TYPES AND λ-CALCULUS

Problem Sheet 4

This week, Questions 3 and 6 will be marked.

* 1. Justify each of the following conversions $M \approx N$ by finding a common reduct P, i.e. such that $M \rhd^* P$ and $N \rhd^* P$.

(a) $(\lambda x.x)y \approx (\lambda xy.x)\ y\ z$

(b) $(\lambda x.M)N \approx M[N/x]$

(c) $\mathsf{fix}\ (\underline{\mathsf{const}}\ \underline{1}) \approx (\lambda x.x\ \underline{2})\ (\underline{\mathsf{const}}\ (\mathsf{pred}\ \underline{2}))$

(d) $z\ (\underline{\mathsf{const}}\ \underline{\mathsf{id}}\ \underline{\mathsf{div}})\ \underline{\mathsf{div}} \approx z\ \underline{\mathsf{id}}\ (\underline{\mathsf{const}}\ \underline{\mathsf{div}}\ \underline{\mathsf{id}})$

Solution

(a) y

(b) $M[N/x]$

(c) $\underline{1}$

(d) $z\ \underline{\mathsf{id}}\ \underline{\mathsf{div}}$

* 2. Define $\underline{\mathsf{Y}} = \lambda f.(\lambda x.f\ (x\,x))(\lambda x.f\ (x\,x))$.

Show that $\underline{\mathsf{Y}}$ is also a fixed point combinator, i.e for all terms M:

$$\underline{\mathsf{Y}}M \approx M\,(\underline{\mathsf{Y}}M)$$

Solution

On the one hand:

$$\begin{aligned}\underline{\mathsf{Y}}\,M &\rhd (\lambda x.M\ (x\,x))(\lambda x.M\ (x\,x))\\ &\rhd M\ ((\lambda x.M\ (x\,x))(\lambda x.M\ (x\,x)))\end{aligned}$$

and, on the other:

$$M\,(\underline{Y}\,M) \;\rhd\; M\;((\lambda x.\, M\;(x\,x))(\lambda x.\, M\;(x\,x)))$$

so, they have a common reduct.

** 3. Prove Lemma 8.1 of the notes, i.e. show all of the following:

Reflexivity For all M: $M \approx M$.

Symmetry For all M, N: $M \approx N$ implies $N \approx M$.

Transitivity For all M, N and P: $M \approx P$ and $P \approx N$ implies $M \approx N$.

Compatibility For all M, N and $C[\,]$: if $M \approx N$ then $C[M] \approx C[N]$.

There is no need for any induction. For compatibility, you will need to use 9(b) from the previous problem sheet.

Solution

We prove each requirement separately:

Reflexivity Let M be a term. Then, there is a 0-step reduction sequence from M to M so, by definition, $M \rhd^* M$. Hence, we can use the definition of convertibility with $P = M$ to obtain $M \approx M$.

Symmetry Let M and N be terms and suppose $M \approx N$. Then, by definition of convertibility, there is a term P such that $M \rhd^* P$ and $N \rhd^* P$. By definition, to show $N \approx M$ we need some common reduct of N and M, so we can use the same witness P again.

Transitivity Let M, N and P be terms and suppose (i) $M \approx P$ and (ii) $P \approx N$. Then, by definition of convertibility there are terms Q_1 and Q_2 such that (a) $M \rhd^* Q_1$, (b) $P \rhd^* Q_1$, (c) $P \rhd^* Q_2$ and (d) $N \rhd^* Q_2$. By confluence applied to (b) and (c), we obtain a common reduct, R, of Q_1 and Q_2. From this, (a) and (d) we obtain that $M \rhd^* Q_1 \rhd^* R$ and $N \rhd^* Q_2 \rhd^* R$ have R as a common reduct and hence, by definition, $M \approx N$.

