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A Model of Population Change and
Return Migration

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ABSTRACT

The paper applies a model of population change derived by
Rogers (1980) to the context of new and return migration in a
two-region system.   The structure of the model is examined
analytically given different assumptions about constancy in numbers
of migrants and in migration rates.   The model is applied to the
numerical example of birthplace - specific movement between Great
Britain and the rest of the world.

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1. INTRODUCTION

1.1 The Study of Return Migration

There are a multitude of papers written about migration, yet relatively few address themselves explicitly to return movement. King has referred to return migration as "a neglected aspect of population geography", (King, 1978, p.175) and A.S. Lee noted a "lack of interest" in the topic in the United States (Lee, 1974, p.283). The reason may be threefold. Since return migration is experienced only by a subset of migrants, who themselves comprise a part of the total population, in numerical terms return migration is a less important element of population change than universally or widely experienced events such as birth, family formation, first migration, and death. Secondly, data requirements for analysis of return migration are more demanding than for some cross-sectional movement studies, in that the location characteristics of two or more moves made by an individual must remain linked. Thirdly, and partly as a consequence of the first two reasons, there are few conceptual frameworks designed for theoretical analysis of return migration in particular.

Nonetheless, the practical and theoretical relevance of return migration is now generally recognised. The social and political implications are several, including the influence of return migrants on population composition, the role played by return migrants in altering the course of economic development in their homelands, and the extent of successful re-integration and further migration following return. A simple example of the theoretical importance of return migration is found in the disaggregation of the age profile of migrants. It is well known that the probability of movement is greatest during late teens and early twenties; a not unexpected corollary is that the age profile of return migrants peaks about three or four years later than the profile for new migrants (Eldridge, 1965). The role of return movement in influencing subsequent place utility assessment and search behaviour (Wolpert, 1965) also is theoretically pertinent.

The study of return migration could be as broad as the study of migration for example, the volume and probabilities of movement, characteristics of migrants, origins, and destinations, and motivations for movement. Further investigation of such topics would increase our understanding not only of return migration itself, but also of migration in general.

1.2 A Brief Review of the Literature

A survey of literature on return migration can be summarized within a framework of five basic types of research.

Papers incorporating the development of typologies, the formulation of broad, structural models, and discussions of social and economic implications and of data potential are perhaps characteristic of early stages of investigation.   King's paper in *Area* (King, 1978) is a useful example.   To some extent, such issues are considered in all papers about return migration (e.g. Richmond, 1967 and 1968), but only a few discussions provide a comprehensive foundation for further investigation or explicitly outline conceptual models of return migration.

Applications of statistical models to the investigation of return migration in particular are better represented in the literature.   The regression models of Vanderkamp (1971), Lianos (1975), and Bell and Kirwan (1979) seek to explain return migration as a function of some combination of income, employment, distance, and previous migration.   The reports indicate firstly that return migrants and new migrants may react quite differently to conditions in the origin and in potential destinations, and secondly that available migration statistics may be exploited in new ways to indicate the volume of return migration, albeit, perhaps, with questionable accuracy (Vanderkamp, 1971 and 1972, in Canada; Bell and Kirwan, 1979, in Britain).

A third category of study lies within the field of mathematical demography. For example, Ledent (1981) uses place of birth to refine life tables, demographic accounts, and associated measures such as multiregional expectations of life.   The influence of prior residence on destinations of migrants is clearly indicated, but the ultimate aim of the work lies more in acquiring an increasingly refined and accurate spatial accounting of the population than in analysis of the volume and characteristics of return migration.   The latter is considered more explicitly by St. John-Jones (1979), who applies demographic methods to assess the volume of re-migration of Canadian immigrants.

A larger number of studies report on empirical explorations of various aspects of return migration : the socio-economic and demographic characteristics of return migrants in contrast with new or non-migrants (Long and Hansen, 1975; Miller, 1977); sociological case studies of motivations for return (Davison, 1968); and more comprehensive investigations of the nature and importance of return migration in particular spatial and temporal contexts (King, 1977b; Hirschman, 1978; Foeken, 1980).   The predominantly inductive nature of the

associated generalization emphasizes, perhaps, the early stage of model-building in the field.

Finally, several papers report on the impact of return migration on the economic development of origin countries, and assess the degree of success of government policies designed to lure migrants home (Toren, 1975; Strachen, 1980).

1.3 The Definition and Measurement of Return Migration

A problem often encountered in comparing empirical results in the literature is the lack of explicitly stated definitions and statistics of measurement. A "conceptual" definition of return migration is as easy or as difficult to state in first principles as is migration itself. Movement back to some original location is indicated, thereby implying the occurrence of at least one previous move. More essential are the "practical" or "operational" definitions involving specification of spatial and temporal units of relevance. As in any study of migration, the level of resolution controls the observation.

Choice of spatial unit occurs on a continuous scale, but certain "marker points" are indicated : dwelling returns, which are rarely considered and arguably theoretically uninteresting to the geographer; neighbourhood returns, infrequently examined due to the lack of large data sets; regional (internal) returns, which have been examined increasingly with the recent availability of suitable census and survey information (e.g. USA studies by Eldridge, 1965; Lee, 1974; Ledent, 1981; and others); and international returns, also frequently considered but subject to problems of data comparability (Appleyard, 1962; Richmond, 1968). The urban/rural nature of locations has been incorporated also (Lee, 1974). It could prove fruitful to compare the level of return movement over several spatial scales.

A clear distinction between continuous and discontinuous time scales of movement observation is critical. Longitudinal, continuous records of every location and move permit the extraction of maximum information. A typology of return movements could be derived, involving such nomenclature as "lifetime", "temporary", "repeated", and "chained" return movement. Theoretically, it does not seem difficult to derive a scheme to extract the desired information from a string of locations, by repeating a limited number of yes/no type questions and working back through time. Unfortunately, suitable longitudinal data sources are few and difficult to access.

Alternatively, return migration can be analyzed on a discontinuous time scale, normally involving comparison of locations at three specific points in time, and capturing only some proportion of the moves. Since data on populations and number of transitions can be summarized in various measures of movement, explicit statement of the statistic and the time interval is essential for comparative analysis. For example, return migration (perhaps better termed "return transition" when discontinuous time information is used) can be expressed as a gross figure, as a proportion of some base population, or as a ratio with non-return movement. Consider, for example, the wide variety of measures obtained from the small data set in Table 1. Frequently, the chosen statistic depends upon the peculiarities of a particular data set (Bell and Kirwan, 1979; Foeken, 1980), rendering comparative analysis even more difficult.

1.4 The Role of this Model

This paper considers the structure and dynamics of a mathematical model of population change which explicitly identifies return movement. The original purpose of the exercise was to provide both time-specific and long-term estimates of birthplace-dependent probabilities of migration between Great Britain and the rest of the world, for use in a multiregional simulation model of migration. As such, it is most closely related to simple numerical accounting models in demography, and it does not seek a causal explanation or socio-economic description of new and return migration. The model uses the discontinuous time scale, noting locations in two regions at three points in time. The rate of return migration is expressed using the population at risk of a return move, in this case residents outwith their place of birth, or the "non-native population". A "native population" comprises those living in their place of birth who are at risk of becoming "new migrants". Even in its simplest form, with only a limited number of crude rates as parameters and no disaggregation of the population by demographic characteristics, the model provides insight into the dynamics of population change given return migration and into some implications of policies of immigration control. Further, the structure of the model permits substantial refinement towards a more accurate and spatially detailed numerical description of return migration.

TABLE 1.   A SAMPLE FIVE-YEAR TRANSITION TABLE WITH ALTERNATIVE
           MEASURES OF RETURN MIGRATION

	Location 1970		
	State of Birth	Outside State of Birth; if Resident Outside State of Birth in 1965, in the Same State	A Third State
Location 1965			
State of Birth	97706040	6499960	—
Outside State of Birth	2871700	30031422	5210878

Source: Miller, 1977, from Table 2, p.4, for Native White Population
             in the U.S.A. aged 5+ at 1970 Census.

Measures of Return Migration Derived from this Data

number of return migrants 1965–1970	2871700
return migrants/total population	0.0202
return migrants/migrants 1965–1970	0.1969
return migrants/population resident outside S.O.B. in 1965	0.0754
return migrants/migrants 1965–1970 resident outside S.O.B. in 1965	0.3553
return migrants/non-return migrants 1965–1970	0.2452

2. THE MODEL

2.1 Definitions and Variables

The model is developed first in the context of two regions (1,2) which together close the world system. The population is divided into four groups on the basis of place of birth and place of residence at time t. Migration is recognized if the spatial boundary is crossed once (or an odd number of times) during the time interval $(t, t + 1)$, with new and return migration differentiated by direction of movement with respect to birthplace (Table 2). The following variables are defined:

- P   population
- M   number of migrants
- b   birth rate
- d   death rate
- m   migration rate = M/P
- 1,2   two regions in the system which, occurring as a variable

  prefix indicates place of birth

  suffix indicates place of current residence

  second suffix, used in migration terms only, refers to destination

- t   time

2.2 The Basis for the Model

A basic demographic growth model is represented by

$$P(t + 1) = (1 + b - d + nm) \times P(t)$$

where nm is net migration rate and other variables are defined above. In considering the dynamics of urban and rural population change, Rogers (1980) disaggregates the population by urban and rural place of birth and by direction of movement, to define four equations which provide the basis for the model examined here.

2.3 The Model Equations

The model is represented by four equations:

$$_1P_1 (t + 1) = (1 + {_1}b_1 - {_1}d_1 - {_1}m_{12}) \times {_1}P_1(t) + {_1}m_{21} \times {_1}P_2(t) + {_2}b_1 \times {_2}P_1(t) \qquad (1)$$

$$_1P_2 (t + 1) = (1 - {_1}d_2 - {_1}m_{21}) \times {_1}P_2(t) + {_1}m_{12} \times {_1}P_1(t) \qquad (2)$$

$$_2P_1 (t + 1) = (1 - {_2}d_1 - {_2}m_{12}) \times {_2}P_1(t) + {_2}m_{21} \times {_2}P_2(t) \qquad (3)$$

$$_2P_2 (t + 1) = (1 + {_2}b_2 - {_2}d_2 - {_2}m_{21}) \times {_2}P_2(t) + {_2}m_{12} \times {_2}P_1(t) + {_1}b_2 \times {_1}P_2(t) \qquad (4)$$

Equations (1) and (4) state that each native population depends on its
own rates of natural increase and out-migration, the rate of return of
native-born migrants who had previously left the region, and the births
occurring to non-native residents.   Each non-native population depends
on its death and outmigration rates and on the rate of entry of new migrants.

2.4 Approaches to the Solution of the Model

The dynamics of this two-region model are examined both analytically and
numerically.   Birth and death rates are held temporally constant, and
several migration control assumptions are considered in turn : first,
constant migration rates;   secondly, constant numbers of migrants;   and
thirdly, some mixed control conditions.

TABLE 2. <u>CLASSIFICATION OF POPULATIONS AND MIGRANTS IN A TWO-REGION SYSTEM</u>

<u>Birthplace 1</u>

Location and Population at t		Location and Population at t + 1	
		1 $_1P_1$	2 $_1P_2$
1	$_1P_1$	Native Stayer	New Migrant $_1M_{12}$
2	$_1P_2$	Return Migrant $_1M_{21}$	Non-Native Stayer

<u>Birthplace 2</u>

Location and Population at t		Location and Population at t + 1	
		1 $_2P_1$	2 $_2P_2$
1	$_2P_1$	Non-Native Stayer	Return Migrant $_2M_{12}$
2	$_2P_2$	New Migrant $_2M_{21}$	Native Stayer

3. INTRODUCTION TO THE NUMERICAL EXAMPLE

3.1 Preliminary Comments

While an analytical solution is most useful in elucidating the structure
and behaviour of the model, inclusion of a numerical example based on
real data serves firstly to identify data needs and availability and hence
the model's practical utility, and secondly to provide an alternative means
of illustration where analytical solution is difficult.

Numerical exploration, undertaken using programs implemented on the
University of Leeds Amdahl VM470 computer, involved iteration of appropriate
forms of the model equations (1) to (4), and also calculation of particular
solutions for related equations derived analytically.   The population
projections resulting from temporal iteration of the equations are not
intended to provide realistic forecasts of population size.   The assumption
of constant crude birth and death rates, thereby ignoring changes in age
structure resulting from the natural increase regime and from age-selective
migration, renders the model inadequate as a forecasting tool, although the
implications of current rates over the long-term are indicated.   Interest
in the projections lies primarily with their illustration of the model's
form and behaviour.   The numerical results are potentially realistic only over
short periods, but are useful also in comparing population development and
relationships under different natural increase and migration conditions,
and in providing a foundation for assessment of model refinements.

The main example used throughout this paper is based on 1970-71
information for Great Britain (Region 1, with subscript G), and the Rest
of the World (Region 2, with subscript R).   A brief review of the method
used to estimate the required parameters illustrates some of the difficulties
encountered in extracting birthplace-specific rates and in reconciling conflicting
migration data.   After estimating the four populations and their associated
birth and death rates for 1970-71, some hypothetical birth and death rate
regimes are chosen to further elucidate the model's behaviour, and finally
three distinct migration estimates for movement between Great Britain and
the Rest of the World in 1970-71 are identified.

To obtain more accurate numerical results, two corrections in the British
populations should be made, but the level of confidence in the data and the
purpose for which it is used does not warrant their application here.   First,
the 1971 populations should be "back-survived" to 1970 populations-at-risk
before the 1970-71 migration rates are applied.   Secondly, the census-date
British populations should be adjusted to midyear populations relevant for

the application of the annual birth and death rates. (See Rees, 1980, for application).

3.2 Estimates for Populations, Birth Rates, and Death Rates

3.2.1 Native and Non-Native Populations Resident in Great Britain

Although the Census usually reports place of birth in terms of the United Kingdom, the figures were adjusted to refer to Great Britain (excluding Northern Ireland) in order to facilitate application of the results to other models requiring more detailed statistics for Northern Ireland than are available. Both the "resident" and "visiting" populations are included in these estimates. Birthplace-specific births, deaths, and populations in the United Kingdom can be collected from various sources (OPCS, 1974, Table 3; OPCS, 1977a, Tables 9.1 and 9.2; OPCS, 1977b, Table 20; OPCS, 1976b, Table 4.5; Northern Ireland Information Service, 1980, Tables 7, 11, 12, 20, 21; Northern Ireland General Register Office, 1975, Tables 4 and 7; Scottish General Register Office, 1972, Tables B2.6 and S2.22; Scottish General Register Office, 1974, Table 13).

To adjust birthplace from United Kingdom to Great Britain, three simplifying assumptions were made for both birth and death rates:

(i) $\quad \text{non-UK}^r\text{G} = \text{non-UK}^r\text{NI}$

(ii) $\quad \text{NI}^r\text{NI} = \text{UK}^r\text{NI} \times \dfrac{\text{UK}^r\text{UK}}{\text{UK}^r\text{G}}$

(iii) $\quad \text{UK}^r\text{G} = \text{G}^r\text{UK}$

where

r	is birth or death rate
prefix	is place of birth
suffix	is place of residence
UK	is United Kingdom; non-UK thereby defined
G	is Great Britain
NI	is Northern Ireland

Then, since the size of the "birthplace not stated" element varies among population, births and deaths, it was allocated proportionately for each birthplace given residence in Great Britain. The crude birthplace-specific rates were then calculated by division of number of events by the population at risk (Table 3).

3.2.2 Population Born and Resident in the Rest of the World

The 1971 and 1976 United Nations Demographic Yearbooks (UN, 1972, Table 1; UN, 1977, Table 1) provide the required estimates of total world population and birth and death rates (Table 3). Again the level of accuracy

TABLE 3.   **BIRTHPLACE-SPECIFIC POPULATION AND BIRTH AND DEATH RATE
DATA FOR 1970-1971**

Birthplace	Residence	1971 Census Date Populations : P(0)	Birth and Death Regime 1 : 1970-71 Rates Per Thousand	
			b	d
G	G	$50,595,500^a$	15.3^a	11.9^a
G	R	$3,300,000^b$	15.3^c	11.9^c
R	G	$3,383,000^a$	28.5^a	8.2^a
R	R	$3,706,000,000^d$	32.0^e	13.0^e

Sources:

a   Estimated using: OPCS 1974, Table 3; OPCS 1977a, Tables 9.1
and 9.2; OPCS 1977b, Table 20; OPCS 1978b,
Table 4.5; Nothern Ireland Information
Service 1980, Tables 7, 11, 12, 20, 21;
Northern Ireland General Register Office 1975,
Tables 4 and 7; Scottish General Register
Office 1972, Tables B2.6 and S2.22; Scottish
General Register Office 1974, Table 13.

b   Estimated using: UN 1977, Table 33; Australian Bureau of
Statistics 1975, p.148; New Zealand Department
of Statistics 1974, p.83; Statistics Canada
1977, Table 4.21; OPCS 1978b, Table 4.8.

c   No source : assumed equivalent to rates for $_GP_G$.

d   UN 1972, Table 1.

e   UN 1977, Table 1.

is sufficient for the purposes of the investigation.

