UNIVERSITY OF BRISTOL

January 2019 Examination Period

FACULTY OF ENGINEERING

Third Year Examination for the Degrees
of
Bachelor of Science
Master of Engineering

COMS30009J
Types and Lambda Calculus

TIME ALLOWED:
2 Hours

This paper contains *two* questions, answer *both*.
Credit will be given for partial or partially correct answers.
The maximum for this paper is *50 marks*.

Other Instructions:

You may use any result that you can recall from the lecture notes, as long as it is labelled clearly in your answer.

TURN OVER ONLY WHEN TOLD TO START WRITING

Q1. (a) State the rules defining one-step β-reduction, $M \to_\beta N$, (the names of the rules are not important).

[3 marks]

(b) For each of the following state whether it is true or false (no justification is necessary).

i. $M = N$ implies $M \twoheadrightarrow_\beta N$
ii. $M \to_\beta N$ implies $M \twoheadrightarrow_\beta N$
iii. $M =_\beta N$ implies $M \twoheadrightarrow_\beta N$
iv. $M \twoheadrightarrow_\beta N$ implies $M =_\beta N$

[4 marks]

(c) For each of the following, give an example of a *closed* term M with that property.

i. M is in β-normal form.
ii. M is normalising but *not* strongly normalising.
iii. $M \to_\beta M$
iv. $M \twoheadrightarrow_\beta MM$

[4 marks]

(d) Recall the inductive definition of the subterm relation:

$$\frac{}{M \sqsubseteq M}\,(\mathsf{SubRefl}) \qquad \frac{P \sqsubseteq M}{P \sqsubseteq (\lambda x.\, M)}\,(\mathsf{SubAbs})$$

$$\frac{P \sqsubseteq M}{P \sqsubseteq (MN)}\,(\mathsf{SubAppL}) \qquad \frac{P \sqsubseteq N}{P \sqsubseteq (MN)}\,(\mathsf{SubAppR})$$

Prove, by induction on $M \sqsubseteq N$, that:

If $M \sqsubseteq N$ and M is a redex, then there is some N' such that $N \to_\beta N'$.

[6 marks]

(e) Prove that there cannot be a term M with the property that:

$$M\,(\lambda z.\, z\,(\mathbf{K}\,\mathbf{I}\,\Omega)\,\Omega) =_\beta \ulcorner 0 \urcorner \qquad \text{and} \qquad M\,(\lambda z.\, z\,\mathbf{I}\,(\mathbf{K}\,\Omega\,\mathbf{I})) =_\beta \ulcorner 1 \urcorner$$

[3 marks]

(f) Let M be term. Suppose that the equation $MN =_\beta NMN$ is true for all terms N. Prove that M cannot have a β-normal form, i.e. if $M \twoheadrightarrow_\beta P$ then P is not in β-normal form.

[5 marks]

Q2. (a) State the rules of the type system (the rule names are not important).

[3 marks]

(b) Give an example of a *closed* term *in β-normal form* that is not typable.

[1 mark]

(c) For each of the following terms M, give a type environment Γ and a type A such that $\Gamma \vdash M : A$ (you need not prove it).

i. $(\lambda x.\, yxz)(\lambda z.\, z)$

ii. $(\lambda xy.\, yx)\, x\, z$

[3 marks]

(d) Prove the following by induction on $M \in \Lambda$.
If $\Gamma, x : B \vdash M : C$ and $\Gamma \vdash N : B$ then $\Gamma \vdash M[N/x] : C$

[7 marks]

(e) Prove that $a \to (a \to b) \to b$ is the principal type of $\lambda xy.\, yx$, i.e. that:

- $\vdash \lambda xy.\, yx : a \to (a \to b) \to b$
- and, for any other type A such that $\vdash \lambda xy.\, yx : A$, there is a substitution σ such that $A = (a \to (a \to b) \to b)\sigma$

[5 marks]

(f) Suppose $M =_\beta \lambda x.\, xx$. Prove that M is *not* typable.

[3 marks]

(g) Give two terms M and N and a type A such that $M \to_\beta N$ and, additionally, both of the following are true:

- There are no proof trees for $\vdash M : A$
- There are infinitely many proof trees for $\vdash N : A$

[3 marks]

END OF PAPER