Numerical linear algebra in the Canadian hydrological model

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Math 313
University of Saskatchewan
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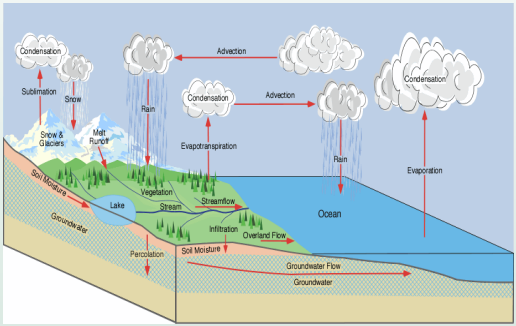
Outline

Hydrology

- The scientific study of the movement, distribution, and quality of water on Earth and other planets.



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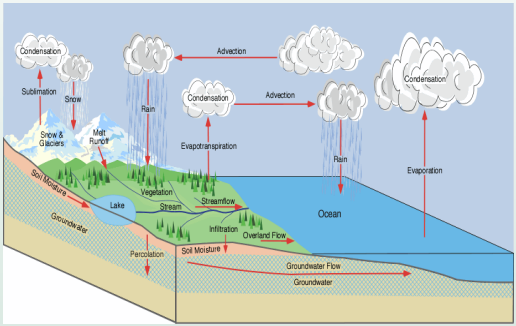

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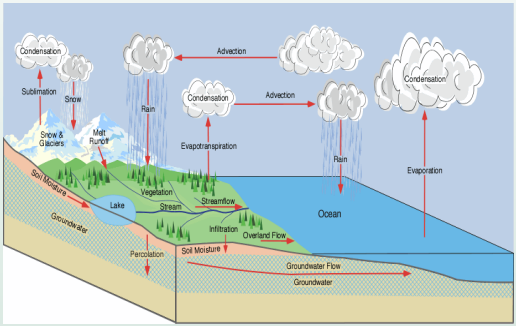

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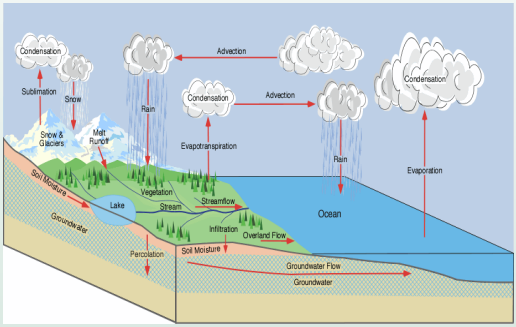
- How to model a given process?



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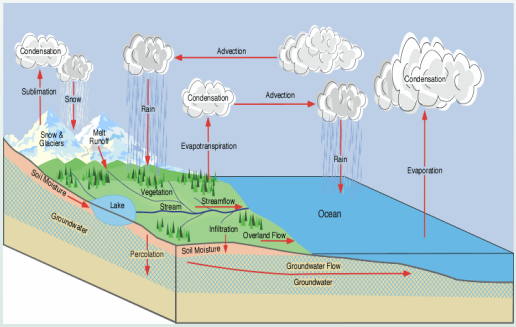
The science isn't always settled:

- How to model a given process?
- How do processes interact?



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What is CHM?

The Canadian hydrological model (CHM) is a novel unstructured mesh-based approach to hydrological modelling. It is designed with a plug-in architecture of modular process representations to assist with the development and testing of hydrological models. The combination of unstructured meshes with plug-in processes allows CHM to bridge spatial scales, temporal scales, and spatial extents.

`https://github.com/Chrismarsh/CHM`

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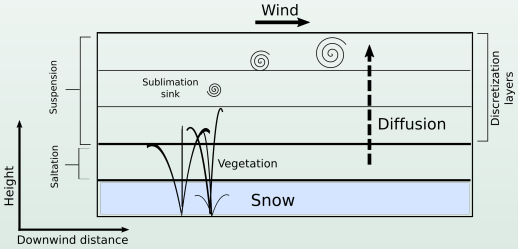
- is designed with the ability to plug in various *types* of other models as well.

Example: Prairie blowing snow model (PBSM3D)

- *Partial differential equation* model for blowing snow.



