Problem Sheet 5

* 1. Give a type derivation/proof tree for the judgements:

(a) $\vdash \lambda xy.\, yxx : a \to (a \to a \to b) \to b$

(b) $x : (b \to b) \to b \to b,\ y : \forall c.\, c \to c \vdash \lambda z.\, x\ (y(\lambda z'.z'))\ (yz) : b \to b$

(c) $\vdash \lambda xyz.\, y(xz) : (a \to b) \to (b \to c) \to a \to c$

Solution

(a) With $\Gamma = \{x : a,\ y : a \to a \to b\}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash y : a \to a \to b}\,(\mathsf{TVar}) \qquad \dfrac{}{\Gamma \vdash x : a}\,(\mathsf{TVar})}{\Gamma \vdash yx : a \to b}\,(\mathsf{TApp}) \qquad \dfrac{}{\Gamma \vdash x : a}\,(\mathsf{TVar})}{\Gamma \vdash yxx : b}\,(\mathsf{TApp})}{x : a \vdash \lambda y.\, yxx : (a \to a \to b) \to b}\,(\mathsf{TAbs})}{\vdash \lambda xy.\, yxx : a \to (a \to a \to b) \to b}\,(\mathsf{TAbs})$$

(b) With $\Gamma' = \{x : (b \to b) \to b \to b,\ y : \forall c.\, c \to c,\ z : b\}$:

$$\dfrac{\dfrac{D \qquad \dfrac{\dfrac{}{\Gamma' \vdash y : b \to b} \qquad \dfrac{}{\Gamma' \vdash z : b}}{\Gamma' \vdash y\ z : b}}{\Gamma' \vdash x\ (y\ (\lambda z'.z'))\ (y\ z) : b}}{x : (b \to b) \to b \to b,\ y : \forall c.\, c \to c \vdash \lambda z.\, x\ (y\ (\lambda z'.z'))\ (y\ z) : a \to b}\,(\mathsf{TAbs})$$

and D is this subderivation:

$$\dfrac{\dfrac{}{\Gamma' \vdash x : (b \to b) \to b \to b} \qquad \dfrac{\dfrac{}{\Gamma' \vdash y : (b \to b) \to b \to b} \qquad \dfrac{\dfrac{}{\Gamma, z' : b \vdash z' : b}}{\Gamma' \vdash (\lambda z'.z') : b \to b}}{\Gamma' \vdash y\ (\lambda z'.z') : b \to b}}{\Gamma' \vdash x\ (y\ (\lambda z'.z')) : b \to b}$$

(c) With $\Gamma = \{x : a \to b,\ y : b \to c,\ z : a\}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash y : b \to c}\,(\mathsf{Var}) \qquad \dfrac{\dfrac{}{\Gamma \vdash x : a \to c}\,(\mathsf{Var}) \quad \dfrac{}{\Gamma \vdash z : a}\,(\mathsf{Var})}{\Gamma \vdash xz : b}\,(\mathsf{App})}{\Gamma \vdash y(xz) : c}\,(\mathsf{App})}{x : a \to b,\ y : b \to c \vdash \lambda z.\, y(xz) : a \to c}\,(\mathsf{Abs})}{x : a \to b \vdash \lambda yz.\, y(xz) : (b \to c) \to a \to c}\,(\mathsf{Abs})}{\vdash \lambda xyz.\, y(xz) : (a \to b) \to (b \to c) \to a \to c}\,(\mathsf{Abs})$$

** 2. Give terms $M \in \Lambda$ that satisfy each of the following (you are *not* required to justify them with a proof tree, but you may wish to so as to check your answer):

(a) $\vdash M : (a \to b) \to a \to b$
(b) $x : (a \to a) \to c \vdash M : c$
(c) $x : \forall ab.\, a \to a \to b \vdash M : a$

Solution

(a) e.g. $\lambda x.\, x$ or perhaps $\lambda xy.\, xy$
(b) e.g. $x\,(\lambda y.\, y)$
(c) e.g. xxx

