UNIVERSITY OF BRISTOL

January 2023 Examination Period

FACULTY OF ENGINEERING

Third Year Examination for the Degrees
of
Bachelor of Science
Master of Engineering

COMS30039J
Types and Lambda Calculus

TIME ALLOWED:
2 Hours

This paper contains *two* questions, answer *both*. Each question is worth 25 marks. The maximum for this paper is *50 marks*. Credit will be given for partially correct answers.

PLEASE WRITE YOUR 7 DIGIT STUDENT NUMBER ON THE FRONT PAGE OF YOUR ANSWERS. YOUR STUDENT NUMBER CAN BE FOUND ON YOUR UCARD.

Other Instructions:

You may bring one page of A4 notes with you, and consult them freely during the examination. You may use any result from the course material, as long as it is labelled clearly.

TURN OVER ONLY WHEN TOLD TO START WRITING

Q1. This question concerns untyped PCF.

(a) For each of the following statements, is it true or false?

i. $(\lambda xy.\, xxy)(\lambda y.\, y)(\lambda z.\, zz) \ \triangleright\ (\lambda y.\, y)(\lambda y.\, y)(\lambda z.\, zz)$

ii. $(\lambda y.\, y)(\lambda y.\, y)(\lambda z.\, zz) \ \triangleright\ (\lambda xy.\, xxy)(\lambda y.\, y)(\lambda z.\, zz)$

iii. $(\lambda xy.\, xxy)(\lambda y.\, y)(\lambda z.\, zz) \ \triangleright^*\ (\lambda y.\, y)(\lambda y.\, y)(\lambda z.\, zz)$

iv. $(\lambda y.\, y)(\lambda y.\, y)(\lambda z.\, zz) \ \triangleright^*\ (\lambda xy.\, xxy)(\lambda y.\, y)(\lambda z.\, zz)$

v. $(\lambda xy.\, xxy)(\lambda y.\, y)(\lambda z.\, zz) \ \approx\ (\lambda y.\, y)(\lambda y.\, y)(\lambda z.\, zz)$

[5 marks]

(b) For each of the following equations, construct a term M that satisfies it for all N:

i. $M\, N \ \approx\ \lambda x.\, x$

ii. $M\, N \ \approx\ N\,(N\, N)$

iii. $M\, N\, M \ \approx\ N$

iv. $M\, N \ \approx\ M\, M$

v. $\underline{\mathsf{id}}\ (\lambda x.\, M\, N) \ \approx\ \underline{\mathsf{const}}\ (\underline{\mathsf{const}}\ N)\ \underline{\mathsf{id}}$

[5 marks]

(c) The factorial function $n!$ is defined by the following equations:

$$0! = 1$$
$$(n+1)! = (n+1) * n!$$

i. Define a term $\underline{\mathsf{fac}}$ that defines the factorial function in PCF. You may recall that there is a term $\underline{\mathsf{mult}}$ satisfying:

$$\underline{\mathsf{mult}}\ \underline{n}\ \underline{m} \ \approx\ \underline{n * m}$$

ii. Prove that your definition works, i.e: for all $n \in \mathbb{N}$, $\underline{\mathsf{fac}}\ \underline{n} \ \approx\ \underline{n!}$.

[6 marks]

(d) Define the size $|M|$ of a term M as the natural number given as follows:

$$|x| = 1$$
$$|c| = 1$$
$$|(P\, Q)| = |P| + |Q|$$
$$|(\lambda x.\, P)| = |P| + 3$$

Show that there is no term $\underline{\mathsf{length}}$ that satisfies the following equation, for all N:

$$\underline{\mathsf{length}}\, N \ \approx\ \underline{|N|}$$

[4 marks]

(e) Prove that there is no term $\underline{\text{halting}}$ such that, for all *closed* terms M:

$$\underline{\text{halting}}\ M \ \approx \begin{cases} \underline{1} & \text{if } M \text{ has a normal form} \\ \underline{0} & \text{otherwise} \end{cases}$$

[5 marks]

Q2. This question is about the type system.

(a) Give a typing derivation for each of the following judgements:

i. $x : \mathsf{Nat} \to \mathsf{Nat}, y : \mathsf{Nat} \to \mathsf{Nat} \vdash x\ (y\ \underline{0}) : \mathsf{Nat}$

ii. $y : \mathsf{Nat} \vdash (\lambda z.\, z)(\mathsf{pred}\ y) : \mathsf{Nat}$

iii. $y : \mathsf{Nat} \to \mathsf{Nat} \vdash (\lambda x.\, x\ y)\ (\lambda z.\, z) : \mathsf{Nat} \to \mathsf{Nat}$

[6 marks]

(b) For each of the following, find a closed term that inhabits the type:

i. $a \to b \to b$

ii. $(a \to a \to c) \to (c \to b) \to a \to b$

iii. $((a \to b \to a) \to c) \to c$

[6 marks]

(c) Prove that $\vdash \lambda xy.\, xyy : C$ implies that C has shape $(A \to A \to B) \to A \to B$ for some types A and B.

[5 marks]

(d) Define the type $A[B/a]$ arising from the substitution of B for all occurrences of type variable a in A as follows:

$$\begin{aligned} a[B/b] &= B \quad \text{if } a = b \\ a[B/b] &= a \quad \text{if } a \neq b \\ \mathsf{Nat}[B/b] &= \mathsf{Nat} \\ (A_1 \to A_2)[B/b] &= A_1[B/b] \to A_2[B/b] \end{aligned}$$

Extend substitution to environments by:

$$\Gamma[B/b] = \{x : A[B/b] \mid x : A \in \Gamma\}$$

Fix a type B and a type variable b. Prove that, for all M, Γ and A:

$\Gamma \vdash M : A$ implies $\Gamma[B/b] \vdash M : A[B/b]$

[5 marks]

(e) Construct a closed term N and a term M with one free variable x, such that:

- There are no types A, B such that $x{:}B \vdash M : A$,
- but the closed term $M[N/x]$ is typable.

[3 marks]

END OF PAPER