UNIVERSITY OF BRISTOL

August/September 2019 Examination Period

FACULTY OF ENGINEERING

Third Year Examination for the Degrees
of
Bachelor of Science
Master of Engineering

COMS30009R
Types and Lambda Calculus

TIME ALLOWED:
2 Hours

Answers to COMS30009R: Types and Lambda Calculus

Intended Learning Outcomes:

Q1. (a) State the rules defining Λ, the set of λ-terms (the names of the rules are not important).

Solution: With or without explicit parentheses is ok.

$$x \in \mathbb{V}\ \frac{}{x \in \Lambda}$$

$$\frac{M \in \Lambda \quad N \in \Lambda}{MN \in \Lambda}$$

$$x \in \mathbb{V}\ \frac{M \in \Lambda}{\lambda x.\, M \in \Lambda}$$

[3 marks]

(b) Write each of the following terms with all λ and parentheses made explicit.

i. $\lambda xyz.\, xz(yz)$

ii. $(\lambda xy.\, x)(\lambda x.\, xx)$

iii. $x(\lambda xy.\, x(\lambda z.\, z))y$

[3 marks]

Solution:

i. $(\lambda x.\, (\lambda y.\, (\lambda z.\, ((xz)(yz)))))$

ii. $((\lambda x.\, (\lambda y.\, x))(\lambda x.\, (xx)))$

iii. $((x(\lambda x.\, (\lambda y.\, (x(\lambda z.\, z)))))y)$

(c) For each of the following, give an example of a *closed* term M that satisfies the equation.

i. $M\,(\lambda x.\, xz) =_\beta zz$.

ii. $M =_\beta M\, M$

iii. $\lambda x.\, M =_\beta M$

iv. $M\, x =_\beta x\, M\, x\, M\, x$

[4 marks]

Solution:

i. $\lambda x.\, xx$

ii. $\lambda x.\, x$

iii. $\mathbf{Y}(\lambda yx.\, y)$

iv. $\mathbf{Y}(\lambda yx.\, xyxyx)$

(d) Prove, by induction on M, that: if $M[P/x] \neq M[Q/x]$ then $x \in \mathsf{FV}(M)$.

[6 marks]

Solution: The proof is by induction on M.

- In case (Var), $M = y$. Assume $M[P/x] \neq M[Q/x]$. There are two sub-cases. If $x \neq y$ then $M[P/x] = y = M[Q/y]$, contradicting the hypothesis. Otherwise, $M = x$ so $x \in \mathsf{FV}(M)$.

- In case (App), M has shape UV. Assume the induction hypotheses:

  (IH1) $U[P/x] \neq U[Q/x]$ implies $x \in \mathsf{FV}(U)$

  (IH2) $V[P/x] \neq V[Q/x]$ implies $x \in \mathsf{FV}(V)$

  By definition, $U[P/x]V[P/x] \neq U[Q/x]V[Q/x]$, so either $U[P/x] \neq U[Q/x]$ or $V[P/x] \neq V[Q/x]$. In the first case the induction hypotheses yields $x \in \mathsf{FV}(U)$ and hence $x \in \mathsf{FV}(UV)$. In the second case the induction hypothesis yields $x \in \mathsf{FV}(V)$ and hence $x \in \mathsf{FV}(UV)$.

- In case (Abs), M has shape $\lambda y.\, N$ and by the variable convention, we can assume that $x \neq y$. The hypothesis on the rule is $N \in \Lambda$, so the induction hypothesis states that $N[P/x] \neq N[Q/x]$ implies $x \in \mathsf{FV}(N)$. Suppose $M[P/x] \neq M[Q/x]$. By definition, therefore $\lambda y.\, N[P/x] \neq \lambda y.\, N[Q/x]$. Therefore, $N[P/x] \neq N[Q/x]$ and $x \in \mathsf{FV}(N)$ follows from the induction hypothesis. Hence, by definition (and since $x \neq y$), $x \in \mathsf{FV}(\lambda y.\, N)$.

