WORKING PAPER 353

EQUILIBRIUM CONDITIONS AND SOLUTION PROCEDURES FOR THE PRODUCTION CONSTRAINED SPATIAL INTERACTION MODEL WITH A GENERAL ATTRACTIVENESS FUNCTION

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EQUILIBRIUM CONDITIONS AND SOLUTION PROCEDURES FOR THE PRODUCTION CONSTRAINED SPATIAL INTERACTION MODEL WITH A GENERAL ATTRACTIVENESS FUNCTION

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Abstract

The consequences of imposing the revenue capacity condition (profit maximization) on the production constrained spatial interaction model with a general attractiveness function are presented. Convergence of the quasi-balancing factor method is shown to be an insufficient condition for providing an equilibrium solution, three different sets of results are obtained for an example retailing system. Two of these results are shown to be non-optimal.

March 1983

1. Introduction

The production-constrained spatial-interaction model was first presented by Huff (1965) and Lakshmanan and Hansen (1965), the statistical theoretical base followed a little later, (Murchland (1966), Wilson (1967)). Since this work the production-constrained spatial-interaction model has been widely used to predict both the flows between locations (consumer behaviour, S_{ij}) and the equilibrium configuration of facility sizes (producer behaviour, W_j), e.g. Harris and Wilson (1978), Clarke (1981). The discussion in this paper will be cast in terms of shopping activities for definiteness but the arguments are quite general.

Harris and Wilson (1978) write the production-constrained spatial-interaction model in the form

$$\begin{aligned} S_{ij} &= A_i . O_i . g(W_j) \ . \ f(C_{ij}) \\ A_i &= \{\sum_k g(W_k) \ . \ f(C_{ik})\}^{-1} \end{aligned} \qquad (1)$$

where

S_{ij}	is the flow of retailing activity measured as a cash flow, say, from residences in i to shops in j
O_i	is the retail demand in i
$g(W_j)$	is the attractiveness of zone j
W_j	is the capacity (say floorspace) at j
C_{ij}	is the travel cost from i to j

The function $g(\cdot)$ converts capacity into attractiveness and $f(\cdot)$ converts travel cost into a representation of impedance.

Harris (1964) also suggested that producers could behave so as to ensure that the capacity provided (W_j) balances the revenue generated ($\sum_i S_{ij}$). This mechanism was interpreted as profit maximization by Harris and Wilson (1978), and can be written

$$\sum_i S_{ij} = k_j W_j \qquad (2)$$

where k_j is a parameter for zone j which converts facility size units into revenue units.

Various functional forms for g $(\cdot)$ have been discussed in the literature, for instance Coelho and Wilson (1976), Harris and Wilson (1978), use the power function

$$g(W_j) \quad = \quad W_j^{\alpha} \tag{3}$$

Poston and Wilson (1977) in an article on catastrophe theory suggest an alternative which would produce a new kind of jump behaviour,

$$g(W_j) \quad = \quad \exp\ (S/(1+ Q.\exp(-\epsilon W_j - r))) \tag{4}$$

However two relatively undocumented outcomes of choosing a particular $g(\cdot)$ are: (1) that the form of S_{ij} may change as a consequence of the capacity revenue condition, this is discussed in section 2 below; and (2) that the log-entropy criterion from which the optimal S_{ij} and W_j are derived may no longer be strictly concave. Convergence of a quasi balancing factor method, such as those reviewed by Phiri (1979) is not sufficient. The solutions obtained can be non-unique and non-optimal. A means of establishing the nature of these solutions is presented in section 3, together with an example.

2. Determining the form of S_{ij}

In order to derive the form of S_{ij} to be used in computingthe equilibrium configuration of W_j, we need to maximize the following objective function,

$$\underset{\{S_{ij},W_j\}}{\text{Max}}\ F \quad = \quad -\sum_{ij} S_{ij} \log\ \{S_{ij}/g(W_j) - 1\} \tag{5}$$

(where F is stirling's approximation to the log-entropy criterion with prior weights $g(W_j)$).