3.2.3 Population Born in Great Britain and Living Abroad

Estimation of the population resident in the Rest of the World
but born in Great Britain proved the most problematic. The United Nations
Demographic Yearbook for 1976, the Yearbooks of three main countries of
immigration, and Office of Population Censuses and Surveys tabulations
(UN,1977, Table 33; Australian Bureau of Statistics,1975, p.148; New
Zealand Department of Statistics,1974, p.83; Statistics Canada,1977,
Table 4.21; OPCS,1978b, Table 4.8) together with a place of birth adjustment
factor from United Kingdom to Great Britain of 0.97 (from OPCS,1978b,
Table 4.8) provided an approximate total for British-born living abroad.
No information could be found for the associated birth and death rates,
and it was assumed that the rates are approximately equivalent to those
of the native population in Great Britain (Table 3).

3.3 Alternative Birth and Death Rate Regimes

To further elucidate the model's behaviour, several alternative birth and
death rate combinations are considered (Table 4). Regimes 2 to 6 represent
a variety of possible long-term natural change conditions; again, the
rates are temporally constant. Regime 7 involves abrupt change in natural
increase conditions at two points in time. A program involving logistic
change in birth and death rates over time was written, but the results are
not pursued here.

3.4 Estimates of the Numbers of Migrants

Both the Census of Great Britain and the International Passenger
Survey (IPS) through the Office of Population Censuses and Surveys (OPCS)
provide estimates of the numbers of migrants during 1970-71 by birthplace
for this system. Again, appropriate factors must be applied to adjust
birthplace and places of origin and destination from the United Kingdom to
Great Britain (estimated using OPCS,1978b, Table 4.8):

    (i)   to change birthplace from UK to GB   x 0.970

    (ii) to change movement in/out of UK to GB x 0.985

    (iii) to change both simultaneously, under
            the assumption of independence   x 0.955

So in 1971, approximately 32% of British immigrants and 66% of emigrants
were born in Great Britain.

The Census and IPS migration estimates are very different, and while the
Census can be considered a more reliable source of information, it does not

TABLE 4. <u>ALTERNATIVE BIRTH AND DEATH RATE REGIMES FOR USE IN NUMERICAL EXPERIMENTS</u>

Regime	Populations	Rates per Thousand		Description of Natural Change
		Birth	Death	
1				See Table 3
2	All four	11	11	no growth, low rate
3	All four	14	14	no growth, higher rate
4	All four	12	11	low rate of growth
5	G^P_G, G^P_R, R^P_G	12	11	low rate of growth in non-$_RP_R$; 1970-71 rates in $_RP_R$
	$_RP_R$	32	13	
6	G^P_G, G^P_R, R^P_G	11	11	no growth in non-$_RP_R$ "low" rate of growth in $_RP_R$
	R^P_R	15	11	
7	0-50 years : All four	Each as 1970-71 rates		change through time towards lower natural increase conditions
	60-100 years			
	G^P_G, G^P_R	12	11	
	R^P_G	15	11	
	R^P_R	18	12	
	100+ years			
	G^P_G, G^P_R, R^P_G	11	11	
	R^P_R	15	11	

TABLE 5. **THREE MIGRATION ESTIMATES BASED ON 1970-1971 INFORMATION FOR GREAT BRITAIN AND THE REST OF THE WORLD**

Migration Estimate	A	B	C
Description	IPS-based[a]	Census-based[b]	Assuming[c] stationary G^P_R
Migrant Flows			
$_G M_{GR}$	155,500	297,660	159,060
$_G M_{RG}$	65,300	119,500	119,500
$_R M_{GR}$	86,000	153,340	81,940
$_R M_{RG}$	134,700	254,000	254,000
Migration Rates			
$_G m_{GR} \times 10^{-3}$	3.073	5.883	3.144
$_G m_{RG} \times 10^{-2}$	1.979	3.621	3.621
$_R m_{GR} \times 10^{-2}$	2.542	4.533	2.422
$_R m_{RG} \times 10^{-5}$	3.635	6.854	6.854

Sources:

a   Estimated using: OPCS 1978b, Tables 1.4, 2.1, 2.5, 3.4, 3.12, 4.8, 4.9

b   OPCS 1978a; OPCS 1978b, Tables 4.1 and 4.2

c   See text for basis of assumption, using Sources b.

provide an emigration statistic. In addition, there is some discrepancy among the various IPS figures; in some of the statistics used below, an approximate value was interpolated from various sources. Three estimates of migrant flows were derived:

Estimate A : An average of three or four IPS/OPCS estimates for each of the four flows and adjusting for birthplace as required. (OPCS, 1978b, Tables 1.4, 2.1, 2.5, 3.4, 3.12, 4.8, 4.9).

Estimate B : Using the Census immigration statistic of 373500 (OPCS, 1978a; OPCS, 1978b, Tables 4.1 and 4.2), and assuming that the ratio of emigration figures of Census and IPS is the same as that for immigration (1.87 = 373500/200000), a "census estimate" of total emigration was derived, 451000, and then disaggregated using the birthplace percentages. Rees uses a similar estimation technique with more precision and detail (Rees, 1980).

Estimate C : Using the Census estimates for immigration to Great Britain, emigration from Great Britain was calculated under the assumption that the population born in Great Britain and living in the Rest of the World is stationary. The number of British-born emigrants, then, must balance the number of British-born residents abroad lost through death or return migration. Return emigration from Great Britain is then estimated using the birthplace disaggregation factor. It is difficult confidently to support or refute this assumption empirically, but the estimate provides an interesting alternative for consideration in the model.

The three estimates of migrant flows and rates are summarized in Table 5. For simplicity in interpretation, the alternative birth/death Regimes 2 to 7 are investigated only with the census-based Migration Estimate B.

3.5 A Comparison of the 1971 New and Return Migration Rates

Using the summary of estimated migration rates for 1971 (Table 5), rates of return migration and new migration are compared for movement in the same direction (Table 6). Despite the variety in the estimates of migration rates, the return/new ratios are consistent. While return movement comprises

about one-third of the total volume of flow in each direction, the
probability of a move is much greater for a non-native than for a native
resident.   The difference is particularly marked in the case of flow from
the Rest of the World to Great Britain, since a huge proportion of the
world's population is physically or economically incapable of such a move.
Different levels of development in origin and destination may be crucial
in determining the relative importance of return migration.

TABLE 6. <u>RATIO OF RETURN TO NEW RATES OF MIGRATION OCCURRING
IN THE SAME DIRECTION</u>

Migration Estimate	A (IPS-based)	B (Census based)	C (Stationary $_G P_R$)
Leaving Great Britain $R^m_{GR}/_G^m{}_{RG}$	8.3	7.7	7.7
Leaving the Rest of the World $_G^m{}_{RG}/R^m_{RG}$	544.4	528.3	528.3

4. INVESTIGATING THE MODEL: CONSTANT MIGRATION RATES

4.1 The Context

Consider first a situation where there are no legal controls on numbers of migrants, and social and economic conditions are such that the attractiveness of migration remains at 1970-71 (t=0) levels for each population. That is, the dynamics of the model can be investigated analytically assuming temporally constant <u>rates</u> of migration, as well as constant birth and death rates, for each population.

4.2 A General Solution for the Populations at Time t

Equation (1) can be rewritten as a difference equation:

$$_1P_1(t+\Delta t) \;=\; {}_1P_1(t) \;+\; ({}_1b_1 - {}_1d_1 - {}_1m_{12}) \;\times\; {}_1P_1(t)\Delta t \;+\; {}_1m_{21}$$
$$\times\; {}_1P_2(t)\Delta t \;+\; {}_2b_1 \;\times\; {}_2P_1(t)\Delta t$$

$$\frac{_1P_1(t+\Delta t) - {}_1P_1(t)}{\Delta t} \;=\; ({}_1b_1 - {}_1d_1 - {}_1m_{12}) \;\times\; {}_1P_1(t) \;+\; {}_1m_{21}$$
$$\times\; {}_1P_2(t) \;+\; {}_2b_1 \;\times\; {}_2P_1(t) \tag{5}$$

The limit of the population change as the time increment approaches zero is the differential $\dfrac{d\,{}_1P_1}{dt}$

$$\lim_{\Delta t \to 0} \frac{_1P_1(t+\Delta t) - {}_1P_1(t)}{\Delta t} \;=\; \frac{d\,{}_1P_1}{dt}$$

so

$$\frac{d\,{}_1P_1}{dt} \;=\; ({}_1b_1 - {}_1d_1 - {}_1m_{12}) \;\times\; {}_1P_1(t) \;+\; {}_1m_{21} \;\times\; {}_1P_2(t)$$
$$+\; {}_2b_1 \;\times\; {}_2P_1(t) \tag{6}$$

With similar derivations:

$$\frac{d\,{}_1P_2}{dt} \;=\; (-{}_1d_2 - {}_1m_{21}) \;\times\; {}_1P_2(t) \;+\; {}_1m_{12} \;\times\; {}_1P_1(t) \tag{7}$$

$$\frac{d_2P_1}{dt} = (-_2d_1-_2m_{12}) \times \ _2P_1(t) + \ _2m_{21} \times \ _2P_2(t) \tag{8}$$

$$\frac{d_2P_2}{dt} = (_2b_2-_2d_2-_2m_{21}) \times \ _2P_2(t) + \ _2m_{12} \times \ _2P_1(t)$$
$$+ \ _1b_2 \times \ _1P_2(t) \tag{9}$$

Each of these equations can be expressed in the form

$$\frac{dP_i}{dt} = a_{11} \times \ _1P_1(t) + a_{12} \times \ _1P_2(t) + a_{13} \times \ _2P_1(t)$$
$$+ \ a_{14} \times \ _2P_2(t) \tag{10}$$

or,

$$\frac{dP_i}{dt} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} \begin{bmatrix} _1P_1 \\ _1P_2 \\ _2P_1 \\ _2P_2 \end{bmatrix} (t) \tag{11}$$

Then a system of differential equations expressed in matrix algebra has been identified:

$$\frac{d\underline{P}}{dt} = \underline{A}\ \underline{P}(t) \tag{12}$$

where

$$\underline{P} = \begin{bmatrix} _1P_1 \\ _1P_2 \\ _2P_1 \\ _2P_2 \end{bmatrix}$$

$$\underline{A} = \begin{bmatrix} _1b_1-_1d_1-_1m_{12} & _1m_{21} & _2b_1 & 0 \\ _1m_{12} & -_1d_2-_1m_{21} & 0 & 0 \\ 0 & 0 & -_2d_1-_2m_{12} & _2m_{21} \\ 0 & _1b_2 & _2m_{12} & _2b_2-_2d_2-_2m_{21} \end{bmatrix}$$

The solution of the differential system to find eigenvalues and eigenvectors follows standard techniques.

$$\text{Let} \quad \underline{P}(t) \quad = \quad \underline{P}(0)e^{\lambda t} \tag{13}$$

where $\underline{P}(0)$ is a vector of the populations at time t=0. Then,

$$\frac{d\underline{P}}{dt} \quad = \quad \frac{d}{dt} \, \underline{P}(0)e^{\lambda t}$$

$$= \quad \lambda \underline{P}(0)e^{\lambda t} \tag{14}$$

Using (13), Equation (14) becomes

$$\frac{d\underline{P}}{dt} \quad = \quad \lambda \underline{P}(t) \tag{15}$$

Equating (12) and (15)

$$\underline{A} \, \underline{P}(t) \quad = \quad \lambda \underline{P}(t)$$

$$(\underline{A} - \lambda \underline{I}) \, \underline{P}(t) \quad = \quad 0 \tag{16}$$

In order to have a non-trivial solution where the populations are not all zero, the determinant $|\underline{A} - \lambda \underline{I}|$ must equal zero. Solution of this equation yields a fourth degree polynomial in λ, with complicated coefficients in this system. Given the four eigenvalues λ_1 to λ_4 (the roots of the equation), the corresponding eigenvectors $\underline{V}_i$ are found by solution of the relation

$$(\underline{A} - \lambda_i \, \underline{I}) \, \underline{V}_i \quad = \quad 0 \tag{17}$$

Each of the four populations can be expressed as linear combinations of the eigenvectors at time t.

$$P_1(t) = c_1 e^{\lambda_1 t} V_{11} + c_2 e^{\lambda_2 t} V_{12} + c_3 e^{\lambda_3 t} V_{13} + c_4 e^{\lambda_4 t} V_{14} \qquad (18)$$

where $c_1, \ldots, c_4$ are constants found by the solution of

$$\underline{W}\,\underline{c} = \underline{P}(0) \qquad (19)$$

where $\underline{W}$ is the matrix of eigenvectors V_1, V_2, V_3, V_4.

Once the c, λ, and V are calculated, normally by the use of a computer library package, a value for each population at any time t can be calculated directly without iteration.

4.3 Stationary Populations Given Constant Rates

Under what conditions could constant birth, death and migration rates result in stationary populations? Identification of the combination of parameters leading to this state requires the application of the general solution, where differential equations (6) to (9) are set to zero. For a non-trivial solution, the determinant of the coefficient matrix $\underline{A}$ in (12) must be zero. Then the constant term of the polynomial in λ in the general case is zero, and $\lambda=0$ is a root. Solution proceeds using the technique outlined above.

Consider also the elements of differential equations (6) to (9) set to zero:

$$(_1b_1 - _1d_1 - _1m_{12}) \times _1P_1(t) + _1m_{21} \times _1P_2(t) + _2b_1 \times _2P_1(t)$$
$$= 0 \qquad (20)$$

$$-(_1d_2 + _1m_{21}) \times _1P_2(t) + _1m_{12} \times _1P_1(t) = 0 \qquad (21)$$

$$-(_2d_1 + _2m_{12}) \times _2P_1(t) + _2m_{21} \times _2P_2(t) = 0 \qquad (22)$$

$$(_2b_2 - _2d_2 - _2m_{21}) \times _2P_2(t) + _2m_{12} \times _2P_1(t) + _1b_2$$
$$\times _1P_2(t) = 0 \qquad (23)$$

By definition in the real system, all rates and populations are positive or zero. In Equations (21) and (22) for non-native populations, the first coefficient is negative or zero, while the second is positive or zero, so $_1P_2$ and $_2P_1$ may be stationary for some combination of parameters.

In Equations (20) and (23), the second and third terms are positive or zero, so a necessary but not sufficient condition for a stationary solution is that (b-d-m) be less than or equal to zero.

One or more of the populations may reach "quasi-stationarity", or the same value over two times followed by further growth or decline. From Equation (21),

$$\frac{_1P_2(t)}{_1P_1(t)} = \frac{_1m_{12}}{_1d_2 + {_1m_{21}}} \tag{24}$$

If the two populations obtain this relationship, $_1P_2$ will be quasi-stationary. Using the example of Great Britain and the Rest of the World, 1970-71 birth and death rates (Regime 1), and Migration Estimate A (Tables 3 and 5), the right hand side of (24) is 0.0970. If $_1P_1 = 50,595,500$ (as at t=0), then $_1P_2(t)$ must be 4,906,246 to remain stationary over the next time. Using (2),

$$_1P_2(t+1) = (0.96831)(4906246) + 155480$$
$$= 4906247$$

So $_1P_2(t) \doteq {_1P_2(t+1)}$. Then from (1),

$$_1P_1(t+1) = 1.000327 \times 50595500 + 0.01979 \times 4906246$$
$$+ 0.0285 \times 3383000$$

Since $_1P_1$ grows, the ratio $_1P_2/_1P_1$ decreases and $_1P_2$ does not remain stationary over the next time interval.

4.4 Numerical Experiments

4.4.1 Introduction

Numerical exploration of the constant migration rates model involved iteration of Equations (1) to (4) using the 1970-71 Migration Estimates (Table 5) and the birth/death Regimes 1 to 7 (Tables 3 and 4). Unless otherwise stated, investigation of the three Migration Estimates proceeds using the 1970-71 birth and death rates (Regime 1) while Regimes 2 to 7 are analyzed using Migration Estimate B. Stability is defined as reached when the difference among the four ratios P(t+1)/P(t) is less than 0.0001 or as otherwise indicated.

4.4.2 Population size, growth and composition

The population sizes after one hundred years resulting from the three Migration Estimates and the seven alternative birth/death Regimes are listed in Table 7. The non-native populations from Census-based Estimate B and 1970-71 birth and death rates are about 36% larger than the IPS Estimate A populations, while the native population resident in Great Britain is only 3.5% larger and the native population resident abroad is marginally smaller. Migration Estimate C restricts the growth of British-born living abroad, but the populations resident in Great Britain are significantly larger than the populations resulting from either Estimate A or B.

The lower natural increase rates illustrated by birth and death Regimes 2 to 7 particularly lessen the growth of the two populations born abroad, as these experience the greatest change in natural increase conditions. Although all births are assigned to native populations, Regimes 2 and 3, each with equal birth and death rates in the four populations, result in growth of the two non-native populations while those born and resident in Britain decline and natives living abroad increase only marginally. The effect is less marked at the higher birth and death rates of Regime 3. The relatively larger natural increase in Regimes 5 and 6 experienced by the population born abroad influences initially the size of the non-native population in Great Britain, and secondarily those born in Great Britain, in a domino effect resulting from population movement.

