Working Paper 287

The dynamics of urban field spatial structure: some exploratory results using difference equations and bifurcation theory.

J.R. Beaumont, M. Clarke and
A.G. Wilson

School of Geography
University of Leeds
LEEDS LS2 9JT

October, 1980

THE DYNAMICS OF URBAN SPATIAL STRUCTURE: SOME EXPLORATORY RESULTS USING DIFFERENCE EQUATIONS AND BIFURCATION THEORY

J.R. Beaumont, M. Clarke and A.G. Wilson

School of Geography,
University of Leeds.

ABSTRACT

It is demonstrated that a simple difference equation model, which exhibits complex bifurcation behaviour, can be used to represent change in urban retailing and residential systems. These submodels are combined to form a rudimentary dynamic model of urban statial structure. A sample of exploratory results are presented for a 149-zone hypothetical system.

1. Introduction.

Cities are made up of populations and economic organisations whose activities take place in physical structures of various kinds. Most progress in urban modelling has been concerned with activity patterns rather than these structures. A common basis for such modelling efforts has been provided by singly-constrained spatial interaction models which also function as location models. In particular, we consider below the application of these techniques to the retailing and residential systems and their combination in a Lowry-like framework. This provides a reasonable, if broad representation of the urban system as a whole. In addition, we show how new equations can be added to the system to represent the evolution of the associated structures - shopping centres and houses.

We show in section 2 below how a simple difference equation exhibits complex dynamical behaviour, and then in sections 4 and 5 how they provide the basis for modelling the evolution of urban spatial structure. In section 3, we present the hypothetical spatial system to which our exploratory results relate. The two sub-systems are combined into a Lowry-like system in section 6.

Although the examples presented in this short paper are particularly simple, it is easy to see that the methods can be applied to more realistic systems. A more detailed treatment is offered in Wilson (1981) and Beaumont, Clarke and Wilson (1981a) Here, we concentrate on the basic principles involved and demonstrate the kinds of results which can be achieved.

2. Mathematical representation of system dynamics.

Differential or difference equations are commonly used to describe changes in system structure. An early example is provided by Forrester's (1969) *Urban dynamics*. In this subsection we follow the work of May (1976) and Wilson (1979) and show that a simple difference equation exhibits a variety of complex dynamic behaviour, and in particular that there are important bifurcation points to be identified. In later sections, this analysis is illustrated by a description of numerical experiments relating to different urban subsystems.

A general single-variable difference equation takes the form

$$X_{t+1} = f(X_t, \psi) \qquad (2.1)$$

where X represents a measure of system structure, f is a function, ψ is a parameter and the subscripts, t and t+1 represent the ends of the time period being considered. Here, we focus on the logistic growth equation given in canonical form by

$$X_{t+1} = \psi X_t(1 - X_t) \qquad (2.2)$$

Although theoretically negative values of X are possible, it is assumed that it only takes values in the range

$$0 < X < 1 \qquad (2.3)$$

ψ must be greater than 1, to avoid the trivial situation, X_{t+1} tends towards zero. Figure 2.1 shows the relationship between X_{t+1} and X_t. The maximum value of X is $\psi/4$, and, because it is assumed X is less than 1, ψ must satisfy the following condition

$$1 < \psi < 4 \qquad (2.4)$$

(where we are now adding the right hand inequality to the one mentioned earlier). By definition, steady-state solutions, that is equilibrium structures, occur when

$$X_{t+1} = X_t \qquad (2.5)$$

If X^* is defined as such a steady-state, it must satisfy (from (2.2))

$$X^* = \psi X^*(1 - X^*) \qquad (2.6)$$

There are two possible solutions: first, the case

$$X^* = 0 \qquad (2.7)$$

which is independent of the value of the ψ parameter, and secondly

$$X^* = (\psi - 1)/\psi \qquad (2.8)$$

which is clearly dependent on the ψ parameter.

For this example, the stability of the second steady-state can be determined by examining the slope of the curve at the steady state, that is

$$\left.\frac{df}{dX}\right|_{X=X^*} = 2 - \psi \qquad (2.9)$$

The conditions for stability are that the slope of the line is between ± 1 specifically

$$-1 < 2 - \psi < 1 \qquad (2.10)$$

on a lattice of equilateral triangles (see figure 3.1). In the case of the actual model the spatial distribution of population (P_i) and demand (e_iP_i) (where e_i is per capita expenditure in zone i) is, for convenience, assumed to be uniform, although of course, it can easily be modified to enable the effects of spatial heterogeneity to be anlysed.

It should be noted that in all the models the term, c_{ij}, representing transport costs between zone i and j, is assumed to be the Euclidean distance metric.

$$c_{ij} = \{(x_i - x_j)^2 + (y_i - y_j)^2\}^{\frac{1}{2}} \tag{3.1}$$

where the centroids of zones i and j are located at (x_i, y_i) and (x_j, y_j) respectively. Obviously, this is not necessary and details of a particular transport system can be realistically reflected in a cost matrix for instance, an algoritm modifying the Euclidean metric to incorportate varying levels of city centre congestion, is documented in a recent paper (Beaumont, et.al., 1980b)

4. The retailing subsystem.

Assume that the city of interest is divided into discrete zones labelled i,j,..., that S_{ij} is the flow of revenue from residents of zone i to shops in zone j, that P_i is the population of zone i, that W_j is the size of the centre in zone j and that c_{ij} is the cost of travel from zone i to zone j, a production constrained shopping model can be written as

$$S_{ij} = \frac{e_iP_iW_je^{-\beta c_{ij}}}{\sum_k W_k e^{-\beta c_{ij}}} \tag{4.1}$$

or

$$1 < \psi < 3 \tag{2.11}$$

When ψ is greater than 3, the steady-state is unstable. This leaves us a region of ψ between 3 and 4 (cf. (2.4) above) to explore.

