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Mathematical Programming Approaches to Activity Location - Some theoretical considerations.

H.C.W.L. Williams.

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School of Geography,
University of Leeds,
Leeds LS2 9JT,
England.

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1. INTRODUCTION.

In the applicaiton of mathematical programming to land use/activity planning, as in any other model based approach to problem solving, there are inevitably important trade-offs to be made between the 'realism' of the representation and the tractability of the resulting model. What concessions are made to the latter will ultimately determine the usefulness of the exercise. As in many analytical approaches progress is made through the successive relaxation of some of the more stringent restrictions of a theory or model. In programming applications, the progression from linear models to non-linear formulations is an example of this tradition of successive refinement and it is an aspect I shall concentrate on below. It is worthwhile to stress right at the start that the incorporation of non-linearities is seldom made at zero cost. The solution procedures for non-linear models are considerably more involved than linear counterparts, and this invariably demands that the problems considered must be restricted in size. This is an important example of the trade-off cited above.

Under the banner of constrained optimisation techniques may be found a wide array of special methods, including : linear; non-linear; integer; geometric; multicriterion; and dynamic programming and a host of heuristic approaches taylored to particular problems. In this note I shall concentrate on linear and non-linear models which include a considerable number of applications. The coverage here is by no means exhaustive.

In Section 2, some problem contexts will be reviewed before a number of theoretical aspects are outlined in Section 3. The latter includes a consideration of <u>dispersion</u> in models; underpinning models with a behavioural basis; and the notion of embedding one program within another to simplify analysis. Two examples which embody these aspects are then given in Section 4.

The design of an activity (land use) system is seldom, if ever, made with disregard for the development of the transportation system. Two features of this complex joint design problem are confronted in Section 5.

An area of work which has enjoyed an increasing degree of popularity in recent years is that of vector maximization or multicriterion programming. Some general comments on this approach are given in Section 6.

A few words on solution methods for non-linear optimisation problems are reserved for Section 7. In appendix 1, some brief notes are included on the solution characteristics (the Kuhn-Tucker conditions) for non-linear programs together with a few statements on non-linear duality which is now much exploited. An example of dual formulation is given in Appendix 2, in which the dual program of that which generates the constrained gravity model is given.

2. PROBLEM CONTEXTS AND FORMULATIONS.

There have now been a considerable number of applications of linear programming models of activity/land use, applied at the urban and regional level. Essentially, these involved the determination of activity levels in spatial units, to maximize (or minimize) some benefit (or cost) function subject to linear demand and/or supply constraints associated with the activity, and linear land use constraints on zonal holding capacities.

It is interesting and important to distinguish the nature of the mechanism and parameters under control in any particular context. In many of the applications the local authority is considered to actually control the <u>level</u> of an activity in a zone, in others - for example, in the Herbert-Stevens type of approach - activity levels (in this case for residential purposes) are the result of decentralized decision making and are subject to bounds on development. Models differ rather significantly in how the supply side is handled and the extent to which it is controlled or controllable by a central agency. We shall elaborate on this below.

TOPAZ - an acronym for Technique for the Optimal Placement of Activities in Zones, developed by Brotchie, was one of the first of so-called second generation land use models. It has a number of characteristics common to conventional linear models together with special features associated with non-linear interactions. The objective function is of the following form :

$$\max_{\underline{x}} \; B_1(\underline{x}) + B_2(\underline{x})$$

and the optimization is subject to zonal capacity, non-negativity and other constraints. The model may be used in a staged development mode.

The term $B_1(\underline{x})$ is the <u>net</u> benefit associated with establishing and operating an activity in a given zone, while $B_2(\underline{x})$ accounts for the interaction between activities in different zones. $B_1(\underline{x})$ is usually specified as a <u>linear</u> function of the level of activity and is thus similar to conventional linear programming approaches. $B_2(\underline{x})$, however, contains non-linearities of at least the second power. In the case of the interaction of residential and industrial and commercial developments, a gravity model is employed which introduces <u>severe non-linearities</u>. As the constraints are linear, iterative linear programming is used to search for a solution.

The approach, which is a generalisation of the Koopmans-Beckmann problem, has been applied many times in Australia, the United States and the Middle East.

It is at this point that a slight detour is desirable in order to introduce a number of theoretical issues which are relevant, both for the interpretation of, and development of, an appropriate objective function and the simplifications which the operation of embedding can bring to the problem.

3. SOME THEORETICAL ISSUES.

There are three particular issues which are discussed before returning to the TOPAZ framework. The first is the incorporation of dispersion into the model, and by dispersion we mean the variability in behaviour as portrayed, for example in spatial interaction models, by the finite β relation

$$T_{ij} = A_i O_i B_j D_j e^{-\beta c_{ij}} \tag{3.1}$$

As $\beta \to \infty$ it is known that $T_{ij}(\beta)$ approaches the solution of the linear programming model

$$\min_{\underline{T}} \sum_{ij} T_{ij} c_{ij} \tag{3.2}$$

subject to

$$\sum_j T_{ij} = O_i \tag{3.3}$$

$$\sum_i T_{ij} = D_j \tag{3.4}$$

$$\underline{T} \geqslant 0 \tag{3.5}$$

If now we are given just two activities associated with the two ends of the journey to work, a simple TOPAZ representation would be

$$\min_{\{\underline{O},\underline{D}\}} \underbrace{\sum_i \sigma_i O_i + \sum_j \eta_j D_j}_{\text{set up cost}} + \underbrace{\sum_{ij} A_i O_i B_j D_j e^{-\beta c_{ij}} \; c_{ij}}_{\text{interaction cost}} \tag{3.6}$$

subject to zone capacity constraints, in which A_i and B_j were determined in the usual way

$$A_i(\underline{O},\underline{D}) = \{\sum_j B_j D_j e^{-\beta c_{ij}}\}^{-1} \tag{3.7}$$

$$B_j(\underline{O},\underline{D}) = \{\sum_i A_i O_i e^{-\beta c_{ij}}\}^{-1} \tag{3.8}$$

The objective function is the sum of linear set up cost terms with σ_i, for example, as the net cost of establishing and operating a residential unit in zone i, and the third term is the total interaction cost.

While the above formulation of the objective function appears a most reasonable proposal, it is now known that when dispersion exists (ie. β is finite) we must include in the objective function a term which is responsible for the dispersion, namely the variation in preferences over the population. That is we must replace

$$\min_{\underline{O},\underline{D}} \quad \{\text{set up cost} + \text{interaction cost}\}$$

by

$$\min_{\{\underline{O},\underline{D}\}} \quad \{\text{set up cost} - \text{interaction benefits}\}$$

in which the interaction benefits are interpreted by the locational surplus (Neuberger, 1971; Coelho and Williams, 1977). The latter is alternatively interpreted as an accessibility benefit (Williams and Senior, 1978; Leonardi, 1978), and for the doubly constrained gravity model is given by (Williams, 1976)

$$LS = \frac{1}{\beta}\left(\sum_i O_i \alpha_i + \sum_j D_j \gamma_j\right) + \text{constant} \tag{3.9}$$

in which $\underline{\alpha}$, $\underline{\gamma}$ are the Lagrange multipliers associated with the program (see Appendix 2).

The land use planning model now becomes

$$\min_{\{\underline{O},\underline{D}\}} \sum_i O_i\left(\sigma_i - \frac{\alpha_i}{\beta}\right) + \sum_j D_j\left(n_j - \frac{\gamma_j}{\beta}\right) \tag{3.10}$$

subject to zone capacity constraints, etc. Note that $\underline{\alpha}$ and $\underline{\gamma}$ are themselves functions of $\underline{O}$ and $\underline{D}$ and the objective function is therefore non-linear. For <u>incremental</u> changes in $\underline{O}$ and $\underline{D}$, $\underline{\alpha}$ and $\underline{\gamma}$ may be treated as constant marginal prices, and a <u>linear</u> allocation model results.

We now turn to the interpretation of locational benefits and the generation of models from random utility theory.

The doubly constrained gravity model may be given an economic interpretation in terms of a competitive labour market and/or a competitive housing market (Cochrane, 1975; Williams and Senior, 1978). In each case the terms

$$\sum_i O_i \alpha_i \quad \text{and} \quad \sum_j D_j \gamma_j$$

are given distinctive interpretations. If a housing market is assumed, the former measures the total consumer surplus and the latter the suppliers' surplus with $\gamma_j - \gamma_{j^*}$ denoting differential rents in the housing market. In the formation of the objective function we must include the benefit accruing to <u>all</u> spatial actors.

Returning to the theme of dispersion, we have here an application in which centralized decisions are invoked to determine the arrangement of activity levels and individual decisions determine the interaction benefits derived from any such activity arrangement.

We note here that the present formulation of the doubly constrained gravity model as an economic model is inconsistent with the notion of travel as a derived demand, because the utility derived from living in a given zone i and working in j is absent from the model formulation. We can appeal to random utility theory to generate both the model and the mean expected utility derived from given location decisions (Cochrane, 1975; Coelho and Williams, 1977; Brotchie, 1977). The results generated within this theoretical framework are consistent with the locational surplus measures given earlier.

We are now in a position to consider the notion of embedding one model within another (Coelho et. al., 1978). Returning to the simple TOPAZ model

$$\min_{\underline{O},\underline{D}} \quad \{\text{set up costs} - \text{interaction benefits}\} \tag{3.11}$$

subject to resource and other constraints.

Now the locational surplus LS, (or interaction benefits) can be written (see Appendix 2) as the dual objective function

$$\frac{1}{\beta}\left(\sum_i O_i\alpha_i + \sum_j D_j\gamma_j\right) \tag{3.12}$$

which is also equal (from primal-dual relations) to the primal objective function at optimality. We can replace the above formulation as

$$\min_{\underline{O},\underline{D},\underline{T}} \quad \{\text{set up costs} - \frac{1}{\beta}\sum_{ij} T_{ij}\ln T_{ij} - \sum_{ij} T_{ij}c_{ij}\} \tag{3.15}$$

subject to

$$\sum_j T_{ij} - O_i = 0 \qquad (3.14)$$

$$\sum_i T_{ij} - D_j = 0 \qquad (3.15)$$

and additional resource constraints.

The interaction benefits are thus derived by nesting a program which generates the interaction model within the broader activity location program.

The original TOPAZ formulation is now converted into a convex minimization problem subject to linear constraints in $\underline{T}$, $\underline{O}$ and $\underline{D}$. The operation of embedding has removed the <u>severe</u> non-linearities in the expression

$$\frac{1}{\beta}\sum_i O_i \alpha_i(\underline{O},\underline{D}) + \frac{1}{\beta}\sum_j D_j \gamma_j(\underline{O},\underline{D}) \qquad (3.16)$$

to produce a <u>convex</u> function but an expanded variable set (including $\underline{T}$). We can perform a dramatic reduction in the dimensionality of the problem by determining the <u>dual</u> of the new primal program. The details are provided in the paper by Coelho and Williams (1977). The procedures followed above can now be summarized:

Step 1 : Formulate TOPAZ program with variables $\underline{O},\underline{D}$

Step 2 : Perform embedding operation : variables $\underline{O},\underline{D},\underline{T}$

Step 3 : Formulate the dual of the program in step 2.

The number of variables and constraints are as follows:

Step 1 : (I + J; R)

Step 2 : (I + J + IJ; R + I + J)

Step 3 : (R + I + J; I + J + non-negativity constraints).

in which I is the number of origin zones, J the number of destination zones, R is the number of planning constraints. The final program is a concave maximization (or convex minimization) problem and, if a feasible solution exists, is characterized by a <u>unique</u> optimum.

