TYPES AND λ-CALCULUS

Problem Sheet 2

Questions 1, 2 6 and 7 will be marked.

* 1. Write these terms using the minimum number of parentheses and λ, according to our conventions.

(a) $(\lambda y.((y\,y)(z\,z)))$
(b) $(\lambda y.(((y\,y)\,y)\,y))$
(c) $((\mathsf{S}\,\mathsf{Z})(\lambda y.(\lambda z.(z\,(\mathsf{S}\,y)))))$

Solution

(a) $\lambda y.\,y\,y(zz)$
(b) $\lambda y.\,y\,y\,y\,y$
(c) One answer is: $\mathsf{S}\,\mathsf{Z}\,\lambda yz.z\,(\mathsf{S}\,y)$. However, in practice it is common to write this term as $\mathsf{S}\,\mathsf{Z}\,(\lambda yz.z\,(\mathsf{S}\,y))$ because most find this easier to read. I will always write it the second way, i.e. I will always parenthesize λ-abstractions when they occur inside a term.

* 2. Write the term $(\lambda xyz.\,xy(xz))(\lambda xy.\,x)$ with all the parentheses and λ, that we will usually omit, tediously put back in.

Solution

$((\lambda x.(\lambda y.(\lambda z.((xy)(xz)))))(\lambda x.(\lambda y.\,x)))$

* 3. List the all the subterms of the following terms (don't bother listing the same subterm more than once even it occurs several times):

(a) The 3 distinct subterms of $\lambda x.\, xx$
(b) The 6 distinct subterms of $(\lambda x.\, xx)(\lambda y.\, y)$
(c) The 8 distinct subterms of $\lambda xyz.\, xy(yx)$
(d) The 13 distinct subterms of fix $(\lambda xy.\, \mathsf{ifz}\ y\ y\ (x\ (\mathsf{pred}\ y)))$

Solution ———————————————————————————

(a) x, xx, $\lambda x.\, xx$
(b) x, xx, $\lambda x.\, xx$, y, $\lambda y.\, y$, $(\lambda x.\, xx)(\lambda y.\, y)$
(c) x, y, xy, yx, $xy(yx)$, $\lambda xyz.(xy(yx))$, $\lambda yz.(xy(yx))$, $\lambda z.(xy(yx))$
(d)
- x
- y
- pred
- pred y
- x (pred y)
- ifz
- ifz y
- ifz y y
- ifz y y (x (pred y))
- $\lambda y.$ ifz y y (x (pred y))
- $\lambda xy.$ ifz y y (x (pred y))
- fix ($\lambda xy.$ ifz y y (x (pred y)))

* 4. Each of the following has two free variables, what are they in each case?
(a) $\lambda xy.\, \lambda u.\, uvxyz$
(b) $\lambda xy.\, z(\lambda u.\, uvxy)$
(c) $\lambda wx.\, z(\lambda u.\, uvwx)$
(d) $\lambda vw.\, z(\lambda z.\, uvvw)$
(e) $\lambda yx.\, z(\lambda u.\, uwyx)$

Solution ———————————————————————————

(a) v, z
(b) z, v

(c) z, v
(d) z, u
(e) z, w

* 5. Which of the following pairs of strings are α-equivalent (and therefore represent the same term):

(a) $\lambda x.xy$ and $\lambda z.zy$
(b) $\lambda x.xy$ and $\lambda z.zx$
(c) $\mathsf{ifz}\ x\ (\mathsf{S}\ x)\ (\mathsf{pred}\ x)$ and $\mathsf{ifz}\ y\ (\mathsf{S}\ y)\ (\mathsf{pred}\ y)$
(d) $\lambda xy.xy$ and $\lambda xy.yx$
(e) $\mathsf{fix}\ (\lambda x.(\lambda y.xy)\ (\mathsf{S}\ x))$ and $\mathsf{fix}\ (\lambda y.(\lambda x.yx)\ (\mathsf{S}\ y))$

Solution

(a) Yes
(b) No
(c) No
(d) No
(e) Yes

* 6. Perform the following substitutions:

(a) $(\mathsf{ifz}\ x\ (\mathsf{S}\ x)\ \mathsf{Z})[\underline{2}/x]$
(b) $\underline{2}[\underline{1}/x]$
(c) $(\lambda x.(\lambda y.xz)z)[(\lambda z.z)/z]$
(d) $(\lambda x.yx)[yz/x]$
(e) $(\lambda x.yz)[yy/z]$
(f) $(\lambda y.xy)[yx/x]$

Solution

(a) $\mathsf{ifz}\ \underline{2}\ (\mathsf{S}\ \underline{2})\ \mathsf{Z}$
(b) $\underline{2}$
(c) $\lambda x.(\lambda y.x(\lambda z.z))(\lambda z.z)$
(d) $\lambda x.yx$

(e) $\lambda x. y\ (y\ y)$
(f) $\lambda z. yxz$

** 7. Complete the following partial proof of the following statement, by filling out (a), (b) (c) and (d). This proof is a particularly good test of how carefully you are keeping track of the proof state.

