WORKING PAPER 266

CALCULATION OF THE EQUILIBRIUM CONFIGURATION OF SHOPPING FACILITY SIZES

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ABSTRACT

This paper presents a range of methods for computing the equilibrium configuration of shopping facility sizes. First, the presently available range of quasi-balancing factor methods is considered and built into a theoretical framework in which further algorithms may be defined. Then we consider the use of the gradient method, which is a general method of solution of non-linear equations. Finally, it is shown that a special case exists in which a closed form solution may be obtained.

1. INTRODUCTION

The dynamic shopping model equations describe the manner in which producers model their retail facilities' stock given that consumers allocate themselves to shopping centres according to a spatial interaction model. The dynamic shopping model, developed by Wilson (1976, 1979) building on the work of Huff (1964), Lakshmanan and Hansen (1965), takes the form

$$\dot{W}_j(t) = \sum_{i=1}^{n} S_{ij}(t) - k_j(t)W_j(t) \qquad (1)$$

where

$$S_{ij}(t) \equiv \frac{P_i(t)W_j^{\alpha(t)}(t)e^{-\beta(t)c_{ij}(t)}}{\sum_{j=1}^{N} W_j^{\alpha(t)}(t)e^{-\beta(t)c_{ij}(t)}} \qquad t\epsilon[0,T]. \qquad (2)$$

The variables are defined as follows:

$S_{ij}(t)$ is the number of shopping trips or cash flow between zone i and zone j;

$P_i(t)$ is the amount of cash spent on shopping goods by residents of zone i;

$W_j(t)$ is the size of the shopping centre in zone j taken as a measure of attractiveness;

$c_{ij}(t)$ is a general measure of impedance to travel between zones i and j which may be taken as actual distance, travel time, transportation costs, or some weighted combination of such factors;

$\alpha(t)$ is related to the consumers' perception of benefits of shopping centre size;

$\beta(t)$ is an elasticity parameter denoting ease of travel.

The argument t has been added to emphasise the dependence of the variables on time, and [0,T] is the period of study. The system comprises of N shopping centres and N residential zones and the 'rest of the world' is the environment. We assume that no changes occur in the environment that affect the system and so the parameters $P_i(t)$, $\alpha(t)$, $\beta(t)$, $c_{ij}(t)$ and $k_j(t)$ are constants. The shopping centre sizes $\{W_j(t)\}$ are the state variables and activities are carried out in or on structures at those locations.

Harris and Wilson (1978) suggested that producers may behave so as to ensure that the capacity provided $(\{W_j(t)\})$ balances the revenue generated $(\sum_{i=1}^{N} S_{ij}(t))$. Thus, if the more general $k_j(t)$ is replaced by k, where k is a constant which converts facility size units into money units, the balancing mechanism can be interpreted as profit maximising. This is the case when

$$\sum_{i=1}^{N} S_{ij}(t) = kW_j(t) \tag{3}$$

Thus, in equation (1), if $k_j(t) = k$, $j = 1,2, \ldots, N$, $t \varepsilon [0,T]$, the equilibrium condition

$$\dot{W}_j(t) = 0 \tag{4}$$

represents the configuration of shopping facility sizes for which producers' profits are maximised. For a system with N zones, it can be shown that there are 2^N-1 possible equilibrium configurations for which some shopping centres attract no sales. There is at most one configuration for which shopping centres coexist in equilibrium, and it is this configuration that the methods presented in this paper seek to identify. This configuration will hereafter be referred to as the positive $(W_j>0)$ equilibrium solution (or point).

Eilon, Fowler and Tilley (1969) considered the case $\alpha = 1$ and presented a quasi-balancing factor method for computing positive equilibrium points. This method has also been successfully used by White (1977) as a basis for modelling time development by assuming that the iterates generated may represent points on a trajectory in phase space. Harris and Wilson (1978) have shown that the quasi-balancing factor method is applicable to any α in general, and for the case $\alpha = 1$, they presented a matrix method that yields solutions in closed form. The aim of this paper is to present a range of methods for computing positive equilibrium points. First, the quasi-balancing factor methods considered by Eilon, Fowler and Tilley (1969) and Harris and Wilson (1978) are studied and built into a theoretical framework in which, it will be shown, further algorithms may be defined. Thus, in section 2, we consider the mechanism underlying quasi-balancing factor methods, the algorithms themselves being presented in section 3. A digression is made in section 4 to consider the conditions that must be satisfied for positive equilibrium solutions to exist. In particular, it is shown that solutions may be expected to exist if β is large enough. In section 5 we consider the use of the gradient method. This is a general method of solution of systems of non-linear equations and is applicable to any α in general. The iterations proceed by identifying the steepest descent vector along which the gradient of the system equations is minimum. Section 6 is a study of the special case $\alpha = \frac{1}{2}$ for which a closed form solution may be obtained using a similarity transformation. This method is applicable only in the two zone case.

In this paper a two zone example is used as a background against which to illustrate the arguments. The computational methods themselves are applicable in higher dimensions, except the method for computing the equilibrium configuration using a similarity transformation.

2. THE MECHANISM UNDERLYING QUASI-BALANCING FACTOR METHODS

2.1 Basic theory

Given an initial set of trial solutions, quasi-balancing factor methods generate a sequence of points that may converge to the solution of the system equations. Convergence occurs if the algorithm is well posed; it diverges otherwise. This concept is made explicit by considering the solution to the equation

$$y^{\phi_1+\phi_2} = 1 \,, \tag{5}$$

using the iterative scheme

(i) $y^{(0)} = c$, c given. Set $m = 0$.

(ii) $x^{(m)} = 1/y^{\phi_2}$

(iii) $y^{(m+1)} = (x^{(m)})^{1/\phi_1}$

(iv) If $|y^{(m+1)}-y^{(m)}| < \varepsilon$, where ε is a predetermined accuracy limit, go to step (v), else $m := m+1$ and go to step (ii).

(v) End of iteration.

This iterative scheme is essentially a reduction of equation (5) into two constituent equations that define the algorithm. These are $x = 1/y^{\phi_2}$ and $y = x^{1/\phi_1}$. This is akin to the reduction of higher order ordinary differential equations to systems of first order ordinary differential equations. The manner in which the reduced equations relate to each other as the iteration proceeds governs the convergence of the iteration they define.

<u>Table 1.</u> Tableau to the solution of equation (5).

m	Trial Solution	$1/y^{(m)}$	$x^{(m)}$
0	c	$c^{-\phi_2}$	$c^{-\phi_2}$
1	$c^{-\phi_2/\phi_1}$	$c^{\phi_2^2/\phi_1}$	$c^{\phi_2^2/\phi_1}$
2	$c^{\phi_2^2/\phi_1^2}$	$c^{-\phi_2^3/\phi_1^2}$	$c^{-\phi_2^3/\phi_1^2}$
3	$c^{-\phi_2^3/\phi_1^3}$	$c^{\phi_2^4/\phi_1^3}$	$c^{\phi_2^4/\phi_1^3}$
4	$c^{\phi_2^4/\phi_1^4}$	$c^{-\phi_2^5/\phi_1^4}$	$c^{-\phi_2^5/\phi_1^4}$
⋮			

Table 1 shows the construction of the sequence of trial solutions for this iterative scheme. Consider $\{c^{(-\phi_2/\phi_1)^m}\}$, the sequence of trial solutions generated. The iterates converge when

$$|\phi_1| > |\phi_2| , \tag{6}$$

and this inequality is the necessary condition for the iterative scheme to converge. If this criterion does not hold, the terms $(-\phi_2/\phi_1)^m$ increase rapidly with m, the iteration number, and hence, because of the alternation in sign, the successive trial solutions move farther apart. Figures 1 and 2 have been drawn to illustrate, respectively, the divergence and convergence of the iterative scheme when $\phi_1+\phi_2 = 3$ using $y = .5$ as the trial solution. In Figure 1, $\phi_1 = 1$, $\phi_2 = 2$, which do not satisfy condition (6) and hence the iterative scheme diverges. In Figure 2 $\phi_1 = 2$ and $\phi_1 = 1$ and the iterative scheme converges.

