III. DRUDE RESPONSE TO A SINUSOIDAL DRIVE

What is the expected response of a gas to a sinusoidal modulation of the scattering length? In linear response, we can start from Eq. (26) to find $C(t)$. Let's assume a sinusoidal drive,

$$
\begin{aligned}
a^{-1}(t) &= a_0^{-1} + Ak_F \sin[\omega(t-t_0)] \\
v(t) &= v_0 + \frac{Ak_F}{4\pi m}\sin[\omega(t-t_0)] \qquad \text{with} \quad A_v = \frac{Ak_F}{4\pi m} \\
\dot{v}(t) &= \omega A_v \cos[\omega(t-t_0)]
\end{aligned}
\tag{56}
$$

Here A is our usual dimensionless drive; and $a_0^{-1} = 0$ is typically chosen in our recent experiments.

We will use a Drude model the response:

$$
\begin{aligned}
\tilde{\zeta}(\omega) &= \frac{\tilde{S}\tau}{1-i\omega\tau}, \quad \tilde{\zeta}(0) = \tau\tilde{S} \\
\tilde{\chi} &= \frac{S}{1-i\omega\tau}
\end{aligned}
\tag{57}
$$

Here $\tilde{S}$ is defined in Eq. (21); we could also write $(2/\pi)\int_0^\infty d\omega\, \mathrm{Re}\,\tilde{\zeta}(\omega) = \tilde{S}$. In the time domain, these responses are

$$
\begin{aligned}
\tilde{\zeta}(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\tilde{\zeta}(\omega) = \tilde{S}e^{-t/\tau}\Theta(t) \qquad \text{such that} \quad \int_0^\infty dt\, \tilde{\zeta}(t) = \tau\tilde{S} = \tilde{\zeta}(0) \\
\tilde{\chi}(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\tilde{\chi}(\omega) = \frac{\tilde{S}}{\tau}e^{-t/\tau}\Theta(t) \qquad \text{such that} \quad \int_0^\infty dt\, \tilde{\chi}(t) = \tilde{S}
\end{aligned}
\tag{58}
$$

Combining Eqs. (26), (56), and (58), we have

$$
\begin{aligned}
C(t) &= C_{\mathrm{eq}}(t) + \int_{t_0}^{t} dt' \, \left(Se^{(t-t')/\tau}\Theta(t-t')\right)\omega A_v \cos[\omega(t'-t_0)] \\
&= C_{\mathrm{eq}}(t) + \omega A_v \frac{S\tau}{1+\omega^2 t^2}\left\{-e^{-t/\tau} + \cos(\omega t) + \omega\tau\sin(\omega\tau)\right\}
\end{aligned}
\tag{59}
$$

If we are looking at $t \gg \tau$, the transient term, $e^{-t/\tau}$, can be dropped. The time-dependent $C_{\mathrm{eq}}(t) = C_0 + SA_v\sin(\omega t)$; separating into quadratures we have

$$
\frac{\delta C(t)}{A_v} = \left(S - \omega\,\mathrm{Im}\,\tilde{\zeta}\right)\sin(\omega t) - (\omega\,\mathrm{Re}\,\zeta)\cos(\omega t) \qquad \text{with} \quad \mathrm{Re}\,\tilde{\zeta} = \frac{\tau\tilde{S}}{1+\tau^2\omega^2}, \quad \mathrm{Im}\,\tilde{\zeta} = \tau\omega\,\mathrm{Re}\,\tilde{\zeta}
\tag{60}
$$

Compared to the hydrodynamic expression (14), we see that $\mathrm{Im}\,\tilde{\zeta}$ now appears in the amplitude of the in-phase quadrature. For low frequency, ζ is real, and $\mathrm{Re}\,\zeta \to \zeta_0$.

We can rewrite this response in terms of a phase shift ϕ and amplitude A_C:

$$
\delta C(t) = A_C(\omega)\sin[\omega t - \phi(\omega)]
\tag{61}
$$

where the phase lag of the response is

$$
\tan\phi(\omega) = \frac{\omega\,\mathrm{Re}\,\tilde{\zeta}(\omega)}{S - \omega\,\mathrm{Im}\,\tilde{\zeta}(\omega)} = \omega\tau
\tag{62}
$$

and the amplitude of the response is

$$
|\tilde{\chi}(\omega)| = \frac{A_C(\omega)}{A_v} = \frac{S}{\sqrt{1+\omega^2\tau^2}}
\tag{63}
$$

These are plotted in Fig. 1, and compared to predictions that use the HD formula, even at nonzero frequency.

