UNIVERSITY OF BRISTOL

January 2023 Examination Period

FACULTY OF ENGINEERING

Third Year Examination for the Degrees
of
Bachelor of Science
Master of Engineering

COMS30039J
Types and Lambda Calculus

TIME ALLOWED:
2 Hours

Answers to COMS30039J: Types and Lambda Calculus

Intended Learning Outcomes:

Q1. This question concerns untyped PCF.

(a) For each of the following statements, is it true or false?

i. $(\lambda xy.\, xxy)(\lambda y.\, y)(\lambda z.\, zz) \;\triangleright\; (\lambda y.\, y)(\lambda y.\, y)(\lambda z.\, zz)$

ii. $(\lambda y.\, y)(\lambda y.\, y)(\lambda z.\, zz) \;\triangleright\; (\lambda xy.\, xxy)(\lambda y.\, y)(\lambda z.\, zz)$

iii. $(\lambda xy.\, xxy)(\lambda y.\, y)(\lambda z.\, zz) \;\triangleright^*\; (\lambda y.\, y)(\lambda y.\, y)(\lambda z.\, zz)$

iv. $(\lambda y.\, y)(\lambda y.\, y)(\lambda z.\, zz) \;\triangleright^*\; (\lambda xy.\, xxy)(\lambda y.\, y)(\lambda z.\, zz)$

v. $(\lambda xy.\, xxy)(\lambda y.\, y)(\lambda z.\, zz) \;\approx\; (\lambda y.\, y)(\lambda y.\, y)(\lambda z.\, zz)$

[5 marks]

Solution:

i. False

ii. False

iii. True

iv. False

v. True

(b) For each of the following equations, construct a term M that satisfies it for all N:

i. $M\, N \;\approx\; \lambda x.\, x$

ii. $M\, N \;\approx\; N\,(N\, N)$

iii. $M\, N\, M \;\approx\; N$

iv. $M\, N \;\approx\; M\, M$

v. $\underline{\mathrm{id}}\;(\lambda x.\, M\, N) \;\approx\; \underline{\mathrm{const}}\;(\underline{\mathrm{const}}\; N)\; \underline{\mathrm{id}}$

[5 marks]

Solution:

i. $\lambda yx.\, x$

ii. $\lambda x.\, x\,(x\, x)$

iii. $\lambda xy.\, x$

iv. $\mathrm{fix}\,(\lambda f x.\, f\, f)$

v. $\underline{\mathrm{id}}$

(c) The factorial function $n!$ is defined by the following equations:

$$0! = 1$$
$$(n+1)! = (n+1) * n!$$

i. Define a term $\underline{\text{fac}}$ that defines the factorial function in PCF. You may recall that there is a term $\underline{\text{mult}}$ satisfying:

$$\underline{\text{mult}}\ \underline{n}\ \underline{m}\ \approx\ \underline{n * m}$$

ii. Prove that your definition works, i.e: for all $n \in \mathbb{N}$, $\underline{\text{fac}}\ \underline{n}\ \approx\ \underline{n!}$.

[6 marks]

Solution:

i. Define $\underline{\text{fac}} = \text{fix}\ (\lambda f\, x.\, \text{ifz}\ x\ \underline{1}\ (\underline{\text{mult}}\ x\ (f\ (\text{pred}\ x))))$

ii. First observe by that, for any M:

$$\underline{\text{fac}}\ M\ \approx\ \text{ifz}\ M\ \underline{1}\ (\underline{\text{mult}}\ M\ (\underline{\text{fac}}\ (\text{pred}\ M)))$$

Then the proof is by induction on $n \in \mathbb{N}$.

- When $n = 0$:

$$\begin{aligned} \underline{\text{fac}}\ \underline{0}\ &\approx\ \text{ifz}\underline{0}\ \underline{1}\ (\underline{\text{mult}}\ \underline{0}\ (\underline{\text{fac}}\ (\text{pred}\ \underline{0}))) \\ &\approx\ \underline{1} \end{aligned}$$

(using the above conversion). As required, since $0! = 1$.

- When $n = k + 1$, we assume the induction hypothesis:

$$\underline{\text{fac}}\ \underline{k}\ \approx\ \underline{k!}$$

$$\begin{aligned} \underline{\text{fac}}\ \underline{k+1}\ &\approx\ \text{ifz}\ \underline{k+1}\ \underline{1}\ (\underline{\text{mult}}\ \underline{k+1}\ (\underline{\text{fac}}\ (\text{pred}\ \underline{k+1}))) && (1) \\ &\approx\ \underline{\text{mult}}\ \underline{k+1}\ (\underline{\text{fac}}\ (\text{pred}\ \underline{k+1})) && (2) \\ &\approx\ \underline{\text{mult}}\ \underline{k+1}\ (\underline{\text{fac}}\ \underline{k}) && (3) \\ &\approx\ \underline{\text{mult}}\ \underline{k+1}\ \underline{k!} && (4) \\ &\approx\ \underline{(k+1) * k!} && (5) \end{aligned}$$

With the penultimate line following from the induction hypothesis. By definition, $\underline{(k+1) * k!} = \underline{(k+1)!}$, as required.

