TYPES AND λ-CALCULUS

Problem Sheet 3

* 1. Show that Θ is also a fixed point combinator, i.e for all terms M:

$$\Theta\, M =_\beta M\,(\Theta\, M)$$

Solution

$$\begin{array}{rl} & \Theta\, M \\ \to_\beta & (\lambda y.\, y((\lambda x y.\, y(x x y))(\lambda x y.\, y(x x y))y))\, M \\ = & (\lambda y.\, y(\Theta\, y))\, M \\ \to_\beta & M\,(\Theta\, M) \end{array}$$

Hence both $\Theta\, M$ and $M\,(\Theta\, M)$ have $M\,(\Theta\, M)$ as a common reduct.

** 2. In this question you will give an alternative predecessor combinator which, although longer, is more intuitive to explain.

We define the ***Church Pair*** of natural numbers m and n, written $\ulcorner(m,n)\urcorner$, as the term $\lambda z.z\ulcorner m\urcorner\ulcorner n\urcorner$.

(a) Define combinators **Fst** and **Snd** with the property that:

$$\mathbf{Fst}\ulcorner(m,\, n)\urcorner =_\beta \ulcorner m\urcorner \qquad \text{and} \qquad \mathbf{Snd}\ulcorner(m,\, n)\urcorner =_\beta \ulcorner n\urcorner$$

(b) Consider the following Haskell program pred' on natural numbers.

$$\mathsf{pred}'\ n = \mathsf{fst}\ (\mathsf{foldn}\ n\ \mathsf{incr}\ (0,0))$$

$$\mathsf{where}$$

$$\begin{array}{rcl} \mathsf{incr}\ (n,0) & = & (n,1) \\ \mathsf{incr}\ (n,1) & = & (n+1,1) \end{array}$$

$$\begin{array}{rcl} \mathsf{foldn}\ 0\ f\ x & = & x \\ \mathsf{foldn}\ n\ f\ x & = & f\ (\mathsf{foldn}\ (n-1)\ f\ x) \end{array}$$

What is the result of computing foldn 3 incr $(0, 0)$?

(c) Implement pred$'$ as a λ-term operating on Church Numerals.

Solution

(a) Define **Fst** as $\lambda p.\ p\ (\lambda xy.\, x)$ and **Snd** as $\lambda p.\ p\ (\lambda xy.\, y)$.
(b) $(2, 1)$
(c) Let **Incr** be the term:

$$\lambda p.\,\mathbf{IfZero}\ (\mathbf{Snd}\ p)\ (\lambda z.z\ (\mathbf{Fst}\ p)\ \ulcorner 1 \urcorner)(\lambda z.z\ (\mathbf{Succ}\ (\mathbf{Fst}\ p))\ \ulcorner 1 \urcorner)$$

Then the required combinator is $\lambda n.\,\mathbf{Fst}\ (n\ \mathbf{Incr}\ \ulcorner(0, 0)\urcorner)$.

** 3.

(a) Prove that natural number multiplication is λ-definable by programming a combinator **Mult**.

Hint: multiplication is iterated addition.

(b) Prove that your construction works by showing the following using induction on $n \in \mathbb{N}$ or on $m \in \mathbb{N}$ (which one works will depend on how you defined **Mult**):

$$\forall n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ \mathbf{Mult}\ \ulcorner m \urcorner\ \ulcorner n \urcorner =_\beta \ulcorner m * n \urcorner$$

Hint: you may use the following fact without proving it:

$$\ulcorner k + 1 \urcorner =_\beta \mathbf{Add}\ \ulcorner 1 \urcorner\ \ulcorner k \urcorner$$

Solution

(a) Define **Mult** as $\lambda mn.\, n\ (\mathbf{Add}\ m)\ \ulcorner 0 \urcorner$. Then we have:
(b) The proof is by induction on n:

- In case $n = 0$,

$$\begin{aligned} \mathbf{Mult}\ \ulcorner m \urcorner\ \ulcorner 0 \urcorner &=_\beta \ulcorner 0 \urcorner\ (\mathbf{Add}\ \ulcorner m \urcorner)\ \ulcorner 0 \urcorner \\ &=_\beta \ulcorner 0 \urcorner \end{aligned}$$

- In case $n = k+1$, assume the induction hypothesis:

$$\forall m \in \mathbb{N}.\ \mathbf{Mult}\ \ulcorner m \urcorner\ \ulcorner k \urcorner =_\beta \ulcorner m * k \urcorner$$