2.2 Rate of convergence

Equations (1) and (2) suggest that different algorithms can be defined to compute the equilibrium configuration of shopping facility sizes whose range of convergence is determined by α. Given a value of α, we wish to compare the rate of convergence of two convergent iterative schemes. The rate of convergence is defined by the number of iterations required for the iterates to fall within prescribed limits of the true solution. Let ϕ_1 and ϕ_2 define a convergent iterative scheme (A) where $\phi_1+\phi_2$ satisfies equation (5), and let ϕ_1' and ϕ_2' be two numbers such that

$$\phi_1'+\phi_2' = \phi_1+\phi_2 \tag{7}$$

with

$$|\phi_1'| > |\phi_2'| . \tag{8}$$

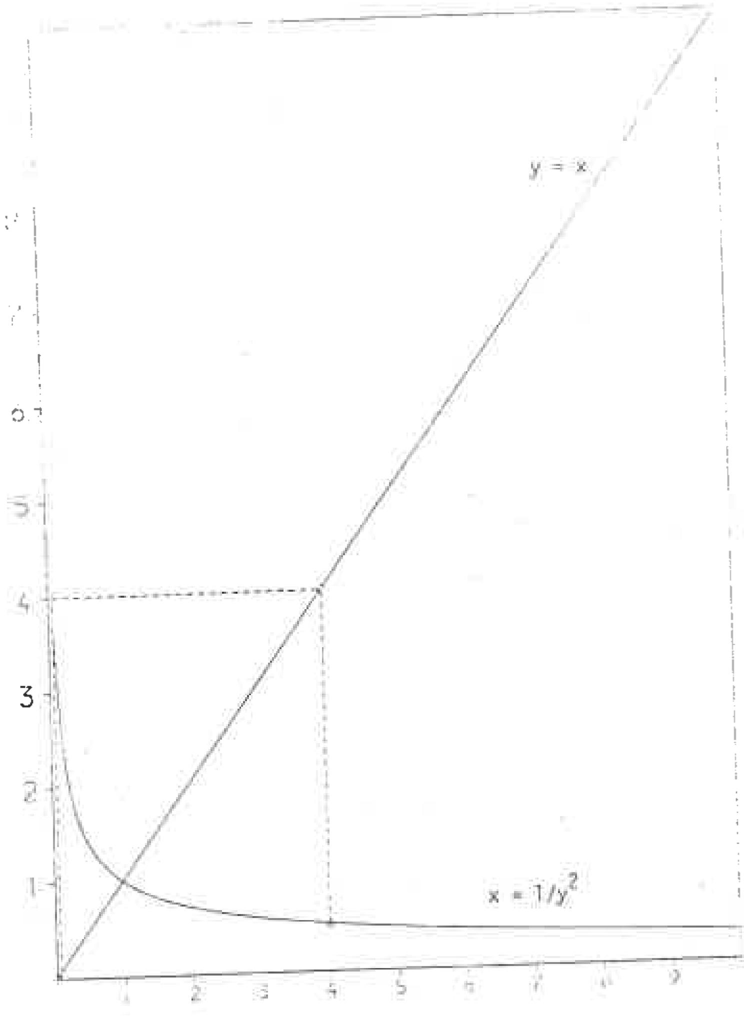
Then ϕ_1' and ϕ_2' may also be used to define a convergent iterative scheme (B) to equation (5). A comparison of the convergence rates of the two iterative schemes (A) and (B) is made by referring to table 1. Let m_1 and m_2, respectively, be the number of iterations required for iterative schemes (A) and (B) to fall within some prescribed limits of the true solution. Then,

$$c^{|(-\phi_2/\phi_1)^{m_1}|} = c^{|(-\phi_2'/\phi_1')^{m_2}|} \tag{9}$$

from which we obtain

$$\frac{m_1}{m_2} = \frac{\ln\left|\frac{\phi_2'}{\phi_1'}\right|}{\ln\left|\frac{\phi_2}{\phi_1}\right|} \tag{10}$$

Equation (10) is the comparison test for the rate of convergence between iterative schemes (A) and (B). It will be used to compare convergence rates of computational algorithms defined in section 3. The theory may be checked using the actual number of iterations taken for the iterates to fall within specified limits of the true solution.



<u>Figure 1.</u> <u>Divergence of iterative scheme 1</u>

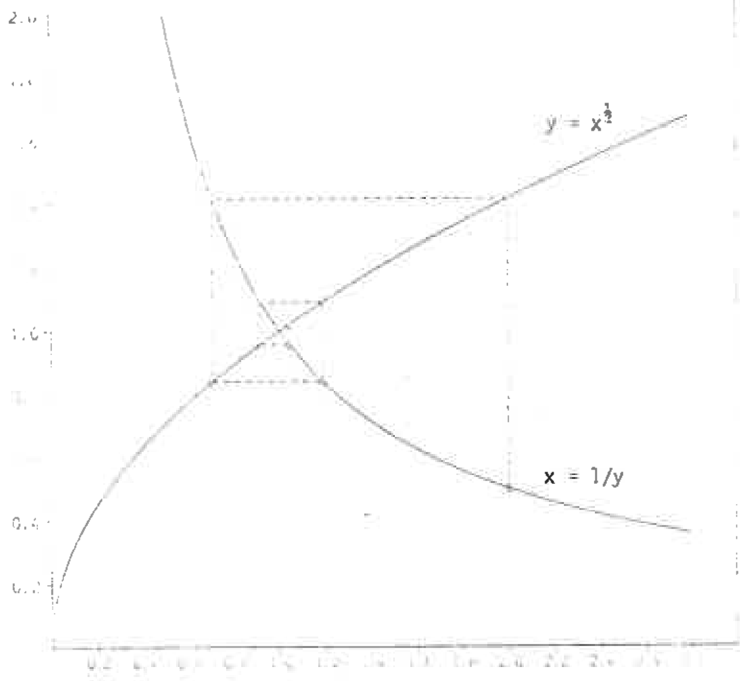


Figure 2. Convergence of iterative scheme 2

3. COMPUTATIONAL ALGORITHMS

At a given time t, the conditions at equilibrium are determined from equations (1), (2) and (3). They are given by

$$k_j W_j = \sum_{i=1}^{N} \frac{P_i W_j^{\alpha} x_{ij}}{\sum_{j=1}^{N} W_j^{\alpha} x_{ij}} \tag{11}$$

where

$$x_{ij} = \exp(-\beta c_{ij}) , \tag{12}$$

and the argument t has been omitted. For a system with N zones it can be shown that there are 2^N-1 possible equilibrium configurations of shopping facility sizes for which some of the shopping centres attract no sales. When N=2 two of these are

$$W_1 = 0 , \qquad W_2 = (P_1+P_2)/k_2, \tag{13}$$

$$W_1 = (P_1+P_2)/k_1, \quad W_2 = 0 . \tag{14}$$

The third configuration is that for which shopping centres coexist in equilibrium. The computational algorithms to be presented seek to identify this fixed point. Three different algorithms are presented for the solution of the system of equations (11). The algorithms are defined against the background of the criterion for the convergence of iterative schemes presented in section 2. The theory is tested by application to the two zone case whose data is given in table 2. Computations are performed for values of α such

Table 2. Data for two zone example

i,j	1	2
P_i	18,138	98,862
k_j	4	4
c_{ij}	11.63	12.10
	18.09	14.31
β = .5		

that $\alpha\varepsilon[-1.75, 1.75]$. Although negative values of α are not geographically meaningful, they have been included here for mathematical completeness. It is shown that each algorithm yields solutions in the range of α values for which it is postulated to converge, and that equation (10) is a valid comparison test for the rate of convergence of the computational algorithms. In all cases the solutions are multiplied by k_j to ensure that the conservation equation

$$\sum_{j=1}^{N} k_j W_j = \sum_{i=1}^{N} P_i \tag{15}$$

is satisfied. This equation is obtained by summing equation (11) over j.

