1 Introduction

We consider the Bidomain equations:

$$\chi C_{\mathrm{m}} \frac{\partial v}{\partial t} + \chi(I_{\mathrm{ion}}(v) + I_{\mathrm{stim}}) = \nabla \cdot (\sigma_{\mathrm{i}} \nabla(v + u_{\mathrm{e}}))$$

$$0 = \nabla \cdot (\sigma_{\mathrm{i}} \nabla(v + u_{\mathrm{e}})) + \nabla \cdot (\sigma_{\mathrm{e}} \nabla u_{\mathrm{e}})$$

We will be considering Neumann boundary conditions only. The domain is $\Omega = [0,1]^n$.

We introduce a new state variable w, and the FitzHugh-Nagumo cell model:

$$I_{\mathrm{ion}}(v, w) = \varepsilon^{-1}(v - \frac{v^3}{3} - w)$$

$$\frac{\partial w}{\partial t} = \varepsilon(v + \beta - \gamma w)$$

When we add this cell model into the original equations, we have the new system:

$$\chi C_{\mathrm{m}} \frac{\partial v}{\partial t} + \chi \varepsilon^{-1}(v - \frac{v^3}{3} - w) + \chi I_{\mathrm{stim}} = \nabla \cdot (\sigma_{\mathrm{i}} \nabla(v + u_{\mathrm{e}}))$$

$$\frac{\partial w}{\partial t} = \varepsilon(v + \beta - \gamma w)$$

$$0 = \nabla \cdot (\sigma_{\mathrm{i}} \nabla(v + u_{\mathrm{e}})) + \nabla \cdot (\sigma_{\mathrm{e}} \nabla u_{\mathrm{e}})$$

We will be splitting the above system of equations into an explicit operator involving I_{ion} and I_{stim}:

$$\chi C_{\mathrm{m}} \frac{\partial v}{\partial t} + \chi \varepsilon^{-1}(v - \frac{v^3}{3} - w) + \chi I_{\mathrm{stim}} = 0$$

$$\frac{\partial w}{\partial t} = \varepsilon(v + \beta - \gamma w)$$

$$0 = 0$$

And an implicit operator involving the Laplacian terms:

$$\chi C_{\mathrm{m}} \frac{\partial v}{\partial t} = \nabla \cdot (\sigma_{\mathrm{i}} \nabla(v + u_{\mathrm{e}}))$$

$$\frac{\partial w}{\partial t} = 0$$

$$0 = \nabla \cdot (\sigma_{\mathrm{i}} \nabla(v + u_{\mathrm{e}})) + \nabla \cdot (\sigma_{\mathrm{e}} \nabla u_{\mathrm{e}})$$

2 Spatial Discretization

Consider the monolithic system of equations, and let $\boldsymbol{q} = \begin{bmatrix} v \\ w \\ u_{\mathrm{e}} \end{bmatrix}$. We consider a

set of basis functions $\boldsymbol{\Phi}_i : \Omega \to \mathbb{R}^3$, where each $\boldsymbol{\Phi}_i = \varphi_{b_i} \hat{\mathbf{e}}_{c_i}$ for some basis functions $\varphi_i : \Omega \to \mathbb{R}$. We make the approximation $\boldsymbol{q} = Q_j \boldsymbol{\Phi}_j$ (Einstein summation) for a vector $\boldsymbol{Q} \in \mathbb{R}^N$, where N is the number of degrees of freedom. Also, we approximate $\frac{\partial \boldsymbol{q}}{\partial t} = \dot{Q}_j \boldsymbol{\Phi}_j$. We multiply the monolithic system of equations by a test function $\boldsymbol{\Phi}_i$ and integrate:

$$\left\langle \boldsymbol{\Phi}_i \;\middle|\; \begin{matrix} \chi C_{\mathrm{m}} \partial_t v \\ \partial_t w \\ 0 \end{matrix} \right\rangle = \left\langle \boldsymbol{\Phi}_i \;\middle|\; \begin{matrix} -\chi \varepsilon^{-1}(v - \frac{v^3}{3} - w) - \chi I_{\mathrm{stim}} \\ \varepsilon(v + \beta - \gamma w) \\ 0 \end{matrix} \right\rangle$$

$$+ \left\langle \boldsymbol{\Phi}_i \;\middle|\; \begin{matrix} \nabla \cdot (\sigma_{\mathrm{i}} \nabla(v + u_{\mathrm{e}})) \\ 0 \\ \nabla \cdot (\sigma_{\mathrm{i}} \nabla(v + u_{\mathrm{e}})) + \nabla \cdot (\sigma_{\mathrm{e}} \nabla u_{\mathrm{e}}) \end{matrix} \right\rangle$$

