

TYPES AND λ -CALCULUS

Problem Sheet 2

Questions 1, 2 6 and 7 will be marked.

- * 1. Write these terms using the minimum number of parentheses and λ , according to our conventions.
- (a) $(\lambda y. ((y\ y)(z\ z)))$
 - (b) $(\lambda y. (((y\ y)\ y)\ y))$
 - (c) $((S\ Z)(\lambda y. (\lambda z. (z\ (S\ y)))))$

Solution _____

- (a) $\lambda y. yy(zz)$
- (b) $\lambda y. yyy y$
- (c) One answer is: $S\ Z\ \lambda yz. z\ (S\ y)$. However, in practice it is common to write this term as $S\ Z\ (\lambda yz. z\ (S\ y))$ because most find this easier to read. I will always write it the second way, i.e. I will always parenthesize λ -abstractions when they occur inside a term.

- * 2. Write the term $(\lambda xyz. xy(xz))(\lambda xy. x)$ with all the parentheses and λ , that we will usually omit, tediously put back in.

Solution _____

$((\lambda x. (\lambda y. (\lambda z. ((xy)(xz))))) (\lambda x. (\lambda y. x)))$

- * 3. List the all the subterms of the following terms (don't bother listing the same subterm more than once even it occurs several times):

- (a) The 3 distinct subterms of $\lambda x. xx$
- (b) The 6 distinct subterms of $(\lambda x. xx)(\lambda y. y)$
- (c) The 8 distinct subterms of $\lambda xyz. xy(yx)$
- (d) The 13 distinct subterms of $\text{fix } (\lambda xy. \text{ifz } y \ y \ (x \ (\text{pred } y)))$

Solution

- (a) $x, xx, \lambda x. xx$
- (b) $x, xx, \lambda x. xx, y, \lambda y. y, (\lambda x. xx)(\lambda y. y)$
- (c) $x, y, xy, yx, xy(yx), \lambda xyz. (xy(yx)), \lambda yz. (xy(yx)), \lambda z. (xy(yx))$
- (d)
 - x
 - y
 - pred
 - $\text{pred } y$
 - $x \ (\text{pred } y)$
 - ifz
 - $\text{ifz } y$
 - $\text{ifz } y \ y$
 - $\text{ifz } y \ y \ (x \ (\text{pred } y))$
 - $\lambda y. \text{ifz } y \ y \ (x \ (\text{pred } y))$
 - $\lambda xy. \text{ifz } y \ y \ (x \ (\text{pred } y))$
 - $\text{fix } (\lambda xy. \text{ifz } y \ y \ (x \ (\text{pred } y)))$

* 4. Each of the following has two free variables, what are they in each case?

- (a) $\lambda xy. \lambda u. uvxyz$
- (b) $\lambda xy. z(\lambda u. uvxy)$
- (c) $\lambda wx. z(\lambda u. uvwx)$
- (d) $\lambda vw. z(\lambda z. uvvw)$
- (e) $\lambda yx. z(\lambda u. uwyx)$

Solution

- (a) v, z
- (b) z, v

- (c) z, v
- (d) z, u
- (e) z, w

* 5. Which of the following pairs of strings are α -equivalent (and therefore represent the same term):

- (a) $\lambda x. xy$ and $\lambda z. zy$
- (b) $\lambda x. xy$ and $\lambda z. zx$
- (c) $\text{ifz } x \text{ (S } x \text{) (pred } x \text{)}$ and $\text{ifz } y \text{ (S } y \text{) (pred } y \text{)}$
- (d) $\lambda xy. xy$ and $\lambda xy. yx$
- (e) $\text{fix } (\lambda x. (\lambda y. xy) \text{ (S } x \text{)})$ and $\text{fix } (\lambda y. (\lambda x. yx) \text{ (S } y \text{)})$

Solution _____

- (a) Yes
- (b) No
- (c) No
- (d) No
- (e) Yes

* 6. Perform the following substitutions:

- (a) $(\text{ifz } x \text{ (S } x \text{) } Z)[\underline{2}/x]$
- (b) $\underline{2}[\underline{1}/x]$
- (c) $(\lambda x. (\lambda y. xz)z)[(\lambda z. z)/z]$
- (d) $(\lambda x. yx)[yz/x]$
- (e) $(\lambda x. yz)[yy/z]$
- (f) $(\lambda y. xy)[yx/x]$

Solution _____

- (a) $\text{ifz } \underline{2} \text{ (S } \underline{2} \text{) } Z$
- (b) $\underline{2}$
- (c) $\lambda x. (\lambda y. x(\lambda z. z))(\lambda z. z)$
- (d) $\lambda x. yx$

(e) $\lambda z. yxz$

- ** 7. Complete the following partial proof of the following statement, by filling out (a), (b) (c) and (d). This proof is a particularly good test of how carefully you are keeping track of the proof state.