4. ANOTHER EXAMPLE.

In the first example we considered an optimal location of basic employment in a region. Before attacking the land use optimization program we need to know the locational surplus associated with a given configuration of basic employment. Coelho and Williams (1977) have proposed an activity location model founded on the economic base. (This latter condition may, in fact, be relaxed). The model can be generated from a program of the following form :

$$\max_{\underline{T},\underline{S}} \left\{ -\frac{1}{\beta_1}\sum_{ij} T_{ij}\ln T_{ij} - \frac{1}{\beta_2}\sum_{ij} S_{ij}\ln S_{ij} - \sum_{ij} T_{ij}c^{w}_{ij} - \sum_{ij} S_{ij}c^{s}_{ij} \right\} \tag{4.1}$$

subject to

$$\sum_j T_{ij} - \lambda_1 \sum_j S_{ij} = 0 \tag{4.2}$$

$$\sum_i T_{ij} - \lambda_2 \sum_i S_{ij} = E^B_j \tag{4.3}$$

in which $\underline{T}$ and $\underline{S}$ are the work and shopping trip matrices, and λ_1 and λ_2 are suitably defined multipliers. The constraint (4.2) ensures that the zonal population P_i generated from the work trip matrix T_{ij} is consistent with that obtained the shopping matrix S_{ij}. Constraint (4.3) is an economic base relation, relating total zonal employment to the basic and non-basic components.

It can be shown that the Primal and Dual objective functions are both precisely the locational surplus associated with the location of activities. We can thus proceed to generate an optimal distribution of basic employment in an exactly analogous way to that appropriate to the doubly constrained gravity model embedded in the conventional TOPAZ system. The details of this process are given in Coelho and Williams (1977).

The problems specified here have been at a highly aggregated state. Higher disaggregation of stock and trip variables may be made - the limits ultimately being determined by the capability of obtaining a solution. It is quite feasible to implement a multi-sector competitive equilibrium model for land use allocation on the Alonso-Wingo-Harris lines, in which the objective functions for each land user embodies a bid rent term. The rent profile emerges from the dual variables in much the same way as in the Herbert-Stevens model (Williams and Senior, 1978).

This basic process of embedding locational surplus quantities has now been used in a variety of applications at Leeds. Current concerns are with explicit consideration of suppliers' surplus functions to generate endogenously thresholds in hierarchically structured models (Wilson, 1978; Beaumont and Clarke, 1979), with particular reference to Central Place hierarchies. These themes of work have recently been coupled to interests in catastrophe theory, by examining the structural properties of land use models as key parameters of the system change (Wilson and Clarke, 1978; Harris and Wilson, 1978).

5. COMMENTS ON THE OPTIMIZATION OF TRANSPORTATION AND LAND USE OPTIONS.

A few words are in order on the joint strategic planning of spatial activities with additions to the transport system - possibly in staged developments. Much theoretical interest has been given to the problem (State-of-the-act considerations can be found in Los, 1978) but until fairly recently little practical interest has been given to an optimization approach. One noteable exception is the regional planning application in Sweden using the TRANSLOC system (Lundqvist, 1975, 1978) which explicitly sought to combine the planning of the transport system with the activity system to incorporate a range of urban interdependencies. The resulting program is a non-linear mixed-integer program quadratic in the activity variables and, of course, integer with respect to the network design additions. For a fixed land use system the problem can be solved by branch and bound methods or by heuristics (Scott, 1971), while for a given network the problem is solvable by quadratic programming.

To keep the problem in bounds a limit (25) of structurally important extensions of the transport system are introduced and the heuristic search methods are confined to these. In the TRANSLOC system the heuristic is segmented into two mutually dependent levels

Generation of building stock location pattern

Generation of transportation network design

Using the arguments presented in Section 3, it is possible to convert this type of program into a mixed-integer concave maximization problem through the process of embedding spatial interaction models within locational surplus welfare functions. The joint problem, however, retains its solution difficulties, and this is particularly true if congestion considerations are added. Again it is possible to embed the usual congested assignment mathematical program within the whole network optimization procedure, but this would be at greatly increased cost.

In British Transportation Planning an example of network design is included in the Coventry Study (1973) in which a heuristic process (called CIDER) is invoked to search for suitable increments to the highway sytem.

We now return to a more general aspect of optimization involving multiple objectives.

6. PROBLEMS WITH MULTIPLE OBJECTIVES.

A characteristic of the large majority of practical optimization procedures, and implicitly those treated so far, is the inclusion of a single objective function in which various components are scaled by explicit weights, and in the case where each component is expressed in cost units, these weights are unity.

