UNIVERSITY OF BRISTOL

January 2021 Examination Period

FACULTY OF ENGINEERING

Third Year Examination for the Degrees
of
Bachelor of Science
Master of Engineering

COMS30039J
Types and Lambda Calculus

TIME ALLOWED:
2 Hours

Answers to COMS30039J: Types and Lambda Calculus

Intended Learning Outcomes:

Q1. This question concerns the pure, untyped λ-calculus.

It will be useful to recall the combinators $\mathbf{I} = \lambda x.\, x$ and $\mathbf{K} = \lambda xy.\, x$

(a) Give the set of free variables for each of the following terms:

i. $\lambda xyz.\, xz(yz)$
ii. $z(\lambda x.\, xy)$
iii. $\lambda z.\, (\lambda x.\, y)(xz)$
iv. $(\lambda x.\, x\, (\lambda y.\, xy))(\lambda x.\, y)$

[4 marks]

Solution:

i. $\emptyset$

ii. $\{z, y\}$

iii. $\{x, y\}$

iv. $\{y\}$

(b) For each of the following, give a term M that satisfies the statement.

i. $M\, x =_\beta xx$
ii. $M \ulcorner n \urcorner =_\beta \ulcorner 2 * n \urcorner$
iii. $M =_\beta MM$
iv. $M\, x =_\beta MM$

[8 marks]

Solution: Two marks each.

i. $\lambda y.\, yy$

ii. $\lambda x.\, \mathbf{Add}\ x\ x$

iii. $\lambda x.\, x$

iv. $\lambda x.\, y$

(c) Recall that there is a combinator **Pred** satisfying:

$$\begin{aligned} \mathbf{Pred}\, \ulcorner 0 \urcorner &=_\beta \ulcorner 0 \urcorner \\ \mathbf{Pred}\, \ulcorner n+1 \urcorner &=_\beta \ulcorner n \urcorner \end{aligned}$$

Define combinators **Sub** and **Succ** as follows:

$$\begin{aligned} \mathbf{Succ} &:= \lambda nfx.\, f\, (n\, f\, x) \\ \mathbf{Sub} &:= \lambda mn.\, n\, \mathbf{Pred}\, m \end{aligned}$$

Prove, by induction on n, that **Sub** satisfies:

$$\mathbf{Sub}\ \ulcorner m \urcorner\ \ulcorner n \urcorner =_\beta \begin{cases} \ulcorner 0 \urcorner & \text{if } m \leq n \\ \ulcorner m - n \urcorner & \text{otherwise} \end{cases}$$

You may use the fact that, for all natural numbers p, $\ulcorner p + 1 \urcorner =_\beta \mathbf{Succ}\ulcorner p \urcorner$.

[7 marks]

Solution: 3 marks for correct form of induction proof. 2 marks for appropriate case split within step case. 2 marks for avoiding any use of elipsis or similar informal reasoning in the step case e.g. by using the given fact.

The proof is by induction on n.

- When $n = 0$, $\mathbf{Sub}\ \ulcorner m \urcorner\ \ulcorner n \urcorner =_\beta \ulcorner 0 \urcorner\ \mathbf{Pred}\ \ulcorner m \urcorner$. By definition of $\ulcorner 0 \urcorner$, this is convertible with $\ulcorner m \urcorner$.

- When $n = k + 1$ for some k, using the given fact we have:

$$\begin{array}{rl} & \mathbf{Sub}\ \ulcorner m \urcorner\ \ulcorner n \urcorner \\ =_\beta & \mathbf{Succ}\ \ulcorner k \urcorner\ \mathbf{Pred}\ \ulcorner m \urcorner \\ =_\beta & \mathbf{Pred}(\ulcorner k \urcorner\ \mathbf{Pred}\ \ulcorner m \urcorner) \\ =_\beta & \mathbf{Pred}(\mathbf{Sub}\ \ulcorner m \urcorner\ \ulcorner k \urcorner) \end{array}$$

  Then we distinguish cases on the relationship between m and n.

  - If $m < n$ we have $m \leq k$, so it follows from the induction hypothesis that $\mathbf{Sub}\ \ulcorner m \urcorner\ \ulcorner k \urcorner =_\beta \ulcorner 0 \urcorner$ whence, by the key property of **Pred**, the whole expression is convertible with $\ulcorner 0 \urcorner$, as required.
  - If $m \geq n$ then $m > k$ and it follows from the induction hypothesis that $\mathbf{Sub}\ \ulcorner m \urcorner\ \ulcorner k \urcorner =_\beta \ulcorner m - k \urcorner$. In this case, $m - k > 0$. Hence, by the key property, $\mathbf{Pred}\ \ulcorner m - k \urcorner =_\beta \ulcorner m - k - 1 \urcorner = \ulcorner m - n \urcorner$, as required.

(d) Prove that there does not exist a term M such that, for all terms N:

$$M\ N =_\beta \begin{cases} \mathbf{I} & \text{if } N \text{ is in } \beta\text{-normal form} \\ \mathbf{K} & \text{otherwise} \end{cases}$$

[3 marks]

Solution: Suppose for contradiction that such an M exists. Then we would have:

$$\mathbf{I} =_\beta M\,\mathbf{I} =_\beta M\,(\mathbf{I}\,\mathbf{I}) =_\beta \mathbf{K}$$

but this is impossible since **I** and **K** are distinct β-normal forms.

(cont.)

(e) Find a finite sequence of *closed* terms $M_1, M_2, \ldots, M_k$ for $k \geq 0$ such that the following two equations are both satisfied:

$$\begin{aligned}(\lambda x.\, x\,\mathbf{I}\,(x\,\mathbf{I}\,\mathbf{I}))\, M_1\, M_2\, \cdots\, M_k\, x\, y &=_\beta x \\ (\lambda x.\, x\,\mathbf{I}\,(x\,x\,\mathbf{I}))\, M_1\, M_2\, \cdots\, M_k\, x\, y &=_\beta y\end{aligned}$$

[3 marks]

Solution: An appropriate sequence is, $k = 6$:

$$(\lambda xyz.\, zxy), (\lambda xy.\, y), (\lambda xy.\, x), (\lambda wxyz.\, y), (\lambda x.\, x), (\lambda wxyz.\, z)$$

Q2. This question concerns type systems.

(a) Give a typing derivation for each of the following judgements:

i. $\vdash \lambda xy.\, x : a \to b \to a$

ii. $\vdash \lambda x.\, x(\lambda y.\, y) : ((a \to a) \to b) \to b$

iii. $y : c \vdash (\lambda x.\, y)(\lambda xz.\, x) : c$

[6 marks]

Solution: 2 marks each.

i.

$$\dfrac{\dfrac{\dfrac{}{x : a, y : b \vdash x : a}}{x : a \vdash \lambda y.\, x : b \to a}}{\vdash \lambda xy.\, x : a \to b \to a}$$

ii.

$$\dfrac{\dfrac{\dfrac{}{x : (a \to a) \to b \vdash x : (a \to a) \to b} \qquad \dfrac{\dfrac{}{x : (a \to a) \to b, y : a \vdash y : a}}{x : (a \to a) \to b \vdash \lambda y.\, y : a \to a}}{x : (a \to a) \to b \vdash x(\lambda y.\, y) : b}}{\vdash \lambda x.\, x(\lambda y.\, y) : ((a \to a) \to b) \to b}$$

iii.

$$\dfrac{\dfrac{\dfrac{}{y : c, x : a \to b \to a \vdash y : c}}{y : c \vdash \lambda x.\, y : (a \to b \to a) \to c} \qquad \dfrac{\dfrac{\dfrac{}{y : c, x : a, z : b \vdash x : a}}{y : c, x : a \vdash \lambda z.\, x : b \to a}}{y : c \vdash \lambda xz.\, x : a \to b \to a}}{y : c \vdash (\lambda x.\, y)(\lambda xz.\, x) : c}$$

(b) For each of the following types, find a closed term that inhabits the type:

i. $a \to b \to b$

ii. $(a \to b) \to (a \to b \to c) \to a \to c$

iii. $c \to ((a \to b \to a) \to c \to d) \to d$

[6 marks]

Solution: 2 marks each.

i. $\lambda xy.\, y$

ii. $\lambda xyz.\, y\, z\, (xz)$

iii. $\lambda xy.\, y\, (\lambda xy.\, x)\, x$

(c) Prove, by induction on the Damas-Miler typing relation, that: if $\Gamma \vdash M : A$ then, for all Γ', $\Gamma \subseteq \Gamma'$ implies $\Gamma' \vdash M : A$.

[6 marks]

Solution: One mark for using the induction principle correctly, one mark per case and an additional mark for constructing Γ' correctly to make use of the induction hypothesis in the abs case.
The proof is by induction on $\Gamma \vdash M : A$.

(TVar) In this case M is a variable x, A is an instantiation $B[\overline{C/b}]$ and we assume $x : \forall \overline{b}.\, B \in \Gamma$. Let Γ' be such that $\Gamma \subseteq \Gamma'$, then also $x : \forall \overline{b}.\, B \in \Gamma'$ and so the result follows immediately by (TVar).

(TApp) In this case M is an application PQ and we assume the induction hypotheses:

(IH1) for all Γ', $\Gamma \subseteq \Gamma'$ implies $\Gamma' \vdash P : B \to A$.

(IH2) for all Γ', $\Gamma \subseteq \Gamma'$ implies $\Gamma' \vdash Q : B$.

Let Γ' be such that $\Gamma \subseteq \Gamma'$. Then we immediately obtain $\Gamma' \vdash P : B \to A$ and $\Gamma' \vdash Q : B$ so that $\Gamma' \vdash PQ : A$ by (TApp).

(TAbs) In this case, M is an abstraction $\lambda x.\, P$ and A is an arrow $B \to C$. Assume the induction hypothesis:

(IH1) for all Γ', $\Gamma \subseteq \Gamma'$ implies $\Gamma' \cup \{x : B\} \vdash P : C$

Let Γ'' be such that $\Gamma \subseteq \Gamma''$, then construct $\Gamma' := \Gamma'' \cup \{x : B\}$. Hence, $\Gamma \subseteq \Gamma'$ and we obtain $\Gamma' \cup \{x : B\} \vdash P : C$ by the induction hypothesis. This is the same as $\Gamma'' \cup \{x : B\} \vdash P : C$ and so we obtain $\Gamma'' \vdash \lambda x.\, P : B \to C$ by (TAbs).

(d) Define the order, $\mathsf{order}(A)$ of a monotype A recursively on the structure of A:

$$\begin{aligned} \mathsf{order}(a) &= 0 \\ \mathsf{order}(B \to C) &= \max(1 + \mathsf{order}(B), \mathsf{order}(C)) \end{aligned}$$

where $\max(m, n)$ is the larger of the two natural numbers m and n. Prove that, if $\mathsf{order}(A) \leq 2$ and $\vdash M : A$, then there is a term N such that:

(i) $M =_\beta \lambda x_1 \ldots x_n.\, N$

(ii) and, moreover, N does not contain any abstraction as a subterm

[7 marks]

Solution: Four marks for point (i) and another three for point (ii)

For point (i), if $\mathsf{order}(A) \leq 2$ then A is of shape:

$$B_1 \to \cdots \to B_k \to b$$

for some $k \geq 0$, some k monotypes B_i and some type variable b. Moreover, we have that $\mathsf{order}(B_i) \leq 1$. Hence, each B_i has shape:

$$c_1 \to \cdots c_m \to c$$

for some $k \geq 0$, some m type variables c_i and some other type variable c.

Then, since $\vdash M : A$, by Strong Normalisation and Subject Reduction, there is a normal form of M, say P with $\vdash P : A$. By Inversion, this P must have shape $\lambda x_1 \ldots x_k. N$ with $x_1 : B_1, \ldots, x_k : B_k \vdash N : b$.

For point (ii) we reason as follows. Suppose, for the purpose of obtaining a contradiction, that N contains an abstraction as a subterm. If N is itself an abstraction, then this contradicts Inversion, since the type would be required to be an arrow. Otherwise some proper subterm of N is an abstraction, but then it must be the argument to some variable since N is in normal form, i.e. there is a subterm of shape:

$$x\, N_1 \,\cdots\, N_k$$

and the abstraction subterm is one of the N_i, say N_j. However, x must be one of the x_i and hence we would have $\cdots \vdash N_j : c$ which contradicts Inversion.

END OF PAPER