Using the fact that $v = Q_j \varphi_{b_j} \delta_{1,c_j}$, $w = Q_j \varphi_{b_j} \delta_{2,c_j}$, and $u_{\mathrm{e}} = Q_j \varphi_{b_j} \delta_{3,c_j}$, as well as $\frac{\partial v}{\partial t} = \dot{Q}_j \varphi b_j \delta_{1,c_j}$ and $\frac{\partial w}{\partial t} = \dot{Q}_j \varphi_{b_j} \delta_{2,c_j}$, we consider each of the forms above:

$$\left\langle \boldsymbol{\Phi}_i \;\middle|\; \begin{matrix} \chi C_{\mathrm{m}} \partial_t v \\ \partial_t w \\ 0 \end{matrix} \right\rangle$$

$$= \delta_{1,c_i} \left\langle \varphi_{b_i} \mid \chi C_{\mathrm{m}} \partial_t v \right\rangle + \delta_{2,c_i} \left\langle \varphi_{b_i} \mid \partial_t w \right\rangle$$

$$= \chi C_{\mathrm{m}} \delta_{1,c_i} \delta_{1,c_j} \left\langle \varphi_{b_i} \mid \dot{Q}_j \varphi_{b_j} \right\rangle + \delta_{2,c_i} \delta_{2,c_j} \left\langle \varphi_{b_i} \mid \dot{Q}_j \varphi_{b_j} \right\rangle$$

$$= (\chi C_{\mathrm{m}} \delta_{1,c_i} \delta_{1,c_j} + \delta_{2,c_i} \delta_{2,c_j}) \left\langle \varphi_{b_i} \mid \varphi_{b_j} \right\rangle \dot{Q}_j$$

$$\left\langle \boldsymbol{\Phi}_i \;\middle|\; \begin{matrix} -\chi \varepsilon^{-1}(v - \frac{v^3}{3} - w) - \chi I_{\mathrm{stim}} \\ \varepsilon(v + \beta - \gamma w) \\ 0 \end{matrix} \right\rangle$$

$$= \delta_{1,c_i} \left\langle \varphi_{b_i} \;\middle|\; -\chi \varepsilon^{-1}(v - \frac{v^3}{3} - w) - \chi I_{\mathrm{stim}} \right\rangle + \delta_{2,c_i} \left\langle \varphi_{b_i} \mid \varepsilon(v + \beta - \gamma w) \right\rangle$$

$$= (-\chi \varepsilon^{-1} \delta_{1,c_i} + \varepsilon \delta_{2,c_i}) \left\langle \varphi_{b_i} \mid v \right\rangle + (\chi \varepsilon^{-1} \delta_{1,c_i} - \varepsilon \gamma \delta_{2,c_i}) \left\langle \varphi_{b_i} \mid w \right\rangle$$

$$+ \frac{1}{3} \chi \varepsilon^{-1} \delta_{1,c_i} \left\langle \varphi_{b_i} \mid v^3 \right\rangle - \chi \delta_{1,c_i} \left\langle \varphi_{b_i} \mid I_{\mathrm{stim}} \right\rangle + \varepsilon \beta \delta_{2,c_i} \left\langle \varphi_{b_i} \mid 1 \right\rangle$$

$$= (-\chi \varepsilon^{-1} \delta_{1,c_i} + \varepsilon \delta_{2,c_i}) \delta_{1,c_j} \left\langle \varphi_{b_i} \mid Q_j \varphi_{b_j} \right\rangle + (\chi \varepsilon^{-1} \delta_{1,c_i} - \varepsilon \gamma \delta_{2,c_i}) \delta_{2,c_j} \left\langle \varphi_{b_i} \mid Q_j \varphi_{b_j} \right\rangle$$

$$+ \frac{1}{3} \chi \varepsilon^{-1} \delta_{1,c_i} \left\langle \varphi_{b_i} \mid v^3 \right\rangle - \chi \delta_{1,c_i} \left\langle \varphi_{b_i} \mid I_{\mathrm{stim}} \right\rangle + \varepsilon \beta \delta_{2,c_i} \left\langle \varphi_{b_i} \mid 1 \right\rangle$$

$$= (-\chi \varepsilon^{-1} \delta_{1,c_i} \delta_{1,c_j} + \varepsilon \delta_{2,c_i} \delta_{1,c_j} + \chi \varepsilon^{-1} \delta_{1,c_i} \delta_{2,c_j} - \varepsilon \gamma \delta_{2,c_i} \delta_{2,c_j}) \left\langle \varphi_{b_i} \mid \varphi_{b_j} \right\rangle Q_j$$

