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Some Comments on the Theoretical and Analytic Structure of Urban and Regional Models.

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To be presented at the 26th North American Meetings of the Regional Science Association, Los Angeles, California, November 9-11, 1979.

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1. Introduction.

It is now widely known that a number of commonly applied location and travel related models may be generated from alternative theoretical premises. Perhaps the best example of this equifinality issue concerns that class of models which involve or are related to the multinomial logit form

$$P_\alpha = \frac{A_\alpha e^{\beta \overline{U}_\alpha(\underline{Z}_\alpha)}}{\sum_\alpha A_\alpha e^{\beta \overline{U}_\alpha(\underline{Z}_\alpha)}} \qquad \alpha = 1, \ldots, N \qquad (1.1)$$

in which P_α, the total occupancy, or probability of occupancy, of a state $\underline{\alpha}$ is expressed as a function of a set of quantities $\{U_1(\underline{Z}_1), \ldots, U_\alpha(\underline{Z}_\alpha), \ldots, U_N(\underline{Z}_N)\}$ – variously described as "generalized costs" or "net utilities" – themselves functions of the attributes $\underline{Z}$ (times, costs; etc.) associated with each alternative, $\alpha = 1, \ldots, N$. The interpretation of the set of constants $\{A_\alpha\}$ and of the parameter β will be considered in later sections.

In this paper we wish to review some of the issues associated with the derivation of this, and more general models, and examine the relationship between their analytic structure and underpinning theoretical statements. More generally we consider the structural similarities between models for different (urban and regional) systems on the one hand, and the similar interpretation of the models and systems they represent which may be traced to their common theoretical base.

We begin with two preliminary sections, the first dealing with the systems of interest, their representation by means of indexed variables, some basic accounting relations and the distinctions between endogenous and exogenous variables particularly with reference to prediction and planning. In the second we discuss the general topics of dispersion, response and aggregation.

The principles of the Entropy maximising and choice theoretic approaches are outlined in sections four and five, respectively.

While we recognize the existence of a broad range of choice theories we concentrate our discussion on recent developments in random utility theory. In section six we discuss how mathematical programming - underpinned by either theoretical approach - provides a convenient framework for the processes of model generation and solution, and the evaluation and design of (transport and land use) plans. In section seven we consider the problems of building dynamic models, and finally, in section eight, offer some concluding remarks.

There is now a voluminous literature relating to the various modelling styles which are being discussed and it is beyond the scope of this paper to attempt a comprehensive review. Accordingly the argument is cast in rather broad terms and a relatively small number of references are cited (which perhaps gives undue weight to our own work in Leeds). However, we hope we have given enough references for the reader who wishes to build a more extensive bibliography to do so by consulting the citation lists of the works mentioned.

2. Representations, Accounting and Constraint Relations.

The general concerns of urban and regional analysis are with the spatial dimensions of the various (demographic, economic and social) processes by which change is propergated through the different systems of interest, and at a superficial level involves the addition of spatial identifiers to the site(s) at which, or between which, a particular process occurs. This introduction of spatial labels to the housing, labour and land systems provide an additional means of segmentation on the demand and supply sides of the economic systems relating to the flow of commodities and performance of activities, and the creation of spatially defined substitutes gives rise to many interesting problems, some of which we shall examine below. We shall be generally concerned with the location of stock, the performance of activities within them and the flow of commodities between these activities (and thereby stock). Typical examples involve the subsystems concerned with the supply of houses and residential location; the supply and use of service centres; and the provision of modal networks and the flows through them. We shall use these as appropriate to illustrate our argument.

The description of the state of a system relates to how it is or might be observed. At the cross section we may observe the number $N(\underline{a}, \underline{b})$ of items (eg. spatial actors - individuals, households, etc.) with characteristics $\underline{a}$ associated with facilities (eg. stock or travel related substitutes) with characteristics $\underline{b}$. $\underline{a}$ and $\underline{b}$ will each consist of a (possibly large) list of attributes. The occupation matrix $\underline{\underline{N}}$ defined over the cross product of states $\underline{a} \otimes \underline{b}$ will often be sparce, particularly if spatial variables are included in the attribute lists. We shall consider the implications of this feature in Section three below.

Examples of this state specification may readily be found in residential location, shopping and other travel related models. Thus, if H_i^{kn} is the number of households of type n living in a house of type k in zone i, then ($\underline{a}$ = n : $\underline{b}$ = i,k). Similarly, with S_{ij}^{hg} as the number of people living in a house of type h using service of type g in zone j, then $\underline{a}$ = ih, $\underline{b}$ = gj. Finally, if T_{ij}^{nkr} is the number of trips by persons of type n living in zone i to zone j by route r on mode k, then we might specify $\underline{a}$ = in; $\underline{b}$ = jkr.

It is clear that the representation of dynamical processes may be introduced by doubling the number of subscripts. It is then possible to form arrays of transition between states. Thus,

$$N_{\underline{ab},\ \underline{a'b'}} \equiv \{\{\underline{a},\ \underline{b}\}_t \rightarrow \{\underline{a}',\ \underline{b}'\}_{t\ +\ \Delta}\} \qquad (2.1)$$

represents the number of items in the state $\underline{ab}$ at time t which make transitions to $\underline{a'b'}$ at t + Δ. For example, if T_{ij} is the number of people travelling from home in zone i to work in zone j, N_{ijpq} may be taken as the number in the (i, j) state at time t who are in the (p, q) state at t + Δ. Such dynamical representations have been considered in different contexts by Rees and Wilson (1977), Byler and Gale (1978), Clarke, Keys and Williams (1979a), and others.

Accounting, conservation or constraint relations may be regarded as general restrictions on the possible values which may be taken by the elements in the matrix $\underline{\underline{N}}$. These will be specific to the problem at hand but independent of the theories of dispersion considered below. They will in general refer either to the independent estimates or restrictions which must be satisfied by partial sums of the matrix (eg. row or column totals) or to logical relationships which exist between different elements of the matrix $\underline{\underline{N}}$. The second form is common in network representation. Both kinds of constraint may co-exist in the same problem, and may refer to either static or dynamical representations. We shall give examples of these below.

The variables, as illustrations so far, represent the outcomes of the interaction of demand and supply: S_{ij}^{kg} tells us how the demand for services at i is met. If we use asterisks - replacing an index to denote summation, we can define aggregates like S_{i*}^{*g} which is the total (satisfied) demand for g at i, and

S^{*g}_{*j} , which is the taken up supply of g at j. We might have defined these terms separately: if e^g_i is the per capita demand for g at i, and if P_j is the population at i, then

$$S^{*g}_{i*} = e^g_i P_i \ , \tag{2.2}$$

and if D^g_j is the total usage at j, then

$$S^{*g}_{*j} = D^g_j \ . \tag{2.3}$$

We may want to relate this to the physical structure supplied, say W^g_j, as

$$D^g_j = KW^g_j \tag{2.4}$$

for some constant K.

This illustrates an important general point that there are many relationships between system variables which can be represented in accounting equations. The array $\{S^{kg}_{ij}\}$ is a set of accounts over space and the right hand sides of Equations (2.2) and (2.3) are sub-totals which may be known independently. It has been an important part of urban modelling strategy to build as much information as possible into such equations and to use them in model building.

Often important structural similarities between different models may be traced to the similar structure of the accounting or conservation relations. An interesting class of conservation relations relate to those systems of allocation or exchange which have a network representation. These include, for example, multicommodity network-flow problems, activity-commodity flow systems, and stock-flow models of transitions in the labour and housing systems (Clarke et al. 1979a).

In the highway assignment problem which has a multicommodity-flow representation, the conservation relations are of the form

$$\sum_{l'} X^i_{ll'} - \sum_{l'} X^i_{l'l} = 0 \qquad \text{for all nodes } l \neq j \tag{2.5}$$

$$= T_{ij} \qquad \text{for } l = j \tag{2.6}$$

in which $X^i_{ll'}$ is the flow from origin i on link ll', and T_{ij} is the number of trips between zones i and j. These well known equations express the conservation of flow (Kirchoff's Laws) which prevents the accumulation of flow at intermediate network nodes l, and the requirement that the flow into a terminal node j corresponds to an independently estimated trip matrix element (see also Section six).

In stock flow models of the labour and housing sectors similar general restrictions may be expressed on the elements $X(\underline{\alpha}, \underline{\beta} \rightarrow \underline{\beta}')$ which represent the transitions in a given time interval, by individuals of type $\underline{\alpha}$ from stock (jobs or houses) of type $\underline{\beta}$ to type $\underline{\beta}'$ (Clarke, Keys and Williams, 1979a, c).

If we subsume in the array of attribute lists $\{\underline{\beta}\}$ a dummy class to allow entry into the system, and distinguish between new and second hand stock by the set lavels N and O, then we may write the following general restrictions on the elements of the $\underline{\underline{X}}$ matrix:

$$\sum_{\underline{\alpha}} \sum_{\underline{\beta}'} X(\underline{\alpha}, \underline{\beta}' \rightarrow \underline{\beta}) - \sum_{\underline{\alpha}} \sum_{\underline{\beta}'} X(\underline{\alpha}, \underline{\beta} \rightarrow \underline{\beta}') - S^V(\underline{\beta}) \leq 0 \quad \underline{\beta}\varepsilon O \tag{2.7}$$

$$\sum_{\underline{\alpha}} \sum_{\underline{\beta}'} X(\underline{\alpha}, \underline{\beta}' \rightarrow \underline{\beta}) \leq S(\underline{\beta}) \qquad \underline{\beta}\varepsilon N \tag{2.8}$$

$$\sum_{\underline{\beta}} \sum_{\underline{\beta}'} X(\underline{\alpha}, \underline{\beta} \rightarrow \underline{\beta}') \leq D(\underline{\alpha}) \tag{2.9}$$

in which $S(\underline{\beta}\varepsilon N)$ denotes the supply of new houses of type $\underline{\beta}$, $S^V(\underline{\beta})$ is the vacant stock of type $\underline{\beta}$ inherited from previous time period(s), and $D(\underline{\alpha})$ is the demand for housing by persons of type $\underline{\alpha}$. Equations (2.7) and (2.8) require that the total <u>satisfied</u> moves to second hand and new stock,* respectively, cannot be greater than the available supply. In the case of the second hand stock, moves to new stock and internal moves create the supply to which moves are made. Equation (2.9) expresses the bound created by demand for housing by persons or households of type $\underline{\alpha}$. The equations (2.7) and (2.8) have the same general structure as equations (2.5) and (2.6). The rather more general nature of the former allow, through inequalities, the accumulation (of vacancies). Slack variables may, of course, be added to render the equations (2.7) - (2.9) equalities. The first set of equations, (2.5) and (2.6), associated with the assignment problems represent a "stock-flow" system at a single point in time - the flow here being physically defined over space - the second set, (2.7) - (2.9), corresponds to a "stock-flow" dynamical system in which the flow refers to the transitions through time.

