

# TYPES AND $\lambda$ -CALCULUS

## Problem Sheet 3

Recall that a *closed* term has no free variables.

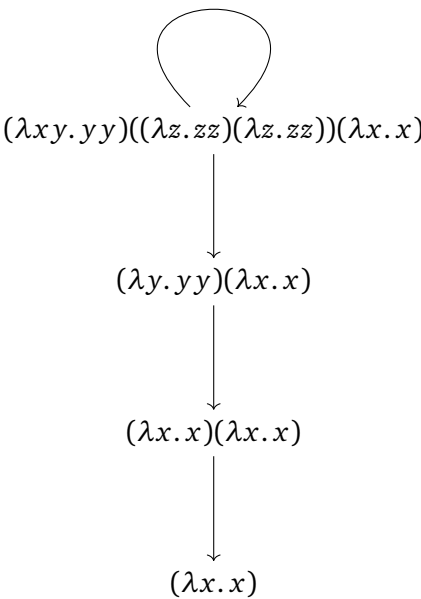
- \* 1. For each of the following specifications, give an example of a *closed* term  $N$  in *normal form* that satisfies it (i.e do some reduction):
- (a)  $\underline{id} \ \underline{id} \triangleright^* N$
  - (b)  $\underline{sub} \ \underline{const} \ \underline{const} \triangleright^* N$
  - (c)  $\text{fix} (\lambda xy. y) \triangleright^* N$
  - (d)  $(\lambda xy. yx) (\underline{const} \ \underline{const}) (\lambda x. xx) \triangleright^* N$

Solution \_\_\_\_\_

- (a)  $\underline{id}$
- (b)  $\underline{id}$
- (c)  $\underline{id}$
- (d)  $\underline{const}$

- \* 2. Draw the reduction graph of the term  $(\lambda xy. yy) ((\lambda z. zz) (\lambda z. zz)) (\lambda x. x)$ . (This graph will have 4 vertices). What I mean by this is to draw a directed graph where:
- The nodes are all  $N$  s.t.  $(\lambda xy. yy) ((\lambda z. zz) (\lambda z. zz)) (\lambda x. x) \triangleright^* N$
  - There is an edge from node  $M$  to node  $N$  iff  $M \triangleright N$

Solution \_\_\_\_\_



- \*\* 3.** Give an example of a *closed* term  $M$  for each of the following properties:
- $M$  is in normal form.
  - $M$  has exactly one reduct.
  - $M$  contains strictly fewer redexes than one of its reducts (here we mean “fewer in number”, the redexes may be quite different).
  - A reduct of  $M$  contains a redex that did not occur anywhere in  $M$ .

Solution \_\_\_\_\_

There are lots of possible answers, here is one set:

- (a)  $\underline{\text{id}}$   
 (b)  $\text{pred } \underline{2}$   
 (c)  $(\lambda x. xxx) (\underline{\text{id}} \underline{1})$   
 (d)  $(\lambda xy. xy) \underline{\text{id}}$

- \* 4. For each of the following reduction steps  $M \triangleright N$ , identify the redex  $P$ , the contraction  $Q$ , and the context  $C[\ ]$  in which the contraction happens, i.e. such that  $M = C[P]$  and  $N = C[Q]$ .
- (a)  $\lambda x. \text{pred} (\text{pred } 2) \triangleright \lambda x. \text{pred } 1$

- (b)  $\underline{\text{id}} (\underline{\text{const}} \underline{\text{div}} \underline{0}) \triangleright \underline{\text{id}} (\underline{\text{const}} (\underline{\text{id}} \underline{\text{div}}) \underline{0})$   
(c)  $\underline{\text{const}} (\underline{\text{id}} \underline{\text{id}}) (S \ x) \triangleright (\lambda y. \underline{\text{id}} \underline{\text{id}}) (S \ x)$

Solution

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- (a)  $P = \text{pred } \underline{2}, Q = \underline{1}, C[] = \lambda x. \text{pred } []$   
(b)  $P = \underline{\text{div}}, Q = \underline{\text{id}} \underline{\text{div}}, C[] = \underline{\text{id}} (\underline{\text{const}} [] \underline{0})$   
(c)  $P = \underline{\text{const}} (\underline{\text{id}} \underline{\text{id}}), Q = \lambda y. \underline{\text{id}} \underline{\text{id}}, C[] = [] (S \ x)$

\*\* 5. Prove the following statement:

For all  $M, N$  and  $C[]$ : if  $M \triangleright N$  then  $C[M] \triangleright C[N]$ .

Note that “if  $M \triangleright N$  then  $C[M] \triangleright C[N]$ ” is subtly different from the definition of  $\triangleright$  which says that  $C[P] \triangleright C[Q]$  whenever  $P$  is a redex and  $Q$  the contraction. Here,  $M$  and  $N$  can be any terms.

You do *not* need to use induction to prove it. You will need to work closely with the definition of  $\triangleright$  : on the one hand you will assume  $M \triangleright N$  and want to know what you get out of it and, on the other hand, you will want to show  $C[M] \triangleright C[N]$  and thus need to know what evidence is required to put into it.