(e) Prove that there cannot be a term M with the property, for all terms N and P:

$$M\ N\ P =_\beta \begin{cases} \ulcorner 0 \urcorner & \text{if } N = P \\ \ulcorner 1 \urcorner & \text{otherwise} \end{cases}$$

[3 marks]

Solution: Suppose for the purposes of obtaining a contradiction that such a term M exists. Then $\ulcorner 1 \urcorner =_\beta M\ (\mathbf{I}\ \mathbf{I})\ \mathbf{I} =_\beta M\ \mathbf{I}\ \mathbf{I} =_\beta \ulcorner 0 \urcorner$. However, $\ulcorner 1 \urcorner \neq_\beta \ulcorner 0 \urcorner$ follows from the definition of $=_\beta$ since $\ulcorner 0 \urcorner$ and $\ulcorner 1 \urcorner$ are distinct β-normal forms and so cannot have a common reduct.

(f) Let us say that a term M is *solvable* just if for all terms $P \in \Lambda$, one can find a sequence of terms $N_1, \ldots, N_k$ such that $(\lambda x_1 \ldots x_m.\, M)\, N_1 \cdots N_k =_\beta P$, where $\mathsf{FV}(M) = \{x_1, \ldots, x_m\}$. Is the membership of following set decidable? Either prove it is not or sketch and justify an algorithm.

$$\{M \mid M \text{ does not have a } \beta\text{-normal form or } M \text{ is solvable}\}$$

(cont.)

[6 marks]

Solution: Suppose M has a normal form. Then it follows that M is solvable. First, observe that $\lambda x_1 \cdots x_m.\, M =_\beta \lambda x_1 \cdots x_m.\, N$ for some normal form N. Second, observe that every normal form has shape $\lambda x_{m+1} \cdots x_n.\, y\ P_1\ \cdots\ P_k$ for some variables $x_{m+1}, \ldots, x_n$ and terms $P_1, \ldots, P_k$. Third, observe that since the original term was closed, y is some x_i. Then Let $P \in \Lambda$. A sequence of the desired form is the length-n squence $\Omega \cdots \Omega\, (\lambda y_1\ \ldots\ y_k.\, P)\, \Omega \cdots \Omega$ in which the non-Ω term occurs in the ith position. Therefore, the set can be decided by the procedure that returns 1 for all inputs.

Q2. (a) State the rules of the type system (the rule names are not important).

[3 marks]

Solution:

$$\frac{x : \forall \overline{a}.\ A \in \Gamma}{\Gamma \vdash x : A[\overline{B}/\overline{a}]}\ (\text{TVar})$$

$$\frac{\Gamma \vdash M : B \to A \quad \Gamma \vdash N : B}{\Gamma \vdash MN : A}\ (\text{TApp})$$

$$x \notin \mathsf{dom}\ \Gamma\ \frac{\Gamma \cup \{x : B\} \vdash M : A}{\Gamma \vdash \lambda x.\, M : B \to A}\ (\text{TAbs})$$

(b) For each of the following terms, state whether or not it is typable. No justification is necessary.

i. $\lambda x.\, yz$

ii. $\lambda x.\, xx$

iii. $\lambda x.\, x(\lambda y.\, y)x$

[3 marks]

Solution:

i. Yes

ii. No

iii. No

(c) Recall that the subterm relation can be defined inductively by the following rules:

$$\frac{}{M \sqsubseteq M}\ (\text{SubRefl}) \qquad \frac{P \sqsubseteq M}{P \sqsubseteq (\lambda x.\, M)}\ (\text{SubAbs})$$

$$\frac{P \sqsubseteq M}{P \sqsubseteq (MN)}\ (\text{SubAppL}) \qquad \frac{P \sqsubseteq N}{P \sqsubseteq (MN)}\ (\text{SubAppR})$$

Prove, by induction on $\Gamma \vdash M : A$, that, in the type system:

If $\Gamma \vdash M : A$ and $N \sqsubseteq M$, then there is some Γ' and A' such that $\Gamma' \vdash N : A'$.

[6 marks]

Solution: The proof is by induction on $\Gamma \vdash M : A$.

- In case (Var), M is a variable x and it must be that the derivation of $N \sqsubseteq M$ is concluded by Refl, whence $M = N$. Hence, the conclusion follows trivially.

