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MULTIPLE BIFURCATION EFFECTS WITH A LOGISTIC ATTRACTIVENESS FUNCTION IN THE SUPPLY SIDE OF A SERVICE SYSTEM

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1. Introduction

The standard Huff (1964) and Lakshmanan and Hansen (1965) shopping models have been used for some time now as a basis for exploring the dynamic nature of urban spatial structures (see Harris and Wilson, 1978; Clarke and Wilson, 1981, 1982-A; Clarke, Wilson & Birkin, 1984; Wilson, 1981-A).

The model can be stated as

$$S_{ij} = A_i e_i P_i W_j^{\alpha} e^{-\beta c_{ij}} \tag{1}$$

where

$$A_i = \frac{1}{\sum_k W_k^{\alpha} e^{-\beta c_{ik}}} \tag{2}$$

to ensure that

$$\sum_j S_{ij} = e_i P_i .$$

The variables are:

S_{ij} = the flow of consumer expenditure from i to j

e_i = the per capita expenditure in zone i

P_i = the population of zone i

W_j = the facility size taken as a measure of attractiveness

c_{ij} = travel cost from i to j (can be measured in units of time or distance or both)

In earlier research, Poston and Wilson (1977) (summarised in Wilson, 1981-B) showed a method of predicting facility size against average

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distance travelled. If per capita benefits to be gained on average increase with the size of the facility to be provided but decrease with the distance which has to be travelled to reach it, then two extreme solutions are possible: firstly, one large facility with people on average travelling a large distance to reach it and secondly, many small facilities with a short associated average distance of travel. In order to strike a balance between the benefits of size and the costs of travel it can be assumed that W (the average size of shopping centre) and c (the average cost of travel) are monotonically related and that there is a functional relationship between W and c: the greater c, the greater W. Poston and Wilson chose then to work with c as the endogenous state variable of the system and used a utility maximising hypothesis to plot the average benefit from facility size (u_1) in relation to the average cost of travel (u_2). They assumed that u_1 takes a logistic form, increasing to some maximum, whilst u_2 is backward-sloping and linear. Total utility (u) is the sum of u_1 and u_2. Figure 1 shows these simple assumptions about u_1 and u_2 whilst Figure 2 shows examples of total utility for different slopes of u_2.

The main exogenous variable for the system is characterised by the gradient of the travel line (u_2) which in this case is $-\beta$. This is taken as a measure of the 'ease of travel' at a particular time. Thus in Figure 2a the u_2 line is shallow which represents relatively easy travel. Here there is a clear non-zero optimal location (c^*). In Poston and Wilson's terminology this is interpreted as a 'supermarket situation', that is a situation where large shopping centres are present. In Figure 2b the u_2 line is far steeper representing more difficult travel. Here the optimal location has switched to the origin or zero. Poston and Wilson term this the 'corner shop solution' since there are many small shopping centres (rather than centres of zero size). c* can be plotted against β with the result as shown in Figure 3, which is an example of Thom's (1975) fold catastrophe.

The aim of this paper is to explore the dynamical analysis of retail structures in association with the Poston-Wilson hypothesis. The first step in the argument is to reveal a difficulty with W_j^α as an attractiveness function and to consider alternatives. A measure of benefit associated with the size of facilities can be obtained as follows.

Note that

$$W_j^\alpha e^{-\beta c_{ij}} = e^{\beta.(\frac{\alpha}{\beta}\log W_j - c_{ij})}$$

Since c_{ij} is a direct measure of cost u_2, this shows that a suitable definition of size benefit is $u_1 = \frac{\alpha}{\beta}\log W_j$. However, as noted in the Poston and Wilson (1977) paper, these definitions do not reproduce the desired hypothesis: log W_j tends to minus infinity when $W_j \to 0$, and so the local maximum at c* = 0 never develops. Consequently there is always a unique maximum which moves slowly as β changes smoothly. This is because the u_2 curve is not a logistic function (as pictured in Figure 1) with a point of inflexion (Wilson, 1981-B). Figure 4 shows the components of utility in the conventional model (from Wilson, 1981-B - note the ij's have been removed) and indicates how the argument breaks down.

The question which can now be asked is: does the real world exhibit such jump behaviour as implied by Figure 3? If it does then a new attractiveness function is required. Empirical evidence has been provided by Wilson and Oulton (1983), which, though somewhat tentative, seems to suggest that there were key or critical years where change from a corner shop based system to a supermarket dominated system was relatively rapid. It may be difficult however to distinguish the pattern change implied by the Poston and Wilson hypothesis from such changes arising from the W_j^α

model and changes in α or β. We pursue this again later in the light of our numerical experiments. First, we seek an alternative attractiveness function which will represent the Poston and Wilson hypothesis.

Poston and Wilson suggest that the W_j^α term could be replaced by a logistic attractiveness term as follows:

$$\frac{\delta}{[1 + \gamma \exp(-\varepsilon W_j)]}$$

where δ, γ and ε are constants to be determined. The interaction model then becomes

$$S_{ij} = A_i e_i P_i \exp\left(\frac{\delta}{[1 + \gamma \exp(-\varepsilon W_j)]} \right) \exp(-\beta c_{ij}) \qquad (3)$$

where

$$A_i = \frac{1}{\sum_j \exp(\delta/[1 + \gamma \exp(-\varepsilon W_j)]) \exp(-\beta c_{ij})} \qquad (4)$$

Using this new attraction term the argument can now be repeated with u_1 and u_2 becoming

$$u_1 = \frac{\delta}{[1 + \gamma \exp(-\varepsilon W)]} \qquad (5)$$

$$u_2 = -\beta C \text{ or } -\beta W \qquad (6)$$

(assuming a monotonic relationship)

We have dropped the i and j subscripts on the right hand side so that we can explore the functional form. However we should recall that they should be there; without them, we are working with some kind of average.

In Figure 5 we plot the new functions for variations in β. Before examining these plots, it is important to note two futher modifications to the logistic function which help to manipulate the shape of the functions. Using the u_1 function in equation 5, smooth changes in the β term produce only smooth changes in the overall system. This is because the point of inflexion associated with the logistic function occurs off the positive x and y coordinates of the graphs and hence the whole curve needs to be 'shifted' to the right. This is done by adding a constant (in this case taking a negative value) to the W_j term of the logistic function (Crouchley, 1983). Hence,

$$u_1 = \frac{\delta}{[1 + \gamma\exp((-\varepsilon(W+C)))]} \tag{5}$$

Similarly it is possible to set an upper and lower bound to the u_1 curve by adding a constant to the whole function. Thus, u_1 becomes,

$$\left(\frac{\delta}{[1 + \gamma\exp(-\varepsilon(W+C))]}\right) + K \tag{6}$$

Hence the upper bound becomes $\delta + K$ and K itself becomes the lower bound.

With these modifications made, Figure 5 shows the plots of the u_1, u_2 and u (total utility) curves for various values of β and specimen values of δ, γ, ε, C and K.

In Figure 5 (a - f) β can be seen steadily increasing, (that is the u_2 line becoming steeper) whilst u_1 is held constant. In Figure 5(a) there is a global maximum at W^* which is indeed a unique maximum. There are also signs of a local minimum forming which by Figure 5(b) has clearly developed (m). Also by Figure 5(b) W^* has become a local maximum and the global maximum has shifted to the origin or where $W = 0$. The local maximum and the unstable minimum gradually merge in plots c and d until in the latter they have coincided to form a point of inflexion. Figures (e) and (f) show this process continuing, so there is not even a local maximum for $W^* > 0$.

The interpretation of these β changes seems relatively straightforward. Low β means that travel is relatively easy and hence supermarkets or any other kind of large store more centrally located will be more attractive to consumers; high β means travel is more expensive or difficult and hence consumers will be more likely to shop locally. From an historical point of view therefore, Figure 5 should read from bottom to top. For example, higher rates of car ownership have made travelling easier to fewer larger facilities. Thus, again in Poston and Wilson terminology, Figure 5(a) represents a supermarket situation whilst Figure 5(f) represents the corner-shop solution.

The new attractiveness term also contains the parameters δ and ε and it is interesting to examine what happens to the u_1 and u_2 curves for smooth changes in these parameters. Figure 6 illustrates the changes for variations in epsilon (ε). This shows that decreasing ε causes a similar response to increasing β. That is, decreasing ε is more likely to cause a situation of small local centres whilst increasing ε generates a transition to larger stores dominating. As Figure 6 a - d show there is a change from a global maximum of W^* with a local minimum at m, to a global maximum at $W = 0$ with a shallower local minimum at m and a local maximum at W^* . By case e the two local turning points have coincided to form a point of inflexion. It is not difficult to imagine a plot of W optimum against ε looking similar to Figure 3, the fold catastrophe.

As Figure 6 also shows it is ε which governs the shape of the logistic curve: the speed with which δ or $\delta + K$ is reached (the latter being the upper limit of the logistic curve) is clearly controlled or governed by ε.

In Figure 7 progressively decreasing values of δ are plotted, keeping the remaining parameters constant as before. As the plots clearly show, δ governs the position of the upper bound relative to the lower. It can be seen that δ works in a similar direction to ε: decreasing δ leads to a situation of smaller more localised facilities and vice versa.

The effect of γ changes can be seen with the aid of a simple diagram (Figure 8 below). γ plays a similar role to ε, determining the pattern of the logistic curve in the sense that low γ means the curve rises quickly from $W = 0$ producing a long 'tailing off' effect whilst a high γ produces a far later rise and a much shorter 'tailing off'.

In the above discussion there are two possible stable equilibrium solutions, W^* is either large and positive or zero. We have interpreted $W = 0$ as the 'corner shop' solution, but it might be better to use a formulation where this is represented by a small non-zero value. This can be achieved by representing attractiveness as a double logistic function. This is shown in Figure 9, where we demonstrate the effects of changing β. As β increases the upper maxima of total utility, u, moves from the W^A to a new maxima W^B, Figure 9d, which represents a much smaller amount of retail facilities, until a high β value finally gives us the maximum utility at the origin W^O. Although the majority of results presented in the next section relate to the single logistic function we do explore the effects of using this different attractiveness function.

2. Numerical Experiments

So far we have seen something of how the parameters associated with the logistic function have operated in theoretical terms. That is, we have seen how increasing δ or ε or decreasing β can generate a sudden transition to a situation where large stores such as supermarkets could function. It is now useful to explore the kinds of spatial patterns which emerge when the actual interaction model is run and the various parameters are in turn, manipulated. We can then attempt to disentangle changes arising from the Poston-Wilson effect from others.

The model to be explored uses the logistic function given in equations (3) and (4). Because we are dealing principally with δ, γ, ε and β changes it should be remembered that a very large number of experiments are possible and hence only a selection can be provided here.

In order to examine the model we use a hypothetical spatial system constructed on a regular grid basis. A symmetrical 729-zone system was designed (as used extensively by Clarke, M.,1984-A, Clarke and Wilson, 1981 etc.). Within this grid (a 27 x 27 array of points) an approximate circle is defined delimiting zones considered endogenous. The population of each origin zone and the amount of expenditure available in each zone were considered to be uniform for our purposes (although in reality of course they would be far from uniform). Euclidean distance was used for calculating the cost matrix, that is,

$$c_{ij} = \sqrt{(x_i-x_j)^2 + (y_i-y_j)^2} \tag{7}$$

(where (x_i,y_i) are the Cartesian coordinates of zone i).

The results or plots presented in the spatial configurations consist of 3-D computer graphical output, with the x and y dimensions corresponding to the spatial system and the z dimension to the amount of retail facilities present in the zone.

Since most urban transport systems are designed to favour travel to the centre we also modify the cost to the centre by taking c(j, 365) = v x euclidian distance, where v takes a series of values less than or equal to one.

To determine the values of $\{W_j\}$ (since we are interested in the modelling of the structural variables which we take to be the W_js) the revenue cost balancing approach is used, first presented in Harris and Wilson (1978). (See also Wilson and Clarke, 1979, and Clarke, 1981, for further modifications).

First, define D_j as the revenue accruing to zone j, so that,

$$D_j = \sum_i S_{ij}. \qquad (8)$$

If we now define K_j as the unit cost of providing retail floorspace in zone j, then K_jW_j is the cost associated with a given shopping centre size. The Harris and Wilson argument was based on the assumption that the suppliers of retail facilities will expand their operation so long as revenue exceeds cost, i.e. if $(D_j-K_jW_j) > 0$, and contract if $(D_j-K_jW_j) < 0$. Therefore, the equilibrium conditions are taken as:

$$D_j = K_jW_j, \qquad (9)$$

where revenue exactly balances cost. Substituting from equations (3) (as modified by (6)), (4) and (8) into (9), we get

$$D_j = K_jW_j = \sum_i \frac{e_iP_i \exp\{\frac{\delta}{[1+\gamma\exp(-\epsilon W_j+C)]} + K\} \exp(-\beta c_{ij})}{\sum_j \exp\{\frac{\delta}{[1+\gamma\exp(-\epsilon W_j+C)]} + K\}\exp(-\beta c_{ij})} \qquad (10)$$

These equations are clearly highly non-linear and are made more complex by the fact that revenue in any one zone is a function of the W_js in all the other zones. It is the combinations of interdependence and particular non-linearities in these equations that gives rise to the interesting dynamic behaviour.

The supply costs for equation (10) are assumed to be constant (see Clarke (1984-A) for initial experiments with variable supply costs).

In Figure 10 we plot the effects of progressively decreasing values of β whilst keeping all other parameters constant. In order to do this a pivotal set of parameter values are used, around which all variations are subsequently made. The pivotal set are:-

$\beta = 2.5$

$\varepsilon = 0.001$

$\delta = 10.00$

$\gamma = 1.00$

$v = 0.85$

C and K = 0

Plot b of Figure 10 shows this pivotal set.

Figure 10 shows that the behaviour of the β parameter is very much as predicted from the analysis of Figure 5. That is, when β is high, much of the spatial system is close to some retail provision representing a situation of more local supply (Figure 10a for example). When β is low however, reflecting far easier travel, then facilities can become concentrated in the centre of the spatial system (Figure 10h for example). It seems that this notion of local and centralized supply is at the heart of the Poston and Wilson 'corner shop', 'supermarket' terminology, although this problem will be returned to later.

The variety of patterns in between the two extremes (a and h of Figure 10) show that discrete changes are occuring for seemingly smooth changes in β. This is indeed highlighted by Figure 11 where one example of a discrete change is plotted. Note how a very small change in the β parameter results in jump behaviour, and in this case, the loss of the inner ring of facilities.

