1 Introduction

Let:

$$\begin{aligned} n &\in \mathbb{N} \\ T &\in \mathbb{R} \\ \Omega &\subseteq \mathbb{R}^n \\ \chi, C_m &\in \mathbb{R} \\ \boldsymbol{\sigma}_\mathrm{i}, \boldsymbol{\sigma}_\mathrm{e} &\in \mathbb{R}^{n\times n} \\ I_\mathrm{ion} &: \mathbb{R} \to \mathbb{R} \\ f, f_\mathrm{e} &: (0,T)\times\Omega \to \mathbb{R} \\ v, u_\mathrm{e} &: (0,T)\times\Omega \to \mathbb{R} \end{aligned}$$

Denote $\boldsymbol{w}(t,\boldsymbol{x}) = \begin{bmatrix} v \\ u_\mathrm{e} \end{bmatrix}$. The Bidomain problem is given as:

$$\chi C_\mathrm{m} \frac{\partial v}{\partial t} + \chi I_\mathrm{ion}(v) = \nabla\cdot(\boldsymbol{\sigma}_\mathrm{i}\nabla(v+u_\mathrm{e})) + f \tag{1}$$

$$0 = \nabla\cdot(\boldsymbol{\sigma}_\mathrm{i}\nabla(v+u_\mathrm{e})) + \nabla\cdot(\boldsymbol{\sigma}_\mathrm{e}\nabla u_\mathrm{e}) + f_\mathrm{e} \tag{2}$$

With initial conditions:

$$\boldsymbol{w}(0,\boldsymbol{x}) = \boldsymbol{w}_0(\boldsymbol{x}) \tag{3}$$

For all $\boldsymbol{x}\in\Omega$ and with boundary conditions:

$$\boldsymbol{\sigma}_\mathrm{i}\nabla(v+u_\mathrm{e})\cdot\hat{\mathbf{n}} = \boldsymbol{\sigma}_\mathrm{e}\nabla u_\mathrm{e}\cdot\hat{\mathbf{n}} = 0 \tag{4}$$

For all $(t,\boldsymbol{x})\in(0,T)\times\partial\Omega$.

Let $R_\mathrm{m}\in\mathbb{R}$. For a passive cell model, we take $I_\mathrm{ion}(v) = \frac{1}{R_\mathrm{m}}v$. We also assume that $\boldsymbol{\sigma}_\mathrm{i} = \sigma_\mathrm{i}\mathbf{I}$ and $\boldsymbol{\sigma}_\mathrm{e} = \sigma_\mathrm{e}\mathbf{I}$ for some $\sigma_\mathrm{i}, \sigma_\mathrm{e}\in\mathbb{R}$, so that $\boldsymbol{\sigma}_\mathrm{i}$ and $\boldsymbol{\sigma}_\mathrm{e}$ can be replaced with σ_i and σ_e in the equations above.

2 Toy Problem

We consider a toy problem with $T = 1$, and $\Omega = [0, 1]^n$:

$$u_\mathrm{e}(t, \boldsymbol{x}) = t^3 \prod_{i=1}^{n} \cos(\pi x_i) \tag{5}$$

$$v(t, \boldsymbol{x}) = -\frac{\sigma_\mathrm{i} + \sigma_\mathrm{e}}{\sigma_\mathrm{i}} u_\mathrm{e}(t, \boldsymbol{x}) \tag{6}$$

It is clear that $v(0, \boldsymbol{x}) = u_\mathrm{e}(0, \boldsymbol{x}) = 0$, so we take $\boldsymbol{w}_0(\boldsymbol{x}) = \mathbf{0}$.

Next, we consider:

$$\begin{aligned}
\sigma_\mathrm{i}\nabla(v + u_\mathrm{e}) &= \sigma_\mathrm{i}(\nabla v + \nabla u_\mathrm{e}) \\
&= \sigma_\mathrm{i}(-\frac{\sigma_\mathrm{i} + \sigma_\mathrm{e}}{\sigma_\mathrm{i}}\nabla u_\mathrm{e} + \nabla u_\mathrm{e}) \\
&= -\sigma_\mathrm{i}\nabla u_\mathrm{e} - \sigma_\mathrm{e}\nabla u_\mathrm{e} + \sigma_\mathrm{i}\nabla u_\mathrm{e} \\
&= -\sigma_\mathrm{e}\nabla u_\mathrm{e}
\end{aligned}$$

