THRESHOLDS AND INSTABILITY IN STREAM HEAD HOLLOWS: A MODEL OF MAGNITUDE AND FREQUENCY FOR WASH PROCESSES

Mike Kirkby

WORKING PAPER 92/20

SCHOOL OF GEOGRAPHY • UNIVERSITY OF LEEDS

WORKING PAPER 92/20

THRESHOLDS AND INSTABILITY IN STREAM HEAD HOLLOWS:
a model of magnitude and frequency for wash processes

Mike Kirkby

School of Geography
University of Leeds
LEEDS
LS2 9JT

THRESHOLDS AND INSTABILITY IN STREAM HEAD HOLLOWS:
a model of magnitude and frequency for wash processes.

Mike Kirkby
School of Geography, Leeds University, England.

Abstract
The Smith & Bretherton instability criterion and the Horton /Wilgoose tractive stress criterion are combined in a comprehensive theory of stream head location. The theory is given for a combination of wash and rainsplash processes only. Both processes are disaggregated on a storm basis to provide explicit long term integrals over the frequency distribution, as well as a magnitude and frequency interpretation of stream head behaviour. The theory takes account of tractive stress thresholds, subsurface flow and storms which are brief relative to overland flow travel times.

This theory forecasts a constant stream head location where instability is dominant, and a strong inverse relationship between distance to stream head and gradient where threshold behaviour is dominant. There is a continuous transition between these two types. Over the distribution of storms, there is also a transition of types along the length of channels. The different behaviours are also associated with a morphological classification of channel types.

Introduction
The scale of fluvially-eroded landscapes is determined mainly by the spacing of significant valleys. Where valleys are very closely spaced, at 5m to 50m apart, we see a characteristic badland topography. At the other extreme, temperate valleys may be more than 1 km wide, so that there are significant variations over at least two orders of magnitude. In any network, half of the channel links are external links (i.e. first order streams). Since links tend to have similar lengths, approximately half of the total channel length lies in these headwaters, and factors which control the position of stream heads therefore have a considerable effect on total channel density and therefore landscape scale. This paper is concerned with the location of stream heads within their unbranched headwater valleys, which are commonly described as stream head hollows. It is important to understand how stream heads are located, both for explaining static spatial patterns of variation, but also for potential sensitivity to land use and to global climate change.

The reasons for the wide range of variation in channel spacing appear to be related in some way to the balance between hillslope and channel processes for water and sediment transport. Closely spaced active channel networks are generally associated with sparse vegetation and high rates of hillslope runoff and sediment loss. Under natural conditions, badland channel densities are commonly found in semi-arid areas on impermeable shales or marls, where sediment yields are highest. Melton (1957) has shown a strong dependence of drainage density on infiltration capacity and proportion of bare (unvegetated) areas for small catchments in the American South West.

In humid temperate areas, channel densities are generally low in association with continuous soil and vegetation cover, but empirical studies generally fail to show strong dependence of density on simple external causal variables, such as infiltration capacity or rainfall intensity. A number of studies in California and Oregon (e.g. Dietrich et al, 1986; Dietrich and Dunne, 1993) have, however, shown a strong inverse dependence of stream head area on valley gradient, which may partially explain the apparent lack of dependence on external factors.

Two theoretical views have been proposed to account for the location of channel heads. One is based on stability theory and the second on erosion thresholds. The first is based on a mathematical stability analysis of the relevant process laws (Smith & Bretherton, 1972: Carson and Kirkby, 1972). If a small random perturbation is made to indent a smooth landscape surface, stability analysis examines the conditions under which the perturbation will grow to macroscopic size, or shrink until it disappears in the original smooth surface. The analysis can, in principle, be made for any continuously varying process or combination of processes. An indentation in the surface concentrates both water and sediment flows by convergence of the flow lines. Unstable growth of the perturbation occurs if the convergence of flow allows more sediment to be transported out of the indentation than the convergence of sediment flow has brought into it. This happens if sediment transport increases more than linearly with water discharge, so that the increased flow is able to carry out more than the increased sediment inflow.

In the alternative, threshold based view, channel heads are defined by an erosion threshold, where tractive stress exceeds material resistance. This is the basis of Horton's (1945) "belt of no erosion" and x_c distance. It is also the basis used by Wilgoose et al (1990, 1993), in the form of an "activator function". This recent formulation generalizes the condition for a stream head to cover a range of possible processes. This approach appears to refer to behaviour at a dominant flow condition, and has not yet incorporated magnitude and frequency concepts, so that there is no explicit spatial or temporal pattern associated with the area around a channel head. In Wilgoose's analysis the activator function separates zones to which different sediment transport laws are applied: stream transport laws below the channel head, and slope process laws above it, although this distinction is not a necessary part of the threshold view.

Here we argue that these views are not necessarily in conflict, but may be combined to give a more general theory. Over a spectrum of conditions which are related to flow regimes and critical erosion thresholds, there is seen to be a progressive change in response from one dominated by stability conditions to one dominated by threshold behaviour, and that these correspond loosely to the semi-arid and humid-temperate cases described above. Here we present the detail for the case corresponding to stream heads created by rainsplash and overland flow. There may be corresponding generalisations to other processes, in particular to stream heads associated mainly with mass movements or to subsurface piping.

<u>Flow Generation</u>
To understand the way in which threshold and instability criteria apply, across the magnitude and frequency spectrum, we require an explicit model for generating overland flow and sediment transport by the flow and as rainsplash. To be reasonably workable, the model should be simple enough to be integrated over the frequency distribution for long periods, and realistic enough to be

relevant. The model used here is based on daily rainfalls, which roughly represent individual single storms, and for which there is widespread observational data.

In the simple view taken here, the soil has a storage capacity h, and any storm rainfall greater than h produces overland flow. A useful and workable approximation to the infiltration equations in the absence of detailed intensity data is to assume that a proportion β of additional rainfall flows overland, so that the total contribution to overland flow from a storm rainfall r is $q(r) = \beta(r-h)$. If r is taken as the daily rainfall total, then the distribution of daily rainfalls can be fitted adequately to the distribution:

$$N(r>r_1)=N_0\exp(-r_1/r_0) \tag{1}$$

where r_0 is the mean rain per rain day
and N_0 is the annual number of rain days $(=R/r_0)$
where R is the total annual rainfall.

If, as is commonly the case, the data are not a perfect fit to this inverse exponential distribution, then the parameters N_0, r_0 and $R=N_0r_0$ should be fitted to it over the relevant range of storm rainfalls (de Ploey at al 1989).
Summing over the distribution of rain days, the total overland flow produced at a point is:

$$q_* = \beta\int_h^\infty (r-h)\, n(r)\, dr = \beta\, r\exp(-h/r_0) \tag{2}$$

where $n(r)$ is the frequency of days with rainfall r $[=1/r_0\exp(-r/r_0)]$,
R is the total annual rainfall
and q_* is the local average annual rate of flow production.

During a period of constant overland flow production, overland flow depth initially increases linearly with time. Local discharge then builds up gradually towards a steady state in which discharge is equal to unit catchment area x intensity. We will first take the simplest case in which flows reach equilibrium everywhere. The total collecting area for overland flow at a point is delimited by neighbouring flow streamlines, ignoring kinetic effects. Defining the equivalent catchment length, a, as the area drained per unit contour length, we can write the topographic identities:

$$a = \frac{1}{w}\int_0^x w\,dx'$$
$$\frac{da}{dx} = 1 + \frac{a}{\rho}$$
$$\frac{1}{\rho} = \frac{1}{w}\frac{dw}{dx}$$

(3)

where ρ is the contour radius of curvature (+ in hollows; - on spurs),
x is horizontal distance measured along the flow direction
and w is the width of the elementary flow strip.

The total overland flow is then given as:

$$Q_* = q_* a = \beta R a \exp(-h/r_0)$$

(4)

where Q_* is the average annual overland flow discharge per unit contour width.

Our second case is for overland flow which does not reach this equilibrium, because of the short duration of rainfall bursts. The time taken to reach this equilibrium stage is short near to the divides, and increases downslope. The overland flow hydrograph may be obtained as a kinematic cascade, using Manning's equation or another appropriate flow law (e.g. Wooding, 1965). The equilibration time at any given point is approximately the time taken for overland flow to travel to the point in question from the divide. Flow velocity depends on gradient and surface roughness, and commonly lies in the range 0.01 to 0.1 ms-1. Equilibration at 5m from the divide therefore occurs after 1 to 8 minutes, but at the base of a 100m slope requires rainfall for 0.3 to 3.0 hours. Comparing with the duration of bursts of intense rainfall, it can be seen that the lower parts of many slopes will not reach equilibrium, particularly under arid and semi-arid conditions. After the end of a burst of intense rain, while intensities are lower or rain has stopped, some of the overland flow previously generated re-infiltrates into the soil. For a single burst of rain, there is a clear transition point between equilibrium conditions and steady state conditions. When overland flow is summed over a distribution of rainstorms varying in duration, the transition is blurred, although the linear increase with distance is preserved near the divide, and the steady state is preserved at long distances downslope. A convenient generalisation for our second case, of non-equilibrium overland flow, takes the form:

$$Q_* = q_* a \left[1 - \left(\frac{a}{x_0}\right)^b\right] \Big/ \left[1 - \left(\frac{a}{x_0}\right)^{b+1}\right] \tag{5}$$

where x_0 is the distance travelled by overland flow during the dominant storm and b is a positive exponent, usually in the range 1 to 4.

