PROOF FROM TRUTH

A proof of A from starting assumptions Γ is a certificate guaranteeing that A is true whenever Γ is true.

TRUTH FROM PROOF

A proof of A from starting assumptions Γ is a method for *constructing* evidence of A from evidence of Γ.

BHK INTERPRETATION OF LOGIC

- There can be no evidence for the truth of false.
- No evidence is needed for the (self-evident) truth of true.
- Evidence for $A \wedge B$ is a pair consisting of evidence for A and evidence for B.
- Evidence for $A \vee B$ is either a piece of evidence of A or a piece of evidence of B.
- Evidence for $A \Rightarrow B$ is a *procedure* for transforming evidence of A into evidence of B.
- Evidence of $\neg A$ is a procedure for transforming any evidence of A into evidence of false.
- Evidence of $\forall x \in X . A$ is a procedure for transforming any y and evidence of $y \in X$ into evidence of $A[y/x]$.
- Evidence of $\exists x \in X . A$ is a triple consisting of a y, evidence of $y \in X$ and evidence of $A[y/x]$.

SUBSTITUTION CLOSURE PROOF

In case (TVar), M is some variable x and A is of shape $A'[\overline{B}/\overline{a}]$ we can assume $x : \forall\overline{a}.\ A' \in \Gamma$. Therefore, by definition (and the assumption allowed by the question), $x : \forall\overline{a}.\ A'[C/c] \in \Gamma[C/c]$. Hence, $\Gamma[C/c] \vdash x : A'[C/c][\overline{B[C/c]}/\overline{a}]$ by (TVar) (i.e. the instance we choose has $B_i[C/c]$ replacing a_i instead of B_i replacing a_i). By the previous question, this is the same as $A'[\overline{B}/\overline{a}][C/c]$, i.e. $A[C/c]$, which was our goal.

SUBSTITUTION CLOSURE PROOF

In case (TApp), M is of shape PQ. Assume the induction hypotheses:

- $\Gamma[C/c] \vdash P : (B \to A)[C/c]$
- $\Gamma[C/c] \vdash Q : B[C/c]$

By definition, $(B \to A)[C/c]$ is just $B[C/c] \to A[C/c]$. Hence, by (TApp), $\Gamma[C/c] \vdash PQ : A[C/c]$ as required.