Although discrete changes are discernible for various parameter values it is difficult from Figure 10 to point to one set of changes which could clearly represent a corner-shop to supermarket transition. Figures 10(c) and (d) (and the discrete jump seen in Figure 11) seem to reflect the most striking change of pattern yet it is difficult to interpret Figure 10(d) as a supermarket situation and Figure 10(c) as a corner shop situation.

The lack of such a clear cut pattern transition is perhaps not so surprising for changes in β. In real terms the β parameter, reflecting ease of travel, may not be the most important in determining such a pattern change. That is, whilst supermarkets and superstores are reliant more on a car-borne society (which is reflected in lower β values) they may be more responsive to changes in scale economies. For example, Guy (1980) explains the development of the supermarket through "allowing economies of scale through increased size and routinisation of day to day management".

One of the striking features of these plots, well demonstrated in Figure 10(h), is the plateau-like distribution of facilities. This type of distribution does not occur when W_j^α is used as the attractiveness factor (see Clarke and Wilson, 1983). The explanation for this result is quite straightforward but very interesting (see Clarke, M., 1984-B). The shape of the attractiveness function was shown in Figure 5 and is repeated in Figure 12. Above a certain point the slope of the curve flattens out towards the upper asymptote. The value at which this asymptote is reached is controlled by the parameter δ in Equation (3). Now for β change the following argument applies. Above a certain point, q_j, $\delta f(W_j)/\delta W_j$, becomes very small compared with below q_j (where f is the attractiveness function). Thus in effect the point q_j represents an upper threshold beyond which W_j is unlikely to grow. Hence other zones can pick up the remaining revenue, up to their own threshold value, q_j (q_j will vary by zone). This provides a simple explanation for Figure 10(h). We can also re-run the model for $\beta = 0.5$ but with different values of δ, which either raise or lower the upper threshold. Some examples of this are given in Figure 13. The effect is entirely as expected. For high δ we get just one centre, for low δ we get many centres, and with the highest value of W_j relatively low. We can now proceed to examine the roles of the other parameters in the attractiveness function.

Figure 14 and Figure 15 show decreasing values of epsilon ε, and discrete changes for ε variation, respectively. According to our analysis in Figure 6, decreasing ε should result in more local facilities, whilst a high ε value would be characteristic of more concentrated facilities. It is immediately clear that changes in the overall spatial pattern occur rapidly with changes in ε. Figure 14 shows how varied the patterns are and Figure 15 shows how quickly discrete change can occur. We might expect ε, γ and δ changes to have such an impact on the overall pattern since these parameters have replaced α in the conventional model as the 'scale economies' parameter. Figure 14(h) shows a low value of ε and, as we might predict, every zone has some retail facilities. As ε increases, facilities become more concentrated and 'rings of facilities' clearly become distinguishable. When ε reaches 0.005 (Figure 14(c)) however, we see a process of infilling and by the time ε reaches 0.01 (Figure 14(a)) our entire landscape is once again covered:- but in a rather different way than in Figure 14(h). (This is shown in Figure 16 where tables (a) and (b) show the actual numerical values, for our spatial system, of the model runs).

Figure 14(g) also clearly needs interpretation. Figure 15 shows discrete changes as we approach Figure 14(g) both from a higher and lower ε value. What seems to be happening is that all supply is becoming concentrated in one centre or zone in this case the city centre. However this is very misleading and is a deficiency of the computer graphical output rather than an anomoly of the model output. The problem boils down to one of scaling. The computer scales facility size in each zone according to the height of the largest centre. Thus when, as in Figure 14(g), the facilities in the city centre are greatly in excess of the facilities present in the remaining zones, only the centre appears to have facilities on the computer plots. Thus in fact Figure 14(g) and

(h) are not too dissimilar except for a much more dominant city centre in the former. This can be seen by referring to plots Figure 16(b) and (c) where the actual numerical values are tabulated for our 729 spatial system.

Figure 17 shows decreasing values for ε without cheap city centre travel, i.e. $v = 1.00$, and, as can be seen, the 'problem' of Figure 14(g) never emerges. It is also interesting from Figure 17 to note how the city centre still dominates for the middle range of ε values.

Figures 18 and 19 show decreasing values of δ and γ and show the parameters working in similar ways. As δ and γ decrease facilities become less concentrated and eventually the entire spatial system becomes covered. Once again however we should note the dominance of the city centre blurring this effect on the graphical output from the computer. This analysis seems to back up our previous findings (i.e. from Figures 7 and 12) that is decreasing δ leads to a situation of smaller more localized facilities.

The effect of using the double logistic function is shown in Figure 20. These results repeat the parameter variation shown in Figure 10, for β change. There are some noticeable differences between the two sets of results; the double logistic function producing more centres in the case of Figure 20(e-g) - and these we would attribute to the existence of an intermediate maximum. The rings in these figures should again be interpreted with caution in relation to the argument of Clarke (1984-B).

Up to now the model has been run keeping the constants C and K (in equation (10)) zero. Earlier we noted how non-zero C and K values changed the shape of the logistic function and hence clearly it is important to see how non-zero values effect the actual model solutions here.

The effect of changing C is clearly seen in Figure 21. The constant C is associated with shifting the position of the logistic slope sideways along the x and y axis. Figure 21 shows that the larger the negative value of this constant the fewer facilities are present. This is simply because a smaller number of zones will reach the threshold beyond which development is possible when C becomes larger (negatively). Thus in effect a large negative value of C means larger individual W_j values are required to achieve the higher benefit.

Whilst changing the constant C does then have an important effect on the spatial patterns the same is not true for the constant K. Numerical experiments have indeed shown that there are no changes to the spatial patterns when K is made non-zero. The explanation for this is quite straightforward; expressing the logistic function as $f(W_j)$, then the overall attractiveness term in the spatial interaction model (from equation (10)), when K is added, becomes

$$\exp(f(W_j) + K) \quad (11)$$

which is equivalent to

$$\exp f(W_j) \times \exp(K). \quad (12)$$

Thus in effect we are multiplying through by a constant term during each iteration. This consequently does not effect the overall nature of the solutions.

So far there has been difficulty in the interpretation of some of the results, especially in terms of 'corner shop' or 'supermarket' solutions postulated by Poston and Wilson. In order to help overcome the 'what is a corner shop, what is a supermarket?' issue, we can firstly work with a general attractiveness function, $F(W_j)$ and secondly compute variations in u_1 and u_2 values for W_j patterns represented in 'sets' - we in effect reintroduce i's and j's into the definitions of 'average' u_1 and u_2

measures. The model can be rewritten as

$$S_{ij} = A_i e_i P_i F(W_j) e^{-\beta c_{ij}}$$
$$= A_i e_i P_i e^{\beta[1/\beta \log F(W_j) - c_{ij}]} \tag{13}$$

so that size benefits become $1/\beta \log F(W_j)$, set against transport costs c_{ij}. We can therefore take aggregate measures of u_1 and u_2 as

$$u_1 = 1/\beta \sum_{ij} S_{ij} \log F(W_j) \tag{14}$$

and

$$u_2 = - \sum_{ij} S_{ij} c_{ij}. \tag{15}$$

With these measures, the next step is to calculate u_1 and u_2 for a representative set of hypothetical $\{W_j\}$s so that we can plot u_1 and u_2 variation for a range of situations from many non-zero centres to a single centre. Our objective is summarised in Figure 22.

A representative index of a pattern $\{W_j\}$, can be characterised by a single $\bar{W}$ index calculated as

$$\bar{W} = \sum_j W_j / N \tag{16}$$

where N is the number of non-zero W_j-zones (and $\sum_j W_j$ is in fact a constant).

The representative set of chosen $\{W_j\}$s can be taken from the equilibrium pattern solutions obtained earlier (and seen in Figures 10, 14 and 18). u_1 and u_2 (equations (14) and (15)) are then calculated for the spatial patterns produced by the numerical experiments. As noted above, the value of $\sum_j W_j$ will be fixed since the model distributes the same amount of supply for all the parameter values since $\Sigma e_i P_i$ is fixed as a constant over all runs. The number of non-zero centres N, can be calculated in each case, thus giving a value for $\bar{W}$ in (16). Thus for each of Figures 10, 14 and 18 we build-up a representative set $\{W_j\}$.

Figure 23 plots the u_1, u_2 curves thus derived for the spatial equilibrium patterns shown in Figure 10. Figures 23 and 10 are now directly comparable. In Figure 10 we can see a pattern of a large number of centres for high β values and fewer centres for lower β values, yet in Figure 23 there is no zero maximum. (As for example in the theoretical plots of Figure 5.) In Poston and Wilson terminology therefore, the 'corner shop' solution never emerges. However for low β values there is a non-zero maximum which is both large and positive. As $\beta \to 3.5$, then the maximum approaches the origin, i.e. $W^* \to 0$ (see Figure 23) as one might expect. The overall number of centres has increased whilst the average size of those centres has decreased. However there are not sufficient numbers of centres, even when β is relatively high, to warrant the label 'corner shop'. The loss of this jump from a non-zero to zero maximum is perhaps not surprising since it is difficult to pinpoint such a jump in the actual spatial pattern changes (Figure 10).

Given the nature of the equilibrium model solutions seen earlier, we might expect this jump to more obviously appear for other parameter variations. In Figure 13 for example, we looked at changes of δ between the extreme values of $\delta = 1$ and $\delta = 20$ - when $\delta = 20$ there was only one facility located at the centre of our 729-zone model (the ultimate 'supermarket' situation), whilst when $\delta = 1$ facilities were located across all the 729 zones (the ultimate 'corner shop' solution). The mechanism of δ change was described earlier (see p. 11). Since we know that these extreme situations are possible it is possible to calculate u_1 and u_2 again for δ variation, which may throw more light on the corner shop to supermarket transition.

Figure 24 shows the effect of δ variation around our 'pivotal set' of other parameter values given earlier. Again it is possible to see a situation where all zones are allocated facilities (Figure 24 e-f) and where there are very few (Figure 24 a especially, but b and c as well). (Figure 24 e is again slightly misleading in that whilst all zones do have facilities they are scaled down because of the increased size of the central zone.) It is interesting that the single zone case seen in Figure 13 does not appear here since β has been increased from 0.5 to 2.5, thereby slightly constraining the centralization effect.

In Figure 25 the u_1, u_2 curves have been calculated from the spatial configurations obtained in Figure 24, so again both figures are comparable. Figure 25 shows that the conventional jump seen in Poston and Wilson has returned as δ varies. It becomes increasingly apparent that whilst the 'corner shop solution' is easy to define, that is facilities concentrated in the majority of zones, the 'supermarket' situation is any solution which is not the 'corner shop' one or in terms of u_1, u_2 where $W^* \neq 0$ (see Figure 25).

3. Conclusions

The Poston and Wilson hypothesis represents a mechanism for bringing about jumps between 'corner shop'-type retail development and a 'supermarket'-type system. It has been shown that these jumps can be reproduced within a spatial-interaction model-based analysis provided a suitable attractiveness function is used. However, when we undertake numerical experiments to demonstrate the effect, complications arise that were not anticipated and further refinement of the original theory is required.

One of the main conclusions that arises from this paper is the value and importance of undertaking numerical experiments to investigate the theoretical properties of models. This forces the researcher to make explicit the assumptions that underpin the models and to refine the initial theoretical concepts. In the case of the 'corner shop' to 'supermarket' transition it has been shown that much extension and reformulation of the original set of models was necessary. One of the main problems in transfering from single-zone analysis to the analysis of the whole system is that due to the nature of the interdependencies the behaviour of any one zone is influenced by the behaviour in every other zone. Nevertheless it has been demonstrated that the mechanisms for change proposed with suitable modifications, are demonstrated by numerical experiments.

A further point has emerged from this work. Sometimes the results produced by the numerical experiments do not appear to accord with what one would expect. This forces the reappraisal of the theoretical properties of the models so as to explain the unusual results. As we have seen this provides very useful insights into the models which can be developed further.

An obvious task of future research is to attempt to determine the types of attraction functions that different forms of retailing activities possess. They may well be even more complicated than we have suggested in this paper, but this will probably not affect our general argument. Empirical work is now underway (G. Clarke, 1984) in relation to retail change in Leeds and the findings from this research should enable us to extend the types of models discussed in this paper.

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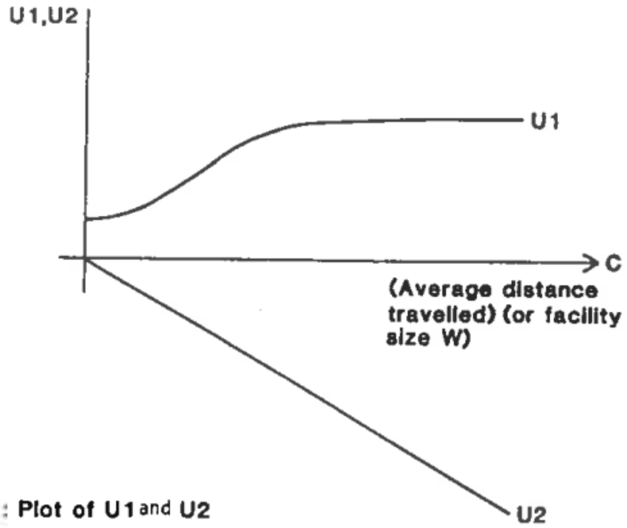


