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Some new sources of instability and oscillation in dynamic models of shopping centres and other urban structures.

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Abstract

May and others have shown that simple non-linear difference equations can exhibit very complicated dynamic behaviour. These results and associated methods are briefly summarised. It is then shown that they offer new insights into the dynamics of shopping centre developments both in respect of these being modelled by difference equations and when they are modelled using differential equations which are then integrated numerically. The methods are applied to two different dynamic shopping models and, in a concluding section, some speculations are presented on the effect of these ideas on more complicated and realistic models.

1. Complicated dynamics and simple non-linear difference equations

May (1974, 1975, 1976, May and Oster, 1976) present a number of interesting results relating to first order difference equations of the form

$$X_{t+1} = F(X_t) \tag{1}$$

The same kinds of results can be obtained for a variety of functions, F, but here we use the main example of his 1976 paper which turns out to be directly applicable to shopping model dynamics. First, however, we comment on the distinctions involved between modelling dynamical systems through difference equations or differential equations. If the main state variables are changing continuously, then differential equations are appropriate; if the events can be considered to be discrete, then difference equations offer the correct formulation. In ecology, if populations have relatively long lives relative to the time periods of analysis, then differential equations represent the correct formulation. If, however, generations do not overlap, but the new populations are still dependent on those of the previous time periods, as with insects, then the model should be formulated in terms of difference equations.

In the shopping centre case, models have been presented in terms of differential equations (Wilson, 1976; Harris and Wilson, 1978) but have been simulated in terms of difference equations (White, 1977, 1978). It turns out that a difference equation formulation may be more appropriate in some circumstances, but we reserve this discussion for later and now concentrate on May's example. Let

$$N_{t+1} = N_t(a-bN_t) \qquad (2)$$

be a first order difference equation describing the growth of a population, N. t is a time subscript and a and b are constants. This is one possible difference-equation equivalent of the logistic equation of growth. By the transformation

$$X = bN/a \qquad (3)$$

it can be written in the more convenient form

$$X_{t+1} = aX_t(1-X_t) \qquad (4)$$

X can then be considered to vary between 0 and 1, though if X ever exceeds 1, it then diverges to $-\infty$ and the negative numbers are unrealistic in many applications. However, we will ignore this complication here and assume the value of a and the initial conditionavoid this. It can always be avoided by using the alternative form of logistic equation

$$X_{t+1} = X_t \exp[r(1-X_t)] \qquad (5)$$

though this is more difficult to handle.

The relation between X_{t+1} and X_t can be plotted as a humped curve, as in Figure 1. It attains a maximum at $X = \frac{1}{2}$ of $a/4$, and since X must remain less than 1, this implies $a < 4$. We also require $a > 1$, or $X_{t+1} \to 0$ for large t. Thus, for non-trivial dynamic behaviour

$$1 < a < 4 \qquad (6)$$

The possible equilibrium values of X are found by putting $X_{t+1} = X_t$ in equation (4). This is equivalent to seeking the intersection of the humped curve and the 45^0 line which is also plotted on Figure 1. Thus P is an equilibrium point.

Let $X = X^*$ be the equilibrium point. For later notational convenience, we also write equation (4) in the form of equation (1) with

$$F(X) = aX(1-X) \tag{7}$$

At equilibrium,

$$X_{t+1} = X_t = X^* \tag{8}$$

and so equation (4) gives

$$X^* = aX^*(1-X^*) \tag{9}$$

which has the non-zero solution (for P)

$$X^* = (a-1)/a \tag{10}$$

The slope of the curve at this point is

$$\left.\frac{dF}{dX}\right|_{X = X^*} = a-2aX^* \tag{11}$$

which is

$$\left.\frac{dF}{dX}\right|_{Z = X^*} = 2-a \tag{12}$$

(substituting from (10)).

Consider points within a small increment $\pm\Delta$ on either side of X^*, as in Figure 2. If the slope of the line is between ± 1, and if

$$X_t = X^*+\Delta \quad \text{, say} \tag{13}$$

then

$$X^* < X_{t+1} < X_t \tag{14}$$

which means that the equilibrium point is stable. Otherwise it is unstable. Since we know the slope to be 2-a from (12), for this to lie between ± 1 we must have

$$1 < a < 3 \tag{15}$$

for stability.

