TYPES AND λ-CALCULUS

Problem Sheet 7

Questions 1 and 3 will be marked.

** 1. Suppose closed term M has a normal form $\underline{1}\,(\lambda x.\,x)$. Prove that all reducts of M are untypable, i.e. $M \,\rhd^*\, N$ implies N untypable.

** 2. Prove both of the following:

(1) $\vdash \underline{n} : A$ implies $A = \mathsf{Nat}$.

(2) $\vdash V : \mathsf{Nat}$ implies V is a numeral.

That is, numerals can only be assigned the type Nat and these are the only closed values of this type. For (1), induction is not necessary, but you will need to analyse the two possible shapes of n (0 or $k+1$). For (2), I suggest appealing to inversion to rule out many possible shapes of V in a one go.

** 3.

(a) Prove the following result by induction on M:

> Let B be a type and N a term. For all terms M, types A and environments Γ: if $\Gamma, x{:}B \vdash M : A$ and $\Gamma \vdash N : B$ then $\Gamma \vdash M[N/x] : A$.

(b) Prove the missing case in the proof of Lemma 12.2 from the notes: if $\Gamma \vdash (\lambda x.\,M)N : A$ then $\Gamma \vdash M[N/x] : A$.

** 4. Find pure terms that inhabit the following types (no need for justification):

(a) $(a \to a \to b) \to a \to b$

(b) $(a \to b) \to (a \to c) \to (b \to c \to d) \to a \to d$

(c) $(((a \to a \to b) \to a \to b) \to c) \to c$

** 5.

(a) Find a *pure* term that inhabits the type $(((a \to b) \to b) \to b) \to a \to b$.

(b) Give the corresponding proof of the corresponding formula.

*** 6. The reason that we don't study full PCF in connection with the Curry-Howard correspondence is the presence of fix.

(a) Use fix to show that every type is inhabited by some *PCF term* (not necessarily pure).

(b) What is the consequence for the Curry-Howard correspondence extended to full PCF?

*** 7. The following property is called Subject Invariance:

$$\text{if } M \approx N \text{ and } \Gamma \vdash M : A \text{ then } \Gamma \vdash N : A$$

Is this property true for our type system? Either prove it or give a counterexample.