UNIVERSITY OF BRISTOL

August/September 2019 Examination Period

FACULTY OF ENGINEERING

Third Year Examination for the Degrees
of
Bachelor of Science
Master of Engineering

COMS30009R
Types and Lambda Calculus

TIME ALLOWED:
2 Hours

This paper contains *two* questions, answer *both*.
Credit will be given for partial or partially correct answers.
The maximum for this paper is *50 marks*.

Other Instructions:

You may use any result that you can recall from the lecture notes, as long as it is labelled clearly in your answer.

YOU MAY START IMMEDIATELY

Q1. (a) State the rules defining Λ, the set of λ-terms (the names of the rules are not important).

[3 marks]

(b) Write each of the following terms with all λ and parentheses made explicit.

i. $\lambda xyz.\, xz(yz)$

ii. $(\lambda xy.\, x)(\lambda x.\, xx)$

iii. $x(\lambda xy.\, x(\lambda z.\, z))y$

[3 marks]

(c) For each of the following, give an example of a *closed* term M that satisfies the equation.

i. $M\,(\lambda x.\, xz) =_\beta zz$.

ii. $M =_\beta M\,M$

iii. $\lambda x.\, M =_\beta M$

iv. $M\,x =_\beta x\,M\,x\,M\,x$

[4 marks]

(d) Prove, by induction on M, that: if $M[P/x] \neq M[Q/x]$ then $x \in \mathsf{FV}(M)$.

[6 marks]

(e) Prove that there cannot be a term M with the property, for all terms N and P:

$$M\,N\,P =_\beta \begin{cases} \ulcorner 0 \urcorner & \text{if } N = P \\ \ulcorner 1 \urcorner & \text{otherwise} \end{cases}$$

[3 marks]

(f) Let us say that a term M is *solvable* just if for all terms $P \in \Lambda$, one can find a sequence of terms $N_1, \ldots, N_k$ such that $(\lambda x_1 \ldots x_m.\, M)\,N_1 \cdots N_k =_\beta P$, where $\mathsf{FV}(M) = \{x_1, \ldots, x_m\}$. Is the membership of following set decidable? Either prove it is not or sketch and justify an algorithm.

$$\{M \mid M \text{ does not have a } \beta\text{-normal form or } M \text{ is solvable}\}$$

[6 marks]

Q2. (a) State the rules of the type system (the rule names are not important).

[3 marks]

(b) For each of the following terms, state whether or not it is typable. No justification is necessary.

i. $\lambda x.\, yz$

ii. $\lambda x.\, xx$

iii. $\lambda x.\, x(\lambda y.\, y)x$

[3 marks]

(c) Recall that the subterm relation can be defined inductively by the following rules:

$$\frac{}{M \sqsubseteq M}\,(\text{SubRefl}) \qquad \frac{P \sqsubseteq M}{P \sqsubseteq (\lambda x.\, M)}\,(\text{SubAbs})$$

$$\frac{P \sqsubseteq M}{P \sqsubseteq (MN)}\,(\text{SubAppL}) \qquad \frac{P \sqsubseteq N}{P \sqsubseteq (MN)}\,(\text{SubAppR})$$

Prove, by induction on $\Gamma \vdash M : A$, that, in the type system:

If $\Gamma \vdash M : A$ and $N \sqsubseteq M$, then there is some Γ' and A' such that $\Gamma' \vdash N : A'$.

[6 marks]

(d) We say that a function $f : \mathbb{N} \to \mathbb{N}$ is *linear* just if $f(x) = a * x + b$ for some natural numbers a and b.

Let us say that a function $f : \mathbb{N} \to \mathbb{N}$ is *simply-definable* just if there is some closed term M which is typable and $M \ulcorner n \urcorner =_\beta \ulcorner f(n) \urcorner$.

Prove that every linear function is simply-definable.

[7 marks]

(e) Suppose $M =_\beta xM$. Show that M cannot have a β-normal form.

[3 marks]

(f) Define exp_k as a tower of 2nd-power exponentials of height k:

$$\begin{aligned} \mathsf{exp}_1 &= 2 \\ \mathsf{exp}_{i+1} &= 2^{\mathsf{exp}_i} \end{aligned}$$

So, for example, $\mathsf{exp}_3 = 2^{2^2} = 16$. Define a term M such that the k-fold application

$$\underbrace{M \cdots M}_{k\text{-times}}$$

is typable and β-convertible with $\ulcorner \mathsf{exp}_k \urcorner$. Justify your answer.

[3 marks]

END OF PAPER