Rf spectrum of two bodies

Zhenhua Yu*
School of Physics and Astronomy, Sun Yat-Sen University (Zhuhai Campus), Zhuhai 519082, China

The correlation function $\rho(\mathbf{r},\mathbf{r}') \equiv \langle\psi_2^\dagger(\mathbf{r})\psi_1^\dagger(0)\psi_1(0)\psi_2(\mathbf{r}')\rangle$ is "Hermitian", i.e., $\rho(\mathbf{r},\mathbf{r}') = \rho^*(\mathbf{r}',\mathbf{r})$. Thus we shall be able to find the corresponding normalized eigen-functions $\{\chi_j\}$ and eigen-values $\{\lambda_j\}$ such that $\int d\mathbf{r}'\rho(\mathbf{r},\mathbf{r}')\chi_j(\mathbf{r}') = \lambda_j\chi_j(\mathbf{r})$ and $\rho(\mathbf{r},\mathbf{r}') = \sum_j \lambda_j\chi_j(\mathbf{r})\chi_j^*(\mathbf{r}')$.

In dilute atomic gases, considering the length scale separation $r_{\rm vdW} \ll d$ with d the mean distance between atoms, the structure of the eigen-functions $\chi_j(\mathbf{r})$ at $r_{\rm vdW} \ll r \ll d$ shall be determined by two-body physics. In another word, when two atoms are close to each other, the possibility to find a third atom in the vicinity is negligible.

The Rf spectrum at frequency $1/md^2 \ll \omega \ll 1/mr_{\rm vdW}^2$ is thought to probe the structure of $\rho(\mathbf{r},\mathbf{r})$ at $r_{\rm vdW} \ll r \ll d$. It is instructive to work out two-body problems in the unitary regime. The Rf spectroscopy is implemented via the operator $M = \int d\mathbf{r}\psi_3^\dagger(\mathbf{r})\psi_2(\mathbf{r}) = \sum_{\mathbf{p}} a_{3,\mathbf{p}}^\dagger a_{2,\mathbf{p}}$. The spectrum function is (the Fermi Golden rule)

$$\Gamma(\omega) = 2\pi\sum_f |\langle f|M|i\rangle|^2\delta(\omega + E_i - E_f). \tag{1}$$

For simplicity, in the following discussion, we assume the interatomic potential $U(r)$ is exactly zero when $r > r_0$. The difference between $U(r)$ and $U_{\rm vdW}(r)$ at $r > r_0$ seems irrelevant for s-wave.

I. WITHOUT FINAL STATE EFFECTS

Let us first consider the case that there is no 1-3 interaction. The final states $|f\rangle = a_{3,\mathbf{p}_f}^\dagger a_{1,-\mathbf{p}_f}^\dagger|0\rangle$ can be labeled by the momentum $\mathbf{p}_f$, and $E_f = p_f^2/m$. Note that for two-body problems, we can stay in the center of mass frame such that $|i\rangle$ and $|f\rangle$ all have zero net momentum.

A. Dimer to free states

Suppose the initial 1-2 state is a shallow s-wave dimer state whose normalized relative wave-function is

$$\phi_d(\mathbf{r}) = \sqrt{\frac{\kappa}{2\pi}}\frac{e^{-\kappa r}}{r}, \tag{2}$$

$$\tilde\phi_d(\mathbf{p}) = \int d\mathbf{r}\phi_d(\mathbf{r})\frac{e^{-i\mathbf{p}\cdot\mathbf{r}}}{\sqrt{V}} = \sqrt{\frac{\kappa}{2\pi V}}\frac{4\pi}{\kappa^2+p^2}, \tag{3}$$

and $E_i = -\kappa^2/2\mu = -\kappa^2/m$. Note only close to the unitarity, $\kappa \to 1/a_s$. Strictly speaking, the above wave-function holds only when $r > r_0$. Since $\kappa r_0 \ll 1$, the weight of $\phi_d(\mathbf{r})$ at $r < r_0$ is negligible. We can take the limit $r_0 \to 0$. The situation is known to be different for higher partial waves. The initial state $|i\rangle$ is given by

$$|i\rangle = \int d\mathbf{R}\int d\mathbf{r}\frac{e^{i\mathbf{Q}\cdot\mathbf{R}}}{\sqrt{V}}\phi_d(\mathbf{r})\psi_2^\dagger(\mathbf{R}+\mathbf{r}/2)\psi_1^\dagger(\mathbf{R}-\mathbf{r}/2)|0\rangle = \sum_{\mathbf{p}}\tilde\phi_d(\mathbf{p})a_{2,\mathbf{p}}^\dagger a_{1,-\mathbf{p}}^\dagger|0\rangle. \tag{4}$$

We have assumed $\mathbf{Q} = 0$. In this case, the spectrum function becomes

$$\Gamma(\omega) = 2\pi\sum_{\mathbf{p}_f}|\tilde\phi_d(\mathbf{p}_f)|^2\delta(\omega + E_i - E_f) \tag{5}$$

$$= \kappa\int\frac{d\mathbf{p}_f}{(2\pi)^3}\left(\frac{4\pi}{\kappa^2+p_f^2}\right)^2\delta(\omega - \kappa^2/m - p_f^2/m) \tag{6}$$

$$= \frac{4m\kappa}{(m\omega)^2}\sqrt{m\omega-\kappa^2}. \tag{7}$$

*huazhenyu2000@gmail.com

On the other hand, when $\kappa r \ll 1$,

$$\phi_d(\mathbf{r}) \approx \sqrt{\frac{\kappa}{2\pi}} \left(\frac{1}{r} - \kappa \right). \tag{8}$$

We can read off the contact as $\mathcal{C} = \kappa/2\pi V$ (V in the denominator comes from the center of mass wave-function $1/\sqrt{V}$) and recast

$$\Gamma(\omega) = \frac{8\pi m V \mathcal{C}}{(m\omega)^2} \sqrt{m\omega - \kappa^2}. \tag{9}$$

Note that the contact defined in the above way can be different from $-\partial E_i/\partial(1/a_s)$ since $\partial\kappa/\partial(1/a_s)$ is not necessarily unity.

B. Continuum to free states

Now we consider the initial 1-2 state is a continuum state. If there is no 1-2 interaction, the relative initial wave-function would be a plane wave $\phi^0_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}/\sqrt{V}$, whose s-wave component is $\phi^0_{\mathbf{k},s}(\mathbf{r}) = \sqrt{\frac{4\pi}{V}} \frac{\sin(kr)}{kr}$. When there is 1-2 s-wave interaction, the s-wave component of the relative wave-function becomes

$$\phi_{\mathbf{k},s}(\mathbf{r}) = A\frac{\sin(kr + \delta_s)}{r}, \tag{10}$$

with δ_s the s-wave scattering phase shift, a function of k. We can choose to normalize $\phi_{\mathbf{k},s}(\mathbf{r})$ inside a sphere of radius ℓ with a hard wall boundary condition such that $k\ell + \delta_s = n\pi$ and the normalization factor

$$A^2 = \frac{1}{2\pi[\ell + \sin(2\delta_s)/2k]}. \tag{11}$$

The Fourier transform is

$$\tilde{\phi}_{\mathbf{k},s}(\mathbf{p}) = \int^{\ell} d\mathbf{r} \phi_{\mathbf{k},s}(\mathbf{r}) \frac{e^{-i\mathbf{p}\cdot\mathbf{r}}}{\sqrt{V}} \tag{12}$$

$$= \frac{1}{\sqrt{V}} \frac{4\pi A}{k^2 - p^2} \left[(-1)^{n+1} \sin(p\ell)\frac{k}{p} - \sin(\delta_s) \right], \tag{13}$$

and the spectrum becomes

$$\Gamma(\omega) = 2\pi V \int \frac{d\mathbf{p}_f}{(2\pi)^3} |\tilde{\phi}_{\mathbf{k},s}(\mathbf{p}_f)|^2 \delta(\omega + k^2/m - p_f^2/m) \tag{14}$$

$$= \frac{8\pi A^2 m}{(m\omega)^2} \sqrt{m\omega + k^2} \left| (-1)^{n+1} \sin(\ell\sqrt{m\omega + k^2}) \frac{k}{\sqrt{m\omega + k^2}} - \sin(\delta_s) \right|^2. \tag{15}$$

When $k^2/m\omega \ll 1$, we have

$$\Gamma(\omega) \to \frac{8\pi A^2 m}{(m\omega)^2} \sqrt{m\omega + k^2} \sin^2(\delta_s). \tag{16}$$

Since when $kr \ll 1$,

$$\phi_{\mathbf{k},s}(\mathbf{r}) \to A\sin(\delta_s) \left(\frac{1}{r} + k\cot\delta_s \right), \tag{17}$$

we can read off the contact as $\mathcal{C} = A^2 \sin^2(\delta_s)/V$ and recast

$$\Gamma(\omega) \to \frac{8\pi m V \mathcal{C}}{(m\omega)^2} \sqrt{m\omega + k^2}. \tag{18}$$