Fig. 1 : Plot of U1 and U2

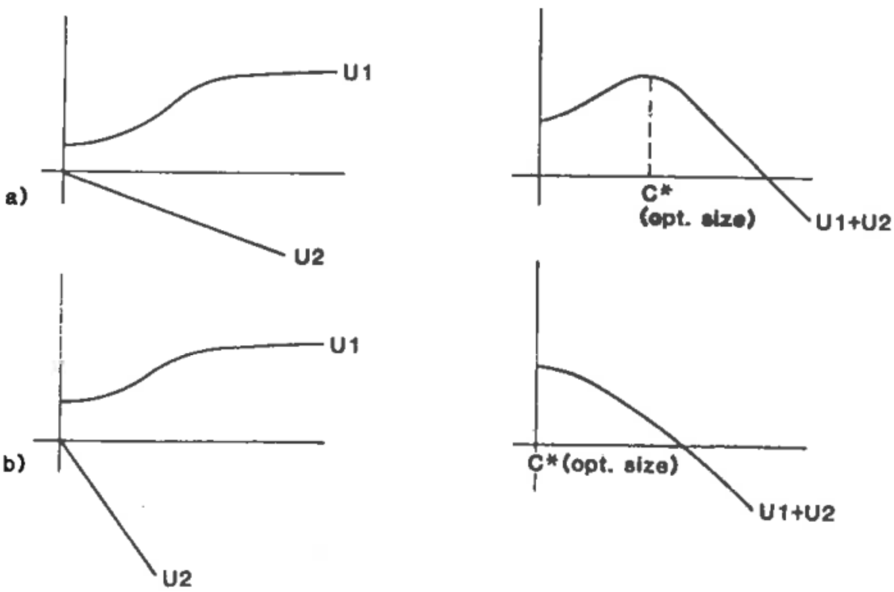


Fig. 2 : The effects of changing U2

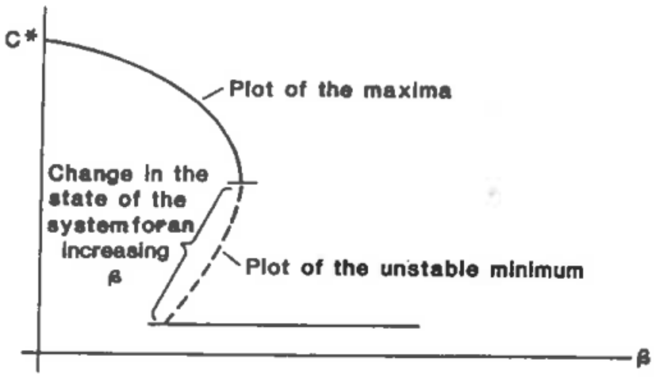


Fig. 3 : The fold catastrophe for optimal values of C (C*)

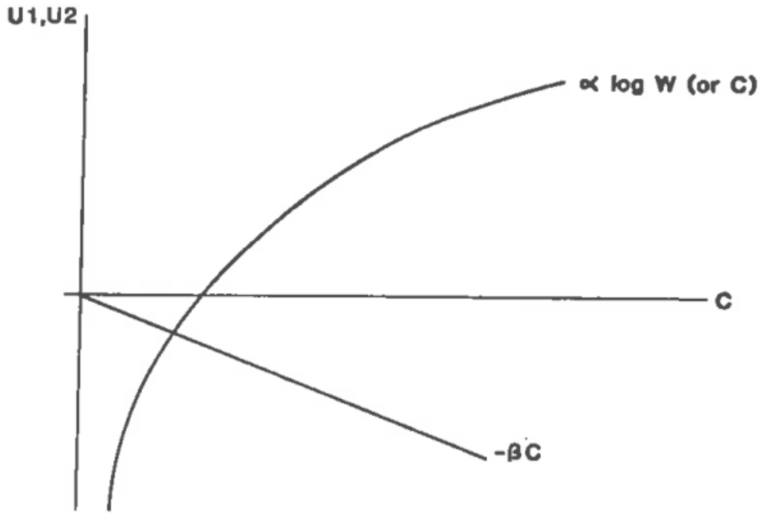


Fig. 4 : Components of utility in the conventional model

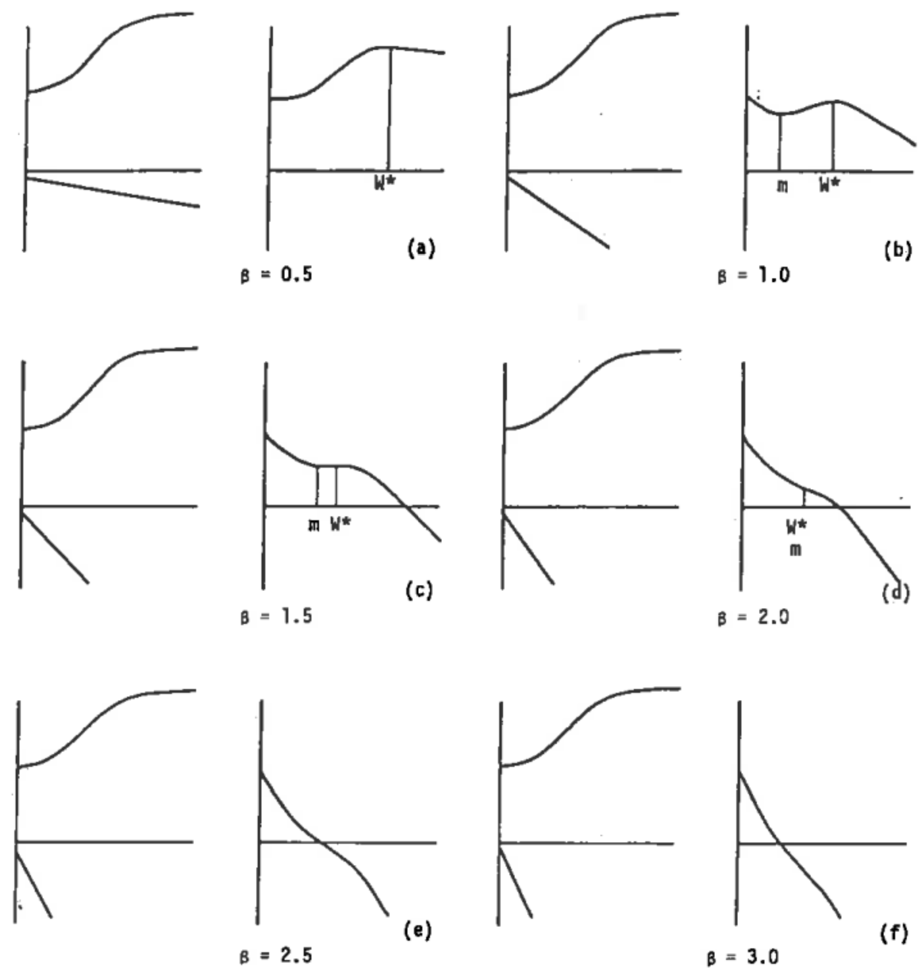


Fig. 5 : Increasing Beta (β)

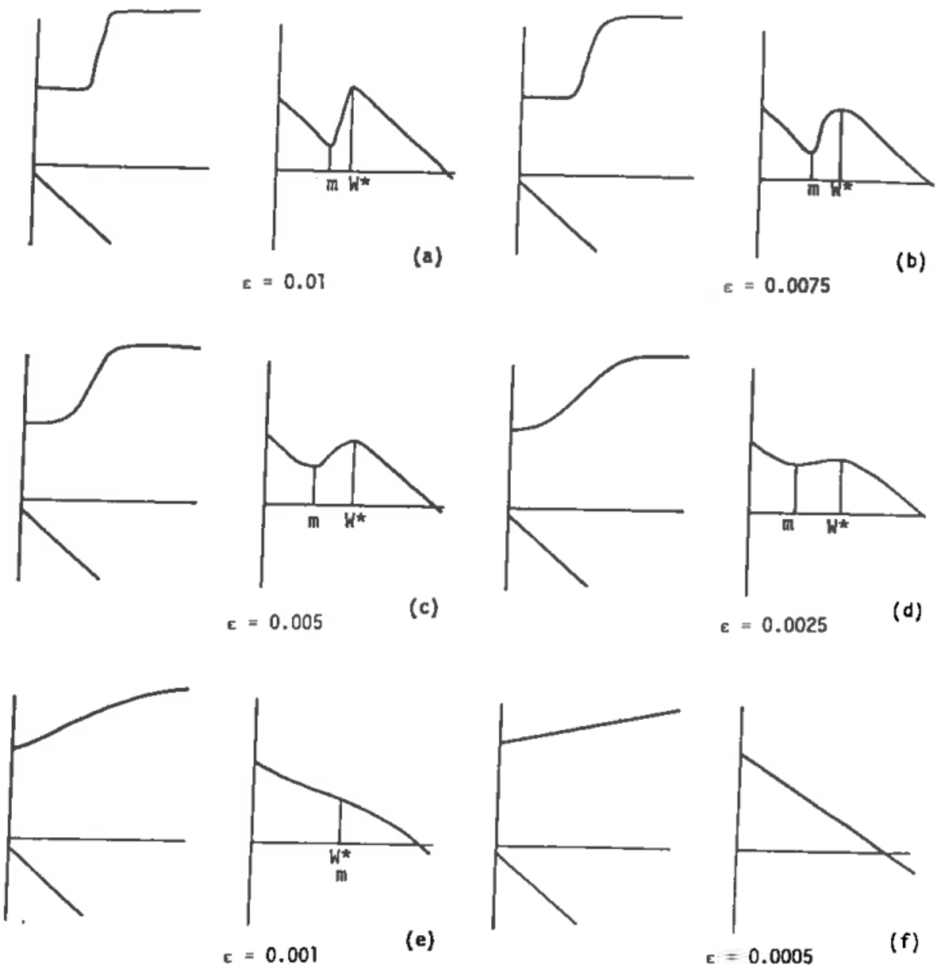


Fig. 6 : Decreasing Epsilon (ε)

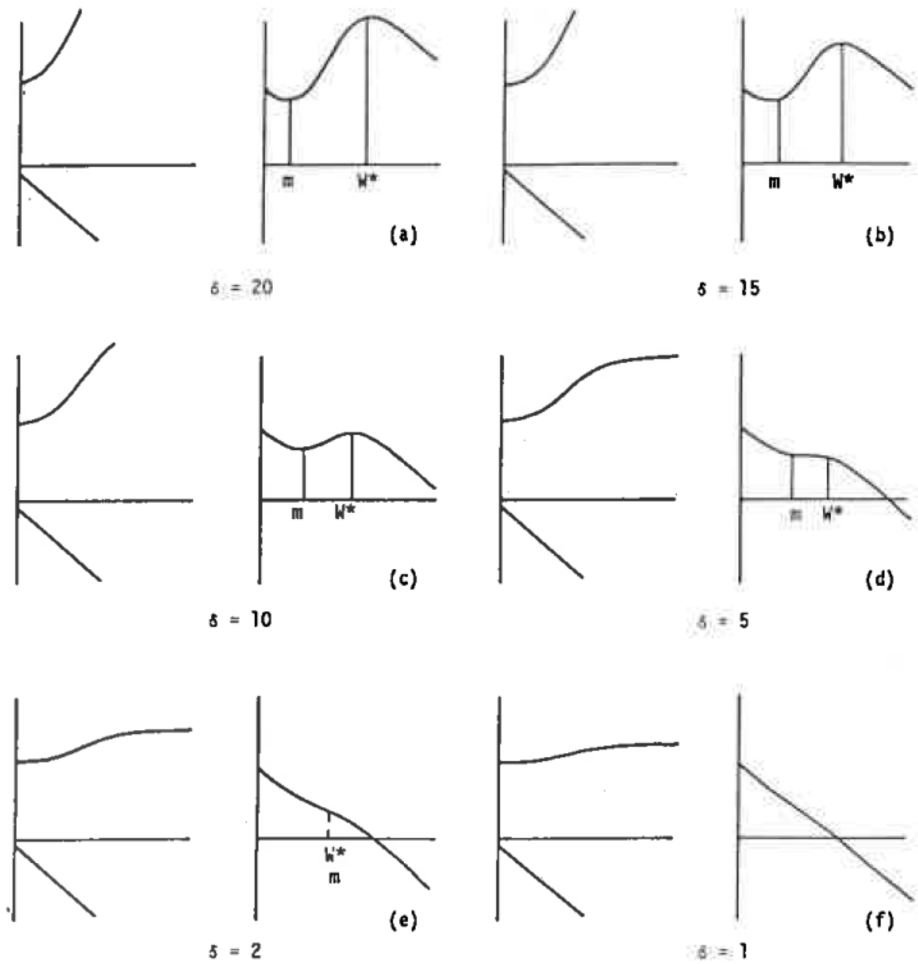


Fig. 7 : Decreasing Delta (δ)

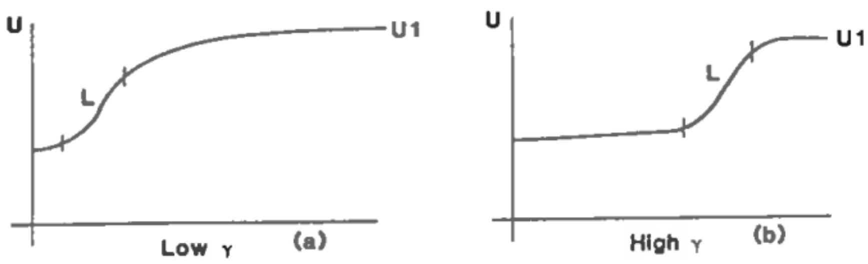


Fig. 8 : The effect of γ

N.B. The shape of the point of inflexion (L) is in the same position in both cases.