$$+ \frac{1}{3} \chi \varepsilon^{-1} \delta_{1,c_i} \left\langle \varphi_{b_i} \mid v^3 \right\rangle - \chi \delta_{1,c_i} \left\langle \varphi_{b_i} \mid I_{\mathrm{stim}} \right\rangle + \varepsilon \beta \delta_{2,c_i} \left\langle \varphi_{b_i} \mid 1 \right\rangle$$

$$\left\langle \boldsymbol{\Phi}_i \left| \begin{array}{c} \nabla \cdot (\sigma_{\mathrm{i}} \nabla (v + u_{\mathrm{e}})) \\ 0 \\ \nabla \cdot (\sigma_{\mathrm{i}} \nabla (v + u_{\mathrm{e}})) + \nabla \cdot (\sigma_{\mathrm{e}} \nabla u_{\mathrm{e}}) \end{array} \right. \right\rangle$$

$$= \delta_{\{1,3\},c_i} \left\langle \varphi_{b_i} \mid \nabla \cdot (\sigma_{\mathrm{i}} \nabla (v + u_{\mathrm{e}})) \right\rangle + \delta_{3,c_i} \left\langle \varphi_{b_i} \mid \nabla \cdot (\sigma_{\mathrm{e}} \nabla u_{\mathrm{e}}) \right\rangle$$

$$= -\delta_{\{1,3\},c_i} \left\langle \nabla \varphi_{b_i} \mid \sigma_i \nabla (v + u_{\mathrm{e}}) \right\rangle - \delta_{3,c_i} \left\langle \nabla \varphi_{b_i} \mid \sigma_{\mathrm{e}} \nabla u_{\mathrm{e}} \right\rangle$$

$$= -\sigma_{\mathrm{i}} \delta_{\{1,3\},c_i} \left\langle \nabla \varphi_{b_i} \mid \nabla (Q_j \varphi_{b_j} \delta_{\{1,3\},c_j}) \right\rangle - \sigma_{\mathrm{e}} \delta_{3,c_i} \left\langle \nabla \varphi_{b_i} \mid \nabla (Q_j \varphi_{b_j} \delta_{3,c_j}) \right\rangle$$

$$= -(\sigma_{\mathrm{i}} \delta_{\{1,3\},c_i} \delta_{\{1,3\},c_j} + \sigma_{\mathrm{e}} \delta_{3,c_i} \delta_{3,c_j}) \left\langle \nabla \varphi_{b_i} \mid \nabla \varphi_{b_j} \right\rangle Q_j$$

So, define:

$$M_{ij} = (\chi C_{\mathrm{m}} \delta_{1,c_i} \delta_{1,c_j} + \delta_{2,c_i} \delta_{2,c_j}) \left\langle \varphi_{b_i} \mid \varphi_{b_j} \right\rangle$$

$$A_{ij} = (-\chi \varepsilon^{-1} \delta_{1,c_i} \delta_{1,c_j} + \varepsilon \delta_{2,c_i} \delta_{1,c_j} + \chi \varepsilon^{-1} \delta_{1,c_i} \delta_{2,c_j} - \varepsilon \gamma \delta_{2,c_i} \delta_{2,c_j}) \left\langle \varphi_{b_i} \mid \varphi_{b_j} \right\rangle$$

$$f_i = -\chi \delta_{1,c_i} \left\langle \varphi_{b_i} \mid I_{\mathrm{stim}} \right\rangle + \varepsilon \beta \delta_{2,c_i} \left\langle \varphi_{b_i} \mid 1 \right\rangle$$

$$B_{ij} = -(\sigma_{\mathrm{i}} \delta_{\{1,3\},c_i} \delta_{\{1,3\},c_j} + \sigma_{\mathrm{e}} \delta_{3,c_i} \delta_{3,c_j}) \left\langle \nabla \varphi_{b_i} \mid \nabla \varphi_{b_j} \right\rangle$$

Then, we rewrite the monolithic equation as:

$$\boldsymbol{M}\dot{\boldsymbol{Q}} = \boldsymbol{A}\boldsymbol{Q} + \frac{1}{3} \chi \varepsilon^{-1} \delta_{1,c_i} \left\langle \varphi_{b_i} \mid v^3 \right\rangle + \boldsymbol{f} + \boldsymbol{B}\boldsymbol{Q}$$

And, the split equations become:

$$\boldsymbol{M}\dot{\boldsymbol{Q}} = \boldsymbol{A}\boldsymbol{Q} + \frac{1}{3} \chi \varepsilon^{-1} \delta_{1,c_i} \left\langle \varphi_{b_i} \mid v^3 \right\rangle + \boldsymbol{f}$$

$$\boldsymbol{M}\dot{\boldsymbol{Q}} = \boldsymbol{B}\boldsymbol{Q}$$

3   Time Integration

As a DAE, complications arise when integrating the Bidomain equations because the mass matrix is singular. Our goal is to solve the first ("explicit") equation using an explicit method such as Forward Euler; we solve the second ("implicit") equation using an implicit method such as Backward Euler.

