Types and λ-calculus

Problem Sheet 3

This week, questions 2, 8 and 9 will be marked.

Recall that a *closed* term has no free variables.

* 1. Perform one step of reduction for each of the following terms:

(a) $\underline{\mathsf{const}}\,\mathsf{pred}\,\mathsf{pred}$

(b) $\underline{\mathsf{sub}}\,\underline{\mathsf{const}}$

(c) $(\lambda x.\,x\,x)(\lambda x.\,x\,x)$

(d) $\underline{\mathsf{const}}\,(\mathsf{pred}\,\mathsf{pred})$

Solution

(a) $(\lambda y.\,\mathsf{pred})\ \mathsf{pred}$

(b) $\lambda yz.\,\underline{\mathsf{const}}\,z\,(y\,z)$

(c) $(\lambda x.\,x\,x)(\lambda x.\,x\,x)$

(d) $\lambda y.\,\mathsf{pred}\,\mathsf{pred}$ or $\underline{\mathsf{const}}\,\mathsf{wrong}$

* 2. For each of the following reduction steps $M \rhd N$, identify the redex P, the contraction Q, and the context $C[]$ in which the contraction happens, i.e. such that $M = C[P]$ and $N = C[Q]$.

(a) $\lambda x.\,\mathsf{pred}\ (\mathsf{pred}\ \underline{2}) \rhd \lambda x.\,\mathsf{pred}\ \underline{1}$

(b) $\underline{\mathsf{id}}\ (\underline{\mathsf{const}}\ \underline{\mathsf{div}}\ \underline{0}) \rhd \underline{\mathsf{id}}\ (\underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{div}})\ \underline{0})$

(c) $\underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{id}})\ (\mathsf{S}\ x) \rhd (\lambda y.\,\underline{\mathsf{id}}\ \underline{\mathsf{id}})\ (\mathsf{S}\ x)$

Solution

(a) $P = \mathsf{pred}\ \underline{2}$, $Q = \underline{1}$, $C[] = \lambda x.\,\mathsf{pred}\ []$

(b) $P = \underline{\mathsf{div}}$, $Q = \underline{\mathsf{id}}\ \underline{\mathsf{div}}$, $C[\,] = \underline{\mathsf{id}}\ (\underline{\mathsf{const}}\ [\,]\ \underline{0})$

(c) $P = \underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{id}})$, $Q = \lambda y.\,\underline{\mathsf{id}}\ \underline{\mathsf{id}}$, $C[\,] = [\,]\ (\mathsf{S}\ x)$

* 3. Let us define the Booleans as follows:

$$\underline{\mathsf{false}} = \underline{0}$$
$$\underline{\mathsf{true}} = \underline{1}$$

Define Boolean conjunction as a term $\underline{\mathsf{and}}$, disjunction as a term $\underline{\mathsf{or}}$ and negation as a term $\underline{\mathsf{not}}$.

Solution

Define:

$$\begin{aligned}
\underline{\mathsf{not}} &= \lambda x.\,\mathsf{ifz}\,x\,\underline{\mathsf{true}}\,\underline{\mathsf{false}} \\
\underline{\mathsf{and}} &= \lambda x y.\,\mathsf{ifz}\,x\,\underline{\mathsf{false}}\,(\mathsf{ifz}\,y\,\underline{\mathsf{false}}\,\underline{\mathsf{true}}) \\
\underline{\mathsf{or}} &= \lambda x y.\,\underline{\mathsf{not}}\,(\underline{\mathsf{and}}\,(\underline{\mathsf{not}}\,x)\,(\underline{\mathsf{not}}\,y))
\end{aligned}$$

* 4. Define terms $\underline{\mathsf{curry}}$ and $\underline{\mathsf{uncurry}}$ with the following behaviour:

$$\begin{aligned}
\underline{\mathsf{curry}}\,M\,N\,P &\rhd^* M\,(N, P) \\
\underline{\mathsf{uncurry}}\,M\,(N, P) &\rhd^* M\,N\,P
\end{aligned}$$