** 3. Prove that $\lambda xy.\, xy(yx)$ is untypable (not typable).

Solution

Suppose that $\lambda xy.\, xy(yx)$ were typable. By definition of typability, there is some A such that $\vdash \lambda xy.\, xy(yx) : A$ is derivable in the type system. This derivation must have the following shape:

$$(\mathsf{TAbs})\,\dfrac{(\mathsf{TAbs})\,\dfrac{(\mathsf{TApp})\,\dfrac{D_2 \quad D_1}{x : B_1,\ y : B_3 \vdash xy(yx) : B_4}}{x : B_1 \vdash \lambda y.\, xy(yx) : B_2}}{\vdash \lambda xy.\, xy(yx) : A}$$

with D_1 the (sub)derivation:

$$\text{(TApp)}\ \dfrac{\text{(TVar)}\ \dfrac{}{x : B_1,\ y : B_3 \vdash y : B_7} \quad \text{(TVar)}\ \dfrac{}{x : B_1,\ y : B_3 \vdash x : B_6}}{x : B_1,\ y : B_3 \vdash yx : B_5}$$

and D_2 the following (sub)derivation:

$$\text{(TApp)}\ \dfrac{\text{(TVar)}\ \dfrac{}{x : B_1,\ y : B_3 \vdash x : B_{10}} \quad \text{(TVar)}\ \dfrac{}{x : B_1,\ y : B_3 \vdash y : B_9}}{x : B_1,\ y : B_3 \vdash xy : B_8}$$

for some types B_1 — B_{10}. We know from the rules of the system that these types have the following relationships:

$$\begin{array}{lrcl}
(1) & A & = & B_1 \to B_2 \\
(2) & B_2 & = & B_3 \to B_4 \\
(3) & B_8 & = & B_5 \to B_4 \\
(4) & B_7 & = & B_6 \to B_5 \\
(5) & B_7 & = & B_3 \\
(6) & B_6 & = & B_1 \\
(7) & B_{10} & = & B_9 \to B_8 \\
(8) & B_{10} & = & B_1 \\
(9) & B_9 & = & B_3
\end{array}$$

Now we can just begin working out all the consequences of these equations by substituting equals for equals. In particular, it must also be true that:

$$\begin{array}{rcl}
B_3 & = & B_6 \to B_5 \\
B_1 & = & B_9 \to B_8
\end{array}$$

by combining (5) and (4), (7) and (8). Therefore, using (6) in the former and (9) in the latter:

$$\begin{array}{rcl}
B_3 & = & B_1 \to B_5 \\
B_1 & = & B_3 \to B_8
\end{array}$$

But this means that $B_1 = (B_1 \to B_5) \to B_8$. This is impossible! No type B_1, which is just a string, can be identical to a type $(B_1 \to B_5) \to B_8$ that properly contains the same string. Since this contradiction is an inevitable consequence of supposing that $\lambda xy.\, yx$ is typable, it must be that it is not typable.

** 4. **(Optional)** Prove: $\forall A$, if $c \notin \{a_1, \ldots, a_n\}$ and $\{a_1, \ldots, a_n\} \cap \mathsf{FV}(C) = \emptyset$ then

$$A[B_1/a_1, \ldots, B_n/a_n][C/c] = A[C/c][B_1[C/c]/a_1, \ldots, B_n[C/c]/a_n]$$

You will want to consider several cases depending on how variables coincide.

Solution

The proof is by induction on A:

(TyVar) When A is a type variable a, we reason as follows. Assume $c \notin \{a_1, \ldots, a_n\}$ and $\{a_1, \ldots, a_n\} \cap \mathsf{FV}(C) = \emptyset$. Then we consider three cases:

- When $a = a_j$ for some $j \in \{1, \ldots, n\}$:

$$\begin{aligned} & a[B_1/a_1, \ldots, B_n/a_n][C/c] \\ &= B_j[C/c] \\ &= a[B_1[C/c]/a_1, \ldots, B_n[C/c]/a_n] \\ &= a[C/c][B_1[C/c]/a_1, \ldots, B_n[C/c]/a_n] \end{aligned}$$

  since, in particular, $c \neq a_j$.

