UNIVERSITY OF BRISTOL

January 2019 Examination Period

FACULTY OF ENGINEERING

Third Year Examination for the Degrees
of
Bachelor of Science
Master of Engineering

COMS30009J
Types and Lambda Calculus

TIME ALLOWED:
2 Hours

Answers to COMS30009J: Types and Lambda Calculus

Intended Learning Outcomes:

Q1. (a) State the rules defining one-step β-reduction, $M \to_\beta N$, (the names of the rules are not important).

Solution:

$$\frac{}{(\lambda x.\, M)N \to_\beta M[N/x]}$$

$$\frac{M \to_\beta N}{MP \to_\beta NP}$$

$$\frac{P \to_\beta Q}{MP \to_\beta MQ}$$

$$\frac{M \to_\beta N}{\lambda x.\, M \to_\beta \lambda x.\, N}$$

[3 marks]

(b) For each of the following state whether it is true or false (no justification is necessary).

i. $M = N$ implies $M \twoheadrightarrow_\beta N$
ii. $M \to_\beta N$ implies $M \twoheadrightarrow_\beta N$
iii. $M =_\beta N$ implies $M \twoheadrightarrow_\beta N$
iv. $M \twoheadrightarrow_\beta N$ implies $M =_\beta N$

[4 marks]

Solution:

i. true
ii. true
iii. false
iv. true

(c) For each of the following, give an example of a *closed* term M with that property.

i. M is in β-normal form.
ii. M is normalising but *not* strongly normalising.
iii. $M \to_\beta M$
iv. $M \twoheadrightarrow_\beta MM$

[4 marks]

Solution:

i. **I**

ii. $\mathbf{K}\,\mathbf{I}\,\Omega$

iii. Ω

iv. $\Theta\,(\lambda x.\, xx)$

(d) Recall the inductive definition of the subterm relation:

$$\frac{}{M \sqsubseteq M}\,(\text{SubRefl}) \qquad \frac{P \sqsubseteq M}{P \sqsubseteq (\lambda x.\, M)}\,(\text{SubAbs})$$

$$\frac{P \sqsubseteq M}{P \sqsubseteq (MN)}\,(\text{SubAppL}) \qquad \frac{P \sqsubseteq N}{P \sqsubseteq (MN)}\,(\text{SubAppR})$$

Prove, by induction on $M \sqsubseteq N$, that:

If $M \sqsubseteq N$ and M is a redex, then there is some N' such that $N \to_\beta N'$.

[6 marks]

Solution: The proof is by induction on $M \sqsubseteq N$.

- In case (SubRefl), $M = N$. Assume M is a redex. Then M has shape $(\lambda x.\, P)Q$. Hence, take witness N' as $P[Q/x]$ and $N \to_\beta N'$ by (Redex).
- In case (SubAbs), N has shape $\lambda x.\, P$. Assume the induction hypothesis: if M is a redex then there is some P' such that $P \to_\beta P'$. Assume M is a redex. It follows from the induction hypothesis that there is such a P'. Therefore, take N' to be $\lambda x.\, P'$ and by (Abs), $\lambda x.\, P \to_\beta \lambda x.\, P'$.
- In case (SubAppL), N has shape PQ. Assume the induction hypothesis: if M is a redex then there is some P' such that $P \to_\beta P'$. Assume M is a redex. Then it follows from the induction hypothesis that there is such a P'. Therefore, take $P'Q$ as N' and, by (AppL), $PQ \to_\beta P'Q$.
- The case (SubAppR) is analogous to (SubAppL).

(e) Prove that there cannot be a term M with the property that:

$$M\,(\lambda z.\, z\,(\mathbf{K}\,\mathbf{I}\,\Omega)\,\Omega) =_\beta \ulcorner 0 \urcorner \qquad \text{and} \qquad M\,(\lambda z.\, z\,\mathbf{I}\,(\mathbf{K}\,\Omega\,\mathbf{I})) =_\beta \ulcorner 1 \urcorner$$

[3 marks]

Solution: Suppose for the purposes of obtaining a contradiction that such a term M exists. We have:

$$\lambda z.\, z\,(\mathbf{K}\,\mathbf{I}\,\Omega)\,\Omega =_\beta \lambda z.\, z\,\mathbf{I}\,(\mathbf{K}\,\Omega\,\mathbf{I})$$

since both reduce to a common term $\lambda z.\, z\,\mathbf{I}\,\Omega$. Call the first of these P and the second Q for short. Then it follows that $\ulcorner 0 \urcorner =_\beta M\,P =_\beta M\,Q =_\beta \ulcorner 1 \urcorner$. However, it follows from the Church-Rosser theorem that $\ulcorner 0 \urcorner \neq_\beta \ulcorner 1 \urcorner$.

