Problem Sheet 2

Questions 1, 2 6 and 7 will be marked.

* 1. Write these terms using the minimum number of parentheses and λ, according to our conventions.
    (a) $(\lambda y.((y\,y)(z\,z)))$
    (b) $(\lambda y.(((y\,y)\,y)\,y))$
    (c) $((\mathsf{S}\,\mathsf{Z})(\lambda y.(\lambda z.(z\,(\mathsf{S}\,y)))))$

* 2. Write the term $(\lambda xyz.\,xy(xz))(\lambda xy.\,x)$ with all the parentheses and λ, that we will usually omit, tediously put back in.

* 3. List the all the subterms of the following terms (don't bother listing the same subterm more than once even it occurs several times):
    (a) The 3 distinct subterms of $\lambda x.\,xx$
    (b) The 6 distinct subterms of $(\lambda x.\,xx)(\lambda y.\,y)$
    (c) The 8 distinct subterms of $\lambda xyz.\,xy(yx)$
    (d) The 12 distinct subterms of fix $(\lambda xy.\,\text{ifz } y\ y\ (x\ (\text{pred } y)))$

* 4. Each of the following has two free variables, what are they in each case?
    (a) $\lambda xy.\lambda u.\,uvxyz$
    (b) $\lambda xy.z(\lambda u.\,uvxy)$
    (c) $\lambda wx.z(\lambda u.\,uvwx)$
    (d) $\lambda vw.z(\lambda z.\,uvvw)$
    (e) $\lambda yx.z(\lambda u.\,uwyx)$

* 5. Which of the following pairs of strings are α-equivalent (and therefore represent the same term):
    (a) $\lambda x. xy$ and $\lambda z. zy$
    (b) $\lambda x. xy$ and $\lambda z. zx$
    (c) ifz x (S x) (pred x) and ifz y (S y) (pred y)
    (d) $\lambda xy. xy$ and $\lambda xy. yx$
    (e) fix $(\lambda x.(\lambda y. xy)$ (S x)) and fix $(\lambda y.(\lambda x. yx)$ (S y))

* 6. Perform the following substitutions:
    (a) (ifz x (S x) Z)$[\underline{2}/x]$
    (b) $\underline{2}[\underline{1}/x]$
    (c) $(\lambda x.(\lambda y. xz)z)[(\lambda z. z)/z]$
    (d) $(\lambda x. yx)[yz/x]$
    (e) $(\lambda x. yz)[yy/z]$
    (f) $(\lambda y. xy)[yx/x]$

** 7. Complete the following partial proof of the following statement, by filling out (a), (b) (c) and (d). This proof is a particularly good test of how carefully you are keeping track of the proof state.

Suppose P and Q are terms, x and y are variables. If $x \neq y$ and $x \notin FV(Q)$ then for all terms M:

$$M[P/x][Q/y] = M[Q/y][P[Q/y]/x]$$

(which tells us how to reorder substitutions in general).

Proof. Suppose P and Q are terms. Suppose x and y are variables. The rest of the proof is by induction on $M \in \Lambda$.

- In case M is some variable z, we argue as follows. Assume $x \neq y$ and $x \neq FV(Q)$. Our goal is to show:

$$z[P/x][Q/y] = z[Q/y][P[Q/y]/x]$$

Now, either $z = x$, $z = y$ or z is neither x nor y. We proceed by a case analysis on this fact:

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– Suppose $z = x$. By our assumption, it follows that $z \neq y$. Then, by definition of substitution, $z[P/x][Q/y] = P[Q/y]$ and

$$z[Q/y][P[Q/y]/x] = z[P[Q/y]/x] = P[Q/y]$$

Hence, the desired equality follows.

– Suppose $z = y$. (a)

– Suppose $z \neq x$ and $z \neq y$. Then $z[P/x][Q/y] = z$ on the left side of the goal and also $z[Q/y][P[Q/y]/x] = z$ on the right side, so the result follows.

- (b)

- In case M is some application $N_1 N_2$ we argue as follows. Assume $x \neq y$ and $x \neq \mathsf{FV}(Q)$. Additionally, assume the induction hypothesis:

(IH1) if $x \neq y$ and $x \neq \mathsf{FV}(Q)$ then $N_1[P/x][Q/y] = N_1[Q/y][P[Q/y]/x]$

(IH2) if $x \neq y$ and $x \neq \mathsf{FV}(Q)$ then $N_2[P/x][Q/y] = N_2[Q/y][P[Q/y]/x]$

Our goal is to show that:

$$(N_1 N_2)[P/x][Q/y] = (N_1 N_2)[Q/y][P[Q/y]/x]$$

(c)

- In case M is of shape $\lambda z.N$, we argue as follows. We can assume by the bound variable convention that z is different from any variable in scope, so $z \neq x$ and $z \neq y$ and $z \notin \mathsf{FV}(P)$ and $z \notin \mathsf{FV}(Q)$. (d)

$\square$