- When $a \notin \{a_1, \ldots, a_n\}$ and $a = c$:

$$\begin{aligned} & a[B_1/a_1, \ldots, B_n/a_n][C/c] \\ &= a[C/c] \\ &= C \\ &= C[B_1[C/c]/a_1, \ldots, B_n[C/c]/a_n] \\ &= a[C/c][B_1[C/c]/a_1, \ldots, B_n[C/c]/a_n] \end{aligned}$$

  with the second to last equation holding since no a_i is free in C.

- When $a \notin \{a_1, \ldots, a_n\}$ and $a \neq c$:

$$\begin{aligned} & a[B_1/a_1, \ldots, B_n/a_n][C/c] \\ &= a[C/c] \\ &= a \\ &= a[B_1[C/c]/a_1, \ldots, B_n[C/c]/a_n] \\ &= a[C/c][B_1[C/c]/a_1, \ldots, B_n[C/c]/a_n] \end{aligned}$$

(Arrow) When A is an arrow $D \to E$, assume the induction hypotheses:

(IH1) If $c \notin \{a_1, \ldots, a_n\}$ and $\{a_1, \ldots, a_n\} \cap \mathsf{FV}(C) = \emptyset$ then:

$$D[B_1/a_1, \ldots, B_n/a_n][C/c] = D[C/c][B_1[C/c]/a_1, \ldots, B_n[C/c]/a_n]$$

(IH2) If $c \notin \{a_1, \ldots, a_n\}$ and $\{a_1, \ldots, a_n\} \cap \mathsf{FV}(C) = \emptyset$ then:

$$E[B_1/a_1, \ldots, B_n/a_n][C/c] = E[C/c][B_1[C/c]/a_1, \ldots, B_n[C/c]/a_n]$$

Assume $c \notin \{a_1, \ldots, a_n\}$ and $\{a_1, \ldots, a_n\} \cap \mathsf{FV}(C) = \emptyset$. Then:

$$\begin{aligned}
&(D \to E)[B_1/a_1, \ldots, B_n/a_n][C/c] \\
&= D[B_1/a_1, \ldots, B_n/a_n][C/c] \to E[B_1/a_1, \ldots, B_n/a_n][C/c] \\
&= D[C/c][B_1[C/c]/a_1, \ldots, B_n[C/c]/a_n] \to E[C/c][B_1[C/c]/a_1, \ldots, B_n[C/c]/a_n] \\
&= (D \to E)[C/c][B_1[C/c]/a_1, \ldots, B_n[C/c]/a_n]
\end{aligned}$$

where the penultimate line follows from the use of both IH.

The following is the induction principle for the type system, viewed as a set of triples or three-place relation $- \vdash - : -$.

> Suppose $\Phi(\Gamma, M, A)$ is a property of triples comprising a type environment, a term and a monotype. Then if the following can all be proven:
>
> - For all type environments Δ, variables x, type schemes $\forall \overline{a}. C$ and sequences of monotypes $\overline{B}$ (of the appropriate length), if $x : \forall \overline{a}.\ C \in \Delta$ then $\Phi(\Delta, x, C[\overline{B}/\overline{a}])$.
> - For all type environments Δ, terms P and Q and monotypes B and C, if $\Phi(\Delta, P, B \to C)$ and $\Phi(\Delta, Q, B)$ then $\Phi(\Delta, PQ, C)$.
> - For all type environments Δ, terms P, variables x and monotypes B and C, if $x \notin \mathsf{dom}(\Gamma)$ and $\Phi(\Delta \cup \{x : B\}, P, C)$ then $\Phi(\Delta, \lambda x. P, B \to C)$.
>
> It follows that $\forall (\Gamma, M, A) \in \vdash.\ \Phi(\Gamma, M, A)$.

** 5. Prove, by induction on $\Gamma \vdash M : A$, that:

If $\Gamma \vdash M : A$ then $\Gamma[C/c] \vdash M : A[C/c]$.

Note: by the variable convention, you may assume $(\forall \overline{a}.\ A)[C/c] = \forall \overline{a}.\ A[C/c]$. Also, you will need to use the result of the previous question.

Solution

The proof is by induction on $\Gamma \vdash M : A$.

