TYPES AND λ-CALCULUS

Problem Sheet 2

* 1. Perform one step of reduction for each of the following terms:

(a) $(\lambda xy.\, x)(\lambda x.\, x)(\lambda z.\, z)$

(b) $(\lambda xyz.\, xz(yz))(\lambda xy.\, x)$

(c) $(\lambda x.\, xx)(\lambda x.\, xx)$

(d) $(\lambda xy.\, x)((\lambda z.\, zz)(\lambda x.\, x))$

* 2. Justify each of the following conversions $M =_\beta N$ by finding a common reduct P, i.e. such that $M \twoheadrightarrow_\beta P$ and $N \twoheadrightarrow_\beta P$.

(a) $(\lambda x.\, x)y =_\beta (\lambda xy.\, x)\ y\ z$

(b) $(\lambda x.\, M)N =_\beta M[N/x]$

(c) $\mathbf{Y}\, M =_\beta M\ (\mathbf{Y}\, M)$

(d) $z\ (\mathbf{K}\, \mathbf{I}\, \Omega)\ \Omega =_\beta z\ \mathbf{I}\ (\mathbf{K}\, \Omega\, \mathbf{I})$

(e) $\Theta\ M =_\beta M\ (\Theta\ M)$

** 3. Choose *one* of the following and try to prove it by induction on n:

- For all $n \in \mathbb{N}$, if $n > 0$ and $P_0 \to_\beta \cdots \to_\beta P_n$ then $MP_0 \to_\beta \cdots \to_\beta MP_n$
- For all $n \in \mathbb{N}$, if $n > 0$ and $P_0 \to_\beta \cdots \to_\beta P_n$ then $P_0M \to_\beta \cdots \to_\beta P_nM$
- For all $n \in \mathbb{N}$, if $n > 0$ and $P_0 \to_\beta \cdots \to_\beta P_n$ then $\lambda x.\, P_0 \to_\beta \cdots \to_\beta \lambda x.\, P_n$

Hint: the base case $n = 0$ will be trivial. In the induction step case, when $n = k + 1$, at some point you will need to case split on whether or not $k > 0$ because the induction hypothesis will require it. You can do this because $k = 0 \vee k > 0$ is an elementary fact about all natural numbers k, and so you can use it as an assumption in your proofs.

** 4. Use the property from the previous question to prove the corresponding statement, called the *compatibility of reduction*:

- If $P \twoheadrightarrow_\beta Q$ then $MP \twoheadrightarrow_\beta MQ$
- If $P \twoheadrightarrow_\beta Q$ then $PM \twoheadrightarrow_\beta QM$
- If $P \twoheadrightarrow_\beta Q$ then $\lambda x.P \twoheadrightarrow_\beta \lambda x.Q$

Since the property in the previous question only applies when $n > 0$, you will likely have to case split on n at some point during the proof.

** 5. Choose one of the following statements, collectively called *the compatibility of conversion*, and prove it.

- if $P =_\beta Q$ then $MP =_\beta MQ$
- if $P =_\beta Q$ then $PM =_\beta QM$
- if $P =_\beta Q$ then $\lambda x.P =_\beta \lambda x.Q$

Hint: using the compatibility of reduction will be essential.

** 6. Prove *all three* of the following. In part (c), look for a place to use confluence.

(a) (Reflexivity) $M =_\beta M$
(b) (Symmetry) if $M =_\beta N$ then $N =_\beta M$
(c) (Transitivity) if $M =_\beta N$ and $N =_\beta P$ then $M =_\beta P$.

These three together give that $=_\beta$ is an equivalence relation. The fact that β-conversion is also compatible means that it is a congruence, and so *equational reasoning* makes sense (i.e. constitutes a valid proof technique).

Using equational reasoning, we can string together a sequence of β-conversions $M_1 =_\beta M_2 =_\beta \cdots =_\beta M_{k-1} =_\beta M_k$ in which each consecutive pair is justified because they only differ in some subterms P and Q which are themselves β-convertible. For example, $\lambda x.x(\mathbf{I}\,x)xx =_\beta \lambda x.xx(\mathbf{K}x\mathbf{I})(\mathbf{I}x)$ because:

$$\begin{array}{rcll} \lambda x.x(\mathbf{I}\,x)xx & =_\beta & \lambda x.xxxx & (\text{since } \mathbf{I}\,x =_\beta x) \\ & =_\beta & \lambda x.xx(\mathbf{K}x\mathbf{I})x & (\text{since } \mathbf{K}x\mathbf{I} =_\beta x) \\ & =_\beta & \lambda x.xx(\mathbf{K}x\mathbf{I})(\mathbf{I}x) & (\text{since } \mathbf{I}x =_\beta x) \end{array}$$

At each line, the compatibility results allow us to promote an equation that holds between subterms to an equation between the toplevel terms, symmetry allows us to use the equations in whichever direction we like, and transitivity allows us to conclude that any two terms in the chain are therefore convertible.