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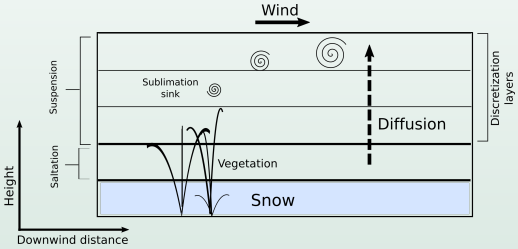
- *Partial differential equation* model for blowing snow.
- Saltation and suspension processes.
  - Multi-layer suspension.



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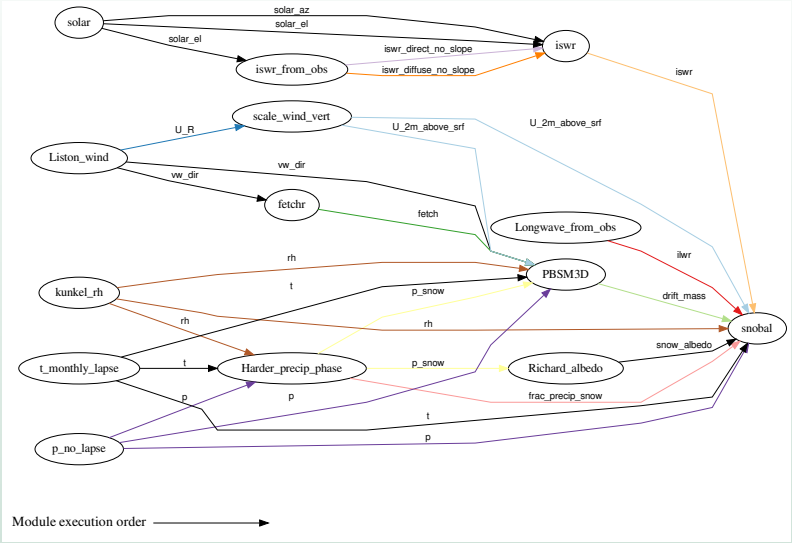
A *basic* simulation with PBSM3D blowing snow



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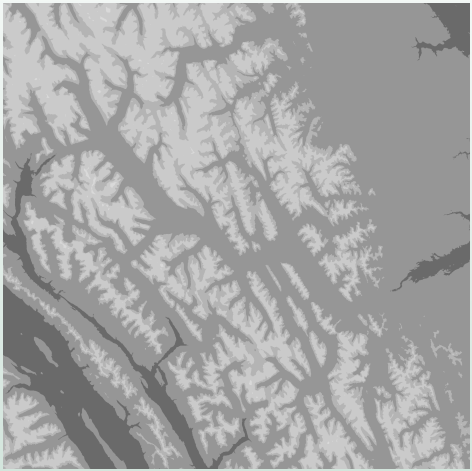
$\rightarrow$ What is $u(x, y)$ for all x and y?

Spatial domains

- Geospatial data: latitude, longitude, elevation *(+ other data)*



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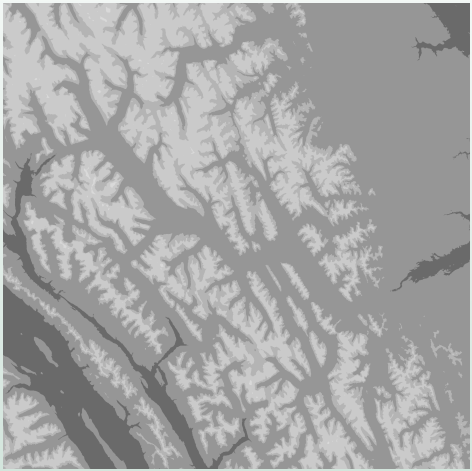
- Geospatial data: latitude, longitude, elevation *(+ other data)*
- Raster format *(data at each "pixel")*



Mesh generation

Goal:

Produce a representation of the spatial domain

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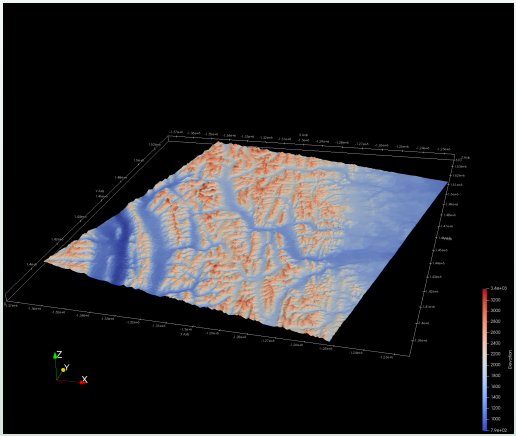
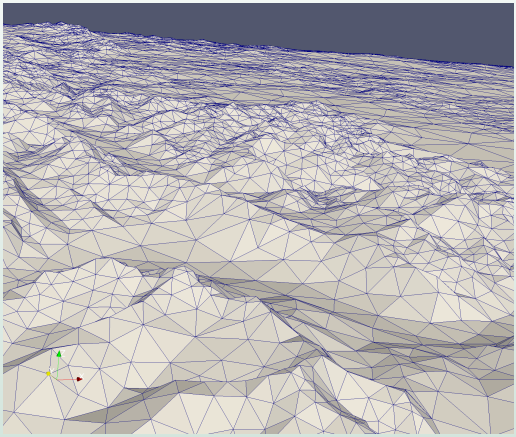
(This domain is $\sim$ 150 km $\times$ 150 km)



Bow Valley, Alberta.

Triangular mesh

Spatial representation is made up of a bunch of planar triangles



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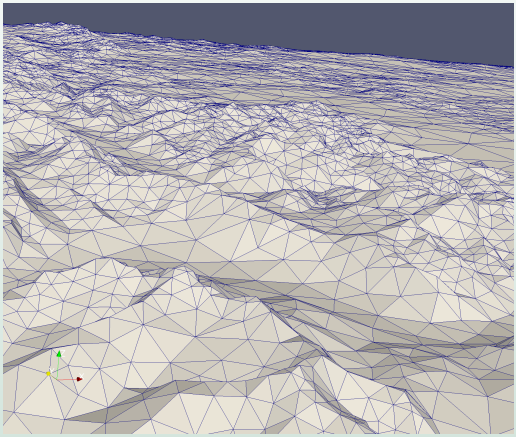

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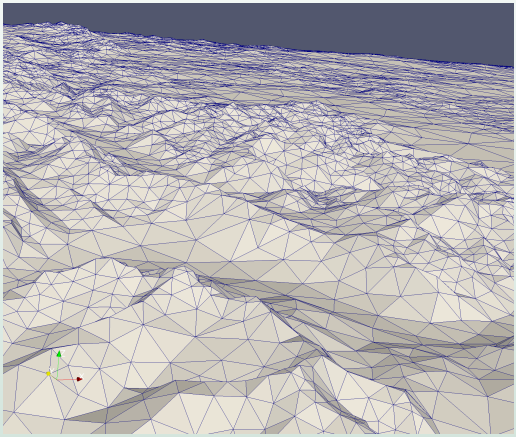

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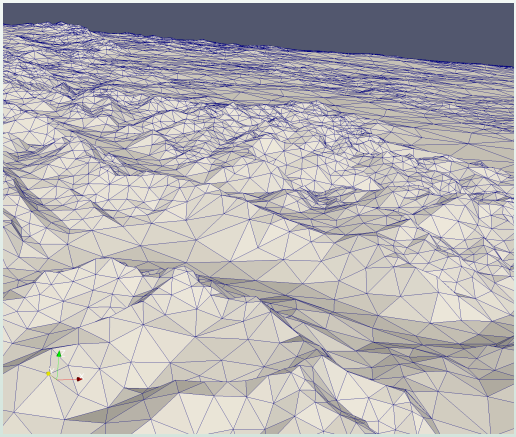

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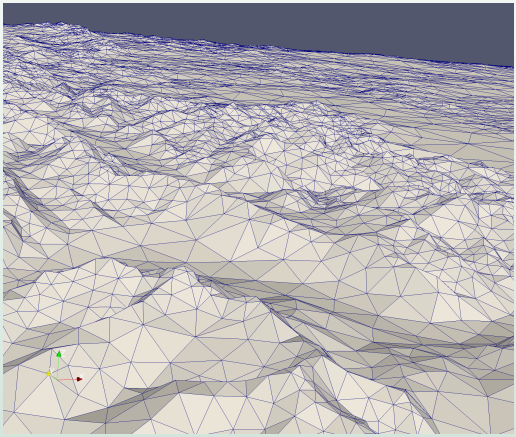
How do we get here?