II. WITH FINAL STATE EFFECTS

If there is 1-3 interaction, the relative wave-function $\phi_f(\mathbf{r})$ is assumed to be

$$\phi_{f,c}(\mathbf{r}) = A_f \frac{\sin(k_f r + \delta_f)}{r} \tag{19}$$

labeled by k_f for the continuum with the sphere hard wall condition $k_f \ell + \delta_f = n_f \pi$, and

$$\phi_{f,d}(\mathbf{r}) = \sqrt{\frac{\kappa_f}{2\pi}} \frac{e^{-\kappa_f r}}{r} \tag{20}$$

for the dimer. The rf spectrum function becomes

$$\Gamma(\omega) = 2\pi \sum_f \left| \int d\mathbf{r} \phi_f^*(\mathbf{r}) \phi_i(\mathbf{r}) \right|^2 \delta(\omega + E_i - E_f). \tag{21}$$

A. Continuum to continuum

If we start from the continuum such that $\phi_i(\mathbf{r}) = A_i \frac{\sin(k_i r + \delta_i)}{r}$ with $k_i \ell + \delta_i = n_i \pi$, the overlap of the relative wave-functions is

$$\begin{aligned} \int d\mathbf{r} \phi_f^*(\mathbf{r}) \phi_i(\mathbf{r}) =& 4\pi A_i A_f \int_0^\ell dr \sin(k_i r + \delta_i) \sin(k_f r + \delta_f) && (22) \\ =& 2\pi A_i A_f \left[\frac{\sin(\delta_i + \delta_f)}{k_i + k_f} - \frac{\sin(\delta_i - \delta_f)}{k_i - k_f} \right] && (23) \\ =& 4\pi A_i A_f \sin\delta_i \cos\delta_f \frac{k_f}{k_i^2 - k_f^2} \left(-1 + \frac{k_i \cot\delta_i}{k_f \cot\delta_f} \right). && (24) \end{aligned}$$

Since $(\ell + d\delta_f/dk_f) dk_f = \Delta n_f \pi = \pi$, we can convert the summation over k_f for the final state into the integral $\sum_{k_f} = \int_0^\infty dk_f (\ell + d\delta_f/dk_f)/\pi$. Thus

$$\begin{aligned} \Gamma(\omega) =& 2\pi (4\pi A_i \sin\delta_i)^2 \int_0^\infty \frac{dk_f}{\pi} (\ell + d\delta_f/dk_f) A_f^2 \cos^2\delta_f \left[\frac{k_f}{k_i^2 - k_f^2} \left(-1 + \frac{k_i \cot\delta_i}{k_f \cot\delta_f} \right) \right]^2 \delta(\omega + k_i^2/m - k_f^2/m) && (25) \\ =& 2\pi (4\pi A_i \sin\delta_i)^2 \int_0^\infty \frac{dk_f}{\pi} \frac{\ell + d\delta_f/dk_f}{2\pi[\ell + \sin(2\delta_f)/2k_f]} \cos^2\delta_f \left[\frac{k_f}{k_i^2 - k_f^2} \left(-1 + \frac{k_i \cot\delta_i}{k_f \cot\delta_f} \right) \right]^2 \delta(\omega + k_i^2/m - k_f^2/m) && (26) \\ =& 8\pi m (A_i \sin\delta_i)^2 \frac{\sqrt{m\omega + k_i^2}}{(m\omega)^2} \left[\frac{\ell + d\delta_f/dk_f}{\ell + \sin(2\delta_f)/2k_f} \cos^2\delta_f \left(-1 + \frac{k_i \cot\delta_i}{k_f \cot\delta_f} \right)^2 \right]_{k_f = \sqrt{m\omega + k_i^2}}. && (27) \end{aligned}$$

Now if we take $\ell \to \infty$ and $k \cot\delta = -1/a$, and resultantly $\cos^2\delta = 1/(1 + k_f^2 a^2)$, we have

$$\begin{aligned} \Gamma(\omega) &= 8\pi m V \mathcal{C}_i \frac{\sqrt{m\omega + k_i^2}}{(m\omega)^2} \frac{1}{1 + a_f^2 (m\omega + k_i^2)} \left[-1 + \frac{1/a_i}{1/a_f} \right]^2 && (28) \\ &= 8\pi m V \mathcal{C}_i \frac{\sqrt{m\omega + k_i^2}}{(m\omega)^2} \frac{1}{1/a_f^2 + m\omega + k_i^2} \left[\frac{1}{a_i} - \frac{1}{a_f} \right]^2, && (29) \end{aligned}$$

which is the Braaten's result.

The above result is nice in the sense that we know if $U_{12}(r) = U_{13}(r)$, $\Gamma(\omega)$ must be $\delta(\omega)$, the same as noninteracting at all. So if $1/a_i = 1/a_f$, the tail does vanish. However, if we expand to the effective range order as $k \cot\delta = -1/a + r_e k^2/2$ in Eq. (27), we would have the difference

$$k_i \cot\delta_i - k_f \cot\delta_f = -(1/a_i - 1/a_f) + (r_{e,i} k_i^2 - r_{e,f} k_f^2)/2, \tag{30}$$

which is not automatically zero if $a_i = a_f$ and $r_{e,i} = r_{e,f}$. This problem probably stems from the calculation of the overlap in Eq. (24) where we neglect the part $r < r_0$. We shall treat the wave-function inside more *carefully.*

B. Continuum to a medianly bound dimer

Close to the 1-2 unitary point, the 1-3 has a dimer of binding energy $|E_b| \approx 4$ MHz. Note $|E_b|/2E_F \approx 200$. Let us calculate the rf spectrum for a 1-2 continuum state to the dimer state. As we understand from the spherical square well model calculation below, if the wave-function overlap for $r < r_0$ is suppressed by the potential difference $U_{12}(r) - U_{13}$, the outside overlap is

$$\int_{r_0} d\mathbf{r}\phi_d^*(\mathbf{r})\phi_i(\mathbf{r}) = 4\pi\mathcal{N}_i\mathcal{N}_d \int_{r_0}^{\infty} dr \sin(p_i r + \delta_i)e^{-\kappa r} \tag{31}$$

$$= 4\pi\mathcal{N}_i\mathcal{N}_d e^{-\tilde{\kappa}} \left[\cos(\tilde{p}_i + \delta_i)\frac{p_i}{\kappa^2 + p_i^2} + \sin(\tilde{p} + \delta_i)\frac{\kappa}{\kappa^2 + p_i^2}\right]. \tag{32}$$

Since from a 1-2 unitary Fermi gas, $p_i \sim k_F$, the first term is negligible compared to the second in the above line. Thus, to lowest order of p_i/κ,

$$\int_{r_0} d\mathbf{r}\phi_d^*(\mathbf{r})\phi_i(\mathbf{r}) \approx 4\pi\mathcal{N}_i\mathcal{N}_d e^{-\tilde{\kappa}} \sin(\tilde{p} + \delta_i)\frac{1}{\kappa} \tag{33}$$

$$= 4\pi\mathcal{N}_i\mathcal{N}_d e^{-\tilde{\kappa}} \frac{1}{\kappa}[\sin\delta_i \cos\tilde{p}_i + \cos\delta_i \sin\tilde{p}_i] \tag{34}$$

$$= 4\pi(\mathcal{N}_i \sin\delta_i)\mathcal{N}_d e^{-\tilde{\kappa}} \frac{1}{\kappa}[1 - 1/\tilde{a}_i + O(\tilde{p}_i^2)] \tag{35}$$

$$= 4\pi(\mathcal{N}_i \sin\delta_i)\mathcal{N}_d e^{-\tilde{\kappa}} \frac{1}{\kappa} \tag{36}$$

The spectrum is

$$\Gamma_{p_i\to d}(\omega) = 2\pi \left|4\pi(\mathcal{N}_i \sin\delta_i)\mathcal{N}_d e^{-\tilde{\kappa}} \frac{1}{\kappa}\right|^2 \delta(\omega + p_i^2/m + \kappa^2/m). \tag{37}$$

Now let us do the frequency integral first

$$I_{p_i\to d} = \int_{-\infty}^{-\kappa^2/m} d\omega \Gamma_{p_i\to d}(\omega) \tag{38}$$

$$= 32\pi^3(\mathcal{N}_i \sin\delta_i)^2\mathcal{N}_d^2 e^{-2\tilde{\kappa}} \frac{1}{\kappa^2} \tag{39}$$

$$= 32\pi^3 V\mathcal{C}_i\mathcal{N}_d^2 e^{-2\tilde{\kappa}} \frac{1}{\kappa^2}. \tag{40}$$

From the spherical square well model below, we have

$$\mathcal{N}_d^2 = \frac{\kappa e^{2\tilde{\kappa}}/2\pi}{1 + \tilde{\kappa}\frac{\eta^2 + \tilde{\kappa}}{\eta^2 - \tilde{\kappa}^2}}. \tag{41}$$

If $\tilde{\kappa} \ll 1$, $\mathcal{N}_d^2 e^{-2\tilde{\kappa}} \approx \kappa/2\pi$; otherwise, to the lowest order of $1/\eta$ ($\eta \gg 1$),

$$\mathcal{N}_d^2 = \frac{\kappa e^{2\tilde{\kappa}}/2\pi}{1 + \tilde{\kappa}}. \tag{42}$$

Thus

$$I_{p_i\to d} \approx (4\pi)^2 V\mathcal{C}_i \frac{1}{\kappa}\frac{1}{1 + \tilde{\kappa}}. \tag{43}$$

