UNIVERSITY OF BRISTOL

January 2021 Examination Period

FACULTY OF ENGINEERING

Third Year Examination for the Degrees
of
Bachelor of Science
Master of Engineering

COMS30039J
Types and Lambda Calculus

TIME ALLOWED:
2 Hours

This paper contains *two* questions, answer *both*. Each question is worth 25 marks. The maximum for this paper is *50 marks*. Credit will be given for partially correct answers.

PLEASE WRITE YOUR 7 DIGIT STUDENT NUMBER (NOT CANDIDATE NUMBER) ON THE ANSWER BOOKLET. YOUR STUDENT NUMBER CAN BE FOUND ON YOUR UCARD.

Other Instructions:

You may use the lecture notes to help you, but you may not collaborate.

YOU MAY START IMMEDIATELY

Q1. This question concerns the pure, untyped λ-calculus.

It will be useful to recall the combinators $\mathbf{I} = \lambda x.\, x$ and $\mathbf{K} = \lambda xy.\, x$

(a) Give the set of free variables for each of the following terms:

i. $\lambda xyz.\, xz(yz)$

ii. $z(\lambda x.\, xy)$

iii. $\lambda z.\, (\lambda x.\, y)(xz)$

iv. $(\lambda x.\, x\,(\lambda y.\, xy))(\lambda x.\, y)$

[4 marks]

(b) For each of the following, give a term M that satisfies the statement.

i. $M\,x =_\beta xx$

ii. $M \ulcorner n \urcorner =_\beta \ulcorner 2 * n \urcorner$

iii. $M =_\beta MM$

iv. $M\,x =_\beta MM$

[8 marks]

(c) Recall that there is a combinator **Pred** satisfying:

$$\begin{aligned} \mathbf{Pred} \ulcorner 0 \urcorner &=_\beta \ulcorner 0 \urcorner \\ \mathbf{Pred} \ulcorner n+1 \urcorner &=_\beta \ulcorner n \urcorner \end{aligned}$$

Define combinators **Sub** and **Succ** as follows:

$$\begin{aligned} \mathbf{Succ} &:= \lambda nfx.\, f\,(n\,f\,x) \\ \mathbf{Sub} &:= \lambda mn.\, n\,\mathbf{Pred}\,m \end{aligned}$$

Prove, by induction on n, that **Sub** satisfies:

$$\mathbf{Sub} \ulcorner m \urcorner \ulcorner n \urcorner =_\beta \begin{cases} \ulcorner 0 \urcorner & \text{if } m \leq n \\ \ulcorner m - n \urcorner & \text{otherwise} \end{cases}$$

You may use the fact that, for all natural numbers p, $\ulcorner p+1 \urcorner =_\beta \mathbf{Succ} \ulcorner p \urcorner$.

[7 marks]

(d) Prove that there does not exist a term M such that, for all terms N:

$$M\,N =_\beta \begin{cases} \mathbf{I} & \text{if } N \text{ is in } \beta\text{-normal form} \\ \mathbf{K} & \text{otherwise} \end{cases}$$

[3 marks]

(e) Find a finite sequence of *closed* terms $M_1, M_2, \ldots, M_k$ for $k \geq 0$ such that the following two equations are both satisfied:

$$\begin{aligned} (\lambda x.\, x\,\mathbf{I}\,(x\,\mathbf{I}\,\mathbf{I}))\, M_1\, M_2 \cdots M_k\, x\, y &=_\beta x \\ (\lambda x.\, x\,\mathbf{I}\,(x\,x\,\mathbf{I}))\, M_1\, M_2 \cdots M_k\, x\, y &=_\beta y \end{aligned}$$

[3 marks]

Q2. This question concerns type systems.

(a) Give a typing derivation for each of the following judgements:

i. $\vdash \lambda xy.\, x : a \to b \to a$

ii. $\vdash \lambda x.\, x(\lambda y.\, y) : ((a \to a) \to b) \to b$

iii. $y : c \vdash (\lambda x.\, y)(\lambda xz.\, x) : c$

[6 marks]

(b) For each of the following types, find a closed term that inhabits the type:

i. $a \to b \to b$

ii. $(a \to b) \to (a \to b \to c) \to a \to c$

iii. $c \to ((a \to b \to a) \to c \to d) \to d$

[6 marks]

(c) Prove, by induction on the Damas-Miler typing relation, that: if $\Gamma \vdash M : A$ then, for all Γ', $\Gamma \subseteq \Gamma'$ implies $\Gamma' \vdash M : A$.

[6 marks]

(d) Define the order, $\mathsf{order}(A)$ of a monotype A recursively on the structure of A:

$$\begin{aligned} \mathsf{order}(a) &= 0 \\ \mathsf{order}(B \to C) &= \max(1 + \mathsf{order}(B), \mathsf{order}(C)) \end{aligned}$$

where $\max(m, n)$ is the larger of the two natural numbers m and n. Prove that, if $\mathsf{order}(A) \leq 2$ and $\vdash M : A$, then there is a term N such that:

(i) $M =_\beta \lambda x_1 \ldots x_n.\, N$

(ii) and, moreover, N does not contain any abstraction as a subterm

[7 marks]

END OF PAPER