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A NOTE ON THE STABILITY OF EQUILIBRIUM SOLUTIONS OF PRODUCTION CONSTRAINED SPATIAL-INTERACTION MODELS.

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1. INTRODUCTION

In a recent paper (Harris and Wilson, 1978) an analysis of the stability of equilibrium solutions of production constrained retailing models was undertaken. It was argued that by examining equilibrium solutions in response to changes in a parameter, k, jumps in the value of state variables, analogous to the fold catastrophe, could be expected. In this note it is shown that this is not necessarily always the case, and two different conditions are identified, one that gives rise to jump behaviours, another that does not. The distinction arises due to the nature of the k parameter.

We first recap the Harris-Wilson argument, then extend the interpretation of the k parameter. Some numerical results illustrate the argument.

2. THE HARRIS-WILSON ARGUMENT

The well-known shopping model can be written as:

$$S_{ij} = A_i e_i P_i W_j^{\alpha} e^{-\beta c_{ij}} \qquad (1)$$

$$A_i = (\sum_j W_j e^{-\beta c_{ij}})^{-1} \qquad (2)$$

where S_{ij} is the flow of revenue from zone i to zone j
e_i is the per capita expenditure on goods in zone i
P_i is the population in zone i
W_j is the size (say floorspace) of retail facilities in zone j. This is taken as a measure of attractiveness
c_{ij} is the cost (suitably defined) of travel between i and j
A_i is the balancing factor
α is a parameter that measures consumer scale economies
β is a parameter reflecting case of travel.

A consumer surplus maximisation of the shopping model has been derived by Coelho and Wilson (1976)

$$\underset{\{\underline{S},\underline{W}\}}{\text{Max}} \; Z = -\frac{1}{\beta} \sum_{ij} S_{ij} \log S_{ij} + \alpha\left(\sum_{ij} S_{ij} \log W_j - \beta\right) + \beta\left(C - \sum_{ij} S_{ij} c_{ij}\right) \qquad (3)$$

subject to

$$\sum_i S_{ij} = e_i P_i \qquad (4)$$

and

$$\sum_j W_j = W \qquad (5)$$

where C is total money available for travel.

Harris and Wilson showed that at optimality that zone revenue, D_j satisfied the condition

$$D_j = kW_j \qquad (6)$$

k was interpreted as a unit cost term, so that at optimality, revenue balanced with cost. Attention then focussed on how the optimal pattern of facility size $\{W_j\}$ varied with changing k. For α greater than 1 the argument proceeded as follows.

If condition (6) is to be satisfied then the solution of (3) must lie on the straight line $D_j = kW_j$. For $\alpha > 1$ the curve $D_j - W_j$ can be shown to have three potential equilibrium points. These are shown in figure 1. Of these three points only the lower point, $W_j = 0$ and the upper point $W_j = W_j^A$ are stable. It was argued that as k changes it is possible that the $D_j - W_j$ curve no longer intersects the straight line thus the upper stable point, W_j^A no longer exists and so the solution must be at $W_j = 0$. This was shown to be analogous to the fold catastrophe, where jumps from $W_j = W_j^A$ to $W_j = 0$ would occur for small changes in the value of k (see Figures 5 and 6 in Harris and Wilson, 1978).

3. EXTENSION OF THE HARRIS-WILSON ARGUMENT

The first thing to note is that k can be interpreted in terms of parameters of the consumer surplus problem (equations 3, 4, 5). The unconstrained Lagrangian version of that program can be written as:

$$\underset{\{\underline{S},\underline{W}\}}{\text{Max}} \; L = -\frac{1}{\beta}\sum_{ij} S_{ij} \log S_{ij} + \alpha\left(\sum_{ij} S_{ij} \log W_j - \beta\right) + \beta\left(C - \sum_{ij} S_{ij} c_{ij}\right)$$
$$+ \sum_i \mu_i \left(\sum_j S_{ij} - e_i P_i\right) + \gamma\left(W - \sum_j W_j\right) \qquad (7)$$

where μ_i and γ are Lagrangian multipliers associated with constraints (4) and (5) respectively. For optimality a necessary condition is that

$$\frac{\partial L}{\partial W_j} = 0 \qquad (8)$$

and from (7)

$$\frac{\partial L}{\partial W_j} = \alpha \sum_i \frac{S_{ij}}{W_j} - \gamma \qquad (9)$$

so at optimality we have

$$\sum_i S_{ij} = \frac{\gamma}{\alpha} W_j \qquad (10)$$

which is the balancing condition (6) but we now have $k = \frac{\gamma}{\alpha}$. γ is the multiplier associated with the total stock constraint and can be interpreted as a marginal rent term. We can now see that there are two potential mechanisms for changing the value of k

(i) change γ and keep α fixed

(ii) change α and keep γ fixed.

