Let us consider the initial and the final state subject to spherical square well potentials $V_\sigma(r) = -V_\sigma$ for $r < r_0$ and $V_\sigma(r) = 0$ otherwise with $\sigma = i, f$.

For a continuum state with energy $E_\sigma = k_\sigma^2/m$, the relative wave-function is

$$\phi_{k_\sigma,>}(\mathbf{r}) = \mathcal{N}_\sigma \frac{\sin(k_\sigma r + \delta_\sigma)}{r}, \text{ for } r > r_0; \tag{1}$$

$$\phi_{k_\sigma,<}(\mathbf{r}) = \mathcal{N}'_\sigma \frac{\sin(\sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2}\tilde{r})}{r}, \text{ for } r < r_0, \tag{2}$$

with $\eta_\sigma^2 \equiv mV_\sigma r_0^2$, $\tilde{r} \equiv r/r_0$ and $\tilde{k}_\sigma \equiv k_\sigma r_0$. The continuity of $\phi(r)$ and $d\phi(r)/dr$ at $r = r_0$ gives

$$\mathcal{N}'_\sigma = \mathcal{N}_\sigma \frac{\sin(\tilde{k}_\sigma + \delta_\sigma)}{\sin(\sqrt{\eta_\sigma^2 + \tilde{k}^2})}. \tag{3}$$

$$\sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2} \cot \sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2} = \tilde{k}_\sigma \cot(\tilde{k}_\sigma + \delta_\sigma). \tag{4}$$

The normalization of the wave-function in a sphere of radius ℓ with a hard wall ($k_\sigma \ell + \delta_\sigma = n_\sigma \pi$) gives

$$\mathcal{N}_\sigma^2 = \frac{1}{2\pi r_0} \left\{ \ell/r_0 - 1 + \frac{\sin(2\tilde{k}_\sigma + \delta_\sigma)}{2\tilde{k}_\sigma} + \frac{\sin^2(\tilde{k}_\sigma + \delta_\sigma)}{\sin^2 \sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2}} \left[1 - \frac{\sin(2\sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2})}{2\sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2}} \right] \right\}^{-1}. \tag{5}$$

If we fix k_σ and take $\ell \to \infty$ ($n_\sigma \to \infty$), we have

$$\mathcal{N}_\sigma^2 \approx \frac{1}{2\pi\ell} = \frac{1}{4\pi} \frac{2}{\ell}. \tag{6}$$

I. EFFECTIVE RANGE EXPANSION

By Eq. (4), we have

$$\cot \delta_\sigma = -\frac{\sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2} + \tilde{k}_\sigma \tan \tilde{k}_\sigma \tan \sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2}}{\sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2} \tan \tilde{k}_\sigma - \tilde{k}_\sigma \tan \sqrt{\eta_\sigma^2 + \tilde{k}_\sigma^2}}. \tag{7}$$

Thus when expanding to the lowest orders of $\tilde{k}_\sigma$ and matching with $\cot \delta = -1/ka + r_e k/2$, we find

$$\tilde{a}_\sigma = 1 - \frac{\tan \eta_\sigma}{\eta_\sigma}, \tag{8}$$

$$\tilde{r}_{e,\sigma} = 1 - \frac{1}{3(1 - \frac{\tan \eta_\sigma}{\eta_\sigma})^2} - \frac{1}{\eta_\sigma^2 (1 - \frac{\tan \eta_\sigma}{\eta_\sigma})}, \tag{9}$$

with $\tilde{a}_\sigma \equiv a_\sigma/r_0$ and $\tilde{r}_{e,\sigma} \equiv r_{e,\sigma}/r_0$.

A. 1-2 channel

For the 1-2 state, since $1/a_{12} = 0$, $\eta_{12} = (n_{12} + 1/2)\pi$ and $r_{e,12} = r_0$.

B. 1-3 channel

On the other hand, given $r_{e,13}/a_{13} \approx 0.47$, we find $1 - \frac{\tan \eta_{13}}{\eta_{13}} \approx 1.94$ (assuming $\eta \gg 1$), consequently $a_{13} = 1.94 r_0$ and $r_{e,13} = 0.91 r_0$.

