WORKING PAPER 254

Some new methods for dynamic geographical modelling: catastrophe theory and bifurcation.

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September, 1979

SOME NEW METHODS FOR DYNAMIC GEOGRAPHICAL MODELLING: CATASTROPHE THEORY AND BIFURCATION

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THE FOUNDATIONS OF THE METHODS

Background

Bifurcation theory is concerned with the way in which the nature of the solutions of differential equations change at certain critical values of the parameters of those equations. It has a long history, going back to the time of Poincaré, for example, but has only been incorporated into a broad framework relatively recently. Catastrophe theory is a newer development - of the last ten years or so - though with earlier pieces of mathematics being important in its development. The theory takes its name from the possibility of systems which it describes exhibiting jump behaviour.

In this brief introductory section, the main ideas of both catastrophe theory and bifurcation theory will be outlined. There is now a voluminous literature on these topics, but the reader is referred only to a number of key works which themselves include much more extensive bibliographies. For an account of bifurcation theory in an applied context, see Hirsch and Smale (1974). In the case of catastrophe theory, it is important to look at the seminal book by Thom (1975), but the more recent books by Zeeman (1977-A) and Poston and Stewart (1978) offer a better introduction. Amson (1975) also provides an excellent introduction in an 'urban studies' context.

Gradient and general systems

Catastrophe theory is concerned with a special class of systems called gradient systems. They can be characterised as follows. Let such a system be described by a set of state variables (or endogenous variables), $\underline{x}$, and a set of parameters (or exogenous variables, sometimes also known as control variables), $\underline{u}$. Then, for a gradient system, there exists a potential function, $F(\underline{x},\underline{u})$ say, which, when maximised (or, if appropriate, with the corresponding changes, minimised), determines the equilibrium state of the system. That is, $\underline{x}$ is given by

$$\max_{\underline{x}} \; F(\underline{x},\underline{u}) \tag{1}$$

If the system is disturbed from equilibrium, its dynamics are described by the differential equations

$$\dot{\underline{x}} = -\frac{\partial F}{\partial \underline{x}} \tag{2}$$

and this type of system takes its name from the gradient vector on the right hand side of the simultaneous equation (2).

More typically, however, the dynamical behaviour of a system is given by a more general set of equations such as

$$\dot{\underline{x}} = \underline{G}(\underline{x},\underline{u}) \tag{3}$$

where the right hand side cannot be written in gradient form.

Catastrophe theory

Let us concentrate on equilibrium states. The maxima of $F(\underline{x},\underline{u})$ will be given by the solutions of

$$\frac{\partial F}{\partial \underline{x}} = 0 \tag{4}$$

These form a set of simultaneous equations in $\underline{x}$ - with as many equations as there are $\underline{x}$-variables - and so can be solved for $\underline{x}$ for each possible value of the parameters, $\underline{u}$. These solutions can be represented geometrically by a surface in $(\underline{x},\underline{u})$-space representing the possible equilibrium states of the system. The two simplest cases with which we shall be mostly concerned involve a single state variable, x, and one or two parameters, given as either u (dropping the subscript, as with the x variable, when there is only one) or as u_1 and u_2. Thus the possible equilibrium states in these cases are represented by either a curve in the two-dimensional (x,u) space or a surface in the three-dimensional (x,u_1,u_2) space. Examples are shown in Figures 1 and 2.

Catastrophe theory is concerned with the singularities of the function F. The maxima, minima and points of inflexion of F are called stationary points. In the example we have considered, the maxima represent stable equilibrium points and the minima and points of inflexion, unstable points. The singularities are *degenerate* stationary points which occur when two or more stationary points coalesce. This gives another clue to the nature of the theory: the interesting features arise out of multiple solutions of (4) which in turn, of course, arise out of the non-linearities in the potential function, F. In the elementary diagrams of the form of Figures 1 and 2, the singular points exhibit themselves where the tangent to the curve or surface is perpendicular to the u-axis (in Figure 1) or the (u_1,u_2)-plane (in Figure 2). This projection of the folds on to this axis or plane gives the set of critical parameter values.

