TYPES AND λ-CALCULUS

Problem Sheet 7

Questions 1 and 3 will be marked.

** 1. Suppose closed term M has a normal form $\underline{1}\,(\lambda x.\,x)$. Prove that all reducts of M are untypable, i.e. $M \vartriangleright^* N$ implies N untypable.

Solution

Assume M has normal form $\underline{1}\,(\lambda x.\,x)$ and suppose $M \vartriangleright^* N$. Then we have $M \vartriangleright^* \underline{1}\,(\lambda x.\,x)$ and $M \vartriangleright^* N$ so, by Confluence, N and $\underline{1}\,(\lambda x.\,x)$ have a common reduct. But $\underline{1}\,(\lambda x.\,x)$ is a normal form, so we must have that $N \vartriangleright^* \underline{1}\,(\lambda x.\,x)$. We claim that N is therefore untypable. To see why, suppose that N were typable, then by Subject Reduction, so is $\underline{1}\,(\lambda x.\,x)$. However, we know from the previous sheet that this is impossible.

** 2. Prove both of the following:

(1) $\vdash \underline{n} : A$ implies $A = \mathsf{Nat}$.

(2) $\vdash V : \mathsf{Nat}$ implies V is a numeral.

That is, numerals can only be assigned the type Nat and these are the only closed values of this type. For (1), induction is not necessary, but you will need to analyse the two possible shapes of n (0 or $k+1$). For (2), I suggest appealing to inversion to rule out many possible shapes of V in a one go.

Solution

We prove each separately.

(1) We analyse the possible shape of n.

- When $n = 0$, suppose $\vdash \mathsf{Z} : A$. By inversion, it must be that $A = \mathsf{Nat}$.

- When n is of shape $k+1$, we suppose $\vdash \underline{k+1} : A$. By inversion it follows that there is a type B such that $\vdash \mathsf{S} : B \to A$. By inversion again, it must be that B and A are both Nat, as required.

(2) Suppose $\vdash V : \mathsf{Nat}$. We analyse the possible shapes of V. If V is of shape $\lambda x.W$, or pred, or S, or fix, or ifz, or ifz $\underline{n}$ or ifz $\underline{n}\,W$ then it follows from inversion that V is assigned an arrow type, which contradicts the assumption. Therefore, the only remaining possibility is that V is a numeral.

** 3.

(a) Prove the following result by induction on M:

> Let B be a type and N a term. For all terms M, types A and environments Γ: if $\Gamma, x{:}B \vdash M : A$ and $\Gamma \vdash N : B$ then $\Gamma \vdash M[N/x] : A$.

(b) Prove the missing case in the proof of Lemma 12.2 from the notes: if $\Gamma \vdash (\lambda x.M)N : A$ then $\Gamma \vdash M[N/x] : A$.

Solution

(a) Suppose B is a type and N a term. The rest of the proof is by induction on M.

- When M is a variable y, we proceed as follows. Let A be a type and Γ an environment. Suppose (i) $\Gamma, x{:}B \vdash y : A$ and (ii) $\Gamma \vdash N : B$. Now we analyse cases on $x = y$?
  - If $x = y$ then, by definition of substitution, $M[N/x] = N$ and, by inversion on (i), we obtain that $B = A$. Then our goal is to show $\Gamma \vdash N : A$, which is exactly (ii).
  - Otherwise, by definition of substitution, $M[N/x] = M = y$. Our goal is to show $\Gamma \vdash y : A$ and we have (i) $\Gamma, x : B \vdash y : A$. By inversion, it must be that $y : A \in \Gamma$ and hence, by (TVar), $\Gamma \vdash y : A$.
- When M is a constant c, we proceed as follows. Let A be a type and Γ an environment. Suppose (i) $\Gamma, x{:}B \vdash c : A$ and (ii) $\Gamma \vdash N : B$. By definition, $c[N/x] = c$ and so our goal is to show $\Gamma \vdash c : A$. By inversion on (i), we have $c : A \in \mathbb{C}$ and so the goal follows immediately from the (TCst) rule.
- When M is of the form PQ, we assume the induction hypotheses:

(IH1) For all A', Γ': if $\Gamma', x{:}B \vdash P : A'$ and $\Gamma' \vdash N : B$ then $\Gamma' \vdash P[N/x] : A'$

(IH2) For all A', Γ': if $\Gamma', x{:}B \vdash Q : A'$ and $\Gamma' \vdash N : B$ then $\Gamma' \vdash Q[N/x] : A'$

Let A be a type and Γ an environment, then suppose (i) $\Gamma, x{:}B \vdash PQ : A$ and (ii) $\Gamma \vdash N : B$. It follows from inversion on (i) that there is a type C such that:

(a) $\Gamma, x{:}B \vdash P : C \to A$

(b) $\Gamma, x{:}B \vdash Q : C$

Hence, we can obtain from (IH1), with $A' := C \to A$ and $\Gamma' := \Gamma$ that $\Gamma \vdash P[N/x] : C \to A$. From (IH2), with $A' := C$ and $\Gamma' = \Gamma$, we obtain $\Gamma \vdash Q[N/x] : C$. Putting these together with (TApp) we obtain $\Gamma \vdash P[N/x]Q[N/x] : A$, and by definition of substitution, this is just $\Gamma \vdash (PQ)[N/x] : A$, which was our goal.