C. Median dimer state

Considering the median dimer state in the final 1-3 channel, for $r > r_0$, the wave-function is $e^{-\kappa r}/r$ and the continuity at $r = r_0$ gives [instead of Eq. (4)]

$$-\tilde{\kappa} = \sqrt{\eta_f^2 - \tilde{\kappa}^2} \cot \sqrt{\eta_f^2 - \tilde{\kappa}^2}. \tag{10}$$

Assuming $\tilde{\kappa} \sim 1$ and $\eta_f \gg 1$ (the atomic potential depth is much larger than 4 MHz), the following solution is quite accurate

$$\eta_f = \eta_{f,0} + \frac{\tilde{\kappa}}{\eta_{f,0}} + \left(\frac{1}{2\eta_{f,0}} - \frac{1}{\eta_{f,0}^3} \right) \tilde{\kappa}^2, \tag{11}$$

$$\eta_{f,0} = \left(n_f + \frac{1}{2} \right) \pi, \; n_f \gg 1 \tag{12}$$

when compared with the numerical solution.

II. RF SPECTRUM

The overlap between the initial and final wave-functions ($k_f \neq k_i$) is given by

$$\int d\mathbf{r} \phi_{k_i}(\mathbf{r}) \phi_{k_f}(\mathbf{r}) = 4\pi r_0 \mathcal{N}_i \mathcal{N}_f \sin(\tilde{k}_i + \delta_i) \sin(\tilde{k}_f + \delta_f) \left[\frac{\sqrt{\eta_f^2 + \tilde{k}_f^2} \cot \sqrt{\eta_f^2 + \tilde{k}_f^2} - \sqrt{\eta_i^2 + \tilde{k}_i^2} \cot \sqrt{\eta_i^2 + \tilde{k}_i^2}}{\eta_i^2 + \tilde{k}_i^2 - \eta_f^2 - \tilde{k}_f^2} \right. \tag{13}$$

$$\left. - \frac{\tilde{k}_f \cot(\tilde{k}_f + \delta_f) - \tilde{k}_i \cot(\tilde{k}_i + \delta_i)}{\tilde{k}_i^2 - \tilde{k}_f^2} \right] \tag{14}$$

$$= 4\pi r_0 \mathcal{N}_i \mathcal{N}_f \sin(\tilde{k}_i + \delta_i) \sin(\tilde{k}_f + \delta_f) \left[\frac{1}{\eta_i^2 + \tilde{k}_i^2 - \eta_f^2 - \tilde{k}_f^2} - \frac{1}{\tilde{k}_i^2 - \tilde{k}_f^2} \right] \times [\tilde{k}_f \cot(\tilde{k}_f + \delta_f) - \tilde{k}_i \cot(\tilde{k}_i + \delta_i)]. \tag{15}$$

For the initial state, we can safely take the limit $\tilde{k}_i \to 0$, and find to lowest order

$$\mathcal{N}_i \sin(\tilde{k}_i + \delta_i) \to \mathcal{N}_i \sin(\delta_i) = \sqrt{\mathcal{C}_i} \tag{16}$$

$$\tilde{k}_i \cot(\tilde{k}_i + \delta_i) \to \tilde{k}_i \cot(\delta_i) = 0 \tag{17}$$

considering $\lim_{\tilde{k}_i \to 0} \cot \delta_i = 0$ (on resonance). Thus we have the simplified form

$$\int d\mathbf{r} \phi_{k_i}(\mathbf{r}) \phi_{k_f}(\mathbf{r}) = 4\pi r_0 \sqrt{\mathcal{C}_i} \mathcal{N}_f \frac{\eta_i^2 - \eta_f^2}{(\eta_i^2 - \eta_f^2 - \tilde{k}_f^2) \tilde{k}_f} \cos(\tilde{k}_f + \delta_f). \tag{18}$$

One crucial point is to notice that the behavior of $\cos(\tilde{k}_f + \delta_f)$ depends sensitively on whether the bound state is shallow or not.

