

TYPES AND λ -CALCULUS

Problem Sheet 6

Questions 1 and 3(c) will be marked.

- * 1. Give a type derivation/proof tree for the judgements:
- (a) $\vdash (\lambda x. x) \underline{2} : \text{Nat}$
 - (b) $x : \text{Nat}, y : \text{Nat} \vdash \text{ifz } y \ x \ (\text{pred } x) : \text{Nat}$
 - (c) $\vdash \lambda x y. y x x : a \rightarrow (a \rightarrow a \rightarrow b) \rightarrow b$
 - (d) $\vdash \lambda x y z. y(xz) : (a \rightarrow b) \rightarrow (b \rightarrow c) \rightarrow a \rightarrow c$
- ** 2. Give terms M in *normal form* that satisfy each of the following (you are *not* required to justify them with a proof tree, but you may wish to so as to check your answer):
- (a) $\vdash M : (a \rightarrow b) \rightarrow a \rightarrow b$
 - (b) $x : (a \rightarrow a) \rightarrow c \vdash M : c$
 - (c) $\vdash M : a \rightarrow b \rightarrow \text{Nat}$
- ** 3. Use inversion to prove that the following terms are not typable:
- (a) $\underline{1} (\lambda x. x)$
 - (b) $\text{pred } (\lambda x. x)$
 - (c) $\lambda x y. x y(yx)$

- ** 4. The following property is called *Weakening*:

For all Γ, Γ' and A : if $\Gamma \vdash M : A$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash M : A$.

We can prove Weakening by induction on M .

Proof. The proof is by induction on M .

- When M is a variable $x \dots$ (a)
- When M is a constant c , let A be a type, Γ and Γ' be type environments such that $\Gamma \subseteq \Gamma'$ and suppose $\Gamma \vdash c : A$. By inversion, it follows that $c:A \in \mathbb{C}$. Therefore, the side condition is fulfilled to use (TCst) to also justify $\Gamma' \vdash c : A$ (this rule does not place any requirements on the environment).
- When M is an application PQ , assume the induction hypotheses:

(IH1) For all Γ'' and Γ''' and A' , if $\Gamma'' \subseteq \Gamma'''$ and $\Gamma'' \vdash P : A'$ then $\Gamma''' \vdash P : A'$.

(IH2) For all Γ'' and Γ''' and A' , if $\Gamma'' \subseteq \Gamma'''$ and $\Gamma'' \vdash Q : A'$ then $\Gamma''' \vdash Q : A'$.

Let A be a type, Γ and Γ' be environments such that $\Gamma \subseteq \Gamma'$. Then suppose $\Gamma \vdash PQ : A$. By inversion, there must be a type B such that $\Gamma \vdash P : B \rightarrow A$ and $\Gamma \vdash Q : B$. It follows from (IH1) with $\Gamma'' := \Gamma$ and $\Gamma''' := \Gamma'$ and $A' := B \rightarrow A$ that $\Gamma' \vdash P : B \rightarrow A$. It follows from (IH2) with $\Gamma'' := \Gamma$, $\Gamma''' := \Gamma'$ and $A' := B$ that $\Gamma' \vdash Q : B$. Therefore, by (TApp), $\Gamma' \vdash PQ : A$.

- When M is an abstraction $\lambda x.P \dots$ (b)

□

Complete the remaining two cases.

*** 5. Find terms M and N such that:

- (i) M is not typable
- (ii) N is typable
- (iii) $M \triangleright N$