Subject to the conventional production constraint

$$\sum_j S_{ij} = O_i, \tag{6}$$

the travel cost constraint

$$C = \sum_{ij} S_{ij} C_{ij}, \tag{7}$$

and the additional revenue-capacity condition

$$K_j W_j = \sum_i S_{ij} \tag{8}$$

We can add these constraints to the objective function and form the Lagrangian Z,

$$\underset{\{S_{ij} W_j\}}{\text{Max}}\ Z = -\sum_{ij} S_{ij} \log\ (S_{ij}/g(W_j) - 1\} - \beta\{\sum_{ij} S_{ij} C_{ij} - C\}$$

$$+ \sum_i \lambda_i\ (O_i - \sum_j S_{ij}) + \sum_j \gamma_j\ (K_j W_j - \sum_i S_{ij}) \tag{9}$$

where β, λ_i and γ_j are the Lagrange multipliers associated with the travel cost, production and revenue-capacity constraints respectively.

The first order uptimality conditions assuming we know β are,

$$\frac{\partial Z}{\partial S_{ij}} = -\log \frac{S_{ij}}{g(W_j)} - \lambda_i - \gamma_j - \beta C_{ij} = 0$$

this gives

$$S_{ij} = g(W_j)\ .\exp(-\lambda_i - \gamma_j - \beta . C_{ij}) \tag{10}$$

and

$$\frac{\partial Z}{\partial W_j} = -\sum_i S_{ij} \frac{\partial \log\ (g(W_j))}{\partial W_j} + K_j \gamma_j = 0 \tag{11}$$

$$\frac{\partial Z}{\partial \lambda_i} = O_i - \sum_j S_{ij} = 0 \tag{12}$$

$$\frac{\partial Z}{\partial \gamma_j} = K_j W_j - \sum_i S_{ij} = 0 \tag{13}$$

where $S_{ij}, W_j > 0$

λ_i is found from (10) and (12),

$$\exp(-\lambda_i) = \frac{O_i}{\sum_j g(W_j) \exp(-\gamma_j - \beta C_{ij})} \tag{14}$$

as $\sum_i S_{ij} = K_j W_j$ equation (13)

we can rewrite (11) as

$$-K_j W_j \frac{\partial \log(g(W_j))}{\partial W_j} + K_j \gamma_j = 0$$

or

$$\gamma_j = W_j \frac{\partial \log(g(W_j))}{\partial W_j} \tag{15}$$

This result is generally ignored and it is assumed that equation (1) holds for all $g(\cdot)$. Equation (1) is only valid when $g(\cdot)$ belongs to that class of functions for which γ_j is a constant. When $g(W_j) = W_j^{\alpha}$

this is the case as

$$\gamma_j = W_j \frac{\alpha}{W_j} = \alpha \tag{16}$$

As a consequence of this the Lagrangian Z can be rewritten

$$\max_{\{S_{ij},W_j\}} Z = -\sum_{ij} S_{ij} \log(S_{ij}/W_j^{\alpha} - 1) - \beta \sum_{ij} {}_{ij}C_{ij}$$

$$+ \sum_i \lambda_i (O_i - \sum_j S_{ij}) + \gamma (\sum_j K_j W_j - \sum_i O_i) \tag{17}$$

with $\gamma = \alpha$, and $S_{ij} = \dfrac{O_i W_j^{\alpha} \exp(-\beta . C_{ij})}{\sum_j W_j^{\alpha} \exp(-\beta . C_{ij})}$ (18)

at the solution.