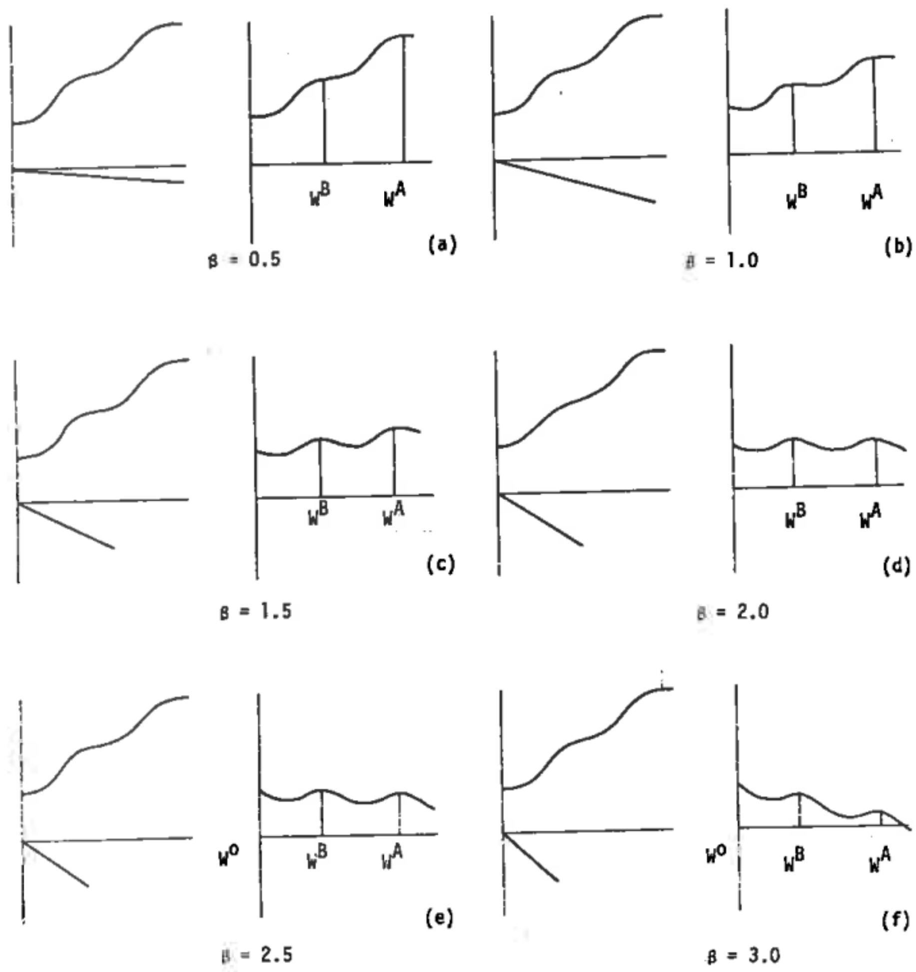


Fig. 9 : Changes in Beta for the double logistic attractiveness function

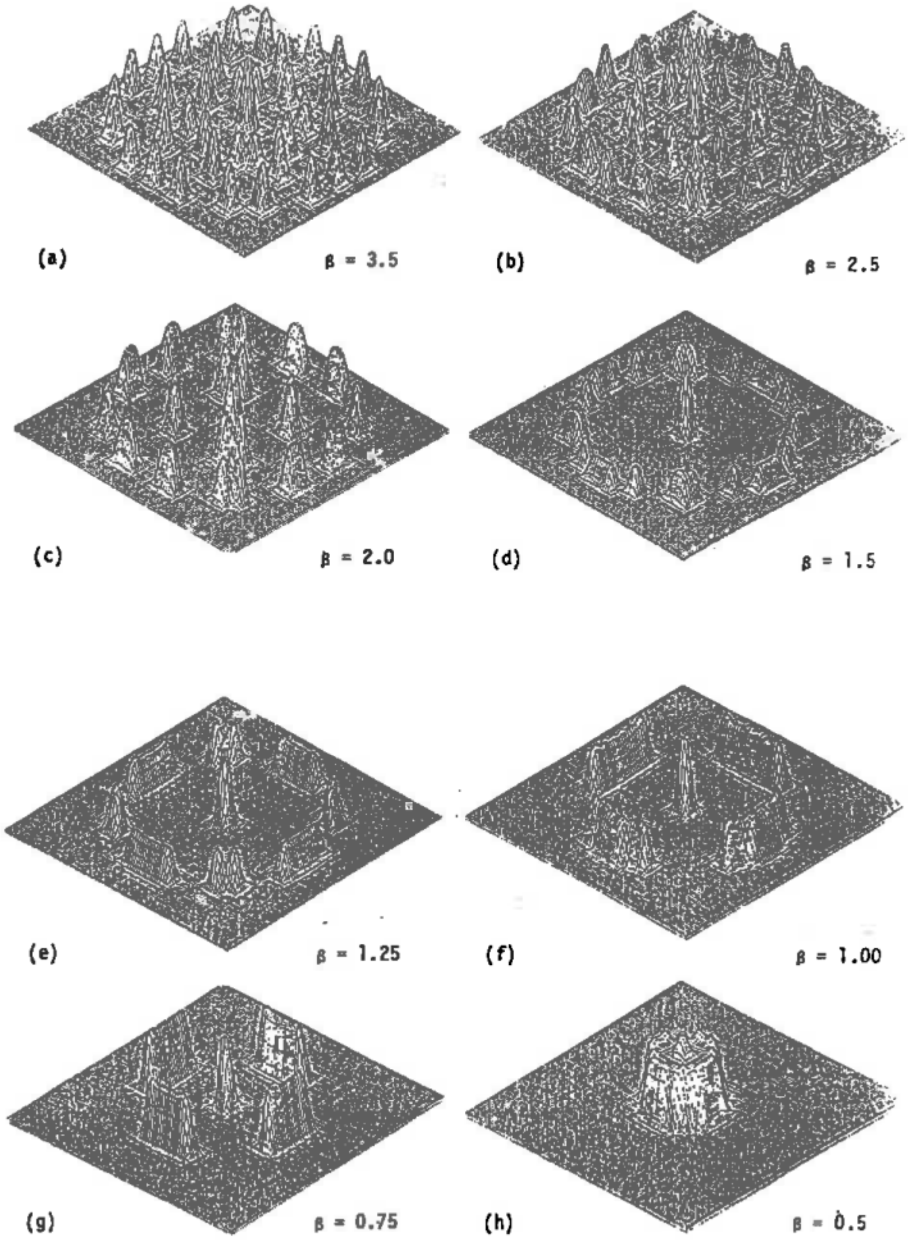
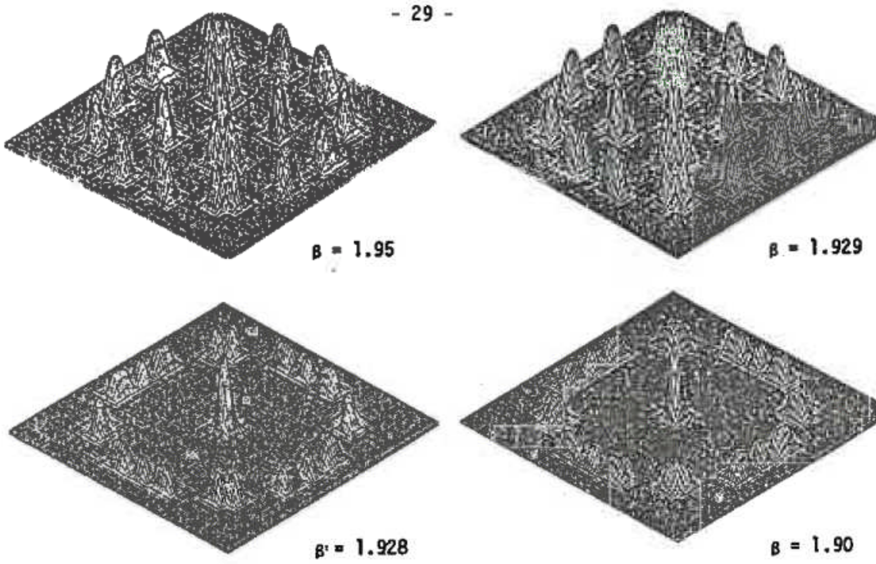


Fig. 10 : Decreasing values of β



β = 1.929

0.	0.	0.	0.	0.	1.	1.	1.	2.	2.	2.	3.	4.	4.	4.	3.	2.	2.	2.	1.	1.	1.	0.	0.	0.	0.	0
0.	0.	0.	0.	1.	1.	2.	3.	3.	4.	4.	4.	5.	6.	5.	4.	4.	4.	3.	3.	2.	1.	1.	0.	0.	0.	0
0.	0.	0.	1.	1.	2.	4.	4.	5.	5.	5.	5.	5.	6.	5.	5.	5.	5.	5.	4.	4.	2.	1.	1.	0.	0.	0
0.	0.	1.	2.	3.	4.	5.	6.	6.	5.	643.	650.	5.	6.	5.	650.	643.	5.	6.	6.	5.	4.	3.	2.	1.	0.	0
0.	1.	1.	3.	4.	5.	5.	6.	6.	6.	6.	6.	6.	7.	6.	6.	6.	6.	6.	6.	5.	5.	4.	3.	1.	1.	0
1.	1.	2.	4.	5.	5.	646.	6.	6.	7.	7.	7.	8.	8.	8.	7.	7.	7.	6.	6.	646.	5.	5.	4.	2.	1.	1
1.	2.	4.	5.	5.	646.	5.	6.	7.	7.	8.	9.	10.	10.	10.	9.	8.	7.	7.	6.	5.	646.	5.	5.	4.	2.	1
1.	3.	4.	6.	6.	6.	6.	6.	7.	7.	8.	9.	10.	11.	10.	9.	8.	7.	7.	6.	6.	6.	6.	6.	4.	3.	1
2.	3.	5.	6.	6.	6.	7.	7.	7.	6.	7.	8.	9.	10.	9.	8.	7.	6.	7.	7.	7.	6.	6.	6.	5.	3.	2
2.	4.	5.	5.	6.	7.	7.	7.	6.	6.	752.	7.	8.	9.	8.	7.	752.	6.	6.	7.	7.	7.	6.	5.	5.	4.	2
2.	4.	5.	643.	6.	7.	8.	8.	7.	752.	6.	6.	7.	8.	7.	6.	6.	752.	7.	8.	8.	7.	6.	643.	5.	4.	2
3.	4.	5.	650.	6.	7.	9.	9.	8.	7.	6.	6.	7.	7.	7.	6.	6.	7.	8.	9.	9.	7.	6.	650.	5.	4.	3
4.	5.	5.	5.	6.	8.	10.	10.	9.	8.	7.	7.	6.	6.	6.	7.	7.	8.	9.	10.	10.	8.	6.	5.	5.	5.	4
4.	6.	6.	6.	7.	8.	10.	11.	10.	9.	8.	7.	6.	****	6.	7.	8.	9.	10.	11.	10.	8.	7.	6.	6.	6.	4
4.	5.	5.	5.	6.	8.	10.	10.	9.	8.	7.	7.	6.	6.	6.	7.	7.	8.	9.	10.	10.	8.	6.	5.	5.	5.	4
3.	4.	5.	650.	6.	7.	9.	9.	8.	7.	6.	6.	7.	7.	7.	6.	6.	7.	8.	9.	9.	7.	6.	650.	5.	4.	3
2.	4.	5.	643.	6.	7.	8.	8.	7.	752.	6.	6.	7.	8.	7.	6.	6.	752.	7.	8.	8.	7.	6.	643.	5.	4.	2
2.	4.	5.	5.	6.	7.	7.	7.	6.	6.	752.	7.	8.	9.	8.	7.	752.	6.	6.	7.	7.	7.	6.	5.	5.	4.	2
2.	3.	5.	6.	6.	6.	7.	7.	7.	6.	7.	8.	9.	10.	9.	8.	7.	6.	7.	7.	7.	6.	6.	6.	5.	3.	2
1.	3.	4.	6.	6.	6.	6.	6.	7.	7.	8.	9.	10.	11.	10.	9.	8.	7.	7.	6.	6.	6.	6.	6.	4.	3.	1
1.	2.	4.	5.	5.	646.	5.	6.	7.	7.	8.	9.	10.	10.	10.	9.	8.	7.	7.	6.	5.	646.	5.	5.	4.	2.	1
1.	1.	2.	4.	5.	5.	646.	6.	6.	7.	7.	7.	8.	8.	8.	7.	7.	7.	6.	6.	646.	5.	5.	4.	2.	1.	1
0.	1.	1.	3.	4.	5.	5.	6.	6.	6.	6.	6.	6.	7.	6.	6.	6.	6.	6.	6.	5.	5.	4.	3.	1.	1.	0
0.	0.	1.	2.	3.	4.	5.	6.	6.	5.	643.	650.	5.	6.	5.	650.	643.	5.	6.	6.	5.	4.	3.	2.	1.	0.	0
0.	0.	0.	1.	1.	2.	4.	4.	5.	5.	5.	5.	5.	6.	5.	5.	5.	5.	5.	4.	4.	2.	1.	1.	0.	0.	0
0.	0.	0.	0.	1.	1.	2.	3.	3.	4.	4.	4.	5.	6.	5.	4.	4.	4.	3.	3.	2.	1.	1.	0.	0.	0.	0.
0.	0.	0.	0.	0.	1.	1.	1.	2.	2.	2.	3.	4.	4.	4.	3.	2.	2.	2.	1.	1.	1.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

β = 1.928

0.	0.	0.	0.	0.	1.	1.	1.	2.	2.	2.	3.	4.	4.	4.	3.	2.	2.	2.	1.	1.	1.	0.	0.	0.	0.	0
0.	0.	0.	0.	1.	1.	2.	3.	3.	4.	4.	4.	5.	6.	5.	4.	4.	4.	3.	3.	2.	1.	1.	0.	0.	0.	0
0.	0.	0.	1.	1.	2.	4.	4.	5.	5.	5.	5.	6.	6.	6.	5.	5.	5.	5.	4.	4.	2.	1.	1.	0.	0.	0
0.	0.	1.	2.	3.	4.	5.	6.	6.	5.	622.	576.	4.	563.	4.	576.	622.	5.	6.	6.	5.	4.	3.	2.	1.	0.	0
0.	1.	1.	3.	4.	5.	5.	6.	7.	7.	6.	6.	6.	6.	6.	6.	6.	7.	7.	6.	5.	5.	4.	3.	1.	1.	0
1.	1.	2.	4.	5.	5.	713.	7.	8.	9.	9.	9.	9.	9.	9.	9.	9.	9.	8.	7.	713.	5.	5.	4.	2.	1.	1
1.	2.	4.	5.	5.	713.	6.	8.	10.	12.	14.	14.	14.	14.	14.	14.	14.	12.	10.	8.	6.	713.	5.	5.	4.	2.	1
1.	3.	4.	6.	6.	7.	8.	11.	14.	18.	20.	21.	21.	21.	21.	21.	20.	18.	14.	11.	8.	7.	6.	6.	4.	3.	1
2.	3.	5.	6.	7.	8.	10.	14.	19.	25.	29.	30.	30.	30.	30.	30.	29.	25.	19.	14.	10.	8.	7.	6.	5.	3.	2
2.	4.	5.	5.	7.	9.	12.	18.	25.	33.	39.	35.	31.	30.	31.	35.	39.	33.	25.	18.	12.	9.	7.	5.	5.	4.	2
2.	4.	5.	622.	6.	9.	14.	20.	29.	39.	35.	28.	24.	23.	24.	28.	35.	39.	29.	20.	14.	9.	6.	622.	5.	4.	2
3.	4.	5.	576.	6.	9.	14.	21.	31.	35.	28.	21.	18.	16.	18.	21.	28.	35.	31.	21.	14.	9.	6.	576.	5.	4.	3
4.	5.	6.	4.	6.	9.	14.	21.	30.	31.	24.	18.	14.	12.	14.	18.	24.	31.	30.	21.	14.	9.	6.	4.	6.	5.	4
4.	6.	6.	563.	6.	9.	14.	21.	30.	30.	22.	16.	12.	****	12.	16.	23.	30.	30.	21.	14.	9.	6.	563.	6.	6.	4
4.	5.	6.	4.	6.	9.	14.	21.	30.	31.	24.	18.	14.	12.	14.	18.	24.	31.	30.	21.	14.	9.	6.	4.	6.	5.	4
3.	4.	5.	576.	6.	9.	14.	21.	31.	35.	28.	21.	18.	16.	18.	21.	28.	35.	31.	21.	14.	9.	6.	576.	5.	4.	3
2.	4.	5.	622.	6.	9.	14.	20.	29.	39.	35.	20.	24.	23.	24.	28.	35.	39.	29.	20.	14.	9.	6.	622.	5.	4.	2
2.	4.	5.	5.	7.	9.	12.	18.	25.	33.	39.	35.	31.	30.	31.	35.	39.	33.	25.	18.	12.	9.	7.	5.	5.	4.	2
2.	3.	5.	6.	7.	8.	10.	14.	19.	25.	29.	31.	30.	30.	30.	31.	29.	25.	19.	14.	10.	8.	7.	6.	5.	3.	2
1.	3.	4.	6.	6.	7.	8.	11.	14.	18.	20.	21.	21.	21.	21.	21.	20.	18.	14.	11.	8.	7.	6.	6.	4.	3.	1
1.	2.	4.	5.	5.	713.	6.	8.	10.	12.	14.	14.	14.	14.	14.	14.	14.	12.	10.	8.	6.	713.	5.	5.	4.	2.	1
1.	1.	2.	4.	5.	5.	713.	7.	8.	9.	9.	9.	9.	9.	9.	9.	9.	9.	8.	7.	713.	5.	5.	4.	2.	1.	1
0.	1.	1.	3.	4.	5.	5.	6.	7.	7.	6.	6.	6.	6.	6.	6.	6.	7.	7.	6.	5.	5.	4.	3.	1.	1.	0
0.	0.	1.	2.	3.	4.	5.	6.	6.	5.	622.	576.	4.	563.	4.	576.	622.	5.	6.	6.	5.	4.	3.	2.	1.	0.	0
0.	0.	0.	1.	1.	2.	4.	4.	5.	5.	5.	5.	6.	6.	6.	5.	5.	5.	5.	4.	4.	2.	1.	1.	0.	0.	0
0.	0.	0.	0.	1.	1.	2.	3.	3.	4.	4.	4.	5.	6.	5.	4.	4.	4.	3.	3.	2.	1.	1.	0.	0.	0.	0
0.	0.	0.	0.	0.	1.	1.	1.	2.	2.	2.	3.	4.	4.	4.	3.	2.	2.	2.	1.	1.	1.	0.	0.	0.	0.	0

