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CRITICALITY AND URBAN RETAIL STRUCTURE: ASPECTS OF CATASTROPHE THEORY AND BIFURCATION

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1. Introduction: criticality and development

There is now much experience of building models of the location of population and economic *activities* in cities and associated *flows.* (see Wilson, 1974, for a review). It has proved more difficult to model the evolution of the underlying *structure.* A specific example will be used to illustrate both the achievements and the problems in this paper: we model the flows of cash from people at their residences to shopping centres and this enables us to predict the revenue attracted to particular locations. The structural problem is the study of the evolution and dynamics of the location and size of the shopping centres.

It will be shown that there are surfaces of critical values (in parameter space) of locational parameters on one 'side' of which development at that location is possible and on the other side, not.

It is suggested that such studies of criticality can form the basis of a theory of evolution of urban structures. When the evolution of the whole system is investigated on this basis, the structures which emerge have a pattern which can be related to 'order from fluctuations' arguments. The methods used for exploring criticality have an informal relationship to the theory of catastrophes and bifurcation.

2. Catastrophe theory and bifurcation

Catastrophe theory is concerned with the structural singularities of a function f of state variables $\underline{x}$ and parameters (or control variables) $\underline{u}$. These are singularities on the surface

$$\frac{\partial f}{\partial \underline{x}} = 0 \tag{1}$$

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and occur when the Hessian matrix of second derivatives of f is singular. The values of the parameters $\underline{u}$ at which such singularities occur are *critical* values of those parameters.

To fix ideas, we shift to an applied viewpoint and suppose some system of interest described by $\underline{x}$ and $\underline{u}$ has equilibrium states which arise from the minimisation of f:

$$\min_{\underline{x}} \; f(\underline{x},\underline{u}) \tag{2}$$

so that the equilibrium surface, in $(\underline{x},\underline{u})$ space, is given by the solution of equation (1).

Catastrophe theory is concerned with the possible shapes of equilibrium surface in the neighbourhood of a degenerate singularity. For a number of cases, Thom (1975) has shown that there are relatively few types and the reader is referred to the now voluminous literature for further details of this (such as Poston and Stewart, 1978, and Zeeman, 1978). In this paper, we will be concerned with three cases; where there is one state variable and one, two or three parameters respectively. In the first case, Thom's theorem tells us that the structural singularities are, at worst folds; for two parameters, at worst folds or cusp points; and for three parameters, at worst folds, cusp points or swallow.

We will show the existence of folds in our particular example, and use Thom's theorem implicitly to bear in mind that cusps and swallowtails may complicate the analysis. However, in this paper, catastrophe theory is only employed indirectly. Perhaps the most common mode of use of the theory is to use the result that the function which is being minimised in a particular problem has singularities which can be topologically transformed into the appropriate canonical form (determined by the number of state variables and number of parameters, which are related to the maximum degeneracy involved, the co-rank, and the co-dimension respectively). Often, the transformation is not carried out explicitly - and indeed the function may not even be specified explicitly. Further, there are problems with the *local* nature of Thom's theorem: we do not know the range in $\underline{u}$-space over which it holds. Here, we use the theory more indirectly: simply to make ourselves aware that structural singularities (in the forms specified by Thom) may occur in our problem and to be aware of the corresponding unusual types of system behaviour near critical parameter values - in particular, of *jump* effects. These are demonstrated below in relation to the particular example.

The dynamics of a system governed by (2) is customarily assumed to be given by

$$\dot{\underline{x}} = -\frac{\partial f}{\partial \underline{x}} \qquad (3)$$

(or more generally by

$$\dot{x}_i = -x_i^n \frac{\partial f}{\partial x_i} \qquad (4)$$

if the problem of zero x's can be handled - and this equation then provides alternative pictures of the path back to equilibrium). This is known as a gradient system, because the return to equilibrium follows the gradient in the right hand side of (3).

More generally, a dynamical system may be governed by equations

$$\dot{\underline{x}} = -\underline{g}(\underline{x},\underline{u}) \qquad (5)$$

where the vector function $\underline{g}$ is not derivable as a gradient, $\frac{\partial f}{\partial \underline{x}}$. Such systems also exhibit bifurcation properties at critical parameter values. This is when the qualitative character of the equilibrium solution of the equations, which is

$$\underline{g}(\underline{x},\underline{u}) = 0 \qquad (6)$$

changes. In the examples below, most of the results refer to a non-gradient system, though alternative forms of the main equations will be described briefly which show a similarity of the equilibrium surface in this case to that of a gradient system.