A. Shallow bound state

If the 13 bound state is shallow, it is a good approximation

$$\cos^2(\tilde{k}_f + \delta_f) \approx \frac{1}{1 + (\tilde{k}_f/\tilde{\kappa})^2}. \tag{19}$$

Figure 1 shows $\cos^2(\tilde{k}_f + \delta_f)$ if $\tilde{\kappa} = 0.01$; $\cos^2(\tilde{k}_f + \delta_f)$ decreases to a very small number before bouncing up. Figure 1 shows $\cos^2(\tilde{k}_f + \delta_f)$ (blue) is well approximated by $1/[1 + (\tilde{k}_f/\tilde{\kappa})^2]$ (orange) in a substantial range; if one goes to smaller $\tilde{\kappa}$, the difference between the blue and orange will be smaller. Since we know at the unitary point, i.e., $\tilde{\kappa} \to 0$, $\lim_{k_f \to 0} \cos^2(\tilde{k}_f + \delta_f) = \cos^2(\pi/2) = 0$, the minimum value of $\cos^2(\tilde{k}_f + \delta_f)$ decreases towards zero as κ does; $1/1 + (\tilde{k}_f/\tilde{\kappa})^2$ is a good fit to the decreasing part of $\cos^2(\tilde{k}_f + \delta_f)$.

Thus for Eq. (18), in the range $\tilde{\kappa}^2 \ll \tilde{k}_f^2 \ll |\eta_f^2 - \eta_i^2|$, the overlap scales $\sim 1/k_f^2$, which would give rise to $\omega^{-5/2}$ in the rf tail.

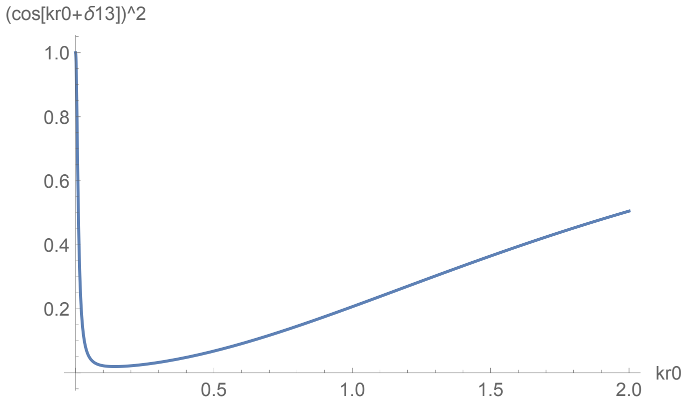

FIG. 1:

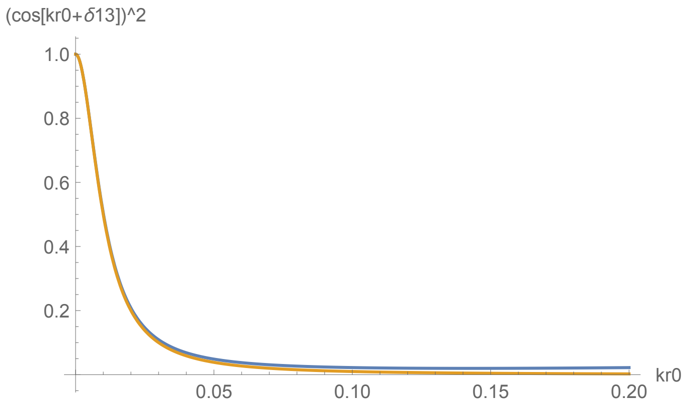

FIG. 2:

B. Median bound state

However, in our case, the bound state is median, and $\tilde{\kappa} \approx 0.76$. Figure 3 shows the minimum value of $\cos^2(\tilde{k}_f + \delta_f)$ (blue) with $\tilde{\kappa} \approx 0.76$ is not close to zero at all! The orange curve is of the form $1/(1+\tilde{k}_f^2/F^2)$ with $F = 1.05$ (differing about 50% from 0.76).

By Eq. (18), to make the f-sum rule convergent, it seems one needs to go into the regime $\tilde{k}_f^2 \gg |\eta_f^2 - \eta_i^2|$ which can be very large. Unless fine tuned such that $|\eta_f^2 - \eta_i^2| \sim \tilde{\kappa}$. This fine tuning may not be that far from the true situation; 13 and 12 are both interacting primarily through the spin triplet potential. So if we assume $\eta_f = \eta_i + \frac{\tilde{\kappa}}{\eta_i} + \left(\frac{1}{2\eta_i} - \frac{1}{\eta_i^3}\right)\tilde{\kappa}^2$ with $\eta_i = (n_i + 1/2)\pi$ (on resonance), we have

