UNIVERSITY OF BRISTOL

January 2021 Examination Period

FACULTY OF ENGINEERING

Third Year Examination for the Degrees
of
Bachelor of Science
Master of Engineering

COMS30039J
Types and Lambda Calculus

TIME ALLOWED:
2 Hours

This paper contains *two* questions, answer *both*. Each question is worth 25 marks. The maximum for this paper is *50 marks*. Credit will be given for partially correct answers.

PLEASE WRITE YOUR 7 DIGIT STUDENT NUMBER (NOT CANDIDATE NUMBER) ON THE ANSWER BOOKLET. YOUR STUDENT NUMBER CAN BE FOUND ON YOUR UCARD.

Other Instructions:

You may use the lecture notes to help you, but you may not collaborate.

YOU MAY START IMMEDIATELY

Q1. This question concerns the pure, untyped λ-calculus.

It will be useful to recall the combinators $\underline{\mathsf{id}} = \lambda x.\, x$ and $\underline{\mathsf{const}} = \lambda xy.\, x$

(a) Give the set of free variables for each of the following terms:

i. $\lambda xyz.\, xz(yz)$

ii. $z(\lambda x.\, xy)$

iii. $\lambda z.\, (\lambda x.\, y)(xz)$

iv. $(\lambda x.\, x\,(\lambda y.\, xy))(\lambda x.\, y)$

[4 marks]

(b) For each of the following, give a term M that satisfies the statement.

i. $M\,x \;\approx\; xx$

ii. $M\,\underline{n} \;\approx\; \underline{2 * n}$

iii. $M \;\approx\; MM$

iv. $M\,x \;\approx\; MM$

[8 marks]

(c) Define combinator $\underline{\mathsf{sub}}$ as follows:

$$\underline{\mathsf{sub}} \quad := \quad \mathsf{fix}\,(\lambda f\,mn.\, \mathsf{ifz}\; n\; m\; (f\; (\mathsf{pred}\; m)\; (\mathsf{pred}\; n)))$$

Prove, by induction on n, that $\underline{\mathsf{sub}}$ satisfies:

$$\underline{\mathsf{sub}}\; \underline{m}\; \underline{n} \;\approx\; \begin{cases} \underline{0} & \text{if } m \leq n \\ \underline{m - n} & \text{otherwise} \end{cases}$$

[6 marks]

(d) Prove that there does not exist a term M such that, for all terms N:

$$M\,N \;\approx\; \begin{cases} \underline{\mathsf{id}} & \text{if } N \text{ is in normal form} \\ \underline{\mathsf{const}} & \text{otherwise} \end{cases}$$

[3 marks]

(e) Find a finite sequence of *closed* terms $M_1, M_2, \ldots, M_k$ for $k \geq 0$ such that the following two equations are both satisfied:

$$(\lambda x.\, x\,\underline{\mathsf{id}}\,(x\,\underline{\mathsf{id}}\,\underline{\mathsf{id}}))\; M_1\, M_2 \cdots M_k\; x\, y \quad \approx \quad x$$
$$(\lambda x.\, x\,\underline{\mathsf{id}}\,(x\,x\,\underline{\mathsf{id}}))\; M_1\, M_2 \cdots M_k\; x\, y \quad \approx \quad y$$

[3 marks]

Q2. This question concerns type systems.

(a) Give a typing derivation for each of the following judgements:

i. $\vdash \lambda xy.\, x : a \to b \to a$

ii. $\vdash \lambda x.\, x(\lambda y.\, y) : ((a \to a) \to b) \to b$

iii. $y : c \vdash (\lambda x.\, y)(\lambda xz.\, x) : c$

[6 marks]

(b) For each of the following types, find a closed pure term that inhabits the type:

i. $a \to b \to b$

ii. $(a \to b) \to (a \to b \to c) \to a \to c$

iii. $c \to ((a \to b \to a) \to c \to d) \to d$

[6 marks]

(c) Prove, by induction M, that: for all M, A, Γ, Γ', if $\Gamma \vdash M : A$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash M : A$.

[7 marks]

(d) i. Prove that the only closed normal form of type $a \to b \to a$ is $\lambda xy.\, x$.

ii. Give a closed normal form of type $a \to b \to a$ which is not itself an abstraction.

[7 marks]

END OF PAPER