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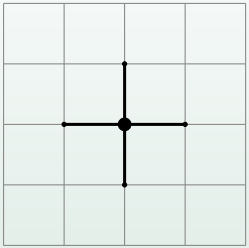
Discretizing a PDE

Finite difference method

- approximate spatial derivatives with finite differences (only really useful for highly structured meshes)



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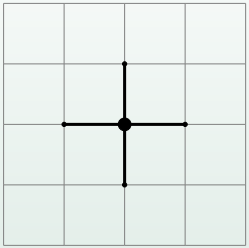

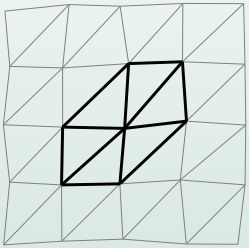

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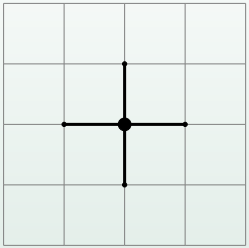
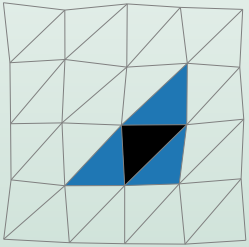
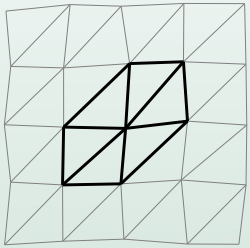
Finite volume method

- what flows out of one element (triangle), flows into its neighbour







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Direct methods

Direct "inversion" of linear systems.

- Obtain numerically precise solution
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P, Q – *permutation matrices*

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Pro/Con:

- Time to solution dependent *entirely* on matrix structure
  - *might* be substantially better than worst-case

Permutations

Free to rearrange our equations and unknowns.

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Want $\tilde{A}$ to have a *better* structure.

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$\rightarrow$ No loss of stability for certain PDE operators.

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How? *(NP-Hard) problem*

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We use RCM for reduced bandwidth order.

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How? *(Heuristics, again)*

- Algorithms based on Multilevel Nested Dissection (ND) [George1973]

Specific problem (from PBSM3D)

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(Requires much more, but we're just going to stick to this)

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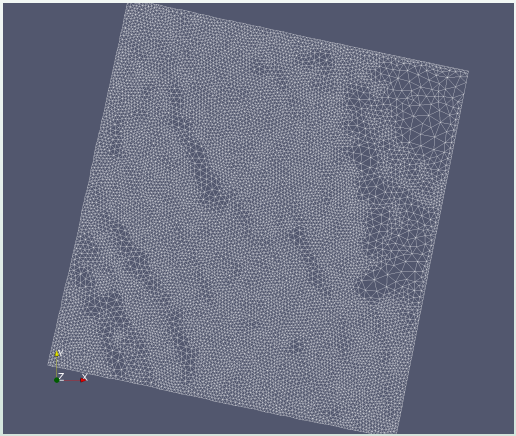
Use known solution $u^* = 1^{N\times 1}$, $b^* = Au^*$ for testing, i.e., solve $Au = b^*$.

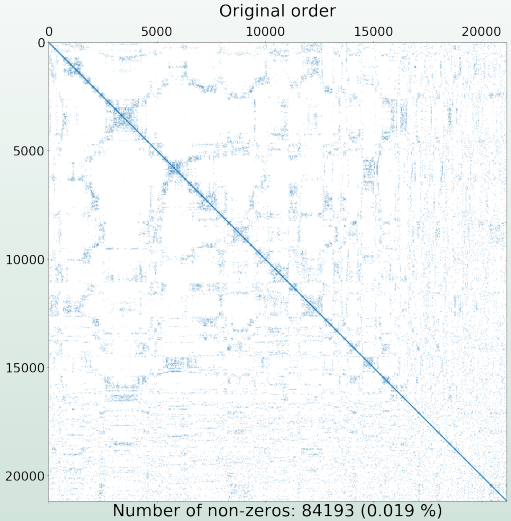
Mesh: coarse resolution

Low resolution (21168 elements)



