Types and λ-calculus

Problem Sheet 4

* 1. Justify each of the following conversions $M \approx N$ by finding a common reduct P, i.e. such that $M \rhd^* P$ and $N \rhd^* P$.

(a) $(\lambda x. x)y \approx (\lambda x y. x)\ y\ z$

(b) $(\lambda x. M)N \approx M[N/x]$

(c) $\mathsf{fix}\ (\underline{\mathsf{const}}\ \underline{1}) \approx M\ (\underline{\mathsf{const}}\ \mathsf{pred}\ 2)$

(d) $z\ (\underline{\mathsf{const}}\ \underline{\mathsf{id}}\ \underline{\mathsf{div}})\ \underline{\mathsf{div}} \approx z\ \underline{\mathsf{id}}\ (\underline{\mathsf{const}}\ \underline{\mathsf{div}}\ \underline{\mathsf{id}})$

Solution

(a) y

(b) $M[N/x]$

(c) $\underline{1}$

(d) $z\ \underline{\mathsf{id}}\ \underline{\mathsf{div}}$

* 2. Define $\underline{\mathsf{Y}} = \lambda f. (\lambda x. f\ (x\,x))(\lambda x. f\ (x\,x))$.

Show that $\underline{\mathsf{Y}}$ is also a fixed point combinator, i.e for all terms M:

$$\underline{\mathsf{Y}} M \approx M (\underline{\mathsf{Y}} M)$$

Solution

On the one hand:

$$\begin{aligned} \underline{\mathsf{Y}}\ M &\rhd (\lambda x. M\ (x\,x))(\lambda x. M\ (x\,x)) \\ &\rhd M\ ((\lambda x. M\ (x\,x))(\lambda x. M\ (x\,x))) \end{aligned}$$

and, on the other:

$$M\,(\underline{\mathrm{Y}}\,M) \rhd M\,((\lambda x.\,M\,(x\,x))(\lambda x.\,M\,(x\,x)))$$

so, they have a common reduct.

** 3. Prove Lemma 7.1 of the notes, i.e. show all of the following:

Reflexivity For all M: $M \approx M$.

Symmetry For all M, N: $M \approx N$ implies $N \approx M$.

Transitivity For all M, N and P: $M \approx P$ and $P \approx N$ implies $M \approx N$.

Compatibility For all M, N and $C[]$: if $M \approx N$ then $C[M] \approx C[N]$.

There is no need for any induction. For compatibility, you will need to use a result from the previous problem sheet.

Solution

We prove each requirement separately:

Reflexivity Let M be a term. Then, there is a 0-step reduction sequence from M to M so, by definition, $M \rhd^* M$. Hence, we can use the definition of convertibility with $P = M$ to obtain $M \approx M$.

Symmetry Let M and N be terms and suppose $M \approx N$. Then, by definition of convertibility, there is a term P such that $M \rhd^* P$ and $N \rhd^* P$. By definition, to show $N \approx M$ we need some common reduct of N and M, so we can use the same witness P again.

Transitivity Let M, N and P be terms and suppose (i) $M \approx P$ and (ii) $P \approx N$. Then, by definition of convertibility there are terms Q_1 and Q_2 such that (a) $M \rhd^* Q_1$, (b) $P \rhd^* Q_1$, (c) $P \rhd^* Q_2$ and (d) $N \rhd^* Q_2$. By confluence applied to (b) and (c), we obtain a common reduct, R, of Q_1 and Q_2. From this, (a) and (d) we obtain that $M \rhd^* Q_1 \rhd^* R$ and $N \rhd^* Q_2 \rhd^* R$ have R as a common reduct and hence, by definition, $M \approx N$.

Compatibility Let M, N be terms and $C[]$ a context. Suppose $M \approx N$, so that, by definition, there is a common reduct of M and N, say P, i.e. $M \rhd^* P$ and $N \rhd^* P$. Then by 6(b) of the previous problem sheet, $C[M] \rhd^* C[P]$ and $C[N] \rhd^* C[P]$. So $C[P]$ is a common reduct of $C[M]$ and $C[N]$. Hence, by definition, $C[M] \approx C[N]$.