Fig. 11 : Discrete change for the β parameter

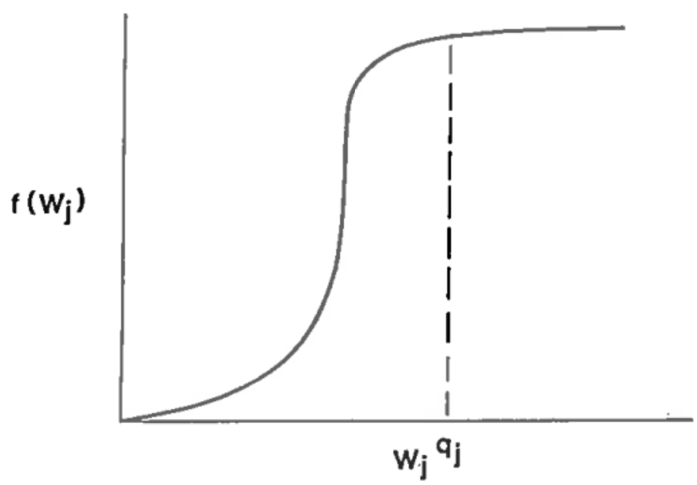
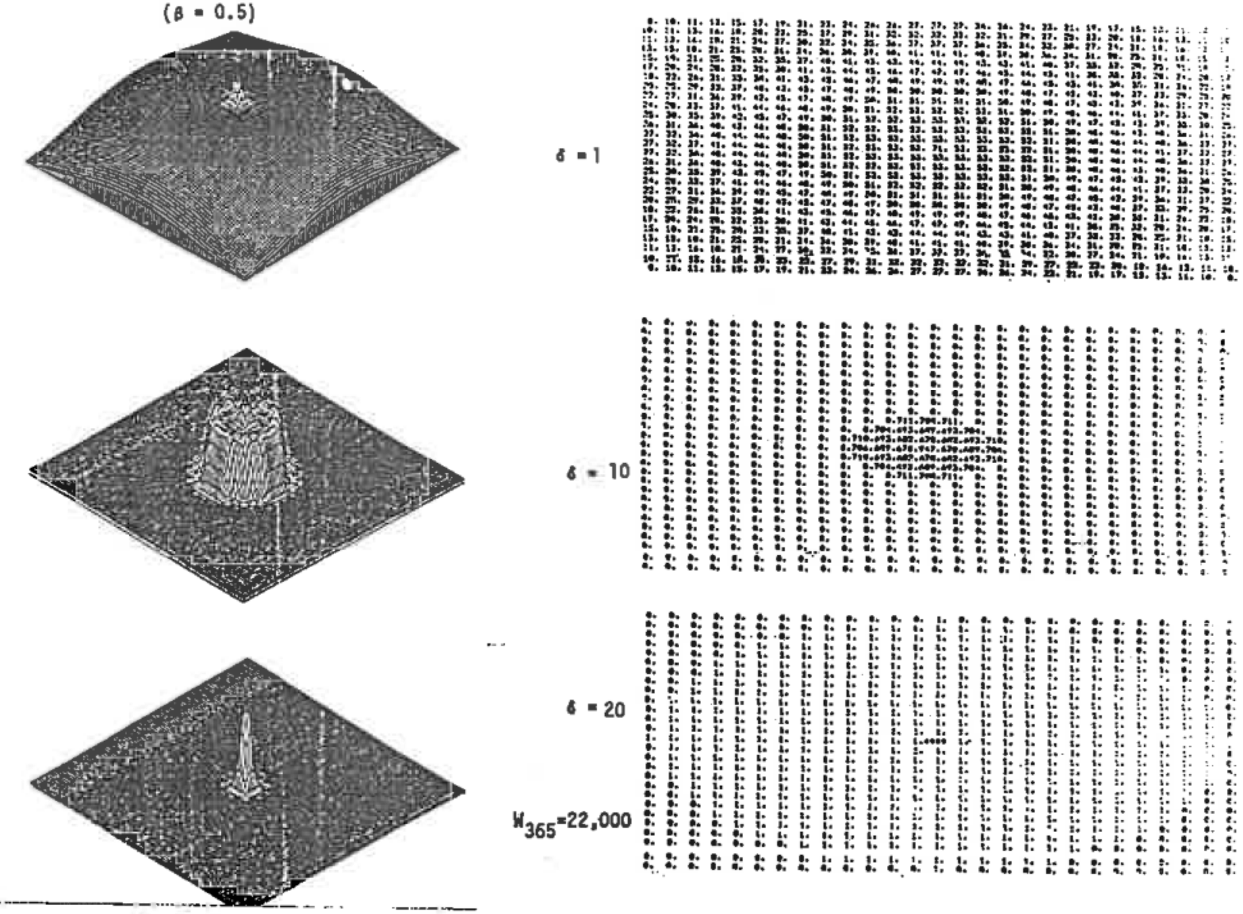


Figure 12 : Logistic function and the upper threshold point q_j

Fig. 13 : The effect of varying δ

(β = 0.5)



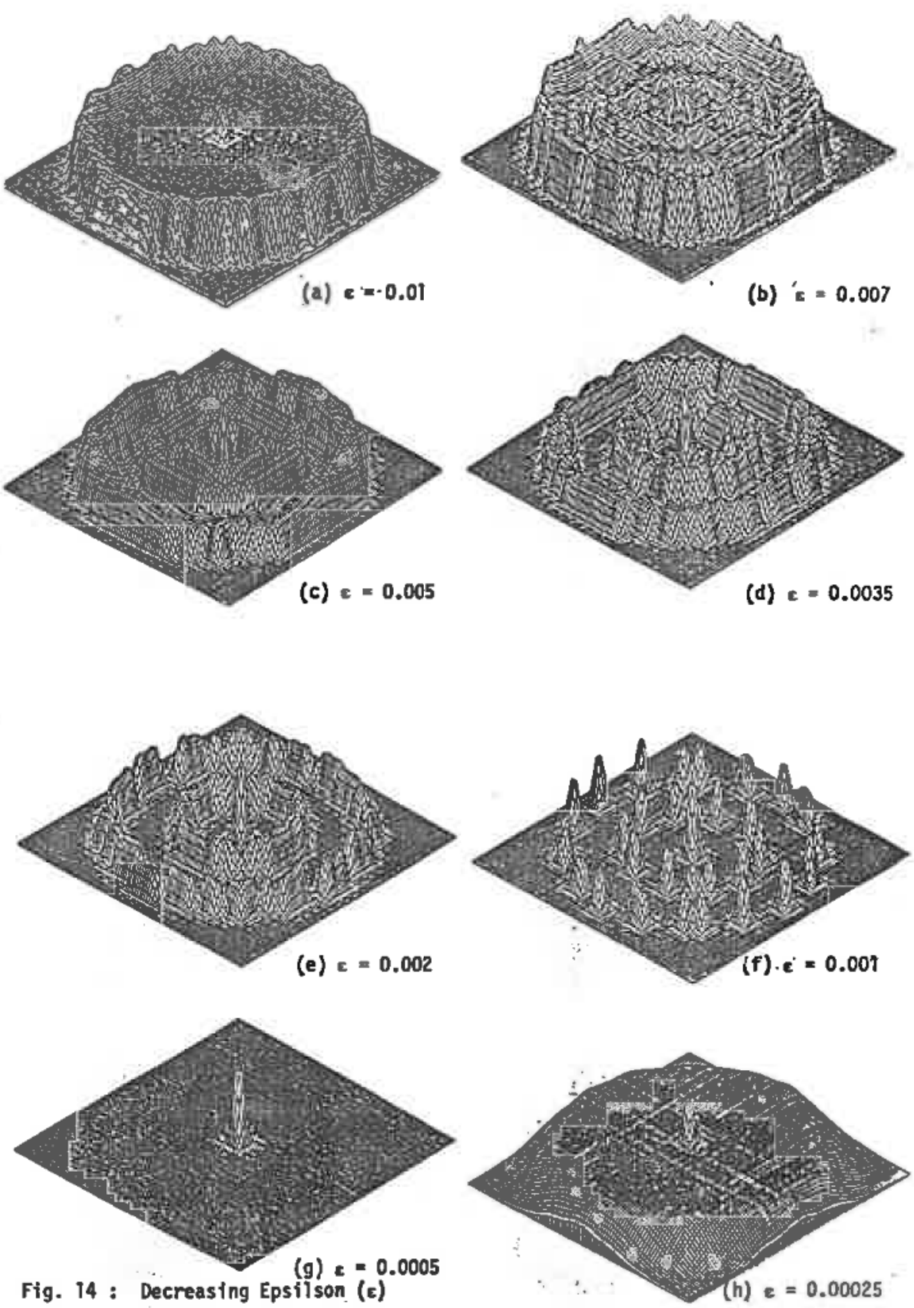


Fig. 14 : Decreasing Epsilson (ε)

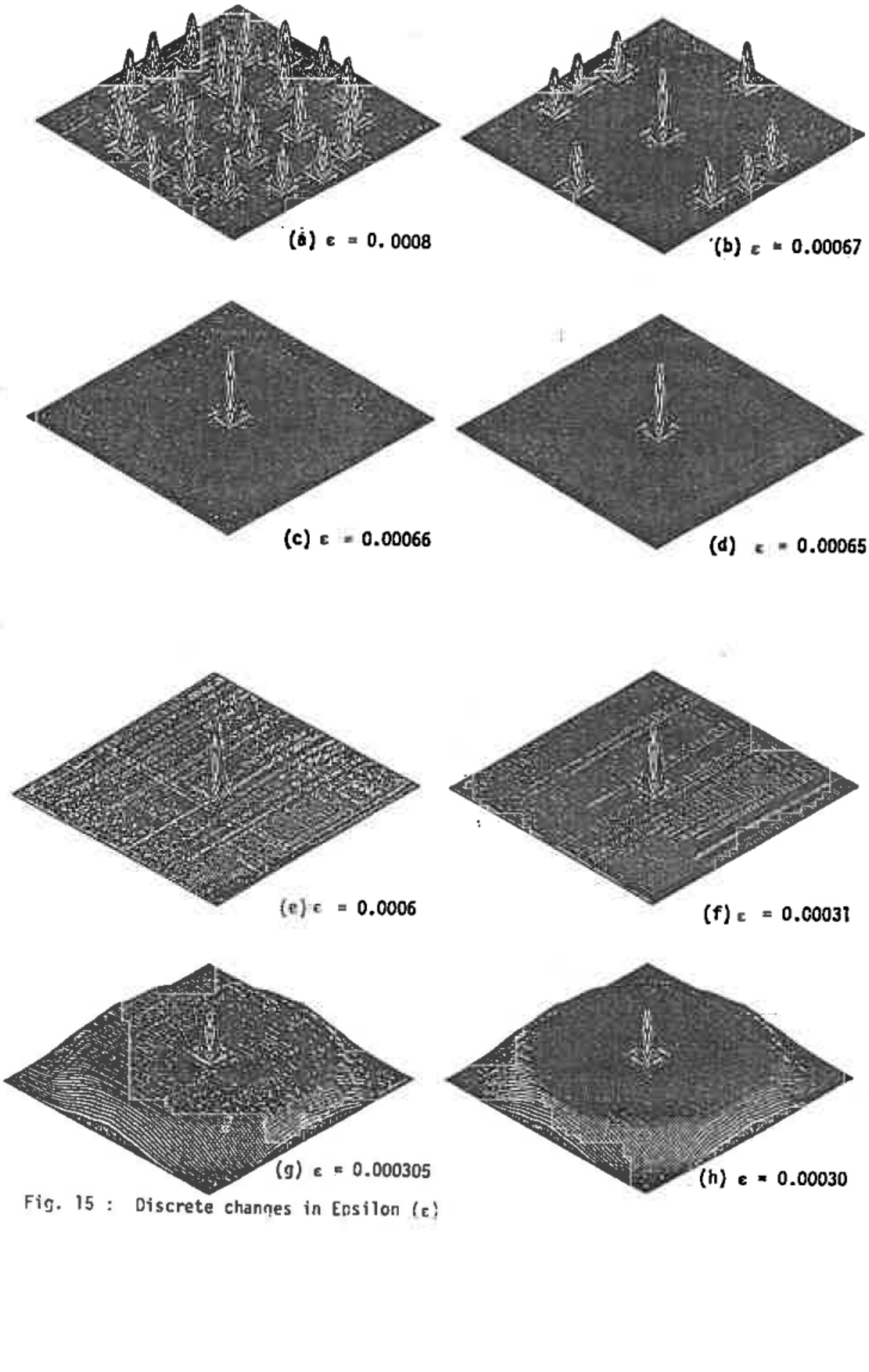


Fig. 15 : Discrete changes in Epsilon (ε)