Thom's main results on catastrophe theory are concerned with stability in two senses. First, they specify the nature of the singularities for a wide class of systems, and it is at these points that the system can be unstable in that it can exhibit 'jumps' in its type of behaviour. Secondly, he is concerned with the structural stability of the system as a whole and the form which this can take. For our present purposes, it will suffice to note that Thom's work tells us what the worst possible forms of singularity are for systems described by up to two state variables and four parameters (though, in practice, the number of state variables can be very much greater, but not the number of parameters). For a single state variable and one parameter, the worst kind of singularity is a fold - as shown in Figure 1; for one state variable and two parameters, the worst possible singularities are folds and cusp points - as shown in Figure 2. In the latter case, for $u_1<0$, a section of the equilibrium surface parallel to the (x,u_2)-plane is folded, and the cusp point occurs where the fold in the surface ends. The possible shapes of the surface for the remaining elementary catastrophes are given in the texts cited earlier.

The behaviour of a system is characterised by a trajectory on the equilibrium surface, which is a surface of *possible* states. We can then immediately see by reference to the three sample trajectories on Figure 2 that unusual kinds of behaviour are possible. Suppose that the parameters are changing in such a way as to generate trajectory (1): the value of x changes smoothly until the fold in the surface is reached, and then it changes discontinuously because it must 'jump' to the lower surface. Trajectory (2) shows the reverse change, but the jump is not forced until a later point and so (1) and (2) together demonstrate a hysteresis effect. (This effect depends on a further assumption known as the 'perfect delay' convention.) Finally, trajectory (3) approaches the cusp point from above and a small change in either direction can take the system smoothly onto the upper or lower surface as shown.

The representations of systems in Figures 1 and 2 are canonical forms: they are generated from potential functions which are a standard form of polynomial. What Thom has shown is that, given some not-very-restrictive conditions, all gradient systems with one or two parameters have equilibrium systems which can be smoothly transformed into the shapes of Figures 1 and 2 respectively. This, potentially, is a powerful result for applied purposes.

Bifurcation theory

Differential equations such as (3) can have many types of solution. These include (i) stable equilibrium points, (ii) unstable equilibrium points, (iii) saddle points - a special types of unstable equilibrium, (iv) closed orbit periodic solutions, (v) limit cycles - which are convergent oscillations, (vi) divergent oscillations or (vii) chaotic behaviour. Bifurcation theory, is like catastrophe theory in one important respect: it is concerned with critical values of system parameters at which some unusual system behaviour can occur - in this case, the transition from one type of solution to another. There is one particular case when the results may appear similar to those of catastrophe theory: in the special cases involving the appearance or disappearance of stable equilibrium points. Indeed, it could be said that catastrophe theory is a special case of bifurcation theory in the sense of the latter used here, but it is useful to distinguish it because of the special results available.

The basis of applied work

Zeeman (1977) usefully identifies six steps for building dynamic models in the sort of framework described here. They can be summarised as follows. *First*, construct the surface of possible equilibrium states and examine its singularities. This has the effect of making what has usually been called comparative statics more interesting than is often thought: because the behaviour of state variables can be discontinuous for small and smooth changes in parameters. *Secondly*, construct the 'fast equations'. These are of the form (4) and represent the return to equilibrium of the state variables, $\underline{x}$, for fixed values of the parameters. *Thirdly*, specify the 'slow equations': the differential equations for the parameters, $\underline{u}$. The solution of these determines the particular trajectory or the equilibrium surface. The *fourth* step involves building in any feedback between the fast and the slow variables, and this has obvious connections to systems analytical approaches to model building. *Fifthly*, recognise the existence of noise and possibly make the models stochastic. And, *sixthly*, seek to build in diffusion effects, possibly by taking two or three space coordinates, together with time, as the 'parameters' of the model.

We will see in relation to examples below how far it has been possible for geographers to progress along this list (which is in developing order of difficulty). We also note one useful distinction which has been introduced implicitly here: that the state variables are the fast variables, and the parameters the slow. The way in which we categorise variables in this respect, therefore, is determined at least partly by their relative rates of change in time.

APPLICATIONS IN GEOGRAPHY: GENERAL CONSIDERATIONS

Do appropriate phenomena exist?

We can begin by asking whether the unusual types of behaviour discussed above are in fact observed for geographical systems. Brunet (1968), in work which must have largely preceded any knowledge of catastrophe theory, documented a range of examples of discontinuities in geographical phenomena - both physical and human. Amson (1975) identified a number of possibilities which include clustering in a previously dispersed society, the rapid increase in population of a city, the depopulation of inner city areas, the reversals of ethnic zoning patterns (the 'tipping' phenomenon discussed, for example, by Goering, 1978; and Woods, 1977). Atkinson (1976) and Wagstaff (1976) both explored the possibilities at this general level and the former even added a couple of examples of possible divergence phenomena (divergence from zero population growth) to the lists of possible jumps and hysteresis effects (the last of which, in Atkinson's paper included a hysteresis-representation of spiralling, as for retail systems in Agergard, Olsen and Allpass, 1970). So we can conclude that there are certainly possibilities, especially if we recognise that discontinuous change, in applied work, need not be interpreted to mean instantaneous, but simply very rapid relative to 'before and after' rates of change.

Methods of application

There has been much criticism of catastrophe theory, notably by Zahler and Sussman (1977) and although they in turn can be criticised (Zeeman, 1977-B), much of the response is in terms of applications in the natural sciences and there are some justified doubts about the nature of the applied work in the social sciences (Stewart, 1977). Wagstaff has also been criticised from a Marxist perspective (Day and Tivey, 1977), though they in turn may note the advocacy of the positive use of catastrophe theory in dialectical thought (Zwick, 1978). Many of the criticisms turn on the representation of systems in canonical form - as in Figures 1 and 2 - while the real systems may differ from such forms by very complicated transformations - and in failures to recognise the 'local' nature of Thom's theorem. Many of the applications of catastrophe theory in geography reviewed briefly below are open to this kind of criticism. What we can advocate is that catastrophe theory (and bifurcation theory which does not suffer from the same kinds of criticism) can be interestingly used in geography in at least two ways: firstly, to remind us that unusual forms of dynamic behaviour exist; and secondly, to attempt to construct directly the mechanisms of change within models which it may then be possible to interpret in terms of catastrophe theory.

Scale

It is nearly always important in defining geographical systems for modelling purposes to take care in the treatment of scale. Dynamic modelling is no exception. Indeed, in the case of applications of catastrophe theory, most examples in geography are at either micro or macro scales, and mostly the latter, because it is at coarse scales that it is more likely to be possible to characterise the system in terms of a small number of variables. The interesting geographical scale if often at the meso level, where spatial structure is distinguished in some detail, and where the problems of multiplicity of variables has to be faced. For these reasons, therefore, the account of geographical examples which follow are classified according to scale, beginning with micro and macro, in turn, and ending with meso.

GEOGRAPHICAL EXAMPLES

The micro scale

Many of the social science applications of catastrophe theory have been to individual behaviour (see Zeeman, 1977-A for examples). In the geographical and related literature, however, there are relatively few. Here we mention only two. One, due to the author (Wilson, 1976-A), purports to represent individual modal choice, originally based on the fold catastrophe, but in a later minor extension, (Wilson, 1979-A), employing the cusp. In Figure 2, let x be a variable representing choice of mode so that if x is positive, this represents mode 1, if negative, mode 2. u_1 can be taken as a habit factor: when it is negative, habit effects occur and increasingly so as the modulus of u_1 increases; u_2 is taken as a measure of cost differences between the two modes (c_2-c_1, say). Then, when no habit effect occurs, any switch of mode will occur as soon as one mode becomes cheaper than another; but when u_1 is negative, the change will only occur after some interval in favour of the cheaper mode. This is equivalent to trajectory 1 on the figure. But if there is a reversal, and perfect delay applies, this will follow trajectory 2 and the habit factor has created a hysteresis effect. Such an effect has also been proposed from different theoretical foundations by Goodwin (1977) and by Williams and Ortuzar (1979). Empirical evidence for its existence has been discovered by Blase (1979). In this case, the example was developed to help the author to learn something about catastrophe theory rather than as a model with serious pretensions, but this does then have the useful effect of uncovering a phenomenon, in this case hysteresis, which can be built into other models.

The second example is due to Dendrinos (1978). He shows that the well-known speed flow curve for traffic, which has a fold, can be considered to arise from assumptions about the utility maximising behaviour of drivers in balancing the effects of speed and congestion.

The macro scale

Most of the examples at the macro scale are concerned fairly directly with modelling urban development, measured by appropriate state variables, in terms of a number of parameters.

The earliest work in this field seems to be that of Amson (1972-A, 1972-B, 1973, 1974, 1977) which is also reviewed by Kilminster (1976). He uses a continuous representation of the urban system and is thus able to reduce its characteristics to densities, rents and the like. His work is important in that he does construct the mechanisms of change directly. He offers alternative models of the equilibrium surfaces of his variables and is then able to interpret the results in terms of catastrophe theory. One particularly important feature is that one of his models is a non-canonical cusp surface where it is possible to give the transformation explicitly to transform it into standard form.

There are a number of authors who use standard catastrophe equilibrium surfaces without specifying in detail the way in which they are derived. Casti and Swain (1975) have as a state variable the level of a settlement in a central place hierarchy and their two parameters, generating a cusp surface, are population and disposable income per capita. Mees (1975) has population as a state variable and four parameters: the difficulty of transport, the average productivity, the difference in productivity between town and country and a crowding factor, and this involves him in using the butterfly catastrophe. Isard (1977) also has population as a state variable and his control variables are increases in productivity and marginal welfare per head. He attempts to interpret the potential function (which he takes in canonical form) as a welfare function in the tradition of economics. The most complicated of the applications of the standard catastrophe surfaces is that due to Dendrinos (1977-B) and he has also worked with more explicit models of disequilibrium (1977-A, 1978-B) following work by authors like Richardson (1974, 1975), Fujita (1976) and Huang, Mueller and Vertinsky (1976). In his 'surface' model, he has two state variables (the quality of the housing stock and the utility of the residents) for a study of slum formation and four control parameters (per capita income, the rate of return on investment, the social rate of discount and the population to capital stock density), and this involves the use of one of the umbilic surfaces - the 'mushroom'.

Wagstaff (1978) applies the cusp catastrophe to a development problem in historical geography, and his study is perhaps most notable for his use of the value of the potential function as the variable he wants to predict in his model. The state variable is used to represent location and the parameters are threat of attack and quality of agricultural land respectively.

It is clear that most of the examples discussed briefly here involve theoretical work of a speculative nature and there is rarely, if ever, adequate data to test the theories. However, considerable ingenuity and variety of technique has been displayed and some of these explorations may prove fruitful. At present, of course, they cannot all be correct: if all the parameters which have been identified and used did play truly independent roles in urban or regional development, then they total many more than four and would thus take us outside the realms of elementary catastrophe theory.

The meso scale.

The heart of geographical theory is concerned with location, flows and networks and these features need to be represented in models in some detail. This means that, inevitably, a large number of variables is involved and, more importantly, the interesting behaviour arises out of the interdependence of these variables. It is this, together with its non-linear nature, which generates bifurcation behaviour.

In this section, we concentrate mainly on one example as an illustration and then discuss its wider implications and some alternative ways of approaching it. There are two main possibilities for the spatial representation of a geographical system: to treat space continuously (as in Amson, 1974), or to use a discrete zoning system. Here, we will use the latter which avoids restrictive assumptions about exogenously given spatial quantities. The example is based on the Huff (1964) and Lakshmanan and Hansen (1965) model and builds on early work by Harris (1965). This argument is presented in detail by Harris and Wilson (1978) and some numerical work and further theoretical developments by Wilson and Clarke (1978).

The standard model can be taken as:

$$S_{ij} = \frac{e_i P_i W_j^{\alpha} e^{-\beta c_{ij}}}{\sum\limits_k W_k^{\alpha} e^{-\beta c_{ik}}} \tag{5}$$

where S_{ij} is the flow of retail sales from it to j, e_i is the per capita income in zone i, P_i is the population of zone i, W_j is the size of shopping facilities in zone j and is taken as a measure of attractiveness, c_{ij} is the cost of travel from i to j and α and β are parameters. The total revenue attracted to j is D_j given by

$$D_j = \sum_i S_{ij} \tag{6}$$

The standard uses of this model are well known - see for example, Wilson (1974), Chapter 4. Here, we are interested in adding a hypothesis which will enable us to analyse the dynamics of the development of the structural variables $\{W_j\}$ - the evolution of the pattern of shopping centres. Suppose for a particular zone j that W_j expands if D_j exceeds some cost of supply, kW_j (where k is a unit cost), and decreases otherwise. Then we have behaviour of the form (Wilson, 1976-B):

$$\dot{W}_j = \varepsilon(D_j - kW_j) \tag{7}$$

for a suitable constant, ε, with equilibrium conditions

$$D_j = kW_j \tag{8}$$

If we substitute for S_{ij} from (6) into (7) and for the resulting D_j into (8), we get a set of non-linear simultaneous equations to solve for the equilibrium values of $\{W_j\}$. These are

$$\sum_i \frac{e_i P_i W_j^{\alpha} e^{-\beta c_{ij}}}{\sum_k W_k^{\alpha} e^{-\beta c_{ik}}} = kW_j \tag{9}$$

The left hand side represents revenue and the right hand side cost. These can be plotted as separate functions of W_j. Their intersection then represents the equilibrium value of W_j and we can use (7) to investigate stability. (An investigation of the equilibrium properties of corresponding terms in an intervening opportunities model has been carried out by Chudzyuska and Slodkowski, 1979.) This process is exhibited in Figure 3 for the case $\alpha>1$. The ringed points are the stable equilibrium points. There are two cases divided by a critical value of k: for low values, the line always intersects the logistic revenue curve; for high values, it does not. The critical value, which is peculiar to this zone and hence is labelled k_j^{crit}, is exhibited when we plot the possible stable optima against k in Figure 4. This is reminiscent of the fold catastrophe in Figure 1.

Since zero is always a possible state, this means that there is not necessarily any development in a zone, but the reverse is not true. (And also note that the 'zero' states have been added to the fold diagram in Figure 4.) If k is higher than k_j^{crit}, then no development is possible. We might call these the DP (development possible) and NDP (no development possible) states respectively. Thus, what we get from this analysis is some insight into the conditions which must be satisfied in a zone as to whether development is possible or not and on the size of the development if it takes place. One can imagine an evolving urban system and that, as certain thresholds are passed, say either costs being lowered until the cost line intersects the revenue curve or population increasing so that the curve moves up to intersect the line as potential revenue increases, and then that development occurs on the lines implied by this model.

The problem turns out to be a very complicated one for a number of reasons. First, k is not the only parameter which might change: α, β, $\{e_i\}$ and $\{P_i\}$, for example, all count as 'parameters', and change in any one of them implies that the curves in Figure 3 shift relative to each other. All these parameters have critical values for the particular zone, analogous to k_j^{crit}. α=1 is also a critical point, and in this case, not a Thom-like cusp, but something more complicated (Amson, 1979). What is worse, however, is that the above analysis was presented on the assumption that all the other W_j's, that is $\{W_k\}$, $k \neq j$, are fixed. This will not be the case: they will all be varying simultaneously and affecting the situation in other zones through competition. It is argued that, for this reason, the basic problem of understanding the dynamics of the evolution of urban structure is a very difficult one, and something like simulation methods are necessary for further progress. In a sense, this can be taken as an encouraging development for theoretical geography, as it is now faced with problems which are just as hard, theoretically, as those of many other sciences.

Two concluding observations can usefully be made. First, this kind of argument can be applied much more widely: for example, to residential location modelling, though the problem is more difficult; and to a comprehensive model which results from stitching the various submodels together and in which retailing and residential 'fields' interact. (This can lead to a 'domino-effect' of the kind also suggested by Isard and Liossatos, 1978 - and see 1979 for a more detailed treatment.) When these investigations can be completed, then it may be possible to have a fully dynamic and evolutionary central place theory (Wilson, 1977-A, 1979).

Secondly, it is appropriate to draw attention to alternative approaches to this particular problem, where other authors have used simulation methods and sometimes different theoretical frameworks, though there remains a family resemblance. White (1974) was an early exponent of this kind of approach to central place theory, and he examined the intersection of a logistic revenue curve and a cost curve though without investigating stability in the manner presented above. He has also carried out a number of interesting simulations (White, 1977, 1978). Another approach altogether is based on the work of Prigogine and his school in Brussels (see, for example, Nicholis and Prigogine, 1977). This involves setting up the appropriate differential equations, adding driving terms and solving them by simulation methods taking particular note to look for bifurcation points. This includes the building in of stochastic are operating - above we have tended to assume perfect delay in the examples mentioned, and more complicated schemes are possible. Thus, the first area of broad concern can be summarised as related to the specification of the detailed dynamical structure of a model.

The second relates to Zeeman's six steps. Perhaps the two where least progress has been made in geographical modelling are (3) and (6): we very rarely specify u-differential equations in any detail, nor have we pursued the construction of models which incorporate diffusion processes.

Finally, one of Thom's ambitions in the development of applied catastrophe theory was the construction of a theory of morphogenesis - the creation of new structures. Some simple examples have been explored in the geographical field (Wilson, 1978) but the underlying theoretical problems are very difficult. This is another case where explorations in other disciplines may offer new insights for geographers.

The next steps

The first important general conclusion to be drawn is that enough is now known about bifurcation phenomenon to make it worthwhile to examine many of the equations which turn up in geographical theory in this respect. For example, the accounting or kinetic equations used by authors like Cordey-Hayes (1972) and Tomlin (1979) could be investigated in this way if their transition coefficients were made functions of other system variables. Then, the interdependencies and non-linearities which generate bifurcation behaviour would be made explicit. Similar comments apply to spatial population models in which the migration flows are made explicit (cf. Rees and Wilson, 1977) and to many of the equations developed in spatial time series analysis, with ultimate extensions to control theory (cf. Bennett and Chorley, 1979, Bennett, 1979).

In general, we can note that the next steps forward will be difficult for a variety of related reasons. First, the techniques involved are difficult, and just as geographers thought they were coping with the results of the last round of quantifications, there is a demand to study dynamics from a topological viewpoint! Secondly, it is difficult to make progress unless some practical work is carried out. This will usually involve simulation methods and various ways of incorporating approximations to theoretical models. But this way, substantial new theoretical problems emerge as well as new insights (as in Wilson and Clarke, 1978). In the first instance, and perhaps for a considerable time, data will be woefully inadequate for these purposes. Not only is time series data needed, but data which is on a sufficiently fine temporal scale for jumps and other such phenomena to be identified. All this is particularly difficult in relation to potential applications in planning. The emphasis on criticality of parameters, the knowledge of which could then be used either to maintain systems in some state or to encourage them to develop, has an obvious value. But while insights as to potential mechanisms and changes of this type may be valuable, what would be even more so is a knowledge of the actual critical values of parameters, and this may be much harder to come by.