Since $\frac{\partial u_\mathrm{e}}{\partial x_k} = -\pi t^3 \sin(\pi x_k) \prod_{i \neq k} \cos(\pi x_i)$, we see that $\frac{\partial u_\mathrm{e}}{\partial x_k} = 0$ when $x_k \in \mathbb{Z}$. Recall that $\Omega = [0, 1]^n$; then, on the x_k-faces of $\partial\Omega$, we have that $\frac{\partial u_\mathrm{e}}{\partial x_k} = 0$ and $\hat{\mathbf{n}} = \pm\hat{\mathbf{e}}_k$. So, $\nabla u_\mathrm{e} \cdot \hat{\mathbf{n}} = 0$ for all $\boldsymbol{x} \in \partial\Omega$, and both boundary conditions in Equation 4 are satisfied.

Then, we determine f and f_e. From above, we have that $\sigma_\mathrm{i}\nabla(v + u_\mathrm{e}) = -\sigma_\mathrm{e}\nabla u_\mathrm{e}$, and so, rewriting Equation 2:

$$\begin{aligned}
f_\mathrm{e} &= -\nabla \cdot (\sigma_\mathrm{i}\nabla(v + u_\mathrm{e})) - \nabla \cdot (\sigma_\mathrm{e}\nabla u_\mathrm{e}) \\
&= \nabla \cdot (\sigma_\mathrm{e}\nabla u_\mathrm{e}) - \nabla \cdot (\sigma_\mathrm{e}\nabla u_\mathrm{e}) \\
&= 0
\end{aligned}$$

When we note that:

$$\begin{aligned}
\frac{\partial^2 u_\mathrm{e}}{\partial x_k^2} &= -\frac{\partial}{\partial x_k}\pi t^3 \sin(\pi x_k) \prod_{i \neq k} \cos(\pi x_i) \\
&= -\pi^2 t^3 \prod_{i=1}^{n} \cos(\pi x_i) \\
&= -\pi^2 u_\mathrm{e} \\
\Delta u_\mathrm{e} &= \sum_{k=1}^{n} \frac{\partial^2 u_\mathrm{e}}{\partial x_k^2} \\
&= -\sum_{k=1}^{n} \pi^2 u_\mathrm{e} \\
&= -n\pi^2 u_\mathrm{e}
\end{aligned}$$

We rewrite Equation 1:

$$
\begin{aligned}
f &= \chi C_{\mathrm{m}} \frac{\partial v}{\partial t} + \chi I_{\mathrm{ion}}(v) - \nabla \cdot (\sigma_{\mathrm{i}} \nabla (v + u_{\mathrm{e}})) \\
&= -\chi C_{\mathrm{m}} \frac{\sigma_{\mathrm{i}} + \sigma_{\mathrm{e}}}{\sigma_{\mathrm{i}}} \frac{\partial u_{\mathrm{e}}}{\partial t} + \chi I_{\mathrm{ion}}(v) - \nabla \cdot (-\sigma_{\mathrm{e}} \nabla u_{\mathrm{e}}) \\
&= -\chi C_{\mathrm{m}} \frac{\sigma_{\mathrm{i}} + \sigma_{\mathrm{e}}}{\sigma_{\mathrm{i}}} \frac{3}{t} u_{\mathrm{e}} - \chi \frac{1}{R_{\mathrm{m}}} \frac{\sigma_{\mathrm{i}} + \sigma_{\mathrm{e}}}{\sigma_{\mathrm{i}}} u_{\mathrm{e}} + \sigma_{\mathrm{e}} \Delta u_{\mathrm{e}} \\
&= -\chi (\frac{3C_{\mathrm{m}}}{t} + \frac{1}{R_{\mathrm{m}}}) \frac{\sigma_{\mathrm{i}} + \sigma_{\mathrm{e}}}{\sigma_{\mathrm{i}}} u_{\mathrm{e}} - n \sigma_{\mathrm{e}} \pi^2 u_{\mathrm{e}} \\
&= -(\chi \frac{\sigma_{\mathrm{i}} + \sigma_{\mathrm{e}}}{\sigma_{\mathrm{i}}} (\frac{3C_{\mathrm{m}}}{t} + \frac{1}{R_{\mathrm{m}}}) + n \sigma_{\mathrm{e}} \pi^2) u_{\mathrm{e}}
\end{aligned}
$$

