TYPES AND λ-CALCULUS

Problem Sheet 3

This week, questions 2, 8 and 9 will be marked.

Recall that a *closed* term has no free variables.

* 1. Perform one step of reduction for each of the following terms:

(a) $\underline{\mathsf{const}}\ \mathsf{pred}\ \mathsf{pred}$

(b) $\underline{\mathsf{sub}}\ \underline{\mathsf{const}}$

(c) $(\lambda x.\, x\, x)(\lambda x.\, x\, x)$

(d) $\underline{\mathsf{const}}\ (\mathsf{pred}\ \mathsf{pred})$

Solution

(a) $(\lambda y.\, \mathsf{pred})\ \mathsf{pred}$

(b) $\lambda y z.\, \underline{\mathsf{const}}\, z\, (y\, z)$

(c) $(\lambda x.\, x\, x)(\lambda x.\, x\, x)$

(d) $\lambda y.\, \mathsf{pred}\ \mathsf{pred}$ or $\underline{\mathsf{const}}\ \mathsf{wrong}$

* 2. For each of the following reduction steps $M \rhd N$, identify the redex P, the contraction Q, and the context $C[\,]$ in which the contraction happens, i.e. such that $M = C[P]$ and $N = C[Q]$.

(a) $\lambda x.\, \mathsf{pred}\ (\mathsf{pred}\ \underline{2}) \rhd \lambda x.\, \mathsf{pred}\ \underline{1}$

(b) $\underline{\mathsf{id}}\ (\underline{\mathsf{const}}\ \underline{\mathsf{div}}\ \underline{0}) \rhd \underline{\mathsf{id}}\ (\underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{div}})\ \underline{0})$

(c) $\underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{id}})\ (\mathsf{S}\ x) \rhd (\lambda y.\, \underline{\mathsf{id}}\ \underline{\mathsf{id}})\ (\mathsf{S}\ x)$

Solution

(a) $P = \mathsf{pred}\ \underline{2}$, $Q = \underline{1}$, $C[\,] = \lambda x.\, \mathsf{pred}\ [\,]$

(b) $P = \underline{\mathsf{div}}$, $Q = \underline{\mathsf{id}}\ \underline{\mathsf{div}}$, $C[\,] = \underline{\mathsf{id}}\ (\underline{\mathsf{const}}\ [\,]\ \underline{0})$
(c) $P = \underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{id}})$, $Q = \lambda y.\, \underline{\mathsf{id}}\ \underline{\mathsf{id}}$, $C[\,] = [\,]\ (\mathsf{S}\ x)$

* 3. Let us define the Booleans as follows:

$$\underline{\mathsf{false}} = \underline{0}$$
$$\underline{\mathsf{true}} = \underline{1}$$

Define Boolean conjunction as a term $\underline{\mathsf{and}}$, disjunction as a term $\underline{\mathsf{or}}$ and negation as a term $\underline{\mathsf{not}}$.

Solution

Define:

$$\underline{\mathsf{not}} = \lambda x.\, \mathsf{ifz}\, x\, \underline{\mathsf{true}}\, \underline{\mathsf{false}}$$
$$\underline{\mathsf{and}} = \lambda x y.\, \mathsf{ifz}\, x\, \underline{\mathsf{false}}\, (\mathsf{ifz}\, y\, \underline{\mathsf{false}}\, \underline{\mathsf{true}})$$
$$\underline{\mathsf{or}} = \lambda x y.\, \underline{\mathsf{not}}\, (\underline{\mathsf{and}}\, (\underline{\mathsf{not}}\, x)\, (\underline{\mathsf{not}}\, y))$$

* 4. Define terms $\underline{\mathsf{curry}}$ and $\underline{\mathsf{uncurry}}$ with the following behaviour:

$$\underline{\mathsf{curry}}\, M\, N\, P \ \rhd^*\ M\, (N, P)$$
$$\underline{\mathsf{uncurry}}\, M\, (N, P) \ \rhd^*\ M\, N\, P$$

Solution

$$\underline{\mathsf{curry}} := \lambda f\, x\, y.\, f\ (x, y)$$
$$\underline{\mathsf{uncurry}} := \lambda f\, p.\, f\ (\underline{\mathsf{proj}}^2_1\, p)\, (\underline{\mathsf{proj}}^2_2\, p)$$