It can be seen that Q_* behaves like q_*a close to the divide, and tends towards an upper limit of q_*x_0 far from the divide, with a transition in the region of $a=x_0$. For actual slopes, lithological differences, surface roughness and topography may modify this general form, often considerably, through different rates of runoff production and flow velocities. The extent of such local variations, and the general trend illustrated by equation 1, is clearly seen in the work of Dunne and his co-workers in Kenya (e.g. Dunne and Aubry, 1986).

Our third case considers overland flow which reaches equilibrium, but in which previous rainfall sets up a downslope pattern of soil moisture, modifying the deficit h in equation (2) in a systematic fashion. 'Saturation' overland flow is produced when the soil can be brought to saturation by the addition of storm rainfall, even of low intensity. Where rainfall exceeds evapotranspiration rates over a period of days or weeks, subsurface flow within the regolith or rock may persist between storms, so that, even at the beginning of a storm, downslope areas and hollows are wetter than divides and spurs. These wetter areas then become saturated during the course of a storm, producing overland flow per unit area at a rate which depends sensitively on topography and soil thickness. Other things being equal, the pattern of overland flow production is concentrated along the margins of channels, especially where there is some flood plain development; and around valley heads. Under extreme storm conditions, this area expands, particularly in hollows (areas of flow convergence) and on concave foot slopes.

The effect of antecedent conditions may be seen for a given hillside profile. For a net rate, j of antecedent subsurface runoff, and an exponential soil moisture store, specified by deficit D, the steady state flow is given (Beven & Kirkby, 1979) as:

$$Q_S = j\,a = q_1\,\Lambda \exp(-D/m) \tag{6}$$

where j is the net rate of antecedent subsurface runoff,
q_1 is the saturated soil flow on unit gradient
and m is a soil moisture parameter with units of depth.

Down the length of the slope, there are consistent changes in a and Λ, which generally reduce the saturation deficit D downslope. Solving for D over a particular topography, and replacing h by D in equation (2) gives the overland flow production at every point, and this may be integrated, in principle, down the length of the flow strip. For a single rainstorm of r, we obtain:

$$Q(r) = \frac{\beta}{w}\int w[r - D(a)]\, dx = ra - E(a) \tag{7}$$

where Q(r) is the total overland flow discharge from a single rainfall of r
and E(a) = 1/w ∫ w D(a) dx is the accumulated deficit downslope.

Integrating over the rainfall distribution, the total annual overland flow discharge is then:

$$Q_+ = \beta\, R\, a \exp\left[-\frac{E(a)}{r_0\, a}\right] \tag{8}$$

This third case is commonly relevant in humid temperate conditions, where there is significant sub-surface flow. Overland flow increases much more than linearly with distance, but only becomes significant in amount within the slope concavity.

Sediment transport

The sediment transport processes considered here are rainsplash, which is driven by rainfall intensity and gradient; and wash processes, which are driven by overland flow discharge, usually to a square power. A threshold for erosion is also included. For both rainsplash and wash, we assume that sediment is carried at its full capacity, that is that sediment removal is transport or flux limited.

The sediment transport model used here for rainsplash is:

$$S_R = \alpha\, r^2\, \Lambda \tag{9}$$

where S_R is the rate of rainsplash sediment transport per unit width
in a storm of rainfall r
and α is an empirical rate constant.

This expression assumes that net sediment transport by splash is proportional to the square of rainfall intensity and to gradient. Integrating over the frequency distribution of rainfalls, we obtain the average annual transport as:

$$S_{R*} = \alpha \frac{N_0}{r_0}\int_0^\infty r^2 \exp(-r/r_0)\, dr = 2\,\alpha\, R r_0\, \Lambda = \kappa\, \Lambda \tag{10}$$

where κ has been written for the overall diffusion rate coefficient
[$=2\alpha R r_0$], measured on an annual basis

For wash transport, we assume that sediment detachment is proportional to unit flow power in excess of a threshold, and to discharge:

$$S_F = \zeta\, Q(r)\,[Q(r)\Lambda - \Theta_c] \tag{11}$$

where S_F is the sediment carried by the flow in a single storm of r,
Θ_c is the power threshold for sediment detachment
and ζ is an empirical rate constant.

This can be integrated in principle over the frequency distribution, to give the average annual wash transport:

$$SF* = \zeta\, \frac{N_0}{r_0} \int_{r_c}^{\infty} Q(r)\,[Q(r)\Lambda - \Theta_c]\, \exp(-r/r_0)\, dr \tag{12}$$

where r_c is the rainfall required to overcome the threshold.

In practice, the integration can readily be completed only for the first case, where the soil deficit h is constant, and overland flow reaches equilibrium. Then we have $r_c = h + \Theta_c/(a\Lambda)$, and:

$$S_{F*} = 2\beta^2 \zeta a^2 R r_0 \Lambda \exp(-h/r_0)\ (1+\phi)\exp(-2\phi) \tag{13}$$

where $\phi = \Theta_c/(2ar_0\Lambda)$ is a threshold dependent function,
which takes the value zero when $\Theta = 0$.

In the case of a zero threshold, the combination of splash and wash may be expressed in the simple form:

$$S_* = \kappa \Lambda \left[1 + \left(\frac{a}{u} \right)^2 \right] \tag{14}$$

where u is a parameter with units of length.

Comparison with equations (10) and (13) give the distance parameter:

$$u = \frac{\exp(h/2r_0)}{(2 R r_0 \beta^2\, \zeta/\kappa)^{1/2}} \tag{15}$$

The parameter u may be interpreted as the distance from the hillslope divide at which wash becomes greater than rainsplash. Empirically this is found to range from 1m to 1000m.

For the more general case, with a threshold, equation (13) may be re-written in the form:

$$S_* = \kappa\Lambda[1+(a/u)^2 (1+\phi)\exp(-2\phi)] \tag{16}$$

where u has the same value as above.

In the arid case of brief storms, the distance a should be replaced in equation (16) by: $a[1-(a/x_0)b]/[1-(a/x_0)b+1]$, following equation (5) above. For the case of humid temperate slopes, equation (16) remains valid, provided that h in equation (15) is replaced by E(a)/a (from equation (6) above) to give the relevant value for u.
The effect of these slope processes may be seen for a ridge in equilibrium with a constant rate, T, of tectonic uplift. The total sediment transport is then necessarily $S_*=Ta$. In the simplest case, with equilibrium overland flow, no subsurface flow and a zero threshold, we may write down an explicit expression for the slope profile obtained. We obtain:

$$\Lambda = \frac{Tx}{\kappa[1+(x/u)^2]}$$
$$z = \frac{u^2 T}{2\kappa}\ln[1+(x/u)^2] \tag{17}$$

where x (= a for a ridge) is distance from the divide,
Λ is the local gradient
and z is the fall from the divide.

This profile is convexo-concave in form, with its steepest gradient at x = u. The distance parameter, u, thus has a clear morphological expression. For non-zero thresholds, we can write down two explicit relationships for distance, x and gradient, Λ in terms of the parameter ϕ (which is inversely proportional to the distance gradient product):

$$\left(\frac{u}{x}\right)^2 = \frac{\phi u^2}{\kappa'\Theta'} - (1+\phi)\exp(-2\phi)$$
$$\Lambda = \frac{\Theta'}{\phi x} \tag{18}$$

where $\Theta' = \Theta_c/(2r_0)$
and $\kappa' = \kappa / T$ each have the dimensions of length

The effect of increasing the erosion threshold is seen in figure 1, which is a log-log plot of gradient against distance. The initial linear rise represents a convex divide. For zero threshold, the maximum gradient occurs at u (10m in this example). As the threshold is increased, the point of inflexion (or maximum gradient) is displaced downslope, due to the lower efficacy of wash. The transition from convex to concave slopes also becomes slightly more abrupt, while the maximum gradient is almost unchanged.

The effect of short storms is to also extend the convexity. For certain parameter values, it is theoretically possible to infer a convexo-concavo-convex profiles, but many reasonable parameter values lead to very long gentle convexities of the type described by Dunne (Dunne & Aubry, 1986) for Kenya. Appreciable sub-surface flow also lengthens the convexity, but terminates it with a very short and abrupt concavity in the slope base region where saturated conditions become dominant.

6 Channel heads

We now use the sediment transport model above to apply the stream head criteria of exceeding the threshold or reaching instability. This is presented both for the individual storm event and for the integrated average effect. The threshold criteria has already been implicitly presented in the form of the wash transport law (equation (11) above). Since only wash, and not rainsplash, is able to cut a channel, it is clear that the power threshold, θ_c, must be exceeded in an individual event to cut a new channel, and on average to cut a permanent channel. Below the threshold, no incision can occur.

The instability criterion for the establishment of a local channel is taken to be that for the growth of an initial small hollow (Smith and Bretherton, 1972; Carson and Kirkby, 1972, p.394). The continuity equation for sediment transport along a flow strip, of variable width , w, and ignoring aerial input of dust etc, is:

$$\frac{\partial z}{\partial t}+\frac{1}{w}\frac{\partial(wS)}{\partial x}=0 \tag{19}$$

Rewriting equation (19), making use of the identities in equations (3):

$$\begin{aligned}
-\frac{\partial z}{\partial t} &= \frac{\partial S}{\partial x}+\frac{S}{w}\frac{dw}{dx}+\frac{\partial S}{\partial \Lambda}\frac{d\Lambda}{dx} \\
&= \frac{\partial S}{\partial a}\frac{da}{dx}+\frac{S}{w}\frac{dw}{dx}+\frac{\partial S}{\partial \Lambda}\frac{d\Lambda}{dx} \\
&= \frac{\partial S}{\partial a}\left(1+\frac{a}{\rho}\right)-\frac{S}{\rho}+\frac{\partial S}{\partial \Lambda}\frac{d\Lambda}{dx} \\
&= \frac{\partial S}{\partial a}+\frac{1}{\rho}\left(a\frac{\partial S}{\partial a}-S\right)+\frac{\partial S}{\partial \Lambda}\frac{d\Lambda}{dx}
\end{aligned} \tag{20}$$

In the neighbourhood of a small irregularity, it is argued that contour curvature is initially changed appreciably, while gradient and unit area are changed to a much smaller extent. The direction of change of erosion rate is therefore given by the sign of the second term on the last line of the right hand side of equation (20). That is to say that small hollows ($1/\rho$ positive) will grow, by increasing their local erosion rate relative to adjacent points, if and only if:

$$\frac{\partial S}{\partial a} > \frac{S}{a} \tag{21}$$

where the partial differentiation keeps gradient constant.

This criterion can be directly applied for simple types of transport limited sediment transport, such as the splash plus wash expression of equation (14) above, which is applicable for semi-arid areas with storms of appreciable duration. Unstable hollow growth occurs, if and only if a > u. For the more complex patterns of humid or arid overland flow production, or if there are significant erosion thresholds (θ_c in equations (11), (12)), then the stability criterion is no longer independent of gradient. Nevertheless the criterion, although applying to the two dimensional form of the hillside contours, can readily be evaluated for a one dimensional slope profile.

There is a strong connection between the form of slope profile (in vertical section) and the stability criterion, and slopes which are unstable with respect to hollow growth are generally concave in profile. To make this relationship more explicit, equation (20) may be rewritten in the form:

$$\frac{\partial S}{\partial \Lambda}\frac{d\Lambda}{dx} = \left(-\frac{\partial z}{\partial t} + \frac{S}{\rho}\right) - \frac{\partial S}{\partial a}\left(1 + \frac{a}{\rho}\right) \tag{22}$$

Writing the local rate of slope lowering $-\partial z/\partial t = T$, and its weighted average down the flow strip to the point of interest as $U = (1/w) \int Tw\, dx = S/a$, gives:

$$\frac{\partial S}{\partial \Lambda}\frac{d\Lambda}{dx} = \left(T + \frac{Ua}{\rho}\right) - \frac{\partial S}{\partial a}\left(1 + \frac{a}{\rho}\right) \tag{23}$$

Since S, $\partial S/\partial \Lambda$ and Qa are always positive, the slope profile is concave ($d\Lambda/dx<0$) if and only if:

$$\frac{\partial S}{\partial a}\left(1+\frac{a}{\rho}\right) > \frac{S}{a}\left(\frac{T}{U}+\frac{a}{\rho}\right)$$

(24)

which should be compared with the stability condition in equation (21) above.

Normally $(1+a/\rho)$ is positive, although there are rare exceptions. For the normal case, equation (24) leads to a series of relationships which clarify the relationship between hollow location and profile concavity. For the special case of a constant downcutting slope (T = U), the conditions are identical, so that, on a profile of constant downcutting, concave sections of the profile are unstable and convex sections stable. For more general circumstances, only more restricted statements can be made, according to the value of the ratio T/U. For mature slopes, with rates of lowering decreasing downslope, then the ratio is less than unity. All convexities must then be stable, and the unstable zone must begin strictly within the concavity. For youthful slopes at an early stage of incision into an uplifted block, T/U is typically greater than unity. In this state all concavities must be unstable, and the unstable zone may extend somewhat into the convexity above. In pronounced hollows with strong flow convergence, where a/ρ is high, the sensitivity of the condition for concavity to local rates of lowering is much less than on a ridge or a nose (spur), so that profile concavity and instability should usually roughly coincide. This analysis can refer meaningfully only to the average position of the instability, since slope profile form changes only slowly over time.

The effect of an erosion threshold, and the role of magnitude and frequency in stream head location may be seen from the single storm version of the sediment transport equations ((9) and (11) above), and their integral (equation (16)). Applying the stability criterion to equation (16), substituting E(a)/a for h to allow for subsurface flow:

$$\frac{\partial S}{\partial a} = \kappa\Lambda\left[\frac{2a}{u^2}(1+\phi)\left(1-\frac{D-h}{r_0}\right) - \frac{a^2}{u^2}(1+2\phi)\frac{d\phi}{da}\right]\exp(-2\phi)$$

(25)

where D is the local saturation deficit
and h is the averaged deficit E(a)/a from equation (7)

The critical stable equivalent catchment length, a_c is given by applying the stability criterion (21) and substituting $-\phi/a$ for $d\phi/da$, to give:

$$\frac{a_c}{u} = \left[(1+\phi)\left(1-\frac{D-h}{r_0}\right)+\phi(1+2\phi)\right]^{-1/2}\exp(\phi)$$

(26)

The significance of equation (26) will first be examined for the case of a fixed soil water deficit h, where subsurface flow is relatively unimportant. Bearing in mind that ϕ has been defined as $\Theta_c/(2ar_1\Lambda)$ in equation (13) above, and contains terms in both distance a and gradient Λ. Equation (26) may thus be read as an expression relating equivalent length to gradient at the critical point, and this expression is drawn in figure 2. It may be seen that, where erosion thresholds are low or gradients high, then the previous model of a fixed stream head distance ,a_c remains appropriate, and stream head position is determined primarily by stability considerations. However, where thresholds are large and/or gradients low, there is a strong inverse relationship between a_c and gradient. Beyond the constant a_c region, this behaves approximately like a power law $\Lambda \propto x_c^{-p}$, with an exponent, p, which falls asymptotically towards 1.0. This relationship broadly matches Dietrich's empirical evidence for a constant product of catchment area and gradient for Northern California. No exact comparison is possible in general since equivalent catchment length, a and catchment area at the stream head are not identical measures.

The critical ratio of gradient to dimensionless threshold required for a gradient dependent stream head may be seen from figure 2 to be around 1.0. The critical threshold required is therefore $\Theta_c > r_0 u\Lambda$, which is the flow power expended at the soil surface during an average rain-day, at distance u from the divide. It may be seen that, as the threshold strength falls, gradient dependence occurs only at lower and lower slope angles. Thus typical stream head gradients are in the constant a_c region where surfaces are weak and thresholds low; and in the gradient dependent region where surfaces are strong. Arguably Low threshold conditions are related to unvegetated surfaces, at least partially covered with easily erodible fine material. High thresholds are then associated with coarse debris, and even more with strong turf cover. Optimum conditions for strong threshold control are therefore argued to be moderate slopes, a sub-humid to humid climate year round to provide a strong turf cover, and some intense rainstorms to erode it, corresponding closely to the areas described by Dietrich in Northern California. Semi-arid conditions, such as those of the Colorado Plateau described by Melton, are similarly thought to lie in the constant a_c domain, with low erosion thresholds. Using climatic parameters from such areas, it is suggested that the critical value of Θ_c should lie close to 0.1 m^2day^{-1}

Where sub-surface flow is important, an exact analysis of equation (26) depends on a complete knowledge of the slope profile. For a linear ridge (parallel contours), one relevant solution is to look at conditions of constant downcutting, for which the unstable position is equivalent to the point of inflexion in the slope profile. Under these conditions there are just two additional parameters which describe the behaviour, both related to the subsurface flow regime. The first is the ratio of the soil parameter ,m (in equation (6)) to the mean rain per rain day, r_0. Ratios greater than 1.0 are associated with areas where subsurface flow is significant, and for humid temperate areas like Britain this may reach values of up to 10. The second relevant parameter is the distance, a_0, at which subsurface flow seeps out at the surface on unit slope gradient. High values of a_0 (>1000m) are associated with highly permeable deep soils and low (but positive) mean net rainfall. High values of m/r_0 and a_0 give conditions which might favour a positive relationship between hollow area and hollow gradient, since lower gradients enhance overland flow, but figure 3 shows that in practice this effect is likely to be very weak, and only evident at realistic gradients where traction thresholds are low.

