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STRUCTURAL DYNAMICS AND SPATIAL ANALYSIS : FROM EQUILIBRIUM BALANCING MODELS TO EXTENDED ECONOMIC MODELS FOR BOTH PERFECT AND IMPERFECT MARKETS

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1. Background and outline of the argument

The model presented by Harris and Wilson (1978) can be represented as follows:

$$S_{ij} = A_i E_i W_j^{\alpha} e^{-\beta c_{ij}} \qquad (1)$$

$$A_i = 1/\sum_k W_k^{\alpha} e^{-\beta c_{ik}} \qquad (2)$$

with $\{W_j\}$ being determined so that

$$D_j = C_j \qquad (3)$$

where

$$D_j = \sum_i S_{ij} \qquad (4)$$

and C_j is usually taken for illustration as

$$C_j = k_j W_j \qquad (5)$$

This model can represent a wide variety of situations. A typical example has E_i as the amount of expenditure on retail goods by the residents of some zone i, W_j as the available retail floorspace in zone j (and taken as an attractiveness factor), c_{ij} as a measure of travel cost from i to j, and then S_{ij} is the flow of expenditure from residents of i to shops in j. α and β are parameters. The A_is are balancing factors calculated in (2) to ensure that

$$\sum_j S_{ij} = E_i \qquad (6)$$

The total revenue attracted to j is the sum D_j given in (4) and it is assumed that the cost of supplying retail facilities is given by (5) - or some more appropriately complicated and realistic function. The structural variables $\{W_j\}$ are then obtained by solving the equilibrium condition (3). By substituting for S_{ij} from (1) and (2) into (4), and then for D_j and C_j from (4) and (5) into (3), this condition can be seen to be

$$\sum_i \frac{E_i W_j^{\alpha} e^{-\beta c_{ij}}}{\sum_k W_k^{\alpha} e^{-\beta c_{ik}}} = k_j W_j \qquad (7)$$

which shows explicitly that the variables $\{W_j\}$ are the solutions to a set of interdependent non-linear simultaneous equations.

Harris and Wilson (1978) showed the nature of the bifurcation properties of this model: that the spatial pattern $\{W_j\}$ changes abruptly at certain critical values of parameters, like α and β (or even $\{E_i\}$ and $\{c_{ij}\}$). Examples and further analyses of such changes have been presented by Wilson and Clarke (1979), Clarke and Wilson (1983). This model has been further explored theoretically and empirically by such authors as Phiri (1980), Leonardi (1981-A, 1981-B), Harris, Choukroun and Wilson (1982), Lombardo and Rabino (1983), Rijk and Vorst (1983-A, 1983-B), Kaashoek and Vorst (1984), Chudzynska and Slodkowski (1984), M. Clarke (1984), Dumain, Saint-Julien and Sanders (1984), Roy and Brotchie (1984), Fotheringham (1985) and G. Clarke (1985).

The model can be interpreted as representing retailers adjusting floorspace provision in a system where consumers gain some external benefits associated with W_j-size (measured by W_j^{α} in the interaction model) relative to the costs of travelling (measured by $e^{-\beta c_{ij}}$). Typically, high α and low β will generate $\{W_j\}$-systems with a small number of large centres; and vice versa.

It is now appropriate to consider extending this model in two ways : first, to attempt to make the model more realistic by also considering adjustment of the prices of goods and of land rent as part of the equilibrating process; and secondly, to use ideas which have been successfully deployed in other contexts to incorporate the consequences of market imperfections on the structure-patterns, $\{W_j\}$. These two developments should obviously be applied in combination, but for clarity and ease of presentation, we first consider them separately. In section 2, therefore, we explore the task of making the Harris and Wilson (1978) model more realistically "economic"; and in section 3, we apply entropy-maximising methods to the problem of representing market imperfections. In section 4, we outline the potential range of application of these ideas. This is wide, partly because of the range of different situations, partly because of different possible levels of aggregation in each case. We also note that in all these situations, both ideas can be incorporated simultaneously though we largely leave such developments as an exercise for the reader. The main aim now, therefore, is to capture the essence of the ideas in the simplest model frameworks in which they can be represented.

2. Prices, land rents and perfect markets

For illustrative purposes, let us focus on four kinds of agents :

(i) consumers;
(ii) retailers;
(iii) land owners/developers; and
(iv) suppliers of goods to retailers.

The last category is itself essentially a chain - from wholesalers via different (linked) manufacturers but we consider it a single type for present purposes. We will also include as a fixed constant the cost of capital, labour and so on. Consider, then, the system of interest implied by these definitions exhibited as Figure 1. There are four markets, or balancing operations :

* Consumer - retailer (the spatial balancing of demand for goods and retail outlets)

* Retailer supply of facilities - retailer demand for facilities (retailer balancing of revenue and costs)

* Retailer demand for facilities - land supply (land market)

* Consumers - suppliers of retail goods (aggregate demand-supply relationships, mediated by retailers).

These are all "markets" in a somewhat unusual sense, which needs further elaboration. The complications arise from the spatial structure of the problem which means that there is no simple aggregate demand-supply relationship, either for goods or for the provision of retail facilities. It should also be noted that it is a reasonable approximation to consider the retail land market in isolation from other uses. Retailing is not an intensive use like residential or agricultural uses; and so the retailer can be considered to pay a "land rent" in proportion to store size, but essentially for the privilege of using that location.

With these preliminary comments, further elaboration can be obtained by resorting to algebraic notation and defining different kinds of prices, as shown in Figure 2. We are assuming that it is reasonable to work with one composite goods for illustrative purposes, so the "prices" can all be regarded as price indexes. The boxes correspond to those on Figure 1. E_i is the demand for goods (measured in the units of goods) by consumer at i and $\hat{p}_i$ is the price of goods at i perceived by these consumers (and we will define it more precisely shortly). p^G is the (non-spatial) price of goods to retailers and λ is the coefficient which represents their "other costs" (which should also be taken to include "normal" profits). r_j is land rent. The portrait of the retailers at j is represented by the variables (D_j, W_j, F_j, P_j, C_j). D_j is total revenue attracted to j and C_j the total cost. W_j is the "attractiveness" of j to consumers and this is now distinguished from floorspace which we denote by F_j. p_j is the price of goods as sold at the retail centre. Retailers are then assumed to fix F_j and p_j by a mechanism to be specified, such that D_j and C_j balance.

There is a problem in constructing $\hat{p}_i$. One possibility is to take

$$\hat{P}_{ij} = P_j + C_{ij} \tag{8}$$

with a j-subscript added on the left-hand side. But this implies simultaneous determination of destination. It may be better to make an approximation and take

$$\hat{P}_i = \frac{\sum_j p_j e^{-\beta c_{ij}}}{\sum_j e^{-\beta c_{ij}}} \tag{9}$$

That is, $\hat{p}_i$ is a sum of P_i's weighted by travel cost, with (probably) β being taken from the spatial interaction model.