There can be another kind of steady-state, two time periods apart, which satisfied

$$X_{t+2} = f|f(X_t)| \tag{2.12}$$

In figure 2.2, X_{t+2} is plotted against X_t, and three different cases are shown.

(i) when ψ is less than three, there is one, stable steady-state.

(ii) when ψ equals three, the line representing $X_{t+2} = X_t$ is at a tangent to the curve,

(iii) when ψ is greater than three, the original steady-state is now unstable because it slope is greater than mod (1), but a, new stable, steady-states exist.

The two cycle periods are stable up to a particular ψ value, and then 4-cycles take over. A number of these are exhibited in figure 2.3. However, beyond ψ equal to 3.8495, the oscillations of definite periods between steady-states cease, and a state is reached in which the oscillations are aperiodic and are termed chaotic. The values of 4 at which the various changes taken place are *critical values* and are bifurcation points. ψ = 3, 3.8495 and 4 are important examples. These results are applied to urban subsystems in sections 4 and 5 below.

3. The spatial system for numerical experiment

To provide the basis for numerical experiments, an idealised, geometric zoning system is employed in which 149 zones are arranged

when α and β are suitably defined parameters. This ensures that

$$\sum_j S_{ij} = e_i P_i \qquad (4.2)$$

As expenditure ($e_i P_i$) in every zone is fixed, the retailing structural variables are $\{W_j\}$. α and β are parameters and the effects of changes in their values can be examined. α is interpreted as a measure of consumer scale economies, the higher it is, the more beneficial it is to go to large centres rather than small ones. β relates to 'ease of travel', and at times of low petrol prices, β will be low relative to a time of high prices.

To consider the dynamical behaviour of the structural variables, it is assumed that their growth (or decline) is dependent on profitability. Formally, we can write

$$W_{jt+1} - W_{jt} = \varepsilon(D_{jt} - KW_{jt})W_{jt} \qquad (4.3)$$

where ε represents the rate of response to change in size to profit or loss, K is the unit cost of supply of centre space, and D_{jt} the revenue for a centre in zone j at time t and is, by definition

$$D_{jt} = \sum_i S_{ijt} = \sum_i \frac{e_i P_i W_{jt} e^{-\beta c_{ij}}}{\sum_k W_{kt} e^{-\beta c_{ik}}} \qquad (4.4)$$

(where S_{ijt} is taken from (4.1) with time subscripts added as appropriate.) The non-trivial steady-state is when aggregate revenue equals aggregate costs - only normal profits are made.

$$D_{jt} = KW_{jt} \qquad (4.5)$$

There are conditions on the stability of equilibrium based on α, β and k. These are described in Harris and Wilson (1978) and Wilson and Clarke (1979). Here, we restrict ourselves to considering other kinds of bifurcation.

Equation (4.3) can be rewritten as

$$W_{jt+1} = \left[(1 + \varepsilon D_{jt}) - \varepsilon K W_{jt} \right] W_{jt} \tag{4.6}$$

and if we take

$$X_j = \frac{\varepsilon K W_{jt}}{1 + \varepsilon D_{jt}} \tag{4.7}$$

this equation takes the canonical form of equation (2.2) with

$$\psi = 1 + \varepsilon D_j \tag{4.8}$$

Thus, we can apply the results described in section 2: for a stable steady-state to exist the following condition must be satisfied

$$1 < 1 + \varepsilon D_{jt} < 3$$

or

$$0 < \varepsilon D_{jt} < 2$$

Clearly, εD_{jt} is non-negative, but it is not necessarily less than two, and, therefore, changes in urban structure can be expected to contribute to periodic and chaotic oscillations. Moreover, when εD_{jt} is greater than three ($\psi > 4$ in the canonical model), divergent behaviour leading to ultimate extincion is found.

It should be carefully noted that D_{jt} is not a constant; it changes over time and is dependent on the spatial competition between centres. In this sense, the positions of the bifurcation points changes in a complicated way. Numerical investigations show that when the value of εD_{jt} passes through a critical value, the expected change in dynamical behaviour does occur, although for various reasons sometimes slightly later than may be expected.

Examples of the different types of bifurcation behaviour are given in figure 4.1 for a simple eight-zone experiment; attention is drawn to the fact that, clearly, changes in dynamical behaviour in one zone have ramifications in other zones.

Figures 4.2 and 4.3 portray evolution of urban spatial structure for different parameter values using the 14[illegible] zone system. In both cases, excluding the outer zones, at $t = 0$, the values of e_i, P_i and W_j are constant for all zones. In figure 4.2 when α equals 1.1, β equals 0.1, and ε equals 0.04 the emergence of a dominant city centre is shown. In contrast, in figure 4.3 when α equals 1.5, β equals 0.5 ε equals 0.06 there is an emergence of two centres on the periphery of thecity centre; between time periods 7 and 20, these two dominant centres move towards the city centre and also other more localised centres emerge.