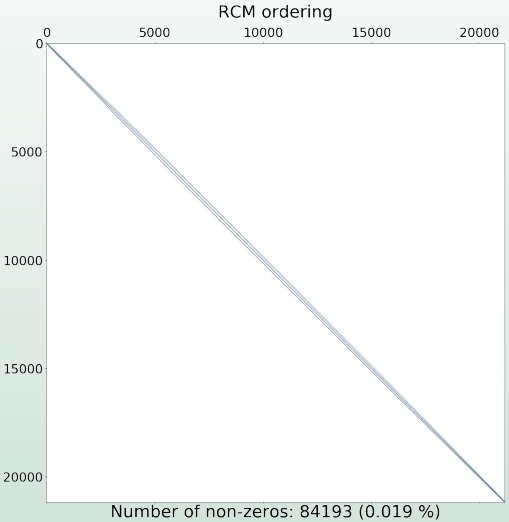
Matrix structure: original order

Bandwidth: 21062



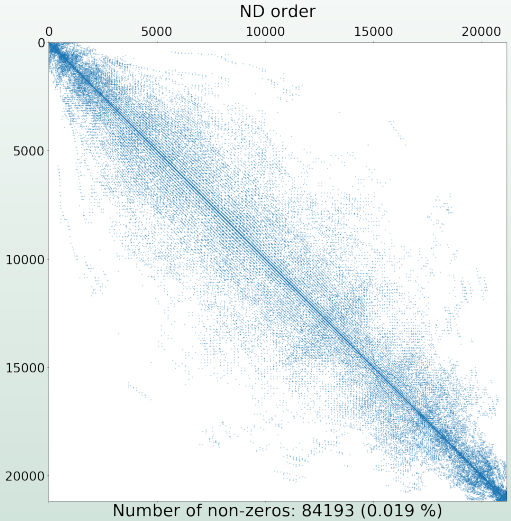
Matrix structure: RCM order

Bandwidth: 168



Matrix structure: ND order

Bandwidth: 12269



Performance: coarse mesh

Not shown here.

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Jupyter notebook

- `https:github.com/uofs-simlab/Math313`
- Can run easily from `colab.research.google.com` if you have a Google account.

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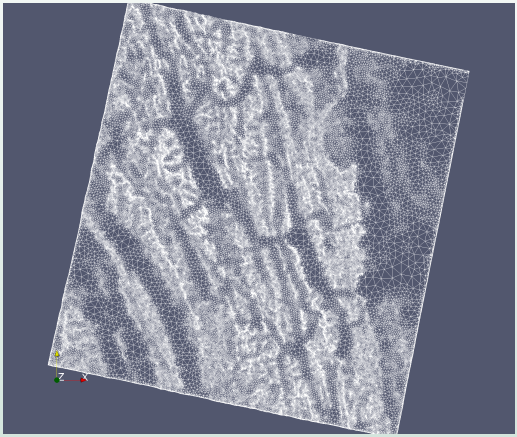
I went bigger...

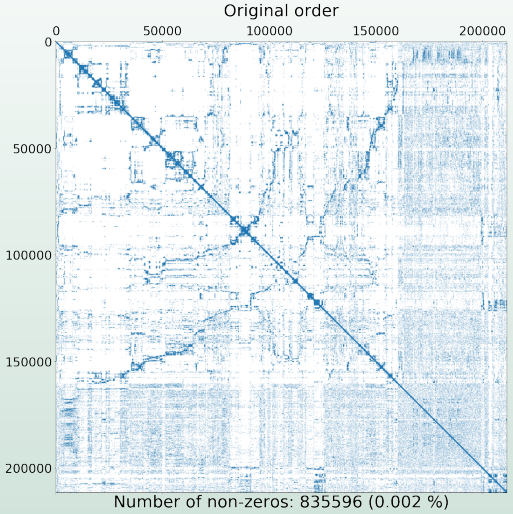
Mesh: fine resolution

High resolution (211415 elements)



