UNIVERSITY OF BRISTOL

August/September 2019 Examination Period

FACULTY OF ENGINEERING

Third Year Examination for the Degrees
of
Bachelor of Science
Master of Engineering

COMS30009R
Types and Lambda Calculus

TIME ALLOWED:
2 Hours

This paper contains *two* questions, answer *both*.
Credit will be given for partial or partially correct answers.
The maximum for this paper is *50 marks*.

Other Instructions:

You may use any result that you can recall from the lecture notes, as long as it is labelled clearly in your answer.

YOU MAY START IMMEDIATELY

Q1. (a) State the grammar of the syntax of PCF.

[3 marks]

(b) Write each of the following terms with all λ and parentheses made explicit.

i. $\lambda xyz.\, xz(yz)$

ii. $(\lambda xy.\, x)(\lambda x.\, xx)$

iii. $x(\lambda xy.\, x(\lambda z.\, z))y$

[3 marks]

(c) For each of the following, give an example of a *closed* term M that satisfies the equation.

i. $M\,(\lambda x.\, xz) \;\approx\; zz.$

ii. $M \;\approx\; M\,M$

iii. $\lambda x.\, M \;\approx\; M$

iv. $M\,x \;\approx\; x\,M\,x\,M\,x$

[4 marks]

(d) Prove, by induction on M, that: if $M[P/x] \neq M[Q/x]$ then $x \in \mathsf{FV}(M)$.

[6 marks]

(e) Prove that there cannot be a term M with the property, for all terms N and P:

$$M\,N\,P \;\approx\; \begin{cases} \underline{0} & \text{if } N = P \\ \underline{1} & \text{otherwise} \end{cases}$$

[3 marks]

(f) Let us say that a *pure* term M is *solvable* just if for all terms $P \in \Lambda$, one can find a sequence of terms $N_1, \ldots, N_k$ such that $(\lambda x_1 \ldots x_m.\, M)\, N_1 \cdots N_k \;\approx\; P$, where $\mathsf{FV}(M) = \{x_1, \ldots, x_m\}$. Prove that every pure term with a normal form is solvable.

[6 marks]

Q2. (a) State the rules of the type system (the rule names are not important).

[3 marks]

(b) For each of the following terms, state whether or not it is typable. No justification is necessary.

i. $\lambda x.\, yz$

ii. $\lambda x.\, xx$

iii. $\lambda x.\, x(\lambda y.\, y)x$

[3 marks]

(c) The subterm relation $M \sqsubseteq N$, "M is a subterm of N", holds exactly when one can construct a proof tree/derivation rooted at $M \sqsubseteq N$, using the following rules:

$$\frac{}{M \sqsubseteq M}\,(\text{SubRefl}) \qquad \frac{P \sqsubseteq M}{P \sqsubseteq (\lambda x.\, M)}\,(\text{SubAbs})$$

$$\frac{P \sqsubseteq M}{P \sqsubseteq (MN)}\,(\text{SubAppL}) \qquad \frac{P \sqsubseteq N}{P \sqsubseteq (MN)}\,(\text{SubAppR})$$

Prove, by induction on M:

For all Γ, A: if $\Gamma \vdash M : A$ and $N \sqsubseteq M$, then there is some Γ' and A' such that $\Gamma' \vdash N : A'$.

[6 marks]

(d) For each of the following, find a closed, pure term that inhabits the type:

i. $(a \to b) \to a \to b$

ii. $(a \to b \to c) \to (b \to a) \to b \to c$

iii. $(a \to c) \to ((c \to c \to c) \to a) \to c$

[6 marks]

(e) Suppose $M \approx xM$. Show that M cannot have a β-normal form.

[3 marks]

(f) Recall that the **Church numeral** for the number n, abbreviated $\ulcorner n \urcorner$, is:

$$\lambda f x.\, \underbrace{f(\cdots(f}_{n\text{-times}}\, x)\cdots)$$

Define exp_k as a tower of 2nd-power exponentials of height k:

$$\begin{aligned} \mathsf{exp}_1 &= 2 \\ \mathsf{exp}_{i+1} &= 2^{\mathsf{exp}_i} \end{aligned}$$

So, for example, $\mathsf{exp}_3 = 2^{2^2} = 16$. Define a term M such that the k-fold application

$$\underbrace{M \cdots M}_{k\text{-times}}$$

is typable and β-convertible with $\ulcorner \mathsf{exp}_k \urcorner$. Justify your answer.

[4 marks]

END OF PAPER