Until growth stability is achieved, the relative sizes of the four populations continue to change, with adjustments in growth rates occurring most rapidly during the first fifty or one hundred years (Figures 1a to 1c). Subsequently, the two populations born in Great Britain continue to approach the rates of growth already experienced by the populations born abroad. Although some of the migration rates are larger than natural change rates, the final stable growth rate is controlled by the rate of natural change of the largest population, that born and resident in the Rest of the World. The populations subjected to 1970-71 birth and death rates converge to a growth rate of 0.0190, while under each of the alternative regimes the populations also converge to the natural change rate of $_R P_R$ (Figures 2a-2f). Migration Estimates A, B and C require 413-453 iterations to reach

Table 7.   **Populations at t=100 given constant migration rates**

MIGRATION ESTIMATE	BIRTH/DEATH REGIME	POPULATIONS (millions)			
		$_G P_G$	$_G P_R$	$_R P_G$	$_R P_R$
A	1	84.78	6.83	16.83	24,316.61
B	1	87.69	9.25	22.97	24,304.12
C	1	107.28	5.84	32.31	24,265.79
B	2	46.80	5.85	4.51	3,706.11
B	3	46.89	5.52	4.28	3,706.59
B	4	51.76	6.34	4.89	4,095.59
B	5	57.85	6.78	22.11	24,293.56
B	6	47.51	5.91	6.27	5,521.03
B	7	68.03	8.04	13.99	12,792.65

Figure 1   The Temporal Pattern of Growth Rates of the
Four Populations Given Constant Migration Rates and
1970 – 71 Birth and Death Rates
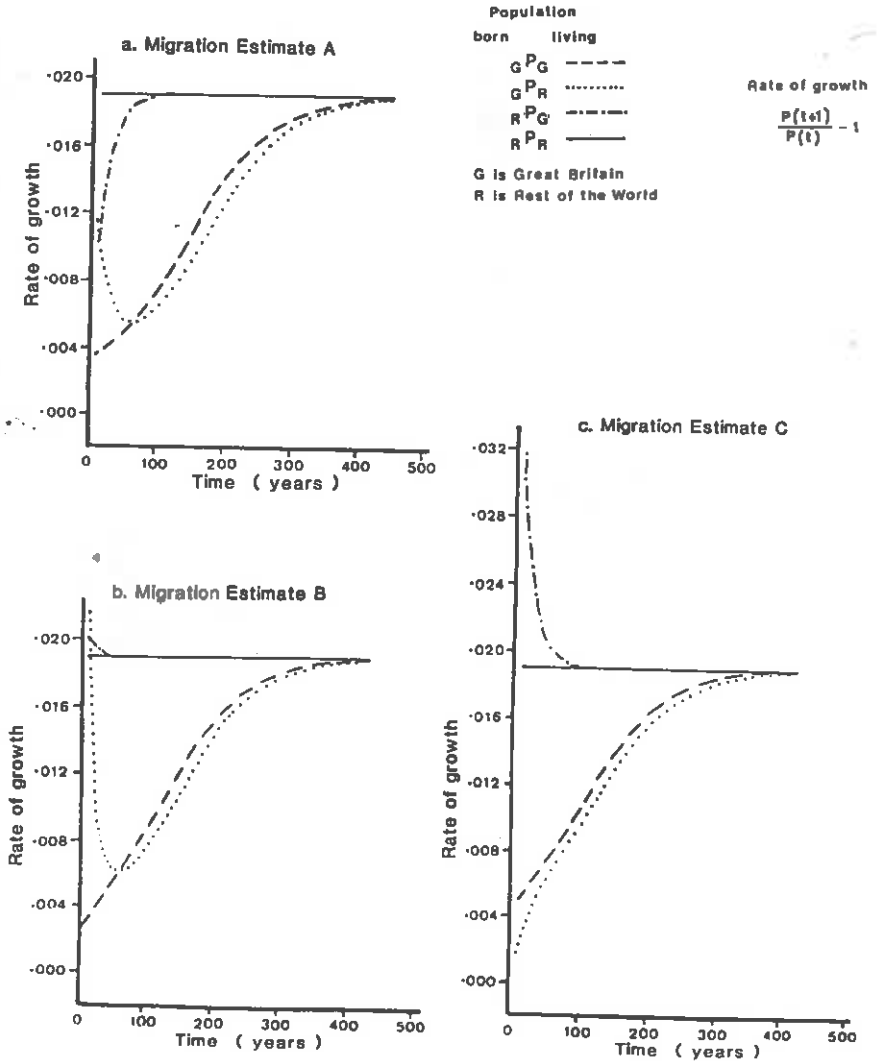
Figure 2   The Temporal Pattern of Growth Rates of the
Four Populations Given Constant Migration
Rates and Alternative Conditions

Population

born	living	
$_G P_G$	----	
$_G P_R$	....	
$_R P_G$	-.-.-.	
$_R P_R$	———	

Rate of growth

$$\frac{P(t+1)}{P(t)} - 1$$

G is Great Britain
R is Rest of the World

a. Migration Estimate B with Regime 2

b Migration Estimate B with Regime 3

c. Migration Estimate B with Regime 4

d Migration Estimate B with Regime 5

Figure 2   continued
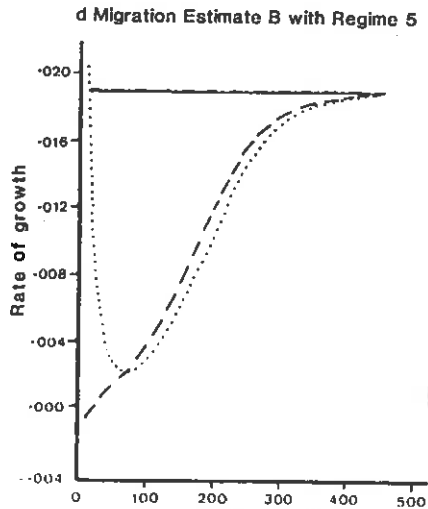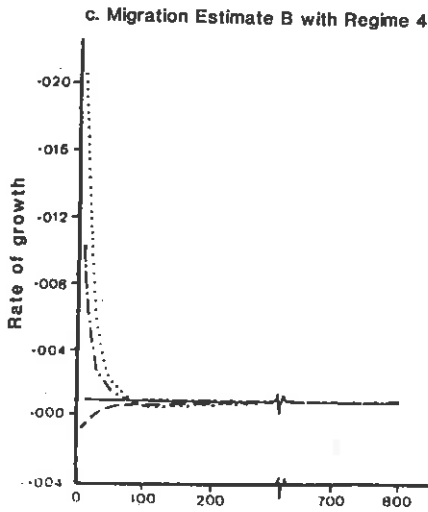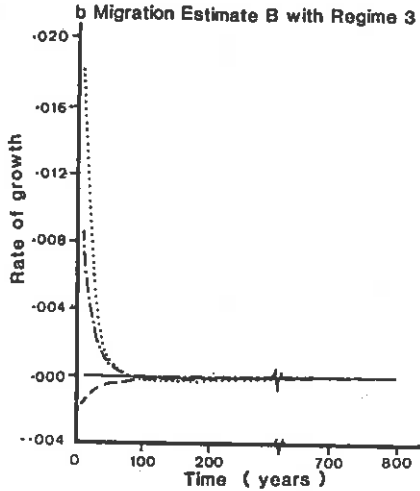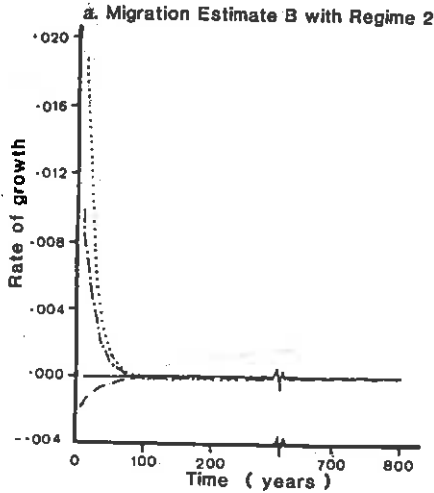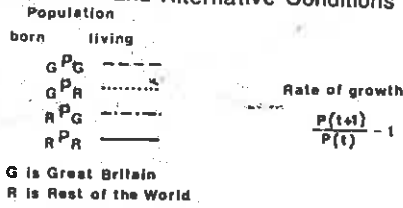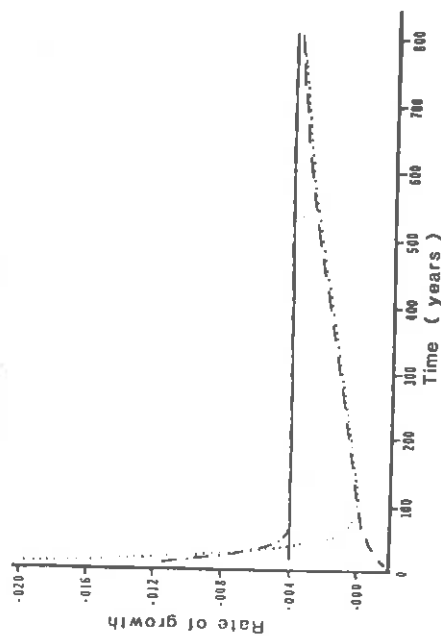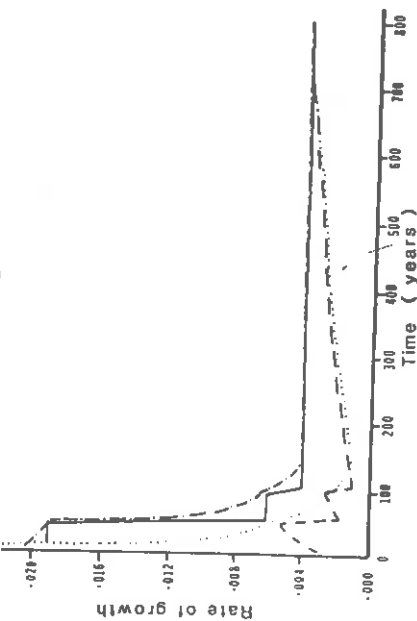
g. No Migration with Regime 1

Rate of growth

Time (years)

e. Migration Estimate B with Regime 6

Rate of growth

Time (years)

f. Migration Estimate B with Regime 7

Rate of growth

Time (years)
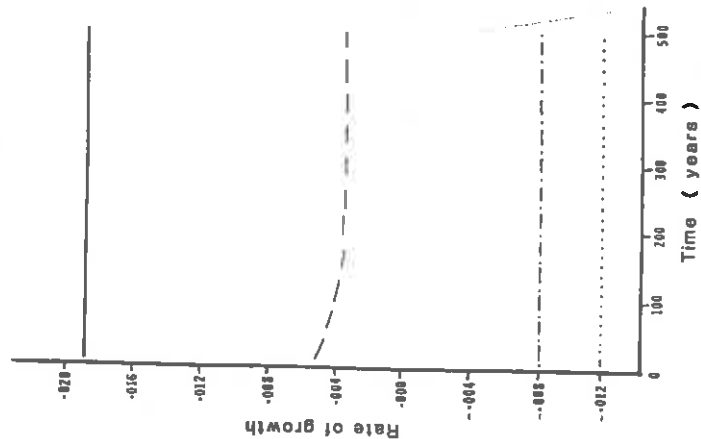
stability with 0.0001 accuracy, while systems with lower or less extreme differences among natural increase rates generally require longer to converge. Nearly all the adjustment occurs during the first one hundred years in Regimes 2 to 4; patterns of change in Regime 5 are similar to those of Regime 1, while in Regimes 6 and 7 the convergence of $_G P_G$ occurs almost linearly. The abrupt changes in natural increase conditions in Regime 7 lead to readjustment towards the final state of stability.

Stability is reached only because the four populations are linked by migration levels dependent upon the populations. Otherwise, the two native populations each converge to their own rates of natural increase while the non-native populations decline according to their death rates (Figure 2g). Once migration occurs, the primary role of differences in migration rate combinations lies in influencing the path of change in growth rates towards the stable rate. The dominant born-abroad, living-abroad population settles almost immediately to the final rate, followed within one hundred years by those born abroad and living in Great Britain. In each Migration Estimate, the growth rate of the population born and resident in Great Britain follows the upper limb of a logistic curve, trailed by about thirty years by the British-born who live abroad. Differences among the Migration Estimates particularly influence patterns of change during the first hundred years when non-native populations are small and more influenced by numbers of migrants. The dependence of the growth of the British-born population living abroad on its migrant source population is particularly notable (Figures 1a to 1c). The impact of the different patterns of growth rates is reflected by the development of the three British-related populations over the first hundred years (Figure 3).

In all the Migration Estimates, the proportion of residents in Great Britain who are born abroad rapidly increases over the first hundred years, and this does not include the effect of second and subsequent generation "non-natives". As it is a question of political relevance, it is interesting to pursue further the composition of the British population. The per cent of residents born in Great Britain is plotted for the first hundred years for the three Migration Estimates with 1970-71 birth and death rates, and for Estimate B with Regime 7 (Figure 4a). In every case the native-born share of the British

Figure 3 The Three British – Related Populations from t = 0 to t = 100 Given Constant Migration Rates·
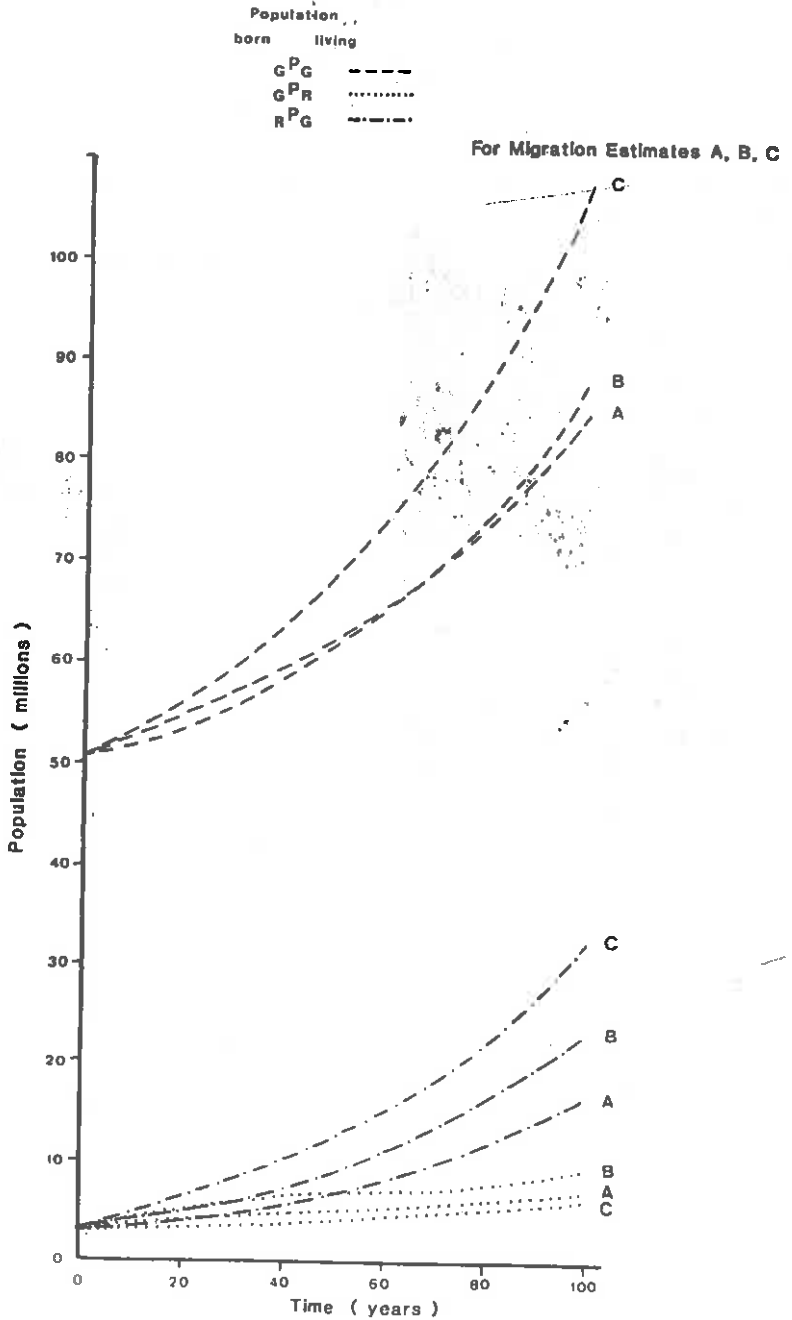
Figure 4   The Birthplace Composition of the Population
            Resident in Great Britain , Given Constant Migration Rates

a. The Percent of Residents Born in Great Britain from t=0 to t=100

For Migration Estimates A, B, C with
1970 - 71 birth and death rates and
Migration Estimate B with Regime 7
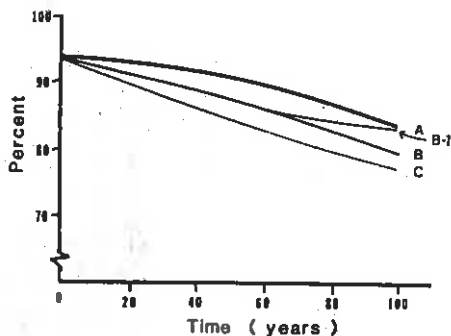
b. The Birthplace Composition at t=0 , t=100 , and at Stability
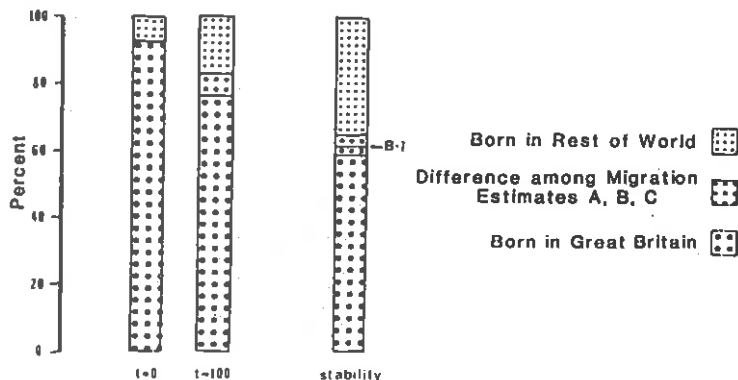
population decreases markedly, with only 1.9% and 4.2% difference in results among the four estimates at ten and at one hundred years respectively. At stability, non-native residents in Great Britain comprise 38% of the population given Regime 1, while in Regime 7 the decline in rates after fifty years reduces the share to 34% (Figure 4b).