Suppose P and Q are terms, x and y are variables. If $x \neq y$ and $x \notin \mathsf{FV}(Q)$ then for all terms M:

$$M[P/x][Q/y] = M[Q/y][P[Q/y]/x]$$

(which tells us how to reorder substitutions in general).

Proof. Suppose P and Q are terms. Suppose x and y are variables. The rest of the proof is by induction on $M \in \Lambda$.

- In case M is some variable z, we argue as follows. Assume $x \neq y$ and $x \neq \mathsf{FV}(Q)$. Our goal is to show:

  $$z[P/x][Q/y] = z[Q/y][P[Q/y]/x]$$

  Now, either $z = x$, $z = y$ or z is neither x nor y. We proceed by a case analysis on this fact:

  - Suppose $z = x$. By our assumption, it follows that $z \neq y$. Then, by definition of substitution, $z[P/x][Q/y] = P[Q/y]$ and

    $$z[Q/y][P[Q/y]/x] = z[P[Q/y]/x] = P[Q/y]$$

    Hence, the desired equality follows.
  - Suppose $z = y$. (a)
  - Suppose $z \neq x$ and $z \neq y$. Then $z[P/x][Q/y] = z$ on the left side of the goal and also $z[Q/y][P[Q/y]/x] = z$ on the right side, so the result follows.

- (b)

- In case M is some application $N_1 N_2$ we argue as follows. Assume $x \neq y$ and $x \neq \mathsf{FV}(Q)$. Additionally, assume the induction hypothesis:

  (IH1) if $x \neq y$ and $x \neq \mathsf{FV}(Q)$ then $N_1[P/x][Q/y] = N_1[Q/y][P[Q/y]/x]$

  (IH2) if $x \neq y$ and $x \neq \mathsf{FV}(Q)$ then $N_2[P/x][Q/y] = N_2[Q/y][P[Q/y]/x]$

Our goal is to show that:

$$(N_1N_2)[P/x][Q/y] = (N_1N_2)[Q/y][P[Q/y]/x]$$

(c)

- In case M is of shape $\lambda z.N$, we argue as follows. We can assume by the bound variable convention that z is different from any variable in scope, so $z \neq x$ and $z \neq y$ and $z \notin \mathsf{FV}(P)$ and $z \notin \mathsf{FV}(Q)$. (d)

$\square$

Solution

(a) By our assumption, it follows that $z \neq x$. Then, by definition of substitution $z[P/x][Q/y] = z[Q/y] = Q$ and also $z[Q/y][P[Q/y]/x] = Q[P[Q/y]/x]$.

(b) In case M is a constant c, we argue as follows. Assume $x \neq y$ and $x \neq \mathsf{FV}(Q)$. Our goal is to show:

$$c[P/x][Q/y] = c[Q/y][P[Q/y]/x]$$

but, by definition, both left and right hand sides are just c, so the result follows immediately. We assumed $x \neq \mathsf{FV}(Q)$, so $Q[P[Q/y]/x] = Q$. The result follows.

(c) It follows by the definition of substitution that:

$$(N_1N_2)[P/x][Q/y] = N_1[P/x][Q/y]N_2[P/x][Q/y]$$

It follows from (IH1) and (IH2) that:

$$N_1[P/x][Q/y]N_2[P/x][Q/y] = N_1[Q/y][P[Q/y]/x]N_2[Q/y][P[Q/y]/x]$$

and this latter term is $(N_1N_2)[Q/y][P[Q/y]/x]$ by definition of substitution.

(d) Assume $x \neq y$ and $x \neq \mathsf{FV}(Q)$. Additionally assume the induction hypothesis:

(IH) If $x \neq y$ and $x \neq \mathsf{FV}(Q)$ then $N[P/x][Q/y] = N[Q/y][P[Q/y]/x]$.

Our goal is to show that:

$$(\lambda z.N)[P/x][Q/y] = (\lambda z.N)[Q/y][P[Q/y]/x]$$

Since $z \notin \mathsf{FV}(xyPQ)$, it follows by definition that:

$$(\lambda z.N)[P/x][Q/y] = \lambda z.N[P/x][Q/y]$$

It follows from (IH) that $\lambda z.N[P/x][Q/y] = \lambda z.N[Q/y][P[Q/y]/x]$. Since $z \notin \mathsf{FV}(xyPQ)$, it follows by definition that this latter term is $(\lambda z.N)[Q/y][P[Q/y]]$.