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IMPROVING SUPPLY SIDE REPRESENTATIONS IN URBAN MODELS, WITH SPECIFIC REFERENCE TO CENTRAL PLACE AND LOWRY MODELS.

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ABSTRACT

A mathematical programming formulation of central place theory is outlined. Concern focuses on the Kuhn-Tucker optimality conditions and interpretation demonstrates a consistency between familiar theoretical concepts, such as the threshold of a good or service, and their underlying economic rationale.

1. INTRODUCTION

In this note we present an approach for examining how the concept of threshold in central place theory (at the intra-urban level) can be given a firmer economic foundation. Attention is drawn to the fact that this concept has direct analogies with the minimum size constraint which is a traditional feature of the Lowry-type urban models. By presenting a model which determines this constraint endogenously within a competitive framework, it proves unnecessary to input *a priori* the value of the minimum size constraint. This is seen as theoretically more satisfying.

Firstly, we discuss some aspects of central place theory and Lowry models, and then we describe a model based on a traditional bid - rent approach, which can be embedded into a wider Lowry-type framework. Finally, we make some remarks about the desirability of a comprehensive structure in which central place theory can be placed.

2. CENTRAL PLACE THEORY : SOME SPECIFIC COMMENTS

It is the aim of this paper to demonstrate how the threshold of a good and service (the total effective demand required to support a particular function) can be determined endogenously through a market mechanism (and not exogenously equated to say, fixed production costs). Moreover, the framework permits the service sector to be analysed as part of the whole urban system, which we believe is conceptually and theoretically more satisfying; scarce urban resources are competed for by a range of consumers, to consider any one in isolation may lead to erroneous conclusions. In fact, it is only within the context of the whole system that the parts can be fully comprehended.

Since Christaller's (1933) desire to find 'a general explanation for the size, number and distribution of towns', central place theory has always held the special attention of geographers; indeed, Bunge (1962) described central place theory as 'geography's finest intellectual product'. Yet doubts continue to be expressed as to the value of the theory as an explanatory tool (see, for example, White's (1978) review of Beavon's (1977) recent book). On a more optimistic note, Batty (1978), while still maintaining certain reservations, encourages the development of more realistic and comprehensive models and speculates that many difficulties could be overcome within the next ten years.

In some ways a basic problem is finding an adequate language to express the theoretical notions of central place theory. Current work on the geometry of central place systems (for example, Aloa, et al., 1977) offers great potential to present new insights. Similarly, mathematical programming formulations can be developed which are consistent with the theoretical foundations of central place systems (Puryear, 1975; Wilson, 1978). A significant facet of a spatial interaction and activity (SIA) representation of a central place system is the employment of a spatial zoning system (as opposed to a continuous spatial approach); centres' locations are described by discrete spatial units which make up the area of analysis. The spacing and size of facilities can, therefore, be readily investigated.

3. THE LOWRY MODEL AND SERVICE PROVISION

Lowry's (1964) attempt to integrate the various industrial, service and housing sectors of the urban system has been extensively built upon (see, for example, Wilson, 1974; Batty, 1976). One of the several shortcomings in the model is that of generating the service sector employment totals by zone. It is typically done in the following way:

$$E^{Rk} = a^k P \tag{1}$$

$$E_j^{Rk} = b^k (q^k \sum_i P_i f(c_{ij}) = d^k E_j^B \tag{2}$$

if

$$E_j^{Rk} > Z^{Rk} \quad , \text{ otherwise } E_j^{Rk} = 0 \qquad (3)$$

where E^{Rk} is employment in retail sector k

P is total population

E_j^{Rk} is employment in retail sector k in zone j

P_i is population in zone i

E_j^B is employment in the basic sector in zone j

c_{ij} is the interzonal tranpsort cost between zones i and j

Z^{Rk} is the minimum employment size in sector k

and a^k, b^k, q^k, d^k are coefficients.

Hence, total employment in sector k is some function of total population (see equation (1)). This is then allocated spatially according to a simple shopping model that incorporates both journey from home and journey from work trips (see equation (2)). If this zonal employment is above some minimum employment size then this is the calculated employment; if not, employment is set equal to zero (see condition (3)).

Our main criticism is that there is little theoretical reason for the input of Z^{Rk} *a priori* in a Lowry model; Z^{Rk} will be a function of supply and demand for goods and thus should be determined *endogenously* within a model of the competitive market situation.

4. CONSIDERATION OF ECONOMIC AND SPATIAL FORCES

That a wide range of economic and spatial forces are at work within the urban system, and that these are prime determinants of spatial structure is generally appreciated. What framework then is required for consideration of service structure and location in an intra-urban context?