0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	45.	43.	43.	49.	55.	49.	43.	43.	45.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	40.	58.	59.	57.	56.	57.	57.	57.	56.	57.	59.	58.	40.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	51.	57.	58.	57.	55.	54.	54.	54.	54.	54.	55.	57.	58.	57.	51.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	43.	59.	59.	57.	55.	54.	53.	52.	52.	52.	52.	52.	53.	54.	55.	57.	59.	59.	43.	0.	0.	0.	0.
0.	0.	0.	0.	60.	60.	57.	54.	53.	52.	51.	51.	51.	51.	51.	51.	51.	52.	53.	54.	57.	60.	60.	0.	0.	0.	0.
0.	0.	0.	53.	61.	58.	55.	52.	51.	51.	50.	50.	50.	50.	50.	50.	50.	51.	51.	52.	55.	58.	61.	53.	0.	0.	0.
0.	0.	0.	59.	59.	55.	53.	51.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	51.	53.	55.	59.	59.	0.	0.	0.
0.	0.	53.	60.	57.	53.	51.	51.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	51.	51.	53.	57.	60.	53.	0.	0.
0.	0.	59.	59.	55.	52.	51.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	51.	52.	55.	59.	59.	0.	0.
0.	46.	59.	57.	54.	51.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	51.	54.	57.	59.	46.	0.
0.	46.	58.	55.	53.	51.	50.	50.	50.	50.	50.	50.	49.	49.	49.	50.	50.	50.	50.	50.	50.	51.	53.	55.	58.	46.	0.
0.	50.	57.	55.	52.	51.	50.	50.	50.	50.	50.	49.	49.	49.	49.	49.	50.	50.	50.	50.	50.	51.	52.	55.	57.	50.	0.
0.	56.	58.	55.	52.	51.	50.	50.	50.	50.	50.	49.	49.	80.	49.	49.	50.	50.	50.	50.	50.	51.	52.	55.	58.	56.	0.
0.	50.	57.	55.	52.	51.	50.	50.	50.	50.	50.	49.	49.	49.	49.	49.	50.	50.	50.	50.	50.	51.	52.	55.	57.	50.	0.
0.	46.	58.	55.	53.	51.	50.	50.	50.	50.	50.	50.	49.	49.	49.	50.	50.	50.	50.	50.	50.	51.	53.	55.	58.	46.	0.
0.	46.	59.	57.	54.	51.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	51.	54.	57.	59.	46.	0.
0.	0.	59.	59.	55.	52.	51.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	51.	52.	55.	59.	59.	0.	0.
0.	0.	53.	60.	57.	53.	51.	51.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	51.	51.	53.	57.	60.	53.	0.	0.
0.	0.	0.	59.	59.	55.	53.	51.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	51.	53.	55.	59.	59.	0.	0.	0.
0.	0.	0.	53.	61.	58.	55.	52.	51.	51.	50.	50.	50.	50.	50.	50.	50.	51.	51.	52.	55.	58.	61.	53.	0.	0.	0.
0.	0.	0.	0.	60.	60.	57.	54.	53.	52.	51.	51.	51.	51.	51.	51.	51.	52.	53.	54.	57.	60.	60.	0.	0.	0.	0.
0.	0.	0.	0.	43.	59.	59.	57.	55.	54.	53.	52.	52.	52.	52.	52.	53.	54.	55.	57.	59.	59.	43.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	51.	57.	58.	57.	55.	54.	54.	54.	54.	54.	55.	57.	58.	57.	51.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	40.	58.	59.	57.	56.	57.	57.	57.	56.	57.	59.	58.	40.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	45.	43.	43.	49.	55.	49.	43.	43.	45.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

(a) $\varepsilon = 0.01$ (corresponds to Fig. 14(a))

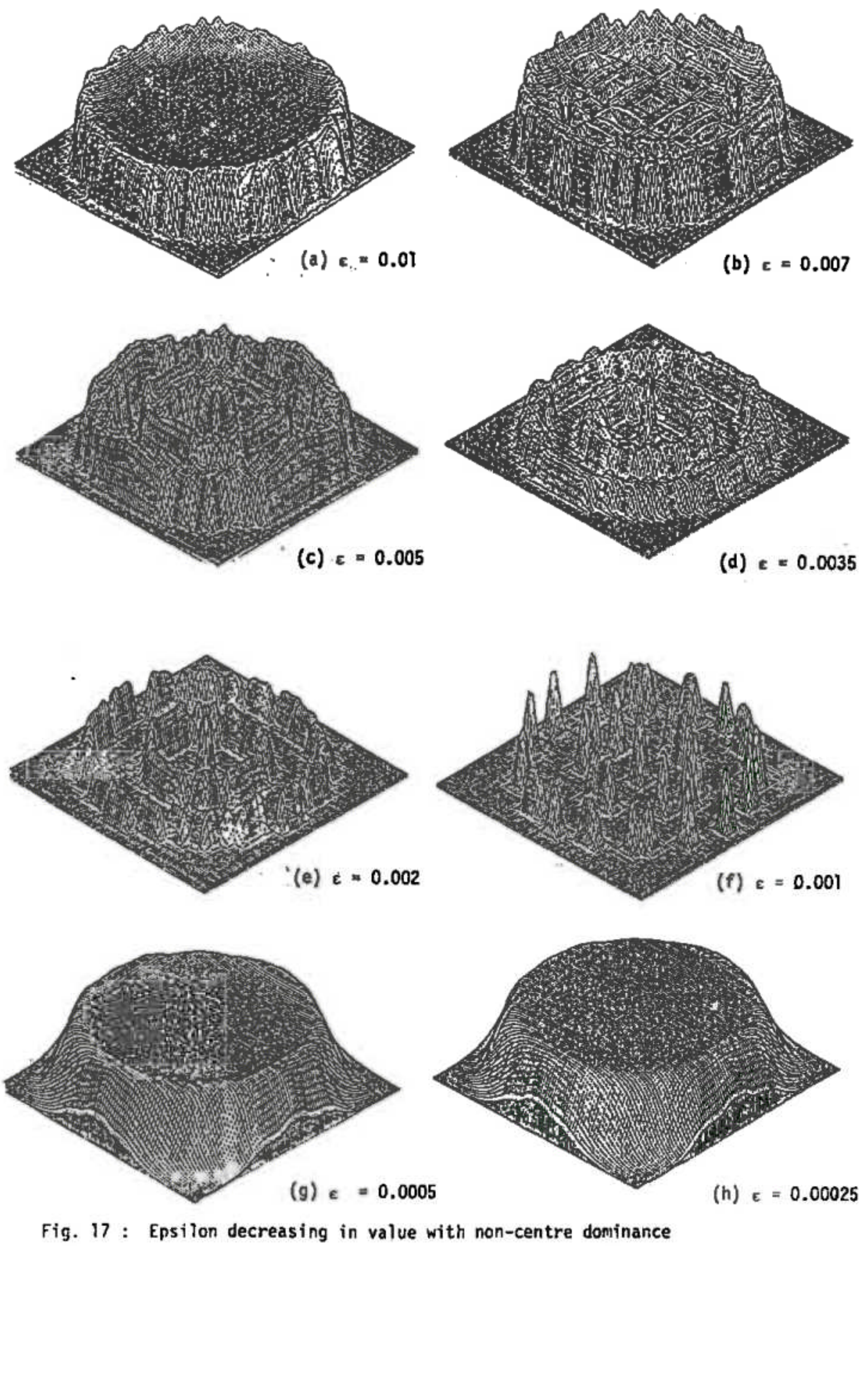
0.	0.	0.	0.	1.	2.	3.	6.	11.	13.	14.	15.	20.	21.	20.	15.	14.	13.	11.	6.	3.	2.	1.	0.	0.	0.	0.
0.	0.	1.	1.	2.	4.	8.	16.	24.	30.	32.	33.	37.	42.	37.	33.	32.	30.	24.	16.	8.	4.	2.	1.	1.	0.	0.
0.	1.	1.	3.	6.	11.	19.	28.	42.	49.	50.	51.	52.	53.	52.	51.	50.	49.	42.	28.	19.	11.	6.	3.	1.	1.	0.
0.	1.	3.	7.	15.	24.	33.	44.	51.	54.	54.	53.	53.	53.	53.	53.	54.	54.	51.	44.	33.	24.	15.	7.	3.	1.	0.
1.	2.	6.	15.	26.	40.	48.	51.	52.	52.	52.	51.	51.	51.	51.	51.	52.	52.	52.	51.	48.	40.	26.	15.	6.	2.	1.
1.	4.	11.	24.	40.	50.	52.	52.	51.	51.	51.	50.	50.	50.	50.	50.	51.	51.	51.	52.	52.	50.	40.	24.	11.	4.	1.
2.	7.	18.	33.	48.	53.	52.	51.	51.	50.	50.	50.	50.	50.	50.	50.	50.	50.	51.	51.	52.	53.	48.	33.	18.	7.	2.
4.	12.	25.	42.	51.	52.	51.	51.	50.	50.	49.	49.	48.	48.	48.	49.	49.	50.	50.	51.	51.	52.	51.	42.	25.	12.	4.
7.	18.	34.	49.	52.	52.	51.	50.	50.	49.	47.	45.	43.	42.	43.	45.	47.	49.	50.	50.	51.	52.	52.	49.	34.	18.	7.
11.	25.	43.	52.	52.	51.	50.	50.	49.	46.	42.	38.	35.	34.	35.	38.	42.	46.	49.	50.	50.	51.	52.	52.	43.	25.	11.
13.	31.	49.	53.	52.	51.	50.	49.	47.	42.	36.	30.	25.	24.	25.	30.	36.	42.	47.	49.	50.	51.	52.	53.	49.	31.	13.
15.	33.	51.	53.	52.	50.	50.	49.	45.	38.	30.	22.	17.	15.	17.	22.	30.	38.	45.	49.	50.	50.	52.	53.	51.	33.	15.
20.	37.	52.	53.	51.	50.	50.	48.	43.	35.	25.	17.	12.	10.	12.	17.	25.	35.	43.	48.	50.	50.	51.	53.	52.	37.	20.
21.	42.	53.	53.	51.	50.	50.	48.	42.	34.	24.	15.	10.	****	10.	15.	24.	34.	42.	48.	50.	50.	51.	53.	53.	42.	21.
20.	37.	52.	53.	51.	50.	50.	48.	43.	35.	25.	17.	12.	10.	12.	17.	25.	35.	43.	48.	50.	50.	51.	53.	52.	37.	20.
15.	33.	51.	53.	52.	50.	50.	49.	45.	38.	30.	22.	17.	15.	17.	22.	30.	38.	45.	49.	50.	50.	52.	53.	51.	33.	15.
13.	31.	49.	53.	52.	51.	50.	49.	47.	42.	36.	30.	25.	24.	25.	30.	36.	42.	47.	49.	50.	51.	52.	53.	49.	31.	13.
11.	25.	43.	52.	52.	51.	50.	50.	49.	46.	42.	38.	35.	34.	35.	38.	42.	46.	49.	50.	50.	51.	52.	52.	43.	25.	11.
7.	18.	34.	49.	52.	52.	51.	50.	50.	49.	47.	45.	43.	42.	43.	45.	47.	49.	50.	50.	51.	52.	52.	49.	34.	18.	7.
4.	12.	25.	42.	51.	52.	51.	51.	50.	50.	49.	49.	48.	48.	48.	49.	49.	50.	50.	51.	51.	52.	51.	42.	25.	12.	4.
2.	7.	18.	33.	48.	53.	52.	51.	51.	50.	50.	50.	50.	50.	50.	50.	50.	50.	51.	51.	52.	53.	48.	33.	18.	7.	2.
1.	4.	11.	24.	40.	50.	52.	52.	51.	51.	51.	50.	50.	50.	50.	50.	51.	51.	51.	52.	52.	50.	40.	24.	11.	4.	1.
1.	2.	6.	15.	26.	40.	48.	51.	52.	52.	52.	51.	51.	51.	51.	51.	52.	52.	52.	51.	48.	40.	26.	15.	6.	2.	1.
0.	1.	3.	7.	15.	24.	33.	44.	51.	54.	54.	53.	53.	53.	53.	53.	54.	54.	51.	44.	33.	24.	15.	7.	3.	1.	0.
0.	1.	1.	3.	6.	11.	19.	28.	42.	49.	50.	51.	52.	53.	52.	51.	50.	49.	42.	28.	19.	11.	6.	3.	1.	1.	0.
0.	0.	1.	1.	2.	4.	8.	16.	24.	30.	32.	33.	37.	42.	37.	33.	32.	30.	24.	16.	8.	4.	2.	1.	1.	0.	0.
0.	0.	0.	0.	1.	2.	3.	6.	11.	13.	14.	15.	20.	21.	20.	15.	14.	13.	11.	6.	3.	2.	1.	0.	0.	0.	0.