* 5. For each of the following specifications, give an example of a *closed* term N *in normal form* that satisfies it (i.e do some reduction):

(a) $\underline{\mathsf{id}}\ \underline{\mathsf{id}}\ \triangleright^* N$
(b) $\underline{\mathsf{sub}}\ \underline{\mathsf{const}}\ \underline{\mathsf{const}}\ \triangleright^* N$
(c) $\mathsf{fix}\ (\lambda xy.\, y) \triangleright^* N$
(d) $(\lambda xy.\, yx)\ (\underline{\mathsf{const}}\ \underline{\mathsf{const}})\ (\lambda x.\, xx) \triangleright^* N$

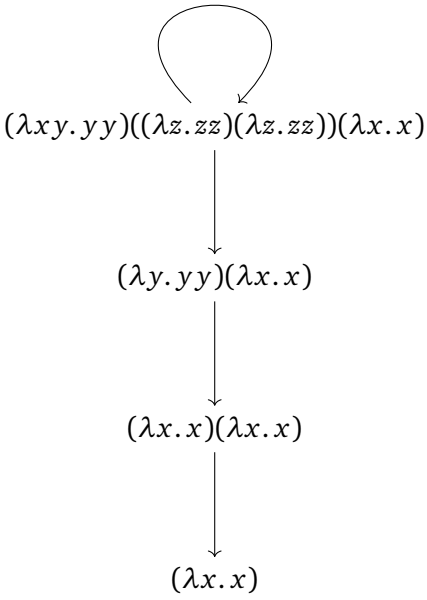
Solution

(a) $\underline{\mathsf{id}}$
(b) $\underline{\mathsf{id}}$
(c) $\underline{\mathsf{id}}$
(d) $\underline{\mathsf{const}}$

* 6. Draw the reduction graph of the term $(\lambda xy.\, yy)\ ((\lambda z.\, zz)\ (\lambda z.\, zz))\ (\lambda x.\, x)$. (This graph will have 4 vertices). What I mean by this is to draw a directed graph where:

- The nodes are all N s.t. $(\lambda xy.\, yy)\ ((\lambda z.\, zz)\ (\lambda z.\, zz))\ (\lambda x.\, x) \triangleright^* N$
- There is an edge from node M to node N iff $M \triangleright N$

Solution



** 7. Give an example of a *closed* term M for each of the following properties:

(a) M is in normal form.

(b) M has exactly one reduct.

(c) M contains strictly fewer redexes than one of its reducts (here we mean "fewer in number", the redexes may be quite different).

(d) A reduct of M contains a redex that did not occur anywhere in M.

Solution

There are lots of possible answers, here is one set:

(a) $\underline{\mathsf{id}}$

(b) $\mathsf{pred}\ \underline{2}$

(c) $(\lambda x.\,xxx)\ (\underline{\mathsf{id}}\ \underline{1})$

(d) $(\lambda xy.\,xy)\ \underline{\mathsf{id}}$

** 8. Prove the following statement:

For all M, N and $C[\,]$: if $M \rhd N$ then $C[M] \rhd C[N]$.

Note that "if $M \rhd N$ then $C[M] \rhd C[N]$" is subtly different from the definition of $\rhd$ which says that $C[P] \rhd C[Q]$ whenever P is a redex and Q the contraction. Here, M and N can be any terms.

You do *not* need to use induction to prove it. You will need to work closely with the definition of $\rhd$: on the one hand you will assume $M \rhd N$ and want to know what you get out of it and, on the other hand, you will want to show $C[M] \rhd C[N]$ and thus need to know what evidence is required to put into it.

Look again at the definition of $\rhd$ using contexts. In the definition, "just if" means the same as "iff", so the definition of $\rhd$ can be seen as a pair of implications: one direction tells you what follows from $M \rhd N$ when you have it as an assumption (forwards reasoning) and the other tells you what you need in order to deduce $M \rhd N$ (backwards reasoning).