One view, albeit simplified, can be seen as an outcome of three processes:

(i) land use competition which is reflected in a spatial rent surface generated by some bid rent process (disaggregated by use). Adoption of the bid rent approach to examine the urban land market follows the work of Alonso (1964) and Herbert and Stevens (1960), which has similarities with Von Thünen's (1826) well-known analysis of agricultural location. Simply stated, a bid rent, B_j, is the maximum amount of rent that a consumer of land is willing to pay for a unit of land in zone j; analysis is commonly formulated as a linear programming problem between competing consumers.

(ii) competition between producers of different services for profits; this competition is both intra- and inter-sectoral (industrial, service, and housing).

(iii) consumer behaviour in the service and housing markets. One approach is that consumers attempt to maximise some utility function (and there are well-known techniques for handling this (Coehlo and Wilson, 1976; Williams, 1977).

5. AN EXAMPLE OF SERVICE FACILITY LOCATION AND SIZE

We begin by investigating competition within a single sector, that of services. (The recent work of Leonardi (1978a; 1978b), analysing facility location by using a log-accessibility function (see also Williams and Senior, 1978) and of Coehlo and Wilson (1976), who considered consumer surplus in their study of the optimum location and size of shopping centres, is especially significant in this direction). The size and distribution of facilities can be ascertained for a profit-maximising type behaviour. Formally, interest is centred on maximising total net profit (T.N.P.), subject to land and revenue constraints.

$$\max_{\{W_j\}} \quad T.N.P. = \sum_j \left(\sum_i S_{ij} - W_j B_j\right) \tag{4}$$

subject to the land constraints,

$$W_j \leq \mu L_j \tag{5}$$

to the constraint balancing revenue and facility size (Harris, 1964; Harris and Wilson, 1979),

$$\sum_i S_{ij} = kW_j \tag{6}$$

and to the non-negativity constraints

$$W_j \geq 0 \tag{7}$$

Here W_j is the size of service facilities in zone j (which is usually taken as a measure of attractiveness); B_j is the bid rent for zone j; L_j is the maximum amount of land available for services in zone j; k is a constant which converts from capacity units to revenue units; and μ is a suitably defined coefficient. S_{ij} is the flow of expenditure from residents in zone i to the facilities in zone j, and it can be determined by using the well-known Huff (1964) and Lakshmanan-Hansen (1965) shopping model

$$S_{ij} = A_i e_i P_i W_j e^{-\beta c_{ij}} \tag{8}$$

where

$$A_i = (\sum_j W_j e^{-\beta c_{ij}})^{-1} \tag{9}$$

given that e_i is the per capita expenditure on goods and services by residents of zone i; P_i is the population of zone i; c_{ij} is the travel cost, suitably defined, from zone i to zone j; and β is a travel impedance parameter.

An intuitive interpretation would be that service provision only occurs when profit is accruing to producers. A threshold arises when marginal profitability is zero. This situation is demonstrated by consideration of the Kuhn-Tucker (1951) conditions necessary for a solution to the model described by equations (4), (5), (6) and (7). To form a new unconstrained problem, we formulate the associated Lagrangian, L,

$$L = \text{T.N.P.} \;+\; \sum_j \lambda_j (\mu L_j - W_j) \;+\; \sum_j \gamma_j (\, kW_j - \sum_i S_{ij}) \qquad (10)$$

The parameters, λ_j and γ_j, are the Lagrangian multipliers associated with constraints (5) and (6), respectively; such association facilitates interpretation later.

The Kuhn-Tucker conditions for maximisation are

$$\frac{\partial L}{\partial W_j} \leq 0 \qquad (11)$$

$$W_j \cdot \frac{\partial L}{\partial W_j} = 0 \qquad (12)$$

and

$$\frac{\partial L}{\partial \lambda_j} \geq 0 \qquad (13)$$

$$\lambda_j \; \frac{\partial L}{\partial \lambda_j} = 0 \qquad (14)$$

$$\frac{\partial L}{\partial \gamma_j} = 0 \qquad (15)$$

which give rise to the constraints (5) and (6). Also we must have

$$W_j \geq 0 \qquad (16)$$

Using economic rationale, provision of services increases as long as marginal revenue is greater than marginal cost. Facility size of a service of a given type is in competitive equilibrium when marginal cost equals marginal revenue. Analysis is focused on two situations : when the service is not provided from zone j (because the threshold conditions have not been met), that is $W_j = 0$, and when the service is provided from zone j (because the threshold conditions have been met), that is $W_j \neq 0$.