Solution

$$\begin{aligned}
\underline{\mathsf{curry}} &:= \lambda f\,x\,y.\,f\ (x,\ y) \\
\underline{\mathsf{uncurry}} &:= \lambda f\,p.\,f\ (\underline{\mathsf{proj}}_1^2\,p)\,(\underline{\mathsf{proj}}_2^2\,p)
\end{aligned}$$

* 5. For each of the following specifications, give an example of a *closed* term N *in normal form* that satisfies it (i.e do some reduction):

(a) $\underline{\mathsf{id}}\ \underline{\mathsf{id}} \ \rhd^* \ N$
(b) $\underline{\mathsf{sub}}\ \underline{\mathsf{const}}\ \underline{\mathsf{const}} \ \rhd^* \ N$
(c) $\mathsf{fix}\,(\lambda xy.\,y) \ \rhd^* \ N$
(d) $(\lambda xy.\,yx)\,(\underline{\mathsf{const}}\ \underline{\mathsf{const}})\,(\lambda x.\,xx) \ \rhd^* \ N$

Solution

(a) $\underline{\mathsf{id}}$
(b) $\underline{\mathsf{id}}$
(c) $\underline{\mathsf{id}}$
(d) $\underline{\mathsf{const}}$

* 6. Draw the reduction graph of the term $(\lambda xy.\,yy)\,((\lambda z.zz)\,(\lambda z.zz))\,(\lambda x.\,x)$. (This graph will have 4 vertices). What I mean by this is to draw a directed graph where:

- The nodes are all N s.t. $(\lambda xy.\,yy)\,((\lambda z.zz)\,(\lambda z.zz))\,(\lambda x.\,x) \ \rhd^* \ N$
- There is an edge from node M to node N iff $M \ \rhd \ N$

Solution

$$
\begin{tikzcd}
(\lambda xy.\,yy)((\lambda z.zz)(\lambda z.zz))(\lambda x.\,x) \arrow[loop above] \arrow[d] \\
(\lambda y.\,yy)(\lambda x.\,x) \arrow[d] \\
(\lambda x.\,x)(\lambda x.\,x) \arrow[d] \\
(\lambda x.\,x)
\end{tikzcd}
$$

** 7. Give an example of a *closed* term M for each of the following properties:

(a) M is in normal form.

(b) M has exactly one proper reduct.

(c) M contains strictly fewer redexes than one of its reducts (here we mean "fewer in number", the redexes may be quite different).

(d) A reduct of M contains a redex that did not occur anywhere in M.

Solution

There are lots of possible answers, here is one set:

(a) $\underline{\mathsf{id}}$

(b) $\mathsf{pred}\ \underline{2}$

(c) $(\lambda x.xxx)\ (\underline{\mathsf{id}}\ \underline{1})$

(d) $(\lambda xy.xy)\ \underline{\mathsf{id}}$

** 8. Prove the following statement:

For all M, N and $C[]$: if $M \rhd N$ then $C[M] \rhd C[N]$.

Note that "if $M \rhd N$ then $C[M] \rhd C[N]$" is subtly different from the definition of $\rhd$ which says that $C[P] \rhd C[Q]$ whenever P is a redex and Q the contraction. Here, M and N can be any terms.

You do *not* need to use induction to prove it. You will need to work closely with the definition of $\rhd$: on the one hand you will assume $M \rhd N$ and want to know what you get out of it and, on the other hand, you will want to show $C[M] \rhd C[N]$ and thus need to know what evidence is required to put into it.

Look again at the definition of $\rhd$ using contexts. In the definition, "just if" means the same as "iff", so the definition of $\rhd$ can be seen as a pair of implications: one direction tells you what follows from $M \rhd N$ when you have it as an assumption (forwards reasoning) and the other tells you what you need in order to deduce $M \rhd N$ (backwards reasoning).