Compatibility Let M, N be terms and $C[\,]$ a context. Suppose $M \approx N$, so that, by definition, there is a common reduct of M and N, say P, i.e. $M \rhd^* P$ and $N \rhd^* P$. Then by 9(b) of the previous problem sheet, $C[M] \rhd^* C[P]$ and $C[N] \rhd^* C[P]$. So $C[P]$ is a common reduct of $C[M]$ and $C[N]$. Hence, by definition, $C[M] \approx C[N]$.

* 4. At the end of the lecture we defined addition, $\underline{\mathsf{add}}$, as:

$$\mathsf{fix}\ (\lambda f x y.\, \mathsf{ifz}\ x\ y\ (\mathsf{S}\ (f\ (\mathsf{pred}\ x)\ y)))$$

Give a complete reduction sequence from $\underline{\mathsf{add}}\ \underline{2}\ \underline{3}$ to $\underline{5}$.

Solution

$$
\begin{array}{ll}
& \underline{\mathsf{add}}\ \underline{2}\ \underline{3} \\
= & \mathsf{fix}\ (\lambda f x y.\, \mathsf{ifz}\ x\ y\ (\mathsf{S}\ (f\ (\mathsf{pred}\ x)\ y)))\ \underline{2}\ \underline{3} \\
\rhd & (\lambda f x y.\, \mathsf{ifz}\ x\ y\ (\mathsf{S}\ (f\ (\mathsf{pred}\ x)\ y)))\ \underline{\mathsf{add}}\ \underline{2}\ \underline{3} \\
\rhd & (\lambda x y.\, \mathsf{ifz}\ x\ y\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ x)\ y)))\ \underline{2}\ \underline{3} \\
\rhd & (\lambda y.\, \mathsf{ifz}\ \underline{2}\ y\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{2})\ y)))\ \underline{3} \\
\rhd & \mathsf{ifz}\ \underline{2}\ \underline{3}\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{2})\ \underline{3})) \\
\rhd & \mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{2})\ \underline{3}) \\
\rhd & \mathsf{S}\ (\underline{\mathsf{add}}\ \underline{1}\ \underline{3}) \\
= & \mathsf{S}\ ((\lambda f x y.\, \mathsf{ifz}\ x\ y\ (\mathsf{S}\ (f\ (\mathsf{pred}\ x)\ y)))\ \underline{\mathsf{add}}\ \underline{1}\ \underline{3}) \\
\rhd & \mathsf{S}\ ((\lambda x y.\, \mathsf{ifz}\ x\ y\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ x)\ y)))\ \underline{1}\ \underline{3}) \\
\rhd & \mathsf{S}\ ((\lambda y.\, \mathsf{ifz}\ \underline{1}\ y\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{1})\ y)))\ \underline{3}) \\
\rhd & \mathsf{S}\ (\mathsf{ifz}\ \underline{1}\ \underline{3}\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{1})\ \underline{3}))) \\
\rhd & \mathsf{S}\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{1})\ \underline{3})) \\
\rhd & \mathsf{S}\ (\mathsf{S}\ (\underline{\mathsf{add}}\ \underline{0}\ \underline{3})) \\
\rhd & \mathsf{S}\ (\mathsf{S}\ ((\lambda x y.\, \mathsf{ifz}\ x\ y\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ x)\ y)))\ \underline{0}\ \underline{3})) \\
\rhd & \mathsf{S}\ (\mathsf{S}\ ((\lambda y.\, \mathsf{ifz}\ \underline{0}\ y\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{0})\ y)))\ \underline{3})) \\
\rhd & \mathsf{S}\ (\mathsf{S}\ (\mathsf{ifz}\ \underline{0}\ \underline{3}\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{0})\ \underline{3})))) \\
\rhd & \mathsf{S}\ (\mathsf{S}\ (\underline{3})) \\
= & \underline{5}
\end{array}
$$

* 5.