(Of course there is a third case when γ and α both vary differently but this does not affect the original.) It is now shown that these two cases give rise to different behaviour.

CASE 1.

When α is fixed and γ is changed the Harris-Wilson argument would suggest that we would expect changes in not only the value of W_j but also 'jumps' from the upper to the lower stable points (and vice versa). This is not however the case.

To solve the program (7) we can employ an iterative method that has the following structure:

(i) Set all the W_j's equal to some initial starting value.

(ii) Solve

$$\sum_i S_{ij} = \sum_i \frac{e_i P_i W_j^{\alpha} e^{-\beta c_{ij}}}{\sum_j W_j^{\alpha} e^{-\beta c_{ij}}}$$

(iii) Let

$$D_j = \sum_i S_{ij}$$

(iv) If

$$D_j = \frac{\gamma}{\alpha} W_j \qquad \forall j$$

then we have an optimal solution. If not then set $W_j = \frac{\alpha}{\gamma} D_j$ and return to (ii).

It can easily be seen from inspection that the role of γ in this model is merely one of proportioning the relationship between demand and supply, it plays no role in the spatial arrangement of facility size. Consequently, if we change k by increasing or decreasing γ we will not experience the type of behaviour anticipated by Harris and Wilson. If for example γ is increased the amount of stock in each zone will decrease but the relative size of each W_j will remain constant. This is shown in Figure 2.

Thus no jumps will occur, and providing an upper stable point exists the lower stable point, $W_j = 0$ will only occur when $k \to \infty$ and total stock, $W \to 0$.

CASE 2.

Fortunately the same argument does not apply for k variation due to changing α. This is because α plays a direct role in the determination of revenue, D_j, whereas γ does not. For example the value of

$$\sum_i S_{ij} = \sum_i \frac{e_i P_i W_j^{\alpha} e^{-\beta c_{ij}}}{\sum_j W_j^{\alpha} e^{-\beta c_{ij}}}$$

will vary as α is changed and so the equilibrium values of W_j will not only experience changes in values for changing α but may experience jump behaviour too, according to criterion put forward by Harris and Wilson.

CASE 3. An alternative formulation

It has been pointed out (Wilson, 1980) that a mechanism can still be devised which would give zonal jumps for a kind of k-change which does not involve α. This involves defining a k for each zone, say k_j, assuming some zonal variation in cost. This can be incorporated in the mathematical programming formulation in such a way that γ, in effect, becomes γ_j. A change in one γ_j could then produce jumps.

4. AN EXAMPLE

To illustrate that the behaviour of W_j in response to k changes does depend on which parameter (either γ or α) is responsible for the change a series of numerical experiments can be performed. The solution to the program (7) was sought under identical conditions with identical k variations, but in one case γ was the source of variation, in the second α. The results are presented for a single zone (though the model was run for 149 zones) in Table 1. It is clear that γ variation produced merely a proportional change while α variation produces a (non-proportional) change and also a jump from the upper to the lower stable point. Note that under identical conditions the two cases produce, as we would expect, identical values for W_j.

5. CONCLUSION

It has been demonstrated that by showing that a single parameter has two components of change the argument of Harris and Wilson is more complex than it first appeared. This argues strongly for the careful analysis and interpretation of parameters of dynamic urban models.

REFERENCES

Coelho, J.D., and Wilson, A.G. (1976) The optimum location and size of shopping centres. *Regional Studies, 10,* 413-21.

Harris, B. and Wilson, A.G. (1978) Equilibrium values and dynamics of attractiveness terms in production constrained spatial interaction models. *Environment and Planning A, 10,* 371-88.

Wilson, A.G. (1980) Private communication.