The implications of these <u>structural</u> similarities will be further explored in Section six.

Similar types of dynamic stock-flow accounting relation are found, for example, in the dynamics of households and firms and are represented in the differential or difference equations for the rate of change of the occupation matrix with elements $N_{\underline{\rho}}(t)$, which have the form

$$\frac{\Delta N_\rho(t)}{\Delta t} = \sum_{\rho'} N_{\rho'\rho}(t, t + \Delta) - \sum_{\rho'} N_{\rho\rho'}(t, t + \Delta). \tag{2.10}$$

This is an identity in which $N_{\rho\rho'}(t, t + \Delta)$ is the total number of transitions made between the states $\underline{\rho}$ and $\underline{\rho}'$ in the interval $(t, t + \Delta)$. The addition of the commonly adopted Markovian assumption allows the system of equations to be written in the general form

$$\frac{\Delta N_\rho(t)}{\Delta t} = \sum_{\rho'} A_{\rho\rho'}(t)\, N_{\rho'}(t') \tag{2.11}$$

and this provides the basis for many dynamical urban models, as we shall discuss further in Section seven.

The argument so far can be summarized as follows: it is useful and important to characterize systems of interest by appropriate arrays of (superscripted and subscripted) variables and to link these, in effect using known information, by accounting, conservation or constraint relations. We conclude with a number of remarks about which of these variables should be taken as endogenous in the context of the models being discussed in this paper. Typically, the variables which represent peoples allocation to activities as flows will be the endogenous variables of the models. One degree of complication is that, often, one such array of variables will form an input to another sub-model - and hence be exogenous for that; but all such variables will be called endogenous. The remaining, exogenous variables, will usually be in one of two categories which are defined in relation to the forecasting uses of the models. First, there are those for which trend forecasts have to be supplied and secondly, those which can be controlled and planned in some way. For example, in the 'use of services' model mentioned above, the array $\{e_i^g\}$ will usually be exogenous in the first sense and the array $\{W_j^g\}$ exogenous in the second. These may be called 'trend' and 'planning' variables respectively. It is also important to note that activity or interaction models may then be embedded within a planning-optimizing framework (cf. section six below) and so be estimated, but in the terminology used here, this will not make them 'endogenous' variables. (We should also note the other possible semantic confusion that some behavioural sub-models are built on the basis of hypothesised optimizing behaviour, and a sharp distinction should be maintained between behaviour optimisation and planning optimisation). The final complication is that it is not always clear which exogenous variables are trend and which are planning variables. Different definitions are appropriate in different circumstances, but the distinctions should be made carefully and explicitly in any given set of circumstances.

3. Dispersion, Response and the Aggregation Problem.

Before encountering the specific theoretical approaches to model formation in Secions four and five a few words on dispersion, the response properties of models, and the aggregation problem are in order.

It is important at the outset to record the role played by the parameter set $\underline{\theta}$ in the general model form

$$P_\rho = f_\rho(\overline{U}_1(\underline{Z}_1), \ldots., \overline{U}_\rho(\underline{Z}_\rho), \ldots., \overline{U}_N(\underline{Z}_N) : \underline{\theta}) \tag{3.1}$$

and the parameter β in the multinomial logit form (1.1) in particular. The parameters $\underline{\theta}$ appear in a number of analytic properties of the models among which are included (Thrift and Williams, 1979):

(i) Moment characteristics:

$$\sum_\rho P_\rho \overline{U_\rho^n} = L_n(\underline{\theta}) \tag{3.2}$$

in which the functional form L_n depends on both the moment order n, but also on the analytic function f.

(ii) Dispersion characteristics:

$$\frac{P_\rho}{P_{\rho'}} = \frac{f_\rho}{f_{\rho'}} = G_{\rho\rho'}(\overline{U}_1, \ldots., \overline{U}_N : \underline{\theta}). \tag{3.3}$$

It is well known that for the multinomial logit model the ratio $G_{\rho\rho'}$, given by

$$G_{\rho\rho'}(\underline{\theta}) = e^{\beta(\overline{U}_\rho - \overline{U}_{\rho'})}, \tag{3.4}$$

embodies the stochastic version of the "independence from irrelevant alternatives" axiom, which will be encountered again in Section five.

(iii) Elasticity Characteristics:

$$\mathcal{E}(\rho;\, \rho'\,(\ell)) = \frac{Z^{\ell}_{\rho'}}{P_\rho} \cdot \frac{\partial P_\rho}{\partial Z^{\ell}_{\rho'}} \tag{3.5}$$

$$= \left\{ \frac{Z^{\ell}_{\rho'}}{\overline{U}_{\rho'}} \cdot \frac{\partial \overline{U}_{\rho'}}{\partial Z^{\ell}_{\rho'}} \right\} \cdot \left\{ \frac{\overline{U}_{\rho'}}{P_\rho} \cdot \frac{\partial P_\rho}{\partial \overline{U}_{\rho'}} \right\} \tag{3.6}$$

$$\equiv \mathcal{H}_{\rho'\ell}(\underline{\theta}),\ \mathcal{E}_{\rho\rho'}(\underline{\theta}), \tag{3.7}$$

in which

$$\underline{Z}_\rho = \{Z^1_\rho,\ \ldots\ldots,\ Z^{\ell}_\rho,\ \ldots\ldots\ Z^M_\rho\}. \tag{3.8}$$

For the logit model (1.1) the elasticity matrix elements $\mathcal{E}_{\rho\rho'}$ are given by

$$\mathcal{E}_{\rho\rho'}(\beta) = \beta \overline{U}_{\rho'}(\delta_{\rho\rho'} - P_{\rho'}) \tag{3.9}$$

in which

$$\begin{aligned} \delta_{\rho\rho'} &= 1 \qquad \text{for } \rho = \rho' \\ &= 0 \qquad\quad\ \ \rho \neq \rho'. \end{aligned} \tag{3.10}$$

We emphasise that these analytic properties depend only on the functional form of the model and not on the specific theory from which it is derived. In each theoretical framework an additional and specific interpretation is endowed on the parameters, and we shall record those in the two sections to follow. The basis for modelling the response to policy stimuli in the cross sectional approach is founded on the "intersection" of the above properties. In this indirect approach the parameter set $\underline{\theta}$ obtained from the (related) moment and dispersion properties are inputs to the elasticity functions (3.5) - (3.10). The measurment of dispersion is central to the development of response models from cross sectional data.

Ultimately the variability in behaviour and response to change is recorded at the level of the individual decision unit, be it a firm, individual, household, or government department. Whether one is concerned with micro- or macro- models, a behavioural perspective which aspires to an <u>explanation</u> of response, inevitably involves a consideration of the choices available to individuals, the set of constraints eliminating infeasible options, and some representation of the decision process involved in selecting an option. A "first principles" approach to model development, based on an explicit incorporation of these facets, is an attractive prospect but may, in the process of aggregation over decision making units, involve some assumptions which are unacceptably restrictive, and often demand information which is simply not available. Nevertheless the behavioural approach is characterized by a search for sufficient information in a "smallest unit representation" to allow a statistical estimation of choice/constraint models which will account for the variation in behaviour observable at the cross-section.

While in the micro-approach an emphasis is placed on the analysis of variability in choice contexts, it should be emphasised that this style of work does not have a monopoly of "behavioural" considerations. In both micro-behavioural, and what is generally referred to as the "aggregate" approach, market (or merely population) segmentation is invariably adopted to account for dispersion in the patterns of behaviour, arising from (some of) the observable differences between spatial actors. In the past this process has not always been performed particularly well, especially in the traditional aggregate style.

Before considering the aggregation process in greater detail we shall consider the choice contexts themselves (Thrift and Williams, 1979; Clarke, Keys and Williams, 1979a), and define the context

$$D_i^{\underline{a}} \; \{\{\underline{A}_p(\underline{Z}_p)\}_i : \underline{\psi}_i^{\underline{a}}\} \; \mathcal{P}_i \qquad (3.11)$$

in which is represented

$D_i^{\underline{a}}$: a decision maker i with characteristics $\underline{a}$

$\underline{\Psi}_i^a$	:	the set of constraints to which the decision maker is subject
$\{A_\rho(\underline{Z}_\rho)\}_i$	:	the set of alternatives available to i from which a feasible set is defined
$(\underline{Z}_\rho)_i$	:	the attributes $\{Z_\rho^\ell\}_i$ of A_ρ, as perceived by i, relevant to the decision making process
$\mathfrak{P}_i$	:	the set of objectives or rules governing the choice process.

The characteristics of individual decision makers may be compounded into a vector $\underline{c}^*$ relevant to the explanation of current patterns of behaviour. A sub-set of these, $\underline{c}$, will be observable in a given set of data.

It is clear from the choice contexts that many aspects can contribute to dispersion in observed behaviour, including: misperception of attributes; information limitations on behalf of the decision maker i and the observer, the heterogeneous disposition of constraints, taste variation, etc.

In a search for explanations of behaviour definitional problems associated with the decision unit and accompanying variables are often encountered. These frequently accompany situations in which organizational structure and asset sharing are involved, as for example in the treatment of car availability in activity-travel behaviour of a household. The process of defining an appropriate decision maker, or more generally embracing organizational structure, often involve challenging modelling problems.

