Types and λ-calculus

Problem Sheet 2

* 1. Write these terms using the minimum number of parentheses and λ, according to our conventions.

(a) $(\lambda y.((y\,y)(z\,z)))$

(b) $(\lambda y.(((y\,y)\,y)\,y))$

(c) $((\mathsf{S}\,\mathsf{Z})(\lambda y.(\lambda z.(z\,(\mathsf{S}\,y)))))$

* 2. Write the term $(\lambda xyz.\,xy(xz))(\lambda xy.\,x)$ with all the parentheses and λ that we will usually omit tediously put back in.

* 3. Perform the following substitutions:

(a) $(\lambda x.(\lambda y.\,xz)\,z)[(\lambda z.\,z)/z]$

(b) $(\lambda x.\,yx)[yz/x]$

(c) $(\lambda y.\,xy)[yx/x]$

* 4. Perform one step of reduction for each of the following terms:

(a) $\underline{\mathsf{const}}\,\mathsf{pred}\,\mathsf{pred}$

(b) $\underline{\mathsf{sub}}\,\underline{\mathsf{const}}$

(c) $(\lambda x.\,x\,x)(\lambda x.\,x\,x)$

(d) $\underline{\mathsf{const}}\,(\mathsf{pred}\,\mathsf{pred})$

* 5. Let us define the Booleans as follows:

$$\underline{\mathsf{false}} = \underline{0}$$
$$\underline{\mathsf{true}} = \underline{1}$$

Define Boolean conjunction as a term $\underline{\mathsf{and}}$, disjunction as a term $\underline{\mathsf{or}}$ and negation as a term $\underline{\mathsf{not}}$.

* 6. Define terms $\underline{\mathsf{curry}}$ and $\underline{\mathsf{uncurry}}$ with the following behaviour:

$$\underline{\mathsf{curry}}\, M\, N\, P \;\triangleright\; \cdots \;\triangleright\; M\,(N\,P)$$
$$\underline{\mathsf{uncurry}}\, M\,(N, P) \;\triangleright\; \cdots \;\triangleright\; M\, N\, P$$

** 7.

(a) For all terms M and N, define a *local definition* term:

$$\underline{\mathsf{let}}\ x = N\ \underline{\mathsf{in}}\ M$$

(M and N will occur inside your answer), with the following behaviour:

$$\underline{\mathsf{let}}\ x = N\ \underline{\mathsf{in}}\ M \;\triangleright\; M[N/x]$$

In other words, $\underline{\mathsf{let}}\ x = N\ \underline{\mathsf{in}}\ M$ behaves like M but where we have defined x locally to be N.

(b) Define a family of local definition forms that are specialised to functions of a certain number of arguments, i.e. a family of terms $\underline{\mathsf{let}}\ f\ x_1 \cdots x_n = N\ \underline{\mathsf{in}}\ M$, that behave like M but where we have defined f to be the function that takes arguments $x_1, \ldots, x_n$ and returns N.

** 8. Complete the following partial proof of the following statement, by filling out (a), (b) (c) and (d). This proof is a particularly good test of how carefully you are keeping track of the proof state.

Suppose P and Q are terms, x and y are variables. If $x \neq y$ and $x \notin \mathsf{FV}(Q)$ then for all terms M:

$$M[P/x][Q/y] = M[Q/y][P[Q/y]/x]$$

(which tells us how to reorder substitutions in general).

Proof. Suppose P and Q are terms. Suppose x and y are variables. The rest of the proof is by induction on $M \in \Lambda$.

- In case M is some variable z, we argue as follows. Assume $x \neq y$ and $x \neq \mathsf{FV}(Q)$. Our goal is to show:

  $$z[P/x][Q/y] = z[Q/y][P[Q/y]/x]$$

  Now, either $z = x$, $z = y$ or z is neither x nor y. We proceed by a case analysis on this fact:

  - Suppose $z = x$. By our assumption, it follows that $z \neq y$. Then, by definition of substitution, $z[P/x][Q/y] = P[Q/y]$ and

    $$z[Q/y][P[Q/y]/x] = z[P[Q/y]/x] = P[Q/y]$$

    Hence, the desired equality follows.
  - Suppose $z = y$. (a)
  - Suppose $z \neq x$ and $z \neq y$. Then $z[P/x][Q/y] = z$ on the left side of the goal and also $z[Q/y][P[Q/y]/x] = z$ on the right side, so the result follows.

- (b)

- In case M is some application $N_1 N_2$ we argue as follows. Assume $x \neq y$ and $x \neq \mathsf{FV}(Q)$. Additionally, assume the induction hypothesis:

  (IH1) if $x \neq y$ and $x \neq \mathsf{FV}(Q)$ then $N_1[P/x][Q/y] = N_1[Q/y][P[Q/y]/x]$

  (IH2) if $x \neq y$ and $x \neq \mathsf{FV}(Q)$ then $N_2[P/x][Q/y] = N_2[Q/y][P[Q/y]/x]$

  Our goal is to show that:

  $$(N_1 N_2)[P/x][Q/y] = (N_1 N_2)[Q/y][P[Q/y]/x]$$

  (c)

- In case M is of shape $\lambda z.N$, we argue as follows. We can assume by the bound variable convention that z is different from any variable in scope, so $z \neq x$ and $z \neq y$ and $z \notin \mathsf{FV}(P)$ and $z \notin \mathsf{FV}(Q)$. (d)

$\square$