However when $g(W_j) = \exp(\delta/(1 + Q \exp(-\varepsilon W_j - r)))$

$$\gamma_j = W_j \frac{\exp(r + W_j \varepsilon) . \delta Q \varepsilon}{(Q \exp(r) + \exp(W_j \varepsilon))^2} \tag{19}$$

and instead we have a quasi doubly-constrained spatial-interaction model,

where $S_{ij} = \dfrac{O_i\ g(W_j) \exp(-\gamma_j - \beta C_{ij})}{\sum_j g(W_j) \exp(-\gamma_j - \beta C_{ij})}$ (20)

3. Solution procedures and the second order optimality conditions

The revenue-capacity condition, equation (13), can provide the basis of a solution procedure. If we substitute for S_{ij}, λ_i and γ_j from equations (10), (14) and (15) we are left with n non-linear simultaneous equations in n unknowns (W_j), viz.

$$K_j W_j - \sum_i S_{ij} = 0 \qquad j = 1, \ldots, n \tag{21}$$

This set of simultaneous equations can be solved by using a class of iterative procedures of the form

$$\underline{W}^{\phi+1} = G(\underline{W}^{\phi}) \tag{22}$$

One widely used scheme is the quasi-balancing factor method, see for instance Eilon et al (1969), Leonardi (1977), Harris and Wilson (1978), Phiri (1979) and Clarke (1981).

This scheme follows the form

(1) Set all the $\underline{W}^{\phi}$ equal to some non-zero initial value

(2) Solve $\sum_i S_{ij}$

(3) If $K_j W_j^{\phi} = \sum_i S_{ij}$ for all j we have

a possible solution, if not set $W_j^{\phi} = \sum_i S_{ij}/K_j$

increment ϕ and return to step 1. Other similar schemes can be derived (Ortega and Rheinbolt (1970)), for instance

$$\underline{W}^{\phi+1} = \underline{W}^{\phi} - F(\underline{W}^{\phi}). \tag{23}$$

one such scheme has F as a combination of the Newton and scaled gradient procedure, it is commercially available as routine CØ5NAF, NAG (1982).

It is intuitively obvious that in this optimization problem we shall want to determine the global maxima, otherwise the behavioural and statistical base of our model will be violated. A global maxima implies that the objective function takes on its highest value at that point no matter where else we may search. A local maxima on the other hand, only guarantees that the objective function is a maximum with respect to other points nearby.

For convex functions a local maximum is also a global maximum and we therefore need find only one maximum to obtain the solution. With non-convex functions we may have more than one maximum. This will require determining all the maxima, then evaluating the objective function at each local maximum and choosing the largest as our global solution. What is more, if we should fail to determine all the local maxima the global maximum could be missed.

Convergence of the iterative schemes above is however not a sufficient condition for ever obtaining a local maximum. As these schemes only use the first order optimality conditions they will yield a unique global maximum when Z is strictly concave, Coelho and Wilson (1976) showed that Z was strictly concave for $g(\cdot) = W_j^{\alpha}$, $\alpha < 1$. When Z is not strictly concave different starting values can yield different solutions by these methods, which may be local maxima, minima or saddle points. In order to distinguish the local maxima from the other types of solution we require use of the second order optimality conditions.

The second order optimality conditions can be written in many ways, e.g. Himmelblau (1972), Fiacco and McCormick (1968). One compact representation which can be easily translated into computer code follows Lunenberger (1973). For this we need to introduce the tangent subspace M over the N unknowns $\{S_{ij}, W_j\}$.

i.e. $$M = \{\underline{y} : \nabla h(\underline{x})\, \underline{y} = 0\} \qquad (24)$$

where $\nabla h(\underline{x})$ is the vector of N first order derivatives of the m binding constraints at the candidate solution $\underline{x}$. In our application this will include those S_{ij} and W_j which are on their lower feasible bounds and the two sets of linear constraints given by equations (12) and (13).

We also require the Hessian $L(\underline{x})$ of the Lagrangian Z at the candidate solution. It can be shown that for $\underline{x}$ to be a unique local maximum $L(\underline{x})$ must be negative definite on M, Lunenberger (1973).