In forecasting system behaviour, including the response to particular policy stimuli, we are usually interested in certain aggregate information which will be expressed as $N(\underline{I}, \underline{K})$ - the number of individuals (or households etc.) with characteristics $\underline{I}$, out of a population N, who are associated with a state with characteristics $\underline{K}$. This variable may be modelled directly at the aggregate level, or be derived from disaggregated relations through the following summation process

$$N(\underline{I}, \underline{K}) = \underset{\underline{c}\varepsilon\underline{I}}{S} \; \underset{A_{\underline{\rho}}^{c}\varepsilon\underline{K}}{S_c} \; \phi(\underline{c}, \{\underline{Z}\}) \, P_{\rho}^{\underline{c}} \, (\{\underline{Z}\}|\underline{\theta}(\underline{c})) \qquad (3.12)$$

in which $\phi(\underline{c}, \{\underline{Z}\})$ is the density function for the distribution of the observable characteristics $\underline{c}$ and the attribute set vectors $\{\underline{Z}\}$ over the population, while P_{ρ} is the probability that an individual with attributes and a choice set characterized by $\underline{c}$ and $\{\underline{Z}\}$, respectively will select A_{ρ}. $\underset{A_{\underline{\rho}}\varepsilon\underline{K}}{S}$ and $\underset{\underline{c}\varepsilon\underline{I}}{S}$ denote, respectively, summation (integration over continuous variables, addition over discrete variables) over all alternatives in the class $\underline{K}$ and individuals in the market segment defined by the attributes of $\underline{I}$.

In Equation (3.12) P_{ρ} is expressed as a parametric function of the attribute sets $\{\underline{Z}\}$, in which the vector of parameters $\underline{\theta}$ will, in general, vary over the segments of the demand, and thus depend on the attributes of $\underline{c}$.† P will in practise be of the form (1.1) or more generally (3.1), but it is important to realise that it may equally well be an algorithm or heuristic process by which one performs the aggregation over decision contexts (3.11) associated with individuals <u>with the same observable attributes</u>.

It is often convenient to adopt the decomposition

$$\phi(\underline{c}, \{\underline{Z}\}) = \phi_1(\underline{c}) \; \phi_2(\{\underline{Z}\}|\underline{c}) \qquad (3.13)$$

in which $\phi_1(\underline{c})$ is the distribution of observable personal attributes, and $\phi_2(\{\underline{Z}\}|\underline{c})$ is the conditional distribution of the attributes associated with the alternatives $\{A_1, \ldots, A_{\rho}, \ldots, A_N\}$ available to an individual drawn from class $\underline{c}$.

There are a number of ways by which the aggregation process (3.12) may be approached, ranging from analytic approximations through to numerical approaches based on simulation. These have been considered by Koppleman (1976), McFadden and Reid (1975), Boutheleir and Daganzo (1979) and others. Here we wish to make four additional comments.

† An extensive demonstration of this dependence is given by Southworth (1977, 1978).

Firstly, if the number of individual attributes (c_1, c_2, ...) in the personal attribute vector $\underline{c}$, which are used in market segmentation to account for heterogeneities in the choice making population, is large then the density functions will appear as elements in a (typically) very sparce matrix. In this case it might well prove to be computationally efficient to create a sample of individuals, representative in terms of the distribution $\phi(\underline{c}, \{\underline{Z}\})$, and to perform the whole summation process (3.12) by Monte Carlo simulation. This approach is in the spirit of those adopted by Orcutt et al. (1961, 1976), Kain et al. (1976), Wilson and Pownall (1976), Kreibich (1979), Bonsall (1979), and Clarke, Keys and Williams (1979a,b).

Secondly, it is clear that integration over the majority of personal attributes, through market segmentation or numerical methods, will not prove analytically troublesome. However, when the states $\underline{K}$ are defined in terms of an irregular zoning system, the integration over spatial attributes, which are included in the vector set $\{\underline{Z}\}$, involved in $S_{A_p \epsilon \underline{K}}$, although straight forward in principle, is difficult and data consuming in practise. In spite of certain analytic simplifications which are achievable through the use of probit models (McFadden and Reid, 1975; Bouthelier and Daganzo, 1979) the treatment of location variables remains a significant problem in the comparison of "macro-models" with aggregated micro-models, constructed for "system testing" purposes. Of key interest is the relation between the parameter estimates in a non-linear micro-model of the form (3.1) with those in an analytically similar macro-model expressed in terms of aggregate variables. A straight substitution of micro-parameters into the macro-expression is known to be unsatisfactory. The validity of the compromise adopted in British Transportation Studies (see, for example, Wilson et al. 1969) of transfering the value of travel attributes (eg. value of time, excess time, etc.) from micro- to macro- levels with a rescaling (by a calibration parameter) of the generalized travel costs to accomodate the extra dispersion introduced by the distribution of attributes (times and costs) over the zonal system, as far as the authors are aware, has yet to be tested.

Various approximations and assumptions may be invoked to relate macro- and micro- models (see, for example Cochrane, 1975, for a discussion of the aggregation of logit models). One approach to the aggregation problem is to sacrifice the "realism" of the troublesome discrete zonal system and adopt smooth continuous distributions for the planning variables, in which case the micro-model is expressed in terms of a continuous set of choices. This approach has been considered by Angel and Hyman (1972), Williams and Ortuzar (1976), McFadden (1976) and Watanatada and Ben-Akiva (1978), and others.

The aggregation process rightly attracts much attention and underpins much of the debate regarding transferability of models over space and time.

It is widely felt that current aggregate models are very heavy consumers of data. In contrast the micro-models are extremely data efficient particularly when used in conjunction with sophisticated sampling designs and appropriate estimation techniques (Manski and McFadden, 1977; Manski and Lerman, 1978; Cosslett, 1978) which render them highly suitable for "short-term" policy studies. However, the fruits and efficiency of the micro-approach presuppose an efficient performance of the aggregation process (3.12), and it remains to be seen to what extent the gains achievable from exploiting the full range of variability in a small sample are off-set in the spatial aggregation process involved in Equation (3.12) in order to achieve comparable results. It is further debateable whether, and under what circumstances, the alternative approaches to model formation are, and should be, considered competitors.

We postpone a discussion of the compatibility of micro-model, with macro-restrictions eg. supply limitations until the following sections. We now turn to two theoretical approaches which are used to account for the residual dispersion in behaviour of a population endowed with similar observable attributes. We shall unless it is necessary in a particular context, omit market segmentation variables.

4. The Entropy Maximising Approach

The bases of the entropy maximising method are by now well known and only the briefest summary is needed here. The entropy function itself has always had some mystery associated with its use in the social sciences but can be understood, more or less straightforwardly, in at least three ways (Wilson, 1970a). Firstly it can be taken as a measure of the probability associated with a state of a system: its log is proportional to a count of the number of micro-states which can give rise to that state. The fundamental assumption of the approach is also transparent in this representation: that, given the information available, all micro-states are equally probable. Essentially, this is the approach used in statistical mechanics in the study of the physics of gases and much can be learned from the analogy (even though the argument does not rest on this).

Secondly, it is possible to use Jaynes' (1957) view of entropy as a measure of the amount of information associated with a probability distribution. For model building, this involves transforming variables such as T_{ij} into probabilities defined by $p_{ij} = \frac{T_{ij}}{T}$.

Thirdly, entropy can be given another statistical interpretation connected to Bayes' theorem and the negative of the likelihood function. Maximising entropy is then seen as choosing the form of probability distribution which minimises the likelihood function (though parameters are then estimated to maximise this function). This can be seen as choosing the form of probability distribution which makes the weakest assumption compatible with what is known.

These three views can all be essentially reconciled. Some have particular advantages as aids to interpretation or as the bases for some of the extensions to be discussed below. But very broadly, it can be argued that the perspective which is adopted is a matter of taste. Two other concepts can be mentioned briefly, first, in thermodynamics, as a macro-property of systems, forming the basis of the expression of some macro-laws; and secondly, in ecology and elsewhere as a measure of diversity.

There may be implications for the first of these in dynamic urban modelling, but these will not be pursued here.

Before considering a number of recent extensions of the method we consider some general issues in the context of a simple and well-known example: the derivation of the aggregated production-attraction constrained spatial interaction model. Using a well-known notation, the model may be derived from the following mathematical program

$$\max_{\{\underline{T}\}} \; S = -\sum_{ij} \ln T_{ij}! \tag{4.1}$$

$$= -\sum_{ij} T_{ij}(\ln T_{ij} - 1) \tag{4.2}$$

subject to

$$\sum_j T_{ij} = O_i \tag{4.3}$$

$$\sum_i T_{ij} = D_j \tag{4.4}$$

$$\sum_{ij} T_{ij} c_{ij} = C \tag{4.5}$$

which gives the result

$$T_{ij} = A_i B_j O_i D_j e^{-\beta c_{ij}} \tag{4.6}$$

with the usual expressions for A_i and B_j.

The balancing factors are related to the Lagrange multipliers $\underline{\alpha}, \underline{\gamma}$ associated with the constraint sets (4.3) and (4.4) through the expressions

$$A_i = \frac{1}{O_i} e^{-\alpha_i} \tag{4.7}$$

$$B_j = \frac{1}{D_j} e^{-\gamma_j} \tag{4.8}$$

We may exploit the usual properties of mathematical programs to write

$$\alpha_i = -\log A_i O_i = \frac{\partial S}{\partial O_i} \qquad (4.9)$$

$$\gamma_j = -\log B_j D_j = \frac{\partial S}{\partial D_j} \qquad (4.10)$$

$$\beta = \frac{\partial S}{\partial C} \qquad (4.11)$$

The dispersion parameter β is then directly related to the variation of the system entropy S.

There are two kinds of constraints: (4.3) and (4.4) are essentially accounting constraints of the form discussed in Section 2. The "cost constraint" (4.5) is of a different kinds. It is a representation of a "known" value of the total expenditure on travel (or, by an easy transformation, the mean expenditure). This might be called an 'expectation-value' constraint, and high-order moment constraints are, of course, possible. Thus, for brevity, we call (4.3) and (4.4) A-constraints and (4.5) EV-constraints.