5. The residential system.

In the previous section we discussed the dynamical behaviour of the structural variables assoicated with a production constrained spatial interaction model. In this section we extend the argument to an attraction constrained model, a residential location model. This involves a different approach to the atrractiveness factor. Here we take as given the pattern of employment within our spatial system and the object is to locate workers in zones according to the relative attractiveness of the residential zones in relation to the zone of employment. the traditional residential location spatial interaction model is

$$T_{ij} = B_j E_j H_i^{\gamma} e^{-\beta c_{ij}} \tag{5.1}$$

$$B_j = \left(\sum_i H_i^{\gamma} e^{-\beta c_{ij}} \right)^{-1} \tag{5.2}$$

where T_{ij} is the number of persons living in zone i who work in j

E_j is the number of persons working in zone j

H_i is the residential attractiveness of zone i

β is travel impedance parameter

c_{ij} is cost of travel between i and j

γ is residential attractiveness parameter.

The residential attractiveness of different zones will obviously depend on a variety of factors. In the model we use here we firstly assume there is a disutility associated with travelling to work and that this increases in relation to the cost of travel. This is accommodated in the exponential term in the model. Secondly, it assumed that zones will have an element of attractiveness related to the number of residences. We denote this as $H_i^{\gamma 1}$ where $\gamma 1$ is a parameter. In addition it assumes that individauls will prefer to live in zones near to services, such as shops. We thus input the spatial pattern of

services, $\{W_j\}$, and calculate the relative accessibility for zone i:

$$X_i = \sum_j W_j e^{-\beta c_{ij}} \qquad (5.3)$$

for each zone. We also incorporate this into the attractiveness function raised to a power as $X_i^{\gamma 2}$. Finally we assume that people do not like living at high densities, so we calculate a density factor

$$D_i = \frac{H_i}{L_i - \eta P_i} \qquad (5.4)$$

where L_i is the amount of residential land in zone i and η a parameter that represents the average amount of land used per person. Thus as the density in zone i increases the value of D_i increases. We also raise this to a power as $D_i^{\gamma 3}$. There are various ways in which a composite attractiveness factor can be formed. For the results presented here we combine the elements additively *after* transforming the measures into percentages (and we denote this operation by ^ as in $\hat{H}_i$.

$$H_i^* = \hat{H}_i^{\gamma 1} + \hat{X}_i^{\gamma 2} - \hat{D}_i^{\gamma 3} \qquad (5.5)$$

The left hand side of (5.5) now takes the place of H_i^{γ} in equaiton (5.1). When the T_{ij}'s are calculated it is possible to determine the residential population, P_i in each zone by

$$P_i = a\sum_j T_{ij} \qquad (5.6)$$

where a is an employment multiplier.

The dynamics of this model can then be formulated in a manner analogous to this retail facility model. Given initial values of E_j and P_i at time t, the change in housing stock (which is the structural variable of a zone between t and t+1 is given by

$$H_i(t+1) - H_i(t) = \sigma\left[a\sum_j T_{ij_t} - H_i(t)\right] H_i\ (t) \qquad (5.7)$$

where $\sum_j T_{ij}$ is calculated for the incremental period from (5.1) Thus the housing stock (and, of course, the population) of each zone will either grow or decline each time period depending on what happens in all the other zones and relative to its attractiveness $\ddot{H}_i^*$.

We now present some results using the 149 zone system described earlier, in each case commencing with a uniform distribution of population, and employment is a simple linear function of the size of services in each zone which is also a set of exogenous variables. In figure 5.1 service is constant for all zones, while in figure 5.2 services are concentrated in the central area. Note that this difference, with all parameters identical in either case leads to quite different patterns for the dynamics of the housing system over the twenty iterations.

6. An integrated model: a Lowry-like model.

It is clear that the assumptions contained within the previous two sections of (a) a fixed population in the retail model and (b) a fixed employment pattern in the residential model can be amended and combined with each other: a change in population distribution may give rise to a change in the pattern of retail facilities (witness suburbanisational growth of retail facilities on the perimeters of cities) and a change in the location of population. The first attempts to incorporate this interaction into an urban model were by Lowry (1964). Here we adopt the spatial interaction version of his original model, the structure of which is outlined in figure 6.1. The retail model is now linked to the residential location model through the P_i term and interesting changes in urban structure can be expected. The addition to the Lowry model is provided by the dynamical equations.

7. Conclusion.

In this paper we have briefly outlined an approach to study the evolution of urban spatial patterns. Whilst we have restricted our discussion to specific models, there is every reason to believe that similar types of behaviour are to be found in other models of the urban system.

Relatively straightforward extensions of our appraoch would be to make the model parameters time dependent, for instance the value of β increasing as the real cost of energy increases. In addition, disaggregation of these models would introduce further linkages and perhaps lead to more complex behaviour. This could include the introduction of a modal choice model, a hierarchical disaggregation of retail service type, or the distinguishing of different person and house types.

8. References.

Beaumont, J.R., Clarke, M. and Wilson, A.G. (1981) *The dynamics and evolution of urban spatial structure.* Forthcoming.

Beaumont, J.R., Clarke, M., and Wilson, A.G. (1981) Changing energy parameters and the evolution of urban spatial structure, *Journal of Regional Science and Urban Economics*, (forthcoming).

Forrester, J. (1969) *Urban dynamics*, MIT Press, Cambridge, Mass.