The variables $\{\hat{p}_i\}$, $\{p_j\}$, $\{F_j\}$ and $\{r_j\}$, and possibly p^G, are all to be adjusted within a dynamical model framework. Assume, therefore, that it is reasonable to start with known initial values of them and then to specify the framework of a model, together with an adjustment procedure. An example of a model which will then illustrate all the main ideas is the following:

$$\hat{p}_i = \frac{\sum_j p_j e^{-\beta c_{ij}}}{\sum_j e^{-\beta c_{ij}}} \qquad (10)$$

$$E_i = E_i^0 (\hat{p}_i)^{-\gamma_1} \qquad (11)$$

for some constant E_i^0. This makes consumer demand at i a function of $\hat{p}_i$. The attractiveness of retailing facilities at j can be taken as a combination of consumer scale economies related to floorspace and possible price discounts :

$$W_j = F_j^{\alpha_1} p_j^{-\alpha_2} \qquad (12)$$

then the spatial interaction model is

$$S_{ij} = A_i E_i W_j e^{-\beta c_{ij}} \qquad (13)$$

with

$$A_i = 1/\sum_k W_k e^{-\beta c_{ik}} \qquad (14)$$

Revenues are given by

$$D_j = p_j \sum_i S_{ij} \qquad (15)$$

and costs by

$$C_j = (p^G + \lambda + r_j)F_j \tag{16}$$

For the present, we assume p^G to be fixed and given, though an obvious extension is to integrate it into an aggregate demand-supply model. We then have to add mechanisms to adjust F_j, p_j and r_j. Assume initially that p^G is fixed, say in a national market which is not influenced by this particular area. This is equivalent to making the assumption that an aggregate supply $\sum_i E_i^U(\hat{p}_i)^{-\gamma_1}$ will be available at price p^G to the retailer.

We now have to specify an adjustment procedure. We can use a 3-D "cobweb" mechanism similar in style to that used by Wilson and Birkin (1985) in the context of agricultural location :

$$\Delta F_j = \varepsilon_1(D_j - C_j)\ f_1(F_j) \tag{17}$$

$$\Delta p_j = -\varepsilon_2(D_j - C_j)f_2(p_j) \tag{18}$$

$$\Delta r_j = \varepsilon_3(D_j - C_3)f_3(r_j) \tag{19}$$

The functions f_1, f_2, and f_3 determine the nature of these differentials for very small values of F_j, p_j and r_j respectively. ε_1, ε_2 and ε_3 represent the relative strengths of the agents in different spatial "markets". ε_1 will be the greatest of the three if spatial competition for store size among retailers is the dominant process; ε_3 if land owners dominate and can extract any surplus in rent. ε_2 determines the significance of retailers' attitudes to price adjustment (or possibly consumers' ability to force them down). Note also that ε_2 could, for some services, have an opposite sign: if $D_j > C_j$, prices could be increased - perhaps for "good" restaurants where F_j cannot be expanded.

For the illustrative results below, the following functional forms were used for (17)-(19) :

$$\Delta F_j = \varepsilon_1 (D_j - C_j) \tag{17A}$$

$$\Delta p_j = -\varepsilon_2 (D_j - C_j) p_j \tag{18A}$$

$$\Delta r_j = \varepsilon_3 (D_j - C_j) r_j \tag{19A}$$

Then:

$$F_j^{new} = F_j^{old} + F_j \tag{20}$$

$$p_j^{new} = p_j^{old} + p_j \tag{21}$$

$$r_j^{new} = r_j^{old} + r_j \tag{22}$$

This scheme has the advantage that $\varepsilon_1 = 1$, $\varepsilon_2 = 0$, $\varepsilon_3 = 0$ reproduces the usual iterative scheme for solving the Harris and Wilson (1978) model - when $C_j = F_j$. That scheme (see Clarke and Clarke, 1984) is

$$F_j^{new} = D_j \tag{23}$$

which would arise from

$$\Delta F_j = D_j - F_j^{old} \tag{24}$$

which is (17A) with $\varepsilon_1 = 1$. This has advantages for testing all the relevant computer programmes. A similar exercise shows that if $C_j = KW_j$, then $\varepsilon_1 = K$ has the same property.

The properties of this model have been explored with the 129-zone grid system shown in Figure 3. In the results presented here, equal E_i^o's have been used together with equal starting values for $\{F_j\}$, $\{p_j\}$ and $\{r_j\}$. We set $p_j = 1$ initially for all j. Equation (10) then shows that all the $\hat{p}_i$'s are 1 initially. We took $p^G + \lambda = 0.5$ and $r_j = 0.5$, so the multiplicative constant for F_j was 1 initially though it could change if r_j changed. Then $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 0, 0)$ is essentially the Harris and Wilson (1978) model. An $\{F_j\}$-grid plot is shown for this case for reference on Figure 4.1; Figures 4.2 and 4.3 shown the unadjusted prices (p_j) and rents (r_j) since $\varepsilon_2 = \varepsilon_3 = 0$. Figures 5.1 and 5.2 show the results of a run with the same parameter values, except that $\varepsilon_3 = 0.1 \neq 0$ and rent adjustment is now possible. The surface is as might be expected. In Figure 6, the results of a run with $\underline{\varepsilon} = (1, 0.1, 0.1)$ are shown. In this case, the price variations had bounds put on so they could not be reduced below 0.8 nor increased beyond 1.2 and these are visible, in an expected way, with the results shown.

In Figures 7 and 8, we show the consequences of a change in α and β values. Figure 7.1 shows a very different pattern with only four centres but no central centre. When rent adjustment is allowed, this pattern is modified and there is a very different, but consistent, rent surface as shown in Figure 8.2. With both price and rent adjustment, we get the results shown in Figure 9.

While much more experimentation could be usefully carried out, the results do seem plausible in character. One feature, for example, particularly in the case of rent adjustment, is that the F_j pattern is more dispersed when price and rent adjustments are allowed. This is to be expected in that it is no longer the case that all balancing has to be achieved through floorspace adjustments. This is one kind of "dispersion". We now pursue another through the possibility of entropy functions representing imperfections in markets.

3. Imperfect markets : sub-optional equilibrium models.

The idea of this section is developed and explained in relation to the Harris and Wilson (1978) model as sketched in section 1 above. The economic extensions of section 2 could obviously be added, however. The idea arises from a problem : the $D_j - C_j$ balancing mechanism is "optimal" in the sense that the transportation problem of linear programming is optimal. It ought to be possible, therefore, to seek to develop a suboptimal version analogous to the entropy-maximising trip distribution model which has the transportation problem of linear programming as a limiting case. (c.f. Wilson, 1967, Evans, 1973, Wilson and Senior, 1974, Senior and Wilson, 1974.)

If this development can be achieved, there are at least three potential advantages : (i) the new model would be more general with the Harris and Wilson (1978) model as a special case; (ii) the "sub-optimal" model may generate more realistic patterns in many cases - and there should be an appropriate parameter to calibrate in relation to this; (iii) the new model should generate an explicit formula for F_j (or W_j) which may provide more analytical clues even in relation to the limiting case.

