UNIVERSITY OF BRISTOL

August/September 2019 Examination Period

FACULTY OF ENGINEERING

Third Year Examination for the Degrees
of
Bachelor of Science
Master of Engineering

COMS30009R
Types and Lambda Calculus

TIME ALLOWED:
2 Hours

# Answers to COMS30009R: Types and Lambda Calculus

Intended Learning Outcomes:

**Q1**. (a) State the grammar of the syntax of PCF.

**Solution:** With or without explicit parentheses is ok.

$$\begin{array}{rrcl} \text{(Terms)} & M, N & ::= & x \mid c \mid (\lambda x.\, M) \mid (M N) \\ \text{(Constants)} & c & ::= & \mathsf{fix} \mid \mathsf{Z} \mid \mathsf{S} \mid \mathsf{pred} \mid \mathsf{ifz} \mid \mathsf{wrong} \end{array}$$

*[3 marks]*

(b) Write each of the following terms with all $\lambda$ and parentheses made explicit.

i. $\lambda xyz.\, xz(yz)$

ii. $(\lambda xy.\, x)(\lambda x.\, xx)$

iii. $x(\lambda xy.\, x(\lambda z.\, z))y$

*[3 marks]*

**Solution:**

i. $(\lambda x.\, (\lambda y.\, (\lambda z.\, ((xz)(yz)))))$

ii. $((\lambda x.\, (\lambda y.\, x))(\lambda x.\, (xx)))$

iii. $((x(\lambda x.\, (\lambda y.\, (x(\lambda z.\, z)))))y)$

(c) For each of the following, give an example of a *closed* term $M$ that satisfies the equation.

i. $M\,(\lambda x.\, xz) \;\approx\; zz.$

ii. $M \;\approx\; M\,M$

iii. $\lambda x.\, M \;\approx\; M$

iv. $M\,x \;\approx\; x\,M\,x\,M\,x$

*[4 marks]*

**Solution:**

i. $\lambda x.\, xx$

ii. $\lambda x.\, x$

iii. $\mathsf{fix}\,(\lambda yx.\, y)$

iv. $\mathsf{fix}\,(\lambda yx.\, xyxyx)$

(d) Prove, by induction on $M$, that: if $M[P/x] \neq M[Q/x]$ then $x \in \mathsf{FV}(M)$.

*[6 marks]*

**Solution:** The proof is by induction on $M$.

- In case $M$ is a variable $y$, assume $M[P/x] \neq M[Q/x]$. There are two subcases. If $x \neq y$ then $M[P/x] = y = M[Q/y]$, contradicting the hypothesis. Otherwise, $M = x$ so $x \in \mathsf{FV}(M)$.

- In case $M$ is a constant $c$, assume $M[P/x] \neq M[Q/x]$. However, this is impossible since $M[P/x] = c = M[Q/x]$ by definition. Hence, we have a contradiction and the conclusion follows vacuously.

- In case $M$ has shape $UV$. Assume the induction hypotheses:

(IH1) $U[P/x] \neq U[Q/x]$ implies $x \in \mathsf{FV}(U)$

(IH2) $V[P/x] \neq V[Q/x]$ implies $x \in \mathsf{FV}(V)$

By definition, $U[P/x]V[P/x] \neq U[Q/x]V[Q/x]$, so either $U[P/x] \neq U[Q/x]$ or $V[P/x] \neq V[Q/x]$. In the first case the induction hypotheses yields $x \in \mathsf{FV}(U)$ and hence $x \in \mathsf{FV}(UV)$. In the second case the induction hypothesis yields $x \in \mathsf{FV}(V)$ and hence $x \in \mathsf{FV}(UV)$.

- In case $M$ has shape $\lambda y.\, N$, by the variable convention, we can assume that $x \neq y$. The hypothesis on the rule is $N \in \Lambda$, so the induction hypothesis states that $N[P/x] \neq N[Q/x]$ implies $x \in \mathsf{FV}(N)$. Suppose $M[P/x] \neq M[Q/x]$. By definition, therefore $\lambda y.\, N[P/x] \neq \lambda y.\, N[Q/x]$. Therefore, $N[P/x] \neq N[Q/x]$ and $x \in \mathsf{FV}(N)$ follows from the induction hypothesis. Hence, by definition (and since $x \neq y$), $x \in \mathsf{FV}(\lambda y.\, N)$.