In case 1, when $W_j = 0$, we know, from equations (11) and (12), that

$$\frac{\partial L}{\partial W_j} \leq 0 \tag{17}$$

Here, we address

$$\frac{\partial L}{\partial W_j} < 0 \tag{18}$$

as this is the non-trivial situation; the equality part is considered in case 2 where it is of significance for interpretation. Differentiating equation (10) with respect to W_j, we are left with

$$\frac{\partial L}{\partial W_j} = \sum_i \left[1 - \frac{S_{ij}}{e_i P_i}\right] \frac{S_{ij}}{W_j} - B_j - \lambda_j + \gamma_j \left[k - \sum_i (1 - \frac{S_{ij}}{e_i P_i}) \frac{S_{ij}}{W_j}\right] \tag{19}$$

which, for optimality, from equation (17), must be less than zero. That is,

$$\sum_i \left[1 - \frac{S_{ij}}{e_i P_i}\right] \frac{S_{ij}}{W_j} + \gamma_j \left[k - \sum_i (1 - \frac{S_{ij}}{e_i P_i}) \frac{S_{ij}}{W_j}\right] < B_j + \lambda_j \tag{20}$$

It is the interpretation of equation (20) which is of particular interest with respect to case 1, when there is no service provision from zone j ($W_j = 0$). The first term on the left-hand side represents the marginal revenue for zone j; B_j is the bid rent in zone j, λ_j is the Lagrangian multiplier associated with constraint (5) which ensures that the land occupied by service facilities in zone j does not exceed the amount available; k is a constant which converts from capacity units to revenue units; and γ_j can be seen as a financial subsidy to a facility in zone j (which could be differentiated by good in a disaggregated model). Note that this subsidy can be either positive or negative. Thus, equation (20) demonstrates that a potential facility in zone j cannot supply any service because the cost of a unit will not be covered by its revenue. In this situation, the subsidy value, which will be at least sufficient to enable the aggregate demand to be satisfied, is not enough to permit a facility to locate in zone j.

Given the total supply constraint, one must be aware that the competitive equilibrium may not necessarily satisfy this condition. In this situation, if planners deemed that such a service provision level was necessary, the subsidies would be the transfers needed to bring about the desired state.

Such analysis, therefore, has an explicit spatial representation, and from a welfare standpoint, it is indicative of where financial assistance is required if it is deemed desirable to provide a specific service.

Furthermore, to satisfy equations (13, (14) and (15) we must have

$$\frac{\partial L}{\partial \lambda_j} = \mu L_j - W_j \geqslant 0 \qquad (21)$$

which means that total service floorspace must not exceed that available in each zone j, and

$$\frac{\partial L}{\partial \gamma_j} = kW_j - \sum_i S_{ij} = 0 \qquad (22)$$

which ensures a matching of supply and demand.

In case 2, when $W_j \neq 0$, we know, from equations (11) and (12), that

$$\frac{\partial L}{\partial W_j} = 0 \qquad (23)$$

which, from equation (19), can be written as,

$$\sum_i \left[1 - \frac{S_{ij}}{e_i P_i}\right] \frac{S_{ij}}{W_j} - B_j - \lambda_j + \gamma_j \left[k - \sum_i \left(1 - \frac{S_{ij}}{e_i P_i}\right) \frac{S_{ij}}{W_j}\right] = 0 \qquad (24)$$

After rearrangement it becomes

$$\sum_i \left[1 - \frac{S_{ij}}{e_i P_i}\right] \frac{S_{ij}}{W_j} + \gamma_j \left[k - \sum_i \left(1 - \frac{S_{ij}}{e_i P_i}\right) \frac{S_{ij}}{W_j}\right] = B_j + \lambda_j \qquad (25)$$

Thus, conditions for service provision from zone j, ($W_j \neq 0$), are that the marginal revenue equals the marginal costs, that is, only normal profits accrue to the supplier. The spatially varying terms, such as γ_j, B_j and λ_j, ensure that excess profits in the system are competed away. This raises new theoretical issues with respect to 'marginal hierarchical' goods and services and 'non-marginal hierarchical' goods and services in a development of a disaggregate model and also to the analysis of the emergence or disappearance of facilities in particular locations using normal profits as a benchmark. A final point is that spatial differences in λ_j, which may be reflected in price differentials over space, results in consumers not necessarily having to travel to their nearest supply facility.

