Types and λ-calculus

Problem Sheet 2

* 1. Write these terms using the minimum number of parentheses and λ, according to our conventions.

(a) $(\lambda y.((y\,y)(z\,z)))$

(b) $(\lambda y.(((y\,y)\,y)\,y))$

(c) $((\mathsf{S}\,\mathsf{Z})(\lambda y.(\lambda z.(z\,(\mathsf{S}\,y)))))$

Solution

(a) $\lambda y.\,y\,y(zz)$

(b) $\lambda y.\,y\,y\,y\,y$

(c) One answer is: $\mathsf{S}\,\mathsf{Z}\,\lambda yz.z\,(\mathsf{S}\,y)$. However, in practice it is common to write this term as $\mathsf{S}\,\mathsf{Z}\,(\lambda yz.z\,(\mathsf{S}\,y))$ because most find this easier to read. I will always write it the second way.

* 2. Write the term $(\lambda xyz.\,xy(xz))(\lambda xy.\,x)$ with all the parentheses and λ that we will usually omit tediously put back in.

Solution

$((\lambda x.(\lambda y.(\lambda z.((xy)(xz)))))(\lambda x.(\lambda y.\,x)))$

* 3. Perform the following substitutions:

(a) $(\lambda x.(\lambda y.\,xz)\,z)[(\lambda z.\,z)/z]$

(b) $(\lambda x.\,yx)[yz/x]$

(c) $(\lambda y.\,xy)[yx/x]$

Solution

(a) $\lambda x.\,(\lambda y.\,x(\lambda z.\,z))(\lambda z.\,z)$
(b) $\lambda x.\,y x$
(c) $\lambda z.\,y x z$

* 4. Perform one step of reduction for each of the following terms:

(a) $\underline{\mathsf{const}}\,\mathsf{pred}\,\mathsf{pred}$
(b) $\underline{\mathsf{sub}}\,\underline{\mathsf{const}}$
(c) $(\lambda x.\,x\,x)(\lambda x.\,x\,x)$
(d) $\underline{\mathsf{const}}\,(\mathsf{pred}\,\mathsf{pred})$

Solution

(a) $(\lambda y.\,\mathsf{pred})\ \mathsf{pred}$
(b) $\lambda y z.\,\underline{\mathsf{const}}\,z\,(y\,z)$
(c) $(\lambda x.\,x\,x)(\lambda x.\,x\,x)$
(d) $\lambda y.\,\mathsf{pred}\,\mathsf{pred}$ or $\underline{\mathsf{const}}\,\mathsf{wrong}$

* 5. Let us define the Booleans as follows:

$$\begin{aligned}\underline{\mathsf{false}} &= \underline{0}\\ \underline{\mathsf{true}} &= \underline{1}\end{aligned}$$

Define Boolean conjunction as a term $\underline{\mathsf{and}}$, disjunction as a term $\underline{\mathsf{or}}$ and negation as a term $\underline{\mathsf{not}}$.

Solution

Define:

$$\begin{aligned}\underline{\mathsf{not}} &= \lambda x.\,\mathsf{ifz}\,x\,\underline{\mathsf{true}}\,\underline{\mathsf{false}}\\ \underline{\mathsf{and}} &= \lambda x y.\,\mathsf{ifz}\,x\,\underline{\mathsf{false}}\,(\mathsf{ifz}\,y\,\underline{\mathsf{false}}\,\underline{\mathsf{true}})\\ \underline{\mathsf{or}} &= \lambda x y.\,\underline{\mathsf{not}}\,(\underline{\mathsf{and}}\,x\,y)\end{aligned}$$

* 6. Define terms $\underline{\mathsf{curry}}$ and $\underline{\mathsf{uncurry}}$ with the following behaviour:

$$\underline{\mathsf{curry}}\, M\, N\, P \;\rhd\; \cdots \;\rhd\; M\,(N\,P)$$
$$\underline{\mathsf{uncurry}}\, M\,(N,\, P) \;\rhd\; \cdots \;\rhd\; M\, N\, P$$

Solution

$$\underline{\mathsf{curry}} := \lambda f\, x\, y.\, f\,(x,\, y)$$
$$\underline{\mathsf{uncurry}} := \lambda f\, p.\, f\,(\underline{\mathsf{proj}}_1^2\, p)\,(\underline{\mathsf{proj}}_2^2\, p)$$

** 7.

(a) For all terms M and N, define a *local definition* term:

$$\underline{\mathsf{let}}\ x = N\ \underline{\mathsf{in}}\ M$$

(M and N will occur inside your answer), with the following behaviour:

$$\underline{\mathsf{let}}\ x = N\ \underline{\mathsf{in}}\ M \;\rhd\; M[N/x]$$

In other words, $\underline{\mathsf{let}}\ x = N\ \underline{\mathsf{in}}\ M$ behaves like M but where we have defined x locally to be N.

(b) Define a family of local definition forms that are specialised to functions of a certain number of arguments, i.e. a family of terms $\underline{\mathsf{let}}\ f\ x_1 \cdots x_n = N\ \underline{\mathsf{in}}\ M$, that behave like M but where we have defined f to be the function that takes arguments $x_1, \ldots, x_n$ and returns N.