As Barber (1976) states:

> 'By its very nature land use planning is concerned with the resolution of multiple, conflicting, often incommensurate and occasionally intangible objectives. Some of these objectives and other relevant criteria cannot be quantified or converted to a single dimension; nor is it likely that a single land use plan will simultaneously optimize each objective considered independently. There is thus a need for a technique capable of generating compromise land use plans which are satisfactory in relation to each objective and which illustrate the key trade-offs among the various objectives'.

We can express the multicriterion program as follows:

$$\left.\begin{array}{l} \max\limits_{\underline{x}} \; f_1(\underline{x}) \\ \quad \vdots \\ \max\limits_{\underline{x}} \; f_p(\underline{x}) \end{array}\right\} \max_{\underline{x}} \; \underline{f}(\underline{x}) \qquad (6.1)$$

subject to

$$\begin{aligned} \underline{g}(\underline{x}) &\leqslant \underline{b} \\ \underline{x} &\geqslant 0. \end{aligned} \tag{6.2}$$

There are various approaches to the solution of this problem, the most common being the traditional trade-off method, in which a super objective $F(\underline{x})$ is formed

$$F(\underline{x},\underline{\alpha}) = \sum_i \alpha_i f_i(\underline{x}) \tag{6.3}$$

and the explicit trade-offs are considered by varying $\underline{\alpha}$. In the same way a 'pay-off matrix' may be formed (using extreme weights) showing the extent to which particular costs or benefits may be influenced if all other components are ignored. Ben-Shahar et.al. (1969), for example, generated a set of efficient plans by fixing the values of ρ-1 objective components and optimizing on the ρ^{th} criterion. The final choice among efficient plans was left to the planning authorities.

Alternatively, an a priori specification of goals or target levels for each objective can be obtained from planners and another objective function constructed to minimize the deviations from these desired targets and feasible values of the objectives functions. That is, the 'super-objective' is of the form

$$\min F(\underline{x}) = \sum_i \left| f_i(\underline{x}) - \hat{f}_i \right|. \tag{6.4}$$

Goal programming is one such approach.

The search among efficient plans is in the spirit of search among pareto or non-dominated solutions of the vector maximization problem. A solution $\underline{x}^*$ is said to be non-dominated if there is no other solution $\underline{x}$ such that $f_i(x) \geqslant f_i(\underline{x}^*)$ for all i and further, that $f_i(\underline{x}) > f_i(\underline{x}^*)$ for at least one i. All non-dominated solutions may be generated and presented in some organized manner to the decision maker who selects a globally optimal solution from the set. Unfortunately there are often very many non-dominated solutions.

Another approach to the multicriterion problem which alleviates the above problem involves interactive programming (Geoffrion et. al, 1972; Barber, 1976) which requires a minimum amount of information about the decision maker's preferences. Specifically, he must be able to provide the trade-offs among objectives in the neighbourhood of any feasible solution. Trade-offs are used as inputs to a local objective function for a mathematical programming problem that generates a new feasible solution in the direction of maximum utility. The trade-offs are then defined by the decision maker for this new solution and the process repeated. It is continued until a solution that is accepted as optimal by the decision maker.

Applications of multicriteria programming in land use planning contexts include those of Ben Shahar et. al. (1969); Nijkamp (1975); Barber (1976) and Lundqvist (1978). The latter paper is of particular interest because the author rejects the possibility of establishing a unique social welfare function expressing a proper weighting of interests and ambitions. The information structure of urban development planning is cited as the main argument for focusing on freedom of action as an important planning criterion. The notion of programming solutions is here intimately related to a philosophy of planning in which the potential conflicts between short term efficiency and long term adaptivity are a central focus of concern.

7. CONCLUSION.

In these notes I have touched on a selected number of techniques and models associated with the planning of land use/activity systems. In conclusion, it would be wrong to suggest that improved technical procedures necessarily improve the process of strategic planning. Indeed it can be argued - with some force - that a pre-requisite for technical progress is a simple analytic structure, and that technical progress can be at the expense of an adequate representation of the problem at hand.