We saw earlier that we must have $a > 1$ anyway for non-trivial behaviour, so the interesting condition is $a < 3$. We should note that as a increases, the hump in Figure 1 steepens and it is easy to see that a point will be reached when the modulus of the tangent at the equilibrium point exceeds 1. This occurs when a = 3. When a > 3, the equilibrium solution X^* becomes unstable. It is then possible to see if there is another kind of equilibrium point two time periods apart; that is, satisfying

$$X_{t+2} = F[F(X_t)] \qquad (16)$$

If X_{t+2} is plotted against X_t, the curve has two humps. Three cases are shown in Figure 3. Case (a) has a < 3. There is only one equilibrium point and it is stable. Case (b) shows the 45^0 line touching the curve. This represents the limiting case, a = 3. In case (c), a > 3, the tangent at the original equilibrium point now exceeds 45^0 and is unstable, but there are two new equilibrium points, $X^{(2)*}$ and $X^{(2)**}$. However, there is then another critical value of a which bifurcates into a four-cycle of stable points, then eight and so on, as shown in Figure 4. Beyond a value a_c, the behaviour becomes chaotic. That is, it oscillates without any observable periodic structure.

We now explore how to apply these results, and these method of analysis, to models of shopping centre dynamics.

2. Model 1: linear growth

Consider now a set of shopping centres across a set of discrete zones. Let the size of the centre in zone j be W_j. Then, if D_j is the revenue potentially attracted to j, a suitable differential equation for the growth of W_j is (Wilson, 1976)

$$\dot{W}_j = \varepsilon(D_j - kW_j) \qquad (16)$$

for suitable constants ε and k. The difference equation form which suggest itself is

$$W_{jt+1} - W_{jt} = \varepsilon(D_j - kW_{jt}) \qquad (17)$$

(where, without loss of generality, the time period is taken as one, or as a factor merged into ε). This can be written

$$W_{jt+1} = \varepsilon D_j + (1-\varepsilon k)W_{jt} \tag{18}$$

Although this is a linear first order equation, and therefore does not have the interesting bifurcation properties of May's examples, we can apply his methods. Equation (18) expresses a linear relationship between W_{jt+1} and W_{jt} and an equilibrium will be at the intersection of this line and the 45^0 line

$$W_{jt+1} = W_{jt} \tag{19}$$

Various examples are shown in Figure 5. In relation to stability of equilibrium, the same argument applies as before: the slope of the 'curve' is now of course the gradient of the line and if this is between ±1, then any intersection is stable – the argument of (13) and (14) above still holds. Four cases are distinguished on Figure 5: (a) and (c) are stable equilibrium points; in case (b), there is no equilibrium point with positive W_j; and in case (c), the equilibrium point is unstable. . We can collect these results together in terms of the gradient of the line:

(a) $0 < 1-\varepsilon k < 1$ (20)

(b) $1 < 1-\varepsilon k$ (21)

(c) $-1 < 1-\varepsilon k < 0$ (22)

(d) $-1 > 1-\varepsilon k$ (23)

The interesting and new feature about these relationships is that results about stability are related to general conditions involving two of the parameters in the model (18). Case (b) is immediately seen to be geographically nonsensical: it implies $\varepsilon k < 0$ when both of these parameters should be positive. (a) or (c) will hold provided the product εk is sufficiently small. Indeed, combining (20), (22) and (23), and assuming $\varepsilon k > 0$, the condition can be restated as

$$\varepsilon k < 2 \tag{24}$$

for stability, and

$$\varepsilon k > 2 \tag{25}$$

for instability. This also gives some clue as to the nature of the instabilities in difference equations. They arise because of the time lags involved in responding to a change. The greater the values of ε or k, the more rapid is the change from period to period and the more difficult it is to get back to equilibrium through feedback.

