Response theory for variations of the scattering length

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Consider the Hamiltonian for a gas with contact interaction,

$$\hat{H}(t) = \hat{H}_{\text{kin}} + \hat{C}v(t) \tag{1}$$

with contact operator $\hat{C}$ and coupling strength $v = 1/(4\pi ma)$. The coupling strength shall vary in time as $v(t) = v_0 + \delta v(t)$, starting at time $t = t_0$ (which can for instance be 0, or $-\infty$). Then the contact expectation value deviates from its equilibrium value (at coupling v_0) as

$$C(t) = C_{\text{eq}} + \int_{-\infty}^{\infty} dt'\, \tilde{\chi}(t,t')\, \delta v(t'). \tag{2}$$

This is an exact expression in terms of the nonlinear susceptibility

$$\tilde{\chi}(t,t') = i\langle [C(t), C(t')] \rangle \theta(t-t') = \frac{\partial C(t)}{\partial v(t')}. \tag{3}$$

Instead, if the expectation value is evaluated using the equilibrium density matrix, one obtains the linear response susceptibility which depends only on the time difference,

$$\tilde{\chi}_{\text{eq}}(t-t') = i\langle [C(t), C(t')] \rangle_{\text{eq}} \theta(t-t'), \tag{4}$$

which determines the response $\delta C(\omega) = \tilde{\chi}_{\text{eq}}(\omega)\delta v(\omega) + \mathcal{O}(\delta v^2)$ to linear order in the perturbation.

Our goal is to express the time dependent contact in linear response in terms of the bulk viscosity $\zeta(\omega)$ in equilibrium. We use the tilde to denote the observables rescaled with $(12\pi ma)^2$ to make them independent of scattering length also near unitarity,

$$\begin{aligned}
\tilde{\zeta}(\omega) &= (12\pi ma)^2 \zeta(\omega) && (5)\\
&= \frac{\tilde{\chi}(\omega + i0) - \tilde{\chi}(0)}{i(\omega + i0)} && (6)\\
&= \int dt\, e^{i\omega t} \tilde{\zeta}(t), && (7)
\end{aligned}$$

where the last line defines the Fourier transform to the time dependent viscosity $\zeta(t)$. The constant term

$$\tilde{\chi}(\omega = 0) = \tilde{\chi} \tag{8}$$
$$= \left(\frac{\partial C}{\partial v}\right)_{N,S} \tag{9}$$
$$= \int_{-\infty}^{\infty} \frac{d\omega}{\pi}\,\tilde{\zeta}(\omega) \tag{10}$$

denotes the scale susceptibility, or (rescaled) bulk viscosity sum rule. One can now use these definitions backwards to obtain the time dependent response function

$$\tilde{\chi}(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}[\tilde{\chi} + i\omega\tilde{\zeta}(\omega)] \tag{11}$$
$$= \tilde{\chi}\delta(t) - \partial_t \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\tilde{\zeta}(\omega) \tag{12}$$
$$= -\partial_t\tilde{\zeta}(t) \quad (t > 0). \tag{13}$$

These relations become intuitively clear when one considers the example of the Drude model with scattering time τ:

$$\tilde{\zeta}(\omega) = \frac{\tilde{\chi}\tau}{1 - i\omega\tau}, \quad \tilde{\zeta}(\omega = 0) = \tilde{\chi}\tau, \quad \int \frac{d\omega}{\pi}\,\tilde{\zeta}(\omega) = \tilde{\chi}, \tag{14}$$
$$\tilde{\zeta}(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\tilde{\zeta}(\omega) = \tilde{\chi}e^{-t/\tau}\theta(t), \quad \int_0^{\infty} dt\,\tilde{\zeta}(t) = \tilde{\chi}\tau = \tilde{\zeta}(\omega = 0), \tag{15}$$
$$\tilde{\chi}(\omega) = \frac{\tilde{\chi}}{1 - i\omega\tau}, \tag{16}$$
$$\tilde{\chi}(t) = \frac{\tilde{\chi}}{\tau}\, e^{-t/\tau}\theta(t), \quad \int_0^{\infty} dt\,\tilde{\chi}(t) = \tilde{\chi}. \tag{17}$$

Note that if one inserts the Drude form in the time response (12), the time derivative acts both on the step function, which cancels the separate delta function from the first term, as well as on the exponential decay, which yields the desired response for $t > 0$.

Now, in the time dependent contact one can integrate by parts to obtain

$$C(t) = C_{\text{eq}} + \int_{t_0}^{t} dt'\,\partial_{t'}\tilde{\zeta}(t - t')\,\delta v(t') \tag{18}$$
$$= C_{\text{eq}} + \tilde{\zeta}(t = 0^+)\delta v(t) - \tilde{\zeta}(t - t_0)\delta v(t_0) - \int_{t_0}^{t} dt'\,\tilde{\zeta}(t - t')\delta\dot{v}(t'). \tag{19}$$

Only the first of the two boundary terms is nonzero and yields $(\partial C/\partial v)_{N,S}\delta v(t)$, which can be combined with the equilibrium contact $C_{\mathrm{eq}}[v_0]$ into $C_{\mathrm{eq}}[v(t)] + \mathcal{O}(\delta v^2)$, and we obtain

$$C(t) = C_{\mathrm{eq}}[v(t)] - \int_{t_0}^{t} dt' \, \tilde{\zeta}(t - t')\delta\dot{v}(t'). \tag{20}$$

If the variation $\delta\dot{v}(t')$ occurs on time scales slower than the scattering time τ then one can pull the factor $\delta\dot{v}(t') \approx \delta\dot{v}(t)$ out of the integral, and for $t - t_0 \gg \tau$ the integral bounds can be extended as $\int_{-\infty}^{t} dt' \, \tilde{\zeta}(t-t') = \tilde{\zeta}(\omega = 0)$ to yield the dc bulk viscosity, and we finally obtain

$$C(t) = C_{\mathrm{eq}}[v(t)] - \tilde{\zeta}(\omega = 0)\delta\dot{v}(t). \tag{21}$$