Suppose P and Q are terms, x and y are variables. If $x \neq y$ and $x \notin \text{FV}(Q)$ then for all terms M :

$$M[P/x][Q/y] = M[Q/y][P[Q/y]/x]$$

(which tells us how to reorder substitutions in general).

Proof. Suppose P and Q are terms. Suppose x and y are variables. The rest of the proof is by induction on $M \in \Lambda$.

- In case M is some variable z , we argue as follows. Assume $x \neq y$ and $x \notin \text{FV}(Q)$. Our goal is to show:

$$z[P/x][Q/y] = z[Q/y][P[Q/y]/x]$$

Now, either $z = x$, $z = y$ or z is neither x nor y . We proceed by a case analysis on this fact:

- Suppose $z = x$. By our assumption, it follows that $z \neq y$. Then, by definition of substitution, $z[P/x][Q/y] = P[Q/y]$ and

$$z[Q/y][P[Q/y]/x] = z[P[Q/y]/x] = P[Q/y]$$

Hence, the desired equality follows.

- Suppose $z = y$. (a)
- Suppose $z \neq x$ and $z \neq y$. Then $z[P/x][Q/y] = z$ on the left side of the goal and also $z[Q/y][P[Q/y]/x] = z$ on the right side, so the result follows.

- (b)

- In case M is some application N_1N_2 we argue as follows. Assume $x \neq y$ and $x \notin \text{FV}(Q)$. Additionally, assume the induction hypothesis:

(IH1) if $x \neq y$ and $x \notin \text{FV}(Q)$ then $N_1[P/x][Q/y] = N_1[Q/y][P[Q/y]/x]$

(IH2) if $x \neq y$ and $x \notin \text{FV}(Q)$ then $N_2[P/x][Q/y] = N_2[Q/y][P[Q/y]/x]$

Our goal is to show that:

$$(N_1 N_2)[P/x][Q/y] = (N_1 N_2)[Q/y][P[Q/y]/x]$$

(c)

- In case M is of shape $\lambda z.N$, we argue as follows. We can assume by the bound variable convention that z is different from any variable in scope, so $z \neq x$ and $z \neq y$ and $z \notin \text{FV}(P)$ and $z \notin \text{FV}(Q)$. (d)

□

Solution

- (a) By our assumption, it follows that $z \neq x$. Then, by definition of substitution $z[P/x][Q/y] = z[Q/y] = Q$ and also $z[Q/y][P[Q/y]/x] = Q[P[Q/y]/x]$.
- (b) In case M is a constant c , we argue as follows. Assume $x \neq y$ and $x \neq \text{FV}(Q)$. Our goal is to show:

$$c[P/x][Q/y] = c[Q/y][P[Q/y]/x]$$

but, by definition, both left and right hand sides are just c , so the result follows immediately. We assumed $x \neq \text{FV}(Q)$, so $Q[P[Q/y]/x] = Q$. The result follows.

- (c) It follows by the definition of substitution that:

$$(N_1 N_2)[P/x][Q/y] = N_1[P/x][Q/y] N_2[P/x][Q/y]$$

It follows from (IH1) and (IH2) that:

$$N_1[P/x][Q/y] N_2[P/x][Q/y] = N_1[Q/y][P[Q/y]/x] N_2[Q/y][P[Q/y]/x]$$

and this latter term is $(N_1 N_2)[Q/y][P[Q/y]/x]$ by definition of substitution.

- (d) Assume $x \neq y$ and $x \neq \text{FV}(Q)$. Additionally assume the induction hypothesis:

(IH) If $x \neq y$ and $x \neq \text{FV}(Q)$ then $N[P/x][Q/y] = N[Q/y][P[Q/y]/x]$.

Our goal is to show that:

$$(\lambda z.N)[P/x][Q/y] = (\lambda z.N)[Q/y][P[Q/y]/x]$$

Since $z \notin \text{FV}(x y P Q)$, it follows by definition that:

$$(\lambda z.N)[P/x][Q/y] = \lambda z.N[P/x][Q/y]$$

It follows from (IH) that $\lambda z.N[P/x][Q/y] = \lambda z.N[Q/y][P[Q/y]/x]$. Since $z \notin \text{FV}(xyPQ)$, it follows by definition that this latter term is $(\lambda z.N)[Q/y][P[Q/y]]$.