1 Introduction

We consider the Bidomain equations:

$$\begin{aligned}
\chi C_{\mathrm{m}}\frac{\partial v}{\partial t} + \chi(I_{\mathrm{ion}}(v) + I_{\mathrm{stim}}) &= \nabla\cdot(\sigma_{\mathrm{i}}\nabla(v+u_{\mathrm{e}})) \\
0 &= \nabla\cdot(\sigma_{\mathrm{i}}\nabla(v+u_{\mathrm{e}})) + \nabla\cdot(\sigma_{\mathrm{e}}\nabla u_{\mathrm{e}})
\end{aligned}$$

We will be considering Neumann boundary conditions only. The domain is $\Omega = [0,1]^n$.

We introduce a new state variable w, and the FitzHugh-Nagumo cell model:

$$\begin{aligned}
I_{\mathrm{ion}}(v,w) &= \varepsilon^{-1}(v - \frac{v^3}{3} - w) \\
\frac{\partial w}{\partial t} &= \varepsilon(v + \beta - \gamma w)
\end{aligned}$$

When we add this cell model into the original equations, we have the new system:

$$\begin{aligned}
\chi C_{\mathrm{m}}\frac{\partial v}{\partial t} + \chi\varepsilon^{-1}(v - \frac{v^3}{3} - w) + \chi I_{\mathrm{stim}} &= \nabla\cdot(\sigma_{\mathrm{i}}\nabla(v+u_{\mathrm{e}})) \\
\frac{\partial w}{\partial t} &= \varepsilon(v + \beta - \gamma w) \\
0 &= \nabla\cdot(\sigma_{\mathrm{i}}\nabla(v+u_{\mathrm{e}})) + \nabla\cdot(\sigma_{\mathrm{e}}\nabla u_{\mathrm{e}})
\end{aligned}$$

We will be splitting the above system of equations into an explicit operator involving I_{ion} and I_{stim}:

$$\begin{aligned}
\chi C_{\mathrm{m}}\frac{\partial v}{\partial t} + \chi\varepsilon^{-1}(v - \frac{v^3}{3} - w) + \chi I_{\mathrm{stim}} &= 0 \\
\frac{\partial w}{\partial t} &= \varepsilon(v + \beta - \gamma w) \\
0 &= 0
\end{aligned}$$

And an implicit operator involving the Laplacian terms:

$$\begin{aligned}
\chi C_{\mathrm{m}}\frac{\partial v}{\partial t} &= \nabla\cdot(\sigma_{\mathrm{i}}\nabla(v+u_{\mathrm{e}})) \\
\frac{\partial w}{\partial t} &= 0 \\
0 &= \nabla\cdot(\sigma_{\mathrm{i}}\nabla(v+u_{\mathrm{e}})) + \nabla\cdot(\sigma_{\mathrm{e}}\nabla u_{\mathrm{e}})
\end{aligned}$$

2 Spatial Discretization

Consider the monolithic system of equations, and let $\boldsymbol{q} = \begin{bmatrix} v \\ w \\ u_\mathrm{e} \end{bmatrix}$. We consider a set of basis functions $\boldsymbol{\Phi}_i : \Omega \to \mathbb{R}^3$, where each $\boldsymbol{\Phi}_i = \varphi_{b_i} \hat{\mathbf{e}}_{c_i}$ for some basis functions $\varphi_i : \Omega \to \mathbb{R}$. We make the approximation $\boldsymbol{q} = Q_j \boldsymbol{\Phi}_j$ (Einstein summation) for a vector $\boldsymbol{Q} \in \mathbb{R}^N$, where N is the number of degrees of freedom. Also, we approximate $\frac{\partial \boldsymbol{q}}{\partial t} = \dot{Q}_j \boldsymbol{\Phi}_j$. We multiply the monolithic system of equations by a test function $\boldsymbol{\Phi}_i$ and integrate:

$$\left\langle \boldsymbol{\Phi}_i \mid \begin{matrix} \chi C_\mathrm{m} \partial_t v \\ \partial_t w \\ 0 \end{matrix} \right\rangle = \left\langle \boldsymbol{\Phi}_i \mid \begin{matrix} -\chi \varepsilon^{-1}(v - \frac{v^3}{3} - w) - \chi I_\mathrm{stim} \\ \varepsilon(v + \beta - \gamma w) \\ 0 \end{matrix} \right\rangle$$
$$+ \left\langle \boldsymbol{\Phi}_i \mid \begin{matrix} \nabla \cdot (\sigma_\mathrm{i} \nabla (v + u_\mathrm{e})) \\ 0 \\ \nabla \cdot (\sigma_\mathrm{i} \nabla (v + u_\mathrm{e})) + \nabla \cdot (\sigma_\mathrm{e} \nabla u_\mathrm{e}) \end{matrix} \right\rangle$$

Using the fact that $v = Q_j \varphi_{b_j} \delta_{1,c_j}$, $w = Q_j \varphi_{b_j} \delta_{2,c_j}$, and $u_\mathrm{e} = Q_j \varphi_{b_j} \delta_{3,c_j}$, as well as $\frac{\partial v}{\partial t} = \dot{Q}_j \varphi b_j \delta_{1,c_j}$ and $\frac{\partial w}{\partial t} = \dot{Q}_j \varphi_{b_j} \delta_{2,c_j}$, we consider each of the forms above:

$$\begin{aligned}
&\left\langle \boldsymbol{\Phi}_i \mid \begin{matrix} \chi C_\mathrm{m} \partial_t v \\ \partial_t w \\ 0 \end{matrix} \right\rangle \\
&\quad = \delta_{1,c_i} \langle \varphi_{b_i} \mid \chi C_\mathrm{m} \partial_t v \rangle + \delta_{2,c_i} \langle \varphi_{b_i} \mid \partial_t w \rangle \\
&\quad = \chi C_\mathrm{m} \delta_{1,c_i} \delta_{1,c_j} \left\langle \varphi_{b_i} \mid \dot{Q}_j \varphi_{b_j} \right\rangle + \delta_{2,c_i} \delta_{2,c_j} \left\langle \varphi_{b_i} \mid \dot{Q}_j \varphi_{b_j} \right\rangle \\
&\quad = (\chi C_\mathrm{m} \delta_{1,c_i} \delta_{1,c_j} + \delta_{2,c_i} \delta_{2,c_j}) \left\langle \varphi_{b_i} \mid \varphi_{b_j} \right\rangle \dot{Q}_j
\end{aligned}$$

$$\begin{aligned}
&\left\langle \boldsymbol{\Phi}_i \mid \begin{matrix} -\chi \varepsilon^{-1}(v - \frac{v^3}{3} - w) - \chi I_\mathrm{stim} \\ \varepsilon(v + \beta - \gamma w) \\ 0 \end{matrix} \right\rangle \\
&= \delta_{1,c_i} \left\langle \varphi_{b_i} \mid -\chi \varepsilon^{-1}(v - \frac{v^3}{3} - w) - \chi I_\mathrm{stim} \right\rangle + \delta_{2,c_i} \langle \varphi_{b_i} \mid \varepsilon(v + \beta - \gamma w) \rangle \\
&= (-\chi \varepsilon^{-1} \delta_{1,c_i} + \varepsilon \delta_{2,c_i}) \langle \varphi_{b_i} \mid v \rangle + (\chi \varepsilon^{-1} \delta_{1,c_i} - \varepsilon \gamma \delta_{2,c_i}) \langle \varphi_{b_i} \mid w \rangle \\
&\quad + \frac{1}{3} \chi \varepsilon^{-1} \delta_{1,c_i} \left\langle \varphi_{b_i} \mid v^3 \right\rangle - \chi \delta_{1,c_i} \langle \varphi_{b_i} \mid I_\mathrm{stim} \rangle + \varepsilon \beta \delta_{2,c_i} \langle \varphi_{b_i} \mid 1 \rangle \\
&= (-\chi \varepsilon^{-1} \delta_{1,c_i} + \varepsilon \delta_{2,c_i}) \delta_{1,c_j} \left\langle \varphi_{b_i} \mid Q_j \varphi_{b_j} \right\rangle + (\chi \varepsilon^{-1} \delta_{1,c_i} - \varepsilon \gamma \delta_{2,c_i}) \delta_{2,c_j} \left\langle \varphi_{b_i} \mid Q_j \varphi_{b_j} \right\rangle \\
&\quad + \frac{1}{3} \chi \varepsilon^{-1} \delta_{1,c_i} \left\langle \varphi_{b_i} \mid v^3 \right\rangle - \chi \delta_{1,c_i} \langle \varphi_{b_i} \mid I_\mathrm{stim} \rangle + \varepsilon \beta \delta_{2,c_i} \langle \varphi_{b_i} \mid 1 \rangle \\
&= (-\chi \varepsilon^{-1} \delta_{1,c_i} \delta_{1,c_j} + \varepsilon \delta_{2,c_i} \delta_{1,c_j} + \chi \varepsilon^{-1} \delta_{1,c_i} \delta_{2,c_j} - \varepsilon \gamma \delta_{2,c_i} \delta_{2,c_j}) \left\langle \varphi_{b_i} \mid \varphi_{b_j} \right\rangle Q_j \\
&\quad + \frac{1}{3} \chi \varepsilon^{-1} \delta_{1,c_i} \left\langle \varphi_{b_i} \mid v^3 \right\rangle - \chi \delta_{1,c_i} \langle \varphi_{b_i} \mid I_\mathrm{stim} \rangle + \varepsilon \beta \delta_{2,c_i} \langle \varphi_{b_i} \mid 1 \rangle
\end{aligned}$$

$$
\begin{aligned}
&\left\langle \boldsymbol{\Phi}_i \;\middle|\; \begin{matrix} \nabla\cdot(\sigma_\mathrm{i}\nabla(v+u_\mathrm{e})) \\ 0 \\ \nabla\cdot(\sigma_\mathrm{i}\nabla(v+u_\mathrm{e})) + \nabla\cdot(\sigma_\mathrm{e}\nabla u_\mathrm{e}) \end{matrix} \right\rangle \\
&\quad = \delta_{\{1,3\},c_i} \left\langle \varphi_{b_i} \mid \nabla\cdot(\sigma_\mathrm{i}\nabla(v+u_\mathrm{e}))\right\rangle + \delta_{3,c_i}\left\langle \varphi_{b_i} \mid \nabla\cdot(\sigma_\mathrm{e}\nabla u_\mathrm{e})\right\rangle \\
&\quad = -\delta_{\{1,3\},c_i} \left\langle \nabla\varphi_{b_i} \mid \sigma_i\nabla(v+u_\mathrm{e})\right\rangle - \delta_{3,c_i}\left\langle \nabla\varphi_{b_i} \mid \sigma_\mathrm{e}\nabla u_\mathrm{e}\right\rangle \\
&\quad = -\sigma_\mathrm{i}\delta_{\{1,3\},c_i} \left\langle \nabla\varphi_{b_i} \mid \nabla(Q_j\varphi_{b_j}\delta_{\{1,3\},c_j})\right\rangle - \sigma_\mathrm{e}\delta_{3,c_i}\left\langle \nabla\varphi_{b_i} \mid \nabla(Q_j\varphi_{b_j}\delta_{3,c_j})\right\rangle \\
&\quad = -(\sigma_\mathrm{i}\delta_{\{1,3\},c_i}\delta_{\{1,3\},c_j} + \sigma_\mathrm{e}\delta_{3,c_i}\delta_{3,c_j})\left\langle \nabla\varphi_{b_i} \mid \nabla\varphi_{b_j}\right\rangle Q_j
\end{aligned}
$$

So, define:

$$
\begin{aligned}
M_{ij} &= (\chi C_\mathrm{m}\delta_{1,c_i}\delta_{1,c_j} + \delta_{2,c_i}\delta_{2,c_j})\left\langle \varphi_{b_i} \mid \varphi_{b_j}\right\rangle \\
A_{ij} &= (-\chi\varepsilon^{-1}\delta_{1,c_i}\delta_{1,c_j} + \varepsilon\delta_{2,c_i}\delta_{1,c_j} + \chi\varepsilon^{-1}\delta_{1,c_i}\delta_{2,c_j} - \varepsilon\gamma\delta_{2,c_i}\delta_{2,c_j})\left\langle \varphi_{b_i} \mid \varphi_{b_j}\right\rangle \\
f_i &= -\chi\delta_{1,c_i}\left\langle \varphi_{b_i} \mid I_\mathrm{stim}\right\rangle + \varepsilon\beta\delta_{2,c_i}\left\langle \varphi_{b_i} \mid 1\right\rangle \\
B_{ij} &= -(\sigma_\mathrm{i}\delta_{\{1,3\},c_i}\delta_{\{1,3\},c_j} + \sigma_\mathrm{e}\delta_{3,c_i}\delta_{3,c_j})\left\langle \nabla\varphi_{b_i} \mid \nabla\varphi_{b_j}\right\rangle
\end{aligned}
$$

Then, we rewrite the monolithic equation as:

$$
\boldsymbol{M}\dot{\boldsymbol{Q}} = \boldsymbol{A}\boldsymbol{Q} + \frac{1}{3}\chi\varepsilon^{-1}\delta_{1,c_i}\left\langle \varphi_{b_i} \mid v^3\right\rangle + \boldsymbol{f} + \boldsymbol{B}\boldsymbol{Q}
$$

And, the split equations become:

$$
\begin{aligned}
\boldsymbol{M}\dot{\boldsymbol{Q}} &= \boldsymbol{A}\boldsymbol{Q} + \frac{1}{3}\chi\varepsilon^{-1}\delta_{1,c_i}\left\langle \varphi_{b_i} \mid v^3\right\rangle + \boldsymbol{f} \\
\boldsymbol{M}\dot{\boldsymbol{Q}} &= \boldsymbol{B}\boldsymbol{Q}
\end{aligned}
$$