4.4.3 Numbers and relative importance of types of migration

Since the growth of each of the four numbers of migrants follows an identical pattern to that of its associated population-at-risk, Figures 1 and 2 can be rescaled to identify changes in the four migrant flows. At stability, all the migrant flows and populations grow at the same rate and the share of each flow is constant.

Before stability, the sum of all four migrant flows adjusts differently to the total population, since different migration rates combine with different rates of population growth (Figure 5). Patterns of change in relative importance of types or directions of flow are particularly interesting. The proportion of total flow which is return movement (Figure 6) declines from t=0 in Migration Estimate A, but peaks after t=0 in Estimates B and C. Comparing conditions at year 100 to those at year 0, an even higher per cent of migrants are generated from the non-native populations whose share of total population has declined (Table 8). Emigration from Great Britain (Figure 7) declines from t=0 to t=100 under Migration Estimates A and B, and peaks at t=20 to 30 in Estimate C which originally restricted the growth of British-born living abroad.

Figure 5   Ten Year Growth Rates of Total Population and of
Total Number of Migrants

Total migrants growth rate labelled for Migration Estimates A, B, C
Total population growth rate nearly identical for Estimates A, B, C

$$\text{Growth rate} = \frac{P(t+10)}{P(t)} - 1$$
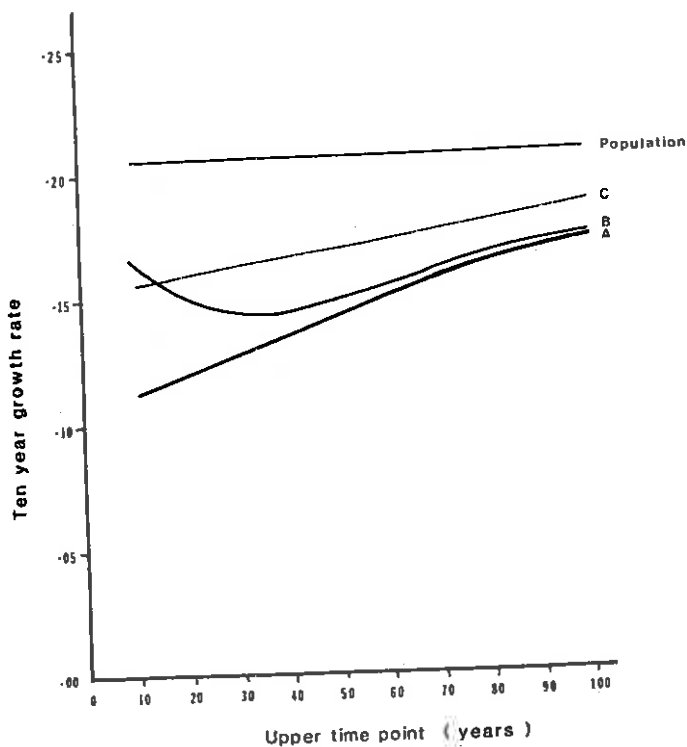
Figure 6   The Percent of Total Migration that is Return Flow from
t=0 to t=100 , Given Constant Migration Rates

For Migration Estimates A, B, C
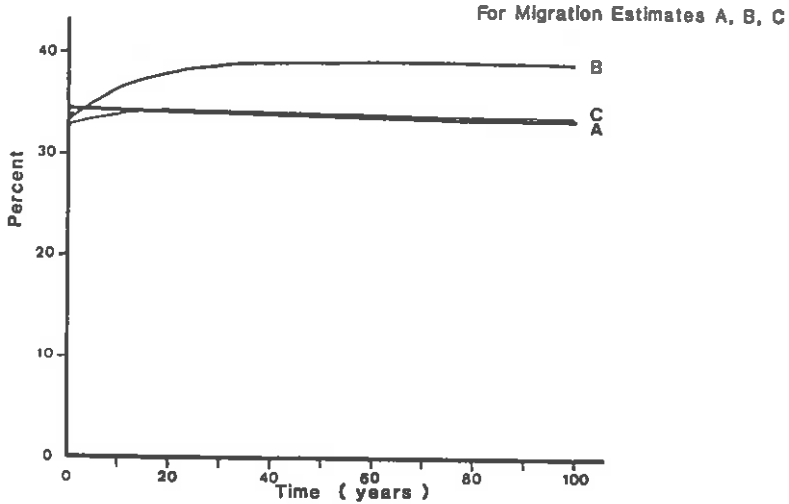
Figure 7   The Percent of Total Migration that is Emigration from
Great Britain from t=0 to t=100 , Given Constant Migration Rates

For Migration Estimates A, B, C
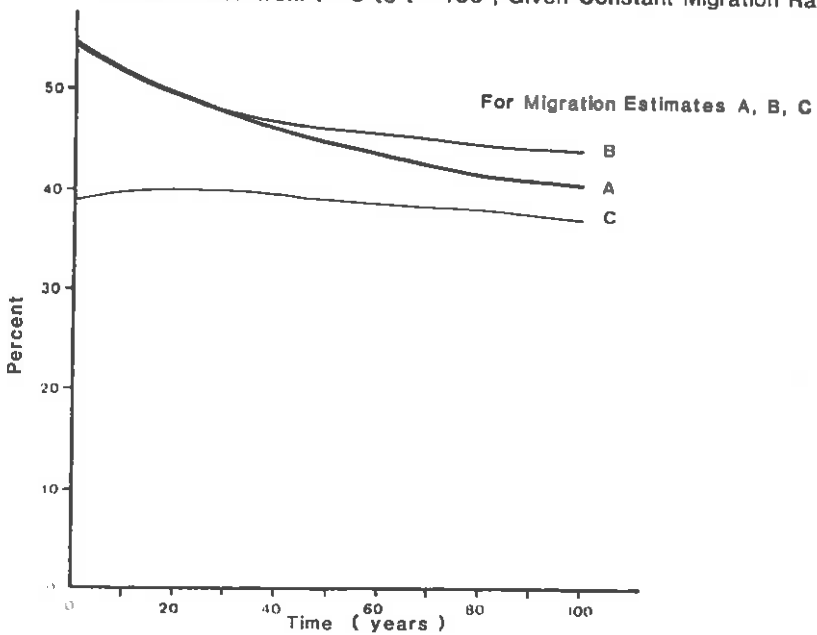
Figure 8   The Three Types of Behaviour of a Native Population
Given Constant Migration Numbers

Exponential growth to infinity

$_1b_1 > _1d_1$

$_1b_1 = _1d_1$ Linear growth to infinity

Population

P(s)

$_1b_1 < _1d_1$ Asymptotic growth to stationary value P(s)

P(0)
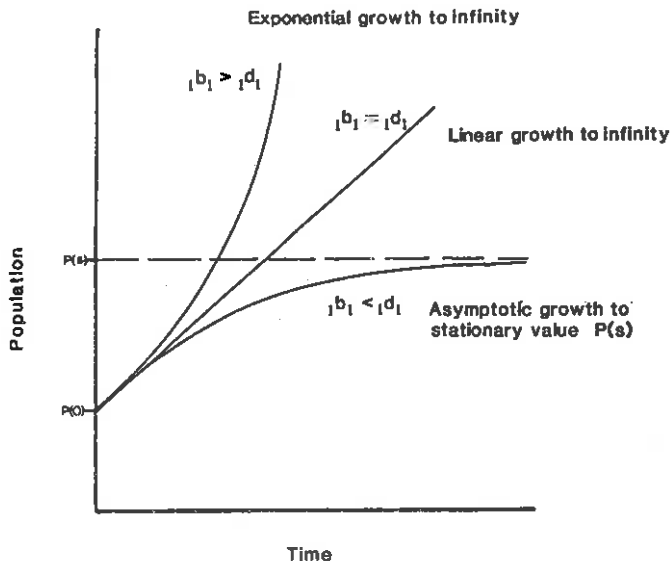
Time

Table 8.   <u>Per cent of return migrants and of non-native populations,
at times t=0 and t=100, given constant migration rates</u>

MIGRATION ESTIMATE	PER CENT OF MIGRANTS WHO ARE RETURN MIGRANTS		PER CENT OF POPULATION WHICH IS RESIDENT OUTSIDE BIRTHPLACE	
	t=0	t=100	t=0	t=100
A	34.3	33.0	0.0178	0.0097
B	33.1	38.7	0.0178	0.0156
C	32.8	33.2	0.0178	0.0132

5. INVESTIGATING THE MODEL: CONSTANT NUMBERS OF MIGRANTS

5.1 The Context

Consider now the case where legal controls are placed on the migrants arriving in each of the two regions, such that the four migrant flows are constant.

5.2 A General Solution for the Populations at Time t

Using the relationship $mP=M$, where the numbers of migrants, M, are constant, Equations (1) to (4) may be rewritten as:

$$_1P_1(t+1) = (1 + {_1}b_1 - {_1}d_1) \times {_1}P_1(t) + {_2}b_1 \times {_2}P_1(t)$$
$$- {_1}M_{12} + {_1}M_{21} \tag{25}$$

$$_1P_2(t+1) = (1 - {_1}d_2) \times {_1}P_2(t) - {_1}M_{21} + {_1}M_{12} \tag{26}$$

$$_2P_1(t+1) = (1 - {_2}d_1) \times {_2}P_1(t) - {_2}M_{12} + {_2}M_{21} \tag{27}$$

$$_2P_2(t+1) = (1 + {_2}b_2 - {_2}d_2) \times {_2}P_2(t) + {_1}b_2 \times {_1}P_2(t)$$
$$- {_2}M_{21} + {_2}M_{12} \tag{28}$$

The four differential equations, derived as in Section 4.2, are

$$\frac{d_1P_1}{dt} = ({_1}b_1 - {_1}d_1) \times {_1}P_1(t) + {_2}b_1 \times {_2}P_1(t)$$
$$+ {_1}M_{21} - {_1}M_{12} \tag{29}$$

$$\frac{d_1P_2}{dt} = -{_1}d_2 \times {_1}P_2(t) + {_1}M_{12} - {_1}M_{21} \tag{30}$$

$$\frac{d_2P_1}{dt} = -{_2}d_1 \times {_2}P_1(t) + {_2}M_{21} - {_2}M_{12} \tag{31}$$

$$\frac{d_2P_2}{dt} = ({_2}b_2 - {_2}d_2) \times {_2}P_2(t) + {_1}b_2 \times {_1}P_2(t)$$
$$+ {_2}M_{12} - {_2}M_{21} \tag{32}$$

In contrast with the differential equations (6) to (9) of the constant migration rate case, these equations contain constant terms. The matrix formulation of the differential system therefore becomes

$$\frac{d\underline{P}}{dt} = \underline{A}\,\underline{P} + \underline{K} \tag{33}$$

where

$$\underline{A} = \begin{bmatrix} {}_1b_1 - {}_1d_1 & 0 & {}_2b_1 & 0 \\ 0 & -{}_1d_2 & 0 & 0 \\ 0 & 0 & -{}_2d_1 & 0 \\ 0 & {}_1b_2 & 0 & {}_2b_2 - {}_2d_2 \end{bmatrix}$$

and

$$\underline{K} = \begin{bmatrix} {}_1M_{21} - {}_1M_{12} \\ {}_1M_{12} - {}_1M_{21} \\ {}_2M_{21} - {}_2M_{12} \\ {}_2M_{12} - {}_2M_{21} \end{bmatrix}$$

Solution of the differential system proceeds using standard techniques. From Equation (33)

$$\frac{d\underline{P}}{dt} = \underline{A}\,(\underline{P} + \underline{A}^{-1}\,\underline{K}) \tag{34}$$

Since $\underline{A}^{-1}\,\underline{K}$ is a constant,

$$\frac{d(\underline{A}^{-1}\,\underline{K})}{dt} = 0$$

$$\frac{d(\underline{P} + \underline{A}^{-1}\,\underline{K})}{dt} = \frac{d\underline{P}}{dt} \tag{35}$$

So

$$\frac{d(\underline{P} + \underline{A}^{-1} \underline{K})}{dt} = \underline{A}(\underline{P} + \underline{A}^{-1} \underline{K}) \tag{36}$$

Replacing $(\underline{P} + \underline{A}^{-1} \underline{K})$ by $\underline{Q}$, say, the problem reduces to that of Equation (12):

$$\frac{d\underline{Q}}{dt} = \underline{A} \underline{Q} \tag{37}$$

Calculation of eigenvalues and eigenvectors can now continue following the method described in Section 4.2, with the final substitution

$$\underline{P} = \underline{Q} - \underline{A}^{-1} \underline{K} \tag{38}$$

Then each population at any time t can be calculated directly.

5.3 A General Solution for the Non-Native Populations

The behaviour of each of the non-native populations can also be investigated individually. For $_1P_2$, in Equation (30) let $_1M_{12} - {}_1M_{21} = M_K$ for convenience.

$$\frac{d_1P_2}{dt} = -_1d_2 \times {}_1P_2 + M_K \tag{39}$$

Following in detail the method outlined above for solution by substitution:

$$\frac{d_1P_2}{dt} = -_1d_2 \times \left({}_1P_2 - \frac{M_K}{_1d_2}\right) \tag{40}$$

Since $\frac{M_K}{_1d_2}$ is constant,

$$\frac{d_1P_2}{dt} = \frac{d\left({}_1P_2 - \frac{M_K}{_1d_2}\right)}{dt}$$

So in (40),

$$\frac{d({_1}P_2 - \frac{M_K}{{_1}d_2})}{dt} = {_{-1}}d_2 \times ({_1}P_2 - \frac{M_K}{{_1}d_2}) \tag{41}$$

Let $\quad {_1}Q_2 = {_1}P_2 - \frac{M_K}{{_1}d_2} \tag{42}$

Then $\dfrac{d{_1}Q_2}{dt} = {_{-1}}d_2 \times {_1}Q_2 \tag{43}$

Now let $\quad {_1}Q_2(t) = {_1}Q_2(0)e^{\lambda t} \tag{44}$

from which

$$\frac{d{_1}Q_2}{dt} = \lambda {_1}Q_2(0)e^{\lambda t} \tag{45}$$

Equating (43) and (45)

$$
{_{-1}}d_2 \times {_1}Q_2 = \lambda {_1}Q_2(0)e^{\lambda t}
$$
$$
{_{-1}}d_2 ({_1}Q_2(0)e^{\lambda t}) = \lambda ({_1}Q_2(0)e^{\lambda t})
$$
$$
\lambda = {_{-1}}d_2 \tag{46}
$$

So in (44)

$${_1}Q_2(t) = {_1}Q_2(0)e^{{_{-1}}d_2 t} \tag{47}$$

Replacing ${_1}Q_2$ by (42),

$${_1}P_2(t) - \frac{M_K}{{_1}d_2} = ({_1}P_2(0) - \frac{M_K}{{_1}d_2}) \, e^{{_{-1}}d_2 t} \tag{48}$$

That is,

$$
{}_1P_2(t) \;=\; \frac{{}_1M_{12} - {}_1M_{21}}{{}_1d_2} \;+\; \left({}_1P_2(0) - \frac{{}_1M_{12} - {}_1M_{21}}{{}_1d_2}\right) e^{-{}_1d_2 t} \tag{49}
$$

Note that at t=0, substitution correctly yields ${}_1P_2(t) = {}_1P_2(0)$.

Since $\lim\limits_{t \to \infty} e^{-{}_1d_2 t} = 0$, ${}_1P_2(t)$ approaches $\dfrac{{}_1M_{12} - {}_1M_{21}}{{}_1d_2}$, a stationary

population (assumed positive) independent of ${}_1P_2(0)$. In practical terms then, the non-native populations become stationary at large t if numbers of migrants are constant:

$$
{}_1P_2(s) \;=\; \frac{{}_1M_{12} - {}_1M_{21}}{{}_1d_2} \tag{50}
$$

$$
{}_2P_1(s) \;=\; \frac{{}_2M_{21} - {}_2M_{12}}{{}_2d_1} \tag{51}
$$

where $P(s)$ is the stationary population, t=s.

Before pursuing the application of this result to the native populations ${}_1P_1$ and ${}_2P_2$, it proves useful to consider the conditions under which each of the four populations would become stationary.