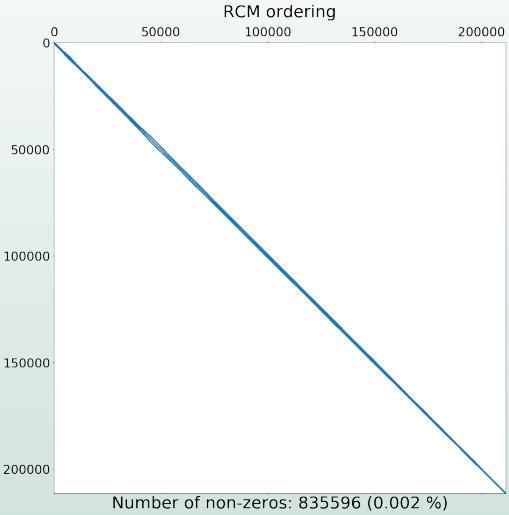
Matrix structure: original order

Bandwidth: 210855



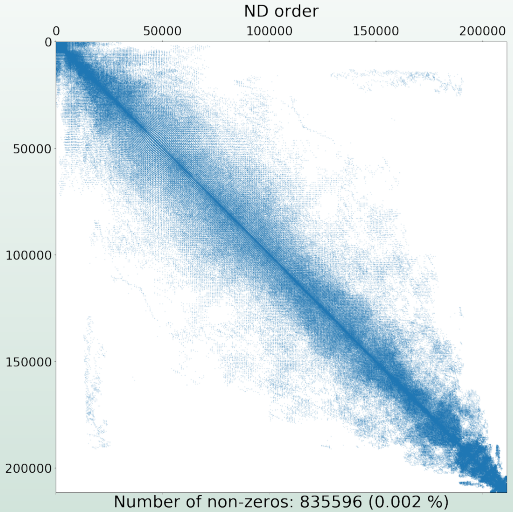
Matrix structure: RCM order

Bandwidth: 1320



Matrix: ND order

Bandwidth: 173136



Performance: LU – Original order

LU factorization

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CPU times:
  user:  8d 12h 41min 15s,
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Wall time:  6h 58min 57s
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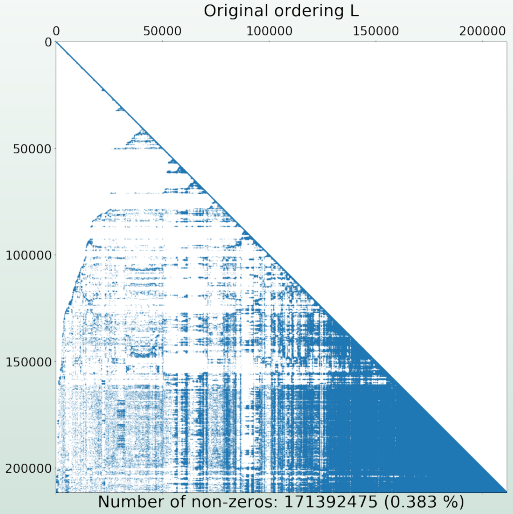

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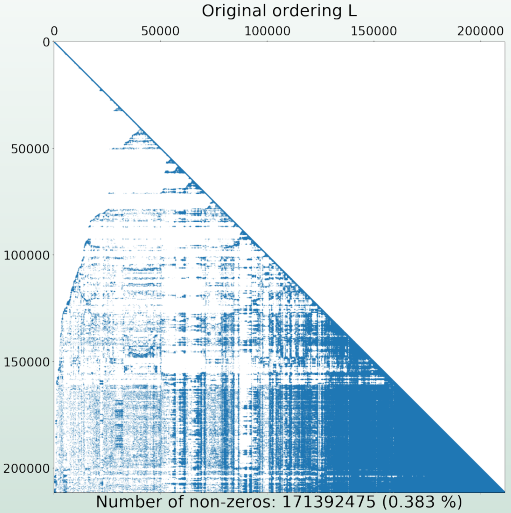

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Wall time:  6h 59min 5s
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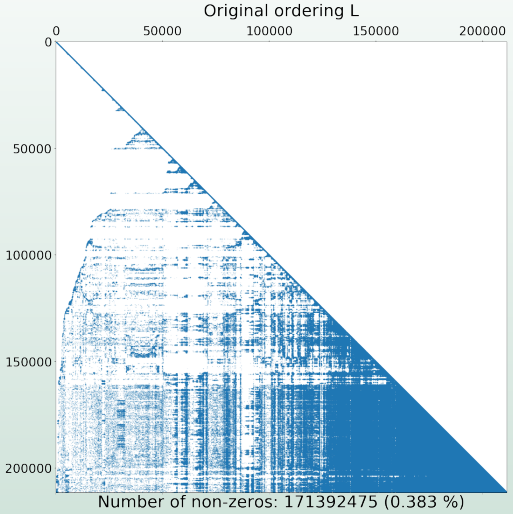

