Types and λ-calculus

Problem Sheet 2

* 1. Perform one step of reduction for each of the following terms:

(a) $(\lambda xy.\,x)(\lambda x.\,x)(\lambda z.\,z)$
(b) $(\lambda xyz.\,xz(yz))(\lambda xy.\,x)$
(c) $(\lambda x.\,xx)(\lambda x.\,xx)$
(d) $(\lambda xy.\,x)((\lambda z.\,zz)(\lambda x.\,x))$

Solution

(a) $(\lambda yx.\,x)(\lambda z.\,z)$
(b) $\lambda yz.\,(\lambda xy.\,x)z(yz)$
(c) $(\lambda x.\,xx)(\lambda x.\,xx)$
(d) $\lambda y.\,(\lambda z.\,zz)(\lambda x.\,x)$ or $(\lambda xy.x.\,)((\lambda x.\,x)(\lambda x.\,x))$

* 2. Justify each of the following conversions $M =_\beta N$ by finding a common reduct P, i.e. such that $M \twoheadrightarrow_\beta P$ and $N \twoheadrightarrow_\beta P$.

(a) $(\lambda x.\,x)y =_\beta (\lambda xy.\,x)\ y\ z$
(b) $(\lambda x.\,M)N =_\beta M[N/x]$
(c) $\mathbf{Y}\,M =_\beta M\ (\mathbf{Y}\,M)$
(d) $z\ (\mathbf{K}\,\mathbf{I}\,\Omega)\ \Omega =_\beta z\ \mathbf{I}\ (\mathbf{K}\,\Omega\,\mathbf{I})$
(e) $\Theta\ M =_\beta M\ (\Theta\ M)$

Solution

(a) y
(b) $M[N/x]$

(c) $M(\lambda x. M(xx))(\lambda x. M(xx))$
(d) $z\,\mathbf{I}\,\Omega$
(e) $M\ (\Theta\ M)$

** 3. Choose *one* of the following and try to prove it by induction on n:

- For all $n \in \mathbb{N}$, if $n > 0$ and $P_0 \to_\beta \cdots \to_\beta P_n$ then $MP_0 \to_\beta \cdots \to_\beta MP_n$
- For all $n \in \mathbb{N}$, if $n > 0$ and $P_0 \to_\beta \cdots \to_\beta P_n$ then $P_0M \to_\beta \cdots \to_\beta P_nM$
- For all $n \in \mathbb{N}$, if $n > 0$ and $P_0 \to_\beta \cdots \to_\beta P_n$ then $\lambda x. P_0 \to_\beta \cdots \to_\beta \lambda x. P_n$

Hint: the base case $n = 0$ will be trivial. In the induction step case, when $n = k+1$, at some point you will need to case split on whether or not $k > 0$ because the induction hypothesis will require it. You can do this because $k = 0 \vee k > 0$ is an elementary fact about all natural numbers k, and so you can use it as an assumption in your proofs.

Solution

We show the first by induction on the length of $P \twoheadrightarrow_\beta Q$, the others are similar:

- If $n = 0$ we proceed as follows. Assume $n > 0$ and we immediately obtain a contradiction, from which anything follows.
- If $n = k+1$, we proceed as follows. We are allowed to assume the induction hypothesis:

$$k > 0 \text{ and } P_0 \to_\beta \cdots \to_\beta P_k \text{ implies } MP_0 \to_\beta \cdots \to_\beta MP_k$$

  Now, suppose $k+1 > 0$ and $P_0 \to_\beta \cdots \to_\beta P_k \to_\beta P_{k+1}$ (*). Since $k+1 > 0$ either $k = 0$ or $k > 0$. We proceed by cases:

  - If $k = 0$ then $P_0 \to_\beta \cdots \to_\beta P_k \to_\beta P_{k+1}$ is just $P_0 \to_\beta P_{k+1}$ with no terms in between. It follows from rule (AppL) that, therefore $MP_0 \to_\beta MP_{k+1}$, which was the goal.
  - If $k > 0$ then we can use the induction hypothesis to obtain the sequence $MP_0 \to_\beta \cdots \to_\beta MP_k$ corresponding to the first k steps of (*). The remaining step is $P_k \to_\beta P_{k+1}$, and it follows from (AppL) that $MP_k \to_\beta MP_{k+1}$. We can glue these two sequences together to obtain the desired goal:

$$MP_0 \to_\beta \cdots \to_\beta MP_k \to_\beta MP_{k+1}$$

** 4. Use the property from the previous question to prove the corresponding statement, called the *compatibility of reduction*:

- If $P \twoheadrightarrow_\beta Q$ then $MP \twoheadrightarrow_\beta MQ$
- If $P \twoheadrightarrow_\beta Q$ then $PM \twoheadrightarrow_\beta QM$
- If $P \twoheadrightarrow_\beta Q$ then $\lambda x.P \twoheadrightarrow_\beta \lambda x.Q$

Since the property in the previous question only applies when $n > 0$, you will likely have to case split on n at some point during the proof.