<u>Method (a)</u>. The first computational algorithm is obtained by solving equation (11) iteratively. The equations that define the algorithm take the form

$$W_j^{(m)} = \frac{1}{k_j} \sum_{i=1}^{N} \frac{P_i W_j^{(m-1)^{\alpha}} x_{ij}}{\sum_{j=1}^{N} W_j^{(m-1)^{\alpha}} x_{ij}} \tag{16}$$

This is the algorithm considered by Eilon, Fowler and Tilley (1969) for $\alpha = 1$ and Harris and Wilson (1978) for general α. By applying inequality (6), the necessary condition for an iterative scheme to converge, the algorithm defined by (16) would be expected to converge when α lies in the range

$$-1 < \alpha < 1 . \tag{17}$$

The results of the computations are shown in table 3; they confirm postulate (6). Note that for the data given in table 2 no positive equilibrium solutions exist when $\alpha > 1$ for this particular value of β. The algorithm yields solution (13) in all cases. This implies that there is an implicit non-negativity constraint on the algorithm. In section 4 below, we will discuss conditions for the existence of positive equilibrium solutions. In particular, it will be shown that these solutions can be obtained if β is large enough. Figure 3 shows the results obtained when computations are performed to define regions in α-β space in which positive equilibrium solutions exist.

TABLE 3

Equilibrium values using Method (a)

Starting values: $(W_1^{(o)}, W_2^{(o)}) = (10^5, 1.7\times10^4)$

α	W_1	W_2	$\Sigma_j k_j W_j$	No. of iterations
-1.75	D	D		
-1.5	D	D		
-1.25	D	D		
-1	9.3337×10^3	1.9916×10^4	1.16999×10^5	144
-.75	8.7176×10^3	2.0532×10^4	1.16998×10^5	49
-.5	7.9554×10^3	2.1295×10^4	1.17001×10^5	26
-.25	6.9974×10^3	2.2253×10^4	1.17002×10^5	14
0	5.7762×10^3	2.3474×10^4	1.17001×10^5	1
.25	4.2090×10^3	2.5041×10^4	1.17000×10^5	14
.5	2.2421×10^3	2.7008×10^4	1.17000×10^5	26
.75	2.9855×10^2	2.8951×10^4	1.16998×10^5	58
1	0	2.9250×10^4	1.17000×10^5	21
1.25	0	2.9250×10^4	1.17000×10^5	11
1.5	0	2.9250×10^4	1.17000×10^5	8
1.75	0	2.9250×10^4	1.17000×10^5	8

D = divergence

A question that arises from the results shown in table 3 is whether there exists a discontinuity in equilibrium values in the range $\alpha\varepsilon[.75,1]$. The results shown in table 4 show that the approach is smooth and no discontinuity exists. In table 5 we present the results obtained in the range $\alpha\varepsilon[-1.2,-.75]$. The number of iterations required to reach the equilibrium values approaches infinity between $\alpha = -1.1$ and $\alpha = -1.075$. This result is not in contrast with the postulated ranges of convergence because the theory implicitly assumes that exact solutions are obtained whereas the computations were performed to an accuracy limit of 10^{-6}.

<u>Method (b)</u>. Method (b) is defined by writing iterative scheme (16) in the form

$$W_j^{(m)^{1-\alpha}} = \frac{1}{k_j} \sum_{i=1}^{N} \frac{P_i x_{ij}}{\sum_{j=1}^{N} W_j^{(m-1)^{\alpha}} x_{ij}} . \qquad (18)$$

Using the condition (6) this algorithm would be expected to converge for values of α that satisfy

$$|1-\alpha| > |\alpha| . \qquad (19)$$

Inequality (19) implies that Method (b) converges when α satisfies

$$\alpha < \tfrac{1}{2} . \qquad (20)$$

This is confirmed by the results obtained that are shown in table 6. They agree with those obtained using Method (a) in the range of α values in which both algorithms converge. When $\alpha = \frac{1}{2}$ the iterates oscillate and no equilibrium point is found. The oscillations are shown in figure 4. Using the values of the iterates obtained for $\alpha = .25$, figure 5 is drawn to illustrate the convergence of this method.

The rates of convergence of Methods (a) and (b) can be compared using equation (10). Table 7 considers only those values of α for which both methods yield solutions. When $\alpha = -1$, $m_1/m_2 = \infty$, an indication that Method (b) has a comparatively much higher rate of convergence, for this value of α, than Method (a). When $\alpha = 0$, $|\alpha| = \frac{|\alpha|}{|\alpha-1|}$ which should give $m_1/m_2 = 1$, in agreement with the

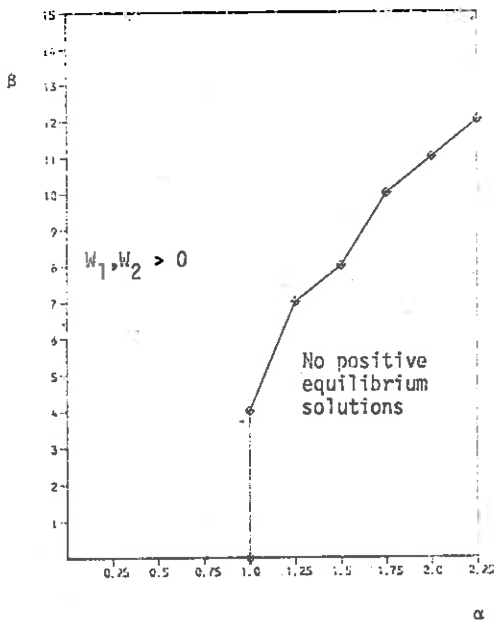


Figure 3. The existence of positive equilibrium solutions for varying α and β.

TABLE 4

Equilibrium values using Method (a) for $\alpha\varepsilon[.75,1]$

Starting values: $(W_1^{(o)}, W_2^{(o)}) = (10^5, 1.7\times10^4)$

α	W_1	W_2	$\Sigma_j k_j W_j$	No. of iterations
.75	2.9855×10^2	2.8951×10^4	1.16998×10^5	58
.775	1.9516×10^2	2.9065×10^4	1.17000×10^5	65
.8	1.0093×10^2	2.9149×10^4	1.17000×10^5	72
.825	4.5771×10^1	2.9204×10^4	1.16999×10^5	80
.85	1.5782×10^1	2.9234×10^4	1.16999×10^5	89
.875	3.5242	2.9246×10^4	1.16998×10^5	97
.9	3.6998×10^{-1}	2.9250×10^4	1.17001×10^5	102
.925	8.7520×10^{-3}	2.9250×10^4	1.17000×10^5	89
.95	6.9541×10^{-5}	2.9250×10^4	1.17000×10^5	42
.975	1.5456×10^{-5}	2.9250×10^4	1.17000×10^5	26
1	0	2.9250×10^4	1.17000×10^5	21