To work out the spectrum function

$$\Gamma(\omega) = \sum_{p_i} \mathcal{P}(p_i)\Gamma_{p_i\to d}(\omega), \tag{44}$$

we need the probability $P(p_i)$. For a 1-2 unitary Fermi gas, if we assume

$$\mathcal{P}(p_i) \propto \sum_{\mathbf{Q}} f(\mathbf{Q}/2 + \mathbf{p})f(\mathbf{Q}/2 - \mathbf{p}) \tag{45}$$

with f the Ferm distribution, at high temperatures $T \to \infty$,

$$\mathcal{P}(p_i) \propto e^{-p_i^2/mk_BT} \tag{46}$$

$$\Gamma(\omega) \propto \int_0^\infty dp_i e^{-p_i^2/mk_BT}\delta(\omega + p_i^2/m + \kappa^2/m)\sin^2\delta_i \tag{47}$$

$$\approx \int_0^\infty dp_i e^{-p_i^2/mk_BT}\delta(\omega + p_i^2/m + \kappa^2/m)\frac{p_i^2}{p_i^2 + 1/a_i^2} \tag{48}$$

$$= \frac{m}{2} e^{(\omega+\kappa^2/m)/k_BT}\frac{\sqrt{(-m\omega - \kappa^2)}}{1/a_i^2 - (m\omega + \kappa^2)}. \tag{49}$$

At $T = 0$,

$$\mathcal{P}(p_i) \propto (2k_F^3 - 3k_F^2 p_i + p_i^3)\theta(k_F - p_i) \tag{50}$$

$$\Gamma(\omega) \propto [2k_F^3 - 3k_F^2\sqrt{(-m\omega - \kappa^2)} + \sqrt{(-m\omega - \kappa^2)}^3]\frac{\sqrt{(-m\omega - \kappa^2)}}{1/a_i^2 - (m\omega + \kappa^2)}\theta(k_F - p_i) \tag{51}$$

III. SPHERICAL SQUARE WELL MODEL

Let us consider a spherical square well model potential $V(r) = -V_0$ for $r < r_0$ and $V(r) = 0$ otherwise.

A. Dimer to free

For a dimer with energy $E_i = -\kappa^2/m$, the relative wave-function is

$$\phi_{d,>}(\mathbf{r}) = \mathcal{N}\frac{e^{-\kappa r}}{r}, \text{ for } r > r_0; \tag{52}$$

$$\phi_{d,<}(\mathbf{r}) = \mathcal{N}'\frac{\sin(\sqrt{\eta^2 - \tilde{\kappa}^2}\tilde{r})}{r}, \text{ for } r > r_0, \tag{53}$$

with $\eta^2 \equiv mV_0r_0^2$, $\tilde{r} \equiv r/r_0$ and $\tilde{\kappa} \equiv \kappa r_0$. The continuity of the wave-function at $r = r_0$ gives

$$\mathcal{N}' = N\frac{e^{-\tilde{\kappa}}}{\sin\sqrt{\eta^2 - \tilde{\kappa}^2}}. \tag{54}$$

The normalization of the wave-function gives

$$\mathcal{N}^2 = \frac{\kappa e^{2\tilde{\kappa}}/2\pi}{1 + \frac{\tilde{\kappa}}{\sin^2(\sqrt{\eta^2 - \tilde{\kappa}^2})}\left[1 - \frac{\sin(2\sqrt{\eta^2 - \tilde{\kappa}^2})}{2\sqrt{\eta^2 - \tilde{\kappa}^2}}\right]}. \tag{55}$$

On the other hand, the first derivative of the wave-function should also be continuous at $r = r_0$, which leads to

$$\frac{\cos\sqrt{\eta^2 - \tilde{\kappa}^2}}{\sin\sqrt{\eta^2 - \tilde{\kappa}^2}} = -\frac{\tilde{\kappa}}{\sqrt{\eta^2 - \tilde{\kappa}^2}}, \tag{56}$$

and

$$\sin\sqrt{\eta^2 - \tilde{\kappa}^2} = -\frac{\sqrt{\eta^2 - \tilde{\kappa}^2}}{\eta} \tag{57}$$

$$\cos\sqrt{\eta^2 - \tilde{\kappa}^2} = \frac{\tilde{\kappa}}{\eta} \tag{58}$$

Thus

$$\mathcal{N}^2 = \frac{\kappa e^{2\tilde{\kappa}}/2\pi}{1 + \tilde{\kappa}\frac{\eta^2 + \tilde{\kappa}}{\eta^2 - \tilde{\kappa}^2}}. \tag{59}$$

Note interatomic potentials are very deep (~ 1 eV? $\sim 10^{14}$ Hz), i.e., $\eta \gg 1$ and $\tilde{\kappa} \ll \eta$. To the lowest order of $1/\eta$,

$$\mathcal{N}^2 \approx \frac{\kappa e^{2\tilde{\kappa}}/2\pi}{1+\tilde{\kappa}}. \tag{60}$$

Now the Fourier transform of the wave-function becomes ($s \equiv \sqrt{\eta^2 - \tilde{\kappa}^2}$, $\tilde{p} \equiv pr_0$)

$$\tilde{\phi}_d(\mathbf{p}) = \int d\mathbf{r} \phi_d(\mathbf{r}) \frac{e^{-i\mathbf{p}\cdot\mathbf{r}}}{\sqrt{V}} \tag{61}$$

$$= \frac{4\pi r_0}{p\sqrt{V}} \left[\mathcal{N}' \int_0^1 d\tilde{r} \sin(s\tilde{r}) \sin(\tilde{p}\tilde{r}) + \mathcal{N} \int_1^\infty d\tilde{r} e^{-\tilde{\kappa}\tilde{r}} \sin(\tilde{p}\tilde{r}) \right] \tag{62}$$

$$= \frac{2\pi r_0}{p\sqrt{V}} \left\{ \mathcal{N}' \left[\frac{\sin(\tilde{p}-s)}{\tilde{p}-s} - \frac{\sin(\tilde{p}+s)}{\tilde{p}+s} \right] - i\mathcal{N} e^{-\tilde{\kappa}} \left[\frac{e^{i\tilde{p}}}{\tilde{\kappa}-i\tilde{p}} - \frac{e^{-i\tilde{p}}}{\tilde{\kappa}+i\tilde{p}} \right] \right\} \tag{63}$$

$$= \frac{4\pi r_0}{p\sqrt{V}} \left\{ \frac{\mathcal{N}'}{\tilde{p}^2 - s^2} \left[s \sin\tilde{p} \cos s - \tilde{p} \cos\tilde{p} \sin s \right] + \frac{\mathcal{N} e^{-\tilde{\kappa}}}{\tilde{\kappa}^2 + \tilde{p}^2} \left[\tilde{p} \cos\tilde{p} + \tilde{\kappa} \sin\tilde{p} \right] \right\} \tag{64}$$

$$= \frac{4\pi r_0 \mathcal{N} e^{-\tilde{\kappa}}}{p\sqrt{V}} \left\{ \frac{1}{\tilde{p}^2 - s^2} \left[s \sin\tilde{p} \cot s - \tilde{p} \cos\tilde{p} \right] + \frac{1}{\tilde{\kappa}^2 + \tilde{p}^2} \left[\tilde{p} \cos\tilde{p} + \tilde{\kappa} \sin\tilde{p} \right] \right\}. \tag{65}$$

Thus the rf spectrum is given by

$$\Gamma(\omega) = 2\pi \frac{V}{(2\pi)^3} \int d\mathbf{p} |\tilde{\phi}_d(\mathbf{p})|^2 \delta(\omega - \kappa^2/m - p^2/m) \tag{66}$$

$$= 2\pi (4\pi r_0 \mathcal{N} e^{-\tilde{\kappa}})^2 \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \left\{ \frac{1}{\tilde{p}^2 - s^2} \left[s \sin\tilde{p} \cot s - \tilde{p} \cos\tilde{p} \right] + \frac{1}{\tilde{\kappa}^2 + \tilde{p}^2} \left[\tilde{p} \cos\tilde{p} + \tilde{\kappa} \sin\tilde{p} \right] \right\}^2 \delta(\omega - \kappa^2/m - p^2/m) \tag{67}$$

$$= 2\pi (4\pi r_0 \mathcal{N} e^{-\tilde{\kappa}})^2 \frac{2\pi m}{(2\pi)^3} \frac{1}{\sqrt{m\omega - \kappa^2}} \left\{ \frac{1}{m\omega r_0^2 - \eta^2} \left[\sin(\sqrt{m\omega r_0^2 - \tilde{\kappa}^2}) s \cot s - \sqrt{m\omega r_0^2 - \tilde{\kappa}^2} \cos(\sqrt{m\omega r_0^2 - \tilde{\kappa}^2}) \right] \right. \tag{68}$$

$$\left. + \frac{1}{m\omega r_0^2} \left[\sqrt{m\omega r_0^2 - \tilde{\kappa}^2} \cos(\sqrt{m\omega r_0^2 - \tilde{\kappa}^2}) + \tilde{\kappa} \sin(\sqrt{m\omega r_0^2 - \tilde{\kappa}^2}) \right] \right\}^2 \theta(\omega - \kappa^2/m). \tag{69}$$