Why choose particular constraints? Essentially, A-constraints play an obvious role and guarentee consistency. The EV-constraints relate to quantities which are thought to determine the distribution of system elements across possible states (i,j) which are "spaced" by the matrix $\{c_{ij}\}$. A good interpretation for such constraints in the present context arises from relatively recent discoveries which are pursued mainly in Section 6 below. Evans (1973) showed that the transportation model of linear programming is a special case of (4.6) arising in the limit as β, tends to infinity. In effect C in Equation (4.5) then achieves a minimum value and the entropy becomes redundant. The left-hand side of (4.5) is in effect the objective function in the limiting case, and this may be seen explicitly if the program (4.2)-(4.5) is written as a cost minimisation problem subject to a constraint on the entropy (or dispersion) in the system (Coelho and Wilson, 1977; Erlander, 1977). This suggests that the EV-constraints refer essentially to components of "utility" and "disutility" and when incorporated as constraints (with terms like C taking sub-optimal values), the outcome of the entropy maximising process is to identify the system states which are most probable in this sub-optimal state.

It should be noted that we refer to sub-optimality only in an analytic sense in which C in Equation (4.5) takes a value greater than that appropriate to the cost minimisation solution of the transportation model of linear programming. The notion of sub-optimality does not necessarily pertain to the behaviour of individuals or households at the micro-level. Such an interpretation could be made but requires additional assumptions outside the framework of the entropy maximising approach.

The EV-constraints thus play the role equivalent to that of terms in an objective function of an optimising mathematical programming model, and the entropy model can then be interpreted as a device for adding dispersion from the optimal base to obtain a 'most-probable' prediction.

As we noted in Section 3 there are many possible reasons for the existence of dispersion. These include imperfections in markets, lack of knowledge on the part of individuals whose behaviour is being modelled, dispersion in preferences, and ignorance on the part of the analyst. The dispersion may possibly be reduced if new information is made available - the framework of the entropy maximising approach is such that additional market segmentation or constraints which represent this information may readily be added.

The model as presented is essentially static. Most of the discussion on dynamics is reserved for Section 7 below, but one point can usefully be made here. If the system is such that, after a disturbance, equilibrium is achieved reasonably quickly, then a comparative static approach is useful and the immediate issue is to note that the terms on the right hand side need not be constant. For example, C in the EV-constraint (4.5) could be separately modelled (Wilson, 1973; Southworth, 1977).

A number of further developments of the method have been made under the various interpretations outlined at the start of the section. We include here a brief summary of two such lines of developments: firstly, relating to appropriate micro-state descriptors for models, including the statistics and combinatorics accompanying the count of micro-states; and secondly, on information theoretic developments.

The derivation of the model (4.6) proceeded on the basis of determining that allocation $\{T_{ij}\}$ which is associated with the maximum number of micro states. That is, the function,

$$S = \frac{T!}{\prod_{ij} T_{ij}!} \tag{4.12}$$

or rather its natural logarithm is maximised. The supply side (eg. of housing stock) is treated explicitly through the constraint system (4.4).

Recently a number of authors (see, for example, Fisk and Brown, 1975; Dacy and Norcliffe, 1976, 1977; and Brotchie, Lesse and Roy, 1979) have formulated models by re-interpreting various problem contexts in such a way that the combinatorial calculations are appropriate to the "statistics" of the situation studied. If, for example, in the location of households to houses we impose the occupancy restriction 0 or 1 at the micro-level, it is appropriate to determine the number of micro-states which are consistent with this restriction. In this case Fermi-Dirac statistics are appropriate and the constraints on stock occupancy may be built directly into the combinatorial calculation underpinning the objective function (with an *inequality* replacing the equality in (4.4)).

In a related context Brotchie et al. (1979) have considered the planning environment to provide the rules which govern the "economic forces" operating within a system, and have deemed the Dermi-Dirac, Maxwell-Botzmann and Einstein-Bose statistics to be appropriate to different planning contexts.

Other aspects to come out of this and other lines of research are the strong formal analogies which exist between the "macro-properties" in both statistical mechanical and economic interpretations of the class of models, and the formal relationship between total entropy and utility under various conditions (Coelho and Williams, 1977; Williams and Senior, 1978; Leonardi, 1978; Brotchie et al., 1979). We shall comment briefly on these relations in Section 8.

We end this section by noting an extension of the entropy concept itself, in the context of information theory, which has arisen out of the work of Kullback (1959). The interpretation and application of Kullback's measure, which is proving to be very useful in urban modelling, has been considered by a number of authors, notably Batty and March (1976); and Snickars and Weibull (1977). It is best presented as an extension of the entropy of a probability distribution

$$S = -\sum_{ij} p_{ij} \ln p_{ij} . \tag{4.13}$$

If prior probabilities are known, say as $p_{ij}^{(o)}$ then the Kullback entropy is defined as

$$S = -\sum_{ij} p_{ij} \ln \frac{p_{ij}}{p_{ij}^{(o)}} \tag{4.14}$$

in which p_{ij} is now interpreted as a posterior probability. S can now be interpreted as a measure of the information added to the prior probabilities by the new constraint equations which are set up and used in the usual way.

Snickars and Weibull (1977) and Los (1979) have used information minimisation as a general approach for the prediction of discrete choices in a disequilibrium context. In these applications, demand functions, treated as prior distributions, are modified by the supply constraints to produce a posteriori the most probable consumption patterns. We shall consider these applications further in Section 7.

The Kullback entropy measure has also been used by Batty and March (1976) to incorporate prior information contributed by a zoning system itself. They suggest taking $p_{ij}^{(o)}$ to be proportional to O_iD_j/c_{ij} which can be interpreted as incorporating the effects of zone size and the amount of attenuation due to two dimensional spatial geometry. We might alternatively take

$$p_{ij}^{(o)} = \frac{O_iD_j}{T} \tag{4.15}$$

which satisfies the constraints

$$\sum_j p_{ij}^{(o)} = \frac{O_i}{T} \tag{4.16}$$

$$\sum_i p_{ij}^{(o)} = \frac{D_j}{T} \tag{4.17}$$

The presence of the expression (4.15) in the entropy function has the beneficial effect of releasing the dual variables from an *explicit* zone size dependence.

Perhaps the most widely adopted, though often unrecognised, use of the Kullback entropy concept is in the so-called bi-proportional matrix or RAS methods in which a matrix is 'adjusted' to known row and column totals. If a base matrix $T_{ij}^{(o)}$ is to be 'updated' to satisfy the constraint

$$\sum_j T_{ij} = O_i \tag{4.18}$$

$$\sum_i T_{ij} = D_j \tag{4.19}$$

according to

$$T_{ij} = A_iB_jT_{ij}^{(o)} , \tag{4.20}$$

then the matrix T_{ij} may readily be shown to be derived from the Kullback measure

$$S = -\sum_{ij} T_{ij} \ln \frac{T_{ij}}{T_{ij}^{(o)}} \tag{4.21}$$

which is maximised subject to (4.18) and (4.19).

5. Choice Theoretic Approaches.

The derivation of travel demand and location models from choice theoretic principles has developed under a number of methodological perspectives including: classical utility theory as portrayed in the work of Niedercorn and Bechdolt (1969), Golob et al. (1973), Neuberger (1971) and others; axiomatic choice theory (Smith, 1975) in the tradition of Luce (1959); and random utility theory (Ben-Akiva, 1973; McFadden, 1973; Domencich and McFadden, 1975; Cochrane, 1975; Lerman, 1975; Williams, 1977; Brotchie, 1979; Daganzo, 1979; and others) in the tradition of Thurstone (1927).

Within the random utility approach the generation of models is achieved through the expression

$$P_\rho = \text{Prob}\{\overline{U}_\rho(\underline{Z}_\rho, \underline{\theta}) + \varepsilon_\rho > \overline{U}_{\rho'}(\underline{Z}_{\rho'}, \underline{\theta}) + \varepsilon_{\rho'}; \text{ for all } A_{\rho'} \epsilon \underline{A}\} \tag{5.1}$$

in which the probability P_ρ that an individual selects an alternative A_ρ from an N-membered choice set $\underline{A}$ is equal to the probability that the utility U_ρ associated with A_ρ is greater than that associated with all other options. The decomposition of U_ρ into representative and random components $\overline{U}_\rho(\underline{Z}_\rho, \underline{\theta})$ and ε_ρ, respectively, allows a parametric relationship to be established between P_ρ and the sets of attributes $(\underline{Z}_1, \ldots \underline{Z}_\rho, \ldots, \underline{Z}_N)$

$$P_\rho = P(\underline{Z}_1, \ldots \underline{Z}_\rho, \ldots, Z_N : \underline{\theta}) \tag{5.2}$$

once the distribution of the residuals is given. The parameter set $\underline{\theta}$ may then be determined by reference to observed choices. The linear specification

$$\overline{U}_\rho(\underline{Z}_\rho, \underline{\theta}) = \sum_\mu \theta_\mu Z_\rho^\mu \tag{5.3}$$

is typically adopted in logit applications, although various transformations of the attributes may be invoked.

It should be recalled that in the random utility approach each individual in choice making population Π is assumed to act rationally and consistently within his own frame of reference, and that the random characteristic reflected in ε_ρ represents an uncertainty on the part of the observer (the modeller) as to which option will be selected by each member of $\mathbb{I}$ endowed with the same observable characteristics. The dispersion in the micro model (5.1) is thus associated with the existence of unobserved attributes.

If the residuals $(\varepsilon_1, \ldots, \varepsilon_\rho, \ldots, \varepsilon_{\mathbb{M}})$ are identically and independently distributed (IID) according to Gnedenko (Weibull) functions, the multinomial logit model is obtained (McFadden, 1973; Cochrane, 1975). The dispersion parameter β in the expression

$$P_\rho = \frac{e^{\beta \overline{U}_\rho}}{\sum\limits_\rho e^{\beta \overline{U}_\rho}} \tag{5.4}$$

is inversely related to the standard deviation of the distributions of the residuals, according to

$$\beta = \frac{\pi}{\sqrt{6}\sigma} \tag{5.5}$$

Although the logit model is renowned for its versitility and ease of implementation, its practical and theoretical limitations are becoming increasingly apparent. These limitations are probably best illustrated with reference to the matrix of elasticity coefficients $\underline{\chi}$ with elements

$$\chi_{\rho\rho'} = \frac{\overline{U}_{\rho'}}{P_\rho} \cdot \frac{\partial P_\rho}{\partial \overline{U}_{\rho'}} \quad \cdot \forall\ A_\rho, A_{\rho'} \varepsilon \underline{A}. \tag{5.6}$$

For the model (5.4), as we remarked in Section three,

$$\chi_{\rho\rho'} = \beta \overline{U}_{\rho'} (\delta_{\rho\rho'} - P_{\rho'}) \tag{5.7}$$

in which $\delta_{\rho\rho'}$ is the Kronecker delta. The single parameter β thus characterizes cross-substitution between all choice options. When choice alternatives are characterized in such a way that a sub-set of options are considered to be "more similar" than others the limited properties of cross substitution can prove to be a severe impediment which may disqualify the application of the simple logit model.

The structure and properties of random utility choice models are now known to relate intimately to the structure of the variance-covariance matrix $\underline{\underline{\Sigma}}$, the elements of which are defined by

$$\Sigma_{\rho\rho'} = E(\varepsilon_\rho \varepsilon_{\rho'}) \tag{5.8}$$

in which E(.) denotes an expectation value. For the IID class of models, the $\underline{\underline{\Sigma}}$ matrix assumes a particularly simple diagonal form

$$\Sigma_{\rho\rho'} = \sigma^2 \delta_{\rho\rho'} \tag{5.9}$$

in which

$$\sigma^2 = E(\varepsilon_\rho{}^2) \qquad \forall A_\rho \varepsilon \underline{A} \tag{5.10}$$.

The notion of similarity between alternatives has a mathematical interpretation in terms of the correlation matrix $\underline{\underline{\Sigma}}$, and much recent research has been directed towards the assembly of a class of models endowed with less restrictive cross-elasticity properties which can be generated from random utility maximization principles. The structure of the cross-elasticity matrix $\underline{\underline{\chi}}$ will mirror the structure of the variance-covariance matrix $\underline{\underline{\Sigma}}$.

The nested-logit model may be considered to be a generalization of the simple multinomial logit form and is characterized by two elasticity parameters β and λ. Defining choice set elements $\underline{A}_{\mu\nu}$ in terms of the two 'dimensions' $\mu\varepsilon X$ and $\nu\varepsilon Y$ (eg. combinations of location and mode, mode and route, tenure and house location) we may write the nested logit model as follows

$$P_{\mu\nu} = \frac{e^{\beta(\overline{U}_\mu + \overline{U}_{\mu*})}}{\sum_\mu e^{\beta(\overline{U}_\mu + \overline{U}_{\mu*})}} \cdot \frac{e^{\lambda \overline{U}_{\mu\nu}}}{\sum_\nu e^{\lambda \overline{U}_{\mu\nu}}} \qquad \begin{matrix} \mu = 1 \dots M, \\ \nu = 1 \dots N. \end{matrix} \tag{5.11}$$

in which the composite utility $\overline{U}_{\mu*}$ is expressed as a function of the utility components $\{\overline{U}_{\mu\nu}; \nu = 1 \dots N\}$. That is

$$\overline{U}_{\mu *} = \overline{U}_{\mu *}(\overline{U}_{\mu 1} \ldots \overline{U}_{\mu \nu} \ldots \overline{U}_{\mu N}). \tag{5.12}$$

In any discussion of the Nested or Hierarchical logit model it is important to stress that the function has been adopted for many years in British Transportation Studies and other research publications (Wilson, 1969; Manheim, 1973; Ben-Akiva, 1974; Ben-Akiva et al. 1976). Its recent derivation within the choice theoretic approach outlined above was achieved independently by Williams (1977), Daly and Zachary (1978) and McFadden (1978). The model may be seen as one member of a class of models derived from utility functions of the form

$$U(\mu,\nu) = U_{\mu} + U_{\mu\nu} \qquad \begin{matrix} \mu = 1 \ldots M \\ \nu = 1 \ldots N \end{matrix} \tag{5.13}$$

$$= \overline{U}_{\mu} + \overline{U}_{\mu\nu} + \varepsilon_{\mu} + \varepsilon_{\mu\nu} \tag{5.14}$$

for which

$$\Sigma_{\mu\nu,\mu'\nu'} = \sigma_X^2 \, \delta_{\mu\mu'} + \sigma_{XY}^2 \delta_{\mu\mu'} . \, \delta_{\nu\nu'} \tag{5.15}$$

with

$$E(\varepsilon_{\mu}\varepsilon_{\mu'}) = \sigma_X^2 \delta_{\mu\mu'}; \; E(\varepsilon_{\mu\nu}\varepsilon_{\mu'\nu'}). \; \sigma_{XY}^2 \delta_{\mu\mu'} . \delta_{\nu\nu'} \tag{5.16}$$

and

$$E(\varepsilon_{\mu}, \varepsilon_{\mu'\nu}) = 0 \text{ for all } \mu, \mu' \text{ and } \nu. \tag{5.17}$$

The generation and interpretation of the nested logit function within the framework of choice theory has added significant additional restrictions to the analytic expressions (5.11)-(5.12) if these are to be consistent with an underpinning theory of utility maximisation. Firstly, the composite function $\overline{U}_{\mu *}$ must be of the following form*

$$\overline{U}_{\mu *} = \frac{1}{\lambda} \log \sum_{\nu} e^{\lambda \overline{U}_{\mu\nu}} + \text{constant} \tag{5.18}$$

* As far as the authors are aware this composite cost, or inclusive price, was first adopted in a nested logit model by Manheim (1973) and Ben-Akiva (1973). A formula of which this is a special case was suggested by Wilson (1970, p.32, eg.2.76) and used in a slightly different form in the SELNEC Transportation Study (Wilson, Hawkins, Hill and Wagon, 1969). Its <u>theoretical</u> significance however was only apparent when the nested logit model was formally derived from utility maximisation.

Secondly, the parameters β and λ must satisfy the inequality

$$\beta \leqslant \lambda. \qquad (5.19)$$

If either of these conditions is violated in a calibrated model, unacceptable response properties may well result (Williams, 1977). Examples of the violation of condition (5.19) are widely apparent in transportation study model results, and are discussed by Williams and Senior (1977). Writing Equation (5.11) in the form

$$P_{\mu\nu} = P_{\mu} \cdot P^{\mu}_{\nu} \qquad (5.20)$$

it may readily be shown that the elasticity matrix $\underline{X}$ is given by

$$X_{\mu\nu,\ \mu'\nu'} = \left\{ \lambda(\delta_{\nu\nu'} - P^{\mu}_{\nu'})\ \delta_{\mu\mu'} + \beta(\delta_{\mu\mu'} P^{\mu}_{\nu'} - P_{\mu'\nu'}) \right\} \cdot \bar{U}_{\mu'\nu'} \qquad (5.21)$$

The two parameter characterization (5.11), which is assymmetric in the indices μ and ν is not theoretically appropriate for applications in which utility functions have the <u>symmetric</u> form

$$U(\mu,\nu) = U_{\mu} + U_{\mu\nu} + U_{\nu} \qquad (5.22)$$

with the corresponding matrix elements

$$\Sigma_{\mu\nu,\ \mu'\nu'} = \sigma^2_X \delta_{\mu\mu'} + \sigma^2_{XY} \delta_{\mu\mu'} \delta_{\nu\nu'} + \sigma^2_Y \delta_{\nu\nu'} \qquad (5.23)$$

More general models have thus been proposed which have less restrictive properties of cross-substitution than the nested logit model. These include the three parameter cross-correlated logit model (Williams, 1977; Williams and Ortuzar, 1979), a class of General Extreme Value models (McFadden, 1978); and the multinomial probit model for which the elements of the vector of residuals $\underline{\varepsilon}$ are considered to be multivariate normally distributed (Domencich and McFadden, 1975; Daganzo et al., 1977; Haussman and Wise, 1978; Daganzo, 1979).

A predictable difficulty in extending the work to a greater degree of generality is the achievement of a model structure with undue theoretical and analytical restrictions which is computationally tractable. While the multinomial logit and its alternative nested extensions may be theoretically inconsistent in certain circumstances it may well be that the errors incurred in adopting such models are acceptable. Recently Williams and Ortuzar (1979) have used Monte Carlo simulation to assess the magnitude of such theoretical errors. Their results appear to confirm Ben-Akiva's (1977a) contention that the choice between the nested and multinomial logit models is not unduly restrictive.

The work on model structure outlined above has been inspired by the deficiencies in the structure of the multinomial logit model resulting from particular assumptions made within the framework of the choice theory. The DOGIT model introduced by Gaudry and Dagenais (1979) is an alternative form not subject to the "independence from irrelevant alternatives " property which characterizes the logit model and now viewed with much suspicion. The model is expressed as follows

$$P_\rho = \frac{e^{\beta\overline{U}_\rho} + g_\rho \sum_\rho e^{\beta\overline{U}_\rho}}{(1 + \sum_\rho g_\rho) \sum_\rho e^{\theta\overline{U}_\rho}} \tag{5.24}$$

in which the parameters $\underline{g}$ may be taken to be functions of the attributes associated with the alternatives.

Recently Ben-Akiva (1977b) has shown that the DOGIT model may be generated from Equation (5.1) simply by relaxing the assumption that all individuals have identical and complete choice sets $\underline{A}$. In fact if the available choice sets are considered to consist of single alternatives (to which an individual might be captive) in addition to the complete set $\underline{A}$, the model (5.24) may be generated by invoking the decomposition of choice models proposed by Manski (1977). Williams and Ortuzar (1979) have extended this work using Monte Carlo simulation to investigate the influence and implications of limited information in choice contexts. Their results indicate that when significant biases exist in the choice process as exemplified by a propensity to search in particular areas in a spatial context, a calibrated logit (or gravity) model may well overestimate the tendency for individuals to respond to policy stimuli.

Because choice models are developed within a framework in which preferences are recorded, benefit measures which are theoretically and analytically consistant with the demand functions (choice probabilities) may be simultaneously generated. It may be shown that for a broad class of models, including the multinomial and nested logit models, unique and exact measures of benefit change accompanying a policy may be defined by means of the computation of expected utilities (Williams, 1977; McFadden, 1978). For the nested logit model (5.11) the expected utility U_{**} associated with the choice process is given by

$$U_{**} = \frac{1}{\beta} \log \sum_{\mu} e^{\beta(\overline{U}_{\mu} + \overline{U}_{\mu *})} \tag{5.25}$$

$$= \frac{1}{\beta} \sum_{\mu} P_{\mu} \log P_{\mu} + \sum_{\mu} P_{\mu} \overline{U}_{\mu}$$

$$+ \sum_{\mu} P_{\mu} \{\frac{1}{\lambda} \sum_{\nu} P^{\mu}_{\nu} \log P^{\mu}_{\nu} + \sum_{\nu} P^{\mu}_{\nu} \overline{U}_{\mu\nu}\}. \tag{5.26}$$

These expected utility or group surplus functions may be used to shed new light on such concepts as composite costs (Williams, 1977) and accessibility benefits associated with land use - transportation plans (Koenig, 1975; Cochrane, 1975; Williams, 1977; Williams and Senior, 1978; Ben-Akiva and Lerman, 1979). We shall note the further implications of these results in the formulation of mathematical programs which are underpinned by choice theoretic principles.

We would remark, in conclusion, that the exact consumer and producer surplus measures which may be derived from certain classes of choice model are consistent with the preferences revealed at the cross section and in response contexts by particular segments of the "market". It is, as ever, necessary to confront distributional issues.

6. The Use of Mathematical Programming in the Formation and Solution of Models.

The recognition that a wide range of urban and regional models corresponded to the optimality (Kuhn-Tucker) conditions of particular mathematical programs has allowed considerable insights into the processes of model formation, solution, system evaluation and design. In this section we wish to review some of these aspects.

Just as analytic models may be distinguished according to the presence or absence of dispersion on the one hand and different theoretical interpretation of the parameters which characterise this dispersion on the other, so too may the mathematical programs which generate the models be distinguished by the structure of their objective and constraint functions, and the interpretation of the extremal principle involved. Concomitantly, the analytic similarity between models may be traced to the structural similarity of the generating programs, while comparisons of the interpretation of model parameters relate to the theory underpinning the optimisation process. To embrace a wide range of examples, and draw out salient methodological points we shall proceed on a somewhat symbolic basis and represent the programs which are used to generate models in the following form

$$\underset{\underline{f}}{\text{Max}}\; Z(\underline{f}) \tag{6.1}$$

subject to

$$\underline{\underline{A}}\,\underline{f} \leqslant \underline{s} \tag{6.2}$$

$$\underline{\underline{B}}\,\underline{f} \leqslant \underline{c} \tag{6.3}$$

$$\underline{f} \geqslant 0 \tag{6.4}$$

in which the vectors of constraints are typically of linear form and often of special structure, as we noted in Section Two. The constraints have been partitioned into a set (Equation 6.2) which have an interpretation of accounting or conservation relations between stocks or resourses $\underline{s}$ and flows or allocations $\underline{f}$; and the set (6.3) which includes the 'expectation' or mean values of particular variables. Through the use of Lagrangian multipliers the set (6.3) may be transfered to the objective function (Wilson and Coelho, 1977) and an equivalent programming model obtained, which will be written

$$\max_{\underline{f}} Z^*(\underline{f}, \underline{\phi}) \tag{6.5}$$

subject to

$$\underline{\underline{A}}\ \underline{f} \leq \underline{s} \tag{6.6}$$

$$\underline{f} \geq 0.$$

By introducing a vector of dual variables $\underline{\alpha}$, with transpose $\underline{\alpha}'$, the Lagrangian corresponding to the program (6.5) - (6.6) may be written

$$L(\underline{f}, \underline{\alpha}) = Z^*(\underline{f}) + \underline{\alpha}'.(\underline{s} - \underline{\underline{A}}\ \underline{f})$$

and the Kuhn-Tucker conditions expressed as follows

$$\underline{f}\ \frac{\partial L}{\partial \underline{f}} = 0 \qquad \frac{\partial L}{\partial \underline{f}} \leq 0 \tag{6.7}$$

$$\underline{\alpha}\ \frac{\partial L}{\partial \underline{\alpha}} = 0 \qquad \frac{\partial L}{\partial \underline{\alpha}} \geq 0 \tag{6.8}$$

in which the complementary slackness conditions hold for each component of the vectors $\underline{f}$ and $\underline{\alpha}$. The solution of these equations may be formally stated in terms of the functional dependence between $\underline{f}$, the stocks $\underline{s}$ and the vector of parameters $\underline{\phi}$

$$\underline{f} = \underline{f}(\underline{s}, \underline{\phi}). \tag{6.9}$$

The technique of generating models from mathematical programs may be underpinned by different theoretical premises. For both the Entropy Maximising and Economic Choice theory approaches the objective function Z^* may be written in the form

$$Z^*(\underline{f}, \underline{\phi}) = \underline{b}'.\underline{f} - G(\underline{f}, \underline{\phi}) \tag{6.10}$$

in which $\underline{b}'$ is a vector with constant coefficients and $G(\underline{f}, \underline{\phi})$ is a non-linear function of the allocation or flow variables $\underline{f}$, typically of the form

$$G(\underline{f}, \underline{\phi}) = \sum_k \frac{1}{\phi_k} \sum_{\gamma} f^k_{\gamma} \ln f^k_{\gamma} \tag{6.11}$$

in which the vector $\underline{f}$ contains elements defined over the subscripts γ and superscripts k.

It can readily be seen from (6.10) and (6.11) that the $\underline{\phi} \to \infty$ limiting solution of the program generated from $Z^*(\underline{f}, \underline{\phi})$ will be equivalent to (one of) the solutions of the linear program generated from

$$Z^*(\underline{f}, \infty) = \underline{b}'.\ \underline{f} \tag{6.12}$$

The presence or absence of dispersion in a model will thus correspond to the finite or infinite value assumed by the elements of the vector $\underline{\phi}$. Often $\underline{\phi}$ will involve a single entry ϕ. The demonstration by Suzanne Evans (1973a)that the transportation problem of linear programming is a limiting form of the double constrained gravity model (see Section Four) may thus be seen as pertaining to one member of a general class of models characterized by the objective function and constraint set. Similar linear programming limits are expected and obtained from other members (Wilson and Senior, 1974; Coelho and Williams, 1977).

In the economic interpretation of the extremal process, Z^* often has the interpretation of a total surplus function embracing contributions from consumers and producers. The maximisation of surplus functions is, under certain conditions, consistent with the generation of probabilistic choice models from random utility theory (see Section Five), in which case choice theoretic principles involving dispersion may be considered to underpin the extremal process (Williams, 1977; Williams and Senior, 1978).

In Section Two we noted that one strategy for generating models was to develope accounting or conservation relations according to the logic of the forecasting context, and treat these as constraints in a mathematical program underpinned by an appropriate theoretical rationale.

This process has been widely adopted for the generation of models underpinned by both Entropy Maximising and Economic Theory approaches. Good examples can be found in the development of models for spatial housing models embracing Alonso's bid-rent concept (Alonso, 1964). The early operational form of the Herbert-Stevens model (Herbert and Stevens, 1960) generalized by Harris (1962) was in the format of a linear programming transportation model. This was later extended to include dispersion by Wilson (1970b),Anas (1973), Gustafsson et al. (1977) and Williams and Senior (1978) adopting Entropy Maximising and Choice theoretic principles. The extension by Clarke et al. (1979c)may be seen as as an attempt to marry within a stock-flow framework dynamical (mobility) characteristics with theories of the matching/allocation process. This model is generated by imposing the stock-flow relations expressing transitions in the housing system as constraints within a mathematical program.

The congested assignment and linear programming allocation models were early examples of the formation of models from extremal principles. In the former case the Kuhn-Tucker relations had a direct interpretation in terms of Wardrop's principle (Wardrop, 1952) characterizing road traffic in (selfish) equilibrium. In the assignment problem, the constraints represent (multicommodity) flow conservation rules, and the shadow prices or dual variables associated with the nodes in the network have a least travel cost connotation. It is interesting to compare the interpretation of the Kuhn-Tucker conditions for those problems which have a comparable network representation. These include the input-output representations of activity-commodity flow systems, and the stock-flow representations of transitions in labour and housing systems (Clarke et al., 1979c). It should be re-emphasized that the formal similarity between the models arises because of the similar structure of the $\underline{\underline{A}}$ matrices involved, which the similar interpretation of the equilibrium (complementary slackness) conditions is allowed when the programs are underpinned by the same economic paradigm.

The complementary slackness conditions may often be given the traditional economic interpretation involving the presence or absence of production or consumption according to the value of the net marginal benefit from so doing. The existence of heterogeneous consumption or production functions over a population, reflected in the entry of a non-linear dispersion function G, will, for the usually adopted functional expressions, ensure that a finite portion of consumers or producers will find it economically advantageous to participate in the relevant activity.

Particularly important areas of application of mathematical programming in the formation and solution of models involves their use in situations where stock-variables $\underline{s}$ and allocation or flow variables $\underline{f}$ are simultaneously manipulated according to extremal principles. Such contexts, which involve the nesting of one problem within another, include: the combination of sub-models in a sequence (eg. the distribution and assignment models); activity planning-commodity flow models; and the important class of planning models which have a location-allocation flavour.