- When M is of the form $\lambda y.\, P$, we assume the induction hypothesis:

  (IH) For all A' and Γ', if $\Gamma', x{:}B \vdash P : A'$ and $\Gamma' \vdash N : B$ then $\Gamma' \vdash P[N/x] : A$.

  We may also assume, by the variable convention, that y does not occur freely in N, is not a subject in Γ and is distinct from x (otherwise, we rename y in the abstraction). Let A be a type and Γ an environment and suppose (i) $\Gamma, x{:}B \vdash \lambda y.\, P : A$ and (ii) $\Gamma \vdash N : B$. It follows from (i) by inversion that there are types A_1 and A_2 such that $A = A_1 \to A_2$ and (*) $\Gamma, x{:}B, y : A_1 \vdash P : A_2$. Then it follows from this by (IH), with $A' = A_2$ and $\Gamma' = \Gamma \cup \{y : A_1\}$, that $\Gamma, y{:}A_1 \vdash P[N/x] : A_2$. From this we can immediately infer $\Gamma \vdash \lambda y.\, P[N/x] : A_1 \to A_2$ using (TAbs). By definition of substitution (recall we assumed $y \neq x$ and $y \notin \mathsf{FV}(N)$) and the identity of A, this is just our goal $\Gamma \vdash (\lambda y.\, P)[N/x] : A$.

(b) Now suppose $\Gamma \vdash (\lambda x.\, M)N : A$. By inversion, it follows that there is some type B such that $\Gamma \vdash \lambda x.\, M : B \to A$ and $\Gamma \vdash N : B$. By inversion on the former, we deduce that $\Gamma, x : B \vdash M : A$. Then the result follows from the previous part.

** 4. Find pure terms that inhabit the following types (no need for justification):

(a) $(a \to a \to b) \to a \to b$

(b) $(a \to b) \to (a \to c) \to (b \to c \to d) \to a \to d$

(c) $(((a \to a \to b) \to a \to b) \to c) \to c$

Solution

(a) $\lambda x y . x\, y\, y$
(b) $\lambda w x y z . y\, (w\, z)\, (x\, z)$
(c) $\lambda x . x\, (\lambda y z . y\, z\, z)$

** 5.

(a) Find a *pure* term that inhabits the type $(((a \to b) \to b) \to b) \to a \to b$.
(b) Give the corresponding proof of the corresponding formula.

Solution

(a) $\lambda x y . x (\lambda z . z y)$
(b) Suppose $(a \Rightarrow b) \Rightarrow b) \Rightarrow b$ (x) and assume a (y). We claim that $(a \Rightarrow b) \Rightarrow b$, the proof is as follows: suppose $a \Rightarrow b$ (z), then applying this to assumption (y) we get b. Returning to our original proof, from this and (x) we obtain b, as required.

*** 6. The reason that we don't study full PCF in connection with the Curry-Howard correspondence is the presence of fix.

(a) Use fix to show that every type is inhabited by some *PCF term* (not necessarily pure).
(b) What is the consequence for the Curry-Howard correspondence extended to full PCF?

Solution

(a) Every type A is inhabited by the term fix $(\lambda x . x)$. We can construct the following derivation.

$$\dfrac{\dfrac{}{\vdash \mathsf{fix} : (A \to A) \to A}\,(\mathsf{TFix}) \quad \dfrac{\dfrac{}{x : A \vdash x : A}\,(\mathsf{TVar})}{\vdash \lambda x . x : A \to A}\,(\mathsf{TAbs})}{\vdash \mathsf{fix}\ (\lambda x . x) : A}\,(\mathsf{TApp})$$

(b) Hence, by the Curry-Howard correspondence, this would lead to a proof system in which every formula was provable – i.e. an inconsistent logic!

*** 7. The following property is called Subject Invariance:

if $M \approx N$ and $\Gamma \vdash M : A$ then $\Gamma \vdash N : A$

Is this property true for our type system? Either prove it or give a counterexample.

Solution

It is not true in our system. To see why, take e.g. $\lambda y.\, y \approx (\lambda x.\, (\lambda y.\, y))(\lambda x.\, xx)$. We have $\vdash \lambda y.\, y : a \to a$ so the hypotheses of the implication are satisfied, but $(\lambda x.\, (\lambda y.\, y))(\lambda x.\, xx)$ is untypable because it includes $\lambda x.\, xx$ as a subterm.