It is perhaps appropriate in a volume which is reviewing progress in the development of quantitative methods to conclude that there is at least one field which, while building on the results of past work, consists of problems which are among some of the most difficult in modern science.

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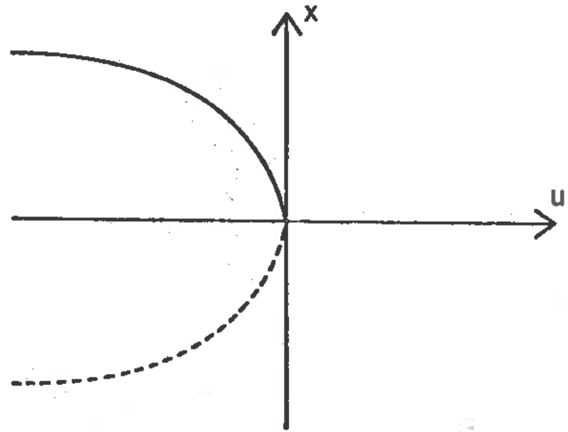
Figure 1.



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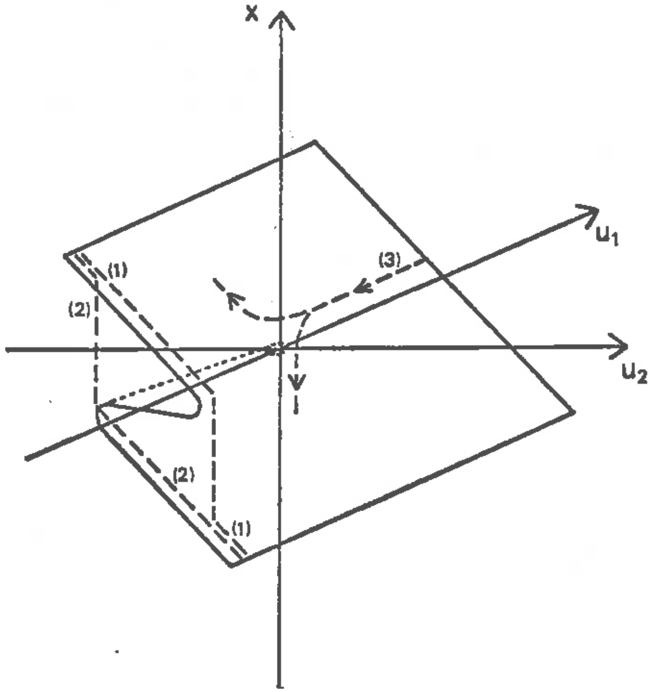

Figure 3.

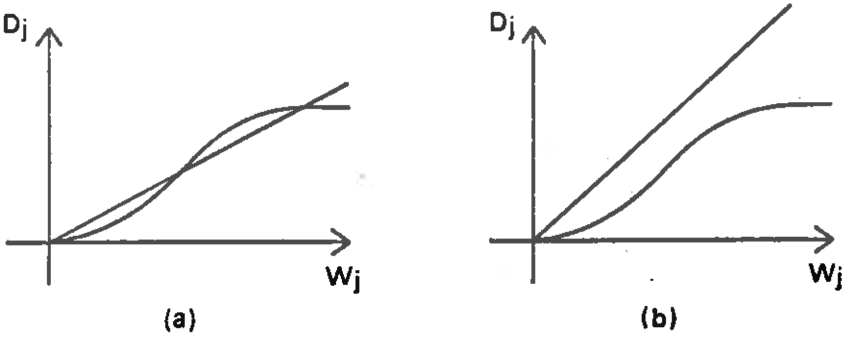

Figure 4.