(d) Define the size $|M|$ of a term M as the natural number given as follows:

$$\begin{aligned} |x| &= 1 \\ |c| &= 1 \\ |(P\,Q)| &= |P| + |Q| \\ |(\lambda x.\, P)| &= |P| + 3 \end{aligned}$$

(cont.)

Show that there is no term $\underline{\mathsf{length}}$ that satisfies the following equation, for all N:

$$\underline{\mathsf{length}}\, N \approx \underline{|N|}$$

[4 marks]

Solution: Suppose there is such a term $\underline{\mathsf{length}}$. Then we would have:

$$\underline{9} = \underline{\mathsf{length}}((\lambda x.\, x)\, y) \approx \underline{\mathsf{length}}\, y \approx \underline{1}$$

However, this is impossible since $\underline{9}$ and $\underline{1}$ are distinct beta normal forms.

(e) Prove that there is no term $\underline{\mathsf{halting}}$ such that, for all *closed* terms M:

$$\underline{\mathsf{halting}}\ M \approx \begin{cases} \underline{1} & \text{if } M \text{ has a normal form} \\ \underline{0} & \text{otherwise} \end{cases}$$

[5 marks]

Solution: Suppose there is a term $\underline{\mathsf{halting}}$ that satisfies this specification. Then consider the term $N = \lambda x.\, \mathsf{ifz}\ (\underline{\mathsf{halting}}\ (x\ x))\underline{\mathsf{div}}\ \underline{1}$. We reason equationally:

$$N\ N \approx \mathsf{ifz}\ (\underline{\mathsf{halting}}\ (N\ N))\ \underline{\mathsf{div}}\ \underline{1}$$

By the specification, we have that, for all terms M, either (i) $\underline{\mathsf{halting}}\ M \approx \underline{1}$ or (ii) $\underline{\mathsf{halting}}\ M \approx \underline{0}$. We consider these two cases with $M = N\ N$:

- In case (i), it must be that $N\ N$ has a normal form. However, by continuing the reduction above we obtain $N\ N \approx \underline{\mathsf{div}}$. This implies that $N\ N$ does not have a normal form! (If it did, then $\underline{\mathsf{div}}$ would have the same normal form by confluence). Hence, we obtain a contradiction.
- In case (ii), it must be that $N\ N$ does not have a normal form. However, by continuing the reduction above we obtain $N\ N \approx \underline{1}$, so $N\ N$ does have a normal form by confluence. Hence, we obtain a contradiction.

Q2. This question is about the type system.

(a) Give a typing derivation for each of the following judgements:

i. $x : \mathsf{Nat} \to \mathsf{Nat}, y : \mathsf{Nat} \to \mathsf{Nat} \vdash x\ (y\ \underline{0}) : \mathsf{Nat}$

ii. $y : \mathsf{Nat} \vdash (\lambda z.\, z)(\mathsf{pred}\ y) : \mathsf{Nat}$

iii. $y : \mathsf{Nat} \to \mathsf{Nat} \vdash (\lambda x.\, x\ y)\ (\lambda z.\, z) : \mathsf{Nat} \to \mathsf{Nat}$

[6 marks]

Solution:

i. With $\Gamma = \{x : a \to b, y : c \to a, z : c\}$:

$$\dfrac{\dfrac{}{\Gamma \vdash x : a \to b} \qquad \dfrac{\dfrac{}{\Gamma \vdash y : c \to a} \quad \dfrac{}{\Gamma \vdash z : c}}{\Gamma \vdash yz : a}}{\Gamma \vdash x(yz) : b}$$

ii. With $\Gamma = \{x : a \to b, y : a\}$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma, z : b \vdash z : b}}{\Gamma \vdash \lambda z.\, z : b \to b} \qquad \dfrac{\dfrac{}{\Gamma \vdash x : a \to b} \quad \dfrac{}{\Gamma \vdash y : a}}{\Gamma \vdash xy : b}}{\Gamma \vdash (\lambda z.\, z)(xy) : b}$$

iii.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y : a, x : a \to a \vdash x : a \to a} \quad \dfrac{}{y : a, x : a \to a \vdash y : a}}{y : a, x : a \to a \vdash xy : a}}{y : a \vdash \lambda x.\, xy : (a \to a) \to a} \qquad \dfrac{\dfrac{}{y : a, z : a \vdash z : a}}{y : a \vdash \lambda z.\, z : a \to a}}{y : a \vdash (\lambda x.\, xy)(\lambda z.\, z) : a}$$

