since $x \neq y$ and y does not occur in N, by definition, the left-hand side of the goal is $\mathsf{FV}(\lambda y.\, P[N/x])$ and, by definition again, this is just $\mathsf{FV}(P[N/x]) \setminus \{y\}$. From (*), this is the same as $((\mathsf{FV}(P) \setminus \{x\}) \cup \mathsf{FV}(N)) \setminus \{y\}$. By simple set theory and since $y \notin \mathsf{FV}(N)$, we can rearrange this expression as $((\mathsf{FV}(P) \setminus \{y\}) \setminus \{x\}) \cup \mathsf{FV}(N)$. By definition, this is just $(\mathsf{FV}(\lambda y.\, P) \setminus \{x\}) \cup \mathsf{FV}(N)$ as required.

The following is the induction principle for the set $\to_\beta$ of pairs of terms (a binary relation on terms).

Suppose $\Phi(M,N)$ is a property of pairs of terms. Then if the following can all be proven:

- For all terms P and Q, $\Phi((\lambda x.\, P)Q, P[Q/x])$.
- For all terms P, Q and Q', if $\Phi(Q,Q')$ then $\Phi(PQ, PQ')$.
- For all terms P, Q and P', if $\Phi(P,P')$ then $\Phi(PQ, P'Q)$.
- For all terms P and P', if $\Phi(P,P')$ then $\Phi(\lambda x.\, P, \lambda x.\, P')$.

It follows that $\forall (M,N) \in \to_\beta .\ \Phi(M,N)$.

** 4. Prove, by induction on $M \to_\beta N$, that: $M \to_\beta N$ implies $\mathsf{FV}(N) \subseteq \mathsf{FV}(M)$.

Solution

The proof is by induction on $M \to_\beta N$, so we are proving $\Phi(M,N)$ the statement: $\mathsf{FV}(N) \subseteq \mathsf{FV}(M)$.

(Redex) In this case, M is of shape $(\lambda x.\, P)Q$ and N is of shape $P[Q/x]$. The goal is to prove $\mathsf{FV}(P[Q/x]) \subseteq \mathsf{FV}((\lambda x.\, P)Q)$. We can consider two sub-cases depending on whether or not x occurs in P:

- If $x \notin \mathsf{FV}(P)$ then the left-hand side $P[Q/x]$ is just P and hence we know that, by Lemma 6.1 of the notes, $\mathsf{FV}(P[Q/x]) = \mathsf{FV}(P) = \mathsf{FV}(P) \setminus \{x\}$. The right hand side is, by definition, $(\mathsf{FV}(P) \setminus \{x\}) \cup \mathsf{FV}(Q)$. Clearly this contains the left-hand side since it adds $\mathsf{FV}(Q)$.
- If $x \in \mathsf{FV}(P)$ then, by the previous question, the left hand side $\mathsf{FV}(P[Q/x])$ is exactly $(\mathsf{FV}(P) \setminus \{x\}) \cup \mathsf{FV}(Q)$. However, this is exactly the definition of $\mathsf{FV}((\lambda x.\, P)Q)$, the right-hand side.

(AppL) In this case, M is of shape PQ and N is of shape $P'Q$. We assume the induction hypothesis:

(IH) $\mathsf{FV}(P') \subseteq \mathsf{FV}(P)$

Then our goal is to prove $\mathsf{FV}(P'Q) \subseteq \mathsf{FV}(PQ)$. By definition $\mathsf{FV}(P'Q) = \mathsf{FV}(P') \cup \mathsf{FV}(Q)$ and by the induction hypothesis and basic set theory, this is contained in $\mathsf{FV}(P) \cup \mathsf{FV}(Q)$. But, this latter expression is exactly the definition of $\mathsf{FV}(PQ)$, so we have $\mathsf{FV}(P'Q) \subseteq \mathsf{FV}(PQ)$, as required.

(AppR) This case is analogous to the above, but with the components of the application swapped.

(Abs) In this case, M is of shape $\lambda x. P$ and N is of shape $\lambda x. P'$. We assume the induction hypothesis:

(IH) $\mathsf{FV}(P') \subseteq \mathsf{FV}(P)$

Then our goal is to prove $\mathsf{FV}(\lambda x. P') \subseteq \mathsf{FV}(\lambda x. P)$. By definition of FV, this amounts to proving:

$$\mathsf{FV}(P') \setminus \{x\} \subseteq \mathsf{FV}(P) \setminus \{x\}$$

Then this follows immediately from the induction hypothesis and simple set theory.