(f) Let M be term. Suppose that the equation $MN =_\beta NMN$ is true for all terms N. Prove that M cannot have a β-normal form, i.e. if $M \twoheadrightarrow_\beta P$ then P is not in β-normal form.

[5 marks]

Solution: Suppose for contradiction that M satisfies this equation and yet has a normal form P. Then, one instance of the equation is $Mx =_\beta xMx$. Since $M \twoheadrightarrow_\beta P$, also $Px =_\beta xPx$. The term xPx is a β-normal form so, by Church Rosser, it must be that $Px \twoheadrightarrow_\beta xPx$ (*). We distinguish two cases for P, either P is an abstraction $\lambda y.\, Q$ or it is not. In the first case, $Px \to_\beta Q[x/y]$ and the latter term must be a normal form. However, $Q[x/y] \neq x(\lambda y.\, Q)x$ because $Q[x/y]$ and Q are strings of the same length. In the second case, Px is already a normal form and, again $Px \neq xPx$. Therefore, it cannot be that $Px \twoheadrightarrow_\beta xPx$, contradicting (*).

Q2. (a) State the rules of the type system (the rule names are not important).

[3 marks]

Solution:

$$\frac{x : \forall \overline{a}.\ A \in \Gamma}{\Gamma \vdash x : A[\overline{B}/\overline{a}]}\ (\text{TVar})$$

$$\frac{\Gamma \vdash M : B \to A \quad \Gamma \vdash N : B}{\Gamma \vdash MN : A}\ (\text{TApp})$$

$$x \notin \text{dom}\,\Gamma\ \frac{\Gamma \cup \{x : B\} \vdash M : A}{\Gamma \vdash \lambda x.\, M : B \to A}\ (\text{TAbs})$$

(b) Give an example of a *closed* term *in β-normal form* that is not typable.

[1 mark]

Solution: $\lambda x.\, xx$

(c) For each of the following terms M, give a type environment Γ and a type A such that $\Gamma \vdash M : A$ (you need not prove it).

i. $(\lambda x.\, yxz)(\lambda z.\, z)$

ii. $(\lambda xy.\, yx)\, x\, z$

[3 marks]

Solution:

i. $y : (a \to a) \to b \to c,\ z : b \vdash (\lambda x.\, yxz)(\lambda z.\, z) : c$

ii. $x : a,\ z : a \to b \vdash (\lambda xy.\, yx)\, x\, z : b$

(d) Prove the following by induction on $M \in \Lambda$.
If $\Gamma, x : B \vdash M : C$ and $\Gamma \vdash N : B$ then $\Gamma \vdash M[N/x] : C$

[7 marks]

Solution: The proof is by induction on $M \in \Lambda$.

- In case (Var), M is a variable y. Assume $\Gamma, x : B \vdash y : C$ and $\Gamma \vdash N : B$. There are two subcases:
  - If $x = y$ then, by Inversion, $B = C$. By definition, $y[N/x] = N$ and it follows from the second assumption that $\Gamma \vdash N : B$.
  - If $x \neq y$ then, $y[N/x] = y$. It follows from the first assumption, by inversion, that $y : B \in \Gamma$. Therefore, by (Var), $\Gamma \vdash y : B$.
- In case (App), M is an application PQ. Assume $\Gamma, x : B \vdash PQ : C$ and $\Gamma \vdash N : B$. Assume the induction hypotheses:

(IH1) if $\Gamma, x : B' \vdash P : C'$ and $\Gamma \vdash N : B'$ then $\Gamma \vdash P[N/x] : C'$

(IH2) if $\Gamma, x : B' \vdash Q : C'$ and $\Gamma \vdash N : B'$ then $\Gamma \vdash Q[N/x] : C'$

By definition $(PQ)[N/x] = P[N/x][Q/x]$. By inversion on the first assumption, there is a type D such that $\Gamma, x : B \vdash P : D \to C$ and $\Gamma, x : B \vdash Q : D$. Therefore, by (IH1) and the second assumption, $\Gamma \vdash P[N/x] : D \to C$. By (IH2) and the second assumption, $\Gamma \vdash Q[N/x] : D$. Therefore, by (App), $\Gamma \vdash P[N/x]Q[N/x] : C$, and $P[N/x]Q[N/x] = (PQ)[N/x]$ by definition.