(e) Prove that there cannot be a term $M$ with the property, for all terms $N$ and $P$:

$$M\,N\,P \;\approx\; \begin{cases} \underline{0} & \text{if } N = P \\ \underline{1} & \text{otherwise} \end{cases}$$

*[3 marks]*

**Solution:** Suppose for the purposes of obtaining a contradiction that such a term $M$ exists. Then $\underline{1} \;\approx\; M\;(\underline{\mathsf{id}}\;\underline{\mathsf{id}})\;\underline{\mathsf{id}} \;\approx\; M\;\underline{\mathsf{id}}\;\underline{\mathsf{id}} \;\approx\; \underline{0}$. However, $\underline{1} \not\approx \underline{0}$ follows from the definition of $\approx$ since $\underline{0}$ and $\underline{1}$ are distinct normal forms and so cannot have a common reduct.

(f) Let us say that a *pure* term $M$ is *solvable* just if for all terms $P \in \Lambda$, one can find a sequence of terms $N_1, \ldots, N_k$ such that $(\lambda x_1 \ldots x_m.\, M)\, N_1 \cdots N_k \;\approx\; P$, where $\mathsf{FV}(M) = \{x_1, \ldots, x_m\}$. Prove that every pure term with a normal form is solvable.

*[6 marks]*

(cont.)

**Solution:** Suppose $M$ has a normal form. Then it follows that $M$ is solvable. First, observe that $\lambda x_1 \cdots x_m.\, M \;\approx\; \lambda x_1 \cdots x_m.\, N$ for some normal form $N$. Second, observe that every pure term in normal form has shape $\lambda x_{m+1} \cdots x_n.\, y\, P_1 \cdots P_k$ for some variables $x_{m+1}, \ldots, x_n$ and terms $P_1, \ldots, P_k$. Third, observe that since the original term was closed, $y$ is some $x_i$. Then Let $P \in \Lambda$. A sequence of the desired form is the length-$n$ squence $\underline{\mathsf{div}} \cdots \underline{\mathsf{div}}\,(\lambda y_1 \ldots y_k.\, P)\,\underline{\mathsf{div}} \cdots \underline{\mathsf{div}}$ in which the non-$\underline{\mathsf{div}}$ term occurs in the ith position.

**Q2.** (a) State the rules of the type system (the rule names are not important).

*[3 marks]*

**Solution:**

$$\frac{x{:}A \in \Gamma}{\Gamma \vdash x : A}\ (\text{TVar}) \qquad \frac{c{:}A \in \mathbb{C}}{\Gamma \vdash c : A}\ (\text{TCst})$$

$$\frac{\Gamma \vdash M : B \to A \quad \Gamma \vdash N : B}{\Gamma \vdash MN : A}\ (\text{TApp}) \qquad x \notin \mathrm{dom}\,\Gamma\ \frac{\Gamma \cup \{x : B\} \vdash M : A}{\Gamma \vdash \lambda x.\, M : B \to A}\ (\text{TAbs})$$

(b) For each of the following terms, state whether or not it is typable. No justification is necessary.

i. $\lambda x.\, yz$
ii. $\lambda x.\, xx$
iii. $\lambda x.\, x(\lambda y.\, y)x$

*[3 marks]*

**Solution:**

i. Yes

ii. No

iii. No

(c) The subterm relation $M \sqsubseteq N$, "$M$ is a subterm of $N$", holds exactly when one can construct a proof tree/derivation rooted at $M \sqsubseteq N$, using the following rules:

$$\frac{}{M \sqsubseteq M}\ (\text{SubRefl}) \qquad \frac{P \sqsubseteq M}{P \sqsubseteq (\lambda x.\, M)}\ (\text{SubAbs})$$

$$\frac{P \sqsubseteq M}{P \sqsubseteq (MN)}\ (\text{SubAppL}) \qquad \frac{P \sqsubseteq N}{P \sqsubseteq (MN)}\ (\text{SubAppR})$$

Prove, by induction on $M$:

> For all $\Gamma$, $A$: if $\Gamma \vdash M : A$ and $N \sqsubseteq M$, then there is some $\Gamma'$ and $A'$ such that $\Gamma' \vdash N : A'$.

*[6 marks]*

**Solution:** The proof is by induction on $M$.

- In case $M$ is a variable $x$, let $\Gamma$ be an environment and $A$ a type. Suppose $\Gamma \vdash M : A$ and $N \sqsubseteq M$. By inspection, it must be that the derivation of $N \sqsubseteq M$ is concluded by Refl, whence $M = N$. Hence, the conclusion follows trivially.

(cont.)