As the constraints are linear we obtain the elements of $L(\underline{x})$ from

$$\frac{\partial^2 Z}{\partial S_{ij}\partial S_{kl}} = \begin{cases} \dfrac{-1}{S_{ij}} & k,l = i,j \\ 0 & k,l \neq i,j \end{cases} \qquad (25)$$

$$\frac{\partial^2 Z}{\partial S_{ij}\partial W_k} = \begin{cases} \dfrac{\partial g(W_j)/dW_j}{gW_j} & k = j \\ 0 & k \neq j \end{cases}$$

$$\frac{\partial^2 Z}{\partial W_j\partial W_k} = \begin{cases} \sum_i S_{ij} \dfrac{\partial^2 \log(g(W_j))}{\partial W_j^2} & k = j \\ 0 & k \neq j \end{cases}$$

$L(\underline{x})$ can be written

$$L(\underline{x}) = \left[\begin{array}{c|c} \dfrac{\partial^2 Z}{\partial S_{ij}\partial S_{kl}} & \dfrac{\partial^2 Z}{\partial S_{ij}\partial W_k} \\ \hline \dfrac{\partial^2 Z}{\partial W_k \partial S_{ij}} & \dfrac{\partial^2 Z}{\partial W_j \partial W_k} \end{array}\right] \tag{26}$$

The matrix $L(\underline{x})$ restricted to the subspace M plays a role analogous to that of the Hessian of the objective function in the unconstrained case. We denote the restriction $L(\underline{x})$ to M as L_M, and in order to determine its representation we introduce an orthonormal basis $e_1, e_2, \ldots, e_{n-m}$ on M and define the matrix E as the n x (n-m) matrix whose column consist of the vectors e_i. Correspondingly the (n-m) x (n-m) matrix E'LE is the matrix representation of L restricted to M. Standard tests for establishing the nature of L_M, such as "slicing the spectrum" are available, Parlett (1980).

By way of example, the quasi-balancing factor method with 3 different starting values gave 3 different sets of results for the spatial demand pattern in figure (1), with $g(\cdot) = W_j^{\alpha}$, $\alpha = 1.5$, $\beta = 0.5$. The results obtained are also shown in figure (1). It must be noted that other results may be possible, as the example is illustrative as exhaustive search of other solutions was not undertaken.

The term C_{ij} in the model representing transport costs between zones i and j is assumed to be the Euclidean distance metric given by

$$C_{ij} = \left[(x_i - x_j)^2 + (y_i - y_j)^2\right]^{\frac{1}{2}} \quad , \quad i \neq j \tag{27}$$

Where the centroids of zones i and j are located at $(x_i\ y_i)$ and $(x_j\ y_j)$ respectively C_{ii} was taken to be 0.5.

Tests on L_M showed answers 2 and 3 not to be local maxima. Answer (1) is the situation where n-m = 0, (considering those S_{ij} and W_j on their lower bounds as binding constraints) here the constraints intersect to give a unique solution. Examination of the function value around the solution showed it to be a local maximum.

Answers (2) or (3) (or answer (1) if it is not the global solution) should not be used for inference on the equilibrium configuration of retail facilities as they violate the behavioural and statistical theory of our model.

In concluding this paper it must also be noted that commercially available routines already exist for solving non-linear objective functions subject to linear and non-linear constraints, NAG (1982), Himmelblau (1972). These routines were centred around one or more of these three basic concepts:

(1) Extension of linear methodology to non-linear programming problems by means of repeated linear approximations;

(2) Transformation of the constrained non-linear problem into a sequence of unconstrained problems though the use of penalty functions.

(3) Use of flexible tollerances to accommodate both feasible and non-feasible solution vectors.

The non-concave objective function is only problematic in as much as different starting values may give different local maxima. With these algorithms there would be no need to distinguish local maxima from other types of solution.

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Demand					(1)					(2)					(3)				
θ	.1	.1	.1	θ	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.1	.1	.1	.1	.1	.	.	.	.	.	.	.	.525	.	.	.	.	.428	.	.
.1	.1	.1	.1	.1	.	.	2.1	.	.	.	.525	.	.525	.	.	.428	.39	.428	.
.1	.1	.1	.1	.1	.	.	.	.	.	.	.	.525	.	.	.	.	.428	.	.
θ	.1	.1	.1	θ	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

θ denotes exogeneous zone

Figure (1). Demand and alternative results for a 5 x 5 grid spatial system