TABLE 5

Equilibrium values using Method (a) for $\alpha \varepsilon [-1.2, -.75]$

Starting values: $(W_1^{(o)}, W_2^{(o)}) = (10^5, 1.7\text{x}10^4)$

α	W_1	W_2	$\Sigma_j k_j W_j$	No. of iterations
-1.2	D	D		
-1.175	D	D		
-1.15	D	D		
-1.125	D	D		
-1.1	D	D		
* -1.075	$9.4957\text{x}10^3$	$1.9754\text{x}10^4$	$1.16998\text{x}10^5$	>10000
-1.05	$9.4427\text{x}10^3$	$1.9807\text{x}10^4$	$1.16998\text{x}10^5$	214
-1.025	$9.3887\text{x}10^3$	$1.9861\text{x}10^4$	$1.16998\text{x}10^5$	172
-1	$9.3337\text{x}10^3$	$1.9916\text{x}10^4$	$1.16999\text{x}10^5$	144
- .975	$9.2775\text{x}10^3$	$1.9972\text{x}10^4$	$1.16998\text{x}10^5$	124
- .95	$9.2203\text{x}10^3$	$2.0030\text{x}10^4$	$1.17001\text{x}10^5$	108
- .925	$9.1618\text{x}10^3$	$2.0088\text{x}10^4$	$1.16999\text{x}10^5$	95
- .9	$9.1022\text{x}10^3$	$2.0148\text{x}10^4$	$1.17001\text{x}10^5$	85
- .875	$9.0414\text{x}10^3$	$2.0209\text{x}10^4$	$1.17002\text{x}10^5$	76
- .85	$8.9793\text{x}10^3$	$2.0271\text{x}10^{44}$	$1.17001\text{x}10^5$	70
- .825	$8.9159\text{x}10^3$	$2.0334\text{x}10^4$	$1.17000\text{x}10^5$	63
- .8	$8.8512\text{x}10^3$	$2.0399\text{x}10^4$	$1.17001\text{x}10^5$	58
- .775	$8.7851\text{x}10^3$	$2.0465\text{x}10^4$	$1.17000\text{x}10^5$	54
- .75	$8.7176\text{x}10^3$	$2.0532\text{x}10^4$	$1.16998\text{x}10^5$	49

D $\equiv$ Divergence

* $\equiv$ Up to 10000 iterations attempted

TABLE 6

Equilibrium values using Method (b)

Starting values: $(W_1^{(o)}, W_2^{(o)}) = (10^5, 1.7 \times 10^4)$

α	W_1	W_2	$\Sigma_j k_j W_j$	No. of iterations
-1.75	1.0617×10^4	1.8634×10^4	1.17000×10^5	41
-1.5	1.0261×10^4	1.8989×10^4	1.17000×10^5	37
-1.25	9.8395×10^3	1.9410×10^4	1.16999×10^5	32
-1	9.3337×10^3	1.9916×10^4	1.16999×10^5	27
-.75	8.7176×10^3	2.0532×10^4	1.16998×10^5	23
-.5	7.9554×10^3	2.1295×10^4	1.17001×10^5	18
-.25	6.9974×10^3	2.2253×10^4	1.17002×10^5	12
0	5.7762×10^3	2.3474×10^4	1.17001×10^5	1
.25	4.2090×10^3	2.5041×10^4	1.17000×10^5	18
* .5	1.7305×10^3	2.0845×10^4		
	2.9051×10^3	3.4994×10^4		
.75	D	D		
1	D	D		
1.25	D	D		
1.5	D	D		
1.75	D	D		

D ≡ Divergence

* ≡ Iteration oscillates. Not an equilibrium point.

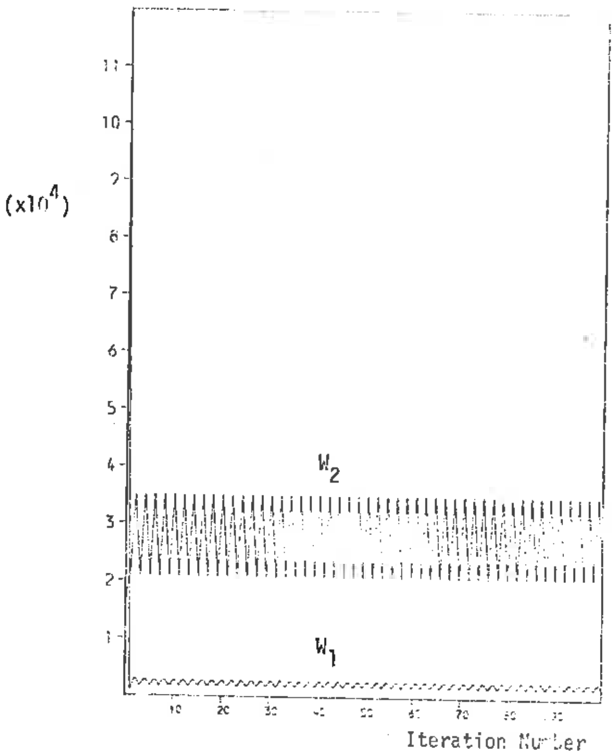


Figure 4. Oscillations in Method (b) when α=.5.

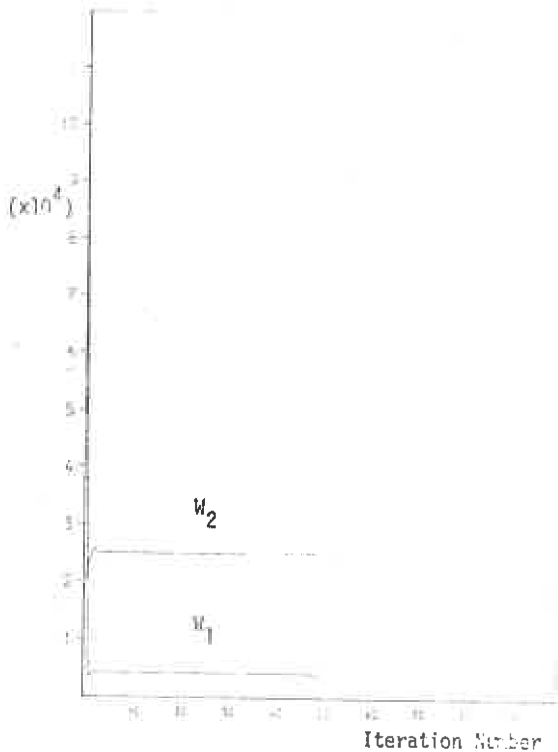


Figure 5. Convergence of Method (b) when α=.[illegible]

Table 7. A comparison of the rates of convergence of Methods (a) and (b)
Accuracy limit $= 10^{-6}$

α	Method (a) $\ln\lvert\alpha\rvert$	Method (a) m_1	Method (b) $\ln\lvert\frac{\alpha}{\alpha-1}\rvert$	Method (b) m_2	m_1/m_2 (From iteration)	m_1/m_2 (From theory)
-1	0	144	-.6931	27	5.333	∞
-.75	-.2877	49	-.8473	23	2.130	2.945
-.5	-.6931	26	-1.0986	18	1.444	1.585
-.25	-1.3863	14	-1.6094	12	1.167	1.161
0	$-\infty$	1	$-\infty$	1	1	Indeterminate
.25	-1.3863	14	-1.0986	18	.778	.793

value of m_1/m_2 obtained by iteration. Since $\alpha = 0$ the result is indeterminate. It is conceivable that if solutions are computed to a much higher accuracy limit than 10^{-6} the values of m_1/m_2 in the last two columns would be in better agreement. This follows because, since Method (a) has a slower rate of convergence, an increase in the accuracy would increase m_1 much more than m_2 (from iteration) with the effect that m_1/m_2 in the penultimate column would approach m_1/m_2 in the last column.

Method (c). The matrix method.

Since Method (a) converges when $-1 < \alpha < 1$ and Method (b) converges when $\alpha < \frac{1}{2}$, we seek an algorithm that complements these two methods in the range of convergence of α values such that the three methods together cover the whole range of real values of α. A matrix inversion procedure enables the matrix method to achieve this. This is the method presented by Harris and Wilson (1978) for $\alpha = 1$ when closed form solutions may be obtained. It is presented here for general α, and it is shown that it converges when $\alpha > \frac{1}{2}$.