Harris, B., and Wilson, A.G. (1978) Equilibrium values and dynamics of attractiveness values in production-constrained spatial interaction models, *Environment and Planning A*, 10, 371-88.

Lowry, I.S. (1964) *A model of metropolis*, RM-4035-RC, Rand Corporation, Santa Monica, California.

May, R.M. (1976) Simple mathematical models with very complicated dynamics, *Nature*, 261, 459-67.

Wilson, A.G. (1979) Some new sources of instability and oscillation in dynamic models of shopping centres and other urban structures, *Sistemi Urbani* (forthcoming).

Wilson, A.G. (1981) *Catastrophe theory and bifurcation applications to urban and regional systems*, Croom Helm, London, forthcoming.

Wilson, A.G. and Clarke, M. (1979) Some illustrations of catastrophe theory applied to urban retailing structures. In M. Breheny (Ed.) *London Papers in Regional Science V 110* Pion London.

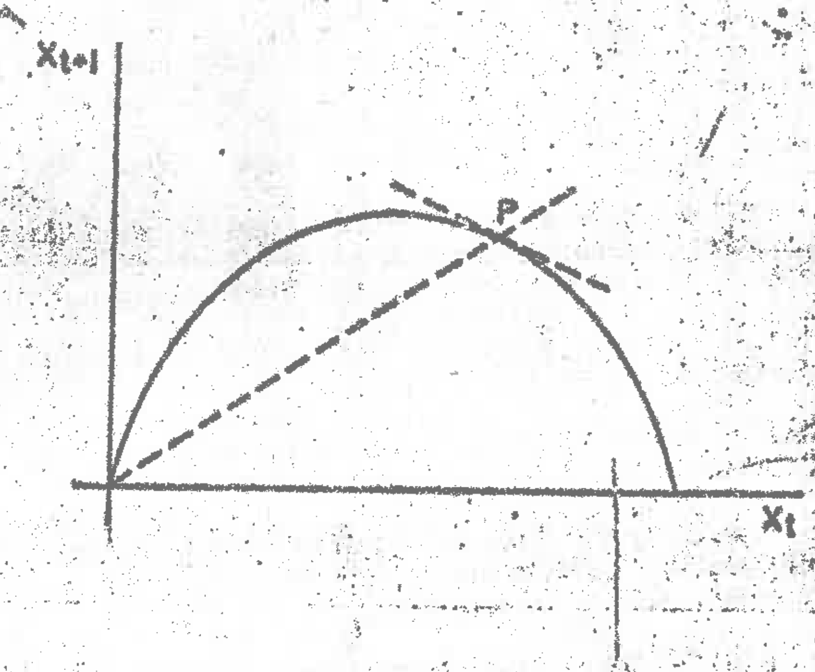


Fig. 2.1

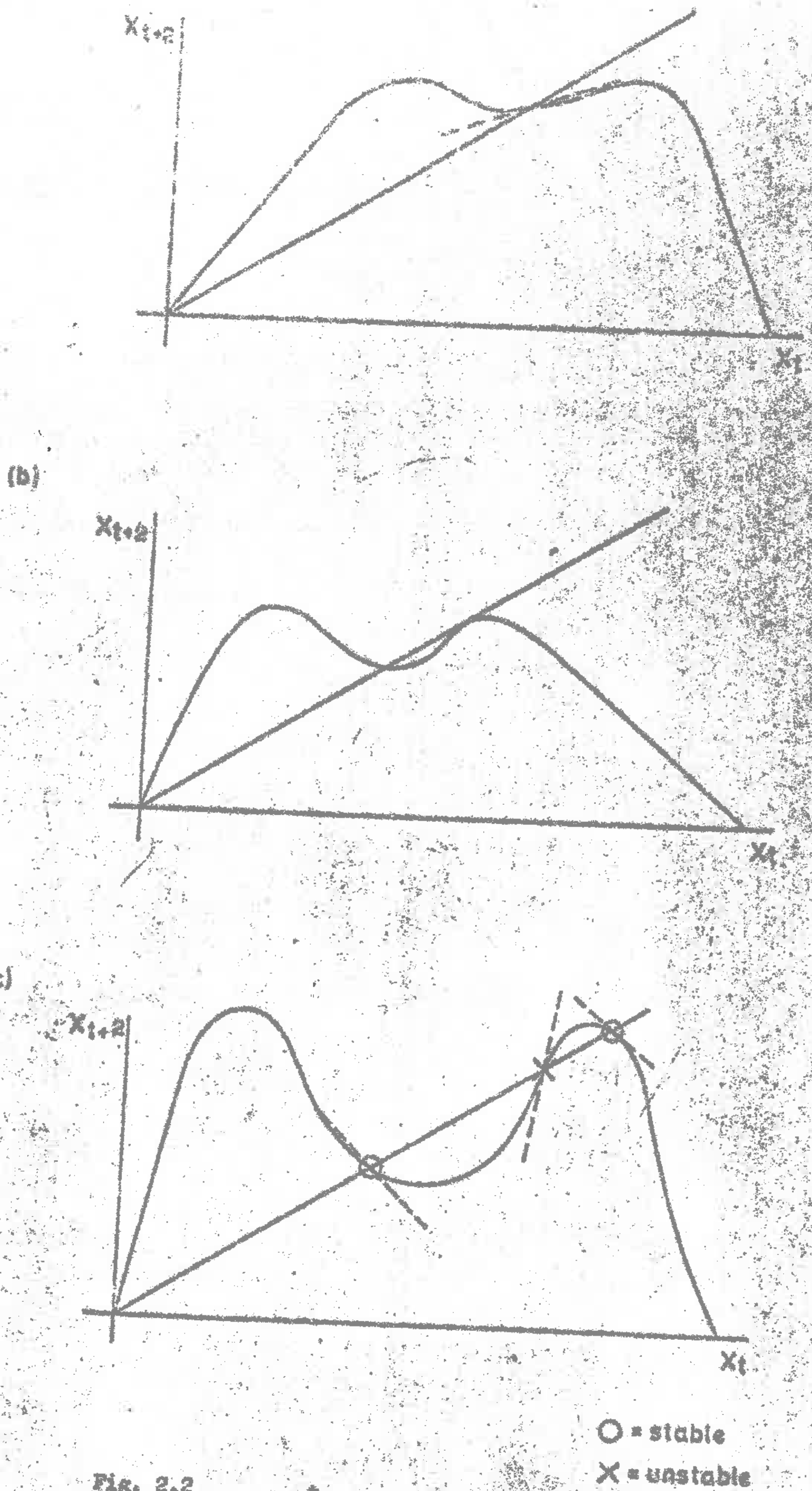
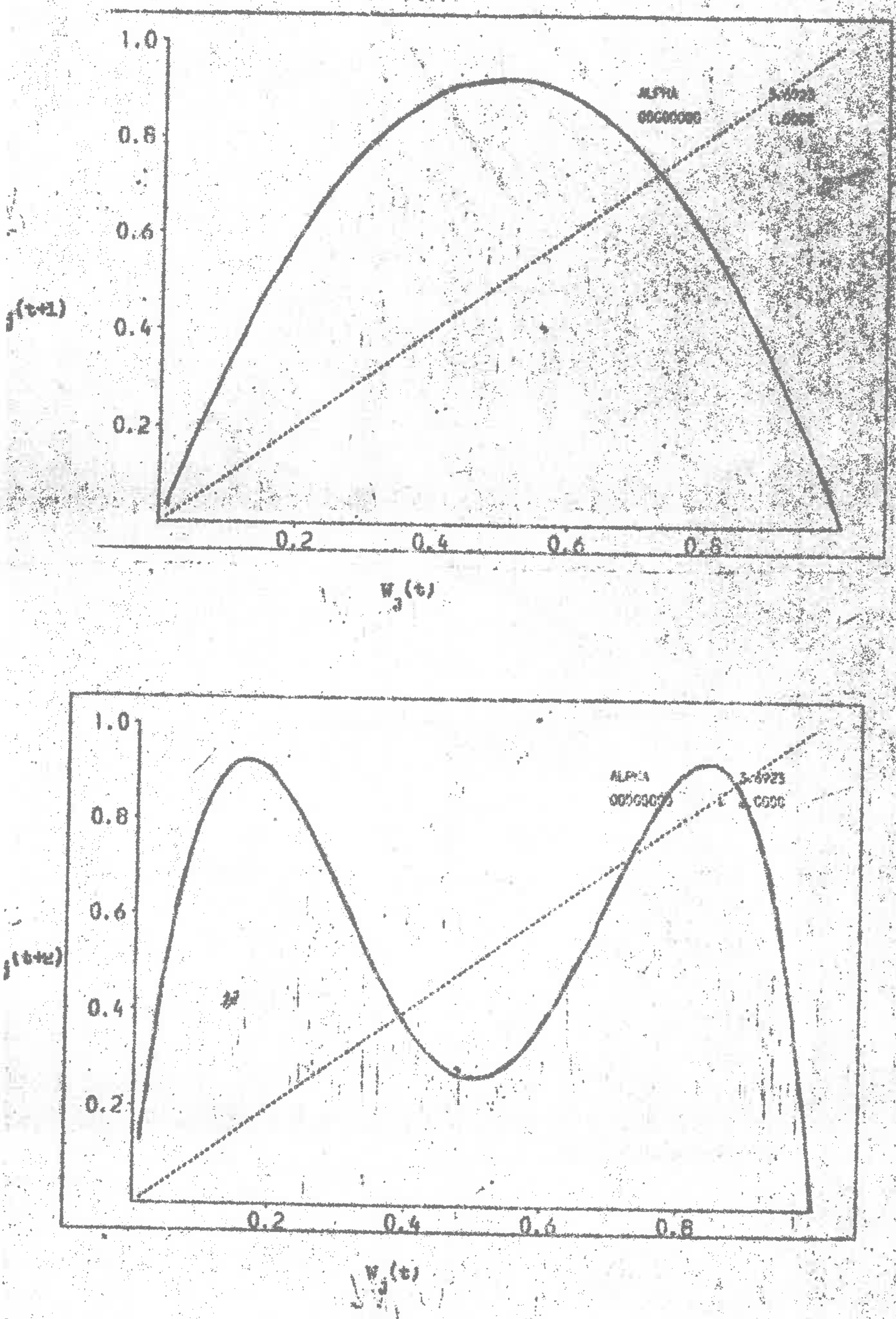
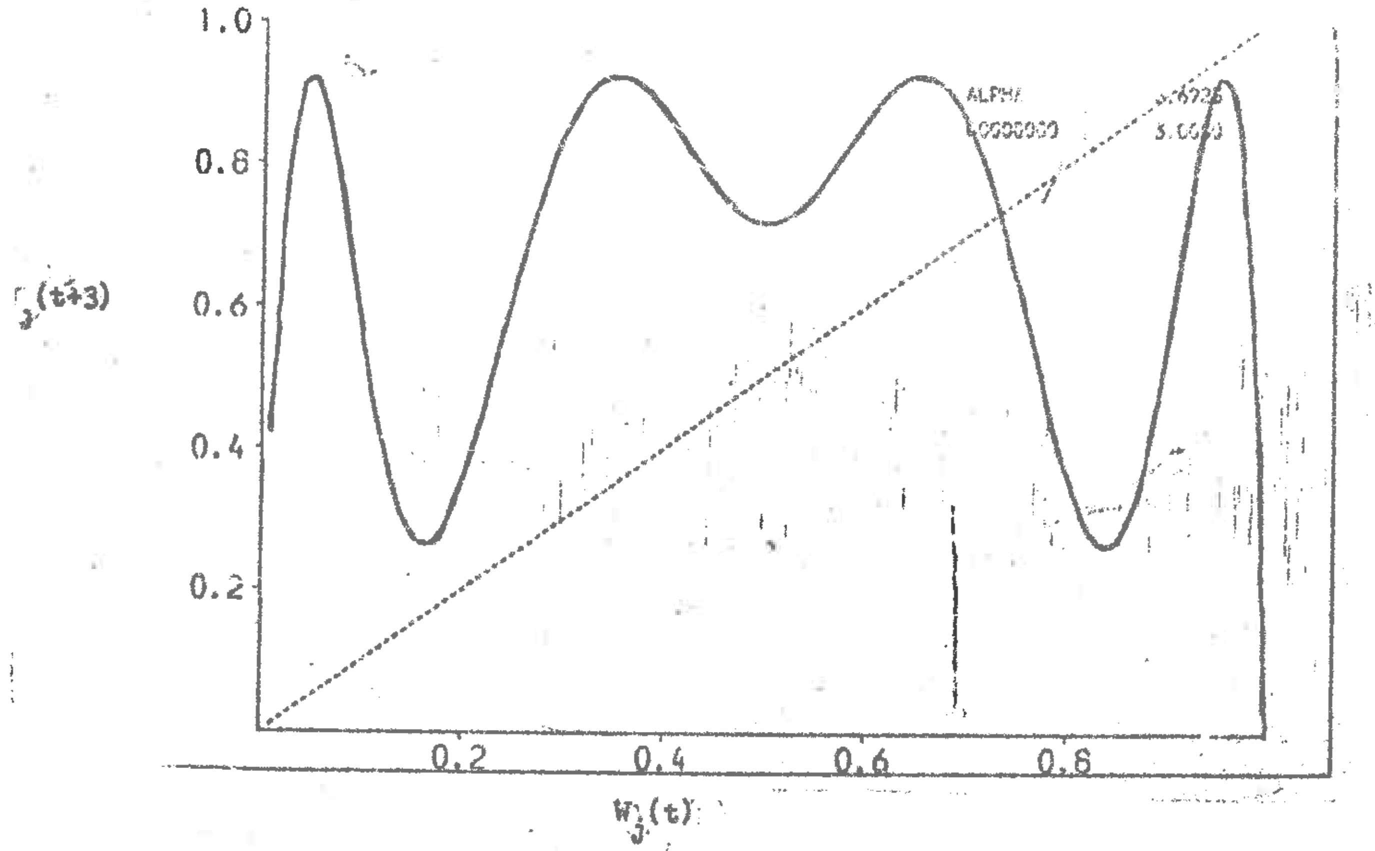