(a) Prove that $\underline{\mathsf{add}}$ satisfies the following equation:

$$\underline{\mathsf{add}} \approx \lambda x y.\, \mathsf{ifz}\ x\ y\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ x)\ y))$$

(b) Prove that $\underline{\mathsf{add}}$ satisfies the following equations. Induction is not necessary.

$$\underline{\mathsf{add}}\ \underline{0}\ \underline{m} \approx \underline{m} \qquad \text{and} \qquad \underline{\mathsf{add}}\ \underline{(n+1)}\ \underline{m} \approx \mathsf{S}\ (\underline{\mathsf{add}}\ \underline{n}\ \underline{m})$$

Hint: In practice, you nearly always want to replace an occurrence of $\underline{\mathsf{add}}$ with the right-hand-side of the equation in (a), rather than by its actual definition (and the same can be said for any recursive function defined using "the recipe").

Solution

(a) The left-hand-side actually reduces to the right-hand-side in two steps.

(b) Then the first equation is true since (using the observation):

$$\underline{\mathsf{add}}\ \underline{0}\ \underline{m} \approx \mathsf{ifz}\ \underline{0}\ \underline{m}\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{0})\ \underline{m})) \approx \underline{m}$$

The second equation holds since:

$$\begin{aligned}\underline{\mathsf{add}}\ \underline{(n+1)}\ \underline{m} &\approx \mathsf{ifz}\ \underline{(n+1)}\ \underline{m}\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{(n+1)})\ \underline{m}))\\ &\approx \mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{(n+1)})\ \underline{m})\\ &\approx \mathsf{S}\ (\underline{\mathsf{add}}\ \underline{n}\ \underline{m})\end{aligned}$$

** 6. Use fix to define the recursive triangular number function: using the "recipe", give a combinator $\underline{\mathsf{Tri}}$ that satisfies:

$$\underline{\mathsf{Tri}}\ \underline{0} \approx \underline{0} \qquad \text{and} \qquad \underline{\mathsf{Tri}}\ \underline{(n+1)} \approx \underline{\mathsf{add}}\ \underline{(n+1)}\ (\underline{\mathsf{Tri}}\ \underline{n})$$

Convince yourself that $\underline{\mathsf{Tri}}\ \underline{2} \approx \underline{3}$ (this is obvious if you believe that your implementation of $\underline{\mathsf{Tri}}$ really satisfies the given equations).

Solution

Define $\underline{\mathsf{Tri}}$ as $\mathsf{fix}\ (\lambda f\, n.\ \mathsf{ifz}\ n\ n\ (\underline{\mathsf{add}}\ n\ (f\ (\mathsf{pred}\ n))))$

** 7. Define multiplication, i.e. construct a term $\underline{\mathsf{mult}}$ that satisfies the following specification:

$$\underline{\mathsf{mult}}\ \underline{0}\ \underline{m} \approx \underline{0} \qquad \text{and} \qquad \underline{\mathsf{mult}}\ \underline{n+1}\ \underline{m} \approx \underline{\mathsf{add}}\ \underline{m}\ (\underline{\mathsf{mult}}\ \underline{n}\ \underline{m})$$

Convince yourself that $\underline{\mathsf{mult}}\ \underline{2}\ \underline{2} \approx \underline{4}$.

Solution

Define $\underline{\mathsf{mult}} = \mathsf{fix}\,(\lambda f\,x\,y.\,\mathsf{ifz}\ x\ \underline{0}\ (\underline{\mathsf{add}}\ y\ (\underline{\mathsf{mult}}\ (\mathbf{pred}\ x)\ y)))$.

** 8. Prove that if $M \approx N$ and N is a normal form, then $M \rhd^* N$.

Therefore, we now know that e.g. $\underline{\mathsf{Tri}}\ \underline{2} \rhd^* \underline{3}$ and $\underline{\mathsf{mult}}\ \underline{2}\ \underline{2} \rhd^* \underline{4}$, so these definitions actually *compute* an output given an input.

Solution

Suppose $M \approx N$ and N is a normal form. It follows from the definition of $\approx$ that there is some common reduct P such that $M \rhd^* P$ and $N \rhd^* P$. Since N is in normal form, $N \rhd^* P$ implies $P = N$. Hence, $M \rhd^* N$.