Let

$$n_i = \frac{P_i}{\sum_{j=1}^{N} W_j^{\alpha} x_{ij}} \tag{21}$$

so that equation (11) becomes

$$\sum_{i=1}^{N} n_i x_{ij} = k_j W_j^{1-\alpha} , \qquad (22)$$

or, in matrix notation,

$$NX = C , \qquad (23)$$

where

$$N = \{n_i\} , \quad X = \{x_{ij}\} ,$$

$$C = \{k_j W_j^{1-\alpha}\} \qquad (24)$$

Then from equation (23)

$$N = CX^{-1} , \qquad (25)$$

and from equation (21),

$$W^{\alpha}X = V , \qquad (26)$$

where

$$W^{\alpha} = \{W_j^{\alpha}\} , \quad V = \{\frac{P_i}{n_i}\} . \qquad (27)$$

Equation (26) is solved to obtain the iterative scheme

$$W^{\alpha}_{(m)} = V^{(m-1)}X^{-1} . \qquad (28)$$

Equations (25) and (28) together with the definitions of the matrices C and V in equations (24) and (27), respectively, show that the factors $k_j W_j^{1-\alpha}$, $j = 1,2, \ldots, N$, appear only in the denominator of the right hand side of equation (28). Thus, the iterative scheme (28) converges when $|\alpha| > |1-\alpha|$, or,

$$\alpha > \tfrac{1}{2} . \qquad (29)$$

Also note that when $\alpha = 1$, $W_j^{1-\alpha} = 1$, $j = 1,2, \ldots, N$, so that a closed form solution is obtained without need for iteration. The matrix method Harris and Wilson (1978) refer to is this special case. In general, however, the matrix method is applicable to any α via the iterative scheme (28), but it may yield solutions only if β is large. This is shown to be the case by applying the method to the two zone case considered earlier. In this case, equation (25) is given by

$$\begin{aligned} n_1 &= (k_1 x_{22} W_1^{1-\alpha} - k_2 x_{21} W_2^{1-\alpha})/J \\ n_2 &= (k_2 x_{11} W_2^{1-\alpha} - k_1 x_{12} W_1^{1-\alpha})/J \;, \end{aligned} \tag{30}$$

where

$$J = x_{11}x_{22} - x_{12}x_{21} \tag{31}$$

The substitution (21) then gives

$$\begin{aligned} W_1^{\alpha} x_{11} + W_2^{\alpha} x_{12} &= \frac{P_1}{n_1} \;, \\ W_1^{\alpha} x_{21} + W_2^{\alpha} x_{22} &= \frac{P_2}{n_2} \;. \end{aligned} \tag{32}$$

Solving for W_1^{α} and W_2^{α} and substituting the expressions for n_1 and n_2 from equation (30) we obtain the iterative scheme:

$$\begin{aligned} W_1^{\alpha\,(m)} &= \frac{P_1}{k_1 W_1^{1-\alpha\,(m-1)} - k_2 z_2 W_2^{1-\alpha\,(m-1)}} - \frac{P_2}{k_2 z_1 W_2^{1-\alpha\,(m-1)} - k_1 W_1^{1-\alpha\,(m-1)}} \\ W_2^{\alpha\,(m)} &= \frac{P_2 z_1}{k_2 z_1 W_2^{1-\alpha\,(m-1)} - k_1 W_1^{1-\alpha\,(m-1)}} - \frac{P_1 z_2}{k_1 W_1^{1-\alpha\,(m-1)} - k_2 z_2 W_2^{1-\alpha\,(m-1)}} \end{aligned} \tag{33}$$

where

$$z_1 = x_{11}/x_{12} \;, \quad z_2 = x_{21}/x_{22}. \tag{34}$$

Note that while equations (33) are valid for all α, the iterative scheme converges when $\alpha > \frac{1}{2}$. It diverges otherwise. When $\alpha = 1$ the equilibrium values can be obtained without iteration. Figure 6 shows the results obtained for varying α and β using data given in table 2. The values obtained can be checked to be correct using Method (a). In general, the results show that as α increases the critical value of β for which the matrix method yields solutions also increases. In particular, note that for given α, as β increases, W_1 and W_2 approach limiting values. In this case

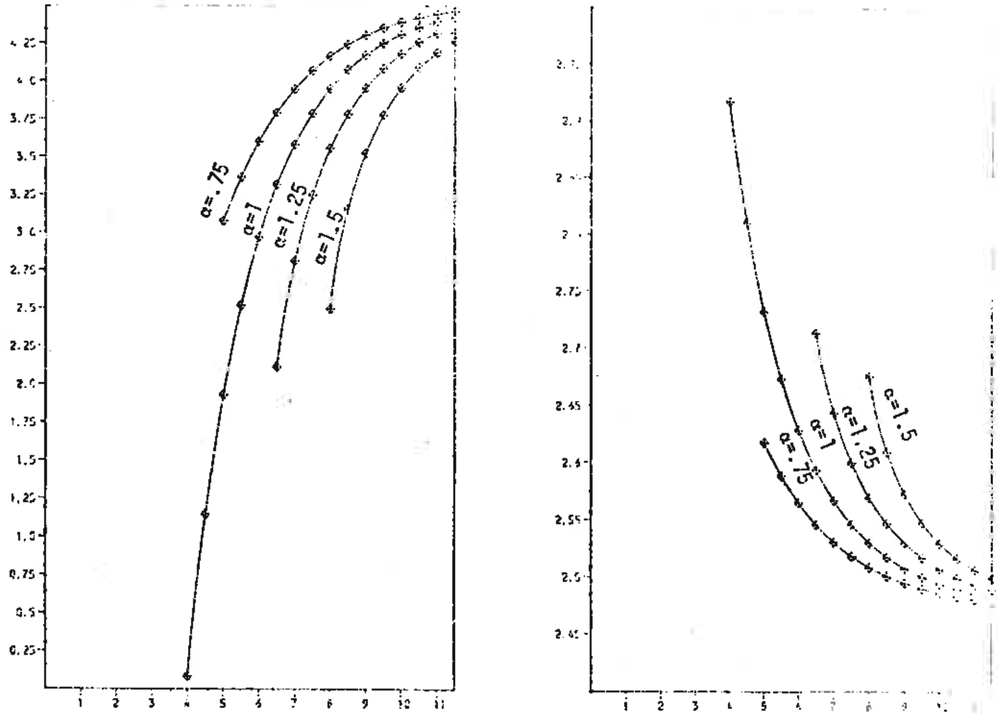


Figure 6. Equilibrium values of W_1 and W_2 for varying α and β using the matrix method.

Starting values in all cases: $(W_1^{(o)}, W_2^{(o)}) = (10^5, 1.7 \times 10^4)$

$$\lim_{\beta\to\infty} W_1(\beta) \to \frac{P_1}{k_1}, \qquad \lim_{\beta\to\infty} W_2(\beta) \to \frac{P_2}{k_2} \qquad (35)$$

where, from table 2, $\frac{P_1}{k_1} = 4.5345\times10^3$ and $\frac{P_2}{k_2} = 2.47155\times10^4$. The bearing of these results on the structure of settlement schemes is discussed in the next section.