5.4 The Assumption of Stationary Populations

Assume that the birth and death rates and migrant numbers are such that all four populations become stationary. In each equation (29) to (32), $\frac{dP}{dt} = 0$; this is equivalent to setting $P(t+1)$ to $P(t)$ for each population. The four equations can be solved algebraically to express each population in terms of known constants and independent of initial populations.

$$
{}_1P_1(s) \;=\; \frac{{}_1M_{12} - {}_1M_{21} - \dfrac{{}_2b_1}{{}_2d_1}\left({}_2M_{21} - {}_2M_{12}\right)}{{}_1b_1 - {}_1d_1} \tag{52}
$$

$$_1P_2(s) \quad = \quad \frac{_1M_{12} - _1M_{21}}{_1d_2} \tag{53}$$

$$_2P_1(s) \quad = \quad \frac{_2M_{21} - _2M_{12}}{_2d_1} \tag{54}$$

$$_2P_2(s) \quad = \quad \frac{_2M_{21} - _2M_{12} - \dfrac{_1b_2}{_1d_2}\left(_1M_{12} - _1M_{21}\right)}{_2b_2 - _2d_2} \tag{55}$$

Alternatively, one can find the stationary migration rates:

$$_1m_{12} \quad = \quad \frac{_1M_{12}\left(_1b_1 - _1d_1\right)}{_1M_{12} - _1M_{21} - \dfrac{_2b_1}{_2d_1}\left(_2M_{21} - _2M_{12}\right)} \tag{56}$$

$$_1m_{21} \quad = \quad \frac{_1M_{21} \times _1d_2}{_1M_{12} - _1M_{21}} \tag{57}$$

$$_2m_{12} \quad = \quad \frac{_2M_{12} \times _2d_1}{_2M_{21} - _2M_{12}} \tag{58}$$

$$_2m_{21} \quad = \quad \frac{_2M_{21}\left(_2b_2 - _2d_2\right)}{_2M_{21} - _2M_{12} - \dfrac{_1b_2}{_1d_2}\left(_1M_{12} - _1M_{21}\right)} \tag{59}$$

The conditions under which the populations, or the migration rates, are undefined, zero, positive, or negative can now be identified. In Equations (52) to (59), for convenience label:

$$_1NUM_1 \quad = \quad _1DEN_1 \quad = \quad _1M_{12} - _1M_{21} - \frac{_2b_1}{_2d_1}\left(_2M_{21} - _2M_{12}\right)$$

$$_2NUM_2 \quad = \quad _2DEN_2 \quad = \quad _2M_{21} - _2M_{12} - \frac{_1b_2}{_1d_2}\left(_1M_{12} - _1M_{21}\right)$$

where NUM are numerators in population equations (52) and (55);
       DEN are denominators in migration rate equations (56) and (59).

Table 9a. **Conditions leading to zero, undefined, negative and positive populations which are assumed stationary**

Character of Population	Populations $_1P_1$	$_1P_2$	$_2P_1$	$_2P_2$
0	$_1\text{NUM}_1 = 0$ i.e. $_1M_1 = _1M_2$ and $_2M_2 = _2M_1$	$_1M_1 = _1M_2$	$_2M_2 = _2M_1$	$_2\text{NUM}_2 = 0$ i.e. $_1M_1 = _1M_2$ and $_2M_2 = _2M_1$
undefined	$_1b_1 = _1d_1$	$_1d_2 = 0$	$_2d_1 = 0$	$_2b_2 = _2d_2$
negative	$_1b_1 > _1d_1$ and $_1\text{NUM}_1 < 0$ OR $_1b_1 < _1d_1$ and $_1\text{NUM}_1 > 0$	$_1d_2 > 0$ and $_1M_1 < _1M_2$	$_2d_1 > 0$ and $_2M_2 < _2M_1$	$_2b_2 > _2d_2$ and $_2\text{NUM}_2 < 0$ OR $_2b_2 < _2d_2$ and $_2\text{NUM}_2 > 0$
positive	$_1b_1 > _1d_1$ and $_1\text{NUM}_1 > 0$ OR $_1b_1 < _1d_1$ and $_1\text{NUM}_1 < 0$	$_1d_2 > 0$ and $_1M_1 > _1M_2$	$_2d_1 > 0$ and $_2M_2 > _2M_1$	$_2b_2 > _2d_2$ and $_2\text{NUM}_2 > 0$ OR $_2b_2 < _2d_2$ and $_2\text{NUM}_2 < 0$

where $_1\text{NUM}_1$ is the numerator of Equation (52)

$_2\text{NUM}_2$ is the numerator of Equation (55)

(birth and death rates are non-negative)

Table 9b. **Conditions leading to zero, undefined, negative and positive migration rates, assuming stationary populations**

Character of Rates	Migration Rates			
	$_1m_{12}$	$_1m_{21}$	$_2m_{12}$	$_2m_{21}$
0	$_1M_1 = 0$ OR $_1b_1 = _1d_1$	$_1M_2 = 0$ OR $_1d_2 = 0$	$_2M_1 = 0$ OR $_2d_1 = 0$	$_2M_2 = 0$ OR $_2b_2 = _2d_2$
undefined	$_1DEN_1 = 0$ i.e. $_1M_1 = _1M_2$ and $_2M_2 = _2M_1$	$_1M_1 = _1M_2$	$_2M_2 = _2M_1$	$_2DEN_2 = 0$ i.e. $_1M_1 = _1M_2$ and $_2M_2 = _2M_1$
negative	$_1b_1 > _1d_1$ and $_1DEN_1 < 0$ OR $_1b_1 < _1d_1$ and $_1DEN_1 > 0$	$0 < _1M_1 < _1M_2$ and $_1d_2 > 0$	$0 < _2M_2 < _2M_1$ and $_2d_1 > 0$	$_2b_2 > _2d_2$ and $_2DEN_2 < 0$ OR $_2b_2 < _2d_2$ and $_2DEN_2 > 0$
positive	$_1b_1 > _1d_1$ and $_1DEN_1 > 0$ OR $_1b_1 < _1d_1$ and $_1DEN_1 < 0$	$0 < _1M_2 < _1M_1$ and $_1d_2 > 0$	$0 < _2M_1 < _2M_2$ and $_2d_1 > 0$	$_2b_2 > _2d_2$ and $_2DEN_2 > 0$ OR $_2b_2 < _2d_2$ and $_2DEN_2 < 0$

where $_1DEN_1$ is denominator of Equation (56)

$_2DEN_2$ is denominator of Equation (59)

(birth and death rates are non-negative)

The sets of conditions are outlined using these abbreviations in
Tables 9a and 9b.

It is interesting to pursue the most realistic case when all four
populations are positive.   The conditions NUM > 0 and NUM < 0 can be
re-expressed in terms of the relationship between ratios:

for $_1NUM_1 > 0$, $\dfrac{_1M_{12} - _1M_{21}}{_2M_{21} - _2M_{12}} > \dfrac{_2b_1}{_2d_1}$

$_1NUM_1 < 0$, $\dfrac{_1M_{12} - _1M_{21}}{_2M_{21} - _2M_{12}} < \dfrac{_2b_1}{_2d_1}$

$_2NUM_2 > 0$, $\dfrac{_1M_{12} - _1M_{21}}{_2M_{21} - _2M_{12}} < \dfrac{_1d_2}{_1b_2}$

$_2NUM_2 < 0$, $\dfrac{_1M_{12} - _1M_{21}}{_2M_{21} - _2M_{12}} > \dfrac{_1d_2}{_1b_2}$

Three cases based on natural change of the native populations can be
identified using Table 9a;   within each, all conditions must hold for
all four stationary populations to be positive.

Case (i) $_1b_1 > _1d_1$ Natural increase of both
                                        native populations (excluding
                $_2b_2 > _2d_2$ non-native births)

                $_1M_{12} > _1M_{21}$ For positive non-native
                                        populations
                $_2M_{21} > _2M_{12}$

                $\dfrac{_2b_1}{_2d_1} < \dfrac{_1M_{12} - _1M_{21}}{_2M_{21} - _2M_{12}} < \dfrac{_1d_2}{_1b_2}$ Combining conditions
                                                                                                                     for positive native
                                                                                                                     populations

Case (ii) (a) $_1b_1 > _1d_1$ Natural increase of one
                                        native population $_1P_1$;
                $_2b_2 < _2d_2$ natural decrease of $_2P_2$

                $_1M_{12} > _1M_{21}$

                $_2M_{21} > _2M_{12}$

$$\frac{_1M_{12} - {_1M_{21}}}{_2M_{21} - {_2M_{12}}} > \frac{_2b_1}{_2d_1}$$

$$\frac{_1M_{12} - {_1M_{21}}}{_2M_{21} - {_2M_{12}}} > \frac{_1d_2}{_1b_2}$$

(b) $_1b_1 < {_1d_1}$ Natural increase of $_2P_2$ and natural decrease of $_1P_1$

$_2b_2 > {_2d_2}$

$_1M_{12} > {_1M_{21}}$

$_2M_{21} > {_2M_{12}}$

$$\frac{_1M_{12} - {_1M_{21}}}{_2M_{21} - {_2M_{12}}} < \frac{_2b_1}{_2d_1}$$

$$\frac{_1M_{12} - {_1M_{21}}}{_2M_{21} - {_2M_{12}}} < \frac{_1d_2}{_1b_2}$$

Case (iii) $_1b_1 < {_1d_1}$ Natural decrease of both native populations (excluding non-native births)

$_2b_2 < {_2d_2}$

$_1M_{12} > {_1M_{21}}$

$_2M_{21} > {_2M_{12}}$

$$\frac{_1d_2}{_1b_2} < \frac{_1M_{12} - {_1M_{21}}}{_2M_{21} - {_2M_{12}}} < \frac{_2b_1}{_2d_1}$$

Do any of these situations hold in the numerical example of Great Britain (G=1) and the Rest of the World (R=2)?   In 1970-71, both native populations experienced natural increase, indicating Case (i), but $_2b_1/_2d_1 > {_1d_2}/_1b_2$ so that the native populations cannot become positive _and_ stationary.   It seems quite likely that the natural increase rate of the native population resident in Great Britain may soon become negative, while that of the Rest of the World remains positive.   From Case (ii)(b) and using 1970-71 non-native birth and death rates, only Migration Estimate C could result in all four _positive_ populations, assuming they were stationary.

5.5 Solution for Native Populations at Time t

The information of the preceding section can be stated more precisely
and extended by applying the result that $_1P_2$ and $_2P_1$ necessarily evolve
to stationary populations, which was shown in Section 5.3.   By definition,
the system of four populations can only become stable if $_1P_1$ and $_2P_2$
also become stationary.   In any case, each of the native populations
can be expressed non-iteratively only once the associated non-native
population has become stationary, so that the number of births (bP) is
constant.   Working with $_1P_1$ from Equation (29), at sufficiently large t
$_2P_1$ can be replaced by its stationary value in (51).

$$\frac{d_1P_1}{dt} = \left(_1b_1 - _1d_1\right) {_1P_1}(t) + \frac{_2b_1}{_2d_1}\left(_2M_{21} - _2M_{12}\right) + _1M_{21} - _1M_{12} \tag{60}$$

For convenience, let

$$_1k_1 = \frac{_2b_1}{_2d_1}\left(_2M_{21} - _2M_{12}\right) + _1M_{21} - _1M_{12} \tag{61}$$

where $_1k_1$ is a constant.   Following the method of Section 5.3,

$$\frac{d_1P_1}{dt} = \left(_1b_1 - _1d_1\right) {_1P_1} + _1k_1$$

$$\frac{d\left(_1P_1 + \frac{_1k_1}{_1b_1-_1d_1}\right)}{dt} = \left(_1b_1 - _1d_1\right)\left(_1P_1 + \frac{_1k_1}{_1b_1-_1d_1}\right) \tag{62}$$

Let

$$_1q_1 = {_1P_1} + \frac{_1k_1}{_1b_1-_1d_1} \tag{63}$$

Let

$$_1q_1(t) = {_1q_1}(0)e^{\lambda t} \tag{64}$$

So

$$\frac{d_1q_1}{dt} = \left(_1b_1 - _1d_1\right) {_1q_1} \tag{65}$$

and

$$\frac{d_1 q_1}{dt} = \lambda_1 \, q_1(0) e^{\lambda t} \tag{66}$$

Therefore

$$(_1 b_1 - _1 d_1) \, _1 q_1(0) e^{\lambda t} = \lambda \, _1 q_1(0) e^{\lambda t}$$

$$\lambda = \, _1 b_1 - \, _1 d_1 \tag{67}$$

Replacing $_1 q_1$ from (63)

$$_1 P_1(t) = - \frac{_1 k_1}{_1 b_1 - _1 d_1} + \left(_1 P_1(0) + \frac{_1 k_1}{_1 b_1 - _1 d_1} \right) e^{(_1 b_1 - _1 d_1) t} \tag{68}$$

If $_1 b_1 > _1 d_1$, $e^{(_1 b_1 - _1 d_1) t}$ increases with time and the population grows to infinity. If $_1 b_1 < _1 d_1$, $e^{(_1 b_1 - _1 d_1) t}$ approaches zero as time increases, and the population approaches a stationary value

$$_1 P_1(s) = - \frac{_1 k_1}{_1 b_1 - _1 d_1}$$

$$= \frac{_1 M_{12} - _1 M_{21} - \frac{_2 b_1}{_2 d_1} (_2 M_{21} - _2 M_{12})}{_1 b_1 - _1 d_1} \tag{69}$$

which is the solution found in Section 5.4 for a stationary population.

The case where $_1 b_1$ equals $_1 d_1$ exactly is very unlikely in reality, but it provides a boundary between the inequality results above. From (68),

$$_1 P_1(t) = \frac{-_1 k_1}{_1 b_1 - _1 d_1} (1 - e^{(_1 b_1 - _1 d_1) t}) + _1 P_1(0) e^{(_1 b_1 - _1 d_1) t} \tag{70}$$

L'Hôpital's Rule is applied to find the limit of $_1P_1(t)$ as $_1b_1-_1d_1$ approaches zero (Spivak, 1967). If $f(x) \to 0$ and $g(x) \to 0$ when $x \to x_o$,

$$\lim_{x \to x_o} \frac{f(x)}{g(x)} = \frac{df/dx}{dg/dx} \tag{71}$$

Here,

$$f(_1b_1 -_1d_1) = -_1k_1 (1 - e^{(_1b_1-_1d_1)t}) \tag{72}$$

$$g(_1b_1 -_1d_1) = _1b_1 - _1d_1 \tag{73}$$

both of which tend to zero as $_1b_1-_1d_1 \to 0$.

$$\frac{df}{d(_1b_1-_1d_1)} = _1k_1 \, te^{(_1b_1-_1d_1)t} \tag{74}$$

$$\frac{dg}{d(_1b_1-_1d_1)} = 1 \tag{75}$$

So

$$\lim_{(_1b_1-_1d_1) \to 0} {}_1P_1(t) = {}_1k_1 \, te^{(0)t} + {}_1P_1(0)$$

$$= {}_1k_1 \, t + P(0) \tag{76}$$

In this special case, $_1P_1$ increases linearly as a function of time (Figure 8).

Similarly,

$$_2P_2(t) = \frac{-_2k_2}{_2b_2-_2d_2} + (_2P_2(0) + \frac{_2k_2}{_2b_2-_2d_2}) \, e^{(_2b_2-_2d_2)t} \tag{77}$$

where

$$_2k_2 = \frac{_1b_2}{_1d_2} (_1M_{12} - _1M_{21}) + _2M_{12} - _2M_{21} \tag{78}$$

$_2P_2(t)$ experiences the same types of growth as $_1P_1$, and

$$_2P_2(s) \quad = \quad \frac{-_2k_2}{_2b_2-_2d_2}$$

$$= \quad \frac{_2M_{21} - _2M_{12} - \dfrac{_1b_2}{_1d_2}\,(_1M_{12} - _1M_{21})}{_2b_2 - _2d_2} \tag{79}$$

Given natural increase of a native population the constant term $\frac{-k}{b-d}$ of Equation (68) or (77) becomes less important with increasing time, and the exponential growth rate of the population approaches its "native" rate of natural increase $(_1b_1-_1d_1 \quad \text{or} \quad _2b_2-_2d_2)$.

5.6 Identifying the Time at Which the Populations Become Stationary

Since the solutions (68) and (77) for native populations are true only when the associated non-native populations become stationary, a theoretical question with implications for the practicality of the model concerns identifying the iteration (or year) when this condition is first met.   From Equation (49) for $_1P_2(t)$, let the stationary value $\frac{_1M_{12}-_1M_{21}}{_1d_2}$ be called $_1P_2(s)$.

$$_1P_2(t) \quad = \quad _1P_2(s) \quad + \quad (_1P_2(0) - _1P_2(s))\, e^{-_1d_2 t} \tag{80}$$

This is reached as time increases and the second element in (80) becomes small.

$$(_1P_2(0) - _1P_2(s))\, e^{-_1d_2 t} \longrightarrow \quad _1v_2$$

where $_1v_2$ is some small, acceptable population increment.   If the initial population is less than the stationary population, the increment $_1v_2$ is approached from the negative side, or $_1v_2 < 0$.

The problem, then, is to solve for $_1t_2(s)$, defined as the time at which stationarity is first reached given the criterion

$$(_1P_2(0) - {}_1P_2(s)) \, e^{-_1d_2x_1t_2(s)} = {}_1v_2 \tag{81}$$

$$\log_e e^{-_1d_2x_1t_2(s)} = \log_e \left(\frac{_1v_2}{_1P_2(0) - {}_1P_2(s)}\right) \tag{82}$$

So

$$_1t_2(s) = -\frac{1}{_1d_2} \log_e \left(\frac{_1v_2}{_1P_2(0) - {}_1P_2(s)}\right) \tag{83}$$

It is easier to choose an acceptable growth rate h as a criterion for stationarity than to identify a suitable population increment. That is,

$$\frac{_1P_2(t+1)}{_1P_2(t)} = {}_1h_2 \tag{84}$$

which equals 1 at some required level of accuracy.