To make progress, we need to recall the mathematical programming version of the Harris and Wilson (1978) model :

$$\begin{aligned} \underset{\{S_{ij}, F_j\}}{\text{Max}} \; Z = & -\sum_{ij} S_{ij}(\log S_{ij} - 1) \\ & + \sum_i a_i (E_i - \sum_i S_{ij}) \\ & + \sum_j \lambda_j (F_j - \sum_i S_{ij}) \\ & + \alpha (\sum_{ij} S_{ij} \log F_j - H) \\ & + \beta (C - \sum_{ij} S_{ij} c_{ij}) \end{aligned} \tag{25}$$

To proceed in the manner of Wilson and Senior (1974), we need to add an entropy term, say, $-\sum_j F_j(\log F_j - 1)$, into the objective function and we need to find a way of relaxing the constraint.

$$F_j = \sum_i S_{ij} \tag{26}$$

which is represented in the Lagrangian (25) as $\sum_j \lambda_j \, (F_j - \sum_i S_{ij})$. If we are responding to market imperfections, then we would expect

$$F_j > \sum_i S_{ij} \tag{27}$$

One way to express this is

$$\sum_j (C_j - D_j)^2 = B \tag{28}$$

The new mathematical programming problem and "relaxed" model can now be written:

$$\begin{aligned}\underset{\{S_{ij},F_j\}}{\text{Max}}\ Z = & -\sum_j F_j(\log F_j - 1) - \sum_{ij} S_{ij}(\log S_{ij} - 1) \\ & + \sum_i a_i (E_i - \sum_j S_{ij}) \\ & + \lambda \sum_j [(C_j - \sum_i S_{ij})^2 - B] \\ & + \alpha \, [\sum_{ij} S_{ij} \log F_j - H] \\ & + \beta \, [C - \sum_{ij} S_{ij} C_{ij}]\end{aligned} \tag{29}$$

If we solve

$$\frac{\partial Z}{\partial S_{ij}} = 0 \tag{30}$$

for S_{ij} and

$$\frac{\partial Z}{\partial F_{ij}} = 0$$

for F_j, keeping λ, α and β as parameters which can in principle be found by solving the relevant constraint equations, we get:

$$S_{ij} = A_i E_i e^{-\lambda(D_j - F_j)} F_j^{\alpha} e^{-\beta c_{ij}} \tag{32}$$

with

$$A_i = 1/\sum_k e^{-\lambda(D_k - F_k)} F_k^{\alpha} e^{-\beta c_{ik}} \tag{33}$$

$$F_j = e^{\lambda(D_j - F_j)} e^{\alpha D_j / F_j} \quad (34)$$

We assumed for simplicity that $C_j = F_j$. The details of the derivation are given in the Appendix, along with some alternative models obtained from different ways of relaxing the $D_j - C_j$ constraint, and which either offer new insights or the possibilities of application in different situations.

Equations (32)-(34) are quite striking. There is a new form of spatial interaction model, through the term $e^{-\lambda(D_j - F_j)}$ and, most important of all, an explicit formula for F_j. The equations have to be solved iteratively. Starting with initial values of $\{F_j\}$ and $\{D_j\}$ - the latter most obviously set to $\{F_j\}$ - (32)-(34) have to be initially solved for S_{ij}, A_i and F_j; and then reiterated until convergence. A reasonable conjecture would then be that large λ would correspond to small β; and as $\lambda \to \infty$, the solutions of (32)-(34) would tend towards those of the Harris and Wilson (1978) model, as presented in Section 1 above. A summarised example of the results of this model are shown as Figure 10. In Figure 10.1, the F_j's do not sum to $\sum_i E_i$, showing that the process has not converged. In Figure 10.2, the results of a run are shown where this condition is enforced. More research is currently needed on finding good iterative procedures for these entropy-structure equations.

4. Dynamics and further extensions.

The perfect market model is set up as a dynamic model; the imperfect market entropy model is an equilibrium model. The first step in this concluding part of the argument, therefore, is to make the entropy model dynamic. This provides the basis for integrating the two sets of ideas. We then consider a number of extensions to the models and finally make some comments on the empirical development of the models.

Equations (32)-(34) for the equilibrium $\{F_j\}$ values in the entropy model clearly have to be solved iteratively. Experience has shown that this iterative process is a sensitively-balanced one and there may be practical as well as theoretical reasons for solving them in steps. We can see how to do this as follows. Rewrite (34) as

$$F_j^{new} = e^{\lambda(D_j - F_j)} e^{\alpha D_j/F_j} \qquad (35)$$

(with F_j as the old value). This represents a stage in the iteration. Hence

$$\Delta F_j = F_j^{new} - F_j^{old} = e^{\lambda(D_j - F_j)} e^{\alpha D_j/F_j} - F_j \qquad (36)$$

and we can then add an ϵ_1 factor which we would expect to be less than 1:

$$\Delta F_j = \epsilon_1 [e^{\lambda(D_j - F_j)} e^{\alpha D_j/F_j} - F_j] \qquad (37)$$

$$F_j^{new} = \Delta F_j + F_j \qquad (38)$$

Equations (32), (33) and (37), (38) then provide an iterative scheme. If the sole interest is in finding equilibrium solutions, then ϵ_1 should be chosen to achieve this in the smallest number of iterations, and experience should show what constitutes good values for these purposes. However, it could also be argued that real processes are dynamic and that the iterative scheme could be taken as the evolution of the system over time. In that case, we need estimates of empirical values of ϵ_1 in relation to whatever time period is chosen.

In solving (32)-(34), we made the assumption that $C_j = F_j$. More generally, if $C_j = C_j(F_j)$, the model can be rewritten, and if we combine this with (37) and (38) above, we get

$$S_{ij} = A_i \epsilon_i e^{-\lambda(D_j - C_j)\frac{\partial C_j}{\partial F_j}} F_j^{\alpha} e^{-\beta C_{ij}} \qquad (39)$$

$$A_i = 1/\sum_k e^{-\lambda(D_j - C_j)\frac{\partial C_j}{\partial F_j}} F_j^{\alpha} e^{-\beta C_{ij}} \qquad (40)$$

$$\Delta F_j = \epsilon_1 [e^{\lambda(D_j - F_j)\frac{\partial C_j}{\partial F_j}} e^{\alpha D_j/F_j} - F_j] \qquad (41)$$

$$F_j^{new} = F_j + \Delta F_j \qquad (42)$$

This model is now in a form where it can be combined with the "economic" model of Section 2. Equations (39) and (40) replace (13) and (14); and equations (41) and (42) replace (17A) and (20). In effect, equations (41) and (42) (with the modification to S_{ij} in (39) and (49)) offer an alternative F_j-adjustment mechanism. C_j in these equations would then be taken from (16).

We thus now have the entropy model formulated as a dynamic model and we have shown how to integrate it into the economic model. For any of these models, as already noted for the dynamic entropy model, we can consider the parameters ε_1, ε_2 and ε_3 as taking values which aid the rapid search for an equilibrium solution or as being empirical parameters which need to be estimated. In either case, we need to be aware of the bifurcation properties of the models with respect to the ε-parameters : see May (1976); Wilson, (1981-A, 1981-B); Clarke and Wilson (1983).

In further theoretical explorations or empirical developments with these models, there are many alternative formulations of particular elements of the models. For example, the ΔF_j adjustment in (17A) could be taken as

$$\Delta F_j = \varepsilon_1 (D_j - C_j)^{\mu} f_1(F_j) \qquad (43)$$

for some parameter, μ. It would also be possible to experiment with alternative rent mechanisms. (19A) and (22), for example, might be replaced by

$$r_j = f(\sum_i S_{ij}/F_j) \qquad (44)$$

That is, the rent could be a function of turnover per square foot directly - what is called a more active determination of land rents in Wilson and Birkin (1985).

