WORKING PAPER 355

LOCATION THEORY: A UNIFIED APPROACH

A.G. WILSON

WORKING PAPER
School of Geography
University of Leeds

LOCATION THEORY: A UNIFIED APPROACH

A.G. Wilson

School of Geography
University of Leeds
LEEDS LS2 9JT

May 1983

LOCATION THEORY: A UNIFIED APPROACH

A.G. Wilson

1. Introduction

There are two directions of approach to location theory (as succinctly and elegantly pointed out by Lowry, 1967). The first focusses on the spatial distribution of activities (type by type) across space; the second on the set of activities, the land-use mix, at a particular place. The customary focus is the first, but both need to be integrated for a location theory to be effective because of the interdependence arising from different activities competing for land. In this paper, we are concerned with the range of theories available for the analysis of various types of activity location and we present a framework within which these can be approached and viewed in a unified way. We also show that inter-dependencies can be introduced into this framework which provide the basis for solving the problem of the land-use mix and related issues.

Typical fields of location theory, essentially defined by the type of activity being located, are: agricultural location, public utilities, industrial location, private services, public services and housing. Some integration has been attempted, usually at rather coarse levels of resolution, with approaches ranging from central place theory to variants of the Lowry (1964) model. An important distinction should be borne in mind at the outset: that introduced by Paelinck and Nijkamp (1975) between land-consuming activities, of which the main examples in the above list are agriculture and housing, and the rest, for which location can be considered, as a first approximation, to be at a point. (In areas of intense economic activity, like central business districts, this approximation breaks down of course.)

The 'classical' approaches to thse problems appear to be very different in style. Consider, for example, von Thunen (1826) on agricultural location, Weber (1909) on industrial location, Burgess (1927) and Hoyt (1939) on residential land use. Or consider later generations of theorists in the same three fields, say Found (1971), Hoover (1967) and Alonso (1960, 1964). These all differ from the early attempts at integrated theory by Christaller (1933) and Losch (1954).

As a further contrast, consider the 'modelling' approach of Lowry (1964), much of it based on the concepts of spatial interaction modelling (Lakshmanan and Hansen, 1965, Wilson, 1967, 1974). At a later stage still, approaches to modelling the supply side have been added, as in Harris and Wilson (1978).

The purposes of this paper are: (i) to review briefly the rules of model design and to show that this provides a basis for the classification and understanding of alternative approaches; (ii) to present a unified approach based on modelling ideas and to show that many other contributions to location theory can then be seen as special cases; and (iii) to provide the basis for further extensions. This is largely achieved by drawing together ideas which have been presented elsewhere in an attempt to create a new synthesis.

We begin (section 2) by reviewing a set of rules for model design and then present, in these terms, the approach to location theory to be adopted in the rest of the paper (section 3). In section 4, we comment on some earlier theories in relation to this framework and show how they can be incorporated in models which illustrate the unified approach. In section 5, we make some concluding comments and outline ongoing research tasks.

2. Rules for model design: the basis of a theory-classification system

We begin by reviewing briefly (and, in one case, expanding) the rules for model design which have been presented elsewhere (eg. in Wilson, 1981-A). It is argued that the model builder takes decisions in relation to the following dimensions:

(i) entitation - the enumeration of individuals, organisations and other components of the (sub) system of interest.

(ii) levels of resolution: sectoral (how to categorise components), spatial and temporal.

(iii) partialness/comprehensiveness: whether to model, at one extreme, a single unit taking everything else as a given environment - a kind of marginal analysis; or whether to be comprehensive and handle numbers of (usually competing) units.

(iv) spatial representation - discrete or continuous.

(v) elements of *theory:*

(a) processes shaping the 'environment';
(b) processes associated with consumer behaviour (the 'demand' side);
(c) processes associated with producer/organisation behaviour (the 'supply' side);
(d) any 'whole system' criteria: planning policies, maximisation of consumers' surplus or whatever;
(e) the representation of interdependence;
(f) the development of explanatory concepts.

(vi) development of techniques for building models which represent the theory.

By asking the question, how did X (von Thunen, Weber, etc.) take these decisions, new insights can be gained into old models; alternative approaches can often be seen; and the basis for a classification is provided. We show examples of this in section 4, but first present the broad outlines of our own approach set against this framework.

3. The general approach advocated

We describe the decisions on model design which constitute the approach to be recommended here in a slightly different order to that used in the previous section. First, we note that it appears to be nearly always more convenient to use a discrete zone system as the basis for characterising location and for modelling. The number of zones determines the fineness of spatial resolution adopted. In general, at least for this author, more powerful mathematical tools seem to be available for building the resulting models. It is also easier to handle land-use constraints. Indeed, at fine levels of resolution at least, discrete zones are more 'natural' since activities are located on units of land of finite size.

Another feature of the representation is determined by sectoral resolution. Typically, sectors are distinguished by zone totals, not individuals or firms. In other words, we work with variables like 'population in an income group in a zone' (P_i^w, say) or 'retail floor space for furniture in a zone' (W_j^g, say). Again, this turns out to be mathematically convenient and to enable processes based on competition to be represented to a reasonable degree of approximation without having to deal with the geographical equivalent of the n-body problem in physics: a large number of individual competing households or firms in continuous space.

These decisions together also imply a third: to attempt to build comprehensive models, treating as many elements as possible endogenously and seeking explicitly to represent competitive and other interacting processes. It is a combination of the discrete zone system and representing the main entities as 'totals by zone' which allow this to be done relative to other approaches.

We can then turn to the various elements of theory. We take the points in the order listed under (v) in section 2 above.

(a) The 'environment'. Once the system of interest is defined, we also try to model where appropriate (as distinct submodels) the main exogenous variables which form a backcloth. For many locational models, for example, this involves providing population and macro-economic backcloths.

(b) Consumer behaviour is usually modelled in terms of spatial interaction between residences and point of supply. The most difficult part of this exercise is to characterise the 'attractiveness' of alternative supply points in realistic detail. These models can be made consistent with utility maximising approaches if appropriate, or other alternative theories. An important element of most models deployed here is that consumers do not necessarily go to the *nearest* facility. This is more in accord with reality than the alternative hypotheses of many economic theories. The specification of the attractiveness function can be seen as incorporating the benefits and costs from the point of view of the consumer including the representation of consumer scale economics. A crucial element of this approach is that 'demand' can be allocated to 'service points' *taking account* of competition, and the summation of flows at each destination is a measure of 'revenue' or its equivalent.

(c) On the supply side, it is easiest to model private sector profit maximising behaviour, as we will see. However, since profit is defined as 'revenue minus costs', we can in principle substitute 'benefits' for revenue and define some appropriate public sector hypotheses. Otherwise, different kinds of hypotheses have to be used. As we saw, the model of consumers' behaviour estimates revenue or benefits. There may also be an element of spatial interaction phenomena associated with costs, for example, in the costing of inputs from different places, taking account of transport costs and again allowing for the effects of competition. Each producer may be assumed to seek the cheapest inputs (possibly with some 'dispersion' built in as with the model of consumers spatial interaction).

(d) There are two kinds of examples of system-wide effects (and we distinguish them from the 'environmental backcloth' of (a) above because they are such an integral part of the system). First, a number of market prices, for many goods for example, will be fixed at this level (or bases for prices to which a locational variation, say for transport costs, is added within the locational model). A good deal of economic and social theory needs to be added to specify how these prices are determined: in the market, in relation to supply and demand curves (and, in turn, production functions and utility functions); or through other processes, such as the struggle between capital and labour. Secondly, there is the exercise of the whole range of policy instruments of government in its various forms (expenditure, regulation, fiscal, form of administrative organisation - see Wilson, 1974, chapter 2) and the way in which these affect the system of interest.

(e) Interdependence is important in many ways. Variables which are exogenous in one subsystem will often be endogenous in another, and this forms an important set of linkages. Another important feature of interdependence is provided by the transport infrastructure of the system as a whole. (The way this develops is also an important field of study in its own right - see Wilson, 1983-A).

(f) An important stage is the development and deployment of particular theoretical concepts as the building blocks of theories. The main principles to be advocated here are two: first, as far as possible, define concepts directly in terms of elements of the system. (It is more effective, for example, to work with a variable which is the proportion of those who live in one zone and shop in another than to invoke a concept of non-overlapping market areas.) Secondly, employ Occam's Razor. The second principle in a sense modifies the first where appropriate. If it is *necessary* to use 'higher-level' abstract concepts, then that is OK; but in a critical spirit. Thereafter, this section must be illustrated by example.

The final step in the argument is to assemble the necessary techniques, usually mathematical in the case of modelling, but not necessarily so, which can be used to build a formal model of a theory - which is to say, a clearly-formulated theory - from all the definitions, ideas, concepts and hypotheses which have been drawn together using the above procedures. This can often be crucial. It is only relatively recently, for example, that the techniques of catastrophe theory and bifurcation theory have become available from mathematics to enable some fundamental problems in locational theory to be essentially solved. Before that, what can be seen with hindsight is a struggle to solve these problems with inadequate tools.

It is useful to show in a formal way what these kinds of models look like. Let i and j be location labels, m and n population groups or economic sectors, and k a product or service of some kind. Let X_i^{mk} be the demand for k by individuals or organisations of type m at i, and let Z_j^{nk} be the amount of k supplied by organisations of type n at j. The allocation variable which matches demand with supply can be taken as Y_{ij}^{mnk}. Any of the indices m, n or k can be taken as lists if more resolution is required. The locational variables in this description are the arrays $\{X_i^{mk}\}$ and $\{Z_j^{nk}\}$. In either case, more aggregated arrays, like $\{Z_j^{n*}\}$ or $\{Z_j^{*k}\}$ may be appropriate (where an asterisk replacing an index denotes summation).

Suppose our main interest is in $\{Z_j^{nk}\}$. $\{X_i^{mk}\}$ may be taken as given for this particular problem - though, as we have implied earlier, they may be the endogenous variables for another problem. Suppose the totals and the flows can all be measured in money (or at least, the same) units. Then the interaction variables can be taken as

$$Y_{ij}^{mnk} = Y_{ij}^{mnk}(X_i^{mk}, W_{ij}^{mnk}, c_{ij}) \qquad (1)$$

where W_{ij}^{mnk} is a measure of the attractiveness of type-n suppliers of k in j to type-m organisations in i; c_{ij} is an index of flow cost (and could be incorporated in W_{ij}^{mnk}, but is shown separately to emphasise the importance of space). Attractiveness is specified in terms of a number of characteristics of suppliers weighted in different ways by appropriate coefficients. To economise on notation, we denote the elements of these as a vector $\underline{h}_{ij}^{nk}$ and coefficients $\underline{\alpha}^{mk}$ and take

$$W_{ij}^{mnk} = W_{ij}^{mnk}(\underline{\alpha}^{mnk}, \underline{h}_{ij}^{nk}) \tag{2}$$

Total revenue for k in n at j (D_j^{nk}) can be calculated as

$$D_j^{nk} = \sum_{im} Y_{ij}^{mnk} \tag{3}$$

Let $\underline{q}_j^{nk}$ be a vector of inputs needed to produce Z_j^{nk} and let $\underline{p}_j^{nk}$ be the corresponding vector of prices paid by n at j. Then

$$C_j^{nk} = \underline{p}_j^{nk} \cdot \underline{q}_j^{nk} \tag{4}$$

is the total cost of producing W_j^{nk}.

A simple profit-maximising hypothesis about suppliers' behaviour would then take the form

$$\dot{Z}_j^{nk} = \varepsilon^{nk}[D_j^{nk} - C_j^{nk}] \; Z_j^{nk} \tag{5}$$

(for logistic growth near the origin) with an equilibrium (which may never be achieved) being specified by

$$D_j^{nk} = C_j^{nk} \tag{6}$$

Note that the elements of the vector $\underline{q}_j^{nk}$ may include input flows from other locations so that these are, potentially, spatial interaction terms in C_j^{nk} as well as in D_j^{nk}. (The ε^{nk} are constants.)

These kinds of equations look relatively straightforward when presented in this formal way. The formality, however, masks enormous complexity, and we can begin to see this by spelling out the issues related to the next two items on the theory-building list. These are the related topics of prices and interdependence and we start with the broader second topic. We take each equation in turn and show how key variables have to be made functions of other variables in order to capture potentially important interrelations.

In equation (1), the demand, X_i^{mk}, will be a function of offered price, the prices of substitutes, and the transport cost to sources of supply (so this will not simply be c_{ij} but a composite measure such as accessibility). So we might write

$$X_i^{mk} = X_i^{mk}(\underline{p},\underline{V}, \ldots) \tag{7}$$

where $\underline{p}$ is a composite vector of prices, $\underline{V}$ a vector of accessibilities and so on. (Other variables would include, for example, the relative wealth of the m-group.)

The characteristics of attractiveness in (2) will obviously include terms like Z_j^{nk}, the availability of k in n at j, but also measures of the competition of other n's and j's - so in this sense the balancing factor is a singly-constrained spatial-interaction model, which can be taken to represent competition, and can be considered formally to play a role in the measurement of attractiveness. The price vector, $\underline{p}$, is relevant again, now in terms of choice of destination rather than scale of demand. The accessibility vector, $\underline{V}$, will also be relevant. For instance, a visit may be made to get k from n at j if j is also highly accessible to other k's. So we write characteristics formally as

$$\underline{h}_{ij}^{nk} = \underline{h}_{ij}^{nk}(Z_j^{nk}, \text{competition}, \underline{p},\underline{V}, \ldots) \tag{8}$$

There are many complexities associated with the vectors $\underline{p}_j^{nk}$ and $\underline{q}_j^{nk}$, the inputs to Z_j^{nk} and their prices. The inputs may well be non-linear functions of the amount produced, to represent scale economies for example:

$$\underline{q}_j^{nk} = \underline{q}_j^{nk}(Z_j^{nk}, \ldots) \tag{9}$$

The prices will be functions of the prices at input sources and the cost of transporting them:

$$\underline{p}_j^{nk} = \underline{p}_j^{nk}(\underline{p},\underline{c}, \ldots) \tag{10}$$

where $\underline{c}$ is a vector of transport costs for different kinds of inputs.

Note that the inputs should include capital, land and labour as well as goods, services and materials. The element of $\underline{p}_j^{nk}$ which is the price of land for the activity Z_j^{nk} is particularly important in relation to interdependence. Many activities compete for land and this competitive process (with other influences such as government policy, the history of past ownership and so on) will determine the land price element of $\underline{p}_j^{nk}$. If, in effect, no land is available for (n,k,j) activity, then this will show itself as a very high price. This will transmit itself to C_j^{nk} via (4), and then into (5). Because the right hand side of (5) will obviously be negative in such an instance, we will have $Z_j^{nk} = 0$. An example is given of this kind of mechanism for residential location and housing in Clarke and Wilson (1983-A).

It is worth trying to pursue the issue of handling land a little further. There are two fundamental difficulties to be dealt with and then a number of consequences follow in relation to the approaches which solve these. The first involves the investigation of the concept of rent, the second, some problems arising from the size of zones in the discrete zone representation. We discuss each of these in turn.

The theory of land rent has its origins in the work of von Thunen (1826) and in its modern form has been much developed by Alonso (1960, 1964) and others. The key idea of relevance here is that of bid rent: that different potential users of a unit of land notionally bid a rent which is the surplus (if normal profits are included in costs) to be gained from economic activity or which is related to utility in the case of housing. The rent which can be gained by the land owner is then the highest bid rent at a particular site. The difficulty posed by this formulation for the present framework is that land rent is not obviously a smooth function of other system variables. If what are specified are schedules of possible profits or consumers' surpluses, then these have to be searched and the market cleared in order to construct the rent surface. In modelling terms, this is an algorithmic process rather than an analytical function of other obvious variables. There are at least two ways of dealing with this in principle. First, to develop an analytic function for rent as an approximation to the process - for example, by making land prices a function of various accessibilities. Secondly, a model of the rent-bidding algorithm could be built in explicitly. This may not be as fearsome as it at first appears since, in a run of the model through time, there would always be the previous period's prices to act as a base; and, initially, empirically-measured prices could be used.

The second problem is concerned with the size of zones. Rent theories work in terms of either price per unit area (and this may in principle vary across a zone) or price for plots of land for particular purposes (which may be parts of zones or may straddle zones). If the first kind of approach is adopted, then an average would have to be assumed for the whole zone; the second is more intractable unless the zone system can continually be redefined in relation to plots (which is impossible in practice). So we can assume a zonal average. The remaining problem is then how to handle a mix of land uses within a zone and to build in appropriate constraints on the total land available in that zone. In a run of the model through time, this is further complicated, as noted above, by the problem of 'converting' old land uses to new ones. We will offer specific mechanisms in the context of examples in section 4 and 5 below.

Vectors like the accessibilities, $\underline{V}$, or the transport costs, $\underline{c}$, are specified in terms of other variables which are already endogenous to the problem. They do introduce new kinds of nonlinearities. What is more difficult to follow through at present is the nature of the elements of the price vector, $\underline{p}$. At a macro level, some of these elements can be considered to be determined by the intersection of supply and demand curves in some 'backcloth' aggregate model. Others might be fixed by government policy. Yet others will be functions of the spatial configurations of activities and the associated transport system. Above all, many prices of goods at locations will be functions of the quantities produced, because of scale economies and other nonlinearities. This, of course, adds a major feedback loop to the model system. Formally, we can write

$$\underline{p} = \underline{p}(\underline{X}, \underline{Z}, \underline{c}, \underline{V}, \ldots) \tag{11}$$

and to complete the equation system show transport costs as determined by infrastructure supply which will be a function of demands, government allocations and so on:

$$\underline{c} = \underline{C}(\underline{X}, \underline{Z}, \text{government allocation to transport} \ldots) \tag{12}$$

The whole system is shown in figure 1 which makes clear many of the important feedbacks which have been built into even a formally-specified model. Note that these are not only feedbacks within the (n,k,j) system shown, but between systems, for all the reasons outlined above and sketched on the figure.

In the next section, we interpret this general model in the terms of various classical contributions to location theory and show how to extend them to fit in with the framework of this section. Meanwhile, we briefly examine the equivalent framework for a public sector system where there are no market prices and so the mechanism implied by equations (5) and (6) is not - at least at first sight - possible. Equation (1) might still describe consumers behaviour and equation (2) attractiveness (though the units of flow will probably be trips and not money). Equation (3) then counts total trips. Equation (4) still represents costs of supply. Versions of equations (7)-(12) still hold.

The substitute mechanism in this case, at least notionally and possibly in practice, would be to find $\{Z_j^{nk}\}$ by some optimisation method which reflected the policies involved. This involves specifying a function which represents the benefit side of the policies, say $B_j^{nk}(\{Z_j^{nk}\}, \ldots)$. Then we can write

$$\max_{\{Z_j^{nk}\}} \quad L = \sum_{jng} (B_j^{nk} - C_j^{nk}) \tag{13}$$

subject to equations (1) and (2) as constraints. It is also likely that other constraints would be added. In particular, following Leonardi (1981), it can be argued that a public authority will maximise net benefits while spending up to its total budget, b, say:

$$\sum_{jnk} C_j^{nk} < B \tag{14}$$

is therefore an additional constraint (with C_j^{nk} then removed from the objective function). There may also be further constraints to represent the operations of this kind of planning system.

Because B_j^{nk}, C_j^{nk} and the various terms in the constraints are functions of $\underline{h}$, $\underline{p}$, $\underline{q}$, $\underline{V}$ and $\underline{c}$, the equations (7)-(12) also, in effect, function as constraints. It can then be seen that the whole system is a highly nonlinear one. In some cases, the nonlinearities can be removed from the constraints and added into the objective function using so-called embedding theorems (Coelho and Wilson, 1977, Coelho, Williams and Wilson, 1978). For example, if (13) is replaced by

$$\underset{\{Z_j^{nk}, Y_{ij}^{mnk}\}}{\text{Max}} \quad L = \sum_{jnk} (B_j^{nk} - C_j^{nk})$$

$$- \sum_{ijmnk} Y_{ij}^{mnk} \log Y_{ij}^{mnk}$$

$$- \beta \sum_{ijmnk} Y_{ij}^{mnk} c_{ij}$$

$$- \alpha \sum_{ijmnk} Y_{ij}^{mnk} \log W_{ij}^{mnk} \tag{15}$$

and the constraint

$$\sum_{jnk} T_{ij}^{mnk} = X_i^{mk} \tag{16}$$

added, then because of the addition of entropy and other relevant terms into the objective function, it can be shown that Y_{ij}^{mnk}, which is now a variable in the optimisation problem, satisfies a suitable spatial interaction model equation.

It remains an open and interesting research question as to whether the two formats, differential equations and optimisation problem, can be linked - perhaps through the iterative procedures which are commonly used to find solutions in each case and by using shadow prices or some other quasi prices (implicit in the benefit measures) in the second. In the private system case, the suppliers budgets are, in effect, determined by the X_i^{mk} terms. So another possibility is to 'transmit' suppliers cost, notionally, to users through an accounting system and then to use these as quasi-budget totals. This can be done through the implicit accounting system provided by the interaction terms. The public model could then possibly be run in the same way as the private model.

4. Examples: from old to new

In this section, we consider four examples of 'classical' theory and show how the ideas can be recast in the framework of section 3. We take, in turn, von Thunen's (1826) model of agricultural location, Weber's (1909) model of industrial location, Alonso's (1960, 1964) model of residential land use and Christaller's (1933) central place theory.

von Thunen's theory is based on continuous space and, as is common in such cases - and as we will see again with central place theory, the geographical analysis task is seen as one of charting boundaries, in this case to demarcate different agricultural land uses. The resolution levels are relatively coarse. The approach appears at first sight to be comprehensive because all land is dealt with. But in practice this is not the case because of the single centre and inelastic demand assumptions: the process of farmers competing to serve different centres is missing. The most striking success of von Thunen's model, however, is in introducing the theoretical concept of bid rent whose uses, as we will see later, extend beyond agricultural location. Meanwhile, we explain these central ideas, but recast in discrete zone format.

Let Z^k_j be the yield of crop k at location j, p^k its unit price, c^k_j the unit production costs at j, r^k the unit transport costs (per unit of distance) and d_{ij} the distance from j to the market at i. Z^k_j can be considered to be a function of the intensity of farming measured by the unit cost of inputs, c^k_j and so could be written $Z^k_j(c^k_j)$. von Thunen's key idea was then to take

$$E^k_j = Z^k_j(c^k_j)(p^k - c^k_j - r^k d_{ij}) \tag{17}$$

as the profit to be derived per unit area from k at j. This can also be interpreted as the maximum rent a farmer would be prepared to bid for k at j. In circumstances of perfect competition, E^k_j would have to be paid as rent to the owner of the land.

von Thunen's analysis can now be implemented. Essentially it involves finding the k (and c^k_j) which maximises E^k_j at each location. He, of course, worked with continuous space. Here, we have added the zone label j. His analysis produces the well known 'rings' of agricultural land use. With the discrete zone system introduced here, such rings would be reproduced in approximate form (if there was one market centre, i): that is, if k (and c_j^k) were assigned to each j such that (17) was maximised in each case. Indeed, Stevens (1968), by shifting to discrete zones, shows how the model can be represented in mathematical programming terms.

This constitutes a route into location theory via rent and land use. The rent function incorporates the main revenues and costs. We can use the approach offered by equations (5) and (6) to collect these together in a different way and then first reproduce von Thunen's results and secondly generalise them. The crucial first step is to return to the discussion of land rent in section 3. Define

$$\rho_j = \operatorname*{Max}_k \; E_j^k / L_j \tag{18}$$

where E_j^k is given by (17) and L_j is the land area in zone j (assuming all other uses can be neglected). The procedure implied by (18) is one of land owners seeking the highest bidder and being able to charge that as rent. Note that ρ_j is independent of k. The bid rents in (17) would have to be calculated on the basis of a known yield function $Z_j^k(c_j^k)$, varying over j and k. For the time being, we also assume a fixed set of cost inputs, $\{c_j^k\}$.

We can now assume that the rent is paid and write down revenue and costs (as in equations (3) and (4)) for farmers in zone j who are considering crop k:

$$D_j^k = Z_j^k p^k \tag{19}$$

$$C_j^k = Z_j^k(c_j^k + \rho_j + d_{ij}r^k) \tag{20}$$

The growth of $\{Z_j^k\}$ will be given by (cf. (5))

$$\dot{Z}_j^k = \varepsilon^k[D_j^k - C_j^k]\, Z_j^k \tag{21}$$

and the equilibrium condition will be

$$D_j^k = C_j^k \tag{22}$$

We will only have $D_j^k > C_j^k$ (for reasonably large values of Z_j^k) when k is the optimal use in the von Thunen sense and (22) will also only be satisfied on the same basis. These results follow from the definition of ρ_j in (18).

At present, we are assumming fixed and given $\{c_j^k\}$, and we are also assuming that everything which is produced can be sold at the market. The simplest assumption is that c_j^k is fixed so that E_j^k in (17) is a maximum with respect to it. In this case, the results we derive from an application of the general framework of section 3 simply reproduce von Thunen's results without adding to them (except that solving (21) from a position of disequilibrium would demonstrate something about the *path* to equilibrium). The next step in the argument is to generalise the von Thunen model.

The most obvious extension is to introduce several markets, i, and prices p_i^k which vary by location of these (perhaps in relation to local demand). Let X_i^k be demand at i, initially fixed, and let Y_{ij}^k be the flow of k from j to i. We might hypothesise that, say

$$Y_{ij}^k = A_i^k X_i^k Z_j^k e^{-\beta^k c_{ij}} \tag{23}$$

where

$$A_i^k = 1/\sum_j Z_j^k e^{-\beta^k c_{ij}} \tag{24}$$

The revenue and cost equations would then become

$$D_j^k = \sum_i Y_{ij}^k p_i^k \tag{25}$$

$$C_j^k = Z_j^k(c_j^k)[c_j^k + \rho_j + \sum_i d_{ij} r^k Y_{ij}^k] \tag{26}$$

The von Thunen surplus from k at j would be

$$E_j^k = Z_j^k(c_j^k)[\sum_i Y_{ij}^k p_i^k - c_j^k - \rho_j - \sum_i d_{ij} Y_{ij}^k] \tag{27}$$

and ρ_j would still be defined by (18). $\{c_j^k\}$ would also be optimised in relation to (27) as before.

The problem as a whole, however, is now more difficult (and interesting) than before, essentially because of the influence of possible markets and the fact that Y_{ij}^k depends on Z_j^k, and on all the other Z_j^k's (in effect representing a competitive process). Thus, solving the equilibrium equation for $\{Z_j^k\}$ is now non-trivial; and, worse, E_j^k in (27), and hence ρ_j, will now depend on the $\{Z_j^k\}$ pattern in what is likely to be quite a sensitive way. The problem in this form can be made obviously more complicated by making X_i^k a function of p_i^k. This is also a matter for further exploration.

There remains the second kind of difficulty (with land use models) mentioned in section 3: the problem of taking land area into account and, in the case of a system of relatively large zones, of handling multiple land use in a zone. We have to find a way of letting costs rise very steeply when the land runs out in a zone. In other words, when $Z_j^k(c_j^k)$ reaches a certain level such that the land used exceeds L_j, then an additional 'cost', $\hat{c}_j^k$ has to be imposed. Let $a^k(c_j^k)$ be the land-used per unit yield at level of intensity c_j^k. Then the additional cost should be

$$\hat{c}_j^k = \begin{cases} b^k & \text{if } a^k(c_j^k)Z_j^k(c_k) > L_j \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

$a^k(c_j^k)$ is the amount of land used to generate a unit of crop k at j when the unit intensity of input is c_j^k. b^k is a suitable constant. If necessary, b^k could be replaced by $b^k Z_j^k$ which would ensure not only a substantial immediate increment, but also rapid further increases.

Where multiple uses are possible, the condition would have to relate to these:

$$\hat{c}_j^k = \begin{cases} b^k & \text{if } \sum_k a^k(c_j^k)Z_j^k(c_j^k) > L_j \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

$\hat{c}_j^k$ should then be added to the c_j^k costs in (26):

$$C_j^k = Z_j^k(c_j^k)(c_j^k + \hat{c}_j^k + \rho_j + \sum_i d_{ij} r^k Y_{ij}^k) \tag{30}$$

and this would prevent further development via (21).

What is now needed is a set of numerical experiments to investigate the range of patterns produced and these are reported elsewhere (Wilson and Birkin, 1983-A).

As a second example, we can examine Weber's approach to industrial location. What is commonly referred to as the Weber model deals with the location of a single firm in continuous space. The approach is therefore obviously highly partial (albeit at a fine level of resolution). Weber was well aware of the limitations of his approach and his 1909 book contains illuminating passages which form the basis for something more comprehensive. Unfortunately, he did not have the mathematical tools available to implement his ideas. Our aim now is to show how the methods of section 3 begin to provide the basis for a solution to many of the problems he formulated and lead the way to a comprehensive model of industrial location. We first present a version of the original Weber problem and then a more modern version of the same problem. We can then extend the latter into a more comprehensive model.

The original model relates a firm to two material sources M_1 and M_2 and a single consumption point C. Let (x,y,z) be the coordinates of the firm and let $a_1(x,y,z)$ be the distance from M_1, $a_2(x,y,z)$ the distance from M_2 and $a_3(x,y,z)$ the distance from C. Then if w_1 is the weight of materials needed from M_1 and w_2 from M_2 to produce w_3 for C, and if transport costs vary with weight and distance only, (x,y,z) is to be found to minimise total transport costs:

$$\underset{(x,y,z)}{\text{Min}} \quad z = \sum_i a_i(x,y,z)w_i \tag{31}$$

There are algorithms for finding (x,y,z) - see Scott (1971) for a review.

The first task is to examine a multi-firm discrete zone version of the model - which is known in the literature as the 'Weber problem on a network' since the discreteness is taken as a set of nodes which, for our purposes, can be taken as zone centroids. The model, which is usually referred to in the literature as the p-median problem, turns out to be more appropriate for certain kinds of public facility location problems rather than industrial location. It can be written:

$$\underset{\{\lambda_{ij},x_j\}}{\text{Min}} \quad z = \sum_{ij} O_i\lambda_{ij}x_jc_{ij} \tag{32}$$

subject to

$$\sum_j \lambda_{ij} = 1 \tag{33}$$

$$\sum_j x_j = p \tag{34}$$

$$\lambda_{ij}, x_j = 0,1 \tag{35}$$

This locates p facilities to minimise total transport costs. Sources O_i are assigned to the nearest facility. As noted above, this problem is more appropriate for certain kinds of services for example when a set of 'needs', $\{O_i\}$ have to be met. (If some of the O_i are taken as consumption points, then it is more like an industrial location model, but in that case connections between inputs and outputs need to be established.) To extend Weber's problem we need to ensure that the correct (if fixed) demands reach consumption points. Let S_ℓ be a set of nodes which supply input material ℓ and let V be a set of consumption points with D_j of product demanded at the node j from this set. Suppose p factories are to be located and there is no constraint on the amount of material to be taken from any material source. Let w_ℓ be the weight of material ℓ needed to produce a *unit* weight of final product. Then consider the following problem:

$$\min_{\{\lambda^\ell_{ij}, \lambda^p_{ij}, x_j, w^p_j\}} Z = \sum_{\ell i \in S_\ell, j} w_\ell \lambda^\ell_{ij} w^p_j x_j c_{ij} + \sum_{jk \in V} w^p_j x_j \lambda^p_{jk} c_{jk} \tag{36}$$

$$\sum_j \lambda^\ell_{ij} = 1, \ i \in V_\ell \tag{37}$$

$$\sum_j \lambda^p_{jk} = 1 \tag{38}$$

$$\sum_j x_j = p \tag{39}$$

$$\sum_{k \in V} w^p_j x_j \lambda^p_{jk} = D_k \tag{40}$$

$$\lambda^\ell_{ij}, \lambda^p_{ij}, x_j = 0, 1 \tag{41}$$

where w_j^p is the total weight of product at j, λ_{ij}^{ℓ} assigns sources to factories and λ_{jk}^{p} assigns factories to consumption points. This is an obvious extension of the p-median problem. Sources still come from the nearest facility, factories send to the nearest consumption point, transport costs are minimised. It would be a straightforward further addition to add inequality constraints on material supply points if there were limits.

It is clear that the model can be easily extended by a shift to a discrete zone system, but, as presented, it is still limited in important respects. The spatial flow patterns are likely to be more dispersed than the 0-1 $\underline{\lambda}$ variables suggest. But above all, the focus on minimising transport costs is restrictive. It relies on assumptions about fixed demand and spatially invariant prices. By switching to the section 3 framework, we can easily make all these extensions. The procedure for doing it will be now be obvious.

The flow array $\{Y_{ij}^{mnk}\}$ in equation (1) of section 3 can be used to describe both inputs and outputs in industrial location. The 'sectors' m and n can be industrial sectors, but also labour (for inputs), final demand (for outputs) and so on. A decision has to be made about how to model prices, and if we assume that all pricing is fob, so that purchasers pay transport costs, then we can specify costs and revenues for the production of k in n at j as follows:

$$C_j^{nk} = \sum_{imk'} Y_{ij}^{mnk'}(p_i^{k'} + c_{ij}^{k'}) \tag{42}$$

where the summation is over all inputs k', and

$$D_j^{nk} = \sum_{im} Y_{ji}^{nmk} p_j^k \tag{43}$$

for the sale of k to all m in all i. Equations (5) and (6) can then be applied directly:

$$\dot{Z}_j^{nk} = \varepsilon^{nk}(D_j^{nk} - C_j^{nk})Z_j^{nk} \tag{44}$$

with equilibrium conditions

$$D_j^{nk} = C_j^{nk} \tag{45}$$

The next step is the spelling out of the spatial interaction array $\{Y_{ij}^{mnk}\}$. For the present, we need simply to note that the elements will be nonlinear functions of the production totals $\{Z_j^{nk}\}$ and should be constructed so that known input-output relations are preserved. Scale economies can now be taken into account because the prices $\{p_j^k\}$ can be made functions of production levels of k in j (though, in this representation, this can only be done in an approximate way because we are dealing with sectors in a zone rather than with individual firms). Agglomeration economies can also be handled through the role of the c_{ij}^k terms in (42) (and hence in (44) and (45)) and through the spatial interaction parameters in the model for $\{Y_{ij}^{mnk}\}$. Thus two key features which could not be incorporated in the Weber approach can now be tackled. The models which result are of immense complexity and the first stage is understanding their capabilities is to carry out numerical experiments. For this example, these are reported elsewhere together with a more extensive review of alternative theories of industrial location when viewed against the section 3 framework (Birkin and Wilson, 1983, Wilson and Birkin, 1983-B).

As a third example, we turn to residential location and housing supply. The 'classical' models are those of the Chicago sociological 'ecologists' school - based on concepts of invasion and succession of territory. Residential structure in cities is then seen as having a ring structure (Burgess, 1927) with a sectoral structure superposed (Hoyt, 1939) and the possibility of multi-nucleated structures (Harris and Ullman, 1945). These theories are essentially descriptive but they do give some idea of the patterns we might seek to reproduce. These patterns have been recorded and reviewed using the techniques of social area analysis which also provides a useful background (cf. Rees, 1979, for a good review). The ecologists approach was based on continuous space and, as in von Thunen's work, focussed on demarcating land uses and identifying boundaries. It operates at a coarse level of resolution, however, and so does not really allow for mixing of different types of residential land use. The approach is, however, comprehensive. The important breakthrough in theory came with Alonso's (1960, 1964) development of von Thunen's ideas in this context: he used the theory of consumer's behaviour and the theory of the firm to calculate the bid rents which householders or firms could make for land to achieve given levels of utility or profit respectively.

The consumer utility functions can be seen as representations of preferences for housing.

Alonso's achievement was not only in developing the bid rent concepts in this context, but also in showing that the market could be cleared: highest bid rents 'accepted' to determine a land price surface; utilities and profits maximised. The approach was also based on continuous space and a fine level of resolution, and this made the theory very difficult to operationalise. An operational version was, however, skillfully constructed by Herbert and Stevens (1960), essentially because they were able to switch to a discrete zone representation and to consider people in groups. They were then able to use a mathematical programming formulation to represent market clearing. Let T_i^{kn} be the number of type n people in type k housing in zone i. Let b^{kn} be the bid rents (or preferences) for type n people for type k houses and let p_i^{kn} be the price to be paid *exclusive of site cost* (but *including* transport costs). Then, if s_{kn} is the amount of land, on average, used by a type n household in a type k house, if there are P^n type n people, and if L_i is the land area in zone i, their model is:

$$\max_{\{T_i^{kn}\}} \quad Z = \sum_{ikn} T_i^{kn}(b^{kn} - p_i^{kn}) \tag{46}$$

subject to

$$\sum_{kn} s_{kn} T_i^{kn} < L_i \tag{47}$$

$$\sum_{ik} T_i^{kn} = P^n \tag{48}$$

At the end of the 60's, disaggregated spatial interaction models of residential location became available (Wilson, 1969) and these were then integrated with the Alonso-Herbert-Stevens approach soon afterwards (Senior and Wilson, 1974). The key new theoretical idea which permitted greater realism to be introduced into economic models was the use of the entropy concept to represent *dispersion* within perfect markets. For present purposes, we simply need to know that, for appropriately defined variables, a spatial interaction model of the form (1) can be built and that this could be given an economic interpretation. If T_{ij}^{kw} is the number of type-w people who live in type k houses in zone i while working in j, then this takes the particular form

$$T_{ij}^{kw} = B_j^w E_j^w (W_j^k)^{\alpha^w} e^{-\beta^w c_{ij}} \quad (49)$$

for job distribution $\{E_j^w\}$ and suitably defined attractiveness factors, $\{W_j^k\}$. B_j^w is a balancing factor and α^w and β^w are parameters. It turns out that $\alpha^w \log W_\ell^k$ can be interpreted as the benefit to be gained from living in a type k house in zone i and this is the basis for reconciling the economic and spatial interaction models. Equation (49) essentially takes the same form as (1), with demand seen to arise from place of employment, though this assumption can be relaxed quite easily.

We can then expect equations (5) and (6) to take the following particular forms:

$$\dot{H}_i^k = \varepsilon^k \left(\sum_{jw} T_{ij}^{kw} q^w - p_i^k H_i^k \right) \quad (50)$$

and

$$\sum_{jw} T_{ij}^{kw} q^w = p_i^k H_i^k \quad (51)$$

where q^w is the average amount a w-household spends on housing and p_i^k the cost of developing a type k house in i. The real situation is immensely more complicated than this. Housing is long-life stock, used many times over. Equation (50) will apply only to new stock. (The inertia of the old will be enough to cause the ring-like growth described by Burgess.) There are many problems of matching expenditure and 'price' of new and old housing. There are problems in defining attractiveness functions, and so on. (It is some of the terms in the attractiveness function, for example, representing property for social classes to group together, which can produce Hoyt's sectoral differentiation.)

These are all addressed in separate papers (Clarke and Wilson, 1983-A, Wilson, 1983-B). For the present, we will assume that some kind of working model can be built which will incorporate the 'old' theory.

As a fourth and final example at this stage we briefly consider Christaller's (1933) central place theory. This provides an early approach to the modelling of service sectors and also differs from most of the early approaches in that it is comprehensive (though at a relatively coarse level of resolution); but like most other early approaches, it uses mainly a continuous space representation. It can be applied (as originally) to settlement structures or to service systems within towns. The main theoretical idea is concerned with market areas. The assumption is made that market areas of service centres do not overlap and that there is a hierarchy of such areas. The concept of the range of a good can be used to calculate the spacing of centres, though once this has been fixed for centres of a given order, it determines, through the market area geometry of the whole system, the spacing of centres in all other orders. The theory is built on other restrictive assumptions which are difficult to relax: a uniform distribution of population, for example, and uniform plain transport frameworks.

Again, a switch to a discrete zone system and the framework of section 3 immediately produces a workable and much more flexible model. Equations (1)-(6) can be interpreted quite directly for service systems (and indeed it is in this field that there has been most experience of running this kind of model). Y_{ij}^{mnk} in equation (1) can be taken as the flow of type m-people in i to shops (or other service facilities) of type n in j for good or service k. There are, as ever, interesting questions about how to define the attractiveness function in (2) for this case. Z_j^{nk} in (5) is the size of facilities of type n in j for good or service k. The equilibrium values can be obtained using the methods of Harris and Wilson (1978) to solve equation (6). The most recent account of the theory underpinning this kind of model is to be found in Wilson (1983-C).

In all four cases, we have shown how a shift of perspective - to discrete zones, to a comprehensive picture or whatever - can create a model which is more powerful and flexible but which nonetheless fits into the framework of section 3. In most cases, the theoretical content can be incorporated within the framework - the obvious exceptions being the use of non-overlapping areas or whatever which do not accord with any reality. In the final section of the paper, we review the research tasks involved in building on this foundation.

5. Concluding comments: towards further research

In this final section, we connect the argument of this paper to a number of other relevant ideas and attempt to point the way towards further productive research. First, it is important to recognise, of course, that there are many alternative approaches to location theory. Some, such as random utility theory (cf. Williams, 1977), provide alternative theories of consumer's behaviour and can easily be incorporated into the approaches of section 3. Others can be seen as more direct descendents of much of the classical theory which we used as a basis for comparison in section 4. These include, for example, what is often termed the 'new urban economics' (cf. Richardson, 1977, for a good review). These mostly use continuous space representations and though they sometimes operate at fine levels of resolution, the price is usually paid in terms of restrictive assumptions and intractable mathematics. However, as might be expected, more progress has been made with price mechanisms and an important avenue of research is the incorporation of these into the kinds of models sketched in this paper.

The second point to make is a comment about the nature of the models - and this comment will apply to the alternatives too when they are properly formulated. It is clear, even in a formally-specified model, that there will be many nonlinearities and a high degree of interdependence. These properties guarantee the existence of multiple solutions (or even types of solution) for particular values of parameters and exogenous variables. This takes us into the realms of catastrophe theory and bifurcation theory (cf. Wilson, 1981-B) and a concern with critical parameter values at which jumps or other transitions can take place. The evolution of an urban or regional system can then be seen as episodic: periods of smooth change interspersed with more rapid structural change. An important field of research becomes the identification, both theoretically and empirically, of this kind of change.

It is relatively difficult to account for structural change analytically in these models. Significant progress has been made for service systems (cf. Harris and Wilson, 1978, Wilson and Clarke, 1979, Harris, Choukroun and Wilson, 1982), but much more work is needed in fields like residential and industrial location where the nonlinearities are different and more complicated. There are some particularly difficult mathematical problems to solve - particularly the so-called 'backcloth' problem (cf. Wilson and Clarke, 1979). When an analysis is carried out for a particular Z_j^k, the results are dependent on values of all other Z_j^k's, each of which is potentially subject to a similar analysis and for which there are possible jumps.

These difficulties give rise to the third point: that the notion of 'numerical experiments' has become particularly valuable. It is very difficult, looking at a set of algebraic equations like (1)-(16), to imagine the range of spatial configurations which can be generated by them and the ways in which these are transformed at critical parameter values. By running different versions of these models over ranges of parameter values, a more systematic understanding of these possibilities can be achieved. An example for service sectors is provided by Clarke and Wilson (1983-B) and for residential location and housing, as noted in another context earlier, by Clarke and Wilson (1983-A). There is an advantage to understanding in conducting these experiments in relation to simple hypothetical systems, which, for example, are symmetric in various ways; but it is also important to understand the nature of change in real systems. What does become clear, even from this very brief account, is that for any particular system there are a large number of alternative paths of development and evolution. Historical 'accidents' will play a large part in determining which actual path is chosen. Two interesting types of research problem result from this observation. First, a new perspective is added to historical geography: to chart the particular path which is chosen, with reasons why this is the case. Secondly, it is interesting to ask whether bundles of alternative paths can be grouped together in such a way that they constitute a *type* of city, and that, for a certain region in parameter space, a city will be of this type whatever perturbations occur.

Fourthly, we note that some technical mathematical points (simpler than the 'backcloth' problem) need to be solved. In particular, the relationship of programming models to the differential equations and equilibrium equations (and the relationship of the iterative solutions of the last to the other two) need to be articulated.

The fifth point is concerned with the application of the methods to the full variety of major subsystems and then the exploration of the major linkages between them. In section 1 we listed the major subsystems:

agriculture, public utilities, industrial location, private services, public services and housing. Public utilities have some of the same character as public services, and so we can say that we have given an illustrative approach to each in section 3 or section 4. These are two main points then to be emphasised at this stage of the argument. First, although we have argued that section 3 offers a unified approach, it is important to note that, for each major subsystem, the particular features demand much research, and these particularities may be the dominant features in the results which emerge. Section 3 provides a framework only. Secondly, it is important to remember that most major systems will be linked, first through the competition for land and secondly through the transport system. It remains important to develop methods of land use accounting which are appropriate for comprehensive dynamic models and to achieve an effective representation of transport demand and supply.

Sixthly and finally, we recognise that the most important ongoing problem is the articulation of mechanisms of price determination and change whether in the neo-classical mode or another. It is at this stage, more broadly, that alternative theories of social processes can be incorporated. There is an implicit contention in the whole argument which is an important one, and provides an appropriate note on which to end: whatever the underpinning social theory, the linked accounts implied by the array definitions will still be relevant concepts and in most cases, the general framework of theory offered here will still be relevant.

References

Alonso, W. (1960) A theory of the urban land market, *Papers, Regional Science Association, 6,* 149-57.

Alonso, W. (1964) *Location and land use,* Harvard University Press, Cambridge, Mass.

Birkin, M. and Wilson, A.G. (1983) Models of industrial location: a review, forthcoming Working Paper, School of Geography, University of Leeds.

Burgess, E.W. (1927) The determination of gradients in the growth of the city, *Publications, American Sociological Society, 21,* 178-84.

Christaller, W. (1933) *Die centralen orte in Suddeutschland,* Jena; English translation by C.W. Baskin, *Central places in southern Germany,* Prentice-Hall, Englewood Cliffs.

Clarke, M. and Wilson, A.G. (1983-A) A highly disaggregated residential location and housing model: formulation and numerical experiments, forthcoming Working Paper, School of Geography, University of Leeds.

Clarke, M. and Wilson, A.G. (1983-B) Dynamics of urban spatial structure: progress and problems, *Journal of Regional Science, 23*(1), 1-18.

Coelho, J.D. and Wilson, A.G. (1977) Some equivalence theorems to integrate entropy maximising submodels within overall mathematical programming frameworks, *Geographical Analysis, 9,* 160-73.

Coelho, J.D., Williams, H.C.W.L. and Wilson, A.G. (1978) Entropy maximising submodels with overall mathematical programming frameworks: a correction, *Geographical Analysis, 9,* 195-201.

Found, W.C. (1971) *A theoretical approach to rural land use patterns,* Arnold, London.

Harris, B., Choukroun, J-M. and Wilson, A.G. (1982) Economies of scale and the existence of supply-side equilibria in a production-constrained spatial-interaction model, *Environment and Planning A, 14*(6), 823-937.

Harris, B. and Wilson, A.G. (1978) Equilibrium values and dynamics of attractiveness terms in production-constrained spatial-interaction models, *Environment and Planning A, 10,* 371-88.

Harris, C.D. and Ullman, E.L. (1945) The nature of cities, *Annals of the American Academy of Political and Social Science, 242,* 7-17.

Herbert, D.J. and Stevens, B.H. (1960) A model for the distribution of residential activity in urban areas, *Journal of Regional Science, 2,* 21-36.

Hoover, E.M. (1967) Some programmed models of industry location, *Land Economics, 43,* 303-11.

Hoyt, H. (1939) *The structure and growth of residential neighbourhoods in American cities,* Federal Housing Administration, Washington DC.

Lakshmanan, T.R. and Hansen, W.G. (1965) A retail market potential model, *Journal of the American Institute of Planners, 31,* 134-43.

Leonardi, G. (1981) A unifying framework for public facility location problems, *Environment and Planning A, 13,* 1001-28; 1085-1108.

Losch, A. (1954) *The economics of location*, Yale University Press, New Haven.

Lowry, I.S. (1964) *A model of metropolis*, RM-4035-RC, Rand Corporation, Santa Monica.

Lowry, I.S. (1968) Seven models of urban development: a structural comparison, *Special Report 97*, Highway Research Board, Washington DC, 121-45.

Paelinck, J.H. and Nijkamp, P. (1975) *Operational theory and method in regional economics*, Saxon House, Farnborough.

Rees, P.H. (1979) *Residential patterns in American cities*, Research Paper 189, Department of Geography, University of Chicago.

Richardson, H.W. (1977) *The new urban economics: and alternatives*, Pion, London.

Scott, A.J. (1971) *Combinatorial programming, spatial analysis and planning*, Methuen, London.