Since

$$\frac{\ln[r\phi_{d,>}(\mathbf{r})]}{dr}|_{r=r_0} = \frac{\ln[r\phi_{d,<}(\mathbf{r})]}{dr}|_{r=r_0}, \tag{70}$$

we have $-\tilde{\kappa} = s \cot s$ with $s \equiv \sqrt{\eta^2 - \tilde{\kappa}^2}$. Thus, in the regime $\tilde{p}_\omega^2 \equiv m\omega r_0^2 - \tilde{\kappa}^2 \ll 1 \ll \eta$, to lowest order of $1/\eta$,

$$\Gamma(\omega) = 2\pi (4\pi r_0 \mathcal{N} e^{-\tilde{\kappa}})^2 \frac{2\pi m}{(2\pi)^3} \frac{1}{\sqrt{m\omega - \kappa^2}} \left\{ \frac{1}{m\omega r_0^2} \left[\sqrt{m\omega r_0^2 - \tilde{\kappa}^2} \cos(\sqrt{m\omega r_0^2 - \tilde{\kappa}^2}) + \tilde{\kappa} \sin(\sqrt{m\omega r_0^2 - \tilde{\kappa}^2}) \right] \right\}^2 \tag{71}$$

$$\approx 2\pi (4\pi \mathcal{N} e^{-\tilde{\kappa}})^2 \frac{2\pi m}{(2\pi)^3} \frac{\sqrt{m\omega - \kappa^2}}{(m\omega)^2} \left\{ 1 - \frac{1}{2}(m\omega r_0^2 - \tilde{\kappa}^2) + O(\tilde{p}_\omega^4) + \tilde{\kappa} \left[1 - \frac{1}{6}(m\omega r_0^2 - \tilde{\kappa}^2) + O(\tilde{p}_\omega^4) \right] \right\}^2 \tag{72}$$

$$= 2\pi (4\pi \mathcal{N} e^{-\tilde{\kappa}})^2 \frac{2\pi m}{(2\pi)^3} \frac{\sqrt{m\omega - \kappa^2}}{(m\omega)^2} \left\{ 1 - \frac{1}{2}\tilde{p}_\omega^2 + O(\tilde{p}_\omega^4) + \tilde{\kappa} \left[1 - \frac{1}{6}\tilde{p}_\omega^2 + O(\tilde{p}_\omega^4) \right] \right\}^2. \tag{73}$$

To lowest order of $\tilde{p}_\omega^2$, WHAT'S THE CONTACT? THE CONTACT CAN ONLY BE DEFINED FOR SHALLOW DIMER STATES!

$$\Gamma(\omega) \to 8\pi m \mathcal{N}^2 \frac{\sqrt{m\omega - \kappa^2}}{(m\omega)^2} \left[e^{-\tilde{\kappa}}(1+\tilde{\kappa}) \right]^2. \tag{74}$$

The factor $e^{-\tilde{\kappa}}(1+\tilde{\kappa}) = 1 + O(\tilde{\kappa}^2)$. Note that the contribution of $\phi_{d,<}(\mathbf{r})$ to $\Gamma(\omega)$ is suppressed by the deep potential $\sim \eta$

B. Continuum to free

For a continuum state with energy $E_i = k^2/m$, the relative wave-function is

$$\phi_{k,>}(\mathbf{r}) = \mathcal{N}\frac{\sin(kr+\delta)}{r}, \text{ for } r > r_0; \tag{75}$$

$$\phi_{k,<}(\mathbf{r}) = \mathcal{N}'\frac{\sin(\sqrt{\eta^2+\tilde{k}^2}\tilde{r})}{r}, \text{ for } r > r_0, \tag{76}$$

with $\eta^2 \equiv mV_0r_0^2$, $\tilde{r} \equiv r/r_0$ and $\tilde{\kappa} \equiv \kappa r_0$. The continuity of the wave-function at $r = r_0$ gives

$$\mathcal{N}' = \mathcal{N}\frac{\sin(\tilde{k}+\delta)}{\sin(\sqrt{\eta^2+\tilde{k}^2})}. \tag{77}$$

The normalization of the wave-function in a sphere of radius ℓ with a hard wall ($k\ell + \delta = n\pi$) gives

$$\mathcal{N}^2 = \frac{1}{2\pi r_0}\left\{\ell/r_0 - 1 + \frac{\sin(2\tilde{k}+\delta)}{2\tilde{k}} + \frac{\sin^2(\tilde{k}+\delta)}{\sin^2\sqrt{\eta^2+\tilde{k}^2}}\left[1 - \frac{\sin(2\sqrt{\eta^2+\tilde{k}^2})}{2\sqrt{\eta^2+\tilde{k}^2}}\right]\right\}^{-1}. \tag{78}$$

If we fix k and take $\ell \to \infty$ ($n \to \infty$), we have

$$\mathcal{N}^2 \approx \frac{1}{2\pi\ell}. \tag{79}$$

Now the Fourier transform of the wave-function becomes ($s \equiv \sqrt{\eta^2+\tilde{k}^2}$, $\tilde{p} \equiv pr_0$; the final free states also obey the hard wall condition such that $p\ell = j\pi$)

$$\tilde{\phi}_k(\mathbf{p}) = \int d\mathbf{r}\phi_d(\mathbf{r})\frac{e^{-i\mathbf{p}\cdot\mathbf{r}}}{\sqrt{V}} \tag{80}$$

$$= \frac{4\pi r_0}{p\sqrt{V}}\left[\mathcal{N}'\int_0^1 d\tilde{r}\sin(s\tilde{r})\sin(\tilde{p}\tilde{r}) + \mathcal{N}\int_1^\ell d\tilde{r}\sin(\tilde{k}\tilde{r}+\delta)\sin(\tilde{p}\tilde{r})\right] \tag{81}$$

$$= \frac{2\pi r_0}{p\sqrt{V}}\left\{\mathcal{N}'\left[\frac{\sin(\tilde{p}-s)}{\tilde{p}-s} - \frac{\sin(\tilde{p}+s)}{\tilde{p}+s}\right] - \mathcal{N}\left[\frac{\sin(\tilde{k}+\delta-\tilde{p})}{\tilde{k}-\tilde{p}} - \frac{\sin(\tilde{k}+\delta+\tilde{p})}{\tilde{k}+\tilde{p}}\right]\right\} \tag{82}$$

$$= \frac{4\pi r_0}{p\sqrt{V}}\left\{\frac{\mathcal{N}'}{\tilde{p}^2-s^2}\left[s\sin\tilde{p}\cos s - \tilde{p}\cos\tilde{p}\sin s\right] + \frac{\mathcal{N}}{\tilde{k}^2-\tilde{p}^2}\left[\tilde{p}\sin(\tilde{k}+\delta)\cos\tilde{p} - \tilde{k}\sin\tilde{p}\cos(\tilde{k}+\delta)\right]\right\} \tag{83}$$

$$\tag{84}$$

Thus the rf spectrum is given by ($\tilde{p}_\omega^2 = m\omega r_0^2 + \tilde{k}^2$)

$$\Gamma(\omega) = 2\pi\frac{V}{(2\pi)^3}\int d\mathbf{p}|\tilde{\phi}_k(\mathbf{p})|^2\delta(\omega + k^2/m - p^2/m) \tag{85}$$

$$= 2\pi(4\pi r_0)^2\frac{4\pi}{(2\pi)^3}\int_0^\infty dp\left\{\frac{\mathcal{N}'}{\tilde{p}^2-s^2}\left[s\sin\tilde{p}\cos s - \tilde{p}\cos\tilde{p}\sin s\right] + \frac{\mathcal{N}}{\tilde{k}^2-\tilde{p}^2}\left[\tilde{p}\sin(\tilde{k}+\delta)\cos\tilde{p} - \tilde{k}\sin\tilde{p}\cos(\tilde{k}+\delta)\right]\right\}^2 \tag{86}$$

$$\times\,\delta(\omega + k^2/m - p^2/m) \tag{87}$$

$$= 2\pi(4\pi r_0)^2\frac{2\pi m}{(2\pi)^3}\frac{1}{\sqrt{m\omega+k^2}}\left\{\frac{\mathcal{N}'}{\tilde{p}_\omega^2-s^2}\left[s\sin\tilde{p}_\omega\cos s - \tilde{p}_\omega\cos\tilde{p}_\omega\sin s\right]\right. \tag{88}$$

$$\left. + \frac{\mathcal{N}}{\tilde{k}^2-\tilde{p}_\omega^2}\left[\tilde{p}_\omega\sin(\tilde{k}+\delta)\cos\tilde{p}_\omega - \tilde{k}\sin\tilde{p}_\omega\cos(\tilde{k}+\delta)\right]\right\}^2. \tag{89}$$

To the zero order of $1/\eta$,

$$\Gamma(\omega) \to 2\pi(4\pi\mathcal{N})^2\frac{2\pi m}{(2\pi)^3}\frac{\sqrt{m\omega+k^2}}{(m\omega)^2}\left[\sin(\tilde{k}+\delta)\cos\tilde{p}_\omega - \frac{\tilde{k}}{\tilde{p}_\omega}\sin\tilde{p}_\omega\cos(\tilde{k}+\delta)\right]^2. \tag{90}$$

In the regime $\tilde{k}/\tilde{p}_\omega \ll 1$ and $\tilde{k} \ll 1$ and $\tilde{p}_\omega \ll 1$

$$\Gamma(\omega) \to 2\pi(4\pi\mathcal{N})^2 \frac{2\pi m}{(2\pi)^3} \frac{\sqrt{m\omega + k^2}}{(m\omega)^2} \sin^2(\delta). \tag{91}$$

C. Continuum to continuum with final state effects

Let us consider the initial and the final state subject to spherical square well potentials $V_\sigma(r) = -V_\sigma$ for $r < r_0$ and $V_\sigma(r) = 0$ otherwise with $\sigma = i, f$.