With reference to the concessions to simplification which are invoked to produce tractable models, it is not particularly unusual that the heavy handed methods of mathematical programming are employed to produce results which are rather transparent. The traditional counter-argument to this, by those who perhaps see planning more of a science than an art, is that

> "... explicit reasoning and calculation are superior merits, making the procedure amenable to simplification, generalization, and improvement. Intuition cannot be improved upon so readily". (Ahmed et. al., 1976).

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Appendix 1 : Some notes on non-linear programs and their duals.

The derivation of solution conditions (the Kuhn-Tucker conditions) for non-linear programs may be conveniently discussed in terms of an extension of classical optimisation methods appropriate to the unconstrained problem

$$\max_{\underline{x}} \; f(\underline{x}) \tag{A1}$$

or more diectly as a development of Lagrangian theory appropriate to the equality constrained problem

$$\max_{\underline{x}} \; f(\underline{x}) \tag{A2}$$

subject to

$$g_m(\underline{x}) = b_m \qquad m = 1, \ldots, M \tag{A3}$$

Recall that the rationale for considering the Lagrangian formulation is the removal of explicit dependencies between $\underline{x}$ variables introduced by the constraints, at the expense of introducing an additional set of variables, the Lagrange multipliers $\lambda_1, \ldots, \lambda_M$, into the problem. A constrained problem may be converted into an unconstrained problem with an extended variable set.

The first order conditions for $\underline{x}$ to solve (A2) - (A3) may be stated in terms of the Lagrangian

$$L(\underline{x},\underline{\lambda}) = f(\underline{x}) + \sum_m \lambda_m (b_m - g_m(\underline{x})), \tag{A4}$$

and are given by

$$\frac{\partial L}{\partial x_n} = 0 \qquad n = 1, \ldots, N \tag{A5}$$

$$\frac{\partial L}{\partial \lambda_m} = 0 \qquad m = 1, \ldots, M. \tag{A6}$$

Consider now the problem, with <u>inequality</u> conditions

$$\max_{\underline{x}} \; f(\underline{x}) \tag{A7}$$

subject to

$$g_m(\underline{x}) \leq b_m \qquad m = 1, \ldots, M \tag{A8}$$

$$\underline{x} \geq 0. \tag{A9}$$

Now (A9) may be converted into <u>equality</u> conditions by the introduction of slack variables s_m and the problem reformulated as

$$\max_{\underline{x}} \; f(\underline{x}) \tag{A10}$$

subject to

$$g_m(\underline{x}) + s_m = b_m \qquad m = 1, \ldots, M \tag{A11}$$

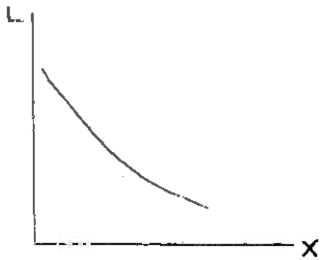
$$\underline{x} \geq 0 \; ; \; \underline{s} \geq 0 \tag{A12}$$

and is now of the form (A2) - (A3) with the additional <u>non-negativity conditions</u> $\underline{x} \geq 0$ and $\underline{s} \geq 0$. The solution conditions are now obtained by examining the behaviour of the Lagrangian

$$L(\underline{x},\underline{\lambda},\underline{s}) = f(\underline{x}) + \sum_m \lambda_m (b_m - g_m(\underline{x}) - s_m). \tag{A13}$$

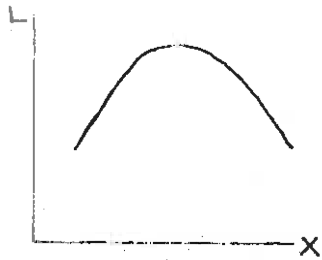
Now for any particular x_n; $n = 1, \ldots, N$ or s_m; $m = 1, \ldots, M$ the variable will <u>at the optimal solution</u> be either zero or some positive value. The optimality conditions must thus account for both possibilities. Essentially, we examine the behaviour of $L(\underline{x},\underline{\lambda},\underline{s})$ at all appropriate boundaries, in addition to its behaviour in the interior space $\underline{x} > 0$, $\underline{s} > 0$.