4. CONDITIONS FOR THE EXISTENCE OF POSITIVE EQUILIBRIUM SOLUTIONS

The results of the computations performed in section 3 show that, for the data given in table 2, positive equilibrium solutions do not exist when $\alpha > 1$. In this section, we derive the conditions on parameter values that must be satisfied for positive equilibrium solutions to exist in the two zone case. The conditions are derived by re-writing equations (33) in the form

$$\begin{aligned} W_1 &= \frac{P_1}{k_1-k_2z_2(\frac{W_2}{W_1})^{1-\alpha}} - \frac{P_2}{k_2z_1(\frac{W_2}{W_1})^{1-\alpha}-k_1}, \\ W_2 &= \frac{P_2z_1}{k_2z_1-k_1(\frac{W_1}{W_2})^{1-\alpha}} - \frac{P_1z_2}{k_1(\frac{W_1}{W_2})^{1-\alpha}-k_2z_2}, \end{aligned} \qquad (36)$$

where z_1 and z_2 are defined in equation (34). The condition W_1, $W_2 > 0$ implies

$$\frac{k_2z_1z_2}{k_1(\tilde{P}_2z_1+\tilde{P}_1z_2)} < \left(\frac{W_1}{W_2}\right)^{1-\alpha} < \frac{k_2}{k_1}(\tilde{P}_2z_2+\tilde{P}_1z_1), \qquad (37)$$

where $\tilde{P}_i = P_i/\sum_{i=1}^{2} P_i$. Expression (37) defines the system of inequalities

$$P_2z_2+P_1z_1 > \frac{k_1}{k_2}\left(\frac{W_1}{W_2}\right)^{1-\alpha} \qquad (38)$$

$$\frac{P_2}{z_2}+\frac{P_1}{z_1} > \frac{k_2}{k_1}\left(\frac{W_2}{W_1}\right)^{1-\alpha} \qquad (39)$$

These are the conditions that must be satisfied for positive equilibrium solutions to exist. Figure 7 has been drawn to depict inequalities (38) and (39) in the z_1-z_2 plane using data given in table 2 when $\alpha = 1$. The annotation in the figure refers to general α. Both inequalities are satisfied in the regions labelled I and II. These regions correspond to large β as follows.

From equations (12) and (34)

$$\frac{z_2}{z_1} = e^{-\beta(c_{21}-c_{22}-c_{11}+c_{12})} \qquad (40)$$

where $\beta(c_{21}-c_{22}-c_{11}+c_{12}) \neq 0$. Suppose $c_{21}-c_{22} > c_{11}-c_{12}$, then

$$\lim_{\beta\to\infty} \frac{z_2}{z_1} \to 0, \qquad (41)$$

and an increase in β represents a shift in equilibrium points from neighbouring regions into region I. Now let $c_{21}-c_{22} < c_{11}-c_{12}$. Then

$$\lim_{\beta\to\infty} \frac{z_2}{z_1} \to \infty \qquad (42)$$

representing a shift of equilibrium points into Region II.

The data in table 2 gives $(\tilde{P}_1,\tilde{P}_2) = (.155, .845)$, $(z_1,z_2) = (.791, 6.619)$, and $k_1/k_2 = 1$, and, in the z_1-z_2 plane, the point (.791, 6.619) does not lie in either region I or II. Hence, when $\alpha = 1$, no positive equilibrium solutions exist.

The analysis in the z_1-z_2 plane has been performed to show that if positive equilibrium solutions do not exist, they may be obtained if β is large. In general, however, if the set of parameters appearing in inequalities (38) and (39) is given, it is sufficient to substitute the data in the expressions. Thus, data in table 2 satisfies (38) but not (39). For positive equilibrium solutions to exist, both conditions must be satisfied.

In geographical terms, moderate to large β values have implications on the structure of settlement schemes in that they tend to promote the prevalence of local facilities. In the limiting cases, for example expression (35), the equilibrium configuration of shopping facility sizes may be modelled to meet the demands only of the consumers in the same zone as the shopping centre. This result

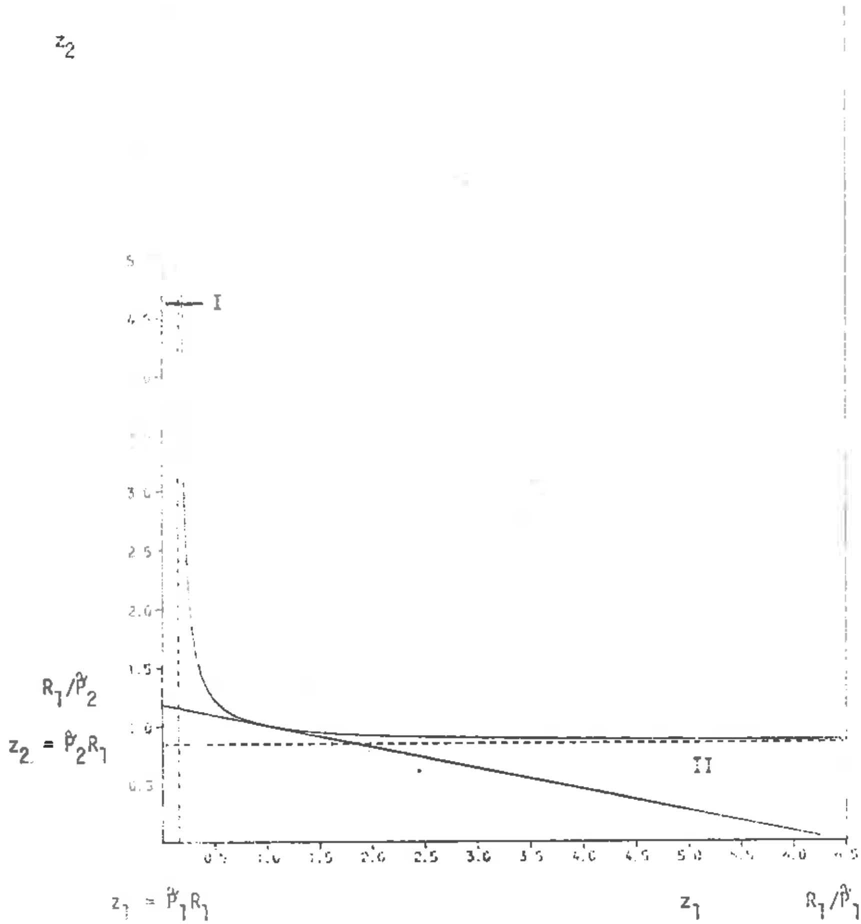


Figure 7. Inequalities (2.43) and (2.44) in the z_1-z_2 plane.

$$R_1 = \frac{k_1}{k_2}\left(\frac{W_1}{W_2}\right)^{1-\alpha}$$

imposes the discrete nature of the zoning methodology. For high values of β competition between shopping centres is non-existent. The disutility to travel by far outweighs the benefits from shopping centre size. Each zone reduces to a discrete entity and spatial interaction does not occur. The possibility of a change in the pattern of distribution of $\{W_j\}$ as β varies has also been reported by Wilson and Clarke (1978). In numerical experiments conducted to investigate the possible occurrence of catastrophe phenomena, they observed that increasing β had the effect of forcing more consumers to patronise local facilities.

5. THE GRADIENT METHOD

This is a general method of solution of systems of non-linear equations. It is applicable to any α in general. The iterations proceed by identifying a steepest descent vector towards which the gradient of the system equations in minimum. Recently, Dixon-Lewis and Greenberg (1975) used the method to determine high temperature equilibrium conditions in gas reactions.