* 4. Recall the definition of $\underline{\mathsf{add}}$:

$$\mathsf{fix}\ (\lambda f\,x\,y.\,\mathsf{ifz}\ x\ y\ (\mathsf{S}\ (f\ (\mathsf{pred}\ x)\ y)))$$

Give a complete reduction sequence from $\underline{\mathsf{add}}\ \underline{2}\ \underline{3}$ to $\underline{5}$.

Solution

$$\begin{aligned}
& \underline{\mathsf{add}}\ \underline{2}\ \underline{3} \\
= {} & \mathsf{fix}\ (\lambda f\,x\,y.\,\mathsf{ifz}\ x\ y\ (\mathsf{S}\ (f\ (\mathsf{pred}\ x)\ y)))\ \underline{2}\ \underline{3} \\
\rhd {} & (\lambda f\,x\,y.\,\mathsf{ifz}\ x\ y\ (\mathsf{S}\ (f\ (\mathsf{pred}\ x)\ y)))\ \underline{\mathsf{add}}\ \underline{2}\ \underline{3} \\
\rhd {} & (\lambda x\,y.\,\mathsf{ifz}\ x\ y\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ x)\ y)))\ \underline{2}\ \underline{3} \\
\rhd {} & (\lambda y.\,\mathsf{ifz}\ \underline{2}\ y\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{2})\ y)))\ \underline{3} \\
\rhd {} & \mathsf{ifz}\ \underline{2}\ \underline{3}\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{2})\ \underline{3})) \\
\rhd {} & \mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{2})\ \underline{3}) \\
\rhd {} & \mathsf{S}\ (\underline{\mathsf{add}}\ \underline{1}\ \underline{3}) \\
= {} & \mathsf{S}\ ((\lambda f\,x\,y.\,\mathsf{ifz}\ x\ y\ (\mathsf{S}\ (f\ (\mathsf{pred}\ x)\ y)))\ \underline{\mathsf{add}}\ \underline{1}\ \underline{3}) \\
\rhd {} & \mathsf{S}\ ((\lambda x\,y.\,\mathsf{ifz}\ x\ y\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ x)\ y)))\ \underline{1}\ \underline{3}) \\
\rhd {} & \mathsf{S}\ ((\lambda y.\,\mathsf{ifz}\ \underline{1}\ y\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{1})\ y)))\ \underline{3}) \\
\rhd {} & \mathsf{S}\ (\mathsf{ifz}\ \underline{1}\ \underline{3}\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{1})\ \underline{3}))) \\
\rhd {} & \mathsf{S}\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{1})\ \underline{3})) \\
\rhd {} & \mathsf{S}\ (\mathsf{S}\ (\underline{\mathsf{add}}\ \underline{0}\ \underline{3})) \\
\rhd {} & \mathsf{S}\ (\mathsf{S}\ ((\lambda x\,y.\,\mathsf{ifz}\ x\ y\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ x)\ y)))\ \underline{0}\ \underline{3})) \\
\rhd {} & \mathsf{S}\ (\mathsf{S}\ ((\lambda y.\,\mathsf{ifz}\ \underline{0}\ y\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{0})\ y)))\ \underline{3})) \\
\rhd {} & \mathsf{S}\ (\mathsf{S}\ (\mathsf{ifz}\ \underline{0}\ \underline{3}\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{0})\ \underline{3})))) \\
\rhd {} & \mathsf{S}\ (\mathsf{S}\ (\underline{3})) \\
= {} & \underline{5}
\end{aligned}$$

* 5. Prove that $\underline{\mathsf{add}}$ satisfies the following equations:

$$\underline{\mathsf{add}}\ \underline{0}\ \underline{m} \approx \underline{m} \qquad \text{and} \qquad \underline{\mathsf{add}}\ \underline{(n+1)}\ \underline{m} \approx \mathsf{S}\ (\underline{\mathsf{add}}\ \underline{n}\ \underline{m})$$

<u>Hint:</u> it will save time to first observe that (why?):

$$\underline{\mathsf{add}} \approx \lambda x\,y.\,\mathsf{ifz}\ x\ y\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ x)\ y))$$

In practice, you nearly always want to replace an occurrence of $\underline{\mathsf{add}}$ with the right-hand-side of this equation, rather than by its actual definition (and the same can be said for any recursive function defined using "the recipe").