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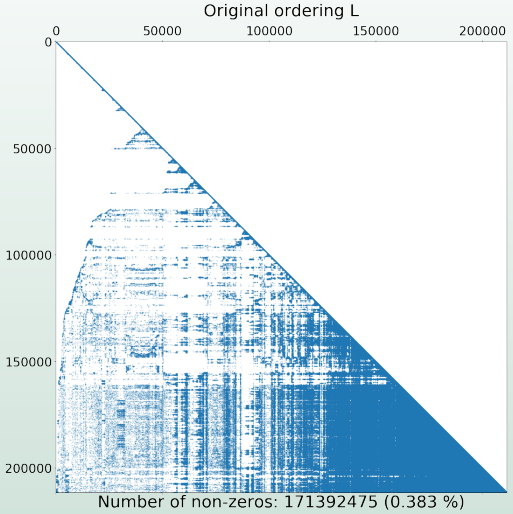

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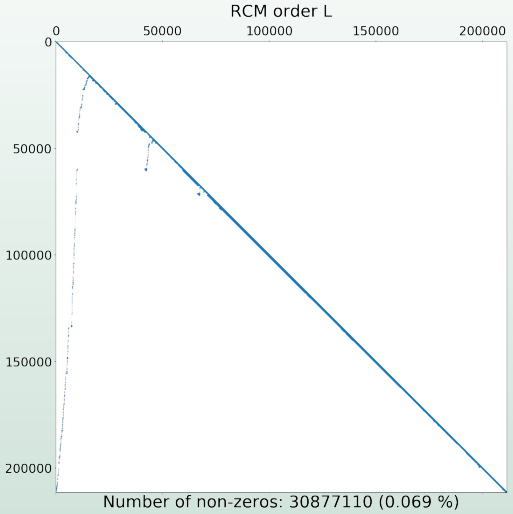

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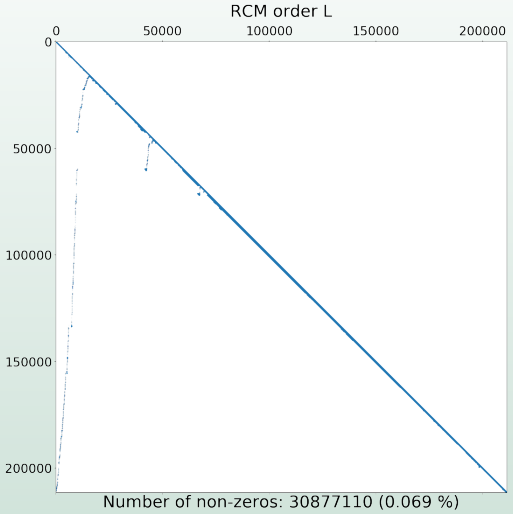

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```
Wall time:  0.741 s
```

Total

```
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```

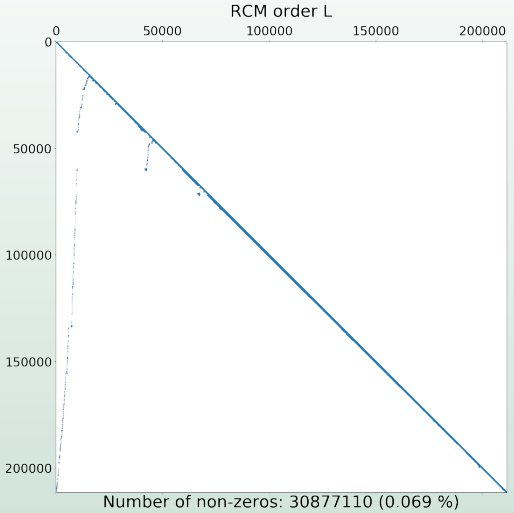

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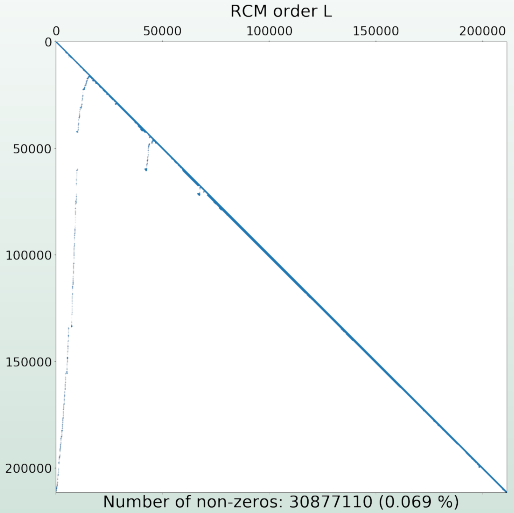

