Types and λ-calculus

Problem Sheet 7

** 1. Suppose closed term M has a normal form $\lambda x.xx$. Prove that all reducts of M are untypable, i.e. $M \;\rhd^*\; N$ implies N untypable.

Solution

Assume M has normal form $\lambda x.xx$ and suppose $M \;\rhd^*\; N$. Then we have $M \;\rhd^*\; \lambda x.xx$ and $M \;\rhd^*\; N$ so, by Confluence, N and $\lambda x.xx$ have a common reduct. But $\lambda x.xx$ is a normal form, so we must have that $N \;\rhd^*\; \lambda x.xx$. We claim that N is therefore untypable. To see why, suppose that N were typable, then by Subject Reduction, so is $\lambda x.xx$. However, we know that this is impossible.

** 2.

(a) Prove the following result by induction on M:

> Let B be a type and N a term. For all terms M, types A and environments Γ: if $\Gamma, x{:}B \vdash M : A$ and $\Gamma \vdash N : B$ then $\Gamma \vdash M[N/x] : A$.

(b) Prove the missing case in the proof of Lemma 12.2 from the notes: if $\Gamma \vdash (\lambda x.M)N : A$ then $\Gamma \vdash M[N/x] : A$.

Solution

(a) Suppose B is a type and N a term. The rest of the proof is by induction on M.

- When M is a variable y, we proceed as follows. Let A be a type and Γ an environment. Suppose (i) Γ, $x{:}B \vdash y : A$ and (ii) $\Gamma \vdash N : B$. Now we analyse cases on $x = y$?

- If $x = y$ then, by definition of substitution, $M[N/x] = N$ and, by inversion on (i), we obtain that $B = A$. Then our goal is to show $\Gamma \vdash N : A$, which is exactly (ii).
- Otherwise, by definition of substitution, $M[N/x] = M$. Our goal is to show $\Gamma \vdash M : A$. This follows immediately from (i) by Weakening (see the previous problem sheet).

- When M is a constant c, we proceed as follows. Let A be a type and Γ an environment. Suppose (i) $\Gamma, x{:}B \vdash c : A$ and (ii) $\Gamma \vdash N : B$. By definition, $c[N/x] = c$ and so our goal is to show $\Gamma \vdash c : A$, which follows immediately from (i) by Weakening.
- When M is of the form PQ, we assume the induction hypotheses:

  (IH1) For all A', Γ': if $\Gamma', x{:}B \vdash P : A'$ and $\Gamma' \vdash N : B$ then $\Gamma' \vdash P[N/x] : A'$

  (IH2) For all A', Γ': if $\Gamma', x{:}B \vdash Q : A'$ and $\Gamma' \vdash N : B$ then $\Gamma' \vdash Q[N/x] : A'$

  Let A be a type and Γ an environment, then suppose (i) $\Gamma, x{:}B \vdash PQ : A$ and (ii) $\Gamma \vdash N : B$. It follows from inversion on (i) that there is a type C such that:

  (a) $\Gamma, x{:}B \vdash P : C \to A$

  (b) $\Gamma, x{:}B \vdash Q : C$

  Hence, we can obtain from (IH1), with $A' := C \to A$ and $\Gamma' := \Gamma$ that $\Gamma \vdash P[N/x] : C \to A$. From (IH2), with $A' := C$ and $\Gamma' = \Gamma$, we obtain $\Gamma \vdash Q[N/x] : C$. Putting these together with (TApp) we obtain $\Gamma \vdash P[N/x]Q[N/x] : A$, and by definition of substitution, this is just $\Gamma \vdash (PQ)[N/x] : A$, which was our goal.
- When M is of the form $\lambda y.\, P$, we assume the induction hypothesis:

  (IH) For all A' and Γ', if $\Gamma', x{:}B \vdash P : A'$ and $\Gamma' \vdash N : B$ then $\Gamma' \vdash P[N/x] : A$.