The equilibrium conditions (11) are written as the system of non-linear equations

$$f_r(W_1,W_2, \ldots, W_N) = Q_r, \; r = 1,2, \ldots, N \qquad (43)$$

and denote by $f_{r,j}(W_1,W_2, \ldots W_N)$ the Jacobian of first partial derivatives $\delta f_r/\partial W_j$. Then in the neighbourhood of the point $(W_1,W_2, \ldots, W_N)$ the changes δf_r in the f_r induced by the changes δW_j in the W_j are given by

$$\delta f_r = \sum_{j=1}^{N} f_{r,j}(W_1,W_2, \ldots, W_N)\delta W_j \quad . \qquad (44)$$

If the point $(W_1^{(0)},W_2^{(0)}, \ldots, W_N^{(0)})$ is considered as the trial solution to (43), then the δf_r in (44) become

$$\sum_{j=1}^{N} f_{r,j}(W_1,W_2, \ldots, W_N)\delta W_j = Q_r - f_r(W_1^{(0)},W_2^{(0)}, \ldots, W_N^{(0)}) \qquad (45)$$

or, in matrix notation

$$J\delta W = B \, , \qquad (46)$$

where J is the Jacobian matrix whose elements are the first partial derivatives $f_{r,j}$. By making use of matrix inversion procedures the set of simultaneous equations (46) is solved to determine the correction vector W to be added to $W^{(0)}$ in order to determine the next approximation

$$W^{(1)} = W^{(0)} + \delta W \, . \qquad (47)$$

This constitutes the Newton-Raphson method for the solution of a system of non-linear equations. In the application of this method it is necessary that at each iteration improved estimates of the equilibrium point are obtained. In order to ensure this, Marquardt (1963) introduced an adjustable parameter ($\lambda^* > 0$) to the Newton Raphson method. The modified form of the method is

$$(\tilde{J}J + \lambda^* D)\delta W = -\tilde{J}B \, , \qquad (48)$$

where ~ stands for transpose, D is a diagonal matrix and $D_{\ell\ell} > 0$, $\ell = 1,2, \ldots, N$. The effect of introducing λ^* is to introduce an adjustable bias towards the steepest descent vector. Note that in this formulation, the Newton-Raphson method is the special case for which $\lambda^* = 0$. Marquardt's method is available as library procedures C05PAA (Algol) and C05PAF (Fortran) of the Numerical Algorithms Group. The inputs are the functions f_r and the Jacobian matrix $f_{r,j}$ given by

$$f_r = \sum_{i=1}^{N} \frac{P_i W_r^{\alpha} x_{ir}}{\sum_{j=1}^{N} W_j^{\alpha} x_{ij}} - k_r W_r \qquad (49)$$

and

$$f_{r,j} = \sum_{i=1}^{N} \frac{\left(\sum_{j=1}^{N} W_j^{\alpha} x_{ij}\right) \alpha x_{ir} P_i W_r^{\alpha-1} \delta_{rj} - \alpha P_i W_r^{\alpha} W_j^{\alpha-1} x_{ir} x_{ij}}{\left(\sum_{j=1}^{N} W_j^{\alpha} x_{ij}\right)^2} - k_r \delta_{rj} \qquad (50)$$

with $Q_r = 0$, $r = 1,2, \ldots, N$. δ_{rj} is the kronecker delta defined as

$$\delta_{rj} \begin{cases} = 1 , & r = j , \\ = 0 , & r \neq j . \end{cases} \tag{51}$$

The results obtained, shown in table 8, agree with those found using Methods (a) and (b) shown in tables 3 and 6, respectively. The sum of squares $\sum_r (f_r - Q_r)^2$ is shown to indicate how closed the final iterates are to the equilibrium point. In certain cases divergence occurs and it is necessary to experiment with different trial solutions before convergence is realised. In general, however, the need to start with initial values close to the solution accentuates the higher the dimensionality of the problem becomes.

6. THE USE OF A SIMILARITY TRANSFORMATION

For the case $\alpha = \frac{1}{2}$, in N dimensions, the equations that describe the equilibrium configuration are a pair of quadratic equations in $(W_1^{\frac{1}{2}}, W_2^{\frac{1}{2}}, \ldots, W_N^{\frac{1}{2}})$. Performing a similarity transformation on these equations casts them into a $\overline{W}_1^{\frac{1}{2}} \ldots \overline{W}_N^{\frac{1}{2}}$ frame of reference where the equations are linear in $\overline{W}_1 \ldots \overline{W}_N$. Thus, for the case N = 2, they can be solved for $\overline{W}_1$ and $\overline{W}_2$, and then transformed to the original frame of reference to obtain the equilibrium values of W_1 and W_2.

6.1 The simultaneous diagonalisation of two symmetric matrices

Let $W^{\frac{1}{2}}$ be an N-dimensional column vector, A and B N-dimensional symmetric matrices, and consider the quadratic forms

$$S_1 = \tilde{W}^{\frac{1}{2}} A W^{\frac{1}{2}} , \tag{52}$$

and

$$S_2 = \tilde{W}^{\frac{1}{2}} B W^{\frac{1}{2}} , \tag{53}$$

where S_1 and S_2 are scalars. We seek the transformation (Williams (1965), Hammarling (1970))

$$W^{\frac{1}{2}} = T\overline{W}^{\frac{1}{2}} , \tag{54}$$

such that the transformed matrices $\tilde{T}AT$ and $\tilde{T}BT$ are both diagonal, in which case each of the forms

TABLE 8

Equilibrium values obtained using the gradient method

α	W_1	W_2	$\sum_\gamma (f_\gamma - Q_\gamma)^2$
-1.75	10.6165×10^3	1.8634×10^4	9.1859×10^{-11}
-1.5	10.26088×10^3	1.8989×10^4	9.6634×10^{-11}
-1.25	9.8395×10^3	1.9410×10^4	3.6380×10^{-11}
-1	9.3337×10^3	1.9916×10^4	2.2737×10^{-13}
-.75	8.7176×10^3	2.0532×10^4	4.5475×10^{-12}
- .5	7.9554×10^3	2.1295×10^4	2.5790×10^{-10}
- .25	6.9974×10^3	2.2253×10^4	1.3512×10^{-10}
0	5.7762×10^3	2.3474×10^4	1.2557×10^{-10}
.25	4.2090×10^3	2.5041×10^4	3.8976×10^{-10}
.5	2.2421×10^3	2.7008×10^4	6.5847×10^{-10}
.75	2.9855×10^2	2.8951×10^4	5.3948×10^{-10}
1	0	2.9250×10^4	9.9713×10^{-10}
1.25	0	2.9250×10^4	8.1855×10^{-10}
1.5	0	2.9250×10^4	7.6489×10^{-10}
1.75	0	2.9250×10	1.4552×10^{-9}

$$S_1 = \tilde{W}^{\frac{1}{2}}\tilde{T}ATW^{\frac{1}{2}} \tag{55}$$

and

$$S_2 = \tilde{W}^{\frac{1}{2}}\tilde{T}BTW^{\frac{1}{2}} \tag{56}$$

is reduced to a linear equation.

Let λ_m and λ_r be two distinct roots associated with the polynomial equation

$$|A-\lambda B| = 0 \tag{57}$$

Then if L_m and L_r are the corresponding vectors, it can be shown (Heading (1958)) that

$$\tilde{L}_m B L_r = \tilde{L}_m A L_r = 0 \ , \tag{58}$$

and that

$$\tilde{L}_m A L_m = \lambda_m \tilde{L}_m B L_m \tag{59}$$

In the expressions (55) and (56) consider the matrices $\tilde{T}AT$ and $\tilde{T}BT$, where

$$\begin{aligned} T &= (L_1 \ \ L_2 \ \ldots \ L_N) \\ &= \begin{bmatrix} L_1^{(1)} & L_2^{(1)} & \ldots & L_N^{(1)} \\ L_1^{(2)} & L_2^{(2)} & \ldots & L_N^{(2)} \\ \vdots & \vdots & & \vdots \\ L_1^{(N)} & L_2^{(N)} & \ldots & L_N^{(N)} \end{bmatrix} \end{aligned} \tag{60}$$

is the matrix obtained by placing side by side the vectors of equation (57). Written out explicitly, using expressions (58) and (59), these matrices are

$$\tilde{T}AT = \begin{bmatrix} \lambda_1\tilde{L}_1BL_1 & 0 & \ldots & 0 \\ 0 & \lambda_2\tilde{L}_2BL_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n\tilde{L}_NBL_N \end{bmatrix} \tag{61}$$

and

$$\tilde{T}BT = \begin{bmatrix} \tilde{L}_1BL_1 & 0 & \dots & 0 \\ 0 & \tilde{L}_2BL_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \tilde{L}_NBL_N \end{bmatrix} \qquad (62)$$