We consider first the formation of a super-program from the combination of the distribution and assignment problems for which a location or interaction pattern is sought which is in equilibrium with the traffic assignment resulting from it. The trip interaction ($\underline{f}_1$) and multi-commodity link flow ($\underline{f}_2$) variables may be seperately generated from programs of the following structure:

$$\underset{\{\underline{f}_i\}}{\mathrm{Max}}\ Z^i(\underline{f}_i, \underline{\phi}_i) \tag{6.13}$$

subject to

$$\underline{A}_i\, \underline{f}_i \leq \underline{s}_i \tag{6.14}$$

$$\underline{f}_i \geq 0. \qquad i = 1,2. \tag{6.15}$$

The stock $\underline{s}_2$ is for terminal nodes of the appropriate network, the trip matrix $\underline{f}_1$ and we shall write

$$\underline{s}_2 = \underline{s}_2(\underline{f}_1). \tag{6.16}$$

The joint distribution-assignment program may now be written in the general form

$$\underset{\{f_2\}}{\text{Max}}\ Z^2(\underline{f}_2, \underline{\phi}_2) \tag{6.17}$$

subject to

$$\underline{A}_2\, f_2 \leq \underline{s}_2(\underline{f}_1) \tag{6.18}$$

$$\underline{f}_1 = \underline{f}_1\ (\underline{s}_1, \underline{f}_2) \tag{6.19}$$

$$\underline{f}_2 \geq 0.$$

Congestion on the network is responsible for the non-linear relationship between $\underline{f}_1$, the trip matrix, and the flow matrix $\underline{f}_2$.

The solution to this program may also be obtained from the super-program

$$\underset{\{\underline{f}_1, \underline{f}_2\}}{\text{Max}}\ \{Z'(\underline{f}_1, \underline{\phi}_1) + Z^2(\underline{f}_2, \underline{\phi}_2)\} \tag{6.20}$$

subject to

$$\underline{A}_1\, \underline{f}_1 \leq \underline{s}_1 \tag{6.21}$$

$$\underline{A}_2\, \underline{f}_2 - \underline{s}_2(\underline{f}_1) \leq 0 \tag{6.22}$$

$$\underline{f}_1, \underline{f}_2 \geq 0$$

which may be demonstrated by examining the Kuhn-Tucker conditions with respect to $\underline{f}_1$. The process of removing the highly non-linear constraint set (6.19) is made at the expense of increasing the variable set over which optimization is performed. The formation of the super-program does however allow existence and uniqueness of the solution to be established for suitably defined constraint sets and objective functions (Coelho and Wilson, 1976) and permit the design of solution algorithms. Specific examples of the formation and solution of such programs may be found in the work of Murchland (1969), Florian and Nguyen (1974), Evans (1976), Boyce and Southworth (1979), and others.

The formation of super-problems for the class of location-allocation problems is of particular interest because the process of dual formulation may often be invoked to provide an enormous simplification in solution methods. Simply stated the location-allocation problem involves a determination of the stock $\underline{s}^*$ which maximises a locational benefit function, subject to the allocation and planning constraints. Here we shall consider land use programming models of the form

$$\max_{\{\underline{s}\}} \{H(\underline{s}) + B(\underline{f})\} \tag{6.23}$$

subject to

$$\underline{f} = \underline{f}(\underline{s}) \tag{6.24}$$

$$\underline{h}(\underline{s}) \leqslant \underline{t} \tag{6.25}$$

$$\underline{s} \geqslant 0$$

in which the objective function contains the net benefit (cost) associated with the establishment and maintenance of the stock $H(\underline{s})$, and $B(\underline{f}(\underline{s}))$ the net benefit associated with the allocation or flow $\underline{f}$ accompanying the configuration $\underline{s}$, according to (6.24). Equation (6.25) represents a set of resource and planning constraints which are usually taken to be linear.

The highly non-linear constraint $\underline{f} = \underline{f}(\underline{s})$ may be removed by noting that the benefit function associated with the allocation $\underline{f}$, namely $B(\underline{f})$ is precisely the quantity $Z(\underline{f}(\underline{s}))$ used to generate the model. This allows the planning model to be written in the form

$$\max_{\{\underline{f},\underline{s}\}} \{H(\underline{s}) + Z(\underline{f})\} \tag{6.26}$$

subject to

$$\underline{\underline{A}}\,\underline{f} - \underline{s} \leqslant 0 \tag{6.27}$$

$$\underline{h}(\underline{s}) \leqslant \underline{t} \tag{6.28}$$

$$\underline{s},\ \underline{f} \geqslant 0.$$

This process of embedding is further discussed by Coelho and Wilson, 1977; and Coelho et al., 1978. For many continuous variable problems of interest this super program involves a concave objective function and convex (often linear) constraints, and if therefore a feasible solution may be found a unique optimal solution is guarenteed.

In order to see how the formation of dual programs can allow simplification of this class of problem, it is useful to firstly consider the formation of duals for the class of spatial interaction models as these are often embedded in the allocation program. We consider the doubly constrained member.

By employing the dual formulations of Wolfe (1961) or Rockafellar (1967) it may readily be shown that the dual of the program (4.2) - (4. 5) which generates the model (4.6) may be written in the unconstrained form

$$\min_{(\underline{\alpha},\underline{\gamma},\beta)} \sum_{ij} e^{-\alpha_i - \gamma_j - \beta c_{ij}} + \sum_i O_i \alpha_i + \sum_j D_j \gamma_j + \beta C \qquad (6.29)$$

as derived by Wilson and Senior, 1974; Evans, 1973b. Champernowne et al. (1976) have used direct minimisation of this function as a means of estimating the parameter set $(\underline{\alpha},\underline{\gamma},\beta)$. This process is not recommended for large problems in which the traditional assymetric treatment of the duals into the sets $(\underline{\alpha},\underline{\gamma})$ and (β) is more efficient.

In general the dual of a non-linear program contains both primal <u>and</u> dual variables. The expression of the dual (6.29) in terms of dual variables alone, is fortuitous and may be traced to the property that the variables $\{T_{ij}\}$ are strictly positive for finite values of the dispersion parameter ϕ, which in the above case equals β. This simplification may be exploited to drastically reduce the dimensions of super-programs such as (6.26) - (6.28), which involve models $\underline{f} = \underline{f}(\underline{s})$ generated by objective functions of the form (6.11), simply by solving the dual of Equations (6.26) - (6.28). Explicit examples of this process are given by Coelho and Williams (1977) and Wilson et al. (1979). Mathematical programs may in this way be used to unify and simplify the processes of formation and solution, of models adopted for evaluation and design of land use plans which entail a consideration of location or spatial interaction behaviour.

We end this section on the use of mathematical programs with a note of caution. In Section Five we reported that the indiscriminate use of the multinomial logit model may prove hazardous due to the restrictive properties of cross substitution between alternative choice options. The arbitrary use of the generating function (6.11) in model formation can give rise to precisely similar problems, and suitable generalizations of the function must be formulated to embody the necessary structural properties of the resultant model.

An example of these difficulties may be given with reference to the congested assignment problem. A finite dispersion variant of this problem may be generated by specifying link-based entropy functions of the form

$$G(\underline{f}) = \frac{1}{\phi} \sum_{i} \sum_{l'} X^{i}_{ll'} \ln X^{i}_{ll'}, \qquad (6.30)$$

in which $f \rightarrow X$ is the flow on link (ll') from origin i. The flow model which results from this inclusion is however unacceptable (for exactly the same reason as is the assignment produced by Dials' double pass algorithm, Dial, 1971) because the cross-substitution matrix pertaining to different routes is characterised solely by the parameter ϕ and ignores the complexities arising from common sections of route (a discussion of this problem is given by Williams, 1977; Sheffi, 1978).

The necessary generalization of Equation (6.11) to accomodate nested models and nested logit models in particular may readily be written down. If the required model structure is of the form

$$P_{\mu\nu} = P_{\mu}(\beta)\ P^{\mu}_{\nu}(\lambda) \qquad (6.31)$$

in which P_{μ} and P^{μ}_{ν} are logit functions with parameters β and λ, the necessary generating function is

$$G(f) = \sum_{\mu} Z_{\mu}(p_{\mu},\beta) + \sum_{\mu} p_{\mu}\ Z^{\mu}_{\nu}(p^{\mu}_{\nu},\lambda) \qquad (6.32)$$

in which

$$Z(p,\sigma).\ bp - \frac{1}{\sigma}\, p \ln p. \qquad (6.33)$$

It is precisely that quantity expressed in Equation (5.26).

It can be seen that the distribution and assignment problem outlined above is a special case in which the route choice dispersion parameter is usually taken as the $\Lambda \to \infty$ limit.

7. Dynamics

Most of the models discussed so far are static. In this section we offer a few comments on the formation, structure and solution of dynamical forms. More general aspects of the construction of dyanmic urban and regional models have been presented elsewhere (see, for example, Wilson, 1974; Williams and Wilson, 1978; Wilson and Macgill, 1978; and Macgill and Wilson, 1979).

While it may be argued that dynamical models by their very nature are superior to comparative static forms - containing the latter as a special case - the data requirements of the former may, and have, often prevented their construction. It must also be said that while we can point to a number of systems parameters which characterise time scales over which dis-equilibrium and stochastic effects may be observed, general guide-lines on the size of errors made in *not* applying dynamical models in a forecasting context are yet to be laid down. These will no doubt depend on the context and output requirements of the application, and to whether the dynamical response of a system is the product of a transient perturbation (eg. the opening or closure of a road) or varying and sustained external "drive" conditions (eg. demographic processes).

In Section 2 we noted the use of accounting relations in the formation of dyanmical equations which typically involve the Markovian assumption. The general solution of this type of equation

$$\frac{dN_c(t)}{dt} = A_{cc'}(t)N_{c'}(t) \tag{7.1}$$

has been considered by Rosen (1970). The signs of the eigenvalues of the matrix $\underline{\underline{A}}$ are important determinants of the qualitative aspects of system evolution.