  Then we reason equationally:

$$\begin{array}{lll}
\mathbf{Mult}\ \ulcorner m \urcorner\ \ulcorner k+1 \urcorner & =_\beta & \ulcorner k+1 \urcorner\ (\mathbf{Add}\ \ulcorner m \urcorner)\ \ulcorner 0 \urcorner \\
 & =_\beta & (\mathbf{Add}\ \ulcorner 1 \urcorner\ \ulcorner k \urcorner)\ (\mathbf{Add}\ \ulcorner m \urcorner)\ \ulcorner 0 \urcorner \\
 & =_\beta & (\lambda f x. \ulcorner 1 \urcorner\ f\ (\ulcorner k \urcorner\ f\ x))\ (\mathbf{Add}\ \ulcorner m \urcorner)\ \ulcorner 0 \urcorner \\
 & =_\beta & \ulcorner 1 \urcorner\ (\mathbf{Add}\ \ulcorner m \urcorner)\ (\ulcorner k \urcorner\ (\mathbf{Add}\ \ulcorner m \urcorner)\ \ulcorner 0 \urcorner) \\
 & =_\beta & \mathbf{Add}\ \ulcorner m \urcorner\ (\ulcorner k \urcorner\ (\mathbf{Add}\ \ulcorner m \urcorner)\ \ulcorner 0 \urcorner) \\
 & =_\beta & \mathbf{Add}\ \ulcorner m \urcorner\ (\mathbf{Mult}\ \ulcorner m \urcorner\ \ulcorner k \urcorner) \\
 & =_\beta & \mathbf{Add}\ \ulcorner m \urcorner\ \ulcorner m * k \urcorner \\
 & =_\beta & \ulcorner m + m * k \urcorner = \ulcorner m * (k+1) \urcorner
\end{array}$$

  The penultimate line follows from the induction hypothesis.

** 4. Use **Y** to define the recursive triangular number function: using the "recipe", give a combinator **Tri** that satisfies:

$$\mathbf{Tri}\ \ulcorner 0 \urcorner =_\beta \ulcorner 0 \urcorner \qquad \text{and} \qquad \mathbf{Tri}\ \ulcorner n+1 \urcorner =_\beta \mathbf{Add}\ \ulcorner n+1 \urcorner\ (\mathbf{Tri}\ \ulcorner n \urcorner)$$

Convince yourself that **Tri** $\ulcorner 2 \urcorner =_\beta \ulcorner 3 \urcorner$ (this is obvious if you believe that your implementation of **Tri** really satisfies the given equations).

Solution

Define **Tri** as **Y** $(\lambda f n.\ \mathbf{IfZero}\ n\ n\ (\mathbf{Add}\ n\ (f\ (\mathbf{Pred}\ n))))$

** 5. Prove that if $M =_\beta N$ and N is a normal form, then $M \twoheadrightarrow_\beta N$.

Therefore, we now know that e.g. **Tri** $\ulcorner 2 \urcorner \twoheadrightarrow_\beta \ulcorner 3 \urcorner$, so these definitions actually *compute* an output given an input.

Solution

Suppose $M =_\beta N$ and N is a normal form. It follows from the definition of $=_\beta$ that there is some common reduct P such that $M \twoheadrightarrow_\beta P \twoheadleftarrow_\beta N$. Since N is in normal form, $N \twoheadrightarrow_\beta P$ implies $P = N$. Hence, $M \twoheadrightarrow_\beta N$.

** 6. Show that β-normal forms are unique, i.e. show that if a term has two β-normal forms N_1 and N_2, then they are actually the same term.

Therefore, we now know that e.g. $\mathbf{Tri}\ \ulcorner 2 \urcorner \not\twoheadrightarrow_\beta \ulcorner 4 \urcorner$, so there is at most one output for each input.

Solution

Suppose $M \twoheadrightarrow_\beta N_1$ and $M \twoheadrightarrow_\beta N_2$ and N_1, N_2 are both β-normal forms. Then it follows from Confluence that there is some term Q and $N_1 \twoheadrightarrow_\beta Q$ and $N_2 \twoheadrightarrow_\beta Q$. Since N_1 and N_2 are normal, they cannot make a β-step. Therefore, $Q = N_1$ and $Q = N_2$. Hence, $N_1 = N_2$.

*** 7. Show that there is no term P that satisfies $P\,(MN) =_\beta N$.

Solution

If there were such an P then $\mathbf{K} =_\beta P(\mathbf{K}(\mathbf{I}\mathbf{I})\mathbf{K}) =_\beta P(\mathbf{I}\mathbf{I}) =_\beta \mathbf{I}$, but this is impossible since $\mathbf{K}$ and $\mathbf{I}$ are distinct normal forms (and hence cannot have a common reduct).