Fig. 2.2



1.0
0.8
0.6
0.4
0.2
(t+1)
ALPHA
0.2
0.4
0.6
0.8
$W_j(t)$
1.0
0.8
0.6
0.4
0.2
(t+2)
ALPHA
0.2
0.4
0.6
0.8
$W_j(t)$

Fig. 2.3(a)


1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
ALPHA
(t+3)
W_j(t)

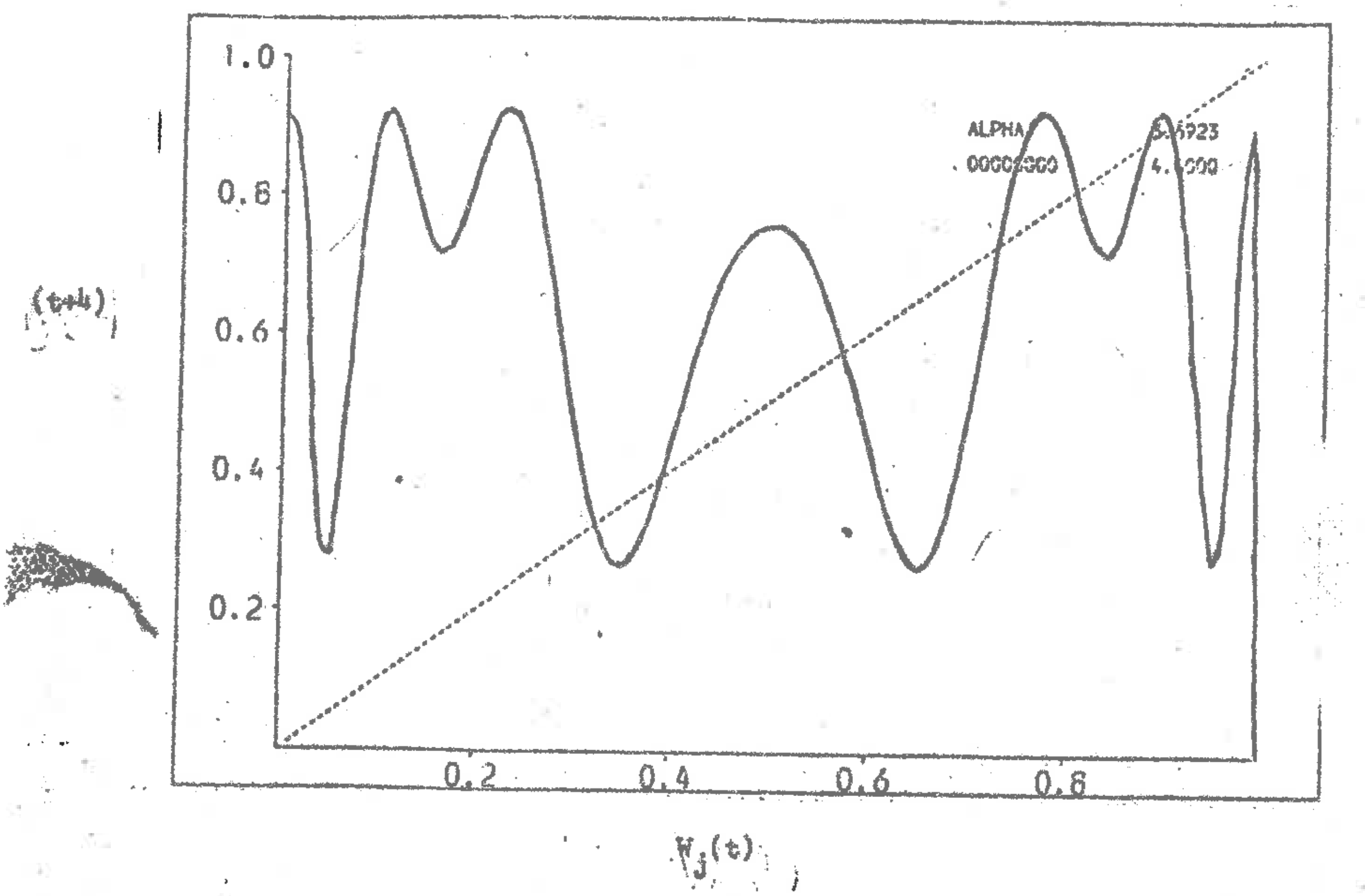

1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
ALPHA
(t+4)
W_j(t)

Fig. 2.3(b)

