TYPES AND λ-CALCULUS

Problem Sheet 1

* 1. Which of the following are terms? For those that are terms, write out a proof tree justification.

(a) $(\lambda x.((xx)x))$

(b) $(\lambda(\lambda x.x))$

(c) $((xy)z)$

Solution

(a)

$$\cfrac{\cfrac{\cfrac{}{x \in \Lambda}\,(\mathsf{Var}) \quad \cfrac{}{x \in \Lambda}\,(\mathsf{Var})}{(xx) \in \Lambda}\,(\mathsf{App}) \quad \cfrac{}{x \in \Lambda}\,(\mathsf{Var})}{\cfrac{((xx)x) \in \Lambda}{(\lambda x.((xx)x)) \in \Lambda}\,(\mathsf{Abs})}\,(\mathsf{App})$$

(b) Not a term.

(c)

$$\cfrac{\cfrac{\cfrac{}{x \in \Lambda}\,(\mathsf{Var}) \quad \cfrac{}{y \in \Lambda}\,(\mathsf{Var})}{(xy) \in \Lambda}\,(\mathsf{App}) \quad \cfrac{}{z \in \Lambda}\,(\mathsf{Var})}{((xy)z) \in \Lambda}\,(\mathsf{App})$$

* 2. Write these terms using the minimum number of parentheses and λ, according to our conventions.

(a) $(\lambda y.((yy)(zz)))$

(b) $(\lambda y.(((yy)y)y))$

(c) $((xy)(\lambda y.(\lambda z.(z(xy)))))$

Solution

(a) $\lambda y.yy(zz)$
(b) $\lambda y.yyyy$
(c) One answer is: $xy\lambda yz.z(xy)$. However, in practice it is common to write this term as $xy(\lambda yz.z(xy))$ because most find this easier to read. I will always write it the second way.

* 3. Write the term $(\lambda xyz.xy(xz))(\lambda xy.x)$ with all the parentheses and λ that we will usually omit tediously put back in.

Solution

$((\lambda x.(\lambda y.(\lambda z.((xy)(xz)))))(\lambda x.(\lambda y.x)))$

** 4. Note that, by the conventions of logic, $A \Rightarrow B \Rightarrow C$ is a shorthand for $A \Rightarrow (B \Rightarrow C)$ and conjunction binds tighter than implication, so $A \wedge B \Rightarrow C$ means $(A \wedge B) \Rightarrow C$.

Give proofs of the following.

(a) $\neg A \Rightarrow A \Rightarrow B$
(b) $(A \wedge B \Rightarrow C) \Rightarrow A \Rightarrow B \Rightarrow C$
(c) $\neg(A \wedge \neg A)$
(d) $(A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$
(e) $\neg A \wedge \neg B \Rightarrow \neg(A \vee B)$

Solution

(a) Assume $\neg A$ then assume A. This yields a contradiction and so, in particular, B follows.
(b) Assume $A \wedge B \Rightarrow C$ (*). Assume A and then assume B. From A and B we have $A \wedge B$ and so from (*) we obtain C.
(c) We assume $A \wedge \neg A$ and then try to obtain a contradiction. We already have A and $\neg A$ which gives the desired contradiction.
(d) Assume $A \Rightarrow B$ (1). Assume $B \Rightarrow C$ (2). Assume A. From (1) and A obtain B. From B and (2) obtain C.

(e) Assume $\neg A$ (1) and $\neg B$ (2). For contradiction suppose that $A \vee B$. We proceed by cases on $A \vee B$:

- If A is true, then this contradicts (1).
- If B is true, then this contradicts (2).

In all cases we obtained a contradiction.