3 Spatial Discretization

We consider a set of basis functions $\mathbf{\Phi}_i : \mathbb{R}^n \to \mathbb{R}^2$, where each $\mathbf{\Phi}_i(\boldsymbol{x}) = \varphi_{b_i}\hat{\mathbf{e}}_{c_i}$ for some set of basis functions $\varphi_i : \mathbb{R} \to \mathbb{R}$ (b_i is the "base" of the index i, and c_i is the "component" of index i). For some fixed time t, we make the approximation that $\boldsymbol{w} = W_j\mathbf{\Phi}_j$ (Einstein summation) where $\boldsymbol{W} \in \mathbb{R}^N$ (N is the number of DoFs). Furthermore, we write that $\frac{\partial \boldsymbol{w}}{\partial t} = X_j\mathbf{\Phi}_j$ for some $\boldsymbol{X} \in \mathbb{R}^N$. Then, we may write:

$$
\begin{aligned}
v &= \boldsymbol{w} \cdot \hat{\mathbf{e}}_1 \\
&= W_j\varphi_{b_j}\delta_{1,c_j} \\
u_\mathrm{e} &= \boldsymbol{w} \cdot \hat{\mathbf{e}}_2 \\
&= W_j\varphi_{b_j}\delta_{2,c_j} \\
\frac{\partial v}{\partial t} &= \frac{\partial \boldsymbol{w}}{\partial t} \cdot \hat{\mathbf{e}}_1 \\
&= X_j\varphi_{b_j}\delta_{1,c_j}
\end{aligned}
$$

Equation 1 becomes:

$$
\chi C_\mathrm{m} X_j\varphi_{b_j}\delta_{1,c_j} + \chi I_\mathrm{ion}(W_j\varphi_{b_j}\delta_{1,c_j}) = \nabla \cdot (\sigma_\mathrm{i}\nabla(W_j\varphi_{b_j}\delta_{1,c_j} + W_j\varphi_{b_j}\delta_{2,c_j})) + f
$$

And, because $\delta_{1,c_j} + \delta_{2,c_j} = 1$:

$$
\chi C_\mathrm{m} X_j\varphi_{b_j}\delta_{1,c_j} + \chi I_\mathrm{ion}(W_j\varphi_{b_j}\delta_{1,c_j}) = \nabla \cdot (\sigma_\mathrm{i}\nabla(W_j\varphi_{b_j})) + f
$$

Similarly, Equation 2 becomes:

$$
\begin{aligned}
0 &= \nabla \cdot (\sigma_\mathrm{i}\nabla(W_j\varphi_{b_j}\delta_{1,c_j} + W_j\varphi_{b_j}\delta_{2,c_j})) + \nabla \cdot (\sigma_\mathrm{e}\nabla(W_j\varphi_{b_j}\delta_{2,c_j})) + f_\mathrm{e} \\
&= \nabla \cdot (\sigma_\mathrm{i}\nabla(W_j\varphi_{b_j})) + \nabla \cdot (\sigma_\mathrm{e}\nabla(W_j\varphi_{b_j}\delta_{2,c_j})) + f_\mathrm{e}
\end{aligned}
$$

Next, we multiply by a test function $\mathbf{\Phi}_i$ and integrate:

$$
\begin{aligned}
&\left\langle \mathbf{\Phi}_i \mid \begin{matrix} \chi C_\mathrm{m} X_j\varphi_{b_j}\delta_{1,c_j} + \chi I_\mathrm{ion}(W_j\varphi_{b_j}\delta_{1,c_j}) \\ 0 \end{matrix} \right\rangle \\
&\quad = \left\langle \mathbf{\Phi}_i \mid \begin{matrix} \nabla \cdot (\sigma_\mathrm{i}\nabla(W_j\varphi_{b_j})) \\ \nabla \cdot (\sigma_\mathrm{i}\nabla(W_j\varphi_{b_j}) + \sigma_\mathrm{e}\nabla(W_j\varphi_{b_j}\delta_{2,c_j})) \end{matrix} \right\rangle + \left\langle \mathbf{\Phi}_i \mid \begin{matrix} f \\ f_\mathrm{e} \end{matrix} \right\rangle
\end{aligned}
$$