Much research needs to be done on the iterative processes associated with these models, whether it is considered as a path to equilibrium only the end state of which constitutes some kind of reality; or as a dynamic trajectory, the whole of which is intended to be realistic - possibly governed by underlying and shifting equilibrium states which are not actually reached. In both these cases, the possibilities of multiple equilibria are important and should be explored. In empirical work, there is the advantage that the starting values - an historical or current postion - are known. Indices could be calculated which show the "distance" from equilibrium and the likely direction of change.

It is clear that these items have, potentially, a wide range of application : both in relation to different kinds of systems and to possibilities of different levels of aggregation in each case. More experience will undoubtedly lead to refinement of the ideas.

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<u>Appendix</u> Derivation of the entropy-structure model; and some alternative models.

First, we derive (32)-(34) from (29)-(31).

$$\frac{\partial Z}{\partial S_{ij}} = -\log S_{ij} - a_i - 2\lambda(C_j - \sum_i S_{ij}) + \alpha \log F_j - \beta c_{ij} = 0 \qquad \text{(A1)}$$

So, absorbing the 2 into the λ changing the sign of λ, and putting $C_j = F_j$,

$$S_{ij} = e^{-a_i} e^{-\lambda(D_j - F_j)} F_j^{\alpha}\, e^{-\beta c_{ij}} \qquad \text{(A2)}$$

(putting $D_j = \sum_i S_{ij}$). Put

$$e^{-a_i} = A_i E_i \qquad \text{(A3)}$$

in the usual way and solve the appropriate constraint equation for A_i. Then

$$S_{ij} = A_i E_i e^{-\lambda(D_j - F_j)} F_j^{\alpha} e^{-\beta c_{ij}} \qquad \text{(A4)}$$

with

$$A_i = 1/\sum_k e^{-\lambda(D_k - F_k)} F_k^{\alpha} e^{-\beta c_{ik}} \qquad \text{(A5)}$$

$$\frac{\partial Z}{\partial F_j} = -\log F_j + 2\lambda(C_j - \sum_i S_{ij}) + \alpha \sum_i S_{ij}/F_j = 0 \qquad \text{(A6)}$$

(so again absorbing 2 into the λ and changing the sign , putting $D_j = \sum_i S_{ij}$ and putting $C_j = F_j$)

$$F_j = e^{\lambda(D_j - F_j)} e^{\alpha D_j / F_j} \qquad \text{(A7)}$$

In the more general case where $C_j = C_j(F_j)$, a factor $\frac{\partial C_j}{\partial F_j}$ appears on the term associated with λ, to give equations (39)-(42) of section 4.

This model seems to be the best of the alternatives considered.For the record, it is worth briefly indicating what these other alternatives were. To devise the above model, the constraint

$$C_j = D_j \qquad \text{(A8)}$$

was replaced by

$$\sum_j (C_j - D_j)^2 = B \tag{A9}$$

An apparently simpler alternative is

$$\sum_j (F_j - D_j) = B \tag{A10}$$

(with $C_j = F_j$ again for simplicity of illustration.) Or,

$$\sum_j (F_j - D_j) = \hat{k}\ F_j \tag{A11}$$

Or, these aggregate alternatives could be replaced by equivalent zonal constraints :

$$F_j = \sum_i S_{ij} + B \tag{A12}$$

or

$$F_j = \sum_i S_{ij} + \hat{k}\ F_j \tag{A13}$$

Call the main model in the text E1. Then the models associated with constraints (A10)-(A13) can be called E2-E5. We quote the results below. The derivations are straightforward and can be accomplished along the lines of the algebra for model E1 presented above.

Model E2

$$S_{ij} = A_i E_i F_j^{\alpha} e^{-\beta c_{ij}} \tag{A14}$$

$$A_i = 1/\sum_k F_k^{\alpha} e^{-\beta c_{ik}} \tag{A15}$$

$$F_j = \frac{F e^{\alpha D_j / F_j}}{\sum_k e^{\alpha D_k / F_k}} \tag{A16}$$

with

$$F = \sum_j F_j = \sum_i E_i + B \tag{A17}$$

This is an attractive-looking model, with the spatial interaction formulae unchanged and a simple expression for F_j. The problem is that the model does not have good limiting properties. As $\hat{B}\to 0$, then $\sum_j F_j = \sum_i E_i$, but unlike model E1 (Figure 11), there remains a positive and negative dispersion around the Harris and Wilson model values. See Figures 11 and 12 for model E3 with floorspace excesses of 5% and 0.5% respectively. Compare Figure 12 with Figure 7 : it is still very dispersed.

Model E3

The same equation holdsbut with

$$F = \sum_j F_j = (\sum_i E_i)/(1 - \hat{k}) \tag{A18}$$

Models E4 and E5

For both models, the following equations hold:

$$S_{ij} = A_i E_i B_j F_j^{\alpha} e^{-\beta c_{ij}} \tag{A19}$$

$$A_i = 1/\sum_k B_k F_k^{\alpha} e^{-\beta c_{ik}} \tag{A20}$$

$$F_j = \frac{1}{B_j} e^{\alpha D_j / F_j} \tag{A21}$$

Then for model E4

$$B_j = \frac{F_j - \hat{B}}{F_j^{\alpha} \sum_i A_i E_i e^{-\beta c_{ij}}} \tag{A22}$$

and for model E5

$$B_j = \frac{(1 - \hat{k}) F_j}{F_j^{\alpha} \sum_i A_i E_i e^{-\beta c_{ij}}} \tag{A23}$$

An example of a run with model E5 is given on Figure 13 - c.f. Figures 7, 10, 11 and 12. In the limit as $\hat{B}\to 0$ or $k\to 0$,

$$B_j = \frac{F_j^{1-\alpha}}{\sum_i A_i E_i e^{-\beta c_{ij}}} \tag{A24}$$

and the model becomes the usual quasi-doubly-constrained model - c.f. Wilson and Crouchley (1983).