$$\eta_f^2 - \eta_i^2 = 2\tilde{\kappa} + (1 - 1/\eta_i^2)\tilde{\kappa}^2 + O(\tilde{\kappa}^3) \tag{20}$$

$$\approx 2\tilde{\kappa} + \tilde{\kappa}^2 \tag{21}$$

In Fig. (4), we log-log plot the quantity (the blue curve)

$$\left[\frac{\eta_i^2 - \eta_f^2}{(\eta_i^2 - \eta_f^2 - \tilde{k}_f^2)\tilde{k}_f}\cos(\tilde{k}_f + \delta_f)\right]^2 = \left[\frac{2\tilde{\kappa} + \tilde{\kappa}^2}{(2\tilde{\kappa} + \tilde{\kappa}^2 + \tilde{k}_f^2)\tilde{k}_f}\cos(\tilde{k}_f + \delta_f)\right]^2. \tag{22}$$

The orange line is $10/\tilde{k}_f^2$, the red line is $1/k^4$, and the green line is $100/\tilde{k}_f^6$. The bend-over of the blue curve from $\sim 1/k^2$ to $\sim 1/k^6$ occurs at about $\tilde{k}_f = 2$. Figure (5) shows up to $\tilde{k}_f = 2$; the bend-over of the blue curve from $\sim 1/k^2$ to $\sim 1/k^4$ occurs at about $\tilde{k}_f = 0.7$ ($\sim \tilde{\kappa}$)

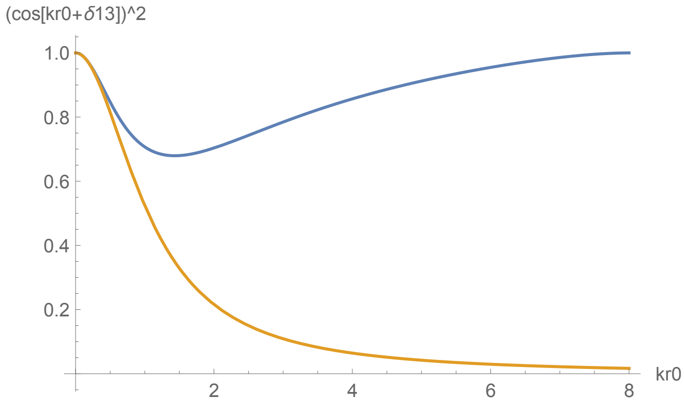

FIG. 3:

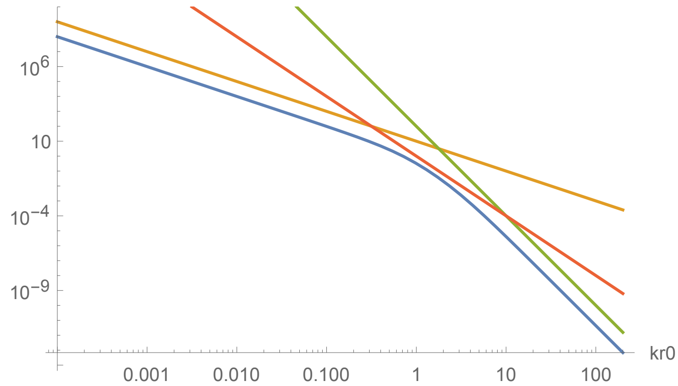

FIG. 4:

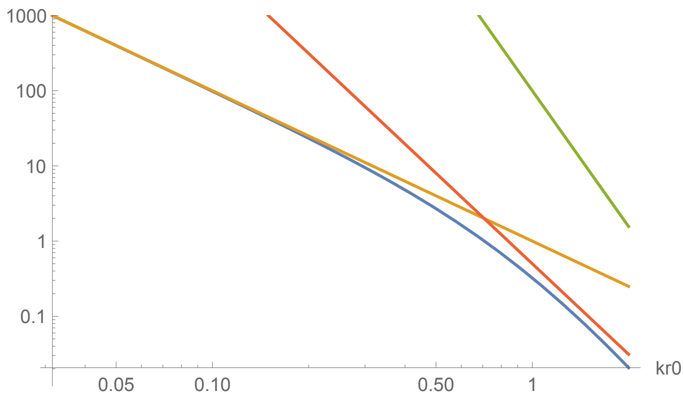

FIG. 5: