TYPES AND λ-CALCULUS

Problem Sheet 3

* 1. Show that Θ is also a fixed point combinator, i.e for all terms M:

$$\Theta\, M =_\beta M\,(\Theta\, M)$$

** 2. In this question you will give an alternative predecessor combinator which, although longer, is more intuitive to explain.

We define the ***Church Pair*** of natural numbers m and n, written $\ulcorner(m,n)\urcorner$, as the term $\lambda z.z\ulcorner m\urcorner\ulcorner n\urcorner$.

(a) Define combinators **Fst** and **Snd** with the property that:

$$\mathbf{Fst}\ulcorner(m,\, n)\urcorner =_\beta \ulcorner m\urcorner \qquad \text{and} \qquad \mathbf{Snd}\ulcorner(m,\, n)\urcorner =_\beta \ulcorner n\urcorner$$

(b) Consider the following Haskell program pred' on natural numbers.

$$\mathsf{pred}'\ n = \mathsf{fst}\ (\mathsf{foldn}\ n\ \mathsf{incr}\ (0,0))$$

$$\text{where}$$

$$\begin{aligned} \mathsf{incr}\ (n,0) &= (n,1) \\ \mathsf{incr}\ (n,1) &= (n+1,1) \end{aligned}$$

$$\begin{aligned} \mathsf{foldn}\ 0\ f\ x &= x \\ \mathsf{foldn}\ n\ f\ x &= f\ (\mathsf{foldn}\ (n-1)\ f\ x) \end{aligned}$$

What is the result of computing $\mathsf{foldn}\ 3\ \mathsf{incr}\ (0,0)$?

(c) Implement pred' as a λ-term operating on Church Numerals.

** 3.

(a) Prove that natural number multiplication is λ-definable by programming a combinator **Mult**.

Hint: multiplication is iterated addition.

(b) Prove that your construction works by showing the following using induction on $n \in \mathbb{N}$ or on $m \in \mathbb{N}$ (which one works will depend on how you defined **Mult**):

$$\forall n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ \mathbf{Mult}\ \ulcorner m \urcorner\ \ulcorner n \urcorner =_\beta \ulcorner m * n \urcorner$$

Hint: you may use the following fact without proving it:

$$\ulcorner k+1 \urcorner =_\beta \mathbf{Add}\ \ulcorner 1 \urcorner\ \ulcorner k \urcorner$$

** 4. Use **Y** to define the recursive triangular number function: using the "recipe", give a combinator **Tri** that satisfies:

$$\mathbf{Tri}\ \ulcorner 0 \urcorner =_\beta \ulcorner 0 \urcorner \qquad \text{and} \qquad \mathbf{Tri}\ \ulcorner n+1 \urcorner =_\beta \mathbf{Add}\ \ulcorner n+1 \urcorner\ (\mathbf{Tri}\ \ulcorner n \urcorner)$$

Convince yourself that $\mathbf{Tri}\ \ulcorner 2 \urcorner =_\beta \ulcorner 3 \urcorner$ (this is obvious if you believe that your implementation of **Tri** really satisfies the given equations).

** 5. Prove that if $M =_\beta N$ and N is a normal form, then $M \twoheadrightarrow_\beta N$.

Therefore, we now know that e.g. $\mathbf{Tri}\ \ulcorner 2 \urcorner \twoheadrightarrow_\beta \ulcorner 3 \urcorner$, so these definitions actually *compute* an output given an input.

** 6. Show that β-normal forms are unique, i.e. show that if a term has two β-normal forms N_1 and N_2, then they are actually the same term.

Therefore, we now know that e.g. $\mathbf{Tri}\ \ulcorner 2 \urcorner \not\twoheadrightarrow_\beta \ulcorner 4 \urcorner$, so there is at most one output for each input.

*** 7. Show that there is no term P that satisfies $P\,(MN) =_\beta N$.