Figure 1

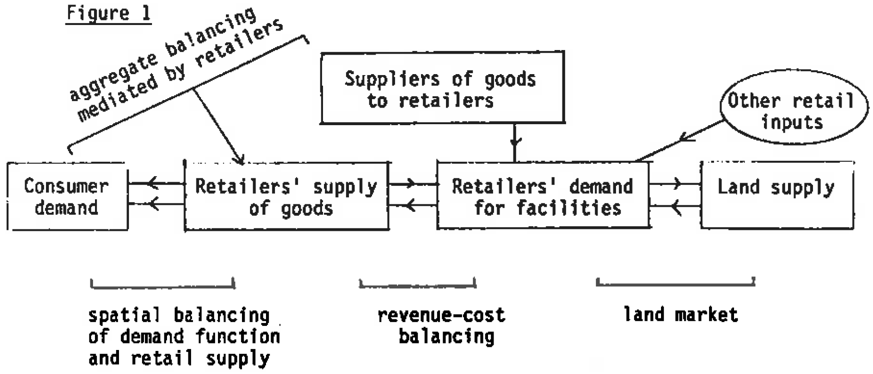

Figure 2

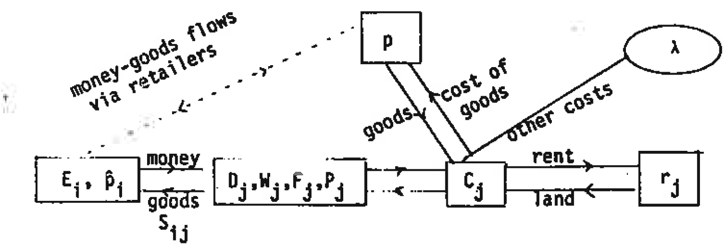

FIGURE 3. The hypothetical grid system

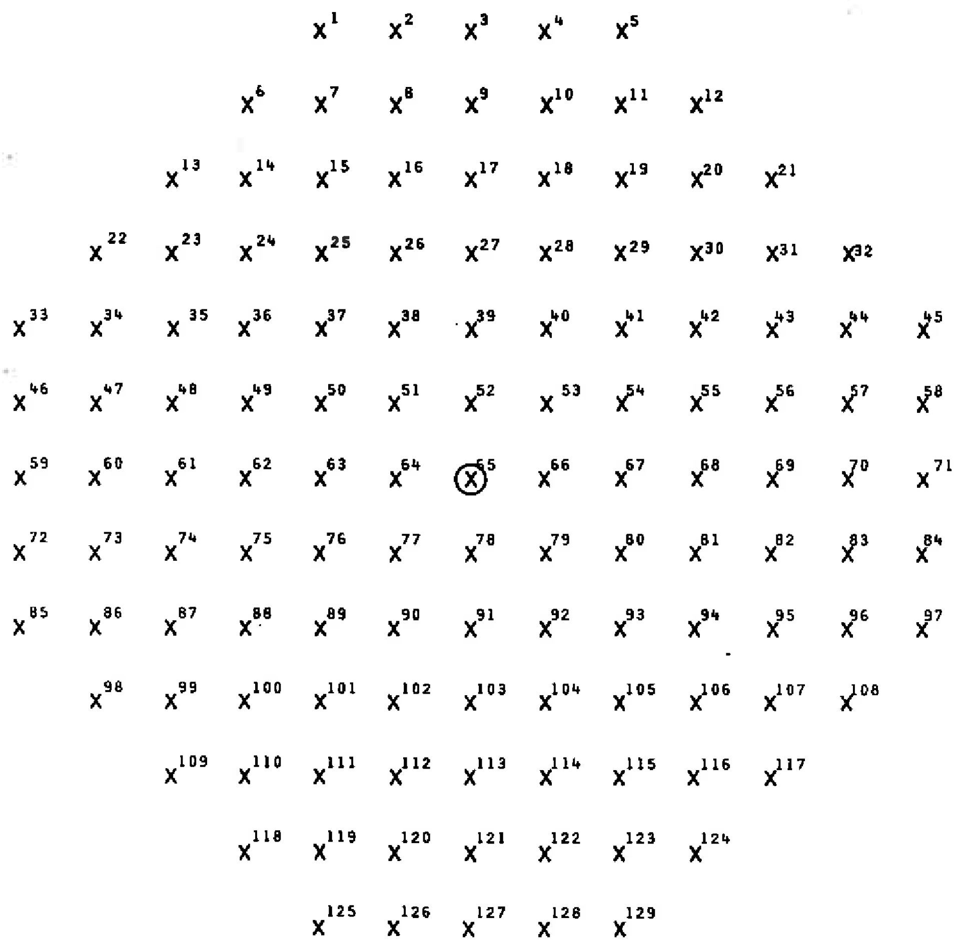

FIGURE 4 : $\alpha = 1.05$, $\beta = 0.25$, $\underline{\varepsilon} = (1,0,0)$

4.1

F GRID PLOT
ITERATION NUMBER 50

0.00000 0.00000 0.00000 0.00000 0.00000
0.00000 0.00000 0.00000 0.00000 0.00000

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.6	2.5	0.6	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.6	38.5	147.3	38.5	0.6	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	2.5	147.3	532.2	147.3	2.5	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.6	38.5	147.3	38.5	0.6	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.6	2.5	0.6	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

4.2

ORIGINAL PRICE
ITERATION NUMBER 50

1.00000 1.00000 1.00000 1.00000 1.00000
1.00000 1.00000 1.00000 1.00000 1.00000

0.0	0.0	0.0	0.0	1.0	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0	0.0	0.0
0.0	0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0	0.0
0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0
0.0	0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0	0.0
0.0	0.0	0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	1.0	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0

4.3

RENT PLOT
ITERATION NUMBER 50

0.50000 0.50000 0.50000 0.50000 0.50000
0.50000 0.50000 0.50000 0.50000 0.50000

0.0	0.0	0.0	0.0	0.5	0.5	0.5	0.5	0.5	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.0	0.0	0.0
0.0	0.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.0	0.0
0.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.0
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.0
0.0	0.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.0	0.0
0.0	0.0	0.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.5	0.5	0.5	0.5	0.5	0.0	0.0	0.0	0.0

FIGURE 5 : $\alpha = 1.05$, $\beta = 0.25$, $\underline{\epsilon} = (1,0,0.1)$

5.1

F GRID PLOT
ITERATION NUMBER 50

0.00000 0.00000 0.00000 0.00000 0.00000
0.00000 0.00000 0.00000 0.00000 0.00000

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	1.0	2.5	1.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.2	10.0	22.0	25.2	22.0	10.0	0.2	0.0	0.0	0.0
0.0	0.0	0.0	10.0	27.1	33.8	35.7	33.8	27.1	10.0	0.0	0.0	0.0
0.0	0.0	1.0	22.0	33.8	39.0	40.5	39.0	33.8	22.0	1.0	0.0	0.0
0.0	0.0	2.5	25.2	35.7	40.5	42.0	40.5	35.7	25.2	2.5	0.0	0.0
0.0	0.0	1.0	22.0	33.8	39.0	40.5	39.0	33.8	22.0	1.0	0.0	0.0
0.0	0.0	0.0	10.0	27.1	33.8	35.7	33.8	27.1	10.0	0.0	0.0	0.0
0.0	0.0	0.0	0.2	10.0	22.0	25.2	22.0	10.0	0.2	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	1.0	2.5	1.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

5.2

RENT PLOT
ITERATION NUMBER 50

0.38279 0.38310 0.38319 0.38310 0.38279
0.38279 0.38310 0.38319 0.38310 0.38279

0.0	0.0	0.0	0.0	0.4	0.4	0.4	0.4	0.4	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.0	0.0	0.0
0.0	0.0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.0	0.0
0.0	0.4	0.4	0.4	0.4	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.0
0.4	0.4	0.4	0.4	0.5	0.5	0.6	0.5	0.5	0.4	0.4	0.4	0.4
0.4	0.4	0.4	0.5	0.5	0.6	0.6	0.6	0.5	0.5	0.4	0.4	0.4
0.4	0.4	0.4	0.5	0.6	0.6	0.6	0.6	0.6	0.5	0.4	0.4	0.4
0.4	0.4	0.4	0.5	0.5	0.6	0.6	0.6	0.5	0.5	0.4	0.4	0.4
0.4	0.4	0.4	0.4	0.5	0.5	0.6	0.5	0.5	0.4	0.4	0.4	0.4
0.0	0.4	0.4	0.4	0.4	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.0
0.0	0.0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.0	0.0
0.0	0.0	0.0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.4	0.4	0.4	0.4	0.4	0.0	0.0	0.0	0.0