Substituting Equation (80) at (t+1) and at (t) into (84),

$$\frac{_1P_2(s) + (_1P_2(0) - {}_1P_2(s)) \, e^{-_1d_2(t+1)}}{_1P_2(s) + (_1P_2(0) - {}_1P_2(s)) \, e^{-_1d_2t}} = {}_1h_2 \tag{85}$$

$$\frac{_1P_2(s) + (_1P_2(0) - {}_1P_2(s)) \, e^{-_1d_2t} \, e^{-_1d_2}}{_1P_2(s) + (_1P_2(0) - {}_1P_2(s)) \, e^{-_1d_2t}} = {}_1h_2 \tag{86}$$

Substituting $_1v_2$ from (81) into (86)

$$\frac{_1P_2(s) + {}_1v_2 \, e^{-_1d_2}}{_1P_2(s) + {}_1v_2} = {}_1h_2 \tag{87}$$

So

$$_1v_2 = \frac{_1P_2(s) \; (_1h_2 - 1)}{e^{-_1d_2} - _1h_2} \tag{88}$$

Finally, the time at which stationarity is reached given a growth rate criterion $_1h_2$ is found by substituting (88) in (83)

$$_1t_2(s) = -\frac{1}{_1d_2} \; \log_e \left(\frac{_1P_2(s) \; (_1h_2 - 1)}{(e^{-_1d_2} - _1h_2)(_1P_2(0) - _1P_2(s))} \right) \tag{89}$$

If a native population $_1P_1$ or $_2P_2$ approaches a stationary state (i.e. b-d <0), then an identical derivation for time at which stationarity is reached can be used. For $_1P_1(s)$,

$$_1t_1(s)* = \frac{1}{_1b_1 - _1d_1} \; \log_e \left(\frac{_1v_1}{_1P_1(0)* - _1P_1(s)} \right) \tag{90}$$

$$_1v_1(s) = \frac{_1P_1(s) \; (_1h_1 - 1)}{e^{_1b_1 - _1d_1} - _1h_1} \tag{91}$$

$$_1t_1(s)* = \frac{1}{_1b_1 - _1d_1} \; \log_e \left(\frac{_1P_1(s) \; (_1h_1 - 1)}{(e^{_1b_1 - _1d_1} - _1h_1)(_1P_1(0)* - _1P_1(s))} \right) \tag{92}$$

Since Equation (68) only becomes valid when $_2P_1$ is stationary,

$$_1P_1(0)* = _1P_1 \, (_2t_1(s)) \tag{93}$$

$_1t_1(s)*$ is the time since the equation became valid. Therefore, the total time before stability is reached is

$$_1t_1(s) = _1t_1(s)* + _2t_1(s) \tag{94}$$

For $_2P_2(s)$,

$$_2t_2(s)* = \frac{1}{_2b_2-_2d_2} \log_e \left(\frac{_2P_2(s) \, (_2h_2-1)}{(e^{_2b_2-_2d_2} - _2h_2)(_2P_2(0)*-_2P_2(s))} \right) \quad (95)$$

$$_2P_2(0)* = _2P_2 \, (_1t_2(s)) \quad (96)$$

$$_2t_2(s) = _2t_2(s)* + _1t_2(s) \quad (97)$$

In some cases the population increment $_1v_2$ calculated by (88) may be larger than the difference between $_1P_2(0)$ and $_1P_2(s)$, leading to a negative value for time. To achieve $_1t_2(s)=0$ in this case, the growth rate criterion $_1h_2$ must be at least as accurate as

$$_1h_2 = e^{-_1d_2} + \frac{_1P_2(s)}{_1P_2(0)} (1 - e^{-_1d_2}) \quad (98)$$

$_1v_2$ is then the exact difference $_1P_2(0)-_1P_2(s)$, and if $_1P_2(s)=_1P_2(0)$, $_1h_2=1$ exactly.

5.7 Numerical Experiments

5.7.1 Population and migration rate change

Equations (29) to (32) have been programmed iteratively to reveal temporal patterns of change in populations, growth rates, and migration rates for Great Britain and the Rest of the World. Growth rates of the four populations given 1970-71 birth/death rates and each of the three Migration Estimates are plotted in Figures 9a to 9c. In each case, the two native populations tend towards their native rates of natural increase, which is achieved almost immediately by the Rest of the World population and more slowly by the British population, and the two new migration rates become small and approach zero. As expected, the non-native populations reach a stationary condition with constant migration rates. Figure 10 illustrates the population growth rate trends using Migration Estimate B conditions with $_Gb_G < _Gd_G$. Here, the population born and resident in Great Britain also tends toward a stationary value, although very slowly.

Figure 9   The Temporal Pattern of Growth Rates of the Four
Populations Given Constant Numbers of Migrants and
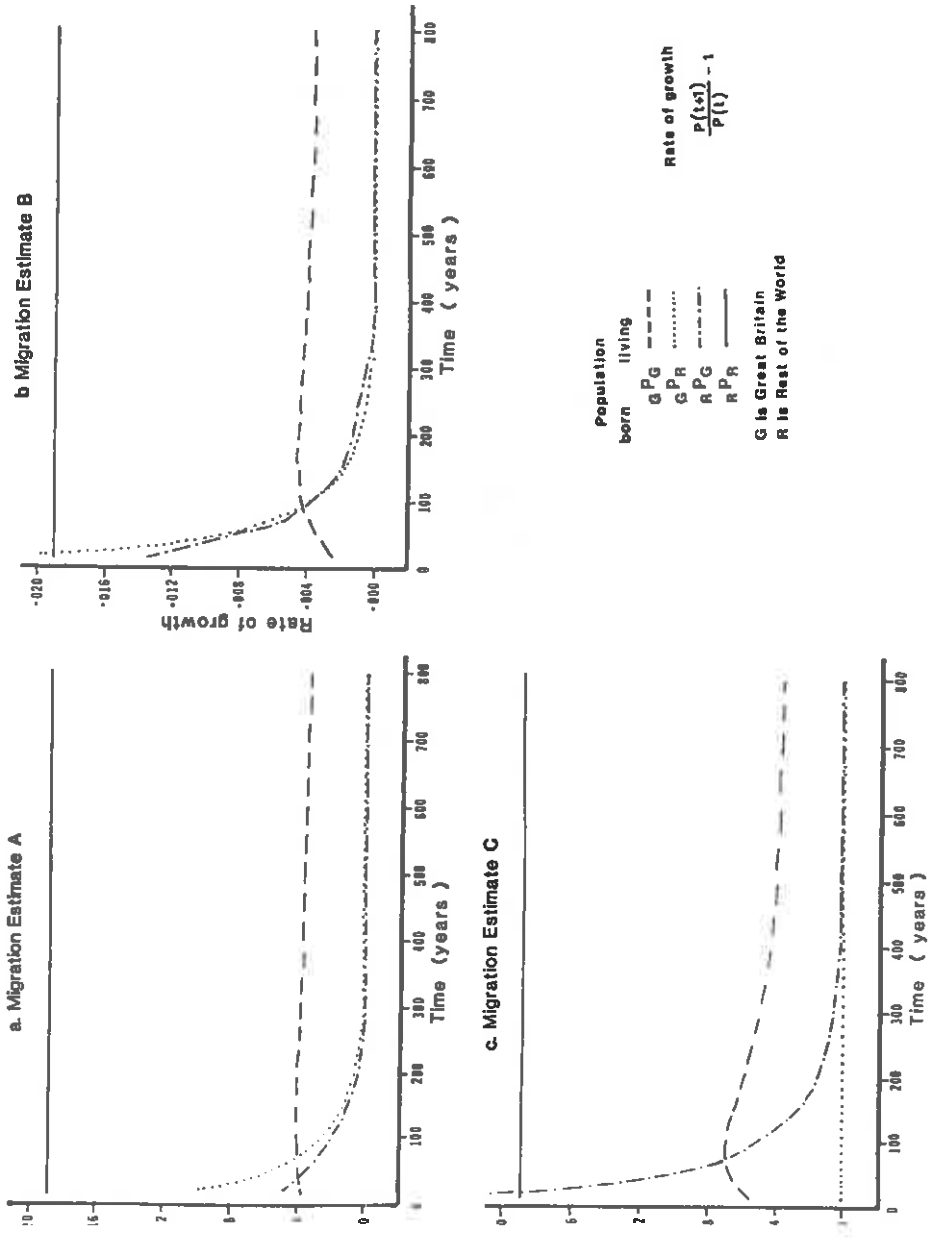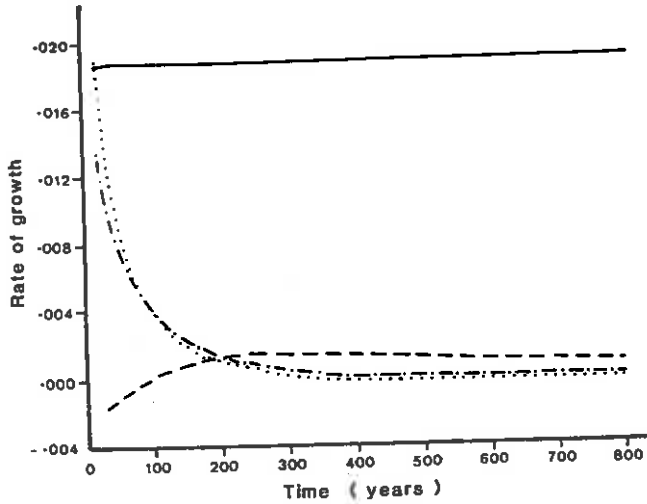1970 – 71 Birth and Death Rates

Figure 10   The Temporal Pattern of Growth Rates of the Four
            Populations Given Constant Numbers of Migrants and
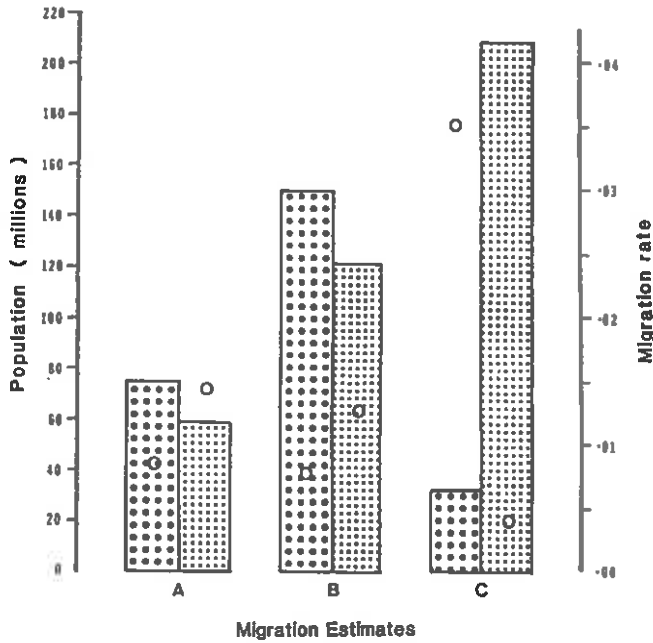   Natural Decrease of a Native Population

Figure 11. Stationary Non–Native Populations and Migration Rates Given Constant Numbers of Migrants and 1970 – 71 Birth and Death Rates

The patterns of population change, then, are less dramatic than in the constant migration rate case (Figures 1a-1c), although some parallels can be noted in relative levels of change of the non-native populations among the three Migration Estimates.

The stationary non-native populations, together with their constant return migration rates, reveal the impact of different migrant numbers (Figure 11). The Census-based Migration Estimate B leads to non-native populations about twice the size that result from IPS Estimate A. In both cases, the stationary non-native population resident in Great Britain is about 0.8 times the stationary non-native population in the Rest of the World, but the rate of return to Great Britain is about 0.63 times the return flow from Great Britain to the Rest of the World.

5.7.2 Time Required to Reach Stationary Values

The times required for the non-native populations to reach stationary values emphasize the slow convergence of the system (Table 10). Migration Estimate C, with $_{G}P_{R}(s) = 3324368$, produces a negative time, which is calculated accurately only if the growth rate criterion is closer to 1 than 1.0000874. In Estimates A and B, the stationary non-native population resident in the Rest of the World is larger than the non-native population in Great Britain, yet requires about one hundred fewer iterations with $h = 1.0001$; a lower death rate extends the time during which the population changes (Figure 12). In the examples considered, requesting a growth ratio accuracy ten times greater ($h = 1.0001$ vs. $h = 1.001$) approximately doubles the convergence time. The required time initially rises rapidly with increasing difference between the stationary and initial populations, and the difference becomes less important after five million (Figure 13).

Considering natural decrease in the native British population ($_{G}b_{G} = 0.010$, $_{G}d_{G} = 0.011$, and using Migration Estimate B and Regime 1 otherwise), Equation (68) for $_{G}P_{G}(t)$ becomes approximately valid after 500 iterations, the value of $_{R}t_{G}(s)$ with $_{R}h_{G} = 1.0001$. Using this equation, $_{G}P_{G}(800) = 117,089,093$, compared to the iterative value of 101,541,694. This population requires 1795 iterations after the non-native population in Britain becomes stationary, to reach its stationary value of 171,345,618 : a total of 2295 years.

TABLE 10. THE TIME AT WHICH NON-NATIVE POPULATIONS $_G P_R$ AND $_R P_G$ REACH A STATIONARY CONDITION GIVEN CONSTANT NUMBERS OF MIGRANTS

MIGRATION ESTIMATE	GROWTH RATE CRITERION $h=P(t+1)/P(t)$	TIME AT WHICH STATIONARY CONDITION IS REACHED	
		$_G t_R(s)$	$_R t_G(s)$
A	1.001 1.0001	166.4 353.8	167.4 435.6
B	1.001 1.0001	193.5 380.9	230.9 499.1
C	1.001 1.0001	-198.7* - 11.3*	248.7 516.9

* Given value of h, stationary condition already exists at t=0.

Figure 12   The Relationship Between Time at Which Stationarity
is Reached and Death Rate
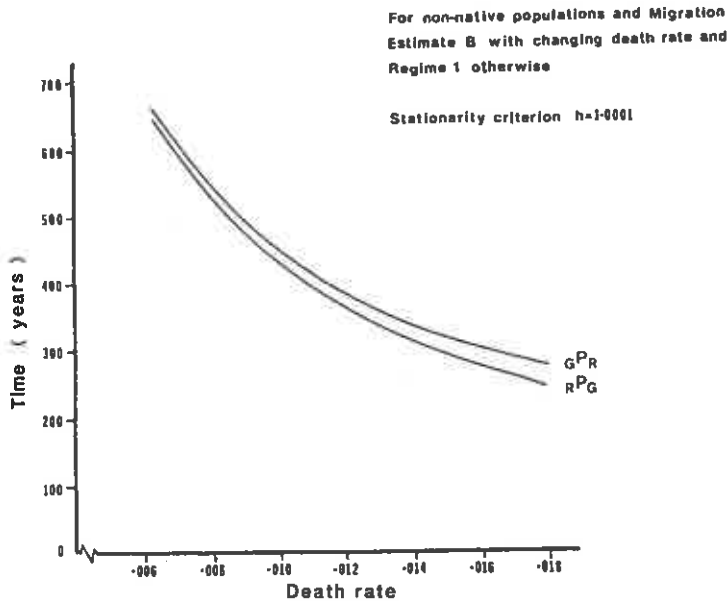
Figure 13   The Relationship Between Time at Which Stationarity
is Reached and Difference Between Stationary and
Initial Populations
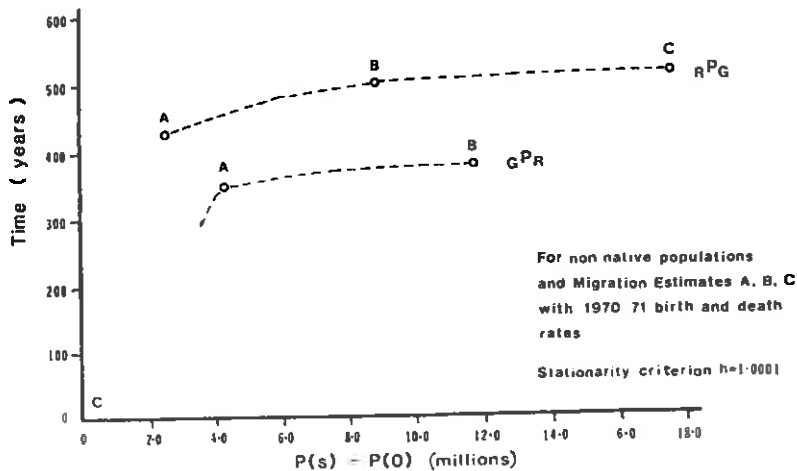
6. SOME MIXED CONTROL MODELS

6.1 Alternative Migration Control Conditions

The model of Equations (1) to (4) permits analysis of situations in which certain of the migrant numbers are fixed by legal control, while other movement is unrestricted but occurring at a constant rate. Two particular examples are outlined here.