Look again at the definition of  $\triangleright$  using contexts. In the definition, “just if” means the same as “iff”, so the definition of  $\triangleright$  can be seen as a pair of implications: one direction tells you what follows from  $M \triangleright N$  when you have it as an assumption (forwards reasoning) and the other tells you what you need in order to deduce  $M \triangleright N$  (backwards reasoning).

Solution

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Let  $M, N$  be terms and let  $C[]$  be an evaluation context. Suppose  $M \triangleright N$ . Then it follows from the definition of  $\triangleright$  that there must exist an evaluation context  $C'[]$  and a redex/contraction pair  $P, Q$  such that  $M = C'[P]$  and  $N = C'[Q]$ . So,  $M \triangleright N$  is really:

$$C'[P] \triangleright C'[Q]$$

Now, by these equations,  $C[M] = C[C'[P]]$  and  $C[N] = C[C'[Q]]$  (\*). Our aim is to prove that  $C[M] \triangleright C[N]$  so, by definition of  $\triangleright$ , we have to find a context  $C''[]$  and a redex/contraction pair  $P', Q'$  such that  $C[M] = C''[P']$  and  $C''[Q'] = C[N]$ . By (\*) we can take  $C''[] = C[C'[]]$  and  $P' = P$  and  $Q' = Q$ .

\*\* 6.

(a) Complete the following proof by filling in (a):

For all  $P, C[\ ]$ , for all  $n \in \mathbb{N}$ : for all  $Q$ , if  $P \triangleright^n Q$  then  $C[P] \triangleright^n C[Q]$ .

*Proof.* Let  $P$  be a term and  $C[\ ]$  a context. We show that, for all  $n \in \mathbb{N}$ , for all  $Q$ ,  $P \triangleright^n Q$  implies  $C[P] \triangleright^n C[Q]$  by induction on  $n$ :

- When  $n = 0$ , let  $Q$  be a term and suppose  $P \triangleright^0 Q$ . Then, by definition,  $P = Q$  and hence  $C[P] = C[Q]$ . By definition, therefore  $C[P] \triangleright^0 C[Q]$ .
- When  $n$  is of shape  $k + 1$ , we can assume the induction hypothesis:  
**(IH)** for all  $Q, P \triangleright^k Q$  implies  $C[P] \triangleright^k C[Q]$ .  
 ... (a) ...

□

(b) Deduce that (i.e. give a short proof of): For all  $P, Q, C[\ ]$ : if  $P \triangleright^* Q$  then  $C[P] \triangleright^* C[Q]$ .

Solution

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- (a) Now, let  $Q$  be a term and suppose  $P \triangleright^{k+1} Q$ . Then, by definition, there is some  $U$  such that (i)  $P \triangleright^k U$  and (ii)  $U \triangleright Q$ . We can use (IH) on (i) to obtain  $C[P] \triangleright^k C[U]$  (\*). We can use the statement of the previous question on (ii) to obtain  $C[U] \triangleright C[Q]$  (\*\*). By definition of  $\triangleright^{k+1}$ , we can use (\*) and (\*\*) to obtain  $C[P] \triangleright^{k+1} C[Q]$ .
- (b) Let  $P$  and  $Q$  be terms and  $C[\ ]$  a context. Suppose  $P \triangleright^* Q$ . By definition, there is some  $n$  such that  $P \triangleright^n Q$ . It follows from the previous result that, therefore,  $C[P] \triangleright^n C[Q]$ . Hence, by definition of  $\triangleright^*$ ,  $C[P] \triangleright^* C[Q]$ .

\*\*\* 7. Show that there is no term  $P$  that satisfies: for all  $M$  and  $N$ ,  $P(MN) \triangleright^* N$ . In other words, prove that we cannot write a PCF program that extracts the argument of an application.

Solution

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Suppose such a  $P$  exists and we look to obtain a contradiction. Then we know that this  $P$  satisfies  $\forall MN. P(MN) \triangleright^* N$ , let's call this (\*). Then, instantiating  $M$  by const (id id) and  $N$  by const in (\*), we have:

$$P(\text{const } (\text{id id}) \text{ const}) \triangleright^* \text{const}$$

However, by simply performing the reductions, we also have

$P(\underline{\text{const}}(\underline{\text{id}}\ \underline{\text{id}})\ \underline{\text{const}}) \triangleright^* P(\underline{\text{id}}\ \underline{\text{id}})$ . By instantiating (\*) with  $M = \underline{\text{id}}$  and  $N = \underline{\text{id}}$  we get  $P(\underline{\text{id}}\ \underline{\text{id}}) \triangleright^* \underline{\text{id}}$ . So, overall, we also have:

$$P(\underline{\text{const}}(\underline{\text{id}}\ \underline{\text{id}})\ \underline{\text{const}}) \triangleright^* \underline{\text{id}}$$

However, with the other reduction sequence inset above, this contradicts the unique normal forms theorem, since  $\underline{\text{id}}$  and  $\underline{\text{const}}$  are distinct normal forms of  $P(\underline{\text{const}}(\underline{\text{id}}\ \underline{\text{id}})\ \underline{\text{const}})$ , so we have our contradiction.