Solution

Let M, N be terms and let $C[]$ be an evaluation context. Suppose $M \rhd N$. Then it follows from the definition of $\rhd$ that there must exist an evaluation context $C'[]$ and a redex/contraction pair P, Q such that $M = C'[P]$ and $N = C'[Q]$. So, $M \rhd N$ is really:

$$C'[P] \rhd C'[Q]$$

Now, by these equations, $C[M] = C[C'[P]]$ and $C[N] = C[C'[Q]]$ (*). Our aim is to prove that $C[M] \rhd C[N]$ so, by definition of $\rhd$, we have to find a context $C''[]$ and a redex/contraction pair P', Q' such that $C[M] = C''[P']$ and $C''[Q'] = C[N]$. By (*) we can take $C''[] = C[C'[]]$ and $P' = P$ and $Q' = Q$.

** 9.

(a) Complete the following proof by filling in (a):

For all P, $C[]$, for all $n \in \mathbb{N}$: for all Q, if $P \rhd^n Q$ then $C[P] \rhd^n C[Q]$.

Proof. Let P be a term and $C[]$ a context. We show that, for all $n \in \mathbb{N}$, for all Q, $P \rhd^n Q$ implies $C[P] \rhd^n C[Q]$ by induction on n:

- When $n = 0$, let Q be a term and suppose $P \rhd^0 Q$. Then, by definition, $P = Q$ and hence $C[P] = C[Q]$. By definition, therefore $C[P] \rhd^0 C[Q]$.
- When n is of shape $k+1$, we can assume the induction hypothesis:

  (IH) forall Q, $P \rhd^k Q$ implies $C[P] \rhd^k C[Q]$.

  ... (a) ...

$\square$

(b) Deduce that (i.e. give a short proof of): For all P, Q, $C[\,]$: if $P \rhd^* Q$ then $C[P] \rhd^* C[Q]$.

Solution

(a) Now, let Q be a term and suppose $P \rhd^{k+1} Q$. Then, by definition, there is some U such that (i) $P \rhd^k U$ and (ii) $U \rhd Q$. We can use (IH) on (i) to obtain $C[P] \rhd^k C[U]$ (*). We can use the statement of the previous question on (ii) to obtain $C[U] \rhd C[Q]$ (**). By definition of $\rhd^{k+1}$, we can use (*) and (**) to obtain $C[P] \rhd^{k+1} C[Q]$.

(b) Let P and Q be terms and $C[\,]$ a context. Suppose $P \rhd^* Q$. By definition, there is some n such that $P \rhd^n Q$. It follows from the previous result that, therefore, $C[P] \rhd^n C[Q]$. Hence, by definition of $\rhd^*$, $C[P] \rhd^* C[Q]$.

** 10. Show that there is no term P that satisfies: for all M and N, $P\,(MN) \rhd^* N$. In other words, prove that we cannot write a PCF program that extracts the argument of an application.

Solution

Suppose such a P exists and we look to obtain a contradiction. Then we know that this P satisfies $\forall MN.\, P\,(MN) \rhd^* N$, let's call this (*). Then, instantiating M by $\underline{\mathsf{const}}$ ($\underline{\mathsf{id}}$ $\underline{\mathsf{id}}$) and N by $\underline{\mathsf{const}}$ in (*), we have:

$$P\ (\underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{id}})\ \underline{\mathsf{const}}) \rhd^* \underline{\mathsf{const}}$$

However, by simply performing the reductions, we also have
P ($\underline{\mathsf{const}}$ ($\underline{\mathsf{id}}$ $\underline{\mathsf{id}}$) $\underline{\mathsf{const}}$) $\rhd^*$ P ($\underline{\mathsf{id}}\,\underline{\mathsf{id}}$). By instantiating (*) with $M = \underline{\mathsf{id}}$ and $N = \underline{\mathsf{id}}$ we get P ($\underline{\mathsf{id}}\,\underline{\mathsf{id}}$) $\rhd^*$ $\underline{\mathsf{id}}$. So, overall, we also have:

$$P\ (\underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{id}})\ \underline{\mathsf{const}}) \rhd^* \underline{\mathsf{id}}$$

However, with the other reduction sequence inset above, this contradicts the unique normal forms theorem, since $\underline{\mathsf{id}}$ and $\underline{\mathsf{const}}$ are distinct normal forms of P ($\underline{\mathsf{const}}$ ($\underline{\mathsf{id}}$ $\underline{\mathsf{id}}$) $\underline{\mathsf{const}}$), so we have our contradiction.