FIGURE 6 : $\alpha = 1.05$, $\beta = 0.25$, $\underline{\varepsilon} = (1,0,0.01)$

6.1

F GRID PLOT
ITERATION NUMBER 50

0.00000 0.00000 0.00000 0.00000 0.00000
0.00000 0.00000 0.00000 0.00000 0.00000

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.(
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.(
0.0	0.0	0.0	0.0	0.0	0.4	1.1	0.4	0.0	0.0	0.0	0.0	0.(
0.0	0.0	0.0	0.1	6.1	18.7	22.6	18.7	6.1	0.1	0.0	0.0	0.(
0.0	0.0	0.0	6.1	25.0	33.1	35.3	33.1	25.0	6.1	0.0	0.0	0.(
0.0	0.0	0.4	18.7	33.1	38.9	40.7	38.9	33.1	18.7	0.4	0.0	0.(
0.0	0.0	1.1	22.6	35.3	40.7	42.3	40.7	35.3	22.6	1.1	0.0	0.(
0.0	0.0	0.4	18.7	33.1	39.0	40.7	39.0	33.1	18.7	0.4	0.0	0.(
0.0	0.0	0.0	6.1	25.0	33.1	35.3	33.1	25.0	6.1	0.0	0.0	0.(
0.0	0.0	0.0	0.1	6.1	18.7	22.6	18.7	6.1	0.1	0.0	0.0	0.(
0.0	0.0	0.0	0.0	0.0	0.4	1.1	0.4	0.0	0.0	0.0	0.0	0.(
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.(
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.(

6.2

ORIGINAL PRICE
ITERATION NUMBER 50

1.20000 1.20000 1.20000 1.20000 1.20000
1.20000 1.20000 1.20000 1.20000 1.20000

0.0	0.0	0.0	0.0	1.2	1.2	1.2	1.2	1.2	0.0	0.0	0.0	0.0
0.0	0.0	0.0	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.0	0.0	0.0
0.0	0.0	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.0	0.0
0.0	1.2	1.2	1.2	1.2	1.1	1.0	1.1	1.2	1.2	1.2	1.2	0.0
1.2	1.2	1.2	1.2	1.0	0.9	0.9	0.9	1.0	1.2	1.2	1.2	1.2
1.2	1.2	1.2	1.1	0.9	0.9	0.8	0.9	0.9	1.1	1.2	1.2	1.2
1.2	1.2	1.2	1.0	0.9	0.8	0.8	0.8	0.9	1.0	1.2	1.2	1.2
1.2	1.2	1.2	1.1	0.9	0.9	0.8	0.9	0.9	1.1	1.2	1.2	1.2
1.2	1.2	1.2	1.2	1.0	0.9	0.9	0.9	1.0	1.2	1.2	1.2	1.2
0.0	1.2	1.2	1.2	1.2	1.1	1.0	1.1	1.2	1.2	1.2	1.2	0.0
0.0	0.0	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.0	0.0
0.0	0.0	0.0	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.0	0.0	0.0
0.0	0.0	0.0	0.0	1.2	1.2	1.2	1.2	1.2	0.0	0.0	0.0	0.0

6.3

RENT PLOT
ITERATION NUMBER 50

0.38278 0.38307 0.38317 0.38307 0.38278
0.38278 0.38307 0.38317 0.38307 0.38278

0.0	0.0	0.0	0.0	0.4	0.4	0.4	0.4	0.4	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.0	0.0	0.0
0.0	0.0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.0	0.0
0.0	0.4	0.4	0.4	0.4	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.0
0.4	0.4	0.4	0.4	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4
0.4	0.4	0.4	0.5	0.5	0.6	0.6	0.6	0.5	0.5	0.4	0.4	0.4
0.4	0.4	0.4	0.5	0.5	0.6	0.6	0.6	0.5	0.5	0.4	0.4	0.4
0.4	0.4	0.4	0.5	0.5	0.6	0.6	0.6	0.5	0.5	0.4	0.4	0.4
0.4	0.4	0.4	0.4	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4
0.0	0.4	0.4	0.4	0.4	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.0
0.0	0.0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.0	0.0
0.0	0.0	0.0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.4	0.4	0.4	0.4	0.4	0.0	0.0	0.0	0.0

FIGURE 7 : α = 1.5, β = 0.75, $\underline{\varepsilon}$ = (1,0,0)

7.1

F GRID PLOT
ITERATION NUMBER 20

0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	322.5	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	322.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	322.5	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	322.5	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 8 : α = 1.5, β = 0.75, $\underline{\varepsilon}$ = (1,0,0.01)

8.1

F GRID PLOT
ITERATION [illegible]

104.02386	0.00061	0.00003	0.00061	104.02382
104.02386	0.00061	0.00003	0.00061	104.02382

0.0	0.0	0.0	0.0	104.0	0.0	0.0	0.0	104.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	[illegible]	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
104.0	[illegible]	0.0	[illegible]	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	104.0
0.0	[illegible]	0.0	[illegible]	[illegible]	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	[illegible]	0.0	[illegible]	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	[illegible]	0.0	0.0	0.0	0.0	0.0	0.0	0.0
104.0	0.0	0.0	[illegible]	[illegible]	0.0	0.0	0.0	0.0	0.0	0.0	0.0	104.0
0.0	0.0	0.0	[illegible]	[illegible]	[illegible]	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	[illegible]	0.0	[illegible]	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	[illegible]	104.0	[illegible]	0.0	[illegible]	104.0	0.0	0.0	0.0	0.0

8.2

[illegible] PLOT
ITERATION [illegible]

1.049[illegible]2	0.5[illegible]14	[illegible]240	0.53145	1.04982
1.049[illegible]2	0.5[illegible]	[illegible]240	0.53145	1.04982

0.0	0.0	[illegible]	[illegible]	[illegible]	0.4	0.4	0.4	1.0	0.0	0.0	0.0	0.0
0.0	[illegible]	[illegible]	[illegible]	0.4	0.4	0.4	0.4	0.4	0.4	0.0	0.0	0.0
0.0	0.0	[illegible]	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.0	0.0
0.0	0.4	[illegible]	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.0
1.0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	1.0
0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
0.4	0.4	[illegible]	[illegible]	[illegible]	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
0.4	[illegible]	[illegible]	[illegible]	[illegible]	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
1.0	[illegible]	[illegible]	[illegible]	[illegible]	0.4	0.4	0.4	0.4	0.4	0.4	0.4	1.0
0.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.4	[illegible]	[illegible]	[illegible]	[illegible]	0.4	0.0
0.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.4	[illegible]	[illegible]	[illegible]	[illegible]	0.0	0.0
0.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.4	[illegible]	[illegible]	[illegible]	[illegible]	0.0	0.0
0.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.4	[illegible]	[illegible]	[illegible]	[illegible]	0.0	0.0