This analysis has been conducted as though D_j was fixed. In practice, of course, it is not and is given by

$$D_j = \sum_i S_{ij} \tag{26}$$

$$= \sum_i \frac{e_i P_i W_j^{\alpha} e^{-\beta c_{ij}}}{\sum_k W_k^{\alpha} e^{-\beta c_{ik}}} \tag{27}$$

since S_{ij}, the flow of revenue from residents of i to shops in j, is given by

$$S_{ij} = \frac{e_i P_i W_j^{\alpha} e^{-\beta c_{ij}}}{\sum_k W_k^{\alpha} e^{-\beta c_{ik}}} \tag{28}$$

e_i is per capita expenditure at i, P_i the population of i and c_{ij} the cost of travel from i to j. α and β are constants. In a previous analysis of equilibrium and stability, the focus has been on the stability of the equilibrium value, once it has been achieved (Harris and Wilson, 1978). Equations (16) or (17) show that the equilibrium point is

$$W_j^* = D_j/k \tag{29}$$

It was shown by Harris and Wilson that the stability of equilibrium depends on the values of the parameters like α, β and k. Here, we have seen that if D_j can be assumed constant, there is an additional condition (24). This can perhaps be interpreted as follows: if an equilibrium value of D_j is calculated using Harris and Wilson (1978) methods, say as D_j^{equil}, then the ability to achieve a stable equilibrium in a simulation will require (24) to hold. Since it is a condition on parameters which are not j-dependent, this presumably means there will be difficulties in simulation in any cases where it is not satisfied. White (1979), for example, has reported simulations of this type which have not converged.

3. Model 2: logistic growth

The model given by (16) and (17) implies a steep rate of growth for W_j from a $W_j = 0$ starting point. This can be slowed down at the origin, but still bounded above, by adding a factor W_j. Equation (16) then becomes

$$\dot{W}_j = \varepsilon(D_j - kW_j)W_j \qquad (30)$$

This does not, of course, change the position of the equilibrium point which is still given by (29). We saw in section 1 that there are at least two versions of difference equations which approximate logistic growth and we work with the one given by equations (2) and (4). The obvious modification of equation (30) is to give

$$W_{jt+1} - W_{jt} = (D_j - kW_{jt})W_{jt} \qquad (31)$$

which can be written

$$W_{jt+1} = [(1+\varepsilon D_j) - \varepsilon kW_{jt}]W_{jt} \qquad (32)$$

This is of the same form as equation (2), and if we write

$$X_j = \frac{\varepsilon kW}{(1+\varepsilon D_j)} \qquad (33)$$

then the equation takes the canonical form (4) with

$$a = 1+\varepsilon D_j \qquad (34)$$

We can then immediately apply May's results on stability. Note that while D_j has the 'dimension' of money, equation (31) shows ε to have the dimension of $(\text{money})^{-1}$, and so εD_j is a dimensionless constant. The 'hump' of the curve in Figure 1 will be steeper for increasing values of either ε or D_j.

A recap of section 1 shows that we require

$$1 < 1+\varepsilon D_j < 3 \qquad (35)$$

for a stable single equilibrium point (using (b)), which is obviously

$$0 < \varepsilon D_j < 2 \qquad (36)$$

Clearly εD_j is always positive, but not necessarily less than 2. As it exceeds 2, then there is first a two period cycle, then a four-period one up to a chaotic regime which sets in at $a = 3.8495$, or $\varepsilon D_j = 2.8495$. We should also recall that the system goes into divergent oscillations if $a > 4$, or $\varepsilon D_j > 3$.

As with model 1, D_j has been treated as a constant in this analysis. Again, a suitable first guess at it would be D_j^{equil} as predicted by the Harris and Wilson (1978) procedure. It is also more interesting in this case that the stability condition is j dependent, and that through D_j^{equil} it is dependent on the effects of any changes in other zones. This suggests the possibility of very complicated dynamic behaviour for a whole system which is evolving through the difference equation (32).

The periodic, chaotic or divergent behaviour which results from εD_j exceeding 2 can arise in two ways which would need to be sorted out in particular empirical cases. First, since ε implicitly contain the time step length, it means that if this is too large there will be problems arising from such a (technical) choice. This means that special care will have to be taken if discrete simulation involving the logistic equation are used - as for example in the work of Allen and Sanglier (1979). Secondly, the instabilities arise in a real sense because the implied feed-back of the decision maker which is represented in the discrete nature of the different equation formulation and it becomes a matter of empirical investigation as to whether these exist or not.