The substitution of the diagonal forms (61) and (62) in expressions (55) and (56) yields a system of equations that is linear in $\overline{W}_1 \dots \overline{W}_N$, and is given by

$$S_1 = \sum_{r=1}^{N} \lambda_r \nu_r \overline{W}_r \;,$$
$$S_2 = \sum_{r=1}^{N} \nu_r \overline{W}_r \;, \qquad (63)$$

where

$$\nu_r = \tilde{L}_r B L_r \;. \qquad (64)$$

The system of equations (63) can be solved only if $N = 2$. In that case,

$$\overline{W}_1 = (\nu_2 S_1 - \lambda_2 \nu_2 S_2)/J_0 \;,$$
$$\overline{W}_2 = (\lambda_1 \nu_1 S_2 - \nu_1 S_1)/J_0 \qquad (65)$$

where

$$J_0 = \nu_1 \nu_2 (\lambda_1 - \lambda_2) \;. \qquad (66)$$

This solution is recast into the original frame of reference by making use of the transformation (54). The required equilibrium values are given by

$$W_1 = (L_1^{(1)} \overline{W}_1^{\frac{1}{2}} + L_2^{(1)} \overline{W}_2^{\frac{1}{2}})^2$$
$$W_2 = (L_1^{(2)} \overline{W}_1^{\frac{1}{2}} + L_2^{(2)} \overline{W}_2^{\frac{1}{2}})^2 \;. \qquad (67)$$

6.2 Application of method

From equation (11), for $\alpha = \frac{1}{2}$, the equilibrium conditions are given by

$$k_j W_j^{\frac{1}{2}} = \sum_{i=1}^{2} \frac{P_i x_{ij}}{\sum_{j=1}^{N} W_j^{\frac{1}{2}} x_{ij}} \qquad (68)$$

Let

$$n_i = \frac{P_i}{\sum_{j=1}^{2} W_j^{\frac{1}{2}} x_{ij}} . \tag{69}$$

Then

$$k_j W_j^{\frac{1}{2}} = \sum_{i=1}^{2} n_i x_{ij} , \tag{70}$$

which can be solved for n_i to give

$$\begin{aligned} n_1 &= (k_1 x_{22} W_1^{\frac{1}{2}} - k_2 x_{21} W_2^{\frac{1}{2}})/J \\ n_2 &= (k_2 x_{11} W_2^{\frac{1}{2}} - k_1 x_{12} W_1^{\frac{1}{2}})/J , \end{aligned} \tag{71}$$

where

$$J = x_{11} x_{22} - x_{12} x_{21} . \tag{72}$$

Eliminating n_1 and n_2 between equations (71) and (69), the resulting expression can be written in the form

$$\begin{aligned} S_1 &= \tilde{W}^{\frac{1}{2}} A W^{\frac{1}{2}} , \\ S_2 &= \tilde{W}^{\frac{1}{2}} B W^{\frac{1}{2}} , \end{aligned} \tag{73}$$

where

$$A = \begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix} , \quad B = \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix} , \tag{74}$$

and

$$\begin{aligned} a_1 &= k_1 x_{11} x_{22} , \\ b_1 &= \tfrac{1}{2}(k_1 x_{12} x_{22} - k_2 x_{11} x_{21}) , \\ c_1 &= -k_2 x_{12} x_{21} , \\ S_1 &= J P_1 , \\ a_2 &= -k_1 x_{12} x_{21} , \\ b_2 &= \tfrac{1}{2}(k_2 x_{11} x_{21} - k_1 x_{12} x_{22}) , \\ c_2 &= k_2 x_{11} x_{22} , \\ S_2 &= J P_2 , \\ \tilde{W}^{\frac{1}{2}} &= (W_1^{\frac{1}{2}} \; W_2^{\frac{1}{2}}) . \end{aligned} \tag{75}$$

For the values of the matrices A and B given in equation (74), the roots of equation (57) are

$$\lambda_{1,2} = \psi + \gamma, \quad \psi - \gamma , \tag{76}$$

where

$$\psi = \frac{(a_2c_1+a_1c_2-2b_1b_2)}{2(a_2c_2-b_2^2)} \quad , \tag{77}$$

$$\gamma = \frac{[(a_2c_1+a_1c_2-2b_1b_2)^2-4(a_2c_2-b_2^2)(a_1c_1-b_1)]^{\frac{1}{2}}}{2(a_2c_2-b_2^2)}$$

The column vectors corresponding to the roots $\lambda = \lambda_1$ and $\lambda = \lambda_2$ are, respectively

$$L_1 = \begin{bmatrix} L_1^{(1)} \\ L_2^{(2)} \end{bmatrix}, \quad L_2 = \begin{bmatrix} L_2^{(1)} \\ L_2^{(2)} \end{bmatrix} , \tag{78}$$

where

$$\begin{aligned} L_1^{(1)} &= -(b_1-\lambda_1b_2)/(a_1-\lambda_1a_2) \quad , \\ L_1^{(2)} &= 1 \quad , \\ L_2^{(1)} &= -(b_1-\lambda_2b_2)/(a_1-\lambda_2a_2) \quad , \\ L_2^{(2)} &= 1 \quad . \end{aligned} \tag{79}$$

These expressions are substituted into equations (64), (65), (66) and (67) in turn to obtain the required equilibrium values. The results of the computations are shown in table 9. The equilibrium point is the same as that shown in tables 3 and 8 obtained using Method (a) and the gradient algorithm, respectively, when $\alpha = \frac{1}{2}$. Thus, for $\alpha = \frac{1}{2}$, this method can be used as an alternative to Method (a) and the gradient method.

Table 9. Calculation of equilibrium point using a similarity transformation for data given in table 2.

m	1	2
λ_m	-5.30960	-0.18834
$L_1^{(m)}$	3.76647	1
$L_2^{(m)}$	-0.26550	1
ν_m	-2.89093×10^{-5}	1.08201×10^{-5}
$\overline{W}_m$	5.09207×10^{2}	2.01012×10^{4}
W_m	2.24214×10^{3}	2.70079×10^{4}

The successful implementation of this method is due to the structure of the equations that define the equilibrium conditions when $\alpha = \frac{1}{2}$. The substitution (69) transforms the equilibrium conditions into a system of equations that is quadratic in $(W_1^{\frac{1}{2}}, W_2^{\frac{1}{2}})$ from which the symmetric matrices A and B are defined. Obtaining the roots, and the corresponding vectors to equation (57), and substituting in equations (64)-(67) yields the desired results.

7. CONCLUSION

This paper has presented a range of methods for computing the equilibrium configuration of shopping facility sizes. In the first part of the paper (sections 2, 3 and 4), presently available methods have been studied and built into a new theoretical framework which, it has been shown, embodies further algorithms. Thus, all algorithms of the quasi-balancing factor family have been placed in their proper mathematical perspective. The latter part of the paper consists of methods whose theoretical basis lies in the field of Engineering Mathematics. The application of these methods was achieved because of the structure of the conditions that define the equilibrium configuration: the gradient method is, in general, applicable to any system of non-linear equations, and it is possible to use the similarity transformation because, for $\alpha = \frac{1}{2}$, the equilibrium conditions can be recast into a form suitable for the simultaneous diagonalisation of two symmetric matrices.

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