For both catastrophe theory and differential equation bifurcation theory the interest arises from nonlinearities in f (or $\underline{g}$) which generate multiple stable equilibrium solutions for particular values of $\underline{u}$. The critical values, or structural singularities usually occur when one or more of these 'merge' or when one disappears.

3. Urban retail structure: the problem

Consider a city divided into discrete spatial zones labelled $i,j=1,2, \ldots, n$. Let S_{ij} be the flow of retail sales from shops in zone j to residents of zone i; e_i, the per capita expenditure on retail goods by residents of zone i; P_i, the population of zone i; W_j the size of shopping facilities in j (which we take as a measure of attractiveness); c_{ij} the travel cost, in suitable units, from i to j). Then a suitable flow model can be shown to be

$$S_{ij} = A_i e_i P_i W_j^{\alpha} e^{-\beta c_{ij}} \tag{7}$$

where

$$A_i = 1/\sum_k W_k^{\alpha} e^{-\beta c_{ik}} \tag{8}$$

and α and β are parameters.

If we define

$$D_j = \sum_i S_{ij} \tag{9}$$

we see that the model also predicts the locational variable D_j which is the total revenue attracted to shops in zone j.

This can be shown to be (Wilson, 1970) an 'equilibrium statistical mechanics' – or entropy maximising – prediction for $\{S_{ij}\}$ which arises from

$$\underset{\{S_{ij}\}}{\text{Max}} \quad z = -\sum_{ij} S_{ij} \log S_{ij} \tag{10}$$

subject to

$$\sum_j S_{ij} = e_i P_i \tag{11}$$

$$\sum_{ij} S_{ij} \log W_j = B \tag{12}$$

$$\sum_{ij} S_{ij} c_{ij} = C \tag{13}$$

where B and C are total benefits (assumed measured on a logarithmic scale) and costs respectively. This model assumes a set of fixed and given structural variables $\{W_j\}$.

This problem can be extended (Coelho and Wilson, 1976) to include $\{W_j\}$ as variables by changing the range of variables in the objective function:

$$\max_{\{S_{ij}, W_j\}} \quad z = -\sum_{ij} S_{ij} \log S_{ij} \tag{14}$$

subject to (11)-(13) as before and the additional constraint on total shopping facility size, W_j:

$$\sum_j W_j = W \tag{15}$$

The new problem can then be formulated in Lagrangian form as

$$\begin{aligned} \max_{\{S_{ij}, W_j\}} L = & -\sum_{ij} S_{ij} \log S_{ij} \\ & +\alpha\left(\sum_{ij} S_{ij} \log W_j - B\right) \\ & +\beta\left(C - \sum_{ij} S_{ij} c_{ij}\right) \\ & +\gamma\left(W - \sum_j W_j\right) \\ & +\sum_i \mu_i\left(\sum_{ij} S_{ij} - e_i P_i\right) \end{aligned} \tag{16}$$

This gives some insights into the parameters: α, β and γ are the Lagrangian multipliers associated with constraints (12), (13) and (15) respectively; the $\{A_i\}$ in (8) are transformations of the multipliers μ_i associated with (11).

It can easily be checked that, with $\{W_j\}$ now varying, $\{S_{ij}\}$, which are the solutions of

$$\frac{\partial L}{\partial S_{ij}} = 0 \tag{17}$$

still satisfy (7) with $\{A_i\}$, obtained by solving for μ_i from (11), by (8). The equations for $\{W_j\}$ are

$$\frac{\partial L}{\partial W_j} = \frac{\alpha \sum_i S_{ij}}{W_j} - \gamma = 0 \tag{18}$$

which can be written

$$\sum_i S_{ij} = \frac{\gamma}{\alpha} W_j \tag{19}$$

or as

$$D_j = \frac{\gamma}{\alpha} W_j \tag{20}$$

using (9).

Equation (19) can be solved for $\{W_j\}$, but only numerically, as the $\{S_{ij}\}$, through (7) and (8), are complicated nonlinear functions of $\{W_j\}$.

If we set

$$\frac{\gamma}{\alpha} = k \tag{21}$$

then (20) can be written as

$$D_j = kW_j \tag{22}$$

and this will play a significant role in our explorations shortly.

We have now provided a procedure for calculating equilibrium values of $\{W_j\}$ and because of the nonlinearities in (19) (or (22)), we can expect the surface of such values, traced out as the parameters α, β and γ (and also $\{e_i\}$, $\{P_i\}$ and $\{c_{ij}\}$, as these are also assumed to be given) vary to exhibit structural singularities. We explore the nature of these in the next section. Meanwhile, as a final preliminary, we explore the nonequilibrium dynamics of the structural variables.

