Types and λ-calculus

Additional Problem on (Un)Decidability

1. For each of the following, is it decidable? Justify your answer.
   (a) The property of reducing to a numeral, i.e. $\{ M \in \Lambda \mid \exists n \in \mathbb{N}.\ M \rhd^* \underline{n} \}$.
   (b) The property of being normal, i.e. $\{ M \in \Lambda \mid M \text{ is in normal form} \}$.
   (c) The set $\{ M \in \Lambda \mid M \approx \underline{\mathsf{div}} \text{ and } M \text{ normalising} \}$.

Solution

(a) No. This set is nontrivial since it contains $\underline{1}$ and it does not contain $\underline{\mathsf{div}}$. It is behavioural too. To see this, suppose M is in the set and $M \approx N$. Then there is some n such that $N \approx M \approx \underline{n}$. Since $\underline{n}$ is a normal form, in fact $N \rhd^* \underline{n}$. Therefore, N is also in the set. Hence, undecidability follows from the Scott-Curry Theorem.

(b) Yes. The normal forms are terms that do not contain a redex. We can easily write a parser for terms that additionally checks whether or not they contain a redex.

(c) Yes. We claim that there is no term M that is convertible with $\underline{\mathsf{div}}$ and yet normalising. To see, this, suppose for contradiction that there is some term M that is normalising and convertible with $\underline{\mathsf{div}}$. Since M is normalising, it has some normal form N. Therefore, $M \approx N$ and hence $N \approx \underline{\mathsf{div}}$. Since N is a normal form, it must be that $\underline{\mathsf{div}} \rhd^* N$. Therefore, $\underline{\mathsf{div}}$ has a normal form. However, the only reducts of $\underline{\mathsf{div}}$ are $\underline{\mathsf{div}}$ and $\underline{\mathsf{id}}\ \underline{\mathsf{div}}$, neither of which are normal forms: contradiction. Hence, the given set is empty, and membership can be decided by the function that returns 0 on every input.