6. EXTENSIONS AND COMPLICATIONS

The above system of equations (4), (5), (6) and (7) forms only a partial model of the interacting urban markets. Taking into account our previous comments, we need to disaggregate the model and incorporate an explicit and realistic consideration of employment, residential and service supply demand inter-relationships. One approach that is potentially opposite is the mathematical programming version of the Lowry model, as presented by Coehlo and Williams (1977); we can adapt this model to take account of our proposals.

The model, incorporating the bid rent approach discussed above, determines the housing and service stocks' configuration to maximise total locational surplus for the aggregate population, P. One facet of the Lowry-type models is extended in this representation. The usual iterative procedure can be recognised as a causal sequence in which residential location is dependent on the residential zones' attractiveness and the transport costs associated with work-trip and service facility location is dependent on their attractiveness and the transport costs associated with shopping trips (see Wilson, 1974). Thus, the selection of service facilities is founded on residential location, yet residential location is independent of accessibility to the service facilities. The Coehlo and Williams (1977) model, outlined above, overcomes this assumption by representing, at the individual level, the joint decision process of residential and service location.

The model is formally written as

$$\max_{\{\underline{T},\, \underline{S},\, \underline{H},\, \underline{W}\}} Z = -\frac{1}{\beta^W} \sum_{ijk} T_{ij}\left(\log \frac{T_{ij}}{H_i^k} - 1\right) - \sum_{ij} T_{ij} c_{ij}^W$$

$$-\sum_k \frac{1}{\beta^s} \sum_{ij} S_{ij}^k \left(\log \frac{S_{ij}^k}{W_j^k} - 1\right) - \sum_{ij} S_{ij}^k\, c_{ij}^{sk}$$

$$-\sum_{jk} B_j^k W_j^k - \sum_{ik} B_i^k H_i^k \tag{26}$$

subject to

$$\psi \sum_k \theta^k \sum_j \left(T_{ij} + \frac{P_i^B}{\psi}\right) - \sum_{jk} \frac{n}{\rho^k} S_{ij}^k = 0 \tag{27}$$

$$\sum_i T_{ij} - \sum_k \frac{1}{\rho^k} \sum_i S_{ij}^k = n E_j^B \tag{28}$$

$$A_j - A_j^u - A_j^B - \sum_k H_j^k - \sum_k W N_j^k \geq 0 \tag{29}$$

$$\underline{T} \geq 0,\ \underline{S} \geq 0,\ \underline{H} \geq 0,\ \underline{W} \geq 0 \tag{30}$$

where β^w is the travel dispersion (elasticity) parameter for works trips; T_{ij} is the number of people living in zone i, working in zone j; H_i^k is the attractiveness weight (or size) of residential activity of type k in zone i; c_{ij}^w is the interaction cost component for work trips from zone i to zone j; β^s is the travel dispersion (elasticity) parameter for service trips; S_{ij}^k is the flow of expenditure from residents in zone i to facilities of type k in zone j; c_{ij}^{sk} is the interaction cost component for services of type k from zone i to zone j; B_j^k is the maximum amount of rent that a consumer type k of land is willing to pay for a unit of land in j; P_i^B is the basic population in zone i; E_j^B is the basic employment in zone j; A_j is the total land area in zone j; A_j^u is the unusable land in zone j; A_j^B is the land used by the basic sector in zone j; and ψ, θ^k, ρ^k, η are suitably defined parameters.

Previous discussion of the services as a closed system meant that transfer of benefits between sectors did not occur. This feature has to be considered when both residential and service sectors are included, and it should be noted that care has to be taken in formulating the objective function in order not to double count the benefits.

Constraint (27) describes the required accounting consistency between the residential population predicted through the work trip matrix, T_{ij}, and the service trip matrix for each facility k, S_{ij}^k, scaled by the parameters ψ, θ^k, η and ρ^k. The economic base foundation of the model, with the basic employment generating the total employment (as seen through the T_{ij} matrix), is portrayed by constraint (28). Constraint (29) ensures land availability for particular uses is satisfied, and the non-negativity constraints (30) are given in standard vector notation.

It should be noted that the dimensionality of the primal function is restrictive in analyses when the number of zones is large or when the model is highly disaggregated. Employment of the associated dual function, following

Coehlo and Williams (1977), results in the reduction of the dimensionality of the problem and also provides a number of dual variables which have an economic interpretation.

7. DISCUSSION

Whilst it is readily appreciated that this is only a preliminary outline of a topic which deserves and requires much more extensive research, sufficient details have been provided to demonstrate the potential of pusuing future analyses in this direction. It is hoped that new theoretical insights may be generated and this may be enhanced by performing a range of numerical analyses (in a similar way to that followed by Wilson and Clarke (1979) investigating catastrophe theory in urban retailing structure); indication of suitable frameworks for empirical studies should also result from this work. Moreover, present concern with the dynamics of facility location and size, such as the employment of difference and differential equations by White (1977) and by Wilson (1978), is wholly compatible with the mechanisms described here.