CASE 1

1	2	3	4	5	6	7
k	γ	α	W_j	k^{n-1}/k^n	W_j^n/W_j^{n-1}	5/6
0.909	1.0	1.1	4743	-	-	-
0.833	0.916	1.1	5176	1.09	1.09	1.0
0.769	0.848	1.1	5606	1.083	1.083	1.0
0.714	0.785	1.1	6042	1.077	1.077	1.0
0.666	0.733	1.1	6474	1.071	1.071	1.0
0.624	0.686	1.1	6914	1.068	1.068	1.0
0.588	0.647	1.1	7331	1.060	1.060	1.0
0.555	0.610	1.1	7762	1.058	1.058	1.0

CASE 2

1	2	3	4	5	6	7
k	γ	α	W_j	k^{n-1}/k^n	W_j^n/W_j^{n-1}	5/6
0.909	1.0	1.1	4743	-	-	-
0.833	1.0	1.2	6430	1.09	1.35	0.807
0.769	1.0	1.3	7477	1.083	1.16	0.933
0.714	1.0	1.4	10200	1.077	1.36	0.792
0.666	1.0	1.5	11013	1.071	1.079	0.993
0.624	1.0	1.6	11864	1.068	1.077	0.992
0.588	1.0	1.7	0	1.060	-	-
0.555	1.0	1.8	0	1.058	-	-

Zone 21, $\beta = 3.0$

Table 1. Numerical Illustration of the two cases

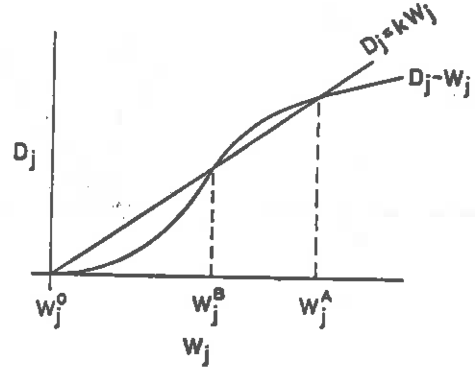

FIGURE 1.

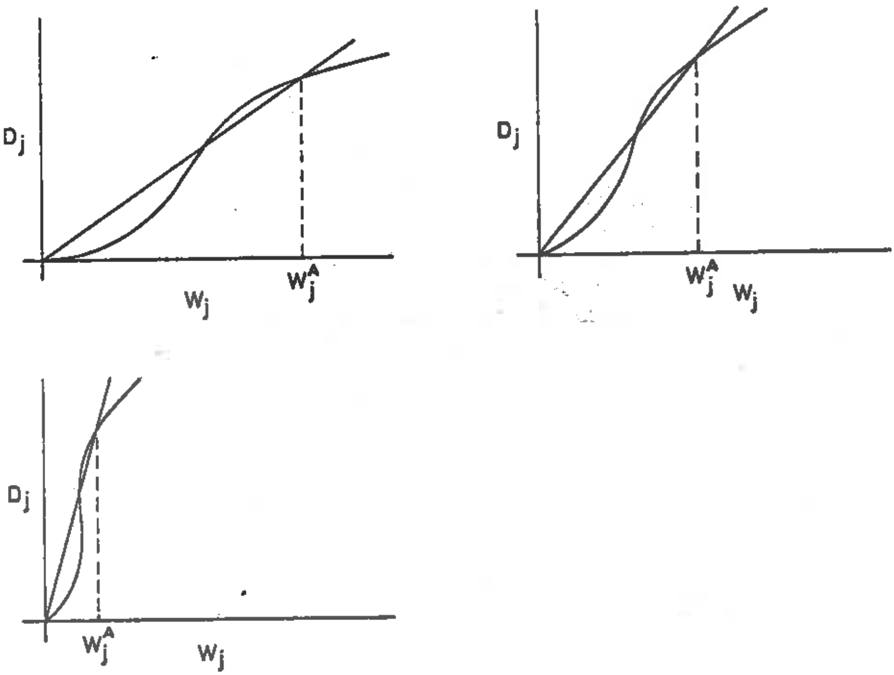

FIGURE 2. The behaviour of W_j^A as k increases ($\alpha > 1$, β fixed)