Recall that matrix form of the explicit equation from the previous section:

$$\boldsymbol{M}\dot{\boldsymbol{Q}} = \boldsymbol{A}\boldsymbol{Q} + \frac{1}{3}\chi\varepsilon^{-1}\delta_{1,c_i}\left\langle \varphi_{b_i} \mid v^3 \right\rangle + \boldsymbol{f}$$

To integrate this system, we write the equation as:

$$\dot{\boldsymbol{Q}} = \boldsymbol{F}_{\mathrm{E}}(t, \boldsymbol{Q})$$
$$= \boldsymbol{M}^{-1}(\boldsymbol{A}\boldsymbol{Q} + \frac{1}{3}\chi\varepsilon^{-1}\delta_{1,c_i}\left\langle \varphi_{b_i} \mid v^3 \right\rangle + \boldsymbol{f})$$

The solution $\boldsymbol{Q}_{t+1}$ at time step $t+1$ is then given by $\boldsymbol{Q}_t$ plus a weighted sum of stages $\boldsymbol{F}_{\mathrm{E}}(t^{(i)}, \boldsymbol{Q}_t^{(i)})$, where each $\boldsymbol{Q}_t^{(i)}$ is itself a weighted sum of previously computed stages. However, $\boldsymbol{M}$ is singular (because no time derivatives of u_{e} appear in the Bidomain equations), and all components of the explicit equation's right-hand side are zero (because the operator has been split such that the explicit equation does not involve u_{e}).

Denote the set of all DoF indices by $\mathcal{I} = [0, N)$. Then, let $\mathcal{J} = \{\, i \in \mathcal{I} \mid c_i \in \{\, 1, 2 \,\} \,\}$ be the set of indices corresponding to the components v and w. Then:

$$\boldsymbol{M}_{\mathcal{J}}\dot{\boldsymbol{Q}}_{\mathcal{J}} = \boldsymbol{A}_{\mathcal{J}}\boldsymbol{Q}_{\mathcal{J}} + \frac{1}{3}\chi\varepsilon^{-1}\delta_{1,c_i}\left\langle \varphi_{b_i} \mid v^3 \right\rangle + \boldsymbol{f}_{\mathcal{J}}$$

Where a matrix or vector subscripted with $\mathcal{J}$ denotes that only those elements with indices in $\mathcal{J}$ are included. Because there are no longer elements corresponding to the u_{e} component of the system, $\boldsymbol{M}_{\mathcal{J}}$ is non-singular, and we may write:

$$\boldsymbol{F}_{\mathrm{E}}(t, \boldsymbol{Q}_{\mathcal{J}}) = \boldsymbol{M}_{\mathcal{J}}^{-1}(\boldsymbol{A}_{\mathcal{J}}\boldsymbol{Q}_{\mathcal{J}} + \frac{1}{3}\chi\varepsilon^{-1}\delta_{1,c_i}\left\langle \varphi_{b_i} \mid v^3 \right\rangle + \boldsymbol{f}_{\mathcal{J}})$$

Now, we recall the matrix form of the implicit equation:

$$\boldsymbol{M}\dot{\boldsymbol{Q}} = \boldsymbol{B}\boldsymbol{Q}$$

Rather than formulating the solution of each time step as a sum of terms, we shall derive an explicit equation that can be solved, corresponding to the Backward Euler and Crank Nicolson methods. First, we again rearrange the equation:

$$\dot{\boldsymbol{Q}} = \boldsymbol{F}_{\mathrm{I}}(t, \boldsymbol{Q})$$
$$= \boldsymbol{M}^{-1}\boldsymbol{B}\boldsymbol{Q}$$

For Backward Euler:

$$\boldsymbol{Q}_{t+1} = \boldsymbol{Q}_t + h\boldsymbol{F}_{\mathrm{I}}(t + h, \boldsymbol{Q}_{t+1})$$
$$\boldsymbol{Q}_{t+1} - \boldsymbol{Q}_t = h\boldsymbol{M}^{-1}\boldsymbol{B}\boldsymbol{Q}_{t+1}$$
$$\boldsymbol{M}(\boldsymbol{Q}_{t+1} - \boldsymbol{Q}_t) = h\boldsymbol{B}\boldsymbol{Q}_{t+1}$$
$$(\boldsymbol{M} - h\boldsymbol{B})\boldsymbol{Q}_{t+1} = \boldsymbol{M}\boldsymbol{Q}_t$$