We can obtain some further insight into the stream head relationships by disaggregating the sediment transport into daily rainfall components. Equations (7) and (11) provide an adequate basis for disaggregating the rill transport, and equation (9) for the splash term in the sediment transport equation. Using the same notation as above, the total sediment transport in a day with rainfall r is:

$$S = \left(\frac{\kappa}{2Rr_0}\right)\left\{\frac{(r-h)x}{\beta^2 u^2}\exp(h/r_0)\left[(r-h)x\Lambda - \Theta_c\right] + \Lambda r^2\right\} \tag{27}$$

where the first term is for rillwash (with zero replacing negative values) and the second is for rainsplash.

Applying the stability criterion as before, instability should occur where two conditions are met: first the instability criterion $x\ \partial S/\partial x > S$, and second the threshold condition for non-zero rillwash $(r-h)x\Lambda > \Theta_c$. For the case of constant deficit h, the instability criterion simplifies to: $x/u > r/(r-h)\exp(-h/2r_0)$. These two expressions are sketched in figure 5a. In general they cross at the storm rainfall $r = (\Theta_c/u\Lambda)\exp(h/2r_0)$, and this intersection is in the relevant range (i.e. $r>h$) if $\Theta_c > u\Lambda h\exp(-h/2r_0)$. Plainly both conditions must be met for a channel to form, so that the upper curve is the control at any given storm rainfall. In other words there are two regimes where thresholds are appreciable, and only one, controlled by the instability criterion, for low thresholds. In both cases the critical distance falls with storm rainfall, supporting the intuitive view that channels incise headwards in severe storms.

This intersection gradient value level is necessarily greater than the critical gradient required for threshold behaviour to be dominant on the average, as defined by equation (26) above. Their ratio is $r_0/h\exp(h/2r_0)$, which has a minimum value of $e/2 \approx 1.36$. Thus we will always have the two regimes in conditions where threshold response is dominant. Where instability response is dominant overall, we may or may not have two regimes, depending on the value of the ratio h/r_0. For any particular case, the average stream head (ASH) is as shown in figure 2. The crossover point between regimes is defined by:

$$\frac{x_c}{u} = \frac{1}{\exp\left(\frac{h}{2r_0}\right) - \frac{h}{r_0}\left(\frac{ur_0\Lambda}{\Theta_c}\right)} \tag{28}$$

where the bracketed term $ur_0\Lambda/\Theta_c$ is the dimensionless ratio of gradient to threshold which was plotted in figure 2.

Figure 5b sketches the possible relationships between regimes, using arithmetic scales, but with the same vertical axis as in figure 2. The horizontal line is taken the stream head gradient, for a given threshold, below which threshold behaviour dominates on average. The curve shows the intersection gradient, referred to above, and applies for each point on the slope profile. Since the stream head is usually close to the locus of maximum gradient (exactly so for the constant downcutting case), then almost all points on the slope plot below the horizontal line in cases where the integral behaviour is threshold dominated. It may be seen that there are three possible regimes for the valley head, as follows:

(i) Instability dominated integral behaviour for ASH
   (a) Storm behaviour is always instability controlled
   (b) Storm behaviour is threshold controlled in very large storms only

(ii) Threshold dominated integral behaviour for ASH. The change to storm threshold control occurs upstream from the ASH

At a down-valley site (x_c larger than the value at P in fig 5a), instability as defined by equation (27) appears to set in before rillwash begins. However, the first term in equation (27) is zero because the flow power $\underline{(r-h)x\Lambda}$ has not yet exceeded the traction threshold, θ_c. Thus the relevant criterion for hollow enlargement is the rillwash threshold. Once the threshold is reached, instability is already well developed. That is to say, the newly effective rill growth is already many times greater than the local rate of infilling by interrill processes. The field relationships associated with this regime are therefore thought to be well defined channel banks, with sharp margins, with relatively little sediment flowing across those margins to soften their morphology. Below the ASH position, this morphology will generally be more or less permanent; but above the ASH the sharp banks will be formed in major storms. For subsequent smaller storms there will be no rill transport, and the stream extensions will gradually degrade by splash infilling and bank collapse.

This sharp-bank morphology is usually clearly defined at downstream sites, but only at some stream heads. Generally it is very clear where there is a strong turf cover, as is the case for Dietrich's sites in N. California, and at the wide variety of other sites with clear headcuts. Plainly there are many cases where sharp stream heads are enhanced or created by subsurface piping. Piping is outside the scope of this discussion, but rapid subsurface flow can be seen as another factor which tends to increase the flow required to overcome traction thresholds. Both piping and strong turf covers have the effect of providing a higher threshold at the soil surface than at shallow depth, so that stream heads, and to a lesser extent banks, are additionally kept sharp by undermining and collapse.

At an upvalley site (x_c less than the value at P in figure 5a), for which the erosion threshold is reached before instability sets in, rillwash is actively eroding and transporting material, but hollow enlargement is prevented in moderate storms by active splash infilling. For large enough storms, however, the rill flow becomes large enough to dominate, and rills can progressively begin to incise the surface faster than interrill processes can fill them in. In this regime, channel margins are receiving material over their banks as fast as it is carried away by sediment transport along the length of the channel. Channels therefore have less distinct banks. Out of the channels, at sites above the rill threshold but below the instability level, exceedance of the traction threshold also leads to effective transport of material, and some tendency to significant sorting and armouring of surface materials. The combination of this fuzzy-bank morphology and surface armouring is common, particularly in many semi-arid environments where there is an incomplete vegetation cover at the surface and appreciable stone content in the regolith. On the long unchannelled slopes described by Dunne & Aubry (1986) for Kenya, the area above permanent stream heads is associated with an increasing cross-slope roughness (Dunne, personal communication), which may also be diagnostic of this morphological regime.

The spatial expression of the three possible morphologies can be summarised as follows. Where the ASH is threshold dominated [Case (ii) above], then storm extensions above the ASH always show sharp headcuts, though these will degrade over time. Above the sharp headcut there may be further fuzzy extensions. Where the ASH is instability dominated, there will may or may not be a transition to sharply defined channel banks at some point downstream of the stream head [Cases (i)a and (i)b above]. Above the stream head, extensions will always be fuzzy. In all cases, the headward extensions are associated with increases in drainage density. We may therefore expect not only linear headward extensions, but also some fanning out into multiple stream heads. This is well shown in figure 6 for the East Twin catchment (Weyman, 1974), and conforms with field descriptions of channel extensions after severe storms. The cumulative effect of the radiating channel extensions above the stream head is to erode the spoon shaped area commonly found there. Where subsurface flow is important, there is a strong positive feedback between the formation of this bowl and the conditions for overland flow and channel extension, and this interaction is much weaker in semi-arid climates.

Figure 7 illustrates the complex behaviour which may occur where subsurface flow is important. This illustrates a slope profile evolving under the wash processes discussed here, and is for a one-dimensional profile without flow convergence in plan. No landslides have been included to degrade the initially steep slopes. To begin with, the profiles may be seen to depart very strongly from the condition of constant downcutting, and the boundary between stable and unstable zones, defining the ASH, is within the convex part of the profile. Once the slope profile reaches something like 'maturity', with no vestige of the initial plateau surface, the ASH corresponds more or less with the locus of maximum gradient. In interpreting the distance to the ASH, it should be remembered that the distance $\underline{u}$ in equation (15) is also dependent on gradients through the average deficit, so that $\underline{u}$ is tending to decrease as the slope gradients decline. Thus there is are two periods in the slope evolution shown during which the distance to the ASH, x_c, increases as gradient falls, as is expected with threshold dominance. Between these two periods is a period of increasing x_c, in which the distance $\underline{u}$ is falling rapidly in response to reduced saturation deficits, so that x_c is falling, while the ratio $x_c/\underline{u}$ continues to rise. Thus drainage density may not have a simple unidirectional history, even though the processes are idealised in a relatively simple way. Furthermore, there may be conditions, as in this example, where the relationship between gradient and x_c appears to reverse, because of the influence of subsurface flow, even though threshold effects are dominant.

Conclusions

It has been shown that the combination of the Smith & Bretherton instability criterion and the Horton /Wilgoose tractive stress criterion can be combined in a single comprehensive theory of stream head location. This theory forecasts a rather fixed drainage density and stream head location where instability is dominant, which is thought to occur typically in semi-arid environments with sparse vegetation, stony regolith and without strong subsurface piping. Where tractive thresholds are dominant, there is a strong inverse relationship between distance to stream head and gradient, but this can be partially masked where subsurface flow is strong. These conditions are best met in temperate conditions, with a strong turf cover.

A morphological classification is proposed on the basis of channel type. Channels may be either have sharply defined or fuzzy banks. Banks are fuzzy when the sediment flux across them at the time of formation is similar to that along their length, so that there is no clean incision. The headmost storm extensions are always fuzzy. Where the average stream head is fuzzily defined, all its extensions are fuzzy, and there may be a transition to sharp banks downstream. Where the average stream head is sharp, it will have storm extensions which are fuzzy at their distal ends, and become sharp before reaching the average stream head. These sharp extensions will gradually degrade over time between major events. Above the average stream head position, the zone of storm extensions is typically associated with an increased channel density.