FIGURE 9 : $\alpha = 1.5$, $\beta = 0.75$, $\underline{c} = (1, 0.01, 0.01)$

9.1

F GRID PLOT
ITERATION NUMBER 41

0.00000 0.00000 0.00000 0.00000 0.00000
0.00000 0.00000 0.00000 0.00000 0.00000

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	15.8	284.5	15.8	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	16.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	16.0	0.0	0.0
0.0	0.0	282.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	283.7	0.0	0.0
0.0	0.0	16.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	15.9	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	15.8	284.5	15.8	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

9.2

ORIGINAL PRICE
ITERATION NUMBER 41

1.10405 1.10428 1.10432 1.10428 1.10405
1.10405 1.10428 1.10432 1.10428 1.10405

0.0	0.0	0.0	0.0	1.1	1.1	1.1	1.1	1.1	0.0	0.0	0.0	0.0
0.0	0.0	0.0	1.1	1.1	1.1	1.1	1.1	1.1	1.1	0.0	0.0	0.0
0.0	0.0	1.1	1.1	1.1	1.2	0.8	1.2	1.1	1.1	1.1	0.0	0.0
0.0	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	0.0
1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
1.1	1.1	1.2	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.2	1.1	1.1
1.1	1.1	0.8	1.1	1.1	1.1	1.1	1.1	1.1	1.1	0.8	1.1	1.1
1.1	1.1	1.2	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.2	1.1	1.1
1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
0.0	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	0.0
0.0	0.0	1.1	1.1	1.1	1.2	0.8	1.2	1.1	1.1	1.1	0.0	0.0
0.0	0.0	0.0	1.1	1.1	1.1	1.1	1.1	1.1	1.1	0.0	0.0	0.0
0.0	0.0	0.0	0.0	1.1	1.1	1.1	1.1	1.1	0.0	0.0	0.0	0.0

9.3

RENT PLOT
ITERATION NUMBER 41

0.45195 0.45205 0.45207 0.45205 0.45195
0.45195 0.45205 0.45207 0.45205 0.45195

0.0	0.0	0.0	0.0	0.5	0.5	0.5	0.5	0.5	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.0	0.0	0.0
0.0	0.0	0.5	0.5	0.5	0.1	0.6	0.1	0.5	0.5	0.5	0.0	0.0
0.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.0
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.5	0.5	0.1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.1	0.5	0.5
0.5	0.5	0.6	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.6	0.5	0.5
0.5	0.5	0.1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.1	0.5	0.5
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.0
0.0	0.0	0.5	0.5	0.5	0.1	0.6	0.1	0.5	0.5	0.5	0.0	0.0
0.0	0.0	0.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.5	0.5	0.5	0.5	0.5	0.0	0.0	0.0	0.0

FIGURE 10.1 : Entropy model E1

F GRID PLOT
ITERATION NUMBER 10

11.74884 13.48128 13.54690 13.48125 11.74894
11.74884 13.48128 13.54690 13.48125 11.74894

0.0	0.0	0.0	0.0	11.7	13.5	13.5	13.5	11.7	0.0	0.0	0.0	0.0
0.0	0.0	0.0	12.8	24.6	23.6	44.3	23.6	24.6	12.8	0.0	0.0	0.0
0.0	0.0	12.1	22.8	27.3	364.6	454.1	364.8	27.3	22.8	12.1	0.0	0.0
0.0	12.8	22.8	15.1	17.7	10.9	9.1	10.9	17.7	15.1	22.8	12.8	0.0
11.7	24.6	27.3	17.7	7.8	5.9	6.8	6.9	7.8	17.7	27.3	24.6	11.7
13.5	23.6	364.9	10.9	6.9	6.5	5.9	6.5	5.9	10.9	364.6	23.6	13.5
13.5	44.3	454.3	9.1	6.8	5.9	5.0	5.9	5.8	9.1	454.3	44.3	13.5
13.5	23.6	364.9	10.9	6.9	6.5	5.5	5.5	5.9	10.9	364.9	23.6	13.5
11.7	24.6	27.3	17.7	7.8	6.9	5.8	6.9	7.8	17.7	27.3	24.6	11.7
0.0	12.8	22.8	15.1	17.7	10.9	9.1	10.9	17.7	15.1	22.8	12.8	0.0
0.0	0.0	12.1	22.8	27.3	364.9	454.3	364.9	27.3	22.8	12.1	0.0	0.0
0.0	0.0	0.0	12.8	24.6	23.6	44.3	23.6	24.6	12.8	0.0	0.0	0.0
0.0	0.0	0.0	0.0	11.7	13.5	13.5	13.5	11.7	0.0	0.0	0.0	0.0

FIGURE 10.2 : Entropy model - totals normalised (E1)

F GRID PLOT
ITERATION NUMBER 7

5.08719 5.42373 6.77634 5.42373 5.08717
5.08719 5.42373 6.77634 5.42373 5.08717

0.0	0.0	0.0	0.0	5.1	6.4	6.8	6.4	5.1	0.0	0.0	0.0	0.0
0.0	0.0	0.0	5.9	9.3	11.2	11.6	11.2	9.3	5.9	0.0	0.0	0.0
0.0	0.0	5.9	9.5	11.8	12.5	12.7	12.5	11.8	9.5	5.9	0.0	0.0
0.0	5.9	9.5	11.9	12.1	12.1	12.1	12.1	12.1	11.9	9.5	5.9	0.0
5.1	9.3	11.8	12.1	11.8	11.4	11.2	11.4	11.9	12.1	11.9	9.3	5.1
6.4	11.2	12.5	12.1	11.4	10.9	10.7	10.9	11.8	12.1	12.5	11.2	6.4
5.8	11.6	12.7	12.1	11.2	10.7	10.5	10.7	11.2	12.1	12.7	11.6	6.8
6.4	11.2	12.5	12.1	11.4	10.9	10.7	10.9	11.4	12.1	12.5	11.2	6.4
5.1	9.3	11.8	12.1	11.8	11.4	11.2	11.4	11.8	12.1	11.8	9.3	5.1
0.0	5.9	9.5	11.9	12.1	12.1	12.1	12.1	12.1	11.9	9.5	5.9	0.0
0.0	0.0	5.9	9.5	11.8	12.5	12.7	12.5	11.9	9.5	5.9	0.0	0.0
0.0	0.0	0.0	5.9	9.3	11.2	11.6	11.2	9.3	5.9	0.0	0.0	0.0
0.0	0.0	0.0	0.0	5.1	6.4	6.8	6.4	5.1	0.0	0.0	0.0	0.0