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REFERENCES

Alao, N., Dacey, M. F., Davies, O., Denike, K. G., Huff, J., Parr, J. B., and Webber, M. J. 1977. Christaller Central Place Structures an Introductory Statement. Department of Geography, Studies in Geography No. 22 Northwestern University, Evanston, Illinois.

Alonso, W. 1964. Location and Land Use. Harvard University Press, Cambridge, Mass.

Batty, M. 1976. Urban Modelling : Algorithms, Calibrations, Predictions. Cambridge Urban and Architectural Studies, Cambridge University Press : Cambridge.

Batty, M. 1978. Urban Models in the Planning Process. In Herbert D. T. and Johnston R. J. (1978) (eds) Geography and the Urban Environment Progress in Research and Applications. Volume 1. pp 63-134.

Beavon, K. S. O. 1977. Central Place Theory : A Reinterpretation. Longman : London.

Bunge, W. 1962. Theoretical Geography. Lund : C.W. Gleerup.

Christaller, W. 1933. Central Places in Southern Germany. (translated by Baskin, C. W. (1966)). Prentice-Hall : Englewood Cliffs, N. J.

Coehlo, J. D. and Williams, H. C. W. L. 1977. On the design of Land Use plans through locational surplus maximisation. Papers of the Regional Science Association, forthcoming. (presented at the 17th European Congress Regional Science Association, Krakow, Poland).

Coehlo, J. D. and Wilson, A. G. 1976. The Optimum Location and size of Shopping Centres. Regional Studies, 10. pp. 413-421.

Harris, B. 1964. A Model of Locational Equilibrium for Retail Trade. Penn-Jersey Transportation Study, Philadelphia.

Harris, B. and Wilson, A. G. 1978. 'Equilibrium Values and Dyamics of Attractiveness Terms in Production - Constrained Spatial Interaction Models' Environment and Planning A, 10. pp. 371-388.

Herbert, D. J. and Stevens, B. H. 1960. 'A Model for the Distribution of Residential Activity in Urban Areas'. Journal of Regional Science, 2. pp. 21-36.

Huff, D. L. 1964. 'Defining and Estimating a Trading Area' Journal of Marketing, 28. pp. 37-48.

Kuhn, H. W. and Tucker, A. W. — 1951. 'Nonlinear Programming'. in Neyman, J. (ed) Proceedings of the 2nd Berkeley Symposium on Mathematical Statistics and Probability. pp. 481-492.

Lakshmanan, T. R. and Hansen, T. R. — 1965. 'A Retail Market Potential Model' Journal of the American Institute of Planners, 3. pp. 134-143.

Leonardi, G. — 1978a. Optimum Facility Location by Accessibility Maximising. Environment and Planning A, 10,p 1287-1306.

Leonardi, G. — 1978b. Some Mathematical Programming Ideas within a generalized spatial interaction and Activity Framework. Presented : R.S.A. Conference, London. September. 1978.

Puryear, D. — 1975. A Programming Model of Central Place Theory. Journal of Regional Science, 15, pp. 307-316.

von Thunen, J. H. — 1826. The Isolated State. (translated by Wartenburg, C. M. with an introduction by Hall, P. 1966). Oxford University Press : Oxford.

White, R. W. — 1977. Dynamic Central Place Theory : Results of a Simulation Approach. Geographical Analysis, 9, pp. 514-532.

White, R. W. — 1978. Book Review -: Beavon, K. S. O. (1977) Central Place Theory : A Reinterpretation, Longman : London. Environment and Planning, A, 10.

Williams, H. C. W. L. — 1977. 'On the Formation of Travel Demand Models and Economic Evaluation Measures of User Benefit'. Environment and Planning A, 9, pp. 285-344.

Wilson, A. G. — 1974. Urban and Regional Models in Geography and Planning. John Wiley and Sons : London.

Wilson, A. G. — 1978. Spatial Interaction and Settlement Structure : Towards an Explicit Central Place Theory, in Karlqvist, A., et al. (eds) Spatial Interaction Theory and Planning Models. North Holland : Amsterdam. pp. 137-156.

Wilson, A. G. and Clarke, M. C. — 1979. Some Illustrations of Catastrophe Theory Applied to Urban Retailing Structures. in M. Breheny (ed) London Papers in Regional Science (forthcoming).