Senior, M.L. and Wilson, A.G. (1973) Disaggregated residential location models: some tests and further theoretical developments, *Geographical Analysis, 6*, 209-38.

Stevens, B.H. (1968) Location theory and programming models: the von Thunen case, *Papers, Regional Science Association, 21*, 19-34.

von Thunen, J.H. (1826) *Der isolierte staat in beziehung auf land wirstchaft und national okonomie*, Hamburg; English translation, *The isolated state*, C.M. Wartenburg with a translation by P. Hall, Oxford, 1966.

Weber, A. (1909) *Uber den standort der industrien*, Tubingen; translation by Freidrich, C.J. (1929) *Alfred Weber's theory of location of industries*, University of Chicago Press, Chicago.

Williams, H.C.W.L. (1977) On the formation of travel demand models and economic evaluation measures of user benefit, *Environment and Planning A, 9*, 285-344.

Wilson, A.G. (1967) A statistical theory of spatial distribution models, *Transportation Research, 1*, 253-69; also reprinted in R.E. Quandt (ed.) *The demand for travel - theory and measurement*, Heath Lexington, Boston, 1970, 55-82; and in S. Angel and G.M. Hyman (eds.) *Urban fields*, Pion, London, 1976, 162-78.

Wilson, A.G. (1970) Disaggregating elementary residential location models, *Papers, Regional Science Association, 24*, 103-25.

Wilson, A.G. (1974) *Urban and regional models in geography and planning*, John Wiley, Chichester and New York.

Wilson, A.G. (1981-A) The evolution of urban spatial structure: a review of progress and research problems using SIA models, *Bulletin of the Institute of Mathematics and its Applications, 18*, 90-100.

Wilson, A.G. (1981-B) *Catastrophe theory and bifurcation: applications to urban and regional systems*, Croom Helm, London and University of California Press, Berkeley.

Wilson, A.G. (1983-A) Transport and the evolution of urban spatial structure, forthcoming Working Paper, School of Geography, University of Leeds.

Wilson, A.G. (1983-B) The complexities of residential location and housing: the possibilities of model development, forthcoming Working Paper, School of Geography, University of Leeds.

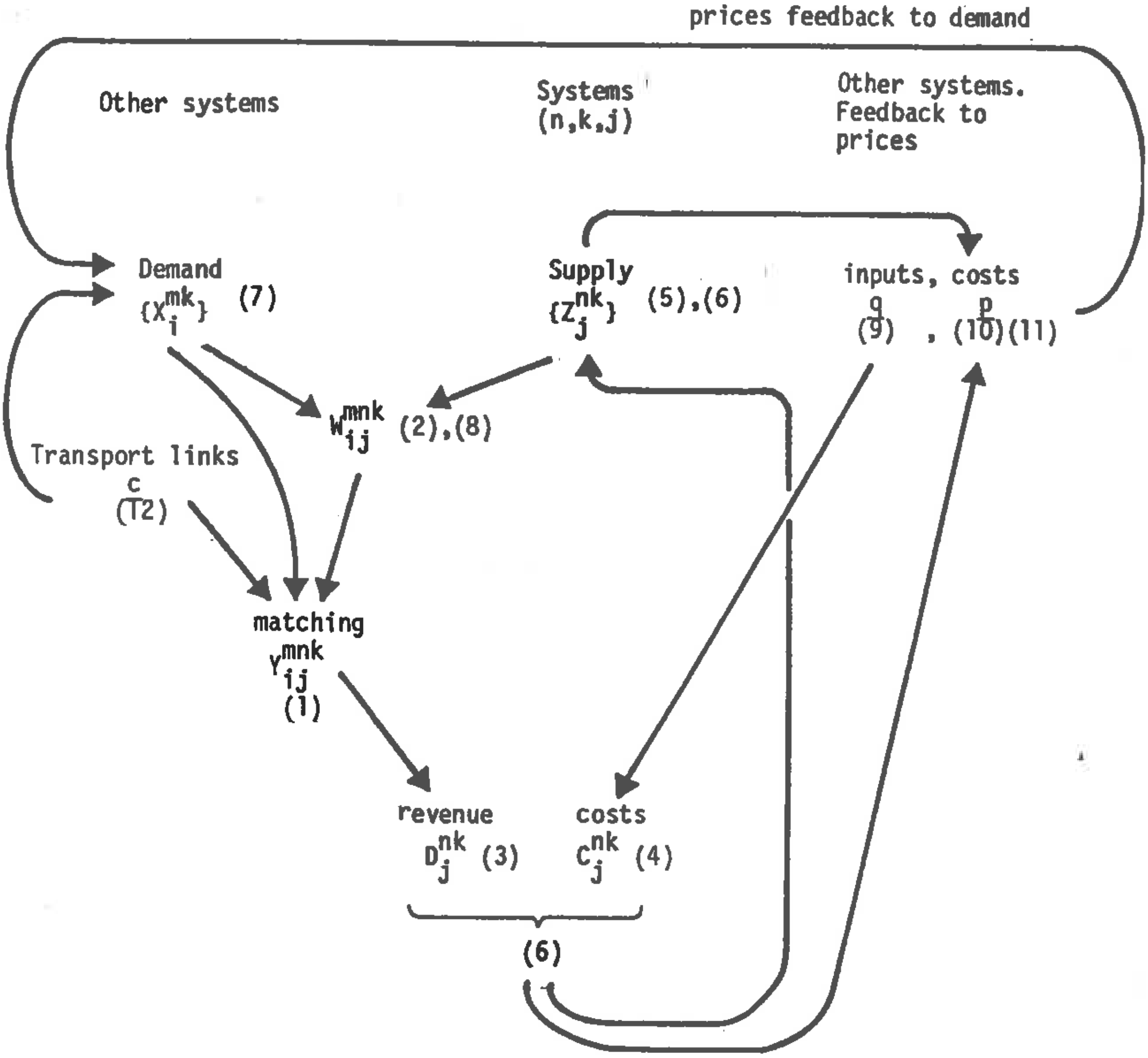
Wilson, A.G. (1983-C) A generalised and unified approach to the modelling of service-supply structures, Working Paper 352, School of Geography, University of Leeds.

Wilson, A.G. and Birkin, M. (1983-A) Models of agricultural land use: some numerical experiments, forthcoming Working Paper, School of Geography, University of Leeds.

Wilson, A.G. and Birkin, M. (1983-B) Models of industrial location: numerical experiments, forthcoming Working Paper, School of Geography, University of Leeds.

Wilson, A.G. and Clarke, M. (1979) Some illustrations of catastrophe theory applied to urban retailing structures, in M.J. Breheny (ed.) *Developments in urban and regional analysis*, Pion, London, 5-27.

Figure 1.



prices feedback to demand
Other systems
Systems (n,k,j)
Other systems. Feedback to prices
Demand $\{X_i^{mk}\}$ (7)
Supply $\{Z_j^{nk}\}$ (5),(6)
inputs, costs
q (9) , p (10)(11)
W_{ij}^{mnk} (2),(8)
Transport links
c (T2)
matching
Y_{ij}^{mnk} (1)
revenue
D_j^{nk} (3)
costs
C_j^{nk} (4)
(6)