- In case (Abs), M is an abstraction $\lambda y.\, P$ and C is an arrow $D \to E$. We can assume by the variable convention that $x \neq y$ and $y \notin \mathsf{FV}(Q)$ and $y \notin \mathsf{ran}(\Gamma)$. Assume $\Gamma, x : B \vdash \lambda y.\, P : D \to E$ and $\Gamma \vdash N : B$. Assume the induction hypothesis IH: if $\Gamma, x : B' \vdash P : C'$ and $\Gamma \vdash N : C'$ then $\Gamma \vdash P[N/x] : C'$. It follows by inversion from the first assumption that $\Gamma, x : B, y : D \vdash P : E$. Therefore, it follows from the induction hypothesis that $\Gamma, y : D \vdash P[N/x] : E$. Therefore, it follows from (Abs) that $\Gamma \vdash \lambda y.\, P[N/x] : D \to E$. By the assumptions on y and definition, $\lambda y.\, P[N/x] = (\lambda y.\, P)[N/x]$.

(e) Prove that $a \to (a \to b) \to b$ is the principal type of $\lambda xy.\, yx$, i.e. that:

- $\vdash \lambda xy.\, yx : a \to (a \to b) \to b$
- and, for any other type A such that $\vdash \lambda xy.\, yx : A$, there is a substitution σ such that $A = (a \to (a \to b) \to b)\sigma$

[5 marks]

Solution: First, observe that $a \to (a \to b) \to b$ is a type of $\lambda xy.\, yx$ because:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : a,\, y : a \to b \vdash y : a \to b} \quad \dfrac{}{x : a,\, y : a \to b \vdash x : a}}{x : a,\, y : a \to b \vdash yx : b}}{x : a \vdash \lambda y.\, yx : (a \to b) \to b}}{\vdash \lambda xy.\, yx : a \to (a \to b) \to b}$$

Next, suppose that A is another type of $\lambda xy.\, yx$. By Inversion, A must have shape $B \to C$ with $x : B \vdash \lambda y.\, yx : C$. By inversion on this judgement, C must have shape $D \to E$ with $x : B, y : D \vdash yx : E$. By inversion on this judgment, there is a type F such that $x : B, y : D \vdash y : F \to E$ and $x : B, y : D \vdash x : F$. By inversion on these final two judgements, we have $D = F \to E$ and $B = F$. Therefore, $\vdash \lambda xy.\, yx : F \to (F \to E) \to E$. We have $(a \to (a \to b) \to b)[F/a, E/b] = F \to (F \to E) \to E$, as required.

(f) Suppose $M =_\beta \lambda x.\, xx$. Prove that M is *not* typable.

[3 marks]

(cont.)

Solution: Suppose for the purpose of obtaining a contradiction that M is typable, i.e. there is a type A such that $\vdash M : A$. Observe that, since $\lambda x.\, xx$ is a β-normal form, it follows from the definition of $=_\beta$ that $M \twoheadrightarrow_\beta \lambda x.\, xx$. By Subject-Reduction, it follows that $\vdash \lambda x.\, xx : A$. However, we know that $\lambda x.\, xx$ is not typable.

(g) Give two terms M and N and a type A such that $M \to_\beta N$ and, additionally, both of the following are true:

- There are no proof trees for $\vdash M : A$
- There are infinitely many proof trees for $\vdash N : A$

[3 marks]

Solution: Take $N = \mathbf{K}\,\mathbf{I}\,\mathbf{I}$ and $M = \mathbf{K}\,N\,\Omega$ and $A = a \to a$. Then, clearly $M \to_\beta N$. M is untypable because it contains Ω as a subterm. On the other hand, there are infinitely many proof trees for $\vdash \mathbf{K}\,\mathbf{I}\,\mathbf{I} : a \to a$ because the following is a proof tree for all types B:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash \mathbf{K} : (a \to a) \to (B \to B) \to a \to a\end{array} \quad \begin{array}{c}\vdots\\ \vdash \mathbf{I} : a \to a\end{array}}{\vdash \mathbf{K}\,\mathbf{I} : (B \to B) \to a \to a} \qquad \begin{array}{c}\vdots\\ \vdash \mathbf{I} : B \to B\end{array}}{\vdash \mathbf{K}\,\mathbf{I}\,\mathbf{I} : a \to a}$$

END OF PAPER