Consider the variation of L with respect to <u>one</u> variable say x_n, holding all other variables constant (at their optimal values). A maximum will occur on the boundary $x_n = 0$ or in the interior $x_n > 0$ if L behaves as follows:



Maximum at

(i) $x_n = 0$ when $\frac{\partial L}{\partial x_n} < 0$



Maximum at

(ii) $x_n > 0$ when $\frac{\partial L}{\partial x_n} = 0$

That is, a maximum will occur if either

$$x_n = 0, \quad \frac{\partial L}{\partial x_n} < 0 \tag{A14}$$

or

$$x_n > 0, \quad \frac{\partial L}{\partial x_n} = 0. \tag{A15}$$

The full conditions for (A10) - (A12) can be written

$$x_n \frac{\partial L}{\partial x_n} = 0 \quad \frac{\partial L}{\partial x_n} \leq 0 \qquad n = 1, \ldots, N \tag{A16}$$

$$x_m \frac{\partial L}{\partial s_m} = 0 \quad \frac{\partial L}{\partial s_m} \leq 0 \qquad m = 1, \ldots, M \tag{A17}$$

$$\frac{\partial L}{\partial \lambda_m} = 0 \qquad m = 1, \ldots, M \tag{A18}$$

These are equivalent to the conditions

$$x_n \frac{\partial L}{\partial x_n} = 0 \qquad \frac{\partial L}{\partial x_n} < 0 \qquad n = 1, \ldots, N \tag{A19}$$

$$\lambda_m \frac{\partial L}{\partial \lambda_m} = 0 \qquad \frac{\partial L}{\partial \lambda_m} > 0 \qquad m = 1, \ldots, M \tag{A20}$$

$$\underline{x} \geqslant 0, \quad \underline{\lambda} \geqslant 0 \tag{A21}$$

for L defined by equation (A4). These are the Kuhn-Tucker conditions. Note that (A19) is equivalent to (A14) and (A15). Equation (A20) embodies the famous 'complementary slackness' condition which requires that:

$$\begin{array}{lll} \text{(i)} & \text{when } s_m > 0 & \lambda_m = 0 \\ \text{(ii)} & \text{when } s_m = 0 & \lambda_m > 0 \end{array} \tag{A22}$$

This can be given the usual economic interpretation in terms of shadow price valuation of scarce resources.

In summary, if we can find vectors $\underline{x}$ and $\underline{\lambda}$ which satisfy (A19) - (A21), we know that the values $(x_1, \ldots, x_N)$ characterise LOCAL OPTIMUM of the program (A7) - (A9). The GLOBAL OPTIMUM is that LOCAL OPTIMUM which yields the maximum value of $f(\underline{x})$ over the set of all local optima. For certain special problems we can be assured that only one local optimum exists and that this is therefore the required global solution. These problems are characterised as follows:

Problem	Objective function $f(\underline{x})$	Constraint set $\underline{g}(\underline{x}) \leqslant \underline{b}$
max $f(\underline{x})$	concave	convex
min $f(x)$	convex	convex

Duality

Corresponding to the PRIMAL program (A7) - (A9) - taking $f(\underline{x})$ as concave and the constraint set as convex - we can define a DUAL as follows (Wolfe, 1961):

$$\text{DUAL:} \quad \min_{(\underline{x},\underline{\lambda})} D(\underline{x},\underline{\lambda}) \tag{A23}$$

subject to

$$\frac{\partial L(\underline{x},\underline{\lambda})}{\partial x_n} \leqslant 0 \qquad n = 1, \ldots, N \tag{A24}$$

$$\underline{\lambda} \geqslant 0 \tag{A25}$$

with

$$D(\underline{x},\underline{\lambda}) = L(\underline{x},\underline{\lambda}) + \sum_n x_n \frac{\partial L}{\partial x_n} \tag{A26}$$

and

$$L(\underline{x},\underline{\lambda}) = f(\underline{x}) + \sum_m \lambda_m (b_m - g_m(\underline{x})). \tag{A27}$$

The program (A23) - (A25) will in general contain the mutually dual variables $\underline{x}$ and $\underline{\lambda}$. At optimality the solution of the primal and dual programs are mutually complementary and, the $(\underline{x},\underline{\lambda})$ which solve (A23) - (A25) satisfy the Kuhn-Tucker conditions. Solution of the primal or dual will provide the solution of the dual and primal respectively and at optimality the value of the primal objective function will equal the value of the dual function.