Since $\mathbf{\Phi}_i(\boldsymbol{x}) = \varphi_{b_i}\hat{\mathbf{e}}_{c_i}$:

$$
\begin{aligned}
&\chi C_\mathrm{m} X_j \left\langle \varphi_{b_i} \mid \varphi_{b_j} \right\rangle \delta_{1,c_i}\delta_{1,c_j} + \frac{\chi}{R_\mathrm{m}} W_j \left\langle \varphi_{b_i} \mid \varphi_{b_j} \right\rangle \delta_{1,c_i}\delta_{1,c_j} \\
&\quad = \sigma_\mathrm{i} W_j \left\langle \varphi_{b_i} \mid \nabla \cdot \nabla\varphi_{b_j} \right\rangle \delta_{1,c_i} \\
&\quad + \sigma_\mathrm{i} W_j \left\langle \varphi_{b_i} \mid \nabla \cdot \nabla\varphi_{b_j} \right\rangle \delta_{2,c_i} + \sigma_\mathrm{e} W_j \left\langle \varphi_{b_i} \mid \nabla \cdot \nabla\varphi_{b_j} \right\rangle \delta_{2,c_i}\delta_{2,c_j} \\
&\quad + \left\langle \varphi_{b_i} \mid f \right\rangle \delta_{1,c_i} + \left\langle \varphi_{b_i} \mid f_\mathrm{e} \right\rangle \delta_{2,c_i}
\end{aligned}
$$

Then, since $\left\langle \varphi_{b_i} \mid \nabla \cdot \nabla \varphi_{b_j} \right\rangle = -\left\langle \nabla \varphi_{b_i} \mid \nabla \varphi_{b_j} \right\rangle$, and since $\delta_{1,c_i} + \delta_{2,c_i} = 1$:

$$\begin{aligned}\chi C_\mathrm{m} X_j \left\langle \varphi_{b_i} \mid \varphi_{b_j} \right\rangle \delta_{1,c_i}\delta_{1,c_j} = &-\frac{\chi}{R_\mathrm{m}} W_j \left\langle \varphi_{b_i} \mid \varphi_{b_j} \right\rangle \delta_{1,c_i}\delta_{1,c_j} \\ &- \sigma_\mathrm{i} W_j \left\langle \nabla \varphi_{b_i} \mid \nabla \varphi_{b_j} \right\rangle - \sigma_\mathrm{e} W_j \left\langle \nabla \varphi_{b_i} \mid \nabla \varphi_{b_j} \right\rangle \delta_{2,c_i}\delta_{2,c_j} \\ &+ \left\langle \varphi_{b_i} \mid f \right\rangle \delta_{1,c_i} + \left\langle \varphi_{b_i} \mid f_\mathrm{e} \right\rangle \delta_{2,c_i}\end{aligned}$$

So, denote:

$$\begin{aligned}A_{ij} &= -\frac{\chi}{R_\mathrm{m}} \left\langle \varphi_{b_i} \mid \varphi_{b_j} \right\rangle \delta_{1,c_i}\delta_{1,c_j} - (\sigma_\mathrm{i} + \sigma_\mathrm{e}\delta_{2,c_i}\delta_{2,c_j}) \left\langle \nabla \varphi_{b_i} \mid \nabla \varphi_{b_j} \right\rangle \\ M_{ij} &= \chi C_\mathrm{m} \left\langle \varphi_{b_i} \mid \varphi_{b_j} \right\rangle \delta_{1,c_i}\delta_{1,c_j} \\ f_i &= \left\langle \varphi_{b_i} \mid f \right\rangle \delta_{1,c_i} + \left\langle \varphi_{b_i} \mid f_\mathrm{e} \right\rangle \delta_{2,c_i}\end{aligned}$$

Then:

$$\boldsymbol{M}\boldsymbol{X} = \boldsymbol{A}\boldsymbol{W} + \boldsymbol{f} \tag{7}$$

We note that: 1) Although we have formulated the problem as an affine equation, this is only the case because $I_\mathrm{ion}(v) = \frac{1}{R_\mathrm{m}} v$, and if we were not using a passive cell model, we may not be able to represent $\boldsymbol{A}$ as a matrix; 2) $\boldsymbol{M}$ and (in this case) $\boldsymbol{A}$ are constant matrices, while $\boldsymbol{X}$, $\boldsymbol{W}$, and $\boldsymbol{f}$ each depend on the current time step.