(b) For each of the following, find a closed term that inhabits the type:

i. $a \to b \to b$

ii. $(a \to a \to c) \to (c \to b) \to a \to b$

iii. $((a \to b \to a) \to c) \to c$

[6 marks]

Solution:

i. $\lambda xy.\, y$

ii. $\lambda xyz.\, y(xzz)$

iii. $\lambda x.\, x(\lambda yz.\, y)$

(c) Prove that $\vdash \lambda xy.\, xyy : C$ implies that C has shape $(A \to A \to B) \to A \to B$ for some types A and B.

[5 marks]

Solution: By inversion twice we have that, necessarily:

$$C = D \to E \quad \text{and} \quad E = F \to G$$

and $x : D, y : F \vdash xyy : G$. By inversion, it follows that there is a type H and a type I such that $x : D, y : F \vdash xy : H \to G$, $x : D, y : F \vdash y : H$. By inversion,

(cont.)

$x : D, y : F \vdash x : I \to H \to G$ and $x : D, y : F \vdash y : I$. Finally, by inversion:

$$F = H \quad \text{and} \quad F = I \quad \text{and} \quad D = I \to H \to G$$

Hence: $C = (F \to F \to G) \to F \to G$, as required.

(d) Define the type $A[B/a]$ arising from the substitution of B for all occurrences of type variable a in A as follows:

$$\begin{aligned} a[B/b] &= B \quad \text{if } a = b \\ a[B/b] &= a \quad \text{if } a \neq b \\ \mathsf{Nat}[B/b] &= \mathsf{Nat} \\ (A_1 \to A_2)[B/b] &= A_1[B/b] \to A_2[B/b] \end{aligned}$$

Extend substitution to environments by:

$$\Gamma[B/b] = \{x : A[B/b] \mid x : A \in \Gamma\}$$

Fix a type B and a type variable b. Prove that, for all M, Γ and A:

$\Gamma \vdash M : A$ implies $\Gamma[B/b] \vdash M : A[B/b]$

[5 marks]

Solution: The proof is by induction on M.

- When M is a variable x, we reason as follows. Let Γ be an environment and A a type. Suppose $\Gamma \vdash x : A$. Then, by inversion, there is $x : A \in \Gamma$. So, by definition, there is $x : A[B/b] \in \Gamma[B/b]$. Hence, $\Gamma[B/b] \vdash x : A[B/b]$ follows by (TVar).

- When M is an application PQ, we reason as follows. Assume the induction hypotheses. Let Γ be an environment and A a type. Suppose $\Gamma \vdash PQ : A$. Then, by inversion, there is a type C such that $\Gamma \vdash P : C \to A$ and $\Gamma \vdash Q : C$. It follows from the induction hypotheses that $\Gamma[B/b] \vdash P : (C \to A)[B/b]$ and $\Gamma[B/b] \vdash Q : C[B/b]$. By definition, $(C \to A)[B/b] = C[B/b] \to A[B/b]$ and so it follows from (TApp) that $\Gamma[B/b] \vdash P\,Q : A[B/b]$.

- When M is an abstraction $\lambda x.\, P$, we reason as follows. Assume the induction hypothesis. Let Γ be an environment and A a type. Suppose $\Gamma \vdash \lambda x.\, P : A$. By inversion, it follows that A is of shape $A_1 \to A_2$ for some A_1 and A_2, and $\Gamma,\, x : A_1 \vdash P : A_2$. Then, it follows from the induction hypothesis that $(\Gamma,\, x : A_1)[B/b] \vdash P : A[B/b]$. By definition $(\Gamma,\, x : A_1)[B/b] = \Gamma[B/b],\, x : A_1[B/b]$ and so it follows from (TAbs) that $\Gamma[B/b] \vdash \lambda x.\, P : A_1[B/b] \to A_2[B/b]$.

(e) Construct a closed term N and a term M with one free variable x, such that:

- There are no types A, B such that $x{:}B \vdash M : A$,
- but the closed term $M[N/x]$ is typable.

[3 marks]

Solution: Take, for example, $M = \lambda y.\, y\,(x\,\underline{\mathrm{id}})\,(x\,\underline{3})$. It can be shown by inversion that this term is not typable, because it would require x to have both function type and natural type. However, $\vdash M[\underline{\mathrm{id}}/x] : ((a \to a) \to \mathsf{Nat} \to b) \to b$.

END OF PAPER