6.2 Restricted Immigration to One Region

Consider the case where both new and return immigration to Region 1 is controlled numerically, while emigration is unrestricted. More explicitly, the rates $_1m_{12}$ and $_2m_{12}$ and the numbers of migrants $_1M_{21}$ and $_2M_{21}$ are constant. Equations (1) to (4) can be rewritten incorporating the constants:

$$_1P_1(t+1) = (1+_1b_1-_1d_1-_1m_{12})\, x_1P_1(t) +_1M_{21} +_2b_1\, x_2P_1(t) \tag{99}$$

$$_1P_2(t+1) = (1-_1d_2)\, x_1P_2(t) -_1M_{21} +_1m_{12}\, x_1P_1(t) \tag{100}$$

$$_2P_1(t+1) = (1-_2d_1-_2m_{12})\, x_2P_1(t) +_2M_{21} \tag{101}$$

$$_2P_2(t+1) = (1+_2b_2-_2d_2)\, x_2P_2(t) -_2M_{21} +_2m_{12}\, x_2P_1(t) +_1b_2\, x_1P_2(t) \tag{102}$$

The associated differential equations are

$$\frac{d_1P_1}{dt} = (_1b_1-_1d_1-_1m_{12})\, x_1P_1(t) +_2b_1\, x_2P_1(t) +_1M_{21} \tag{103}$$

$$\frac{d_1P_2}{dt} = -_1d_2\, x_1P_2(t) +_1m_{12}\, x_1P_1(t) -_1M_{21} \tag{104}$$

$$\frac{d_2P_1}{dt} = -(_2d_1+_2m_{12})\, x_2P_1(t) +_2M_{21} \tag{105}$$

$$\frac{d_2P_2}{dt} = (_2b_2-_2d_2)\, x_2P_2(t) +_2m_{12}\, x_2P_1(t) +_1b_2\, x_1P_2(t) -_2M_{21} \tag{106}$$

A chain of dependence among populations requires that the solution for each population at time t be found in the order $_2P_1$, $_1P_1$, $_1P_2$, $_2P_2$ (Table 11). The general solutions are derived using methods illustrated earlier. First,

TABLE 11.   THE CHAIN OF DEPENDENCE AMONG POPULATIONS GIVEN
CONSTANT NUMBERS OF MIGRANTS TO REGION 1 AND
CONSTANT RATES OF MIGRATION TO REGION 2.

POPULATION	DESCRIPTION	DEPENDENT ON
$_2P_1$	Non-native population resident in region restricting immigration	only itself
$_1P_1$	Native population resident in region restricting immigration	$_2P_1$ and itself
$_1P_2$	Non-native population resident in non-restricting region	$_1P_1$, $_2P_1$, and itself
$_2P_2$	Native population resident in non-restricting region	$_1P_2$, $_2P_1$, and itself

$$_2P_1(t) = \frac{_2M_{21}}{_2d_1 + _2m_{12}} + \left(_2P_1(0) - \frac{_2M_{21}}{_2d_1 + _2m_{12}}\right) e^{-(_2d_1 + _2m_{12})t} \tag{107}$$

As t increases to infinity, the non-native population $_2P_1(t)$ always approaches the stationary value

$$_2P_1(s) = \frac{_2M_{21}}{_2d_1 + _2m_{12}} \tag{108}$$

The solution for $_1P_1(t)$ only then becomes valid:

$$_1P_1(t) = -\frac{_1M_{21} + _2b_1 \times _2P_1(s)}{_1b_1 - _1d_1 - _1m_{12}} + \left(_1P_1(0) + \frac{_1M_{21} + _2b_1 \times _2P_1(s)}{_1b_1 - _1d_1 - _1m_{12}}\right)$$

$$\times e^{(_1b_1 - _1d_1 - _1m_{12})t} \tag{109}$$

If $_1b_1 - _1d_1 - _1m_{12} > 0$, the native population $_1P_1(t)$ grows exponentially to infinity, and Equations (100) and (102) for populations $_1P_2$ and $_2P_2$ cannot be simplified similarly to find a general solution at time t. In the unlikely situation of $_1b_1 - _1d_1 - _1m_{12} = 0$ exactly, $_1P_1(t)$ grows linearly with time, since

$$\lim_{(_1b_1 - _1d_1 - _1m_{12}) \to 0} {}_1P_1(t) = (_1M_{21} + _2b_1 \times _2P_1(s))t + _1P_1(0) \tag{110}$$

Finally, if $_1b_1 - _1d_1 - _1m_{12} < 0$,

$$\lim_{t \to \infty} {}_1P_1(t) = _1P_1(s) = -\frac{_1M_{21} + _2b_1 \times _2P_1(s)}{_1b_1 - _1d_1 - _1m_{12}} \tag{111}$$

which is positive and stationary. If and when this last situation is encountered, then:

$$_1P_2(t) = \frac{_1m_{12} \times _1P_1(s) - _1M_{21}}{_1d_2} + \left(_1P_2(0) - \frac{_1m_{12} \times _1P_1(s) - _1M_{21}}{_1d_2}\right)$$

$$\times e^{-_1d_2 t} \tag{112}$$

For all positive $_1d_2$, the non-native population $_1P_2$ becomes stationary.

$$\lim_{t \to \infty} {}_1P_2(t) = {}_1P_2(s) = \frac{{}_1m_{12} \times {}_1P_1(s) - {}_1M_{21}}{{}_1d_2} \tag{113}$$

Next,

$$_2P_2(t) = - \frac{{}_2m_{12} \times {}_2P_1(s) + {}_1b_2 \times {}_1P_2(s) - {}_2M_{21}}{{}_2b_2 - {}_2d_2} + \left({}_2P_2(0) \right.$$

$$+ \frac{{}_2m_{12} \times {}_2P_1(s) + {}_1b_2 \times {}_1P_2(s) - {}_2M_{21})}{{}_2b_2 - {}_2d_2} \left. \right) e^{({}_2b_2 - {}_2d_2)t} \tag{114}$$

If $_2b_2 - {}_2d_2 > 0$, the native population $_2P_2$ grows exponentially to infinity; if $_2b_2 - {}_2d_2 = 0$, $_2P_2$ grows linearly to infinity; and if $_2b_2 - {}_2d_2 < 0$, $_2P_2$ becomes stationary:

$$_2P_2(s) = - \frac{{}_2m_{12} \times {}_2P_1(s) + {}_1b_2 \times {}_1P_2(s) - {}_2M_{21}}{{}_2b_2 - {}_2d_2} \tag{115}$$

In this model, the emigration rates enter the exponential factor in the equations for the two populations resident in the region that controls immigration, thereby becoming important controls on rates of population change, stationary populations, and the times at which stationarity is reached:

$$_2t_1(s) = - \frac{1}{{}_2d_1 + {}_2m_{12}} \quad \log e \left(\frac{{}_2P_1(s) \; ({}_2h_1 - 1)}{(e^{-({}_2d_1 + {}_2m_{12})} - {}_2h_1)({}_2P_1(0) - {}_2P_1(s))} \right) \tag{116}$$

Then, ONLY IF $_1b_1 - {}_1d_1 - {}_1m_{12} < 0$:

$$_1t_1(s) = {}_2t_1(s) + \left[\frac{1}{{}_1b_1 - {}_1d_1 - {}_1m_{12}} \quad \log e \left(\frac{{}_1P_1(s) \; ({}_1h_1 - 1)}{(e^{{}_1b_1 - {}_1d_1 - {}_1m_{12}} - {}_1h_1)({}_1P_1({}_2t_1(s)) - {}_1P_1(s))} \right) \right]$$

$$\tag{117}$$

Table 12a. <u>DETAILS OF POPULATIONS GIVEN IMMIGRATION CONTROL BY GREAT BRITAIN</u>

<u>MIGRATION ESTIMATE A</u>

   <u>CONSTANTS</u> $_G m_{GR}$ = 0.003073 $_R m_{GR}$ = 0.02542

                $_G M_{RG}$ = 65,300 $_R M_{RG}$ = 134,700

   <u>POPULATION CONDITIONS</u>

          $_R P_G(s)$ = 4,006,544 reached at t = 117

          $_G b_G - _G d_G - _G m_{GR}$ = 0.000327 using Regime 1, so
                $_G P_G$ grows to infinity and $_G P_R$ and $_R P_R$ are
                calculated iteratively

<u>MIGRATION ESTIMATE B</u>

   <u>CONSTANTS</u> $_G m_{GR}$ = 0.005883 $_R m_{GR}$ = 0.04533

                $_G M_{RG}$ = 119,500 $_R M_{RG}$ = 254,000

   <u>POPULATION CONDITIONS</u>

          $_R P_G(s)$ = 4,745,003 reached at t = 94

          $_G b_G - _G d_G - _G m_{GR}$ = -0.002483 < 0 using Regime 1,

          $_G P_G(s)$ = 102,590,650   reached at t = 1035

          $_G P_R(s)$ = 40,675,097 reached already

          $_R b_R - _R d_R$ = 0.019 using Regime 1, so $_R P_R$ grows to
                    infinity

<u>MIGRATION ESTIMATE C</u>

   <u>CONSTANTS</u> $_G m_{GR}$ = 0.003144 $_R m_{GR}$ = 0.02422

                $_G M_{RG}$ = 119,500 $_R M_{RG}$ = 254,000

   <u>POPULATION CONDITIONS</u>

          $_R P_G(s)$ = 7,834,670 reached at t = 161

          $_G b_G - _G d_G - _G m_{GR}$ = 0.000256 using Regime 1, so
                $_G P_G$ grows to infinity and $_G P_R$ and $_R P_R$ are
                calculated iteratively.

Table 12b. <u>SUMMARY OF POPULATION CONDITIONS GIVEN IMMIGRATION CONTROL</u>
<u>BY GREAT BRITAIN</u>

MIGRATION ESTIMATE	POPULATIONS			
	$_RP_G$	$_GP_G$	$_GP_R$	$_RP_R$
A	4,006,544 $_Gt_R(s) = 117$	→+∞	iterative solution	
B	4745003 $_Rt_G(s) = 94$	102590650 $_Gt_G(s) = 1035$	40675697 already reached	→+∞
C	7,834,670 $_Gt_R(s) = 161$	→+∞	iterative solution	

$$_1t_2(s) = {}_1t_1(s) + \left[-\frac{1}{{}_1d_2} \log e \left(\frac{{}_1P_2(s) \ ({}_1h_2 - 1)}{(e^{-{}_1d_2} - {}_1h_2)({}_1P_2({}_1t_1(s)) - {}_1P_2(s))} \right) \right] \quad (118)$$

Then, ONLY IF $_2b_2 - {}_2d_2 < 0$:

$$_2t_2(s) = {}_1t_2(s) + \left[\frac{1}{{}_2b_2 - {}_2d_2} \log e \left(\frac{{}_2P_2(s) \ ({}_2h_2 - 1)}{(e^{{}_2b_2 - {}_2d_2} - {}_2h_2)({}_2P_2({}_1t_2(s)) - {}_2P_2(s))} \right) \right]$$

$$(119)$$

This mixed system can be applied to the Great Britain and Rest of the World example, with Great Britain as the immigration control Region 1 and the Rest of the World as Region 2. Results from Migration Estimates A, B, and C using 1970-71 birth and death rates are summarized in Tables 12a and 12b. Census-based Migration Estimate B leads to a stationary population born in the Rest of the World and resident in Great Britain which is 1.2 times larger than that resulting from IPS-based Estimate A, but achieves this larger population in a shorter time (110 years as opposed to 117). Estimate B is the only case of the three, given 1970-71 birth and death rates, which also leads to a stationary native population in Great Britain, and hence a stationary non-native population in the Rest of the World.

6.3 Restricted New Migration and Unrestricted Return Migration, Both Regions

Generally, return migration is relatively unrestricted, while new migration increasingly is being limited numerically by most countries. That is, in the present system, the numbers of new migrants $_1M_{12}$ and $_2M_{12}$, and the return migration rates, $_1m_{21}$ and $_2m_{21}$ are constant. Equations (1) to (4) become:

$$_1P_1(t+1) = (1 + {}_1b_1 - {}_1d_1) \times {}_1P_1(t) - {}_1M_{12} + {}_1m_{21} \times {}_1P_2(t) + {}_2b_1 \times {}_2P_1(t)$$

$$(120)$$

$$_1P_2(t+1) = (1 - {}_1d_2 - {}_1m_{21}) \times {}_1P_2(t) + {}_1M_{12}$$

$$(121)$$

$$_2P_1(t+1) = (1-_2d_1-_2m_{12}) \times {}_2P_1(t) + {}_2M_{21} \tag{122}$$

$$_2P_2(t+1) = (1 + {}_2b_2-_2d_2) \times {}_2P_2(t) - {}_2M_{21} + {}_2m_{12} \times {}_2P_1(t) + {}_1b_2 \times {}_1P_2(t) \tag{123}$$

The associate differential equations are:

$$\frac{d_1P_1}{dt} = (_1b_1-_1d_1) \; {}_1P_1(t) + {}_1m_{21} \times {}_1P_2(t) + {}_2b_1 \times {}_2P_1(t) - {}_1M_{12} \tag{124}$$

$$\frac{d_1P_2}{dt} = - (_1d_2 + {}_1m_{21}) \times {}_1P_2(t) + {}_1M_{12} \tag{125}$$

$$\frac{d_2P_1}{dt} = - (_2d_1 + {}_2m_{12}) \times {}_2P_1(t) + {}_2M_{21} \tag{126}$$

$$\frac{d_2P_2}{dt} = (_2b_2-_2d_2) \times {}_2P_2(t) + {}_2m_{12} \times {}_2P_1(t) + {}_1b_2 \times {}_1P_2(t) - {}_2M_{21} \tag{127}$$

Unlike the previous mixed control model, the non-native populations have identical relationships and can be solved individually.

$$_1P_2(t) = \frac{_1M_{12}}{_1d_2 + {}_1m_{21}} + (_1P_2(0) - \frac{_1M_{12}}{_1d_2 + {}_1m_{21}}) \; e^{-(_1d_2+_1m_{21})t} \tag{128}$$

$$\lim_{t \to \infty} \; {}_1P_2(t) = {}_1P_2(s) = \frac{_1M_{12}}{_1d_2 + {}_1m_{21}} \tag{129}$$

Similarly,

$$_2P_1(t) = \frac{_2M_{21}}{_2d_1 + {}_2m_{12}} + (_2P_1(0) - \frac{_2M_{21}}{_2d_1 + {}_2m_{12}}) \; e^{-(_2d_1+_2m_{12})t} \tag{130}$$

$$\lim_{t \to \infty} {}_2P_1(t) = {}_2P_1(s) = \frac{{}_2M_{21}}{{}_2d_1 + {}_2m_{12}} \tag{131}$$

So with control on the numbers of _new_ migrants, the non-native populations necessarily become stationary. Then,

$${}_1P_1(t) = - (\frac{{}_1m_{21} \times {}_1P_2(s) + {}_2b_1 \times {}_2P_1(s) - {}_1M_{12}}{{}_1b_1 - {}_1d_1})$$

$$({}_1P_1(0) + \frac{{}_1m_{21} \times {}_1P_2(s) + {}_2b_1 \times {}_2P_1(s) - {}_1M_{12}}{{}_1b_1 - {}_1d_1})_e^{({}_1b_1 - {}_1d_1)t} \tag{132}$$

Similarly,

$${}_2P_2(t) = - (\frac{{}_2m_{12} \times {}_2P_1(s) + {}_1b_2 \times {}_1P_2(s) - {}_2M_{21}}{{}_2b_2 - {}_2d_2}) +$$

$$({}_2P_2(0) + \frac{{}_2m_{12} \times {}_2P_1(s) + {}_1b_2 \times {}_1P_2(s) - {}_2M_{21}}{{}_2b_2 - {}_2d_2})_e^{({}_2b_2 - {}_2d_2)t} \tag{133}$$

In each case, the native population grows exponentially to infinity if b-d > 0, linearly to infinity if b-d = 0, and becomes stationary if b-d < 0.

Equations for time at which stationarity is reached are derived as previously indicated, using the appropriate exponential factors. Both ${}_1t_2(s)$ and ${}_2t_1(s)$ are calculated from t = 0. Since Equations (132) and (133) for native populations are valid only when both non-native populations are stationary, values for ${}_1t_1(s)$ and ${}_2t_2(s)$ are calculated from the starting time.

$$t = \max \{{}_1t_2(s), {}_2t_1(s)\} \tag{134}$$

For the Great Britain and Rest of the World example, and using 1970-71 rates, b-d > 0 for each of the native populations ${}_GP_G$ and ${}_RP_R$. The stationary values for the non-native populations are listed in Table 13.

TABLE 13. STATIONARY NON-NATIVE POPULATIONS GIVEN CONSTANT NUMBERS OF NEW MIGRANTS AND CONSTANT RATES OF RETURN MIGRATION

MIGRATION ESTIMATE	CONSTANTS	STATIONARY POPULATIONS AND TIMES			
		${}_G{}^P{}_R(s)$	${}_G{}^t{}_R(s)$	${}_R{}^P{}_G(s)$	${}_R{}^t{}_G(s)$
A	${}_G M_{GR} = 155,500$ ${}_G m_{RG} = 0.01979$ ${}_R m_{GR} = 0.02542$ ${}_R M_{RG} = 134,700$	4,906,911	146	4,006,544	117
B	${}_G M_{GR} = 297,660$ ${}_G m_{RG} = 0.03621$ ${}_R m_{GR} = 0.04533$ ${}_R M_{RG} = 254,000$	6,187,071	112	4,745,003	94
C	${}_G M_{GR} = 159,060$ ${}_G m_{RG} = 0.03621$ ${}_R m_{GR} = 0.02422$ ${}_R M_{RG} = 254,000$	3,306,173	-2.7*	7,834,670	161

* (already reached to this level of accuracy).