4 Time Integration

Let us discretize the time domain $(0, T)$ into k fixed-size time steps, so denote the step size $h = \frac{T}{k}$. We make the approximation:

$$\frac{\partial \boldsymbol{w}_{t+1}}{\partial t} = \frac{\boldsymbol{w}_{t+1} - \boldsymbol{w}_t}{h}$$

Then, recall that we have already made the approximation:

$$\begin{aligned} \boldsymbol{w} &= W_j \varphi_j \\ \frac{\partial \boldsymbol{w}}{\partial t} &= X_j \varphi_j \end{aligned}$$

This implies:

$$X_{t+1,j} \varphi_j = \frac{W_{t+1,j} - W_{t,j}}{h} \varphi_j$$

To keep these approximations consistent, we may simply define:

$$\boldsymbol{X}_{t+1} = \frac{1}{h}(\boldsymbol{W}_{t+1} - \boldsymbol{W}_t)$$

We wish to integrate using the Crank-Nicolson scheme, which we apply to Equation 7 as follows:

$$\begin{aligned} \frac{1}{h}\boldsymbol{M}(\boldsymbol{W}_{t+1} - \boldsymbol{W}_t) &= \frac{1}{2}(\boldsymbol{A}\boldsymbol{W}_t + \boldsymbol{f}_t) + \frac{1}{2}(\boldsymbol{A}\boldsymbol{W}_{t+1} + \boldsymbol{f}_{t+1}) \\ \boldsymbol{M}(\boldsymbol{W}_{t+1} - \boldsymbol{W}_t) &= \frac{h}{2}\boldsymbol{A}(\boldsymbol{W}_{t+1} + \boldsymbol{W}_t) + \frac{h}{2}(\boldsymbol{f}_{t+1} + \boldsymbol{f}_t) \end{aligned}$$

As mentioned in the previous section, although this equation is affine, there are cases in which the vector function $\boldsymbol{A}$ may be nonlinear; so, we will use Newton's method to solve for $\boldsymbol{W}_{t+1}$. So, define:

$$\boldsymbol{R}_{t+1}(\boldsymbol{W}) = \boldsymbol{M}(\boldsymbol{W} - \boldsymbol{W}_t) - \frac{h}{2}\boldsymbol{A}(\boldsymbol{W} + \boldsymbol{W}_t) - \frac{h}{2}(\boldsymbol{f}_{t+1} + \boldsymbol{f}_t)$$

Where $\boldsymbol{W}_t$ above has been determined by the initial conditions or by the previous time step. The iteration proceeds as follows:

$$\begin{aligned} \boldsymbol{W}_{t+1}^{(0)} &= \boldsymbol{W}_t \\ \mathbf{J}[\boldsymbol{R}_{t+1}]\Delta \boldsymbol{W}_{t+1}^{(\ell+1)} &= -\boldsymbol{W}_{t+1}^{(\ell)} \\ \boldsymbol{W}_{t+1}^{(\ell+1)} &= \boldsymbol{W}_{t+1}^{(\ell)} + \alpha_{t+1}\Delta \boldsymbol{W}_{t+1}^{(\ell+1)} \end{aligned}$$

Where $\mathbf{J}[\boldsymbol{R}_{t+1}]$ is the Jacobian of $\boldsymbol{R}_{t+1}$ and α_{t+1} is a step size to be chosen by the Newton solver. Once the residual $||\boldsymbol{W}_{t+1}^{(\ell)}||$ is sufficiently small, we take $\boldsymbol{W}_{t+1} = \boldsymbol{W}_{t+1}^{(\ell)}$ as the solution for the time step.

5 Linear Formulation

We can also solve the problem using the assumption that $I_{\text{ion}}(v) = \frac{1}{R_{\text{m}}} v$. If we solve using Backward Euler, we have that:

$$\frac{1}{h}\boldsymbol{M}(\boldsymbol{W}_{t+1} - \boldsymbol{W}_t) = \boldsymbol{A}\boldsymbol{W}_{t+1} + \boldsymbol{f}_{t+1}$$
$$(\frac{1}{h}\boldsymbol{M} - \boldsymbol{A})\boldsymbol{W}_{t+1} = \frac{1}{h}\boldsymbol{M}\boldsymbol{W}_t + \boldsymbol{f}_{t+1}$$