Equation (22) suggests that if $D_j > kW_j$, shops at j are profitable and should expand; and vice versa. Thus, a suitable differential equation is

$$\dot{W}_j = \varepsilon(D_j - kW_j) \tag{23}$$

for some suitable constant ε. (And here we make the assumption that, after a disturbance, consumers move into their equilibrium so rapidly that we retain the equilibrium equations (7) and (8) rather than use differential equations in $\dot{S}_{ij}$.)

In the context of our earlier brief review of bifurcation theory, it is of interest to see whether the dynamical system represented by (23) (together with associated equations for $\{D_j\}$, $\{S_{ij}\}$ and $\{A_i\}$) is a gradient system. It can easily be checked that it is not the most obvious gradient system associated with (16), since we have already calculated $\frac{\partial L}{\partial W_j}$ in (18) and this is not equal to $\dot{W}_j$. In fact,

$$\dot{W}_j = W_j \frac{\partial L}{\partial W_j} \tag{24}$$

However, it can be shown that there is another Lagrangian which generates (23). This involves replacing the log W_j term in the second term of L by W_j and the constraint (15) by

$$\Sigma W_j^2 = \overline{W^2} \tag{25}$$

There would be a corresponding change in the $\{S_{ij}\}$ equations: W_j^α would be replaced by $e^{\alpha W_j}$. Neither this result nor the constraint (25) appear helpful: the first modification implies that the benefits of shopping centre size increase linearly rather than logarithmically (or something less than linearly), and the constraint (25) seems impossible to interpret. However, we will shortly make brief use of the fact that this dynamical system is a gradient system, but will basically consider bifurcation in the non-gradient system given by the equilibrium formulation (16) and the differential equation (23). Because the two systems have essentially the same equilibrium points - the solutions of (22) - we can assume they have similar bifurcation properties.

It may be useful to conclude this section by writing both the equilibrium conditions for W_j and the differential equations in full to emphasise the nonlinearity. Take $\{D_j\}$ from (9), $\{S_{ij}\}$ from (7), $\{A_i\}$ from (8) and substitute in turn in (22) and (23). This gives the equilibrium condition as

$$\sum_i \frac{e_i P_i W_j^\alpha e^{-\beta c_{ij}}}{\sum_k W_k^\alpha e^{-\beta c_{ik}}} = kW_j \tag{26}$$

and the differential equations as

$$\dot{W}_j = \varepsilon\left[\sum_i \frac{e_i P_i W_j^{\alpha} e^{-\beta c_{ij}}}{\sum_k W_k^{\alpha} e^{-\beta c_{ik}}} - kW_j\right] \qquad (27)$$

4. Dynamical analysis

Our system of interest is not, strictly speaking, a gradient system; but we have seen that such a system can be set up which has very similar properties. If we consider all exogenous variables to be fixed except α, β and γ (or analogously for γ, k, since $k=\gamma/\alpha$), then this suggests that structural singularities are possible up to the swallowtail. However, we will simply use this as an argument to suggest that we should be on the alert for this; we will not attempt either to translate our mathematical system into canonical form, or to make any assumptions based on classification theorems. Instead, we will try to gain some insights into the mechanics of catastrophic change directly for this system and by this means we show the existence of folds.

The detailed argument is presented in a paper by Harris and Wilson (1978) and only the main ideas and results will be given here. They will then be used as the basis for further explorations and extensions. The argument is presented here for the case $\alpha>1$. When $\alpha=1$, the results are qualitatively similar and when $\alpha<1$ the behaviour is smoother.

The argument proceeds in a number of stages. First, we identify possible equilibrium points and secondly their stability. We then relax the assumptions on which this analysis was based and proceed towards a full dynamical analysis which includes an account of the evolution of structure.