- In case $M$ is a constant $c$, let $\Gamma$ and $A$ be arbitrary and assume $\Gamma \vdash c : A$ and $N \sqsubseteq c$. Then it must be that the subterm derivation is concluded using Refl, and so $N = c$. Hence, the conclusion follows trivially.

- In case $M$ is an application $PQ$, we assume the induction hypotheses:

(IH1) For all $\Gamma''$, $A''$: if $\Gamma'' \vdash P : A''$ and $N \sqsubseteq P$ then there is some $\Gamma'$ and $A'$ such that $\Gamma' \vdash N : A'$

(IH2) For all $\Gamma''$, $A''$: if $\Gamma'' \vdash Q : A''$ and $N \sqsubseteq Q$ then there is some $\Gamma'$ and $A'$ such that $\Gamma' \vdash N : A'$

Let $\Gamma$ and $A$ be arbitrary and assume $\Gamma \vdash PQ : A$. Since $M$ has shape $PQ$, the derivation of $N \sqsubseteq PQ$ can only be ending in (SubRefl), (SubAppL) or (SubAppR). In the first case, the conclusion is immediate. In either of the other cases, the result follows immediately from the induction hypotheses via inversion.

- In case $M$ is an abstraction $\lambda x.\, P$, assume the induction hypothesis:

  - Forall $\Gamma''$, $A''$: if $\Gamma'' \vdash P : A''$ $N \sqsubseteq P$ then there is some $\Gamma'$ and $A'$ such that $\Gamma' \vdash N : A'$

  Let $\Gamma$ and $A$ be arbitrary and assume $\Gamma \vdash \lambda x.\, P : A$ and $N \sqsubseteq \lambda x.\, P$. Then, by inversion, $A$ is an arrow $B \to C$. By the variable convention, we can assume that $x \notin \mathsf{FV}(\Gamma)$. Due to the shape of $M$, it can only be that the proof of $N \sqsubseteq M$ is concluded by either (SubRefl) or (SubAbs). In the former case, the result is immediate, in the latter case it follows that $N \sqsubseteq P$ and the result follows by IH.

(d) For each of the following, find a closed, pure term that inhabits the type:

i. $(a \to b) \to a \to b$

ii. $(a \to b \to c) \to (b \to a) \to b \to c$

iii. $(a \to c) \to ((c \to c \to c) \to a) \to c$

*[6 marks]*

**Solution:**

i. <u>id</u>

ii. $\lambda xyz.\, x(yz)z$

iii. $\lambda xy.\, x(y(\lambda xy.\, x))$

(e) Suppose $M \approx xM$. Show that $M$ cannot have a $\beta$-normal form.

*[3 marks]*

**Solution:** Suppose for contradiction that $M$ has a normal form $N$. It follows from the definition of convertibility that $M$ and $xM$ have a common reduct, say $P$, and therefore that $P \rhd^* N$. However, it follows then that $N \approx xN$, which means, by the definition of convertibility, that $N$ and $xN$ have a common reduct. However, both $N$ and $xN$ are in normal form.

(f) Recall that the **Church numeral** for the number $n$, abbreviated $\ulcorner n \urcorner$, is:

$$\lambda f x.\ \underbrace{f(\cdots(f}_{n\text{-times}}\, x)\cdots)$$

Define $\mathsf{exp}_k$ as a tower of 2nd-power exponentials of height $k$:

$$\begin{array}{rcl} \mathsf{exp}_1 & = & 2 \\ \mathsf{exp}_{i+1} & = & 2^{\mathsf{exp}_i} \end{array}$$

So, for example, $\mathsf{exp}_3 = 2^{2^2} = 16$. Define a term $M$ such that the $k$-fold application

$$\underbrace{M \cdots M}_{k\text{-times}}$$

is typable and $\beta$-convertible with $\ulcorner \mathsf{exp}_k \urcorner$. Justify your answer.

*[4 marks]*

**Solution:** Define $M = \ulcorner 2 \urcorner$. To see that the application is convertible with $\underline{\mathsf{exp}_k}$, observe that $\ulcorner m \urcorner \ulcorner n \urcorner \approx \ulcorner n^m \urcorner$. To see that the application is typable, define the following sequence of types:

$$\begin{array}{rcl} N_1 & = & (o \to o) \to o \to o \\ N_{i+1} & = & (N_i \to N_i) \to N_i \to N_i \end{array}$$

Then, it follows that $\vdash \underbrace{M \cdots M}_{k\text{-times}} : N_k$.



**END OF PAPER**