For a continuum state with energy $E_\sigma = k_\sigma^2/m$, the relative wave-function is

$$\phi_{k_\sigma,>}(\mathbf{r}) = \mathcal{N}_\sigma \frac{\sin(k_\sigma r + \delta_\sigma)}{r}, \text{ for } r > r_0; \tag{92}$$

$$\phi_{k_\sigma,<}(\mathbf{r}) = \mathcal{N}'_\sigma \frac{\sin(\sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2}\tilde{r})}{r}, \text{ for } r > r_0, \tag{93}$$

with $\eta_\sigma^2 \equiv mV_\sigma r_0^2$, $\tilde{r} \equiv r/r_0$ and $\tilde{k}_\sigma \equiv k_\sigma r_0$. The continuity of the wave-function at $r = r_0$ gives

$$\mathcal{N}'_\sigma = \mathcal{N}_\sigma \frac{\sin(\tilde{k}_\sigma + \delta_\sigma)}{\sin(\sqrt{\eta_\sigma^2 + \tilde{k}^2})}. \tag{94}$$

The normalization of the wave-function in a sphere of radius ℓ with a hard wall ($k_\sigma \ell + \delta_\sigma = n_\sigma \pi$) gives

$$\mathcal{N}_\sigma^2 = \frac{1}{2\pi r_0} \left\{ \ell/r_0 - 1 + \frac{\sin(2\tilde{k}_\sigma + \delta_\sigma)}{2\tilde{k}_\sigma} + \frac{\sin^2(\tilde{k}_\sigma + \delta_\sigma)}{\sin^2 \sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2}} \left[1 - \frac{\sin(2\sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2})}{2\sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2}} \right] \right\}^{-1}. \tag{95}$$

If we fix k_σ and take $\ell \to \infty$ ($n_\sigma \to \infty$), we have

$$\mathcal{N}_\sigma^2 \approx \frac{1}{2\pi\ell}. \tag{96}$$

The crucial step is to calculate the overlap ($k_f \neq k_i$)

$$\int d\mathbf{r} \phi_{k_i}(\mathbf{r}) \phi_{k_f}(\mathbf{r}) = 4\pi r_0 \mathcal{N}_i \mathcal{N}_f \sin(\tilde{k}_i + \delta_i) \sin(\tilde{k}_f + \delta_f) \left[\frac{\sqrt{\eta_f^2 + \tilde{k}_f^2} \cot \sqrt{\eta_f^2 + \tilde{k}_f^2} - \sqrt{\eta_i^2 + \tilde{k}_i^2} \cot \sqrt{\eta_i^2 + \tilde{k}_i^2}}{\eta_i^2 + \tilde{k}_i^2 - \eta_f^2 - \tilde{k}_f^2} \right. \tag{97}$$

$$\left. - \frac{\tilde{k}_f \cot(\tilde{k}_f + \delta_f) - \tilde{k}_i \cot(\tilde{k}_i + \delta_i)}{\tilde{k}_i^2 - \tilde{k}_f^2} \right] \tag{98}$$

$$= 4\pi r_0 \mathcal{N}_i \mathcal{N}_f \sin(\tilde{k}_i + \delta_i) \sin(\tilde{k}_f + \delta_f) \left[\frac{1}{\eta_i^2 + \tilde{k}_i^2 - \eta_f^2 - \tilde{k}_f^2} - \frac{1}{\tilde{k}_i^2 - \tilde{k}_f^2} \right] \tag{99}$$

$$\times [\tilde{k}_f \cot(\tilde{k}_f + \delta_f) - \tilde{k}_i \cot(\tilde{k}_i + \delta_i)]. \tag{100}$$

We have used the continuity of the wave-functions at $r = r_0$ which gives

$$\sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2} \cot \sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2} = \tilde{k}_\sigma \cot(\tilde{k}_\sigma + \delta_\sigma). \tag{101}$$

The nice property of the above expression for the overlap is that if $V_i = V_f$, i.e., $\eta_i = \eta_f$ and $\delta_i(k) = \delta_f(k)$, the overlap is automatically zero.

The rf spectrum function is

$$\Gamma(\omega) = 2\pi \sum_{k_f} \left| \int d\mathbf{r} \phi_{k_i}(\mathbf{r}) \phi_{k_f}(\mathbf{r}) \right|^2 \delta(\omega + k_i^2/m - k_f^2/m). \tag{102}$$

Since $\delta(\omega+k_i^2/m-k_f^2/m)$ in $\Gamma(\omega)$ picks $k_f^\omega=\sqrt{m\omega+k_i^2}$ and usually $\eta_i^2-\eta_f^2\sim O(1)$ while $m\omega r_0^2\ll 1$, the contribution of the overlap for $r<r_0$ to Eq. (100) is relatively negligible. Therefore

$$\Gamma(\omega)\approx m\ell(4\pi r_0\mathcal{N}_i\mathcal{N}_f)^2\frac{1}{\sqrt{m\omega+k_i^2}}\frac{1}{(m\omega r_0^2)^2}\left[\tilde{k}_f^\omega\cos(\tilde{k}_f^\omega+\delta_f)\sin(\tilde{k}_i+\delta_i)-\tilde{k}_i\cos(\tilde{k}_i+\delta_i)\sin(\tilde{k}_f^\omega+\delta_f)\right]^2 \tag{103}$$

$$=m\ell(4\pi r_0\mathcal{N}_i\mathcal{N}_f)^2\frac{1}{\sqrt{m\omega+k_i^2}}\frac{1}{(m\omega r_0^2)^2}\sin^2(\tilde{k}_i+\delta_i)\sin^2(\tilde{k}_f^\omega+\delta_f)\left[\tilde{k}_f^\omega\cot(\tilde{k}_f^\omega+\delta_f)-\tilde{k}_i\cot(\tilde{k}_i+\delta_i)\right]^2. \tag{104}$$

Let us look at the functions

$$\tilde{k}\cot(\tilde{k}+\delta)=\frac{\tilde{k}(\cot\delta-\tan\tilde{k})}{1+\tan\tilde{k}\cot\delta} \tag{105}$$

$$=\frac{\tilde{k}[-1/\tilde{k}\tilde{a}+\tilde{r}_e\tilde{k}/2+O(\tilde{k}^3)-\tilde{k}]}{1+[\tilde{k}+\tilde{k}^3/3+O(\tilde{k}^5)][-1/\tilde{k}\tilde{a}+\tilde{r}_e\tilde{k}+O(\tilde{k}^3)]} \tag{106}$$

$$=\frac{-1/\tilde{a}+(\tilde{r}_e/2-1)\tilde{k}^2+O(\tilde{k}^4)}{1-1/\tilde{a}+(\tilde{r}_e/2-1/3\tilde{a})\tilde{k}^2+O(\tilde{k}^4)} \tag{107}$$

$$=\frac{1/\tilde{a}}{1/\tilde{a}-1}+O(\tilde{k}^2). \tag{108}$$

$$\sin^2(\tilde{k}+\delta)=[\sin\tilde{k}\cos\delta+\cos\tilde{k}\sin\delta]^2 \tag{109}$$

$$=\sin^2\delta\left[\frac{\sin\tilde{k}}{\tilde{k}}\tilde{k}\cot\delta+\cos\tilde{k}\right]^2 \tag{110}$$

$$=\sin^2\delta\left[1-1/\tilde{a}+(\tilde{r}_e/2+1/6\tilde{a}-1/2)\tilde{k}^2+O(\tilde{k}^4)\right]^2 \tag{111}$$

$$\sin^2\delta=\frac{1}{1+[-1/\tilde{k}\tilde{a}+\tilde{r}_e\tilde{k}/2+O(\tilde{k}^3)]^2} \tag{112}$$

$$=\frac{1}{1+[-1/\tilde{k}\tilde{a}+\tilde{r}_e\tilde{k}/2]^2+O(\tilde{k}^2)} \tag{113}$$

$$=\frac{\tilde{k}^2}{\tilde{k}^2+[-1/\tilde{a}+\tilde{r}_e\tilde{k}^2/2]^2+O(\tilde{k}^4)} \tag{114}$$

$$=\frac{\tilde{k}^2}{\tilde{k}^2+1/\tilde{a}^2-\tilde{r}_e\tilde{k}^2/\tilde{a}+O(\tilde{k}^4)}. \tag{115}$$