Solution

(a) $\underline{\mathsf{let}}\ x = N\ \underline{\mathsf{in}}\ M := (\lambda x.\, M)\, N$

(b) $\underline{\mathsf{let}}\ f\, x_1 \cdots x_n = N\ \underline{\mathsf{in}}\ M := \underline{\mathsf{let}}\ f = \lambda x_1 \ldots x_n.\, N\ \underline{\mathsf{in}}\ M$

** 8. Complete the following partial proof of the following statement, by filling out (a), (b) (c) and (d). This proof is a particularly good test of how carefully you are keeping track of the proof state.

Suppose P and Q are terms, x and y are variables. If $x \neq y$ and $x \notin \mathsf{FV}(Q)$ then for all terms M:

$$M[P/x][Q/y] = M[Q/y][P[Q/y]/x]$$

(which tells us how to reorder substitutions in general).

Proof. Suppose P and Q are terms. Suppose x and y are variables. The rest of the proof is by induction on $M \in \Lambda$.

- In case M is some variable z, we argue as follows. Assume $x \neq y$ and $x \neq \mathsf{FV}(Q)$. Our goal is to show:

  $$z[P/x][Q/y] = z[Q/y][P[Q/y]/x]$$

  Now, either $z = x$, $z = y$ or z is neither x nor y. We proceed by a case analysis on this fact:

  - Suppose $z = x$. By our assumption, it follows that $z \neq y$. Then, by definition of substitution, $z[P/x][Q/y] = P[Q/y]$ and

    $$z[Q/y][P[Q/y]/x] = z[P[Q/y]/x] = P[Q/y]$$

    Hence, the desired equality follows.
  - Suppose $z = y$. (a)
  - Suppose $z \neq x$ and $z \neq y$. Then $z[P/x][Q/y] = z$ on the left side of the goal and also $z[Q/y][P[Q/y]/x] = z$ on the right side, so the result follows.

- (b)

- In case M is some application $N_1 N_2$ we argue as follows. Assume $x \neq y$ and $x \neq \mathsf{FV}(Q)$. Additionally, assume the induction hypothesis:

  (IH1) if $x \neq y$ and $x \neq \mathsf{FV}(Q)$ then $N_1[P/x][Q/y] = N_1[Q/y][P[Q/y]/x]$
  (IH2) if $x \neq y$ and $x \neq \mathsf{FV}(Q)$ then $N_2[P/x][Q/y] = N_2[Q/y][P[Q/y]/x]$

  Our goal is to show that:

  $$(N_1 N_2)[P/x][Q/y] = (N_1 N_2)[Q/y][P[Q/y]/x]$$

  (c)

- In case M is of shape $\lambda z.N$, we argue as follows. We can assume by the bound variable convention that z is different from any variable in scope, so $z \neq x$ and $z \neq y$ and $z \notin \mathsf{FV}(P)$ and $z \notin \mathsf{FV}(Q)$. (d)

$\square$

Solution

(a) By our assumption, it follows that $z \neq x$. Then, by definition of substitution $z[P/x][Q/y] = z[Q/y] = Q$ and also $z[Q/y][P[Q/y]/x] = Q[P[Q/y]/x]$.

(b) In case M is a constant c, we argue as follows. Assume $x \neq y$ and $x \neq \mathsf{FV}(Q)$. Our goal is to show:

$$c[P/x][Q/y] = c[Q/y][P[Q/y]/x]$$

but, by definition, both left and right hand sides are just c, so the result follows immediately. We assumed $x \neq \mathsf{FV}(Q)$, so $Q[P[Q/y]/x] = Q$. The result follows.

(c) It follows by the definition of substitution that:

$$(N_1N_2)[P/x][Q/y] = N_1[P/x][Q/y]N_2[P/x][Q/y]$$

It follows from (IH1) and (IH2) that:

$$N_1[P/x][Q/y]N_2[P/x][Q/y] = N_1[Q/y][P[Q/y]/x]N_2[Q/y][P[Q/y]/x]$$

and this latter term is $(N_1N_2)[Q/y][P[Q/y]/x]$ by definition of substitution.

(d) Assume $x \neq y$ and $x \neq \mathsf{FV}(Q)$. Additionally assume the induction hypothesis:

(IH) If $x \neq y$ and $x \neq \mathsf{FV}(Q)$ then $N[P/x][Q/y] = N[Q/y][P[Q/y]/x]$.

Our goal is to show that:

$$(\lambda z.N)[P/x][Q/y] = (\lambda z.N)[Q/y][P[Q/y]/x]$$

Since $z \notin \mathsf{FV}(xyPQ)$, it follows by definition that:

$$(\lambda z.N)[P/x][Q/y] = \lambda z.N[P/x][Q/y]$$

It follows from (IH) that $\lambda z.N[P/x][Q/y] = \lambda z.N[Q/y][P[Q/y]/x]$. Since $z \notin \mathsf{FV}(xyPQ)$, it follows by definition that this latter term is $(\lambda z.N)[Q/y][P[Q/y]]$.