4. Concluding comments

May (1976) has shown that very simple difference equations exhibit very complicated dynamic behaviour and he suggested a number of fields where the results were potentially applicable. We have shown in this paper that they appear to have a direct application in geographical dynamics. It is perhaps a coincidence that the correspondence of equation (3) with May's example is so exact, and of course this involves the restrictive condition that D_j should be treated as a constant. What will be even more interesting will be to explore the consequences of these kinds of bifurcation phenomena in more complicated economic models. For example, a retail model might be linked to a residential location model (Wilson, 1979) and this would, through the P_i's in equation (27), have an impact on the D_j's. For particular values of ϵ, a 'jump' in D_j resulting from a P_i change could then lead, say, to new periodic behaviour in W_j. It is also clear that, though the main argument has been cast in terms of shopping centres, the methods and principles are more widely applicable to other urban structures. There is also beginning to be an extension of May's ideas to interacting populations in ecology - see for example Lawton, Hassell and Beddington (1975) on the investigation of dynamic complexity in prey-predator equations. There is much scope for numerical and empirical experiment and investigation.

References

P.M. Allen and M. Sanglier (1979) A dynamic model of growth in a central place system. Geographical Analysis, 11, 256-72.

J.R. Beddington, C.A. Free and J.H. Lawton (1975) Dynamic complexity in predator-prey models formed in difference equations. Nature, 255, 58-60.

B. Harris and A.G. Wilson (1978) Equilibrium values and dynamics of attractiveness terms in production-constrained spatial-interaction models. Environment and Planning A, 10, 371-88.

R.M. May (1974) Biological populations with non-overlapping generations: stable points, stable cycles and chaos. Science, 186, 645-7.

R.M. May (1975) Biological populations obeying difference equations: stable points, stable cycles and chaos. Journal of Theoretical Biology, 51, 511-24.

R.M. May (1976) Simple mathematical models with very complicated dynamics. Nature, 261, 459-67.

R.M. May and G.F. Oster (1976) Bifurcation and dynamic complexity in simple ecological models. The American Naturalist, 110, 573-99.

R.W. White (1977) Dynamic central place theory: results of a simulation approach. Geographical Analysis, 9, 226-43.

R.W. White (1978) The simulation of central place dynamics: two sector systems and the rank-size distribution. Geographical Analysis, 10, 201-8.

A.G. Wilson (1976) Retailers' profits and consumers' welfare in a spatial interaction shopping model. In I. Masser (Ed.) _Theory and practice in regional science_. Pion, London, 42-59.

A.G. Wilson (1979) Aspects of catastrophe theory and bifurcation theory in regional science. _Papers, Regional Science Association_, to be published. (Working Paper 249, School of Geography, University of Leeds.)

Figure 1.

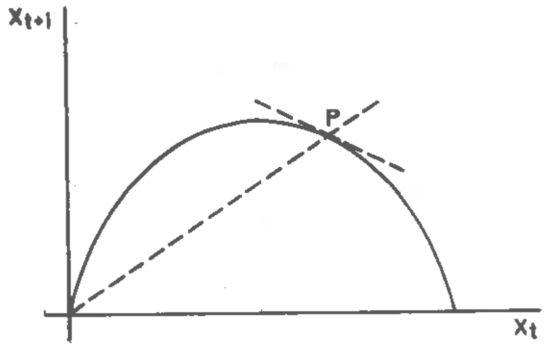
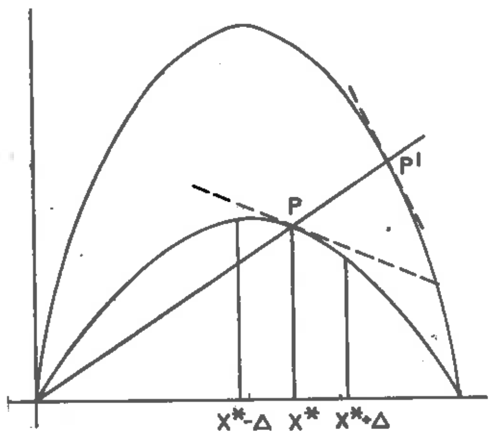


Figure 2.