References cited

Beven, K.J. & Kirkby, M.J., 1979. A physically based, variable contributing area model of basin hydrology. Hydrological Sciences Bulletin 24, 43-69.

Carson, M.A. and Kirkby, M.J.,1972. Hillslope form and process. Cambridge Univ Press. 475pp.

Dietrich, W.E. and Dunne, T., 1993. Chapter 7 in this ms.... in K.J. Beven & M.J. Kirkby (Eds) Network Hydrology, John Wiley, xxx-xxx

Dietrich, W.E., Wilson, C.J. and Reneau, S.L. 1986. Hollows, colluvium and landslides in soil-mantled landscapes. in: Hillslope Processes, (Ed A.D. Abrahams) Allen and Unwin, 361-388.

Dunne, T. and Aubry, B.F., 1986. Evaluation of Horton's theory of sheetwash and rill erosion on the basis of field experiments. in: Hillslope Processes, (Ed A.D. Abrahams) Allen and Unwin 31-53.

Horton, R.E. 1945. Erosional development of streams and their drainage basins: hydrophysical approach to quantitative morphology. Geol Soc Amer, Bull, 56, 275-370.

Melton, M.A., 1957. An analysis of the relations among elements of climate, surface properties, and geomorphology. Technical Report 11, project NR 389-042, Office of Naval Research, Columbia University.

de Ploey, J., Kirkby, M.J. and Ahnert, F., 1991. Hillslope erosion by rainstorms - a magnitude-frequency analysis. Earth Surface Processes and Landforms, 16, 399-409.

Smith, T.R. and Bretherton, F.P.,1972. Stability and the conservation of mass in drainage basin evolution. Water Resources research, 8(6), 1506-2

Weyman, D.R., 1974. Runoff processes, contributing area and streamflow in a small upland catchment. in K.J. Gregory & D.E. Walling (Eds), Fluvial processes in instrumented watersheds, Institute of British Geographers, Special Publication 6, 33-43.

Wilgoose, G.R., Bras, R.L. and Rodriguez-Iturbe, I., 1990. A coupled channel network growth and hillslope evolution model. Water Resources Research.

Wilgoose, G., Bras, R., & Rodriguez-Iturbe, I, 1993. Hydro-geomorphology modelling with a physically based river basin evolution model. This volume, pp xxx-xxx.

Wooding, R.A., 1965. A hydraulic model for the catchment stream problem. I: Kinematic ave theory. J. Hydrology 3, 254-276.

Figure captions.

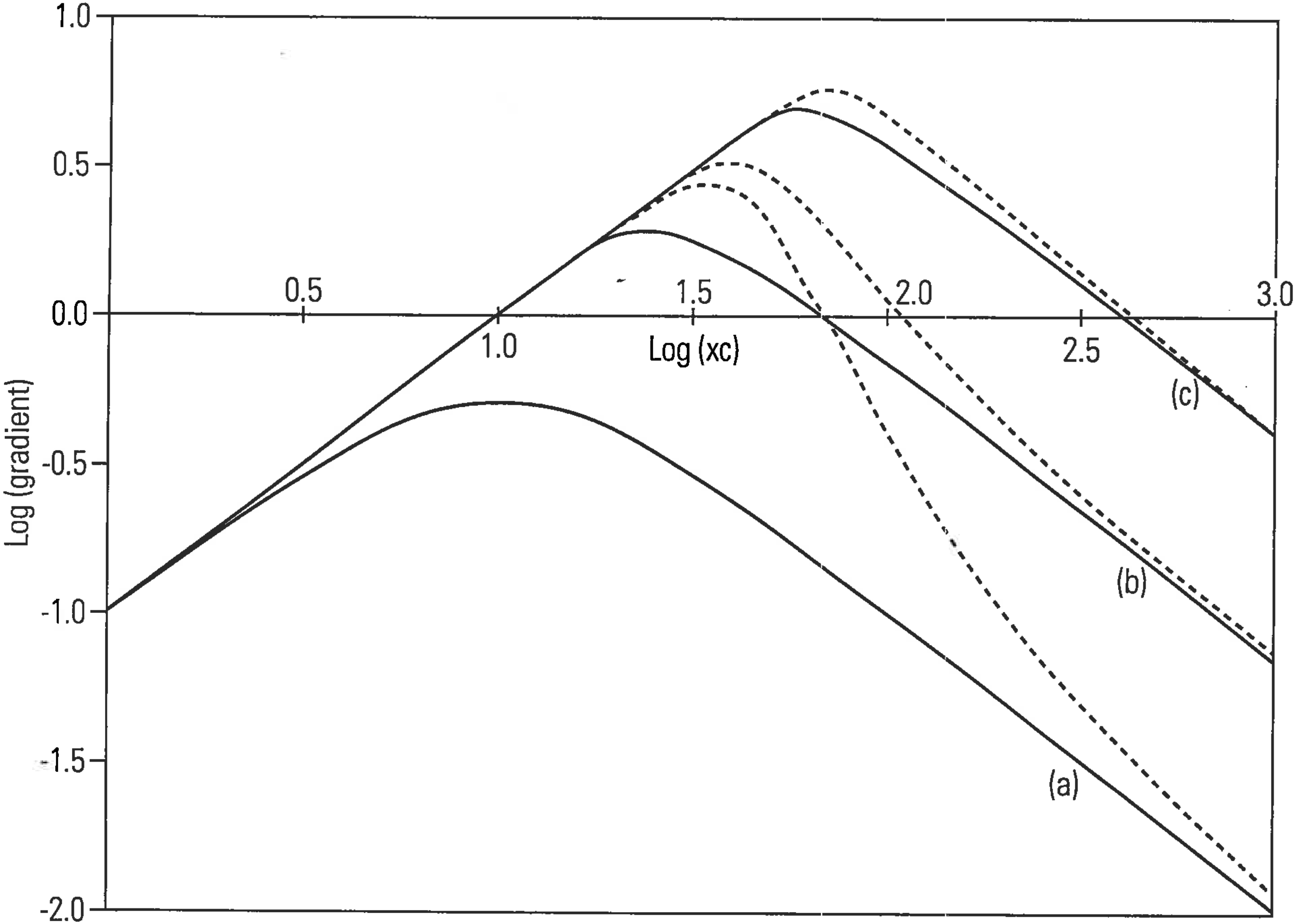
1: Example slope profiles for conditions of constant downcutting, shown as log-log plots of gradient against distance from divide. The conditions are for strong sub-surface flow (u_0=10m; m/r_0=10; a_0=15,000m). The three pairs of curves are for traction thresholds of (a)0, (b)100 and (c)1000m. Solid lines are for conditions of Hortonian overland flow; broken lines with appreciable subsurface flow and saturation overland flow.

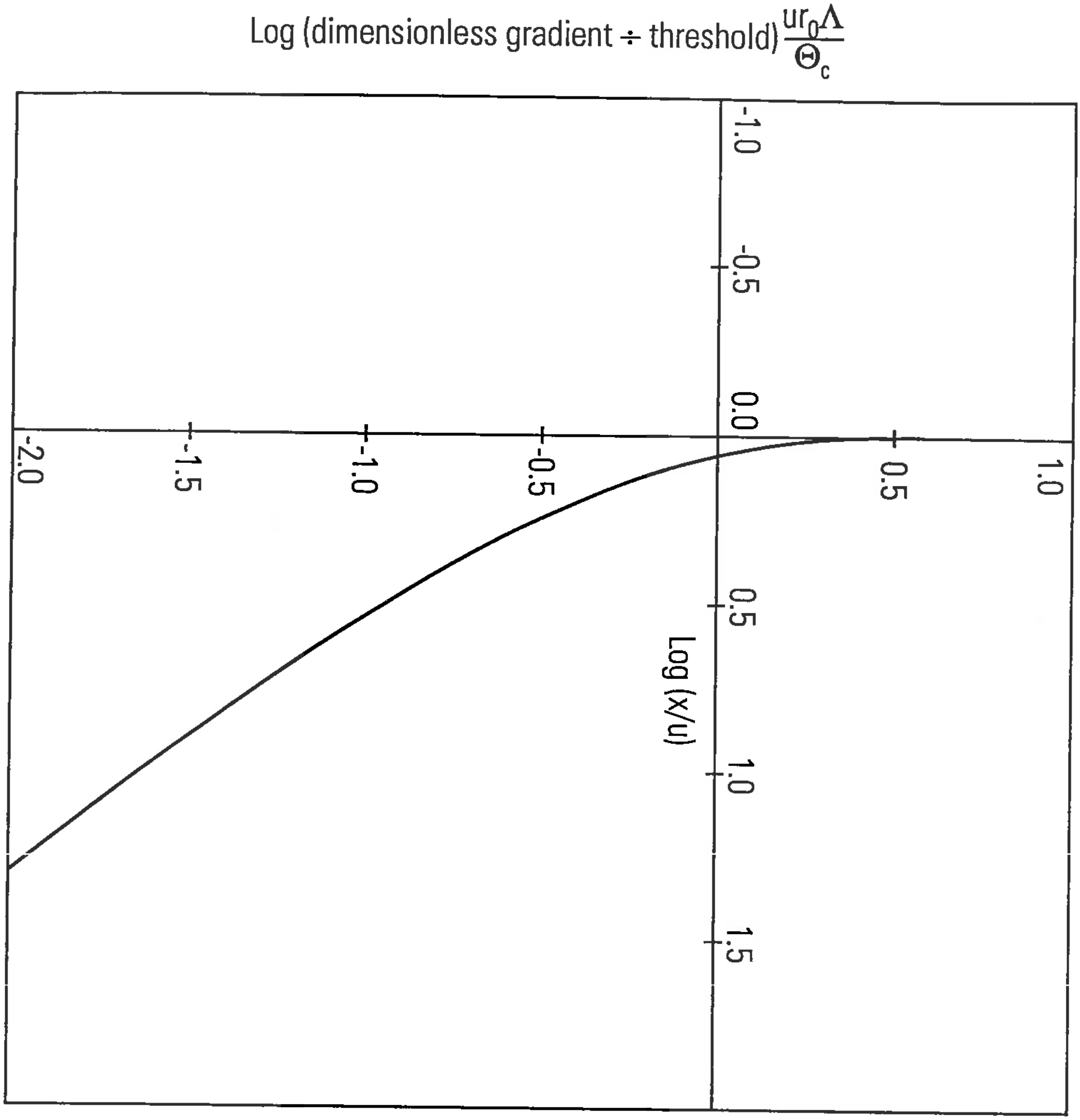
2: The theoretical relationship between average stable (unchannelled) valley length, x_c and local slope gradient Λ in the presence of a slope threshold. The curve is drawn from equation (24) for conditions of splash and wash erosion, without significant saturation overland flow.

