TYPES AND λ-CALCULUS

Problem Sheet 2

Questions 1, 2 6 and 7 will be marked.

* 1. Write these terms using the minimum number of parentheses and λ, according to our conventions.

(a) $(\lambda y.((y\, y)(z\, z)))$

(b) $(\lambda y.(((y\, y)\, y)\, y))$

(c) $((\mathsf{S}\,\mathsf{Z})(\lambda y.(\lambda z.(z\,(\mathsf{S}\,y)))))$

* 2. Write the term $(\lambda xyz.xy(xz))(\lambda xy.x)$ with all the parentheses and λ, that we will usually omit, tediously put back in.

* 3. List the all the subterms of the following terms (don't bother listing the same subterm more than once even it occurs several times):

(a) The 3 distinct subterms of $\lambda x.xx$

(b) The 6 distinct subterms of $(\lambda x.xx)(\lambda y.y)$

(c) The 8 distinct subterms of $\lambda xyz.xy(yx)$

(d) The 13 distinct subterms of $\mathsf{fix}\ (\lambda xy.\,\mathsf{ifz}\ y\ y\ (x\ (\mathsf{pred}\ y)))$

* 4. Each of the following has two free variables, what are they in each case?

(a) $\lambda xy.\lambda u.uvxyz$

(b) $\lambda xy.z(\lambda u.uvxy)$

(c) $\lambda wx.z(\lambda u.uvwx)$

(d) $\lambda vw.z(\lambda z.uvvw)$

(e) $\lambda yx.z(\lambda u.uwyx)$

* 5. Which of the following pairs of strings are α-equivalent (and therefore represent the same term):

(a) $\lambda x.xy$ and $\lambda z.zy$
(b) $\lambda x.xy$ and $\lambda z.zx$
(c) $\mathsf{ifz}\ x\ (\mathsf{S}\ x)\ (\mathsf{pred}\ x)$ and $\mathsf{ifz}\ y\ (\mathsf{S}\ y)\ (\mathsf{pred}\ y)$
(d) $\lambda xy.xy$ and $\lambda xy.yx$
(e) $\mathsf{fix}\ (\lambda x.(\lambda y.xy)\ (\mathsf{S}\ x))$ and $\mathsf{fix}\ (\lambda y.(\lambda x.yx)\ (\mathsf{S}\ y))$

* 6. Perform the following substitutions:

(a) $(\mathsf{ifz}\ x\ (\mathsf{S}\ x)\ \mathsf{Z})[\underline{2}/x]$
(b) $\underline{2}[\underline{1}/x]$
(c) $(\lambda x.(\lambda y.xz)z)[(\lambda z.z)/z]$
(d) $(\lambda x.yx)[yz/x]$
(e) $(\lambda x.yz)[yy/z]$
(f) $(\lambda y.xy)[yx/x]$

** 7. Complete the following partial proof of the following statement, by filling out (a), (b) (c) and (d). This proof is a particularly good test of how carefully you are keeping track of the proof state.

Suppose P and Q are terms, x and y are variables. If $x \neq y$ and $x \notin \mathsf{FV}(Q)$ then for all terms M:

$$M[P/x][Q/y] = M[Q/y][P[Q/y]/x]$$

(which tells us how to reorder substitutions in general).

Proof. Suppose P and Q are terms. Suppose x and y are variables. The rest of the proof is by induction on $M \in \Lambda$.

- In case M is some variable z, we argue as follows. Assume $x \neq y$ and $x \neq \mathsf{FV}(Q)$. Our goal is to show:

$$z[P/x][Q/y] = z[Q/y][P[Q/y]/x]$$

  Now, either $z = x$, $z = y$ or z is neither x nor y. We proceed by a case analysis on this fact:

- Suppose $z = x$. By our assumption, it follows that $z \neq y$. Then, by definition of substitution, $z[P/x][Q/y] = P[Q/y]$ and

  $$z[Q/y][P[Q/y]/x] = z[P[Q/y]/x] = P[Q/y]$$

  Hence, the desired equality follows.
- Suppose $z = y$. (a)
- Suppose $z \neq x$ and $z \neq y$. Then $z[P/x][Q/y] = z$ on the left side of the goal and also $z[Q/y][P[Q/y]/x] = z$ on the right side, so the result follows.

* (b)

* In case M is some application $N_1 N_2$ we argue as follows. Assume $x \neq y$ and $x \neq \mathsf{FV}(Q)$. Additionally, assume the induction hypothesis:

  (IH1) if $x \neq y$ and $x \neq \mathsf{FV}(Q)$ then $N_1[P/x][Q/y] = N_1[Q/y][P[Q/y]/x]$

  (IH2) if $x \neq y$ and $x \neq \mathsf{FV}(Q)$ then $N_2[P/x][Q/y] = N_2[Q/y][P[Q/y]/x]$

  Our goal is to show that:

  $$(N_1 N_2)[P/x][Q/y] = (N_1 N_2)[Q/y][P[Q/y]/x]$$

  (c)

* In case M is of shape $\lambda z.N$, we argue as follows. We can assume by the bound variable convention that z is different from any variable in scope, so $z \neq x$ and $z \neq y$ and $z \notin \mathsf{FV}(P)$ and $z \notin \mathsf{FV}(Q)$. (d)

$\square$