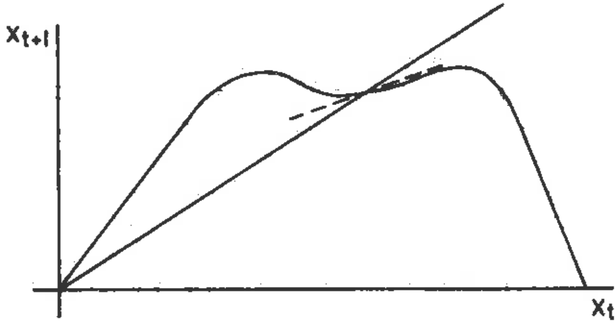


P = stable case

P' = unstable case (higher a)

Figure 3.

(a)



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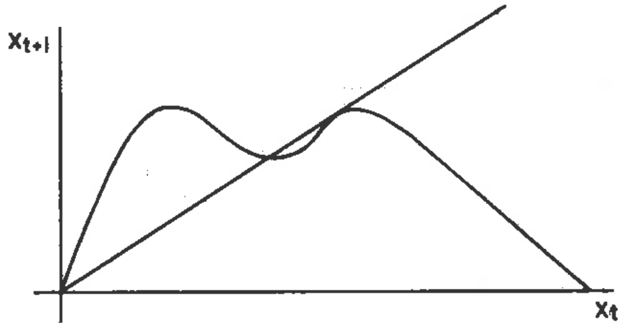

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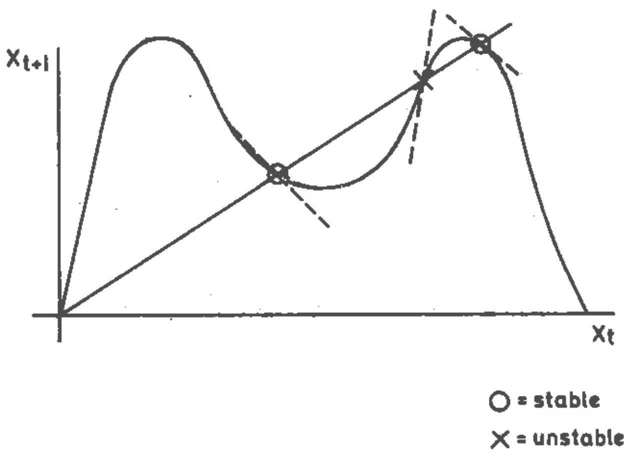


Figure 4.

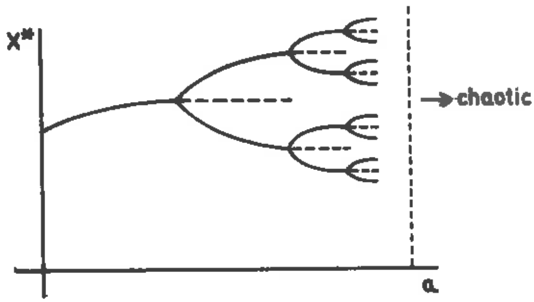
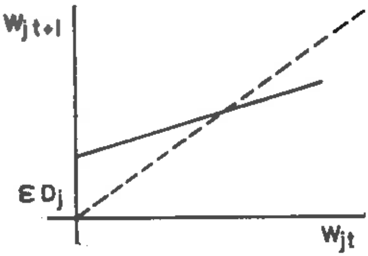


Figure 5.

(a)



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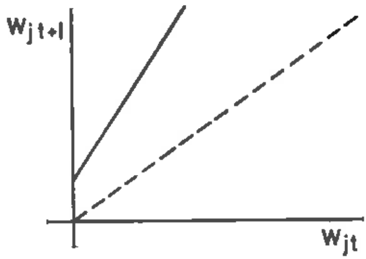

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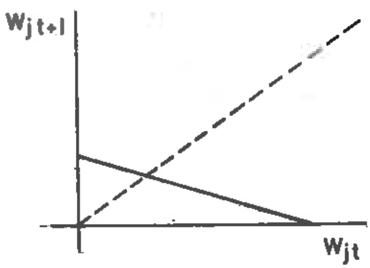

(d)