FIGURE 11: Entropy model E3

$\alpha = 1.5$, $\beta = 0.75$, 5% excess floorspace

F GRID PLOT
ITERATION NUMBER 37

5.81767 6.80244 7.04602 6.80244 5.81767
5.81767 6.80244 7.04602 6.80244 5.81767

0.0	0.0	0.0	0.0	5.8	6.8	7.0	6.8	5.8	0.0	0.0	0.0	0.0
0.0	0.0	0.0	5.4	9.2	11.4	12.0	11.4	9.2	6.4	0.0	0.0	0.0
0.0	0.0	5.5	9.3	12.5	14.3	14.7	14.3	12.5	9.3	6.5	0.0	0.0
0.0	6.4	9.3	12.1	13.2	12.9	12.6	12.9	13.2	12.1	9.3	6.4	0.0
5.8	9.2	12.5	13.2	12.2	11.4	11.2	11.4	12.2	13.2	12.5	9.2	5.8
6.8	11.4	14.3	12.9	11.4	10.7	10.5	10.7	11.4	12.9	14.3	11.4	6.8
7.0	12.0	14.7	12.6	11.2	10.5	10.3	10.5	11.2	12.6	14.7	12.0	7.0
6.8	11.4	14.3	12.9	11.4	10.7	10.5	10.7	11.4	12.9	14.3	11.4	6.8
5.8	9.2	12.5	13.2	12.2	11.4	11.2	11.4	12.2	13.2	12.5	9.2	5.8
0.0	6.4	9.3	12.1	13.2	12.9	12.6	12.9	13.2	12.1	9.3	6.4	0.0
0.0	0.0	5.5	9.3	12.5	14.3	14.7	14.3	12.5	9.3	6.5	0.0	0.0
0.0	0.0	0.0	6.4	9.2	11.4	12.0	11.4	9.2	6.4	0.0	0.0	0.0
0.0	0.0	0.0	0.0	5.8	6.8	7.0	6.8	5.8	0.0	0.0	0.0	0.0

FIGURE 12: Entropy model E3

$\alpha = 1.5$, $\beta = 0.75$, 0.5% excess floorspace

F GRID PLOT
ITERATION NUMBER 10

5.33717 6.27685 6.50994 6.27685 5.33717
5.33717 6.27685 6.50994 6.27685 5.33717

0.0	0.0	0.0	0.0	5.3	6.3	6.5	6.3	5.3	0.0	0.0	0.0	0.0
0.0	0.0	0.0	5.9	8.6	11.0	11.7	11.0	8.6	5.9	0.0	0.0	0.0
0.0	0.0	5.9	8.8	12.1	14.0	14.4	14.0	12.1	8.8	5.9	0.0	0.0
0.0	5.9	8.8	11.7	12.8	12.5	12.3	12.5	12.8	11.7	8.8	5.9	0.0
5.3	8.6	12.1	12.8	11.7	10.9	10.7	10.9	11.7	12.8	12.1	8.6	5.3
6.3	11.0	14.0	12.5	10.9	10.2	9.9	10.2	10.9	12.5	14.0	11.0	6.3
6.5	11.7	14.4	12.3	10.7	9.9	9.7	9.9	10.7	12.3	14.4	11.7	6.5
6.3	11.0	14.0	12.5	10.9	10.2	9.9	10.2	10.9	12.5	14.0	11.0	6.3
5.3	8.6	12.1	12.8	11.7	10.9	10.7	10.9	11.7	12.8	12.1	8.6	5.3
0.0	5.9	8.8	11.7	12.8	12.5	12.3	12.5	12.8	11.7	8.8	5.9	0.0
0.0	0.0	5.9	8.8	12.1	14.0	14.4	14.0	12.1	8.8	5.9	0.0	0.0
0.0	0.0	0.0	5.9	8.6	11.0	11.7	11.0	8.6	5.9	0.0	0.0	0.0
0.0	0.0	0.0	0.0	5.3	6.3	6.5	6.3	5.3	0.0	0.0	0.0	0.0

FIGURE 13. Entropy model E5

F GRID PLOT
ITERATION NUMBER 3

23.74292		28.66304		29.16863		23.56862		28.74290				
23.74292		28.66756		29.16869		23.56862		28.74290				
0.0	0.0	0.0	0.0	27.7	29.7	29.2	28.7	29.7	0.0	0.0	0.0	0.0
0.0	0.0	0.0	28.7	20.6	10.1	7.4	10.1	20.6	28.7	0.0	0.0	0.0
0.0	0.0	28.1	17.1	4.0	1.0	0.6	1.0	4.0	17.1	28.1	0.0	0.0
0.0	28.7	17.1	3.6	0.6	0.4	0.3	0.4	0.9	3.6	17.1	29.7	0.0
29.7	20.6	4.0	0.9	0.5	0.5	0.5	0.5	0.5	0.9	4.0	20.6	28.7
29.7	10.1	1.0	0.4	0.5	0.7	0.8	0.7	0.5	0.4	1.0	10.1	29.7
29.2	7.4	0.5	0.3	0.5	0.8	0.9	0.8	0.5	0.3	0.6	7.4	29.2
29.7	10.1	1.0	0.4	0.5	0.7	0.9	0.7	0.5	0.4	1.0	10.1	29.7
28.7	20.6	4.0	0.9	0.5	0.5	0.5	0.5	0.5	0.9	4.0	20.6	28.7
0.0	28.7	17.1	3.6	0.3	0.4	0.3	0.4	0.8	3.6	17.1	28.7	0.0
0.0	0.0	29.1	17.1	4.0	1.0	0.6	1.0	4.0	17.1	28.1	0.0	0.0
0.0	0.0	0.0	28.7	20.6	10.1	7.4	10.1	20.6	28.7	0.0	0.0	0.0
0.0	0.0	0.0	0.0	28.7	23.7	29.2	29.7	29.7	0.0	0.0	0.0	0.0

Figure 9. Entropy model (vi)

F GRID PLOT
ITERATION NUMBER 10

11.74884		13.48128		13.54690		13.48125		11.74894				
11.74884		13.48129		13.54690		13.48125		11.74894				
0.0	0.0	0.0	0.0	11.7	13.5	13.5	13.5	11.7	0.0	0.0	0.0	0.0
0.0	0.0	0.0	12.8	24.6	23.6	44.3	23.6	24.6	12.8	0.0	0.0	0.0
0.0	0.0	12.1	22.8	27.3	364.8	454.1	364.9	27.3	22.3	12.1	0.0	0.0
0.0	12.8	22.3	15.1	17.7	10.9	9.1	10.9	17.7	15.1	22.2	12.8	0.0
11.7	24.6	27.3	17.7	7.9	5.9	4.3	5.9	7.8	17.7	27.3	24.6	11.7
13.5	23.6	364.9	10.9	6.5	4.5	5.3	4.5	5.9	10.9	364.8	23.6	13.5
13.5	44.3	454.3	9.1	5.3	5.5	5.0	5.0	5.8	9.1	454.3	44.3	13.5
13.5	23.6	364.3	10.9	6.7	4.5	5.3	5.5	5.9	10.9	364.9	23.6	13.5
11.7	24.6	27.3	17.7	7.9	5.9	4.3	5.9	7.8	17.7	27.3	24.6	11.7
0.0	12.8	22.3	15.1	17.7	10.9	9.1	10.9	17.7	15.1	22.9	12.8	0.0
0.0	0.0	12.1	22.9	27.3	364.9	454.3	364.9	27.3	22.3	12.1	0.0	0.0
0.0	0.0	0.0	12.8	24.6	23.6	44.3	23.6	24.6	12.7	0.0	0.0	0.0
0.0	0.0	0.0	0.0	11.7	13.5	13.5	13.5	11.7	0.0	0.0	0.0	0.0

26

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