Solution

The observation is true because the left-hand-side actually reduces to the right-hand-side in two steps. Then the first equation is true since (using the observation):

$$\underline{\mathsf{add}}\ \underline{0}\ \underline{m} \ \approx\ \mathsf{ifz}\ \underline{0}\ \underline{m}\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{0})\ \underline{m})) \ \approx\ \underline{m}$$

The second equation holds since:

$$\begin{aligned}\underline{\mathsf{add}}\ \underline{(n+1)}\ \underline{m} &\approx \mathsf{ifz}\ \underline{(n+1)}\ \underline{m}\ (\mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{(n+1)})\ \underline{m}))\\ &\approx \mathsf{S}\ (\underline{\mathsf{add}}\ (\mathsf{pred}\ \underline{(n+1)})\ \underline{m})\\ &\approx \mathsf{S}\ (\underline{\mathsf{add}}\ \underline{n}\ \underline{m})\end{aligned}$$

** 6. Use fix to define the recursive triangular number function: using the "recipe", give a combinator $\underline{\mathsf{Tri}}$ that satisfies:

$$\underline{\mathsf{Tri}}\ \underline{0} \ \approx\ \underline{0} \qquad \text{and} \qquad \underline{\mathsf{Tri}}\ \underline{(n+1)} \ \approx\ \underline{\mathsf{add}}\ \underline{(n+1)}\ (\underline{\mathsf{Tri}}\ \underline{n})$$

Convince yourself that $\underline{\mathsf{Tri}}\ \underline{2} \ \approx\ \underline{3}$ (this is obvious if you believe that your implementation of $\underline{\mathsf{Tri}}$ really satisfies the given equations).

Solution

Define $\underline{\mathsf{Tri}}$ as $\mathsf{fix}\ (\lambda f\, n.\ \mathsf{ifz}\ n\ n\ (\underline{\mathsf{add}}\ n\ (f\ \ (\mathsf{pred}\ n))))$

** 7. Define multiplication, i.e. construct a term $\underline{\mathsf{mult}}$ that satisfies the following specification:

$$\underline{\mathsf{mult}}\ \underline{0}\ \underline{m} \ \approx\ \underline{0} \qquad \text{and} \qquad \underline{\mathsf{mult}}\ \underline{n+1}\ \underline{m} \ \approx\ \underline{\mathsf{add}}\ \underline{m}\ (\underline{\mathsf{mult}}\ \underline{n}\ \underline{m})$$

Solution

Define $\underline{\mathsf{mult}} = \mathsf{fix}\ (\lambda f\, x\, y.\, \mathsf{ifz}\ x\ \underline{0}\ (\underline{\mathsf{add}}\ y\ (\underline{\mathsf{mult}}\ (\mathbf{pred}\ x)\ y)))$.

** 8. Prove that if $M \approx N$ and N is a normal form, then $M \rhd^* N$.

Therefore, we now know that e.g. $\underline{\mathsf{Tri}}\,\ulcorner 2 \urcorner \rhd^* \ulcorner 3 \urcorner$, so these definitions actually *compute* an output given an input.

Solution

Suppose $M \approx N$ and N is a normal form. It follows from the definition of $\approx$ that there is some common reduct P such that $M \rhd^* P$ and $N \rhd^* P$. Since N is in normal form, $N \rhd^* P$ implies $P = N$. Hence, $M \rhd^* N$.

*** 9. Prove that there is no PCF term $\underline{\mathsf{isNat}}$ that satisfies, for all terms M:

$$\underline{\mathsf{isNat}}\ M \approx \begin{cases} \underline{1} & \text{if } M \text{ is a numeral, i.e. } \underline{n} \text{ for some } n \\ \underline{0} & \text{otherwise} \end{cases}$$

Solution

Suppose there is a term $\underline{\mathsf{isNat}}$ that satisfies the given specification. Then we would have

$$\underline{0} \approx (\lambda x.\, \underline{0})\ \underline{1} \approx (\lambda x.\, \underline{\mathsf{isNat}}\ x)\ \underline{1} \approx \underline{\mathsf{isNat}}\ \underline{1} \approx \underline{1}$$

which is absurd, since $\underline{0}$ and $\underline{1}$ would have a common reduct, but they are distinct normal forms.