(b) $\varepsilon = 0.00025$ (corresponds to Fig. 14(h))

0.	0.	0.	1.	1.	2.	4.	7.	12.	15.	17.	18.	23.	23.	23.	18.	17.	15.	12.	7.	4.	2.	1.	1.	0.	0.	0.
0.	0.	1.	1.	3.	5.	9.	18.	26.	32.	34.	35.	38.	42.	38.	35.	34.	32.	26.	18.	9.	5.	3.	1.	1.	0.	0.
0.	1.	2.	3.	7.	13.	21.	30.	41.	46.	48.	48.	49.	50.	49.	48.	48.	46.	41.	30.	21.	13.	7.	3.	2.	1.	0.
1.	1.	3.	8.	17.	26.	35.	43.	48.	50.	51.	51.	51.	51.	51.	51.	51.	50.	48.	43.	35.	26.	17.	8.	3.	1.	1.
1.	3.	7.	17.	28.	40.	46.	49.	50.	51.	51.	51.	51.	51.	51.	51.	51.	51.	50.	49.	46.	40.	28.	17.	7.	3.	1.
2.	5.	13.	26.	40.	47.	49.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	49.	47.	40.	26.	13.	5.	2.
3.	8.	20.	34.	46.	49.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	49.	46.	34.	20.	8.	3.
5.	14.	27.	42.	48.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	48.	42.	27.	14.	5.
8.	21.	35.	46.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	46.	35.	21.	8.
13.	27.	43.	49.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	49.	43.	27.	13.
16.	33.	46.	50.	51.	50.	50.	50.	50.	50.	50.	50.	49.	49.	49.	50.	50.	50.	50.	50.	50.	50.	51.	50.	46.	33.	16.
18.	35.	48.	51.	51.	50.	50.	50.	50.	50.	50.	49.	49.	49.	49.	49.	50.	50.	50.	50.	50.	50.	51.	51.	48.	35.	18.
23.	38.	49.	51.	51.	50.	50.	50.	50.	50.	49.	49.	48.	48.	48.	49.	49.	50.	50.	50.	50.	50.	51.	51.	49.	38.	23.
23.	42.	50.	51.	51.	50.	50.	50.	50.	50.	49.	49.	48.	101.	48.	49.	49.	50.	50.	50.	50.	50.	51.	51.	50.	42.	23.
23.	38.	49.	51.	51.	50.	50.	50.	50.	50.	49.	49.	48.	48.	48.	49.	49.	50.	50.	50.	50.	50.	51.	51.	49.	38.	23.
18.	35.	48.	51.	51.	50.	50.	50.	50.	50.	50.	49.	49.	49.	49.	49.	50.	50.	50.	50.	50.	50.	51.	51.	48.	35.	18.
16.	33.	46.	50.	51.	50.	50.	50.	50.	50.	50.	50.	49.	49.	49.	50.	50.	50.	50.	50.	50.	50.	51.	50.	46.	33.	16.
13.	27.	43.	49.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	49.	43.	27.	13.
8.	21.	35.	46.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	46.	35.	21.	8.
5.	14.	27.	42.	48.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	48.	42.	27.	14.	5.
3.	8.	20.	34.	46.	49.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	49.	46.	34.	20.	8.	3.
2.	5.	13.	26.	40.	47.	49.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	50.	49.	47.	40.	26.	13.	5.	2.
1.	3.	7.	17.	28.	40.	46.	49.	50.	51.	51.	51.	51.	51.	51.	51.	51.	51.	50.	49.	46.	40.	28.	17.	7.	3.	1.
1.	1.	3.	8.	17.	26.	35.	43.	48.	50.	51.	51.	51.	51.	51.	51.	51.	50.	48.	43.	35.	26.	17.	8.	3.	1.	1.
0.	1.	2.	3.	7.	13.	21.	30.	41.	46.	48.	48.	49.	50.	49.	48.	48.	46.	41.	30.	21.	13.	7.	3.	2.	1.	0.
0.	0.	1.	1.	3.	5.	9.	18.	26.	32.	34.	35.	38.	42.	38.	35.	34.	32.	26.	18.	9.	5.	3.	1.	1.	0.	0.
0.	0.	0.	1.	1.	2.	4.	7.	12.	15.	17.	18.	23.	23.	23.	18.	17.	15.	12.	7.	4.	2.	1.	1.	0.	0.	0.

(c) $\varepsilon = 0.0005$ (corresponds to Fig. 14(g))

Fig. 16 : Numerical values for the 729 spatial system



(a) ε = 0.01
(b) ε = 0.007
(c) ε = 0.005
(d) ε = 0.0035
(e) ε = 0.002
(f) ε = 0.001
(g) ε = 0.0005
(h) ε = 0.00025

Fig. 17 : Epsilon decreasing in value with non-centre dominance

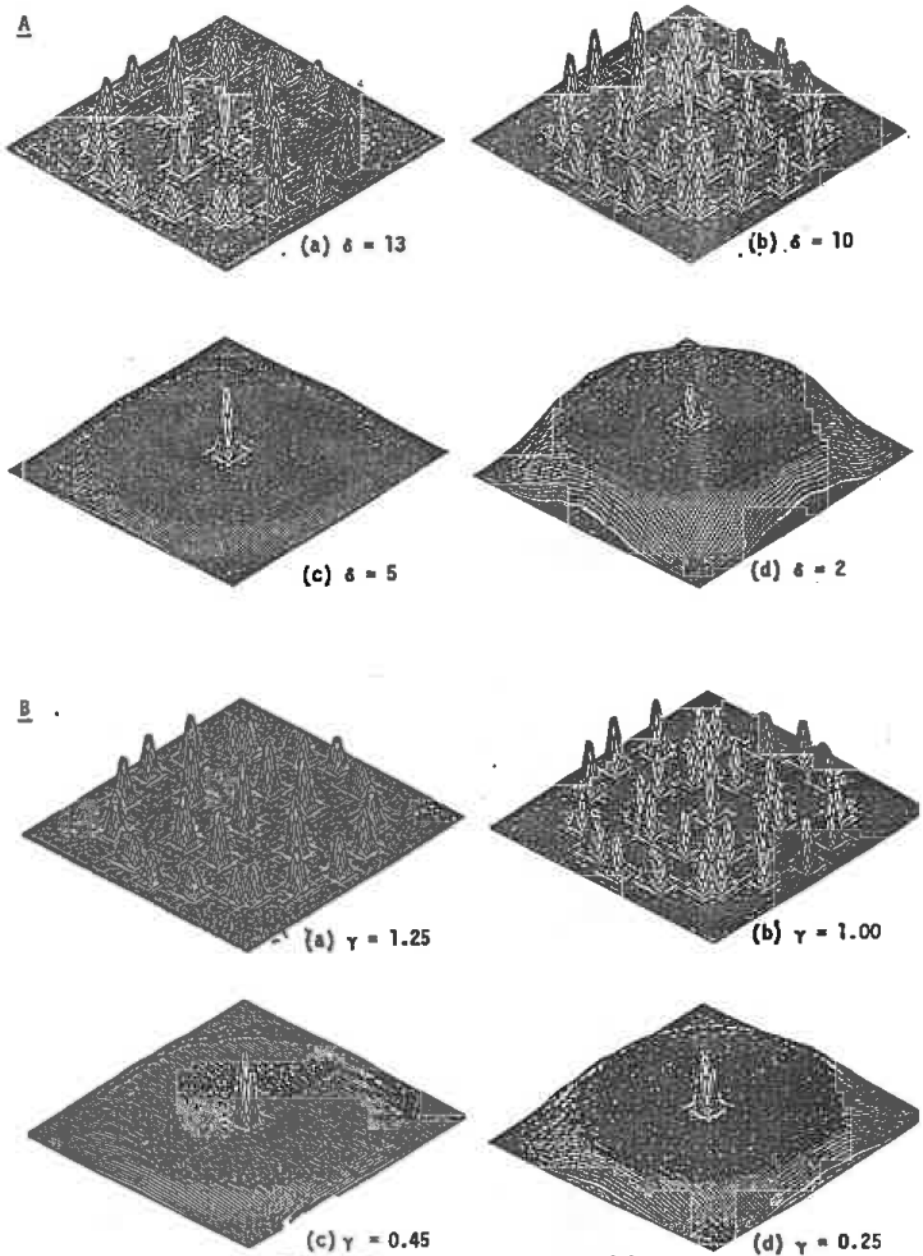


Fig. 18 : A Decreasing Delta (δ) B Decreasing Gamma (γ)

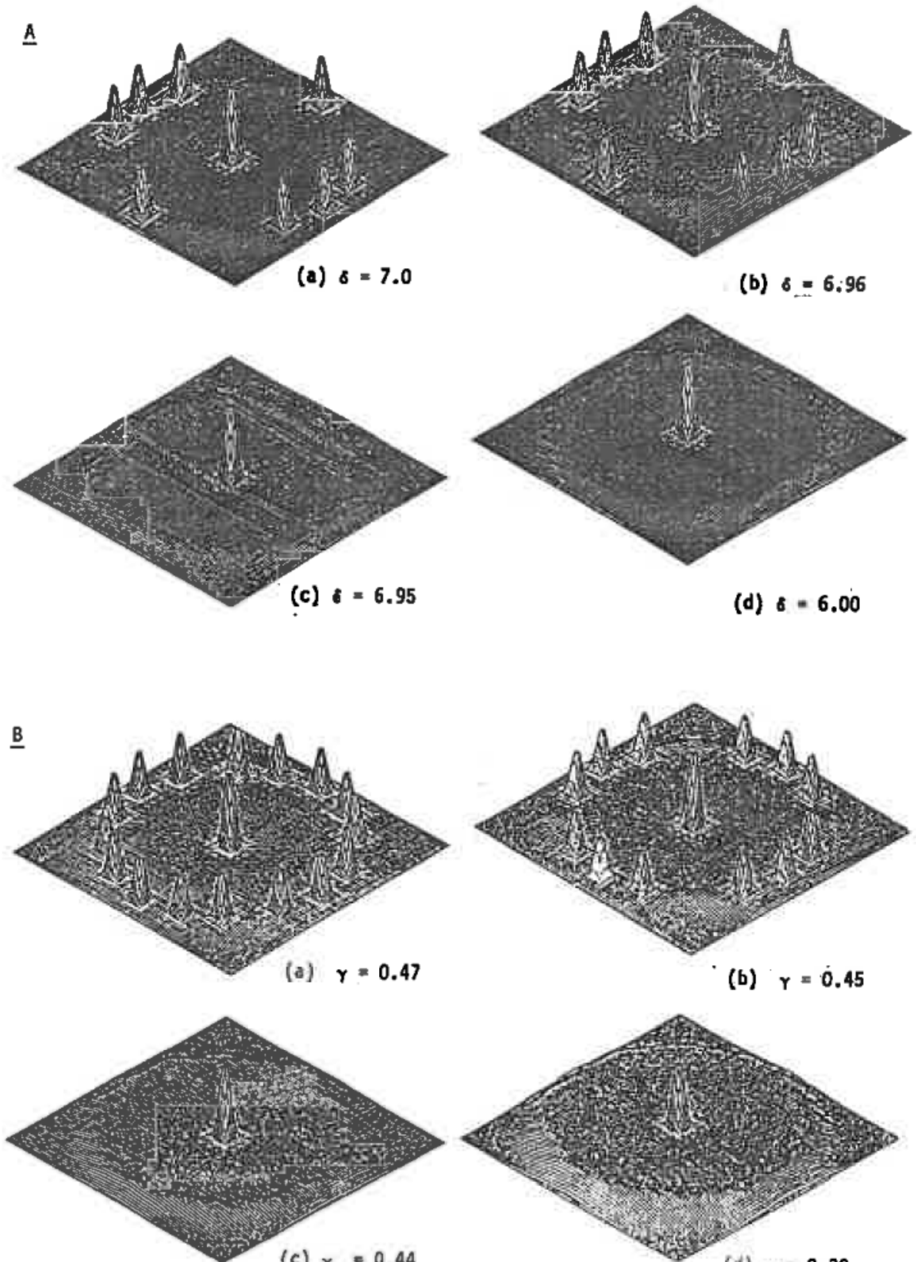


Fig. 19 : A Discrete changes involving Delta (δ)
B Discrete changes involving Gamma (γ)

