

TYPES AND λ -CALCULUS

Problem Sheet 4

This week, Questions 3 and 6 will be marked.

- * 1. Justify each of the following conversions $M \approx N$ by finding a common reduct P , i.e. such that $M \triangleright^* P$ and $N \triangleright^* P$.
- (a) $(\lambda x. x)y \approx (\lambda xy. x) y z$
 - (b) $(\lambda x. M)N \approx M[N/x]$
 - (c) $\text{fix } (\underline{\text{const } 1}) \approx (\lambda x. x \underline{2}) (\underline{\text{const } (\text{pred } 2)})$
 - (d) $z (\underline{\text{const id div}}) \underline{\text{div}} \approx z \underline{\text{id}} (\underline{\text{const div id}})$

- * 2. Define $\underline{Y} = \lambda f. (\lambda x. f (x x))(\lambda x. f (x x))$.

Show that \underline{Y} is also a fixed point combinator, i.e show that, for all terms M :

$$\underline{Y}M \approx M(\underline{Y}M)$$

- ** 3. Prove Lemma 8.1 of the notes, i.e. show all of the following:

Reflexivity For all M : $M \approx M$.

Symmetry For all M, N : $M \approx N$ implies $N \approx M$.

Transitivity For all M, N and P : $M \approx P$ and $P \approx N$ implies $M \approx N$.

Compatibility For all M, N and $C[\]$: if $M \approx N$ then $C[M] \approx C[N]$.

There is no need for any induction. For compatibility, you will need to use 9(b) from the previous problem sheet.

- * 4. Following the recipe, addition add, may be defined as:

$$\text{fix } (\lambda f x y. \text{ifz } x \ y \ (S \ (f \ (\text{pred } x) \ y)))$$

Give a complete reduction sequence from add 2 3 to 5.

- * 5.

- (a) Prove that add satisfies the following equation:

$$\text{add} \approx \lambda x y. \text{ifz } x \ y \ (S \ (\text{add} \ (\text{pred } x) \ y))$$

- (b) Prove that add satisfies the following equations. Induction is not necessary.

$$\text{add } \underline{0} \ \underline{m} \approx \underline{m} \quad \text{and} \quad \text{add } (\underline{n+1}) \ \underline{m} \approx S \ (\text{add } \underline{n} \ \underline{m})$$

Hint: In practice, you nearly always want to replace an occurrence of add with the right-hand-side of the equation in (a), rather than by its actual definition (and the same can be said for any recursive function defined using “the recipe”).

- ** 6. Use fix to define the recursive triangular number function: using the "recipe", give a combinator Tri that satisfies:

$$\text{Tri } \underline{0} \approx \underline{0} \quad \text{and} \quad \text{Tri } (\underline{n+1}) \approx \text{add } (\underline{n+1}) \ (\text{Tri } \underline{n})$$

Convince yourself that Tri 2 \approx 3 (this is obvious if you believe that your implementation of Tri really satisfies the given equations).

- ** 7. Define multiplication, i.e. construct a term mult that satisfies the following specification:

$$\text{mult } \underline{0} \ \underline{m} \approx \underline{0} \quad \text{and} \quad \text{mult } \underline{n+1} \ \underline{m} \approx \text{add } \underline{m} \ (\text{mult } \underline{n} \ \underline{m})$$

Convince yourself that mult 2 2 \approx 4.

** 8. Prove that if $M \approx N$ and N is a normal form, then $M \triangleright^* N$.

Therefore, we now know that e.g. $\underline{\text{Tri}}\ \underline{2} \triangleright^* \underline{3}$ and $\underline{\text{mult}}\ \underline{2}\ \underline{2} \triangleright^* \underline{4}$, so these definitions actually *compute* an output given an input.