3: Examples of the relationship between gradient and distance at the theoretical stream head. The curves are shown for conditions of appreciable saturation overland flow, and with a range of thresholds. The relationship is calculated on the assumption that profiles are in equilibrium with a constant rate of downcutting.

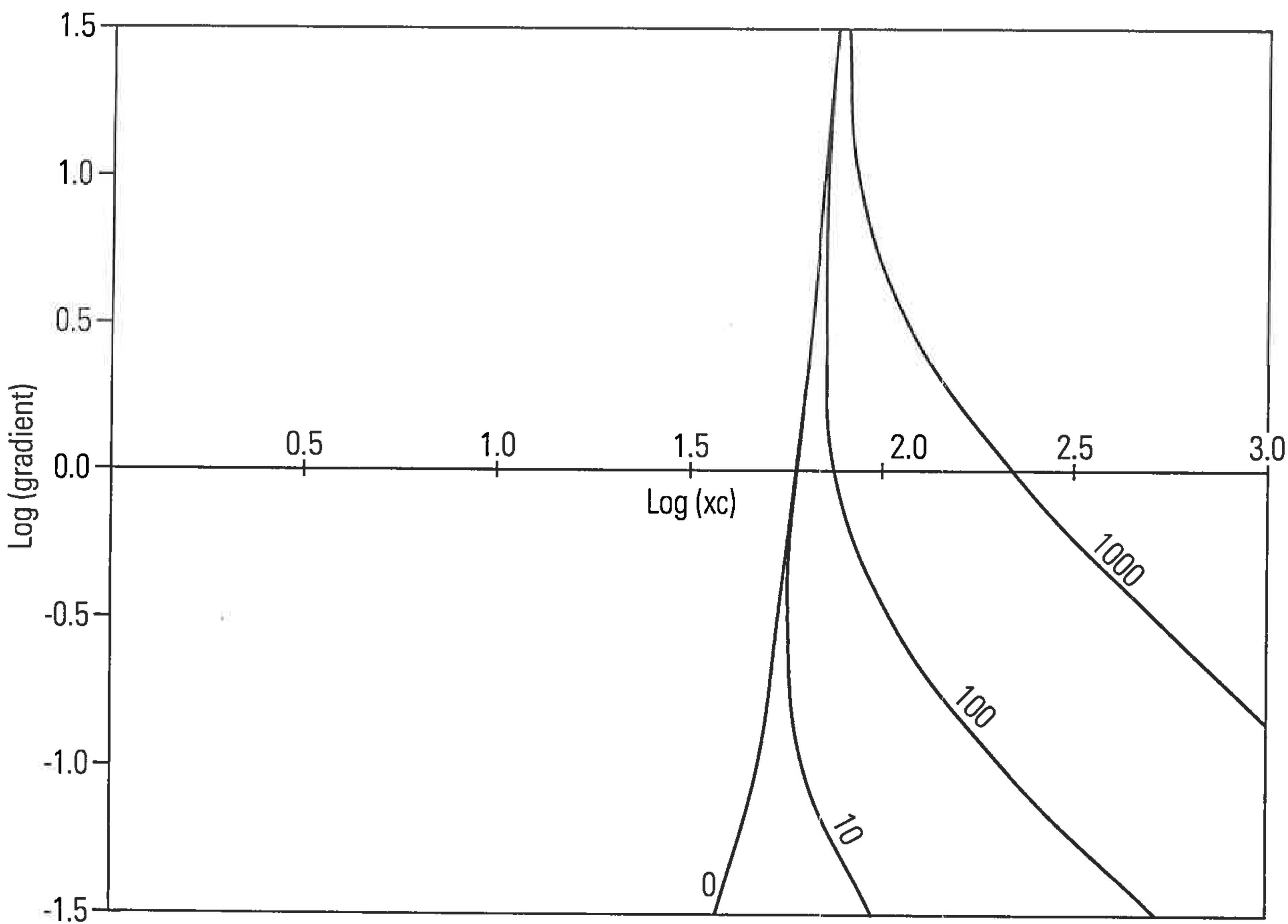
4: Some constant downcutting profile forms for the examples shown in figure 3. The curves are drawn for maximum gradients of 0.1 in each case, and for thresholds of 0, 10 and 100m.

5: Stable (unchannelled) valley length x_c as a function of daily storm rainfall r for an example set of parameter values, and for splash and wash erosion. In general the stability condition is critical at upvalley sites, corresponding to large critical storms; and the rill erosion threshold is critical at downstream sites, for smaller critical storms.

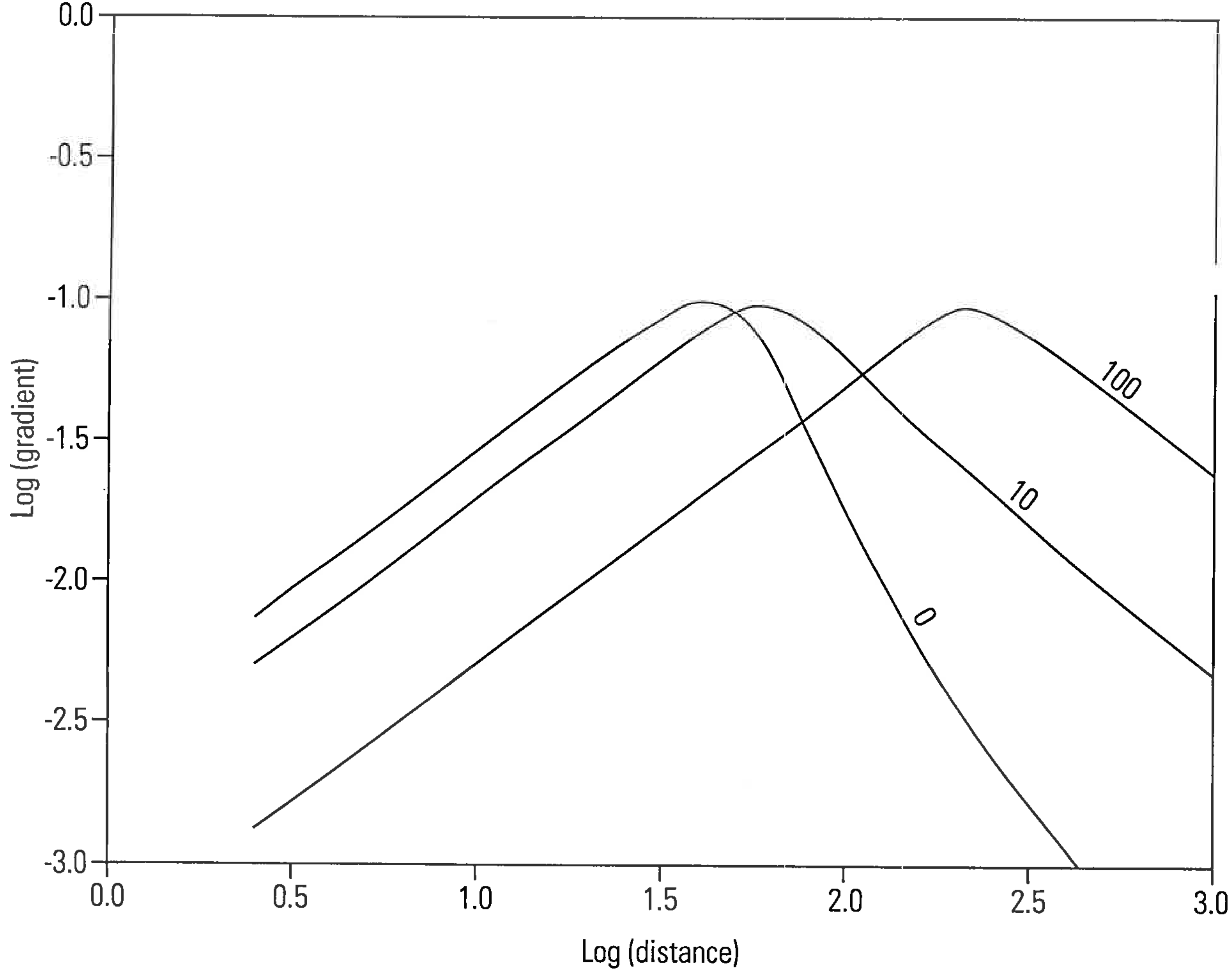

Log (gradient)
Log (xc)
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
0.5
1.0
1.5
2.0
2.5
3.0
(a)
(b)
(c)