Equation (7.1) may readily be seen to be a special case of the more general form

$$\frac{dN_\rho(t)}{dt} = \sum_{\rho'} \int_{-\infty}^{t} dt' A_{\rho\rho'}(t,t') N_{\rho'}(t') \tag{7.2}$$

which in discrete time may be written

$$\frac{\Delta N_\rho(t_i)}{\Delta t} = \sum_{\rho'} \sum_{j=1}^{i} A_{\rho\rho'}(t_i,t_j) N_{\rho'}(t_j) \tag{7.3}$$

Here, the transfer function $\underline{\underline{A}}$ relates events at different points in time. By an appropriate redefinition of states, to incorporate part of the time dimension, the system (7.3) may often be conveniently handled in the Markovian form

$$\frac{\Delta N_\mu(t_i)}{\Delta t} = \sum_{\mu'} A_{\mu\mu'}(t_i) N_{\mu'}(t_i) \tag{7.4}$$

The efficiency of this transformation will depend on the nature of the "lags" in the system and whether any convenient analytic form exists for the transfer function. "Duration of stay" and "accumulation of assets" problems are typical contexts in which the probability of a transition in a given time interval is a function of the history of system elements. These include that class of problems associated with transitions between states (with spatial and financial attributes) in the labour, housing and demographic systems.

If equations of the form (7.2) are not amenable to analytic solution - in either stochastic or deterministic specifications - we may readily appeal to Monte Carlo Simulation as a general solution technique. Disaggregate models of household dynamics are particularly suited to this method (Orcutt, et al., 1961, 1976; Clarke et al., 1979a,b).

While the above considerations relate very strongly to the representation and solution of the dynamical equations, the modelling problem now turns on the general structure and time variation of the transition matrix $\underline{\underline{A}}$. Several authors have sought to embed specific models for the transition matrix $\{A_{\rho\rho'}\}$ within the dynamical framework, based on the decomposition:

Probability of a transition from state ρ to ρ'

= Probability of a transition from ρ ×
Probability of a transition to ρ' given the transition from ρ.

or

$$A_{\rho\rho'} = P_{\rho}.P_{\rho'/\rho} \tag{7.5}$$

Early examples of the use of choice theory and kinetic theory in developing structures for $\{A_{\rho\rho'}\}$ may be found, respectively, in the work of Ginsberg (1971, 1972) and Cordey Hayes and Gleave (1972) on migration, and Tomlin (1969) in the dynamical study of trip distribution patterns. Tomlin considered the dynamics of the trip interaction system, over states $\{ij\}$, by solving the first order equation

$$\frac{dT_{ij}}{dt} = \sum_{pq} T_{pq} a_{pq,ij} \tag{7.6}$$

which represents a continuous time Markov process of the form (7.1). It was shown how to choose the matrix elements $\{a_{pq,ij}\}$ so that the equilibrium conditions are those of the standard static model (4.6) encountered in Section 4. If it is then thought desirable to model the system in disequilibrium after a disturbance, equations (7.6) could be solved numerically given the new initial conditions.

Although there are other ways to examine the dynamics of trip patterns – for example, we might build causal models of residential and workplace relocation – the above example serves to illustrate an important general point, namely, that it is possible to embed a static model within a dynamical framework in such a way that the equilibrium conditions are the same. There may, however, be many ways of doing this. An alternative approach to Tomlin's problem, for example, would be to represent the dynamical properties of the system through the non-linear differential equation

$$\frac{dT_{ij}}{dt} = \varepsilon(A_i B_j O_i D_j e^{-\beta c_{ij}} - T_{ij})T_{ij} \tag{7.5}$$

This model also has (4.6) as an equilibrium condition and represents logistic growth of T_{ij} until this is achieved. The formation of this kind of equation is in the spirit of the work of Allen, et al. (1978).

A recent innovation has been the development of dynamic models to represent the behaviour of the supply side variables which feature in some of the sub-models discussed above, notably service centres (Harris and Wilson, 1978; Wilson and Clarke, 1979) and housing (Wilson, 1979). Since these variables have previously been taken as exogenous, this involves adding additional hypotheses which represent user behaviour. These take the form of differential equations with interesting equilibrium solutions. This seems to be a particularly important development since the life-times of physical structures which are built *are* long and the relaxation times associated with departures from equilibrium will also be larger than those for population activity variables. An interesting feature of the analyses which have been carried out is the emergence of the possibility of "jump" and other bifurcation behaviour of equilibrium solutions and such phenomena could also be expected to be present if the system is not in equilibrium and the differential equations are integrated numerically (cf. Allen, et al., 1978).

The interaction between supply and demand in equilibrium and disequilibrium contexts has raised a number of interesting issues including;

(i) the development of efficient "market clearing" procedures for models in which the number of substitutes is large, and the extension to disequilibrium contexts.

(ii) the relationship between micro-behaviour and macro-restrictions.

Although in *equilibrium* models there is no dynamical significance to the incremental or iterative processes by which equilibria are achieved, both "incremental loading" and "equilibrium tracking" strategies have been made the bases for dynamical extensions. In the former, slices of demand are allocated and prices recomputed before further increments are added. In the latter, which embrace both Marshallian and Walras concepts of stability, "adjustments" are made according to the "distance" from equilibrium. In the Walras process, for example, price adjustments are functions of excess demand or supply. Here again the equilibrium model is "embedded" in the dynamical form.

It is known that incremental strategies are not *in general* satisfactory for deriving *equilibrium* solutions. In the congested assignment problem, for example, such an allocation of trips to routes will usually be inconsistent with Wardrop's principle. The method has however been made the basis for a dynamical extension (Yagar, 1976) in order to investigate the evolution of the peak period.

In the housing system considerable research has been devoted to extending the Herbert-Stevens model (Herbert and Stevens, 1960) firstly, to include dispersion, as we have noted above, and secondly to incorporate dynamics - the traditional view of the housing system being that it is in permanent disequilibrium, but continually tracking a trajectory of equilibrium states.

Various members of the logit family have been employed in choice models of housing demand (see, for example, Quigley, 1976; Lerman, 1976; Kain, et al., 1976; Anas, 1979; Los, 1979), and models which have embedded micro-considerations within macro-frameworks have been developed by Kain, et al. (1976), Anas (1979) and Los (1979). In the NBER model (Ingram, et al., 1972; Kain, et al., 1976) the demand for housing "bundles" is first formed before allocation to spatially distinct supply sites. A linear program is used for clearing the market in each time slice and in the dynamic model, rents in any period are related to lagged adjusted shadow prices derived from the transportation program. Anas (1975, 1979) has employed logit-model demand functions in conjunction with a Walras "tatonament" process for an efficient approach to equilibration.

In a recent paper Los (1979) has adapted and extended the work of Snickars and Weibull (1977) by embedding a discrete choice model within a programming framework based on the information minimisation principle (see Section 4). The probabilistic choice model is treated as an "ex ante" demand and is associated with a "prior" distribution. The "ex post" demand, associated with a "posterior" distribution is determined through the optimisation process to be consistent with aggregate supply constraints and non-market clearing prices. Los (1979, p. 56) argues that this approach provides the basis for dynamical extensions by expressing the "ex ante" demands in a given time period in terms of prices in that period, and the shadow prices derived from previous periods.

8. Conclusion.

Considerable attention has been given recently to the philosophical, theoretical and practical issues involved in forming location and travel-related models within entropy maximising and choice theoretic approaches. We have drawn attention to, and traced the source of, close analytic similarities which exist between particular classes of models, and to the more general relationship between model structures and theoretical assumptions.

How then are we to view the equifinality issue outlined in the introduction. In our opinion little significance should be given to the fact that the multinomial logit function may be generated from an embarassing number of theoretical standpoints - over the past few years it has become a research objective to generate this model, which, after all, often provides an adequate discription of the dispersion in data sets. The issue should not indicate a privileged position for the logit model, or the family of models (eg. gravity models) based on it, and we have noted some of its restrictive properties of cross substitution which assert themselves in response contexts, usually the very contexts for which models are built.

We have shown above, particularly in Section six that the dichotomy between the two modelling styles according to whether they can be set up in program representations or not disappears. A large class of models may be formulated within a programming framework, which may be adapted for model calibration, plan evaluation and design.

There are temptations to see the Entropy maximising and choice theory methodologies, on the one hand, as competitors in all contexts with a view to eliminating one of them. On the other hand, because they may in particular contexts produce similar model forms there is the danger that they may somehow be caste as analytically equivalent model generators. While it is the case that for particular distributions of utility there are formal simularities and relations between the macro-variables in the

Entropy maximization and random utility approaches (Coelho and Williams, 1977; Williams and Senior, 1978; Leonardi, 1978; Brotchie et al., 1979), and while they may in fact be used in conjunction as in the applications of Snickars and Weibull (1977) and Los (1979), they remain distinct philosophical approaches. As "modes of explanation" they each have restrictions as Karlquist (1978) has discussed , and it remains largely a matter of taste which approach is adopted for model generation.

In both approaches to cross-sectional model formation the assumptions associated with model generation are distinct from whose made in forecasting response. While the different theoretical approaches leave open the possibility of employing different forecasting assumptions for the parameters, there remains the fundamental "leap-of-faith" involved in determining population response characteristics from the dispersion in trip and location patterns observable at a cross section (Williams, 1979). Williams and Senior (1977) have shown how apparently reasonable transportation models, judged by measures of statistical fit to base year patterns may generate extremely unlikely response properties (simply because they violated the theoretical restriction(5.19)). Further, Williams and Ortuzar (1979) have demonstrated how a number of alternative theoretical assumptions may be statistically consistent with observed dispersion patterns, but generate very different forecasts of response to policy stimuli.

This gives rise to whether alternative methods of estimating elasticity parameters should be adopted (eg. on "stated" rather than "revealed" preferences) and on whether alternative (eg. longitudinal) data sets should be sought. We believe that it is desirable to develop further both these lines of enquiry. It is particularly important to further embrace dynamical frameworks. What is urgently needed is the empirical basis to guide such development. The issues regarding aggregate and disaggregate models will again be faced and the problems associated with each must be spelt out and confronted.

Finally, we would suggest that in both entropy maximising and choice theoretic approaches it is desirable to gain a greater appreciation of what constitutes the dispersion in data patterns, and what is the influence, in any particular application, of the representation (definition of states) set up by the observer (the model builder) within which the dispersion is recorded and from which so many properties of a model, and the system it represents, are derived.

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