TYPES AND λ-CALCULUS

Problem Sheet 1

1. Consider the following proof (annotated with circled numbers) of:

$$\forall nm \in \mathbb{N}.\ n \leq m \Rightarrow \exists x.\ m = n + x$$

In this proof, we silently assume the basic facts about arithmetic, but the only thing we know about $\leq$ is it's definition:

$$\forall p.\ 0 \leq p \tag{1}$$

$$\forall pq.\ (p+1) \leq q \text{ iff } \exists q'.\ q = q' + 1 \wedge p \leq q' \tag{2}$$

Proof. The proof is by induction on n.

- When $n = 0$, we argue as follows. Let $m \in \mathbb{N}$. 1 Suppose $0 \leq m$. Then let the witness x be m. Then the goal $m = 0 + x$ is just $x = 0 + x$ which is true by arithmetic.

- When n is of shape $k + 1$, we assume the induction hypothesis. Let $m \in \mathbb{N}$ and suppose $k + 1 \leq m$. We can apply the definition of less-than, clause (2), from left to right to obtain some q' such that (i) $m = q' + 1$ and (ii) $k \leq q'$. 2 Then we can apply the induction hypothesis to (ii) to obtain some x' such that (iii) $q' = k + x'$. Then let the witness to the goal also be x'. 3 It follows from (i) that this is just $q'+1 = k+1+x'$; and by (iii), this becomes $k+x'+1 = k+1+x'$ which is true by basic arithmetic.

□

Note that we often apply forwards rules implicitly in this proof, and this is typical.

(a) What is the induction hypothesis in the second case of the proof?

(b) What is the state of the proof at each position 1, 2 and 3?

Solution

The answers don't have to be identical to those given, because we typically take short-cuts here and there which will lead to discrepencies. However, the goals should by equivalent and the assumptions should not be missing anything important.

(a) $\forall m \in \mathbb{N}.\ k \leq m \Rightarrow \exists x.\ m = k + x$

(b) ① Assumptions: (1), (2), $n = 0$, $m \in \mathbb{N}$.
Goal: $0 \leq m \Rightarrow \exists x.\ m = 0 + x$.

② Assumptions: (1), (2), $n = k + 1$, (IH), $m \in \mathbb{N}$, $k + 1 \leq m$, (i), (ii).
Goal: $\exists x.\ m = k + 1 + x$.

③ Assumptions: as above, (iii).
Goal: $m = k + 1 + x$.

** 2. Note that, by the conventions of logic, $A \Rightarrow B \Rightarrow C$ is a shorthand for $A \Rightarrow (B \Rightarrow C)$ and conjunction binds tighter than implication, so $A \wedge B \Rightarrow C$ means $(A \wedge B) \Rightarrow C$.

Give proofs of the following. I recommend you keep track of the proof state on a scrap of paper as you complete the proof, but you need not submit this.

(a) $\neg A \Rightarrow A \Rightarrow B$
(b) $(A \wedge B \Rightarrow C) \Rightarrow A \Rightarrow B \Rightarrow C$
(c) $\neg(A \wedge \neg A)$
(d) $(A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$
(e) $\neg A \wedge \neg B \Rightarrow \neg(A \vee B)$

Solution

(a) Assume $\neg A$ then assume A. This yields a contradiction and so, in particular, B follows.
(b) Assume $A \wedge B \Rightarrow C$ (*). Assume A and then assume B. From A and B we have $A \wedge B$ and so from (*) we obtain C.
(c) We assume $A \wedge \neg A$ and then try to obtain a contradiction. We already have A and $\neg A$ which gives the desired contradiction.
(d) Assume $A \Rightarrow B$ (1). Assume $B \Rightarrow C$ (2). Assume A. From (1) and A obtain B. From B and (2) obtain C.
(e) Assume $\neg A$ (1) and $\neg B$ (2). For contradiction suppose that $A \vee B$. We proceed by cases on $A \vee B$:

- If A is true, then this contradicts (1).
- If B is true, then this contradicts (2).

In all cases we obtained a contradiction.

** 3. The following build on top of each other:

(a) Prove $(A \vee B) \wedge \neg B$ implies A.
(b) Prove $\forall nm \in \mathbb{N}.\ n + m = 0 \Rightarrow m = 0$. Induction is not necessary. You may use Lemma 1.1 from the notes and the following theorem of arithmetic:

$$(i) \qquad \forall pq \in \mathbb{N}.\ p = 0 \vee p = q + 1$$

(c) Prove $\forall nm \in \mathbb{N}.\ n+m=0 \Rightarrow n*m=0$. Induction is not necessary. Multiplication on natural numbers can be defined as follows:

$$p*0=0 \tag{3}$$

$$p*(q+1)=p+(p*q) \tag{4}$$

Solution

(a) Suppose $A \vee B$ and $\neg B$. Then we analyse cases on $A \vee B$ to prove A.

- In case A is true, then the goal is immediate.
- In case B is true, then with $\neg B$ we obtain a contradiction, from which A follows.

Hence, A is true in all eventualities.

(b) Let $n, m \in \mathbb{N}$. Suppose $n+m=0$. Then, by Lemma 1.1, we have that, (ii) $m \neq k+1$. By specialising (i) with $p := m$ and $q := k$ we obtain $m=0 \vee m=k+1$. Hence, we can apply part (a) to this and (ii) to obtain $m=0$.

(c) Let $n, n \in \mathbb{N}$. Suppose $n+m=0$. Then, by part (b), we have that $m=0$. Hence, our goal is really $n*0=0$ which follows by definition of multiplication.