Thus after collecting all, we have

$$\Gamma(\omega)=m\ell(4\pi r_0\mathcal{N}_i\mathcal{N}_f)^2\frac{1}{\sqrt{m\omega+k_i^2}}\frac{1}{(m\omega r_0^2)^2}\sin^2(\tilde{k}_i+\delta_i)\sin^2(\tilde{k}_f^\omega+\delta_f)\left[\tilde{k}_f^\omega\cot(\tilde{k}_f^\omega+\delta_f)-\tilde{k}_i\cot(\tilde{k}_i+\delta_i)\right]^2 \tag{116}$$

$$=m\ell(4\pi r_0\mathcal{N}_i\sin\delta_i\mathcal{N}_f)^2\frac{1}{\sqrt{m\omega+k_i^2}}\frac{1}{(m\omega r_0^2)^2}\frac{(\tilde{k}_f^\omega)^2}{(\tilde{k}_f^\omega)^2+1/\tilde{a}_f^2-\tilde{r}_{e,f}(\tilde{k}_f^\omega)^2/\tilde{a}_f}(1-1/\tilde{a}_i)^2(1-1/\tilde{a}_f)^2 \tag{117}$$

$$\times\left[\frac{1/\tilde{a}_f}{1/\tilde{a}_f-1}-\frac{1/\tilde{a}_i}{1/\tilde{a}_i-1}\right]^2 \tag{118}$$

$$=m\ell(4\pi\mathcal{N}_i\sin\delta_i\mathcal{N}_f)^2\frac{1}{\sqrt{m\omega+k_i^2}}\frac{1}{(m\omega)^2}\frac{(\tilde{k}_f^\omega)^2}{(\tilde{k}_f^\omega)^2+1/\tilde{a}_f^2-\tilde{r}_{e,f}(\tilde{k}_f^\omega)^2/\tilde{a}_f}\left[\frac{1}{a_f}-\frac{1}{a_i}\right]^2 \tag{119}$$

$$=8\pi m(\mathcal{N}_i\sin\delta_i)^2\frac{1}{\sqrt{m\omega+k_i^2}}\frac{1}{(m\omega)^2}\frac{(\tilde{k}_f^\omega)^2}{(\tilde{k}_f^\omega)^2+1/\tilde{a}_f^2-\tilde{r}_{e,f}(\tilde{k}_f^\omega)^2/\tilde{a}_f}\left[\frac{1}{a_f}-\frac{1}{a_i}\right]^2 \tag{120}$$

$$=8\pi mV\mathcal{C}_i\frac{1}{\sqrt{m\omega+k_i^2}}\frac{1}{(m\omega)^2}\frac{(k_f^\omega)^2}{(k_f^\omega)^2+1/a_f^2-r_{e,f}(k_f^\omega)^2/a_f}\left[\frac{1}{a_f}-\frac{1}{a_i}\right]^2. \tag{121}$$

The effective range correction appears for the final states, which is significant only if $r_{e,f}/a_f\sim O(1)$.

IV. NONINTERACTING DIMER APPROXIMATION

If one assume that the resultant dimer in the final state is not interacting with the rest gas, one can approximate the final state as

$$|f\rangle = \int_{\mathbf{R},\mathbf{r}} \frac{e^{i\mathbf{P}\cdot\mathbf{R}}}{\sqrt{V}} \phi_d(\mathbf{r})\psi_1^\dagger(\mathbf{R}+\mathbf{r}/2)\psi_3^\dagger(\mathbf{R}-\mathbf{r}/2)|f'\rangle \tag{122}$$

such that in the final state the dimer has net momentum $\mathbf{P}$ and the rest $1-2$ gas is in the final state $|f'\rangle$. The spectrum becomes

$$\Gamma(\omega) = 2\pi \sum_{f',\mathbf{P}} |\langle f'| \int_{\mathbf{R},\mathbf{r}} \frac{e^{-i\mathbf{P}\cdot\mathbf{R}}}{\sqrt{V}} \phi_d^*(\mathbf{r})\psi_1(\mathbf{R}+\mathbf{r}/2)\psi_2(\mathbf{R}-\mathbf{r}/2)|i\rangle|^2 \delta(\omega + E_i - E_{f'} + |E_d| - P^2/4m). \tag{123}$$

After the integral, we have

$$\int \frac{d\omega}{2\pi}\Gamma(\omega) = \sum_{f',\mathbf{P}} |\langle f'| \int_{\mathbf{R},\mathbf{r}} \frac{e^{-i\mathbf{P}\cdot\mathbf{R}}}{\sqrt{V}} \phi_d^*(\mathbf{r})\psi_1(\mathbf{R}+\mathbf{r}/2)\psi_2(\mathbf{R}-\mathbf{r}/2)|i\rangle|^2 \tag{124}$$

$$= \sum_{\mathbf{P}} \int_{\mathbf{R},\mathbf{r},\mathbf{R}',\mathbf{r}'} \frac{e^{-i\mathbf{P}\cdot\mathbf{R}}}{\sqrt{V}} \phi_d^*(\mathbf{r}) \frac{e^{i\mathbf{P}\cdot\mathbf{R}'}}{\sqrt{V}} \phi_d(\mathbf{r}')\langle i|\psi_2^\dagger(\mathbf{R}'-\mathbf{r}'/2)\psi_1^\dagger(\mathbf{R}'+\mathbf{r}'/2)\psi_1(\mathbf{R}+\mathbf{r}/2)\psi_2(\mathbf{R}-\mathbf{r}/2)|i\rangle \tag{125}$$

$$= \int_{\mathbf{R},\mathbf{r},\mathbf{r}'} \phi_d^*(\mathbf{r})\phi_d(\mathbf{r}')\langle i|\psi_2^\dagger(\mathbf{R}-\mathbf{r}'/2)\psi_1^\dagger(\mathbf{R}+\mathbf{r}'/2)\psi_1(\mathbf{R}+\mathbf{r}/2)\psi_2(\mathbf{R}-\mathbf{r}/2)|i\rangle \tag{126}$$

$$= V\int_{\mathbf{r},\mathbf{r}'} \phi_d^*(\mathbf{r})\phi_d(\mathbf{r}')\langle \psi_2^\dagger(-\mathbf{r}'/2)\psi_1^\dagger(+\mathbf{r}'/2)\psi_1(+\mathbf{r}/2)\psi_2(-\mathbf{r}/2)\rangle \tag{127}$$

$$= V\int_{\mathbf{r},\mathbf{r}'} \phi_d^*(\mathbf{r})\phi_d(\mathbf{r}') \sum_j \lambda_j \chi_j^*(r')\chi_j(r) \tag{128}$$

$$= V\sum_j \lambda_j \left| \int_{\mathbf{r}} \phi_d^*(\mathbf{r})\chi_j(r) \right|^2 \tag{129}$$

Assuming the overlap is mainly contributed from $r_0 < r < 1/k_F$ and we normalize $\chi_j(r) \sim 1/r - 1/a$ in this region, we have

$$\int \frac{d\omega}{2\pi}\Gamma(\omega) = V\mathcal{C}_i \left| \int_{\mathbf{r}} \phi_d^*(\mathbf{r})(\frac{1}{r} - \frac{1}{a}) \right|^2 \tag{130}$$

A. BCS

Let's get an idea how $\Gamma(\omega)$ shall look like within the noninteracting dimer approximation assuming the 1-2 gas is in the BCS state. At the chemical potential μ, the Hamiltonian of the 1-2 gas is given by

$$H = \sum_{\mathbf{k}} \xi_k [a_{1,\mathbf{k}}^\dagger a_{1,\mathbf{k}} + a_{2,-\mathbf{k}}^\dagger a_{2,-\mathbf{k}}] + \bar{g}\int d\mathbf{r}\psi_1^\dagger(\mathbf{r})\psi_2^\dagger(\mathbf{r})\psi_2(\mathbf{r})\psi_1(\mathbf{r}), \tag{131}$$

with $\xi_k = \epsilon_k - \mu$ and $\epsilon_k = k^2/2m$.

To carry out the BCS mean field approximation, we write $\bar{g}\int d\mathbf{r}\psi_1^\dagger(\mathbf{r})\psi_2^\dagger(\mathbf{r}) = \Delta^* + \delta\Delta^*$ with $\Delta^* = \langle \bar{g}\int d\mathbf{r}\psi_1^\dagger(\mathbf{r})\psi_2^\dagger(\mathbf{r})\rangle$ and $\delta\Delta^* = \bar{g}\int d\mathbf{r}\psi_1^\dagger(\mathbf{r})\psi_2^\dagger(\mathbf{r}) - \Delta^*$. By neglecting the second order of fluctuations ($\delta\Delta$) in the Hamiltonian, we obtain the BCS mean field Hamiltonian

$$H_{\rm BCS} = -\frac{V|\Delta|^2}{\bar{g}} + \sum_{\mathbf{k}} \xi_k [a_{1,\mathbf{k}}^\dagger a_{1,\mathbf{k}} + a_{2,-\mathbf{k}}^\dagger a_{2,-\mathbf{k}}] + \sum_{\mathbf{k}} [\Delta^* a_{1,\mathbf{k}} a_{2,-\mathbf{k}} + \Delta a_{2,-\mathbf{k}}^\dagger a_{1,\mathbf{k}}^\dagger] \tag{132}$$

$$= -\frac{V|\Delta|^2}{\bar{g}} + \sum_{\mathbf{k}} \xi_k + \sum_{\mathbf{k}} \begin{bmatrix} a_{1,\mathbf{k}}^\dagger & a_{2,-\mathbf{k}} \end{bmatrix} \begin{bmatrix} \xi_k & -\Delta \\ -\Delta^* & -\xi_k \end{bmatrix} \begin{bmatrix} a_{1,\mathbf{k}} \\ a_{2,-\mathbf{k}}^\dagger \end{bmatrix}. \tag{133}$$