We can get some insight into the nature of the equilibrium values of $\{W_j\}$ by a simple trick. The manipulation which generated this in terms of W_j can be used to show D_j as a function of W_j as

$$D_j = \sum_i \frac{e_i P_i W_j e^{-\beta c_{ij}}}{\sum_k W_k e^{-\beta c_{ik}}} = D_j^{(1)}(W_j) \qquad (28)$$

say, and we also require:

$$D_j = kW_j = D_j^{(2)}(W_j) \tag{29}$$

We can thus plot $D_j^{(1)}$ against W_j, and $D_j^{(2)}$ against W_j and the intersections are the solutions to (26) and the possible equilibrium points. The shape of the $D_j^{(1)}$ curve can be obtained from an analysis of $\frac{\partial D_j}{\partial W_j}$ and $\frac{\partial^2 D_j}{\partial W_j}$. It can be shown that it has essentially a logistic shape (though with the possibility of additional points of inflexion, though we will neglect this complication to deal with only the essentials of the argument here - see Wilson, 1979, for further details). This generates plots such as those in figure 1(a) and figure 1(b). It can immediately be seen that there are two fundamentally different types of case: when the line has the upper intersections W_j^A and W_j^B, and when not. There is a critical case, when the line touches the curve, as shown in figure 1(c). The slope of the line in this case depends on the curve, and is hence j-dependent (ie. location dependent) and is denoted by k_j^{crit}. k is a system-wide parameter, which can be interpreted as the average cost of supplying facilities (and could itself be j-dependent, but again we will neglect this complication here to concentrate on the essentials of the argument.

It can easily be seen by using the differential equations (23) that the origin is always a stable point j in case 1(b), W_j^A is stable and W_j^B is unstable. Hence, if $k<k_j^{crit}$, W_j^A exists and there is the possibility of development at j; if $k>k_j^{crit}$, there is no development at j.

These ideas can be collected together in an alternative graphical form: consider k to be a variable parameter and plot the stable values of W_j (the origin or W_j^A when it exists) against k. This leads to figure 2. This is immediately recognisable as an example of the fold catastrophe, though with the stable state at $W_j=0$ for all k added. The structural singularity occurs, as we have seen, at $k=k_j^{crit}$ and at point, the tangent to the curve is vertical, which can be expressed as:

$$\frac{\partial k}{\partial W_j} = 0 \tag{30}$$

We have so far said little about the practical problem of computing equilibrium values numerically. The equations (26) can be solved numerically by various methods and these turn out to be equivalent to the solution to the mathematical programming problem (16). The details of various methods are presented elsewhere (Harris and Wilson, 1978; Leonardi, 1978 and Wilson and Clarke, 1978).

The analysis so far has been presented in terms of the parameter k. In fact, when $k=k_j^{crit}$, the zone's situation is also critical in all the other parameters. The focus on k provides a useful introduction as the effect of changes is so clearly visible with k being simply the slope of the $D_j^{(2)}$ line. However, if α and β change, the $D_j^{(1)}$ curve moves. The values of α and β at which this curve touches the line can be labelled α_j^{crit} and β_j^{crit} and 'fold' curves, analogous to figure 2, can be constructed for these parameters. Development is more likely in a zone if α is high and β low and so the fold curves are as shown in figures 3 and 4 - in the former $1/\alpha$ is used as the horizontal axis.

This result can also be expressed algebraically. Denote by h_j the following functions:

$$h_j(W_j) = D_j^{(1)}(W_j)-D_j^{(2)}(W_j) \tag{31}$$

At the critical point, $D_j^{(2)}$ touches the $D_j^{(1)}$ curve, and so

$$\frac{\partial D_j^{(1)}}{\partial W_j} = \frac{\partial D_j^{(2)}}{\partial W_j} \tag{32}$$

Thus, differentiating (31) and using (32), we see that

$$\frac{\partial h_j}{\partial W_j} = 0 \tag{33}$$

at the critical point. Now h_j is a function of the parameters α, β, k also - written explicitly as $h_j(W_j,\alpha,\beta,k)$. The implicit function theorem shows that

$$\frac{\partial \alpha}{\partial W_j} = -\frac{\partial h_j/\partial W_j}{\partial h_j/\partial \alpha} = 0 \tag{34}$$

$$\frac{\partial \beta}{\partial W_j} = - \frac{\partial h_j/\partial W_j}{\partial h_j/\partial \beta} = 0 \tag{35}$$

$$\frac{\partial k}{\partial W_j} = - \frac{\partial h_j/\partial W_j}{\partial h_j/\partial k} = 0 \tag{36}$$

where each right hand side is zero because of (33). This is an algebraic representation of the folds in figures 3, 4 and 2 respectively, with the vertical tangent occurring at the critical points.

For any given α and β, there exists, as we have seen, a critical value of k, k_j^{crit}. These associated values of α and β are also critical. There is thus a whole surface traced out in (α,β,k) space of critical parameter values $(\alpha_j^{crit},\beta_j^{crit},k_j^{crit})$ for each zone j. We have demonstrated that the equilibrium surface in (W_j,α,β,k) space has folds at the critical values of α, β and k. Although there are hints from catastrophe theory that more complicated forms of structural singularity - such as cusps and swallowtails - may exist in that surface, it remains an interesting research task to see whether this is so in this particular case.

So far, we have assumed that the other independent variables are fixed in our investigation of what can happen to W_j at zone j as a function of α, β and k. The argument can easily be extended to incorporate $\{e_i\}$, $\{P_i\}$ and $\{c_{ij}\}$. If these are treated as parameters, then the argument of equations (31)-(36) can be extended in an obvious way and thus shows that, in the critical state, they also have critical values. For modelling the evolution of urban development, this is important as changes in $\{P_i\}$ say, as a city grows, are likely to have greater effects than changes in α, β or k. However the formal argument remains the same and we will pick up its implications in this respect in the next section. We should note in this case, however, that by adding many new varying parameters, we go much beyond the limits of elementary catastrophe theory (for the associated gradient system) and the structural singularities may be much more complicated.

A more difficult problem, for zone j, is presented by $\{W_k\}$, $k \neq j$. These also have been assumed fixed. They also, if and when they vary, are critical parameters for zone j when everything else is critical. The difficulty here is that a similar analysis to that used for zone j applies to each other zone. This adds complications which will also be taken up in the next section. There is also the problem that, although we have constructed the $D_j^{(2)}(W_j)$ curve for $\{W_k\}$, $k \neq j$, fixed, this is unlikely to be a realistic assumption for the range of W_j considered. However, the basis of the analysis for zone j can probably be maintained in the following way. Near the value of W_j at which parameters are critical, there will exist a set of $\{W_k\}$, $k \neq j$, which are 'realistic' – perhaps existing values. It is likely that we only need to construct the curve 'near' such a point, as in figure 5, for the argument to hold. The further implications of this will also be considered in the next section. A range of alternative assumptions for $\{W_k\}$, $k \neq j$, are explored in another paper (Wilson and Clarke, 1978).

5. The dynamics and evolution of spatial structure

We can now explore in outline the implications of this analysis for the modelling of the evolution of spatial structure. As cities grow and economies develop, this can be represented by the growth of $\{e_i\}$, $\{P_i\}$ and $\{c_{ij}\}$. We can assume, for simplicity, that either α, β and k are given and do not change, or also change in some specified way. Then, for each zone j, at each point in time, the analysis of the preceding section could be used to determine whether that zone was in a 'no-development-possible' (NDP) or 'development-possible' (DP) state. As the whole system evolves, zones will switch from NDP to DP and W_j will become non-zero for such zones. In principle then, we can now see how to model the evolution of structure in the whole system.

The reality is likely to be different at least in the following sense: when actual development occurs will be determined by entrepreneurial or governmental agencies, and they are unlikely to be doing these precise calculations. Such developments (or lack of them) can be viewed as fluctuations around the equilibrium state evolutionary path at each point in time. However, we have seen that a given zonal development affects the calculation of criticality for all other zones. In this sense, fluctuations will drive the whole system to new states. This is an analytical representation of common sense results: if a large shopping centre is established at j, say, by entrepreneurial whim, then even though that zone at that time may be in the NDP state, and indeed nowhere near the top of the list for development, then from the moment of its construction, all the criticality conditions must be recalculated and future developments will be correspondingly affected. In spite of this randomness, there is likely to be a degree of order in the spatial structure which evolves. For example, the number of centres in particular size groups, and their average spacing, may still be determined by the mechanism of evolution proposed. In this sense, the model proposed here generates 'order from fluctuations' in the same sense as that of Allen et al. (1978), even though their mechanisms are somewhat different.

There is a further complication which arises because of the way in which all the W_k's affect the criticality computation for each zone j. When there is a jump, say from NDP to DP for zone j, this will cause jumps in the critical parameter values for other zones - that is for other zones k, $(\alpha_k^{crit}, \beta_k^{crit}, k_k^{crit})$ jump in value and this may take the current (α, β, k) value for the system from NDP to DP state, or vice versa, for some such zones. Although this is a complication, it does not present formidable new analytical difficulties.

Finally, we should note a further necessary theoretical extension which is of great importance. So far, we have assumed the temporal variation of $\{P_i\}$ to be given. In reality, the evolution of $\{P_i\}$, essentially a residential structural variable, will depend on mechanisms rather like those given for $\{W_j\}$. Further, development in such a residential model will depend on the location of jobs - which will be partly associated with $\{W_j\}$ - and on access to shopping facilities - the $\{W_j\}$ directly, so that the two models will be strongly coupled. Again, we can easily see the principles on which such a more general model can be based, but the detailed working out will be a complex task. In effect, such a model provides the basis of a dynamic central place theory (cf. Wilson, 1978).

6. Concluding comments

It is clear that the methods presented here offer insights but are nontheless only the beginnings of a research programme which is a major task. This will lead to new questions, both theoretical and empirical. For example, the model presented here is an aggregated one. Central place theory has concerned itself more explicitly with hierarchical structure and the presence or absence of certain kinds of function. This style of disaggregation could be added to the models presented here. It is also likely that more complicated differential equation systems can be developed, though the mechanism presented here may offer some insight into the solutions of those. For example, the equations (27) are special cases of some which occur in ecology and which are analysed by Hirsch and Smale (1974). They note bifurcation properties involving a separtrix-crossing jump from a state with all W_j's non-zero to another state with one or more zeros. The 'all-non-zero' state is unique if it exists, and our analysis may contribute knowledge of the conditions in which it does exist.

The empirical task is to match data against the predictions of the model. The biggest difficulty is in the poor availability of relevant data, and this may be a case where the provision of a theory may aid the formulation of plans for data collection and organisation. But that will be a mammoth task for a long time. This also leads us into a final comment on theoretical issues. Perhaps the main tasks involve the interpretation of jump behaviour: identifying the conditions under which structures can suddenly emerge and finding examples (though the theory says much about smooth change also). We have seen that jumps can be triggered by changes in any of the parameters, and that the critical parameters are strongly coupled among all the zones. This alone makes the interpretation of jumps difficult. The situation is further complicated by the existence of other mechanisms for causing jumps behaviour. These include separtrix-crossing jumps mentioned briefly earlier, so-called 'constraint catastrophes' evident from mathematical programming formulations, and alternative models of jump behaviour such as that presented by Poston and Wilson (1977).

It is hoped that the methods presented here offer substantial new insights. It is equally clear that the research programme which must follow is a very extensive one.

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Figure 1

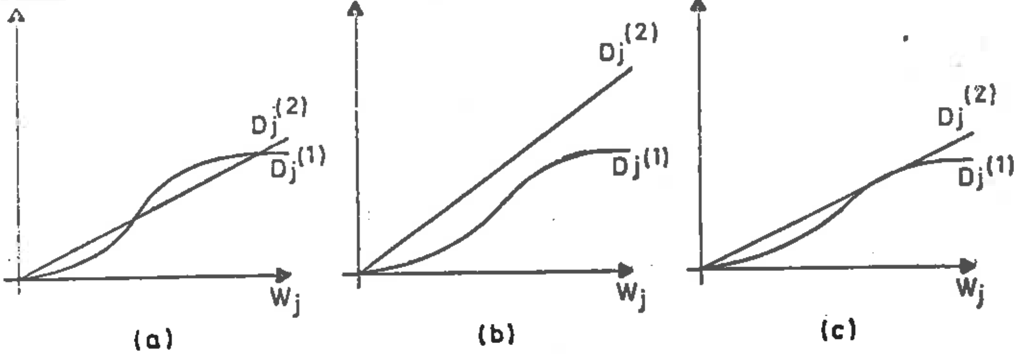

Figure 2

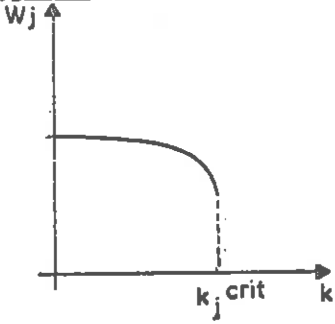

Figure 3

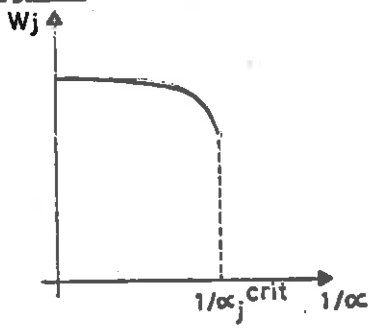

Figure 4

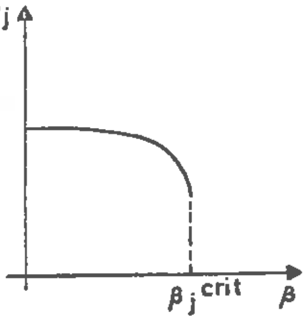

Figure 5