TYPES AND λ-CALCULUS

Problem Sheet 4

* 1. Put in all the implicit parentheses required by the official syntax of types for the following examples:

(a) $a \to b \to a$
(b) $\forall bc.\, (b \to c) \to c$
(c) $\forall ab.\, (c \to c \to d) \to a \to b$
(d) $\forall a.\, (a \to b) \to a \to b$

The following is the induction principle for the set $\mathbb{T}$ of monotypes.

> Suppose Φ is a property of monotypes. Then if the following can both be proven:
>
> - For all type variables a, $\Phi(a)$.
> - For all monotypes B and C, if $\Phi(B)$ and $\Phi(C)$ then $\Phi(B \to C)$.
>
> It follows that $\forall A \in \mathbb{T}.\ \Phi(A)$.

** 2. Prove, by induction on A, that $A(\sigma_1\sigma_2) = (A\sigma_1)\sigma_2$.

** 3. Prove, by induction on M, that:

$$\text{if } x \in \mathsf{FV}(M) \text{ then } \mathsf{FV}(M[N/x]) = (\mathsf{FV}(M) \setminus \{x\}) \cup \mathsf{FV}(N).$$

Hint: You will want to use Lemma 6.1 of the notes.
Hint: In the application case, consider splitting on whether x is free in the operator only, the operand only, or both.

The following is the induction principle for the set $\to_\beta$ of pairs of terms (a binary relation on terms).

> Suppose $\Phi(M, N)$ is a property of pairs of terms. Then if the following can all be proven:
>
> - For all terms P and Q, $\Phi((\lambda x.P)Q, P[Q/x])$.
> - For all terms P, Q and Q', if $\Phi(Q, Q')$ then $\Phi(PQ, PQ')$.
> - For all terms P, Q and P', if $\Phi(P, P')$ then $\Phi(PQ, P'Q)$.
> - For all terms P and P', if $\Phi(P, P')$ then $\Phi(\lambda x.P, \lambda x.P')$.
>
> It follows that $\forall (M, N) \in \to_\beta . \Phi(M, N)$.

** 4. Prove, by induction on $M \to_\beta N$, that: $M \to_\beta N$ implies $\mathsf{FV}(N) \subseteq \mathsf{FV}(M)$.