- In case (App), M is an application PQ. The hypotheses on the (App) are $\Gamma \vdash P : B \to A$ and $\Gamma \vdash Q : B$, therefore the induction hypotheses are:

  (IH1) If $N \sqsubseteq P$ then there is some Γ' and A' such that $\Gamma' \vdash N : A'$

  (IH2) If $N \sqsubseteq Q$ then there is some Γ' and A' such that $\Gamma' \vdash N : A'$

  Assume the induction hypotheses. Since M has shape PQ, the derivation of $N \sqsubseteq PQ$ can only be ending in (SubRefl), (SubAppL) or (SubAppR). In the first case, the conclusion is immediate. In either of the other cases, the result follows immediately from the induction hypotheses.

- In case (Abs), M is an abstraction $\lambda x. P$ and A is an arrow $B \to C$. By the variable convention, we can assume that $x \notin \mathsf{FV}(\Gamma)$. The hypothesis on the (Abs) rule is $\Gamma, x : B \vdash P : C$. Therefore, the induction hypothesis is (IH) if $N \sqsubseteq P$ then there is some Γ' and A' such that $\Gamma' \vdash N : A'$. Due to the shape of M, it can only be that the proof of $N \sqsubseteq M$ is concluded by either (SubRefl) or (SubAbs). In the former case, the result is immediate, in the latter case it follows that $N \sqsubseteq P$ and the result follows by IH.

(d) We say that a function $f : \mathbb{N} \to \mathbb{N}$ is *linear* just if $f(x) = a * x + b$ for some natural numbers a and b.

Let us say that a function $f : \mathbb{N} \to \mathbb{N}$ is *simply-definable* just if there is some closed term M which is typable and $M \ulcorner n \urcorner =_\beta \ulcorner f(n) \urcorner$.

Prove that every linear function is simply-definable.

[7 marks]

Solution: Let N be the simple type $(o \to o) \to o \to o$. Let $\mathbf{plus} = \lambda xyfz.\, x\, f\, (y\; f\; z)$. It follows that:

$$\begin{array}{rcl} \mathbf{plus} & : & N \to N \to N \\ \mathbf{plus} \ulcorner n \urcorner \ulcorner m \urcorner & =_\beta & \ulcorner n + m \urcorner \end{array}$$

Then, observe that every Church numeral can be given the type N. Suppose f is a linear function $a * x + b$. Then we can define f by the term

$$\lambda x.\, \mathbf{plus}\, (\mathbf{plus}\, x\, (\mathbf{plus}\, x\, (\cdots (\mathbf{plus}\, x\, x) \cdots))) \ulcorner b \urcorner$$

with $(a-1)$-many copies of the subterm $\mathbf{plus}\, x$.

(e) Suppose $M =_\beta xM$. Show that M cannot have a β-normal form.

[3 marks]

Solution: Suppose for contradiction that M has a normal form N. It follows from the definition of convertibility that M and xM have a common reduct, say P, and therefore that $P \twoheadrightarrow_\beta N$. However, it follows then that $N =_\beta xN$, which means, by the definition of convertibility, that N and xN have a common reduct. However, both N and xN are in normal form.

(f) Define exp_k as a tower of 2nd-power exponentials of height k:

$$\begin{array}{rcl} \mathsf{exp}_1 & = & 2 \\ \mathsf{exp}_{i+1} & = & 2^{\mathsf{exp_i}} \end{array}$$

So, for example, $\mathsf{exp}_3 = 2^{2^2} = 16$. Define a term M such that the k-fold application

$$\underbrace{M \cdots M}_{k\text{-times}}$$

is typable and β-convertible with $\ulcorner \mathsf{exp}_k \urcorner$. Justify your answer.

[3 marks]

Solution: Define $M = \ulcorner 2 \urcorner$. To see that the application is convertible with $\ulcorner \mathsf{exp}_k \urcorner$, observe that $\ulcorner m \urcorner \ulcorner n \urcorner =_\beta \ulcorner n^m \urcorner$. To see that the application is typable, define the following sequence of types:

$$\begin{array}{rcl} N_1 & = & (o \to o) \to o \to o \\ N_{i+1} & = & (N_i \to N_i) \to N_i \to N_i \end{array}$$

Then, it follows that $\vdash \underbrace{M \cdots M}_{k\text{-times}} : N_k$.

END OF PAPER