Solution

Let M, N be terms and let $C[\,]$ be an evaluation context. Suppose $M \rhd N$. Then it follows from the definition of $\rhd$ that there must exist an evaluation context $C'[\,]$ and a redex/contraction pair P, Q such that $M = C'[P]$ and $N = C'[Q]$. So, $M \rhd N$ is really:

$$C'[P] \rhd C'[Q]$$

Now, by these equations, $C[M] = C[C'[P]]$ and $C[N] = C[C'[Q]]$ (*). Our aim is to prove that $C[M] \rhd C[N]$ so, by definition of $\rhd$, we have to find a context $C''[\,]$ and a redex/contraction pair P', Q' such that $C[M] = C''[P']$ and $C''[Q'] = C[N]$. By (*) we can take $C''[\,] = C[C'[\,]]$ and $P' = P$ and $Q' = Q$.

** 9.

(a) Complete the following proof by filling in (a):

For all P, $C[\,]$, for all $n \in \mathbb{N}$: for all Q, if $P \rhd^n Q$ then $C[P] \rhd^n C[Q]$.

Proof. Let P be a term and $C[\,]$ a context. We show that, for all $n \in \mathbb{N}$, for all Q, $P \rhd^n Q$ implies $C[P] \rhd^n C[Q]$ by induction on n:

- When $n = 0$, let Q be a term and suppose $P \rhd^0 Q$. Then, by definition, $P = Q$ and hence $C[P] = C[Q]$. By definition, therefore $C[P] \rhd^0 C[Q]$.
- When n is of shape $k+1$, we can assume the induction hypothesis:

  (IH) forall Q, $P \rhd^k Q$ implies $C[P] \rhd^k C[Q]$.

  ... (a) ...

□

(b) Deduce that (i.e. give a short proof of): For all P, Q, $C[\,]$: if $P \rhd^* Q$ then $C[P] \rhd^* C[Q]$.

Solution

(a) Now, let Q be a term and suppose $P \rhd^{k+1} Q$. Then, by definition, there is some U such that (i) $P \rhd^k U$ and (ii) $U \rhd Q$. We can use (IH) on (i) to obtain $C[P] \rhd^k C[U]$ (*). We can use the statement of the previous question on (ii) to obtain $C[U] \rhd C[Q]$ (**). By definition of $\rhd^{k+1}$, we can use (*) and (**) to obtain $C[P] \rhd^{k+1} C[Q]$.

(b) Let P and Q be terms and $C[\,]$ a context. Suppose $P \rhd^* Q$. By definition, there is some n such that $P \rhd^n Q$. It follows from the previous result that, therefore, $C[P] \rhd^n C[Q]$. Hence, by definition of $\rhd^*$, $C[P] \rhd^* C[Q]$.

** 10. Show that there is no term P that satisfies: for all M and N, $P\,(MN) \rhd^* N$. In other words, prove that we cannot write a PCF program that extracts the argument of an application.

Solution

Suppose such a P exists and we look to obtain a contradiction. Then we know that this P satsifies $\forall MN.\, P\,(MN) \rhd^* N$, let's call this (*). Then, instantiating M by $\underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{id}})$ and N by $\underline{\mathsf{const}}$ in (*), we have:

$$P\ (\underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{id}})\ \underline{\mathsf{const}}) \rhd^* \underline{\mathsf{const}}$$

However, by simply performing the reductions, we also have
$P\ (\underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{id}})\ \underline{\mathsf{const}}) \rhd^* P\ (\underline{\mathsf{id}}\ \underline{\mathsf{id}})$. By instantiating (*) with $M = \underline{\mathsf{id}}$ and $N = \underline{\mathsf{id}}$ we get $P\ (\underline{\mathsf{id}}\ \underline{\mathsf{id}}) \rhd^* \underline{\mathsf{id}}$. So, overall, we also have:

$$P\ (\underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{id}})\ \underline{\mathsf{const}}) \rhd^* \underline{\mathsf{id}}$$

However, with the other reduction sequence inset above, this contradicts the unique normal forms theorem, since $\underline{\mathsf{id}}$ and $\underline{\mathsf{const}}$ are distinct normal forms of $P\ (\underline{\mathsf{const}}\ (\underline{\mathsf{id}}\ \underline{\mathsf{id}})\ \underline{\mathsf{const}})$, so we have our contradiction.