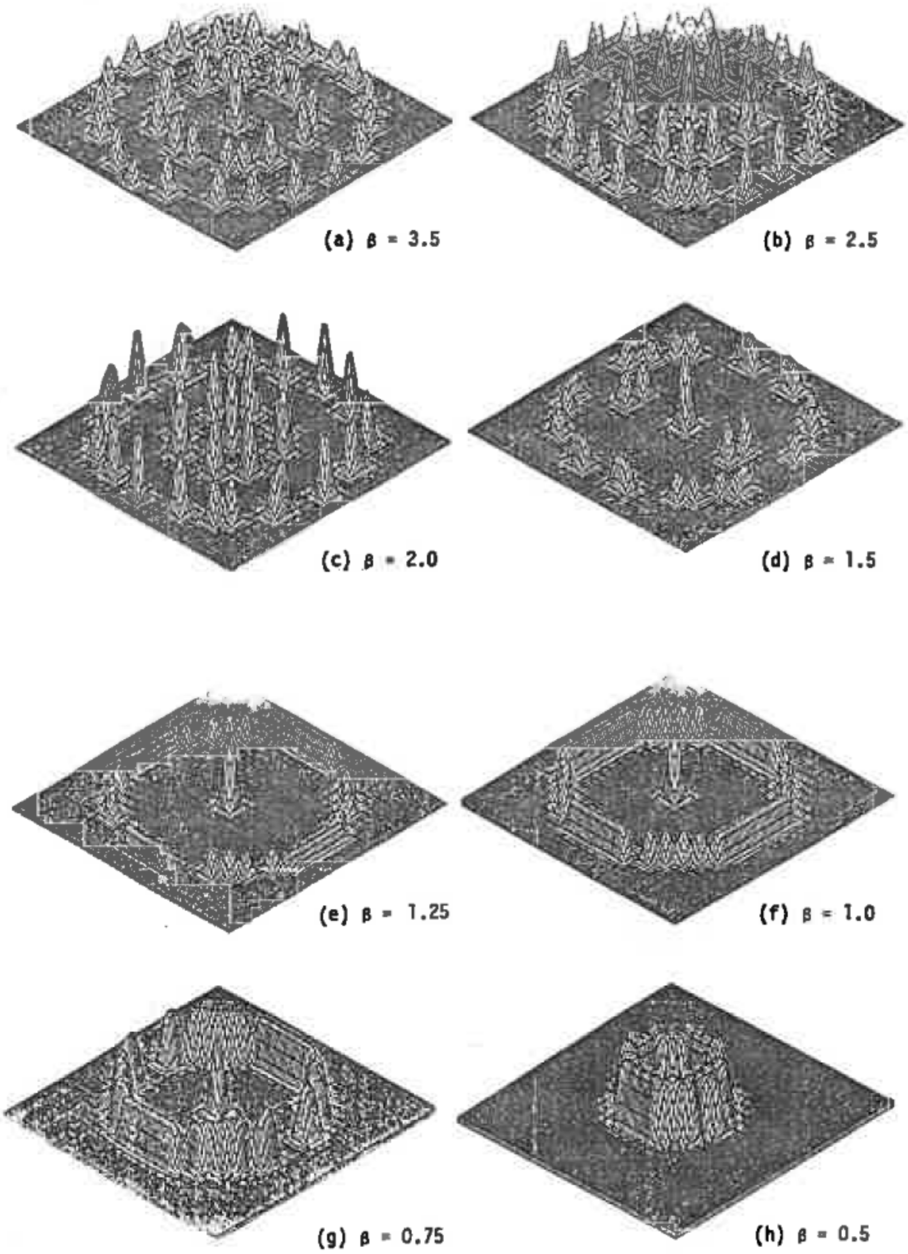


Fig. 20 : Variations of Beta for the double logistic attractiveness function

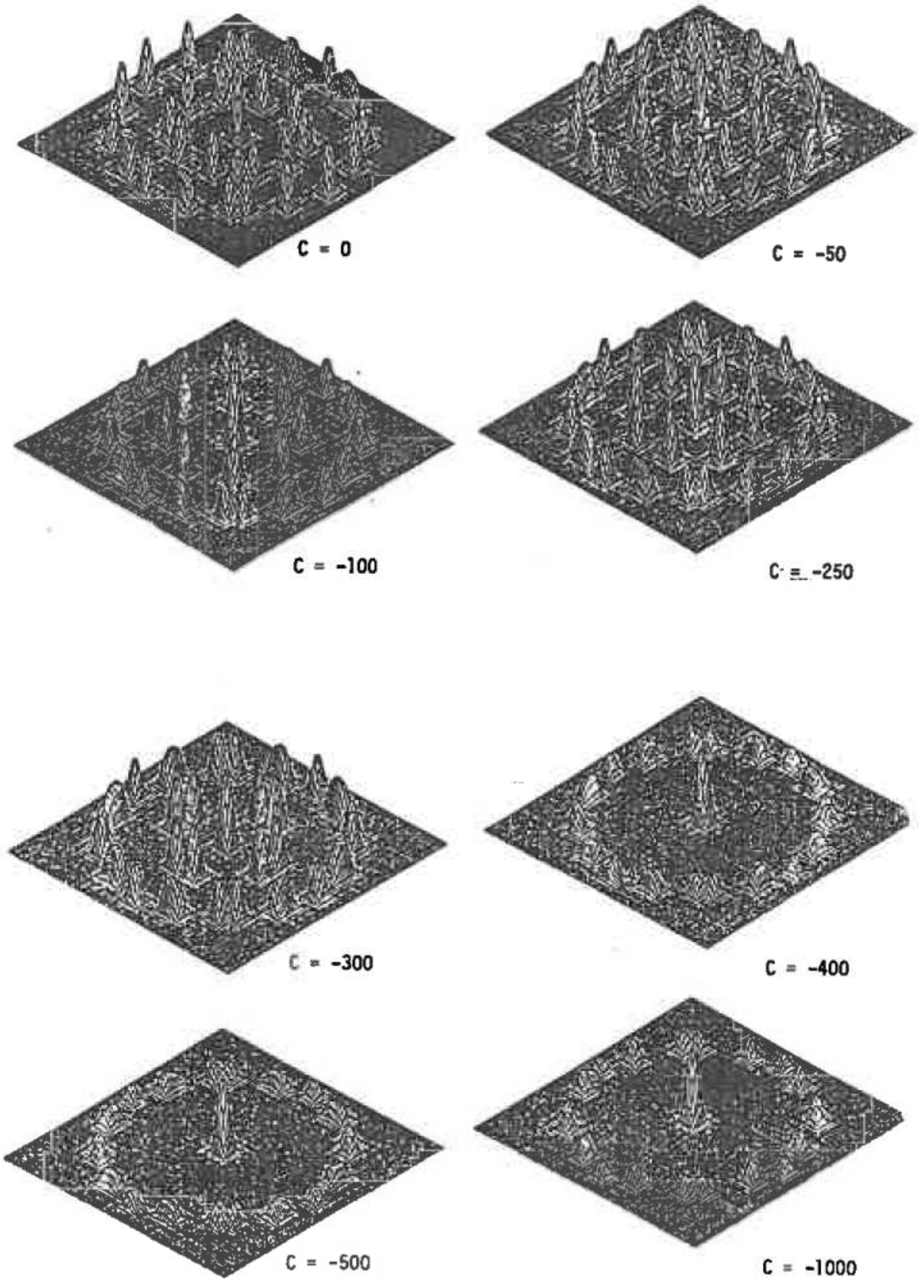


Figure 21 : The effect of varying the parameter C.

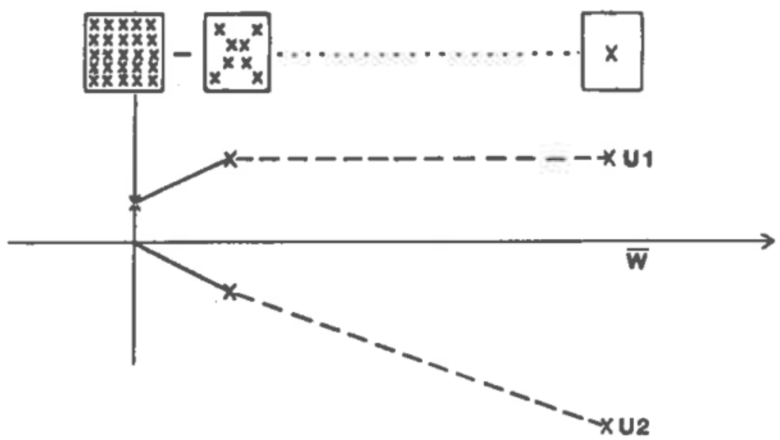


Fig. 22 : Representative set of $\overline{W}$

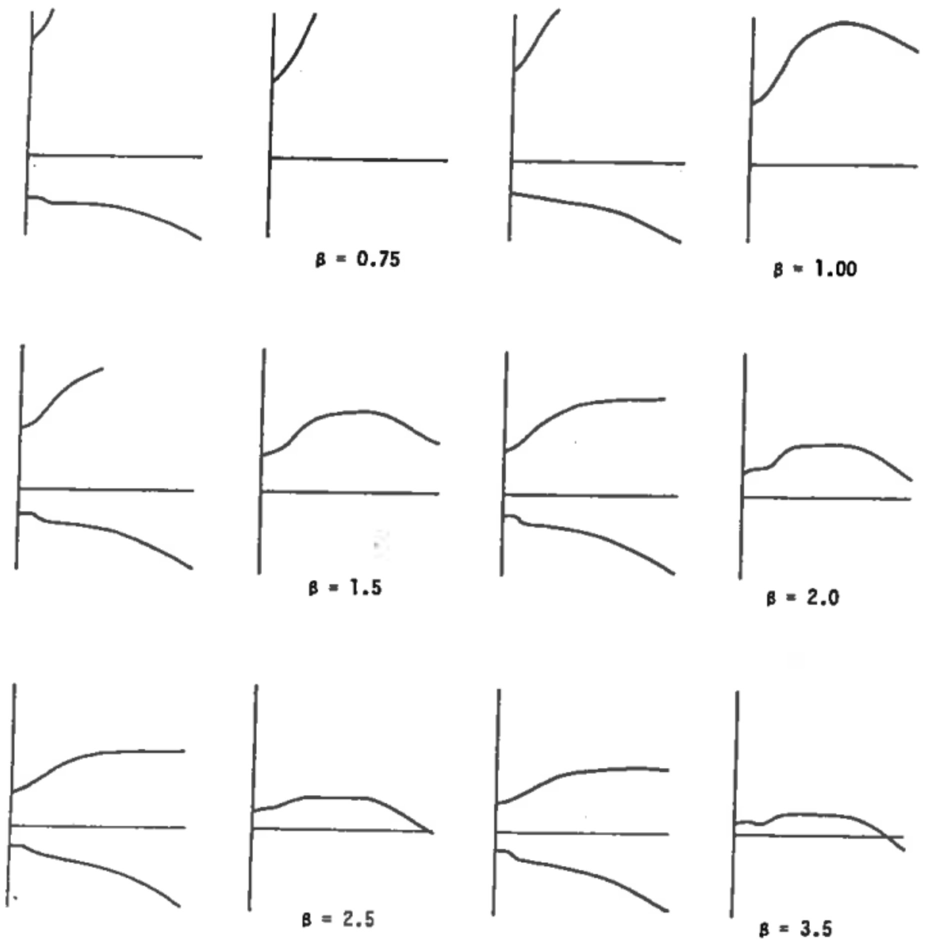
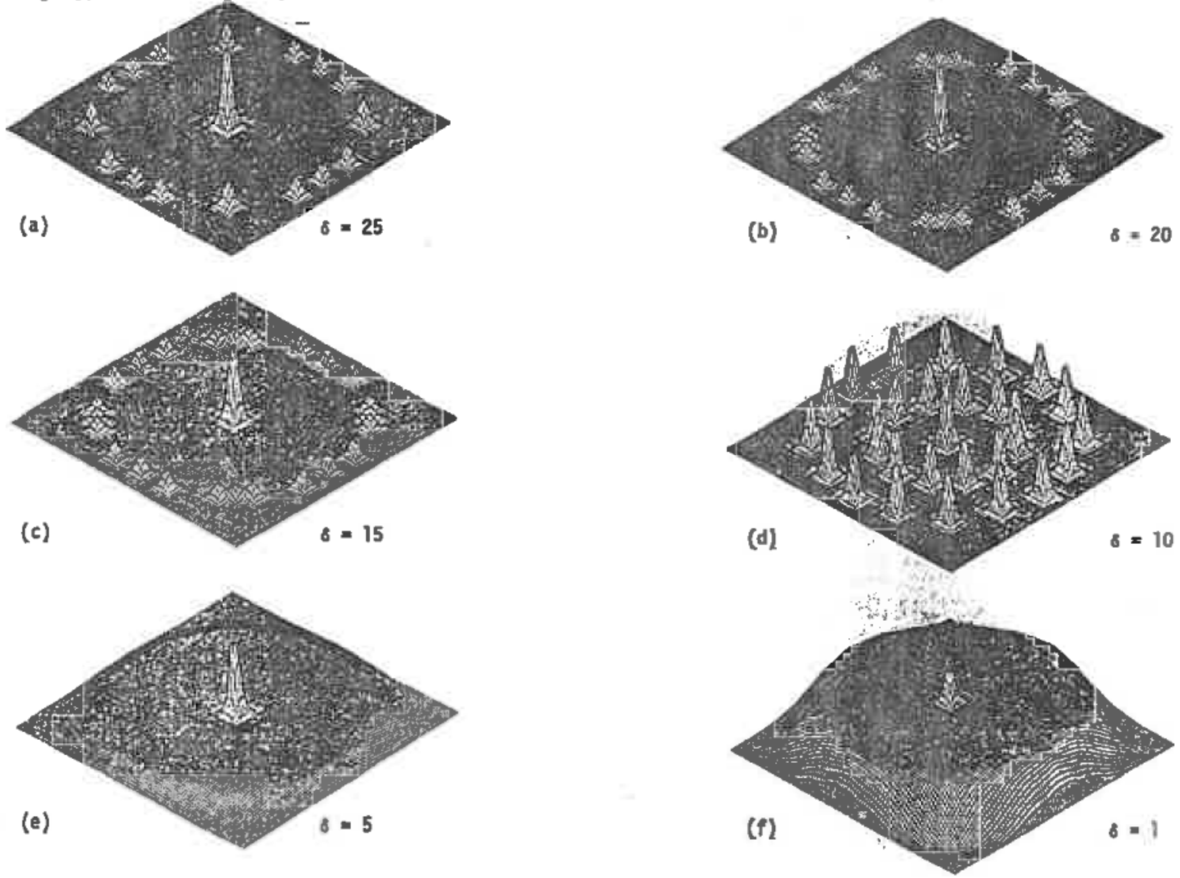


Fig. 23 : Changes for β for W_j set patterns from the spatial equilibrium patterns

Fig. 24 : Variation in δ



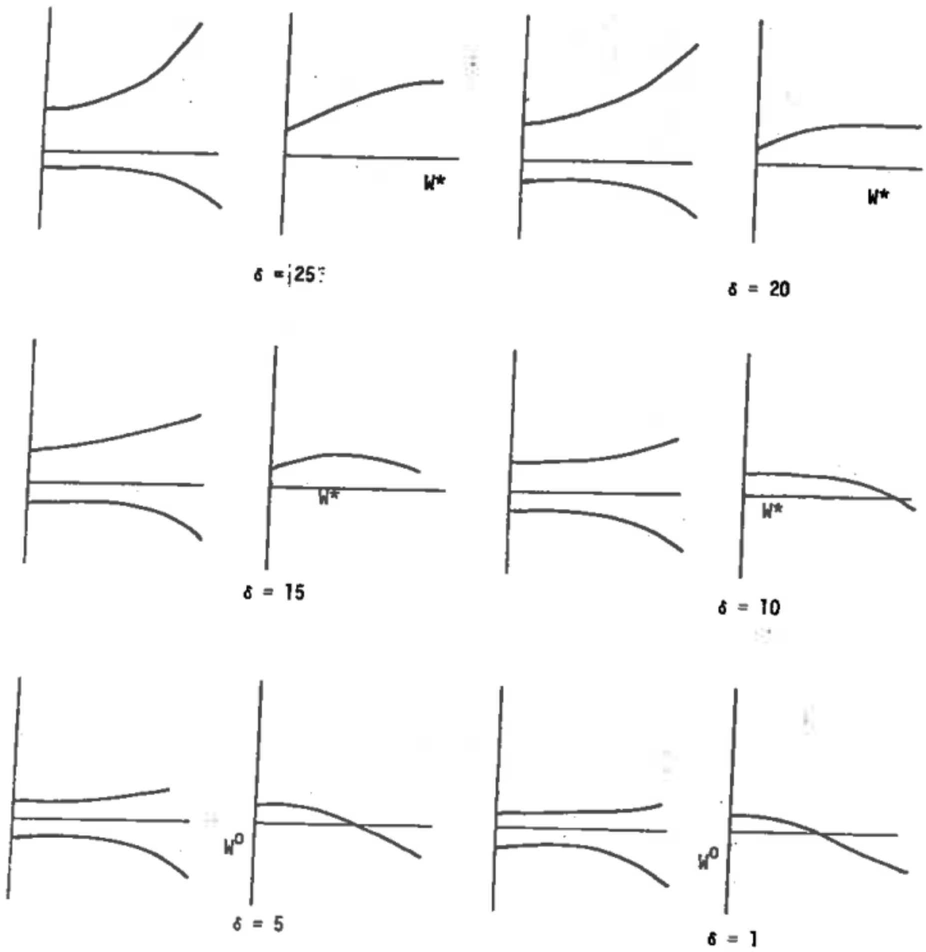


Fig. 25: Changes in δ for W_j set patterns from the spatial equilibrium patterns