For simplicity let us assume Δ real. To diagonalize $H_{\rm BCS}$, we define the new quasi-particle operators

$$\begin{bmatrix} \gamma_{1,\mathbf{k}} \\ \gamma^\dagger_{2,-\mathbf{k}} \end{bmatrix} = \begin{bmatrix} -\cos\theta_k & \sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix} \begin{bmatrix} a_{1,\mathbf{k}} \\ a^\dagger_{2,-\mathbf{k}} \end{bmatrix}, \tag{134}$$

where

$$\cos\theta_k = \frac{E_k + \xi_k}{\sqrt{2E_k(E_k+\xi_k)}} \tag{135}$$

$$\sin\theta_k = \frac{\Delta}{\sqrt{2E_k(E_k+\xi_k)}} \tag{136}$$

with $E_k = \sqrt{\xi_k^2 + \Delta^2}$. Likewise,

$$\begin{bmatrix} a_{1,\mathbf{k}} \\ a^\dagger_{2,-\mathbf{k}} \end{bmatrix} = \begin{bmatrix} -\cos\theta_k & \sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix} \begin{bmatrix} \gamma_{1,\mathbf{k}} \\ \gamma^\dagger_{2,-\mathbf{k}} \end{bmatrix}. \tag{137}$$

Thus

$$H_{\rm BCS} = E_{\rm GND} + \sum_{\mathbf{k}} E_k(\gamma^\dagger_{1,\mathbf{k}}\gamma_{1,\mathbf{k}} + \gamma^\dagger_{2,\mathbf{k}}\gamma_{2,\mathbf{k}}) \tag{138}$$

$$E_{\rm GND} = -V\frac{\Delta^2}{\bar{g}} + \sum_{\mathbf{k}}(\xi_k - E_k). \tag{139}$$

In the following we carry out a "non-conserving" calculation of $\Gamma(\omega)$ using $\gamma_{1/2,\mathbf{k}}$ (neglecting corrections to vertices).

$$\int_{\mathbf{R},\mathbf{r}} \frac{e^{-i\mathbf{P}\cdot\mathbf{R}}}{\sqrt{V}}\phi_d^*(\mathbf{r})\psi_1(\mathbf{R}+\mathbf{r}/2)\psi_2(\mathbf{R}-\mathbf{r}/2) \tag{140}$$

$$= \sum_{\mathbf{q}} a_{1,\mathbf{P}/2+\mathbf{q}}a_{2,\mathbf{P}/2-\mathbf{q}} \int d\mathbf{r}\phi_d^*(r)\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\sqrt{V}} \tag{141}$$

$$= \frac{1}{\sqrt{V}}\sum_{\mathbf{q}} \tilde{\phi}_d^*(q) a_{1,\mathbf{P}/2+\mathbf{q}}a_{2,\mathbf{P}/2-\mathbf{q}} \tag{142}$$

$$= \frac{1}{\sqrt{V}}\sum_{\mathbf{q}} \tilde{\phi}_d^*(q)[-\cos\theta_+\gamma_{1,\mathbf{P}/2+\mathbf{q}} + \sin\theta_+\gamma^\dagger_{2,-(\mathbf{P}/2+\mathbf{q})}][\sin\theta_-\gamma^\dagger_{1,-(\mathbf{P}/2-\mathbf{q})} + \cos\theta_-\gamma_{2,\mathbf{P}/2-\mathbf{q}}] \tag{143}$$

At $T=0$,

$$\frac{1}{\sqrt{V}}\sum_{\mathbf{q}} \tilde{\phi}_d^*(q) a_{1,\mathbf{P}/2+\mathbf{q}}a_{2,\mathbf{P}/2-\mathbf{q}}|\mathrm{GND}\rangle \tag{144}$$

$$= \frac{1}{\sqrt{V}}\sum_{\mathbf{q}} \tilde{\phi}_d^*(q)\sin\theta_-[-\cos\theta_+\gamma_{1,\mathbf{P}/2+\mathbf{q}}\gamma^\dagger_{1,-(\mathbf{P}/2-\mathbf{q})} + \sin\theta_+\gamma^\dagger_{2,-(\mathbf{P}/2+\mathbf{q})}\gamma^\dagger_{1,-(\mathbf{P}/2-\mathbf{q})}]|\mathrm{GND}\rangle \tag{145}$$

$$= \frac{1}{\sqrt{V}}\sum_{\mathbf{q}} \tilde{\phi}_d^*(q)\sin\theta_-[-\cos\theta_+\delta_{\mathbf{P},0} + \sin\theta_+\gamma^\dagger_{2,-(\mathbf{P}/2+\mathbf{q})}\gamma^\dagger_{1,-(\mathbf{P}/2-\mathbf{q})}]|\mathrm{GND}\rangle \tag{146}$$

$$\Gamma(\omega) = 2\pi \sum_{f',\mathbf{P}} |\langle f'| \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} \tilde{\phi}_d^*(q) a_{1,\mathbf{P}/2+\mathbf{q}} a_{2,\mathbf{P}/2-\mathbf{q}} |\mathrm{GND}\rangle|^2 \delta(\omega + E_{\mathrm{GND}} - E_{f'} + |E_d| - P^2/4m) \tag{147}$$

$$= 2\pi \sum_{\mathbf{P}} \left| \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} \tilde{\phi}_d^*(q) \sin\theta_- \cos\theta_+ \right|^2 \delta_{\mathbf{P},0} \delta(\omega + |E_d|) \tag{148}$$

$$+ \frac{2\pi}{V} \sum_{\mathbf{P},\mathbf{q}} \left| \tilde{\phi}_d^*(q) \sin\theta_- \sin\theta_+ \right|^2 \delta\left(\omega - E_{\mathbf{P}/2+\mathbf{q}} - E_{\mathbf{P}/2-\mathbf{q}} + |E_d| - P^2/4m\right) \tag{149}$$

$$= \Gamma_1(\omega) + \Gamma_2(\omega) \tag{150}$$

$$\Gamma_1(\omega) = \frac{2\pi}{V} \left| \sum_{\mathbf{q}} \tilde{\phi}_d^*(q) \sin\theta_q \cos\theta_q \right|^2 \delta(\omega + |E_d|) \tag{151}$$

$$\Gamma_2(\omega) = \frac{2\pi}{V} \sum_{\mathbf{P},\mathbf{q}} \left| \tilde{\phi}_d^*(q) \sin\theta_- \sin\theta_+ \right|^2 \delta\left(\omega - E_{\mathbf{P}/2+\mathbf{q}} - E_{\mathbf{P}/2-\mathbf{q}} + |E_d| - P^2/4m\right) \tag{152}$$

The delta peak at $\omega = -|E_d|$ is due to the condensate of the cooper pairs.

Since $k_F \ll \kappa$, we approximate $\tilde{\phi}_d(q) \approx \tilde{\phi}_d(0)$. Therefore

$$\Gamma_2(\omega) = \frac{2\pi|\tilde{\phi}_d(0)|^2}{V} \sum_{\mathbf{k}_1,\mathbf{k}_2} |\sin\theta_{k_1} \sin\theta_{k_2}|^2 \delta\left(\omega - E_{k_1} - E_{k_2} + |E_d| - (\mathbf{k}_1 + \mathbf{k}_2)^2/4m\right) \tag{153}$$

$$= \frac{2\pi|\tilde{\phi}_d(0)|^2}{V} \left[\frac{V}{(2\pi)^3}\right]^2 \int d\mathbf{k}_1 \int_0^\infty dk_2 k_2^2 2\pi \left|\sin\theta_{k_1} \sin\theta_{k_2}\right|^2 \tag{154}$$

$$\times \int_{-1}^{1} d\cos\theta \delta\left(\omega + |E_d| - E_{k_1} - E_{k_2} - k_1^2/4m - k_2^2/4m + k_1 k_2 \cos\theta/2m\right) \tag{155}$$

$$= \frac{2\pi|\tilde{\phi}_d(0)|^2}{V} \left[\frac{V}{(2\pi)^3}\right]^2 (2\pi)^2 m \Delta^4 \int_0^\infty dk_1 \int_0^\infty dk_2 \frac{k_1 k_2}{E_{k_1}(E_{k_1} + \xi_{k_1}) E_{k_2}(E_{k_2} + \xi_{k_2})} \tag{156}$$

$$\times \left[\Theta\left(\omega - E_{k_1} - E_{k_2} + |E_d| - (k_1 - k_2)^2/4m\right) - \Theta\left(\omega - E_{k_1} - E_{k_2} + |E_d| - (k_1 + k_2)^2/4m\right)\right] \tag{157}$$

Note $\Gamma_2(\omega)$ is nonzero only if $\omega > -|E_d| + 2\Delta$.