- In case (TVar), M is some variable x and A is of shape $A'[\overline{B}/\overline{a}]$ we can assume $x : \forall \overline{a}.\ A' \in \Gamma$. Therefore, by definition (and the assumption allowed by the question), $x : \forall \overline{a}.\ A'[C/c] \in \Gamma[C/c]$. Hence, $\Gamma[C/c] \vdash x : A'[C/c][\overline{B[C/c]}/\overline{a}]$ by (TVar) (i.e. the instance we choose has $B_i[C/c]$ replacing a_i instead of B_i replacing a_i). By the previous question, this is the same as $A'[\overline{B}/\overline{a}][C/c]$, i.e. $A[C/c]$, which was our goal.

- In case (TApp), M is of shape PQ. Assume the induction hypotheses:
  - $\Gamma[C/c] \vdash P : (B \to A)[C/c]$
  - $\Gamma[C/c] \vdash Q : B[C/c]$

  By definition, $(B \to A)[C/c]$ is just $B[C/c] \to A[C/c]$. Hence, by (TApp), $\Gamma[C/c] \vdash PQ : A[C/c]$ as required.

- In case (TAbs), M is of shape $\lambda x.\,P$ and A is of shape $B_1 \to B_2$. Assume the induction hypothesis: $(\Gamma \cup \{x : B_1\})[C/c] \vdash P : B_2[C/c]$. By definition and set reasoning, $(\Gamma \cup \{x : B_1\})[C/c] = \Gamma[C/c] \cup \{x : B_1[C/c]\}$. Therefore, it follows from (Abs) that $\Gamma[C/c] \vdash \lambda x.\,P : B_1[C/c] \to B_2[C/c]$. The result follows because $B_1[C/c] \to B_2[C/c] = (B_1 \to B_2)[C/c]$ by definition.

** 6. **(Optional)** Prove the Subject Reduction Theorem by induction on $M \to_\beta N$:

if $M \to_\beta N$ and $\Gamma \vdash M : A$ then $\Gamma \vdash N : A$.

Three tips:

- You will want to have an induction hypothesis of the form $\forall \Gamma.\ \Phi(M,N)$, which will be useful in the (Abs) case, so set up your goal accordingly.
- You will need to use the substitution lemma from the notes.
- You will need to use the Inversion theorem from the notes.

Solution

We prove $\forall (M,N) \in \to_\beta .\ \forall \Gamma.\ \Gamma \vdash M : A \Rightarrow \Gamma \vdash N : A$. The proof is by induction on $M \to_\beta N$:

(Redex) In this case M is $(\lambda x.\,P)Q$ and N is $P[Q/x]$. Let Γ be a type environment. It follows from Inversion that there is some type B such that $\Gamma \vdash (\lambda x.\,P) : B \to A$ and $\Gamma \vdash Q : B$. Similarly, it follows that $\Gamma \cup \{x : B\} \vdash P : A$. Then by the substitution lemma $\Gamma \vdash P[Q/x] : A$ as required.

(AppL) In this case M is of shape $P\ Q$ and N is of shape $P'\ Q$. We assume the following induction hypothesis: forall Γ', if $\Gamma \vdash P : C$ then $\Gamma \vdash P' : C$. Let Γ be a type environment and suppose $\Gamma \vdash P\ Q : A$. It follows from Inversion that there must be some type B such that $\Gamma \vdash P : B \to A$ and $\Gamma \vdash Q : C$. Consequently, by the induction hypothesis, we have $\Gamma \vdash P' : B \to A$. Then $\Gamma \vdash P'\ Q : A$ follows by (RAppL).

(AppR) This case is symmetrical to the previous.

(Abs) In this case M is of the form $\lambda x.P$ and N is of the form $\lambda x.Q$. Assume the induction hypothesis: for all Γ', if $\Gamma' \vdash P : C$ then $\Gamma' \vdash Q : C$. Let Γ be a type environment. By inversion, A is of shape $D \to E$ and $\Gamma \cup \{x : D\} \vdash P : E$. Hence, by the induction hypothesis, with $\Gamma' = \Gamma \cup \{x : D\}$, we obtain $\Gamma \cup \{x : D\} \vdash Q : E$. Hence, the required goal follows immediately from (TAbs).