Appendix 2 : Spatial interaction models and their duals

The dual programs corresponding to the spatial interaction family may now readily be derived. Here we follow the treatment of Wilson and Senior (1974), and Evans (1973) for the doubly constrained gravity model generated from:

$$\max f(\underline{T}) = -\frac{1}{\beta} \sum_{ij} T_{ij}(\ln T_{ij} - 1) - \sum_{ij} T_{ij}c_{ij} \tag{A28}$$

subject to

$$\sum_j T_{ij} = O_i \tag{A29}$$

$$\sum_i T_{ij} = D_j \tag{A30}$$

$$\underline{T} \geqslant 0. \tag{A31}$$

The Lagrangian is given by

$$L(\underline{T},\underline{\alpha},\underline{\gamma}) = f(\underline{T}) + \sum_i \alpha_i(O_i - \sum_j T_{ij}) + \sum_j \gamma_j(D_j - \sum_i T_{ij}) \tag{A32}$$

and the appropriate Kuhn-Tucker conditions are:

$$T_{ij} \frac{\partial L}{\partial T_{ij}} = 0, \quad \frac{\partial L}{\partial T_{ij}} \leqslant 0 \qquad \text{for all } i \text{ and } j \tag{A33}$$

$$\frac{\partial L}{\partial \underline{\alpha}} = 0, \quad \frac{\partial L}{\partial \underline{\gamma}} = 0. \tag{A34}$$

Now because $\underline{T} > 0$ when β is finite, (A33) may be written $\frac{\partial L}{\partial T} = 0$ and the model may be formed in the usual way from:

$$-\frac{1}{\beta} \ln T_{ij} - \alpha_i - \gamma_j - c_{ij} = 0 \tag{A35}$$

in conjunction with (A34).

Now consider the dual which from (A23) – (A27) is given by

$$\min_{(\underline{T},\underline{\alpha},\underline{\gamma})} \quad D(\underline{T},\underline{\alpha},\underline{\gamma}) \tag{A36}$$

subject to

$$\frac{\partial L}{\partial T_{ij}} \leq 0. \tag{A37}$$

(No <u>inequality</u> conditions exist on $\underline{\alpha}$ and $\underline{\gamma}$ because (A29) and (A30) are strict equalities.) In addition because the Kuhn-Tucker conditions are always satisfied for an interior solution ($\underline{T} > 0$) we can write (A37) as an equality

$$\frac{\partial L}{\partial T} = \frac{1}{\beta} \ln T_{ij} - \alpha_i - \gamma_j - c_{ij} = 0. \tag{A38}$$

After some manipulation the dual can be written

$$\min_{(\underline{T},\underline{\alpha},\underline{\gamma})} \frac{1}{\beta} \sum_{ij} T_{ij} + \sum_i O_i \alpha_i + \sum_j D_j \gamma_j \tag{A39}$$

subject to

$$-\frac{1}{\beta} \ln T_{ij} - \alpha_i - \gamma_j - \beta c_{ij} = 0 \tag{A40}$$

or finally, substituting T_{ij} from (A40) into (A39), we have

$$\text{DUAL:} \quad \min_{(\underline{\alpha},\underline{\gamma})} \frac{1}{\beta} \sum_{ij} e^{-\beta(\alpha_i + \gamma_j + c_{ij})} + \sum_i O_i \alpha_i + \sum_j D_j \gamma_j. \tag{A41}$$

Note that the dual is expressed entirely in terms of dual variables, and that $(\underline{\alpha},\underline{\gamma})$ will solve (A41) if T_{ij} determined from (A40) solves the primal. From duality theory we have

$$f(\underline{T}) \leqslant D(\underline{T},\underline{\alpha},\gamma) \tag{A42}$$

the equality sign holding at optimality.

Perhaps the most important feature about (A41) is that it is an UNCONSTRAINED minimisation problem. The classical Furness method of 'balancing' a doubly constrained gravity model can be viewed as solving the first order conditions for $D(\underline{\alpha},\underline{\gamma})$ to attain a minimum.