7.   SOME EXTENSIONS OF THE MODEL

7.1 Possible Extensions

The model has been examined so far only for the simple case
of two regions, each occupied by a native and a non-native population,
whose movement between regions is defined as new or return according
to their place of birth. Some spatial and temporal extensions of the
model, which are theoretically interesting and might prove analytically
or numerically tractable, are outlined here briefly. It could also be
valuable to specify and investigate model variations that incorporate
temporal change in birth, death, and migration rates, and refinement
by age, sex or other population characteristics.

7.2 Different Time Scales

The prefix of the variables, which has been used to indicate place
of birth, can indicate place of residence at any previous time, thereby
generalizing the operational definition of return migration. Locations
at three points in time are required to implement the model. The
problem in this extension is practical rather than theoretical. For
example, the 1971 Census of Great Britain collected information on
residence at three dates (April 1966, 1970, and 1971), but published
tables do not link the three locations by disaggregating the 1970-71
movement by 1966 residence. Analysis using disaggregation by multiple
previous residences, or by several previous residences considered
separately, could prove informative in comparing return migration rates
over various time scales.

7.3 Multiple Regions

Extension of the model to deal with more than two regions is also
theoretically attractive. However, the number of equations required
for complete expression of the system, without temporal extension, is
N^2 where N is the number of regions and, excluding stayers, there are
$N^2(N-1)$ migration rates or numbers (Table 14).

The use of three regions is a particularly valuable extension from
two regions, since it permits identification of "secondary" movement as
well as new and return (Eldridge, 1965). Secondary migration involves
movement between two regions, neither of which is the birthplace, say.
Table 15 illustrates the possible combinations of residences and
associated movement types.

TABLE 14.   <u>NUMBER OF EQUATIONS AND MIGRATION TERMS IN MULTIPLE</u>
<u>REGION MODELS</u>

ਤੀ

NUMBER OF REGIONS	NUMBER OF EQUATIONS	NUMBER OF MIGRATION TERMS			
		TOTAL	NEW	RETURN	SECONDARY
N	N^2	$N^2(N-1)$	$N(N-1)$	$N(N-1)$	$N(N-1)(N-2)$
0	0	0	0	0	0
1	1	0	0	0	0
2	4	4	2	2	0
3	9	18	6	6	6
4	16	48	12	12	24
5	25	100	20	20	60
6	36	180	30	30	120
7	49	294	42	42	210
8	64	448	56	56	336
9	81	648	72	72	504
10	100	900	90	90	720
50	2500	122500	2450	2450	117600

TABLE 15.   CLASSIFICATION OF POPULATIONS AND MIGRANTS IN A
THREE-REGION SYSTEM

Birthplace 1

Location and Population at t	Location and Population at t + 1		
	1 $_1P_1$	2 $_1P_2$	3 $_1P_3$
1 $_1P_1$	Native Stayer	New Migrant $_1M_{12}$	New Migrant $_1M_{13}$
2 $_1P_2$	Return Migrant $_1M_{21}$	Non-native Stayer	Secondary Migrant $_1M_{23}$
3 $_1P_3$	Return Migrant $_1M_{31}$	Secondary Migrant $_1M_{32}$	Non-Native Stayer

Birthplace 2

Location and Population at t	Location and Population at t + 1		
	1 $_2P_1$	2 $_2P_2$	3 $_2P_3$
1 $_2P_1$	Non-Native Stayer	Return Migrant $_2M_{12}$	Secondary Migrant $_2M_{13}$
2 $_2P_2$	New Migrant $_2M_{21}$	Native Stayer	New Migrant $_2M_{23}$
3 $_2P_3$	Secondary Migrant $_2M_{31}$	Return Migrant $_2M_{32}$	Non-native Stayer

....Cont'd.

Birthplace 3

Location and Population at t	Location and Population at t + 1		
	1 $\quad _3P_1$	2 $\quad _3P_2$	3 $\quad _3P_3$
1 $\quad _3P_1$	Non-Native Stayer	Secondary Migrant $_3M_{12}$	Return Migrant $_3M_{13}$
2 $\quad _3P_2$	Secondary Migrant $_3M_{21}$	Non-Native Stayer	Return Migrant $_3M_{23}$
3 $\quad _3P_3$	New Migrant $_3M_{31}$	New Migrant $_3M_{32}$	Native Stayer

The model equations for a three region system are:

$$_A P_A(t+1) = (1+_A b_A - _A d_A - _A m_{AB} - _A m_{AC}) \times _A P_A(t) + _A m_{BA} \times _A P_B(t)$$

$$+ _A m_{CA} \times _A P_C(t) + _B b_A \times _B P_A(t) + _C b_A \times _C P_A(t) \tag{135}$$

$$_A P_B(t+1) = (1-_A d_B - _A m_{BA} - _A m_{BC}) \times _A P_B(t) + _A m_{AB} \times _A P_A(t) + _A m_{CB} \times _A P_C(t) \tag{136}$$

$$_A P_C(t+1) = (1-_A d_C - _A m_{CA} - _A m_{CB}) \times _A P_C(t) + _A m_{AC} \times _A P_A(t) + _A m_{BC} \times _A P_B(t) \tag{137}$$

where A, B, C are replaced by permutations of regions 1, 2, 3 to produce nine equations.

A cursory investigation suggests that analysis and results of the model parallel the two region case. For example, given constant numbers of migrants, the non-native populations become stationary. For $_A P_B(t)$,

$$\lim_{t \to \infty} {_A P_B(t)} = \frac{_A M_{AB} + _A M_{CB} - _A M_{BA} - _A M_{BC}}{_A d_B} \tag{138}$$

In addition, the three region case may provide greater potential for further analytical and numerical experiments concerning population change and the impact of migration policies.

8. CONCLUSION

8.1 The General Structure and Behaviour of the Models

The models examined in Sections 4, 5 and 6 are specific forms of a
general model of demographic change experienced by populations which are
disaggregated by some past "situation". The contexts to which the model
may be applied are numerous; in particular, Rogers (1980) investigates
urban and rural change, while this paper focusses on inter-regional
migration disaggregated by birthplace, which explicitly differentiates
new and return movement. Given any context, the structure of a general
model remains the same. However, by making different assumptions about
the temporal constancy of the parameters, model variants are identified,
and these may differ in some aspects of structure and population behaviour.
Rogers (1980), in considering the case where all migration rates are constant,
together with constant birth and death rates, specifies the basic model of
Equations (1) to (4). In this paper, additional models have been identified
by assuming that certain migrant numbers are constant. The temporal
behaviour of the populations can now be generalized.

8.1.1 All Four Migration Rates Constant

In the model of Equations (1) to (4) with all four migration rates
constant, the native and non-native populations all converge to the natural
change rate of the dominant population. The particular patterns of change
in the size of the populations and in their growth rates towards the stable
condition depend upon the interrelationships among parameter values and
populations (Figure 14). The populations remain linked through time, and
the structure of the model is examined either using the iterative form of
Equations (1) to (4), or by solving for eigenvalues and eigenvectors.

8.1.2 Models Involving Some Constant Migrant Numbers

In each of the models examined here which involved constancy in
some or all of the migrant flows rather than in the rates, the iterative
expressions for the populations each could be reformulated non-iteratively
in an equation of general form:

$$P(t) = \frac{c}{r} + (P(i) - \frac{c}{r})e^{-rt} \tag{139}$$

where   r   is an expression of constant **rates**

c   is an expression of constant **numbers**

$P(t)$ is the population at time t

$P(i)$ is the initial population, t=i, existing when the
equation becomes valid. $P(i) \neq P(0)$ if c depends
upon another population becoming stationary at t > 0.

**Figure 14   Generalized Population Growth Rate Patterns
Given All Migration Rates Constant**
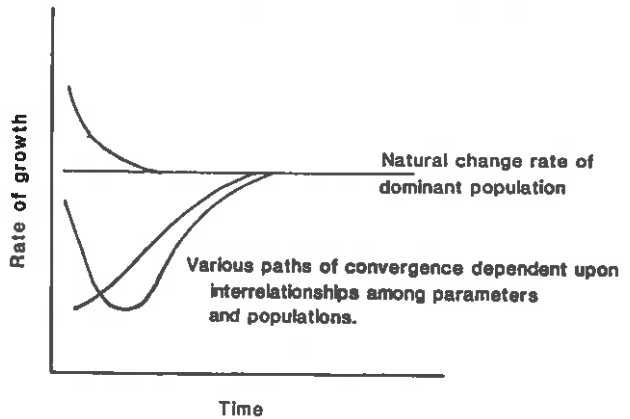
Some general observations can be made about the behaviour of each population $P(t)$ expressed in the form of (139).

In the case of non-native populations, r includes factors of population decline : the death rate and in some model variants also the out-migration rate. Since r is necessarily positive (assuming positive death rate), and therefore e^{-rt} approaches zero, a non-native population necessarily approaches a stationary value of magnitude $\left|\frac{c}{r}\right|$. For the native populations, r includes the rate of natural change of the native-born residents $(b-d)$, and in some model variants also incorporates the out-migration rate of that population $(b-d-m)$. For the native populations, then, r can be positive, zero, or negative.

For both native and non-native populations, c expresses the number of net in-migrants, which may be constant from $t = 0$, or a constant resulting from application of a migration rate to a stationary population. For the native populations c also incorporates a constant number of births occurring to a stationary non-native population. Table 16 summarizes the expressions for c and r, for the three models examined in this paper.

The behaviour with temporal progression of a population expressed using Equation (139) depends on the combination of signs of c and r (Table 17 and Figure 15a, b, c). Since populations must be non-negative, only certain numerical values for the parameters in the expressions for c and r would provide a realistic situation, assuming constant rates and/or numbers.

The precise shape of the curve (or the value of a population at any time) depends on the numerical values of the parameters. In practical application, this implies that legal restriction on the number of migrants conditions the population growth patterns, population sizes, and birthplace compositions of residents. If some ultimate population size or structure is desired, the appropriate number of migrants can be calculated, given assumptions about birth and death rates.

8.2 The Assignment of Non-Native Births

The models of population change considered here are numerically deterministic, but their formulation involves a value judgment : that individuals born to non-natives "belong" to the native population. It is true that birthplace is unequivocal, but behavioural similarity is not certain. The temporal projections of the populations depend upon this assignment; the non-native populations necessarily become stationary because the rate expression r does not contain a birth parameter. The choice in assignment of births should be considered in each context.

TABLE 16. THE EXPRESSIONS FOR c AND r IN THE THREE MODELS INVOLVING CONSTANT NUMBERS OF MIGRANTS

For c and r in Equation (139)

$$P(t) = \frac{c}{r} + \left(P(i) - \frac{c}{r}\right)e^{-rt}$$

Model	Population	c	r
All 4 migrant flow constant	${}_1P_1$	${}_1M_{21} - {}_1M_{12} + {}_2b_1x_2P_1(S)$	$-({}_1b_1 - {}_1d_1)$
	${}_1P_2$	${}_1M_{12} - {}_1M_{21}$	${}_1d_2$
	${}_2P_1$	${}_2M_{21} - {}_2M_{12}$	${}_2d_1$
	${}_2P_2$	${}_2M_{12} - {}_2M_{21} + {}_1b_2x_1P_2(S)$	$-({}_2b_2 - {}_2d_2)$
Constant flows to Region 1	${}_1P_1$	${}_1M_{21} + {}_2b_1x_2P_1(S)$	$-({}_1b_1 - {}_1d_1 - {}_1m_{12})$
	${}_1P_2$	${}_1m_{12}x_1P_1(S) - {}_1M_{21}$	${}_1d_2$
	${}_2P_1$	${}_2M_{21}$	${}_2d_1 + {}_2m_{12}$
	${}_2P_2$	${}_2m_{12}x_2P_1(S) - {}_2M_{21} + {}_1b_2x_1P_2(S)$	$-({}_2b_2 - {}_2d_2)$
Constant flows of new migrants	${}_1P_1$	${}_1m_{21}x_1P_2(S) - {}_1M_{12} + {}_2b_1x_2P_1(S)$	$-({}_1b_1 - {}_1d_1)$
	${}_1P_2$	${}_1M_{12}$	${}_1d_2 + {}_1m_{21}$ 15
	${}_2P_1$	${}_2M_{21}$	${}_2d_1 + {}_2m_{12}$
	${}_2P_2$	${}_2m_{12}x_2P_1(s) - {}_2M_{21} + {}_1b_2x_1P_2(s)$	$-({}_2b_2 - {}_2d_2)$

TABLE 17.   THE FORM OF THE POPULATION PROJECTION FOR COMBINATIONS
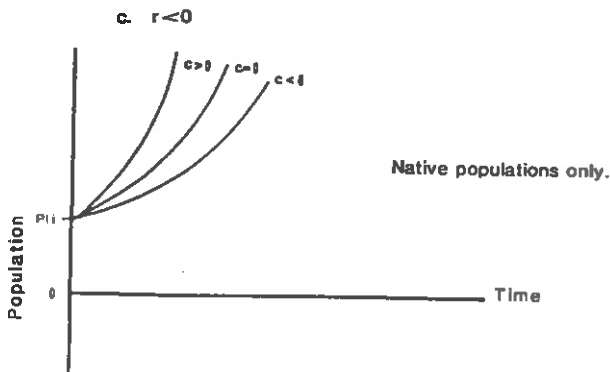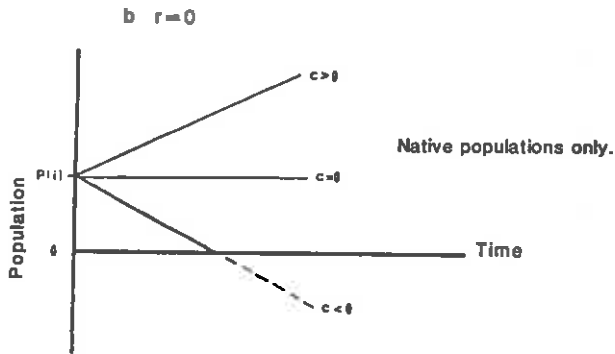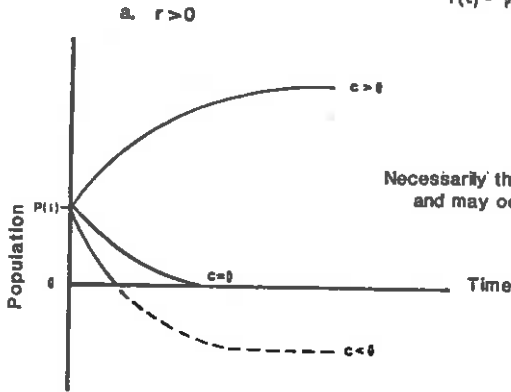   OF SIGN IN c AND r

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For c and r in Equation (139)

$$P(t) = \frac{c}{r} + (P(i) - \frac{c}{r})e^{-rt}$$

r	> 0	0	< 0
c **> 0**	$P(t)$ approaches a positive stationary value $\frac{c}{r}$	linear growth of $P(t)$ to +ve ∞ in form $ct + P(0)$	exponential growth of $P(t)$ to +ve ∞ with $\frac{c}{r}$ negative
0	$P(t)$ approaches a stationary value of 0	$P(t)$ remains stationary at $P(0)$	exponential growth of $P(t)$ to +ve ∞ in form $P(0)e^{-rt}$
< 0	$P(t)$ approaches a negative stationary value $\frac{c}{r}$ (unrealistic)	linear decline of $P(t)$ to -ve ∞ in form $ct + P(0)$ (unrealistic)	exponential growth of $P(t)$ to +ve ∞ with $\frac{c}{r}$ positive

Figure 15   Form of Population Projections Given
Constant Migrant Number Models

For Equation (139)

$$P(t) = \frac{c}{r} + \left(P(t) - \frac{c}{r} \right) e^{-rt}$$

a. $r > 0$

c > 0

Necessarily the case for the non native populations
and may occur for native populations.

c = 0

Time

c < 0

Population

P(t)

0

b $r = 0$

c > 0

Native populations only.

c = 0

Time

c < 0

Population

P(t)

0

c. $r < 0$

c > 0   c = 0   c < 0

Native populations only.

Population

P(t)

0

Time

A potentially fruitful extension of the model lies in distinguishing these second (or subsequent) generation "non-natives", instead of assigning individuals born to non-natives to the native population. Mixed-parentage populations could also be incorporated, by the inclusion of two or more prefixes. In each case, the structure and temporal behaviour of the populations could be examined.

8.3 Population Structure and Behaviour Given Further Model Refinement

Several other model refinements can be envisaged: for example, increasing the number of regions, disaggregating the populations by age and sex, and incorporating temporal functions of birth, death, and migration terms. While the numerical results may be altered significantly due to increased realism of the system (as Rogers (1980) found in age disaggregation), it is possible that the broad trends in the temporal projection of the populations would not be substantially altered, although further structural detail would be revealed.

8.4 A Final Comment on Return Migration

The specific concern with return migration has been somewhat submerged in the investigation of a model suited to various contexts, and the numerical importance of return migration has not been indicated with any confidence. However, the basis of the population disaggregation in the model lies in the explicit recognition of new and return movement in whatever social or spatial context is under consideration. The observed model structure and population behaviour are a direct result of this distinction. Return movement, theoretically and numerically important in its own right, is also influential in this broader context of population change.

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Abbreviations:

   O.P.C.S. Office of Population Censuses and Surveys

   H.M.S.O. Her Majesty's Stationery Office