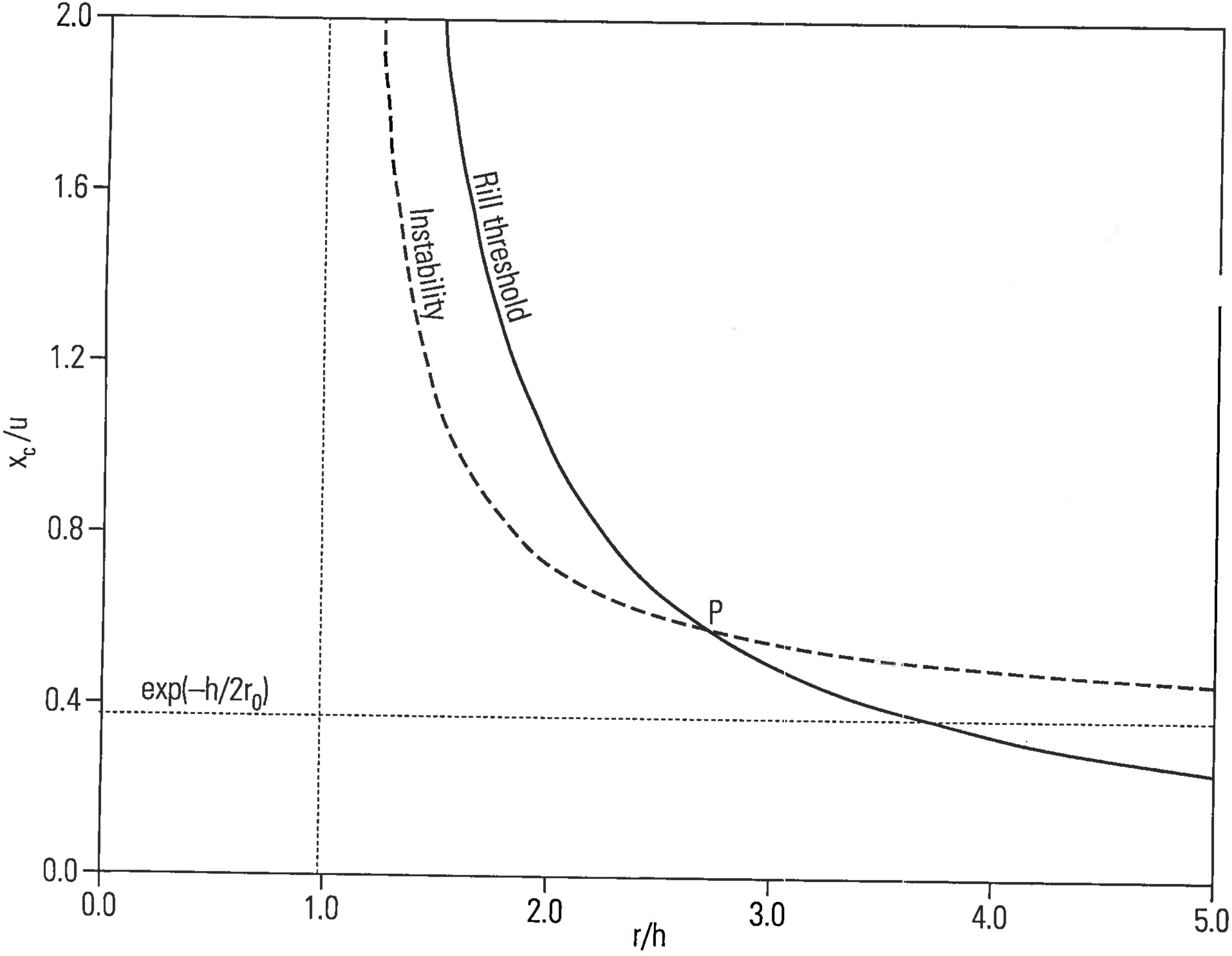
Log (dimensionless gradient ÷ threshold) $\frac{ur_0\Lambda}{\Theta_c}$
Log (x/u)
-1.0
-0.5
0.0
0.5
1.0
1.5
-2.0
-1.5
-1.0
-0.5
0.5
1.0



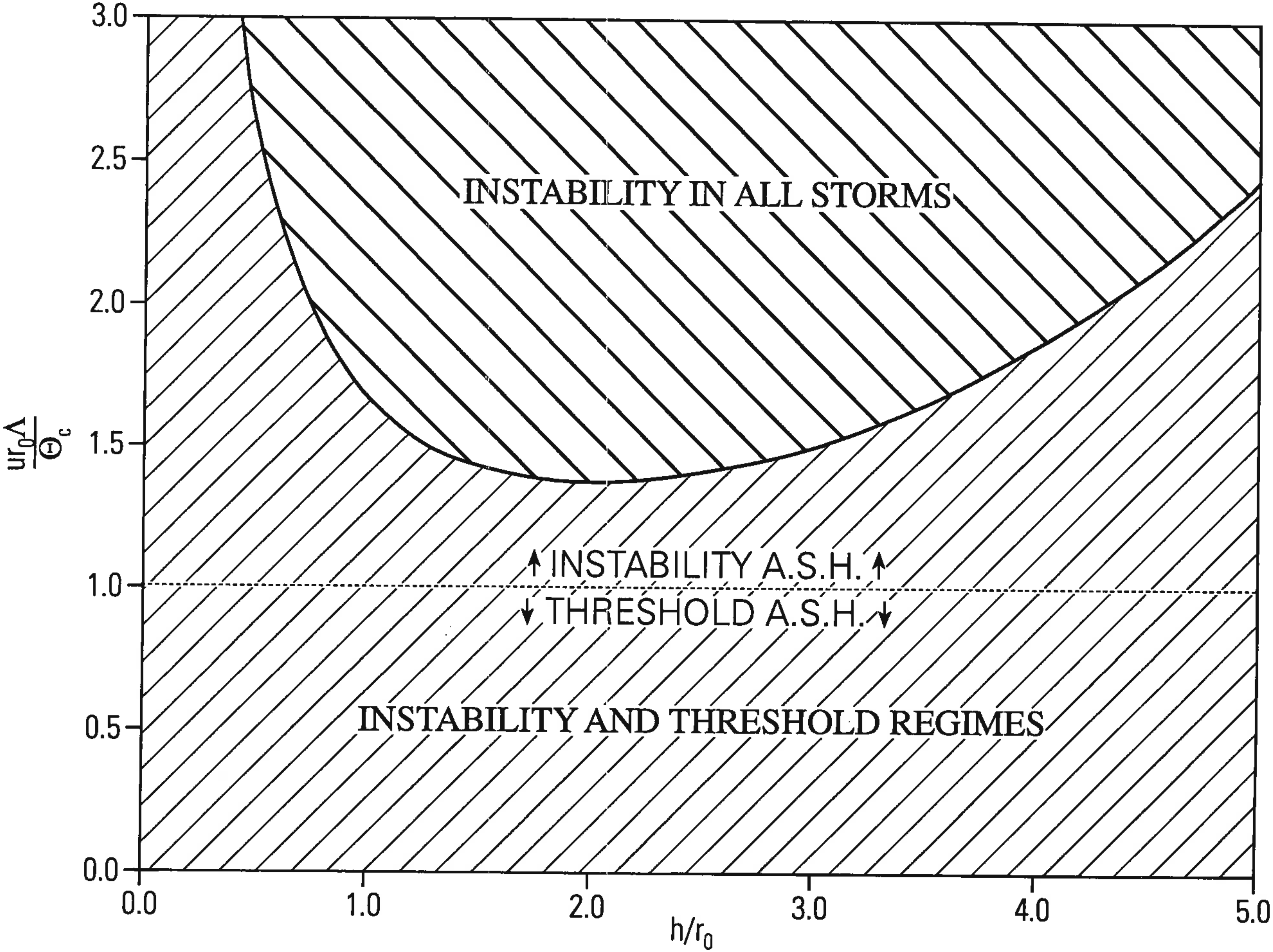
Log (gradient)
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
0.5
1.0
1.5
2.0
2.5
3.0
Log (xc)
0
10
100
1000