At finite temperature T, we need to average over the initial states by the weight $\rho_i = e^{-\beta E_i}/Z$ with the partition function $Z = \sum_i e^{-\beta E_i}$ and $H_{\mathrm{BCS}}|i\rangle = E_i|i\rangle$. Thus

$$\Gamma(\omega) = 2\pi \sum_{i,f',\mathbf{P}} \rho_i \left| \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} \tilde{\phi}_d^*(q) \langle f'| a_{1,\mathbf{P}/2+\mathbf{q}} a_{2,\mathbf{P}/2-\mathbf{q}} |i\rangle \right|^2 \delta(\omega + E_i - E_{f'} + |E_d| - P^2/4m) \tag{158}$$

$$
\begin{aligned}
a_{1,\mathbf{P}/2+\mathbf{q}}a_{2,\mathbf{P}/2-\mathbf{q}} =&[-\cos\theta_+\gamma_{1,\mathbf{P}/2+\mathbf{q}}+\sin\theta_+\gamma^\dagger_{2,-(\mathbf{P}/2+\mathbf{q})}][\sin\theta_-\gamma^\dagger_{1,-(\mathbf{P}/2-\mathbf{q})}+\cos\theta_-\gamma_{2,\mathbf{P}/2-\mathbf{q}}] && (159)\\
=&-\cos\theta_+\sin\theta_-\gamma_{1,\mathbf{P}/2+\mathbf{q}}\gamma^\dagger_{1,-(\mathbf{P}/2-\mathbf{q})}+\sin\theta_+\cos\theta_-\gamma^\dagger_{2,-(\mathbf{P}/2+\mathbf{q})}\gamma_{2,\mathbf{P}/2-\mathbf{q}} && (160)\\
&-\cos\theta_+\cos\theta_-\gamma_{1,\mathbf{P}/2+\mathbf{q}}\gamma_{2,\mathbf{P}/2-\mathbf{q}}+\sin\theta_+\sin\theta_-\gamma^\dagger_{2,-(\mathbf{P}/2+\mathbf{q})}\gamma^\dagger_{1,-(\mathbf{P}/2-\mathbf{q})} && (161)\\
=&-\cos\theta_+\sin\theta_-\delta_{\mathbf{P},0}+\cos\theta_+\sin\theta_-\gamma^\dagger_{1,-(\mathbf{P}/2-\mathbf{q})}\gamma_{1,\mathbf{P}/2+\mathbf{q}}+\sin\theta_+\cos\theta_-\gamma^\dagger_{2,-(\mathbf{P}/2+\mathbf{q})}\gamma_{2,\mathbf{P}/2-\mathbf{q}} && (162)\\
&-\cos\theta_+\cos\theta_-\gamma_{1,\mathbf{P}/2+\mathbf{q}}\gamma_{2,\mathbf{P}/2-\mathbf{q}}+\sin\theta_+\sin\theta_-\gamma^\dagger_{2,-(\mathbf{P}/2+\mathbf{q})}\gamma^\dagger_{1,-(\mathbf{P}/2-\mathbf{q})} && (163)\\
=&-\cos\theta_+\sin\theta_-\delta_{\mathbf{P},0}+\delta_{\mathbf{P},0}\cos\theta\sin\theta[\gamma^\dagger_{1,\mathbf{q}}\gamma_{1,\mathbf{q}}+\gamma^\dagger_{2,-\mathbf{q}}\gamma_{2,-\mathbf{q}}] && (164)\\
&+(1-\delta_{\mathbf{P},0})[\cos\theta_+\sin\theta_-\gamma^\dagger_{1,-(\mathbf{P}/2-\mathbf{q})}\gamma_{1,\mathbf{P}/2+\mathbf{q}}+\sin\theta_+\cos\theta_-\gamma^\dagger_{2,-(\mathbf{P}/2+\mathbf{q})}\gamma_{2,\mathbf{P}/2-\mathbf{q}}] && (165)\\
&-\cos\theta_+\cos\theta_-\gamma_{1,\mathbf{P}/2+\mathbf{q}}\gamma_{2,\mathbf{P}/2-\mathbf{q}}+\sin\theta_+\sin\theta_-\gamma^\dagger_{2,-(\mathbf{P}/2+\mathbf{q})}\gamma^\dagger_{1,-(\mathbf{P}/2-\mathbf{q})} && (166)\\
I =&-\cos\theta\sin\theta\delta_{\mathbf{P},0}+\delta_{\mathbf{P},0}\cos\theta\sin\theta[\gamma^\dagger_{1,\mathbf{q}}\gamma_{1,\mathbf{q}}+\gamma^\dagger_{2,-\mathbf{q}}\gamma_{2,-\mathbf{q}}] && (167)\\
II =&(1-\delta_{\mathbf{P},0})\cos\theta_+\sin\theta_-\gamma^\dagger_{1,-(\mathbf{P}/2-\mathbf{q})}\gamma_{1,\mathbf{P}/2+\mathbf{q}} && (168)\\
III =&(1-\delta_{\mathbf{P},0})\sin\theta_+\cos\theta_-\gamma^\dagger_{2,-(\mathbf{P}/2+\mathbf{q})}\gamma_{2,\mathbf{P}/2-\mathbf{q}} && (169)\\
IV =&-\cos\theta_+\cos\theta_-\gamma_{1,\mathbf{P}/2+\mathbf{q}}\gamma_{2,\mathbf{P}/2-\mathbf{q}} && (170)\\
V =&\sin\theta_+\sin\theta_-\gamma^\dagger_{2,-(\mathbf{P}/2+\mathbf{q})}\gamma^\dagger_{1,-(\mathbf{P}/2-\mathbf{q})} && (171)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Gamma(\omega) =&\sum_{\alpha=I}^{V}\Gamma_\alpha(\omega) && (172)\\
\Gamma_I(\omega) =&2\pi\sum_i\rho_i\left|\sum_{\mathbf{q}}\frac{\phi(q)\cos\theta_q\sin\theta_q}{\sqrt{V}}[1-\langle i|\gamma^\dagger_{1,\mathbf{q}}\gamma_{1,\mathbf{q}}|i\rangle+\langle i|\gamma^\dagger_{2,-\mathbf{q}}\gamma_{2,-\mathbf{q}}|i\rangle]\right|^2\delta(\omega+|E_d|) && (173)\\
=&2\pi\left|\sum_{\mathbf{q}}\frac{\phi(q)\cos\theta_q\sin\theta_q}{\sqrt{V}}[1-2f(E_q)]\right|^2\delta(\omega+|E_d|)+2\pi\sum_{\mathbf{q}}2\left|\frac{\phi(q)\cos\theta_q\sin\theta_q}{\sqrt{V}}\right|^2 f(E_q)f(-E_q)\delta(\omega+|E_d|) && (174)\\
\Gamma_{II}(\omega) =&2\pi\sum_{\mathbf{P},\mathbf{q}}(1-\delta_{\mathbf{P},0})\left|\frac{\phi(q)\cos\theta_+\sin\theta_-}{\sqrt{V}}\right|^2 f(-E_{-\mathbf{P}/2+\mathbf{q}})f(E_{\mathbf{P}/2+\mathbf{q}})\delta(\omega+E_{\mathbf{P}/2+\mathbf{q}}-E_{-\mathbf{P}/2+\mathbf{q}}+|E_d|-P^2/4m) && (175)\\
=&2\pi\sum_{\mathbf{P},\mathbf{q}}\left|\frac{\phi(q)\cos\theta_+\sin\theta_-}{\sqrt{V}}\right|^2 f(-E_{-\mathbf{P}/2+\mathbf{q}})f(E_{\mathbf{P}/2+\mathbf{q}})\delta(\omega+E_{\mathbf{P}/2+\mathbf{q}}-E_{-\mathbf{P}/2+\mathbf{q}}+|E_d|-P^2/4m) && (176)\\
&-2\pi\sum_{\mathbf{q}}\left|\frac{\phi(q)\cos\theta_+\sin\theta_-}{\sqrt{V}}\right|^2 f(-E_{\mathbf{q}})f(E_{\mathbf{q}})\delta(\omega+|E_d|) && (177)\\
\Gamma_{III}(\omega) =&\Gamma_{II}(\omega) && (178)\\
\Gamma_{IV}(\omega) =&2\pi\sum_{\mathbf{P},\mathbf{q}}\left|\frac{\phi(q)\cos\theta_+\cos\theta_-}{\sqrt{V}}\right|^2 f(E_{\mathbf{P}/2+\mathbf{q}})f(E_{\mathbf{P}/2-\mathbf{q}})\delta(\omega+E_{\mathbf{P}/2+\mathbf{q}}+E_{\mathbf{P}/2-\mathbf{q}}+|E_d|-P^2/4m) && (179)\\
\Gamma_V(\omega) =&2\pi\sum_{\mathbf{P},\mathbf{q}}\left|\frac{\phi(q)\sin\theta_+\sin\theta_-}{\sqrt{V}}\right|^2 f(-E_{\mathbf{P}/2+\mathbf{q}})f(-E_{\mathbf{P}/2-\mathbf{q}})\delta(\omega-E_{\mathbf{P}/2+\mathbf{q}}-E_{\mathbf{P}/2-\mathbf{q}}+|E_d|-P^2/4m) && (180)
\end{aligned}
$$

According to the extra number of quasi-particles in $|f'\rangle$ compared with $|i\rangle$, the constituents of $\Gamma(\omega)$ are

$$\Gamma_0(\omega) = 2\pi \left| \sum_{\mathbf{q}} \frac{\phi(q)\cos\theta_q \sin\theta_q}{\sqrt{V}} [1 - 2f(E_q)] \right|^2 \delta(\omega + |E_d|) \tag{181}$$

$$\Gamma_{+-}(\omega) = 2\pi \times 2 \sum_{\mathbf{P},\mathbf{q}} \left| \frac{\phi(q)\cos\theta_+ \sin\theta_-}{\sqrt{V}} \right|^2 f(-E_{-\mathbf{P}/2+\mathbf{q}}) f(E_{\mathbf{P}/2+\mathbf{q}}) \delta(\omega + E_{\mathbf{P}/2+\mathbf{q}} - E_{-\mathbf{P}/2+\mathbf{q}} + |E_d| - P^2/4m) \tag{182}$$

$$\Gamma_{--}(\omega) = \Gamma_{IV} \tag{183}$$

$$\Gamma_{++}(\omega) = \Gamma_{V} \tag{184}$$