Or, using Crank-Nicolson:

$$\frac{1}{h}\boldsymbol{M}(\boldsymbol{W}_{t+1} - \boldsymbol{W}_t) = \frac{1}{2}(\boldsymbol{A}(\boldsymbol{W}_{t+1} + \boldsymbol{W}_t) + \boldsymbol{f}_{t+1} + \boldsymbol{f}_t)$$
$$(\frac{1}{h}\boldsymbol{M} - \frac{1}{2}\boldsymbol{A})\boldsymbol{W}_{t+1} = (\frac{1}{h}\boldsymbol{M} + \frac{1}{2}\boldsymbol{A})\boldsymbol{W}_t + \frac{1}{2}(\boldsymbol{f}_{t+1} + \boldsymbol{f}_t)$$

6 Godunov Splitting

Let us slightly rewrite the original system of equations:

$$\begin{aligned}\chi C_{\mathrm{m}} \frac{\partial v}{\partial t} &= \nabla \cdot (\sigma_i \nabla (v + u_{\mathrm{e}})) - \chi I_{\mathrm{ion}}(v) + f \\ 0 &= \nabla \cdot (\sigma_i \nabla (v + u_{\mathrm{e}})) + \nabla \cdot (\sigma \nabla u_{\mathrm{e}}) + f_{\mathrm{e}}\end{aligned}$$

Which, after spatial discretization, has been written as:

$$\boldsymbol{M}\boldsymbol{X} = \boldsymbol{A}\boldsymbol{W} + \boldsymbol{f}$$

Where $\boldsymbol{M}$ is the mass matrix, $\boldsymbol{A}$ is the stiffness matrix, $\boldsymbol{f}$ is the stiffness right-hand side vector, $\boldsymbol{W}$ is the solution state vector, and $\boldsymbol{X}$ is the state vector for the solution's time derivative (which is typically written as $\boldsymbol{X}_{t+1} = \frac{1}{h}(\boldsymbol{W}_{t+1} - \boldsymbol{W}_t)$).

We split the right-hand side operator as follows:

$$\begin{aligned}\boldsymbol{M}\boldsymbol{X} &= \boldsymbol{f} \\ \boldsymbol{M}\boldsymbol{X} &= \boldsymbol{A}\boldsymbol{W}\end{aligned}$$

And, applying Backward Euler time stepping to each system, we get:

$$\begin{aligned}\boldsymbol{M}\boldsymbol{X}_t^{(*)} &= \boldsymbol{f}_{t+1} \\ \boldsymbol{M}\boldsymbol{X}_{t+1} &= \boldsymbol{A}\boldsymbol{W}_{t+1}\end{aligned}$$

Substituting $\boldsymbol{X}_t^{(*)} = \frac{1}{h}(\boldsymbol{W}_t^{(*)} - \boldsymbol{W}_t)$, we get:

$$\begin{aligned}\frac{1}{h}\boldsymbol{M}(\boldsymbol{W}_t^{(*)} - \boldsymbol{W}_t) &= \boldsymbol{f}_{t+1} \\ \frac{1}{h}\boldsymbol{M}\boldsymbol{W}_t^{(*)} &= \frac{1}{h}\boldsymbol{M}\boldsymbol{W}_t + \boldsymbol{f}_{t+1}\end{aligned}$$

And substituting $\boldsymbol{X}_{t+1} = \frac{1}{h}(\boldsymbol{W}_{t+1} - \boldsymbol{W}_t^{(*)})$, we get:

$$\begin{aligned}\frac{1}{h}\boldsymbol{M}(\boldsymbol{W}_{t+1} - \boldsymbol{W}_t^{(*)}) &= \boldsymbol{A}\boldsymbol{W}_{t+1} \\ (\frac{1}{h}\boldsymbol{M} - \boldsymbol{A})\boldsymbol{W}_{t+1} &= \frac{1}{h}\boldsymbol{M}\boldsymbol{W}_t^{(*)}\end{aligned}$$

So, our time stepping scheme is:

$$\begin{aligned}\frac{1}{h}\boldsymbol{M}\boldsymbol{W}_t^{(*)} &= \frac{1}{h}\boldsymbol{M}\boldsymbol{W}_t + \boldsymbol{f}_{t+1} \\ (\frac{1}{h}\boldsymbol{M} - \boldsymbol{A})\boldsymbol{W}_{t+1} &= \frac{1}{h}\boldsymbol{M}\boldsymbol{W}_t^{(*)}\end{aligned}$$