  We may also assume, by the variable convention, that y does not occur freely in N, is not a subject in Γ and is distinct from x (otherwise, we rename y in the abstraction). Let A be a type and Γ an environment and suppose (i) $\Gamma, x{:}B \vdash \lambda y.\, P : A$ and (ii) $\Gamma \vdash N : B$. It follows from (i) by inversion that there are types A_1 and A_2 such that $A = A_1 \to A_2$ and (*) $\Gamma, x{:}B, y : A_1 \vdash P : A_2$. Then it follows from this by (IH), with $A' = A_2$ and $\Gamma' = \Gamma \cup \{y : A_1\}$, that $\Gamma, y{:}A_1 \vdash P[N/x] : A_2$. From this we can immediately infer $\Gamma \vdash \lambda y.\, P[N/x] : A_1 \to A_2$ using (TAbs). By definition of substitution (recall we assumed $y \neq x$ and $y \notin \mathsf{FV}(N)$) and the identity of A, this is just our goal $\Gamma \vdash (\lambda y.\, P)[N/x] : A$.

(b) Now suppose $\Gamma \vdash (\lambda x.\, M)N : A$. By inversion, it follows that there is some type B such that $\Gamma \vdash \lambda x.\, M : B \to A$ and $\Gamma \vdash N : B$. By inversion on the former, we deduce that $\Gamma, x : B \vdash M : A$. Then the result follows from the previous part.

** 3. Find pure terms that inhabit the following types (no need for justification):

(a) $(a \to a \to b) \to a \to b$

(b) $(a \to b) \to (a \to c) \to (b \to c \to d) \to a \to d$

(c) $(((a \to a \to b) \to a \to b) \to c) \to c$

Solution

(a) $\lambda xy.\, x\, y\, y$

(b) $\lambda wxyz.\, y\,(x\,z)\,(y\,z)$

(c) $\lambda x.\, x\,(\lambda yz.\, y\,z\,z)$

** 4.

(a) Find a *pure* term that inhabits the type $(((a \to b) \to b) \to b) \to a \to b$.

(b) Give the corresponding proof of the corresponding formula.

Solution

(a) $\lambda xy.\, x(\lambda z.zy)$

(b) Suppose $(a \Rightarrow b) \Rightarrow b) \Rightarrow b$ (x) and assume a (y). We claim that $(a \Rightarrow b) \Rightarrow b$, the proof is as follows: suppose $a \Rightarrow b$ (z), then applying this to assumption (y) we get b. Returning to our original proof, from this and (x) we obtain b, as required.

*** 5. The reason that we don't study full PCF in connection with the Curry-Howard correspondence is the presence of fix.

(a) Use fix to show that every type is inhabited by some *PCF term* (not necessarily pure).

(b) What is the consequence for the Curry-Howard correspondence extended to full PCF?

Solution

(a) Every type A is inhabited by the term fix $(\lambda x.\, x)$. We can construct the following derivation.

$$\dfrac{\dfrac{}{\vdash \mathsf{fix} : (A \to A) \to A}\,(\mathsf{TFix}) \quad \dfrac{\dfrac{}{x : A \vdash x : A}\,(\mathsf{TVar})}{\vdash \lambda x.\, x : A \to A}\,(\mathsf{TAbs})}{\vdash \mathsf{fix}\ (\lambda x.\, x) : A}\,(\mathsf{TApp})$$

(b) Hence, by the Curry-Howard correspondence, this would lead to a proof system in which every formula was provable – i.e. an inconsistent logic!

*** 6. The following property is called Subject Invariance:

if $M \approx N$ and $\Gamma \vdash M : A$ then $\Gamma \vdash N : A$

Is this property true for our type system? Either prove it or give a counterexample.

Solution

It is not true in our system. To see why, take e.g. $\lambda y.\, y \approx (\lambda x.\, (\lambda y.\, y))(\lambda x.\, xx)$. We have $\vdash \lambda y.\, y : a \to a$ so the hypotheses of the implication are satisfied, but $(\lambda x.\, (\lambda y.\, y))(\lambda x.\, xx)$ is untypable because it includes $\lambda x.\, xx$ as a subterm.