Solution

Again, we do only the first because the others are similar:

Suppose $P \twoheadrightarrow_\beta Q$. By definition, this means that there is some (possibly singleton) sequence of reducts $P_0 \rightarrow_\beta \cdots \rightarrow_\beta P_n$ with $P = P_0$ and $Q = P_n$. Either $n = 0$ or $n > 0$ and we proceed by cases:

- If $n = 0$ then $P = Q$ and so $MP = MQ$ too. Therefore $MP \twoheadrightarrow_\beta MQ$ by a zero-step reduction sequence.
- If $n > 0$ then the result from question (4) is applicable and so we obtain a corresponding sequence $MP_0 \rightarrow_\beta \cdots \rightarrow_\beta MP_n$ and hence, by definition of reduction, $MP_0 \twoheadrightarrow_\beta MP_n$, but this is just $MP \twoheadrightarrow_\beta MQ$.

** 5. Choose one of the following statements, collectively called *the compatibility of conversion*, and prove it.

- if $P =_\beta Q$ then $MP =_\beta MQ$
- if $P =_\beta Q$ then $PM =_\beta QM$
- if $P =_\beta Q$ then $\lambda x.P =_\beta \lambda x.Q$

Hint: using the compatibility of reduction will be essential.

Solution

Again we only do the first part.

Suppse $P =_\beta Q$. By definition, this means that there is a common reduct R, i.e. $P \twoheadrightarrow_\beta R$ and $Q \twoheadrightarrow_\beta R$. By the compatibility of reduction, $MP \twoheadrightarrow_\beta MR$ and

$MQ \twoheadrightarrow_\beta MR$. Therefore MP and MQ have a common reduct too, so $MP =_\beta MQ$, which was the goal.

** 6. Prove *all three* of the following. In part (c), look for a place to use confluence.

(a) (Reflexivity) $M =_\beta M$
(b) (Symmetry) if $M =_\beta N$ then $N =_\beta M$
(c) (Transitivity) if $M =_\beta N$ and $N =_\beta P$ then $M =_\beta P$.

These three together give that $=_\beta$ is an equivalence relation. The fact that β-conversion is also compatible means that it is a congruence, and so *equational reasoning* makes sense (i.e. constitutes a valid proof technique).

Using equational reasoning, we can string together a sequence of β-conversions $M_1 =_\beta M_2 =_\beta \cdots =_\beta M_{k-1} =_\beta M_k$ in which each consecutive pair is justified because they only differ in some subterms P and Q which are themselves β-convertible. For example, $\lambda x.\, x(\mathbf{I}\ x)xx =_\beta \lambda x.\, xx(\mathbf{K}x\mathbf{I})(\mathbf{I}x)$ because:

$$\begin{array}{rcll} \lambda x.\, x(\mathbf{I}\ x)xx & =_\beta & \lambda x.\, xxxx & (\text{since } \mathbf{I}\ x =_\beta x) \\ & =_\beta & \lambda x.\, xx(\mathbf{K}x\mathbf{I})x & (\text{since } \mathbf{K}x\mathbf{I} =_\beta x) \\ & =_\beta & \lambda x.\, xx(\mathbf{K}x\mathbf{I})(\mathbf{I}x) & (\text{since } \mathbf{I}x =_\beta x) \end{array}$$

At each line, the compatibility results allow us to promote an equation that holds between subterms to an equation between the toplevel terms, symmetry allows us to use the equations in whichever direction we like, and transitivity allows us to conclude that any two terms in the chain are therefore convertible.

Solution

(a) According to the definition of $=_\beta$, we just need to find a term P such that $M \twoheadrightarrow_\beta P$. We can take $P := M$, since $M \twoheadrightarrow_\beta M$.

(b) Suppose $M =_\beta N$. Then, by definition, M and N have a common reduct P, i.e. $M \twoheadrightarrow_\beta P$ and $N \twoheadrightarrow_\beta P$. Therefore, we also have that $N =_\beta M$, which by definition requires the same thing.

(c) Suppose $M =_\beta N$ and $N =_\beta P$. Then, by definition, M and N have a common reduct Q. Similarly, N and P have a common reduct R. We can draw the situation as follows:

$$\begin{tikzcd}
M \arrow[dr, two heads] & =_\beta & N \arrow[dl, two heads] \arrow[dr, two heads] & =_\beta & P \arrow[dl, two heads] \\
& Q & & R &
\end{tikzcd}$$

But then, by confluence, since $N \twoheadrightarrow_\beta Q$ and $N \twoheadrightarrow_\beta R$ it must be that Q and R have a common reduct U, so:

$$\begin{tikzcd}
M \arrow[dr, two heads] & =_\beta & N \arrow[dl, two heads] \arrow[dr, two heads] & =_\beta & P \arrow[dl, two heads] \\
& Q \arrow[dr, two heads] & & R \arrow[dl, two heads] & \\
& & U & &
\end{tikzcd}$$

So U is a common reduct of M and P and hence, by definition, $M =_\beta P$.