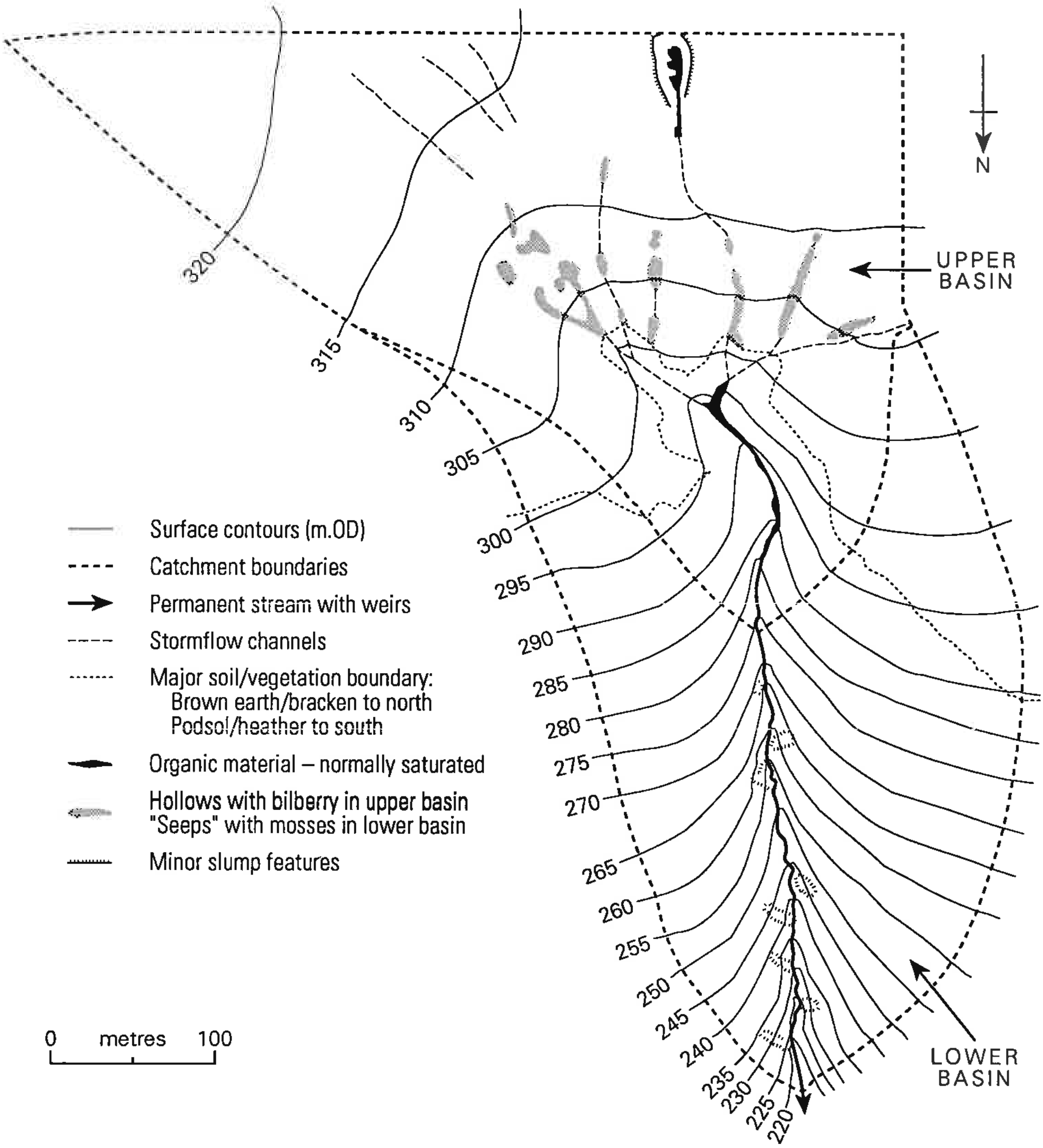


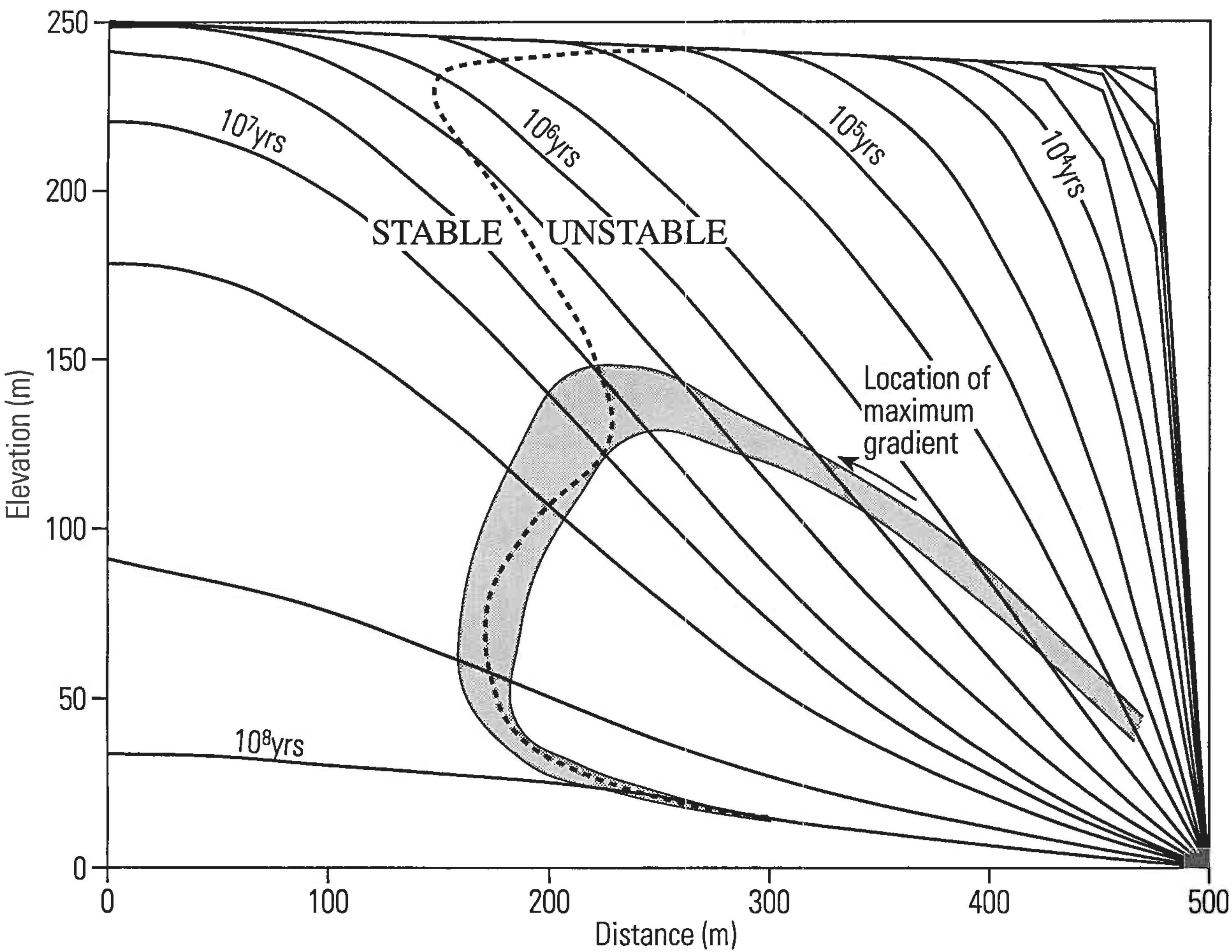
Log (gradient)
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
-3.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Log (distance)
0
10
100



2.0
1.6
1.2
0.8
0.4
0.0
x_c/u
0.0
1.0
2.0
3.0
4.0
5.0
r/h
Rill threshold
Instability
P
exp(−h/2r_0)


INSTABILITY IN ALL STORMS
INSTABILITY A.S.H.
THRESHOLD A.S.H.
INSTABILITY AND THRESHOLD REGIMES
3.0
2.5
2.0
1.5
1.0
0.5
0.0
$\frac{ur_0\Lambda}{\Theta_c}$
0.0
1.0
2.0
3.0
4.0
5.0
h/r_0


N
UPPER BASIN
LOWER BASIN
320
315
310
305
300
295
290
285
280
275
270
265
260
255
250
245
240
235
230
225
220
Surface contours (m.OD)
Catchment boundaries
Permanent stream with weirs
Stormflow channels
Major soil/vegetation boundary:
Brown earth/bracken to north
Podsol/heather to south
Organic material – normally saturated
Hollows with bilberry in upper basin
"Seeps" with mosses in lower basin
Minor slump features
0 metres 100



250
200
150
100
50
0
Elevation (m)
0
100
200
300
400
500
Distance (m)
10^7yrs
10^6yrs
10^5yrs
10^4yrs
10^8yrs
STABLE
UNSTABLE
Location of
maximum
gradient