The phase response, Fig. 1(a), shows a linear increase at low frequency, passing through $\phi = \pi/4$ at $\omega = \tau^{-1}$, and asymptoting to $\pi/2$, exactly out of phase. The amplitude response, Fig. 1(b), shows a linear decrease in $|\chi|/S$, with a response reduced by $\sqrt{2}$ at $\omega = \tau^{-1}$, and continuing to decrease as $\omega^{-1/2}$ thereafter.

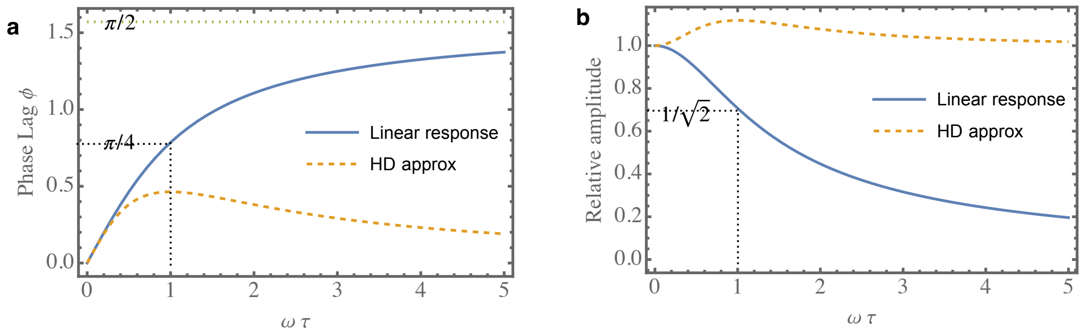

FIG. 1 Dynamical contact response, Drude model. **(a):** The phase shift with respect to the drive, given by Eq. (62), is shown versus drive frequency ω rescaled with τ. The phase lag is $\pi/4$ at $\omega\tau = 1$, and asymptotes to $\pi/2$. By comparison, using the HD prediction for $C(t)$, Eq. (14), has a peak phase shift ≈ 0.46 and asymptotes to zero. **(b)** The response amplitude, $|\chi|/S$ as given by Eq. (63), decreases from unity to $1/\sqrt{2}$ at $\omega = \tau^{-1}$.

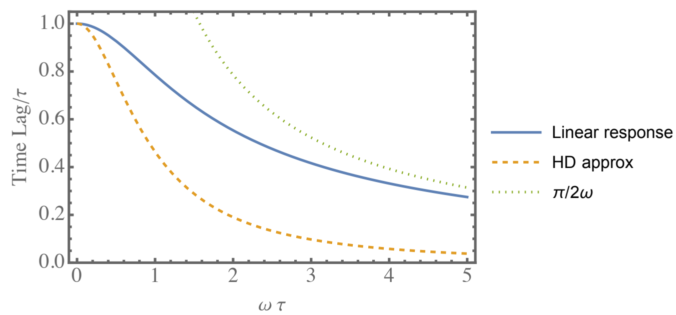

FIG. 2 Time lag of the dynamical response, Drude model. The contact response has a time lag shown here, versus drive frequency, in terms of τ. We see that for low frequencies, one expects a delay that is roughly independent of frequency; whereas at high frequency, the time lag no longer has information about τ.

By comparison, the "naive" HD prediction would be a peak phase lag at $\omega = \tau^{-1}$, and a relatively constant amplitude response. We should be able to distinguish between these alternatives by examining the experimental response at $\omega \approx \tau^{-1}$.

What is the time lag of the contact response predicted by a Drude model? Writing $\delta C(t) = A_C(\omega)\sin[\omega(t-\tau_{\rm lag}(\omega))]$, this is related to the phase delay simply by ϕ/ω. This is plotted in Fig. 2. We see that at low frquency, one can anticipate a time lag that is comparable to τ independent of frequency. At $\omega = \tau^{-1}$, the time lag is $(\pi/4)\tau \approx 0.79\tau$. At high frequencies, however, the time lag no longer contains information about τ: it approaches $(\pi/2)\omega^{-1}$, independent of τ.

In reality, the Drude approximation will break down at high frequencies.