Performance: LU – ND order

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CPU times:
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 Wall time:  1.25 s
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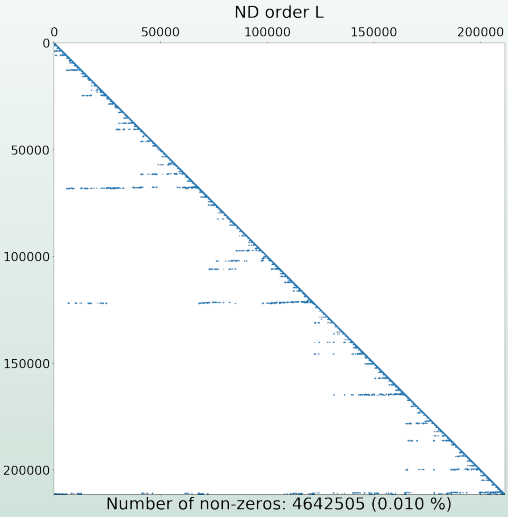

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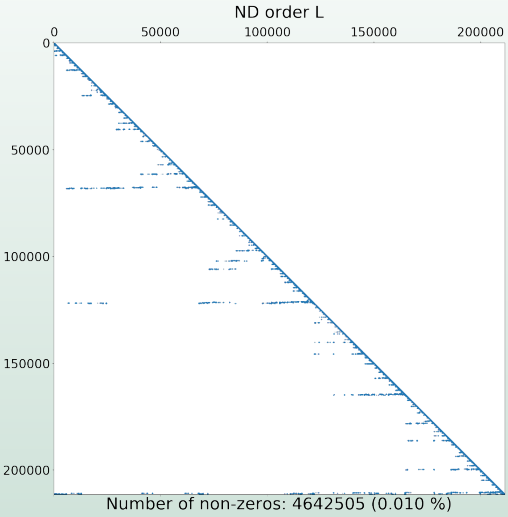

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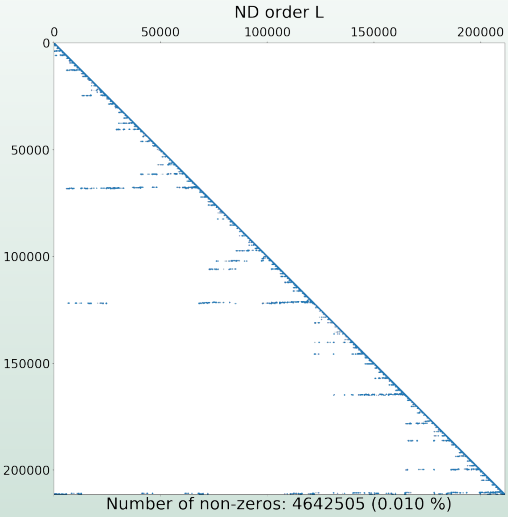
Solve

```
 Wall time:  0.049 s
```

Total

```
 Wall time:  1.30s
```



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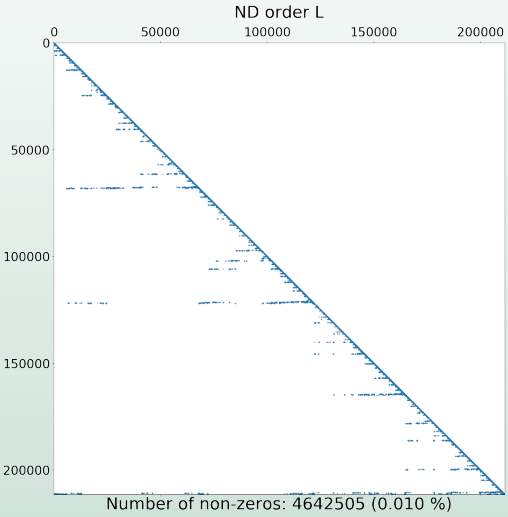

Fine mesh summary

211415 elements $\rightarrow$ 835596 non-zero matrix entries

LU		
Order	Time (s)	Accuracy
original	25145	1.2e-13
RCM	53.3	1.6e-13
ND	1.3	1.5e-13

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  - machine-dependent things *(High-performance computing... if you're interested)*