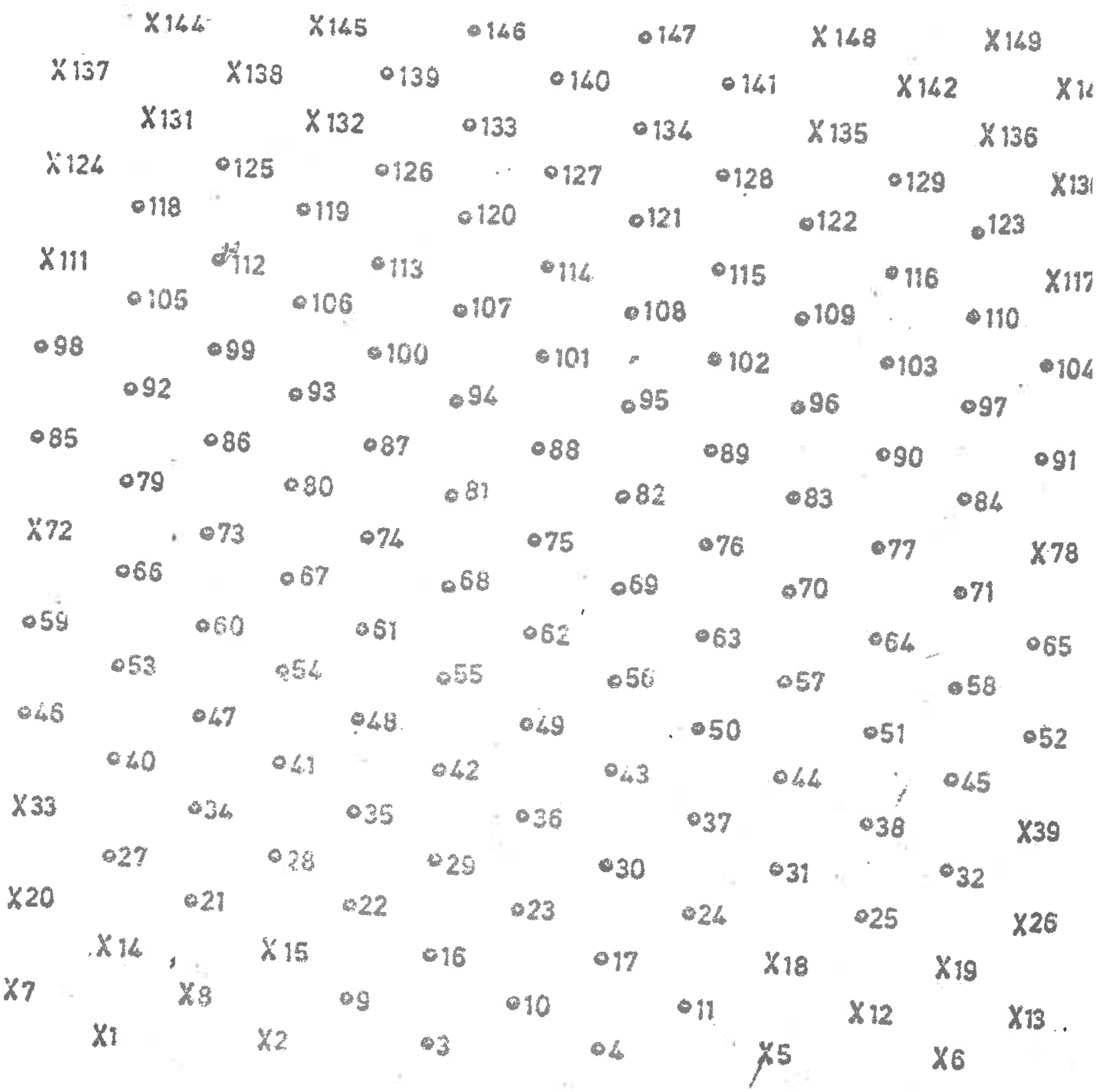
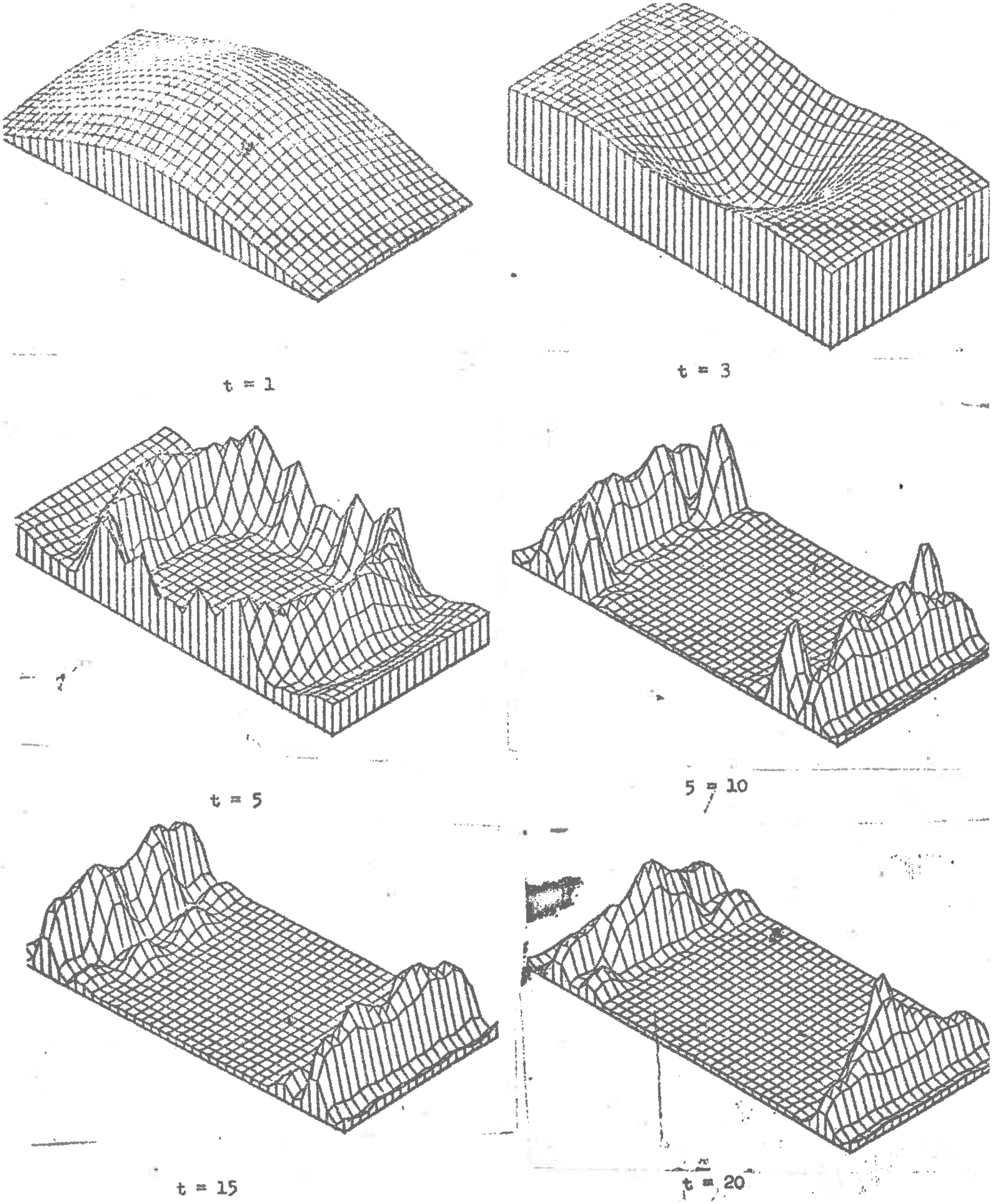


FIGURE 3.1 The zoning system

X indicates exogenous zone



$\sigma = 0.04$ $\beta = 0.5$

$\gamma_1 = 1.2$ $\eta = 1.0$

$\gamma_2 = 1.0$ $a = 1.5$

$\gamma_3 = 1.0$

Fig. 5.1

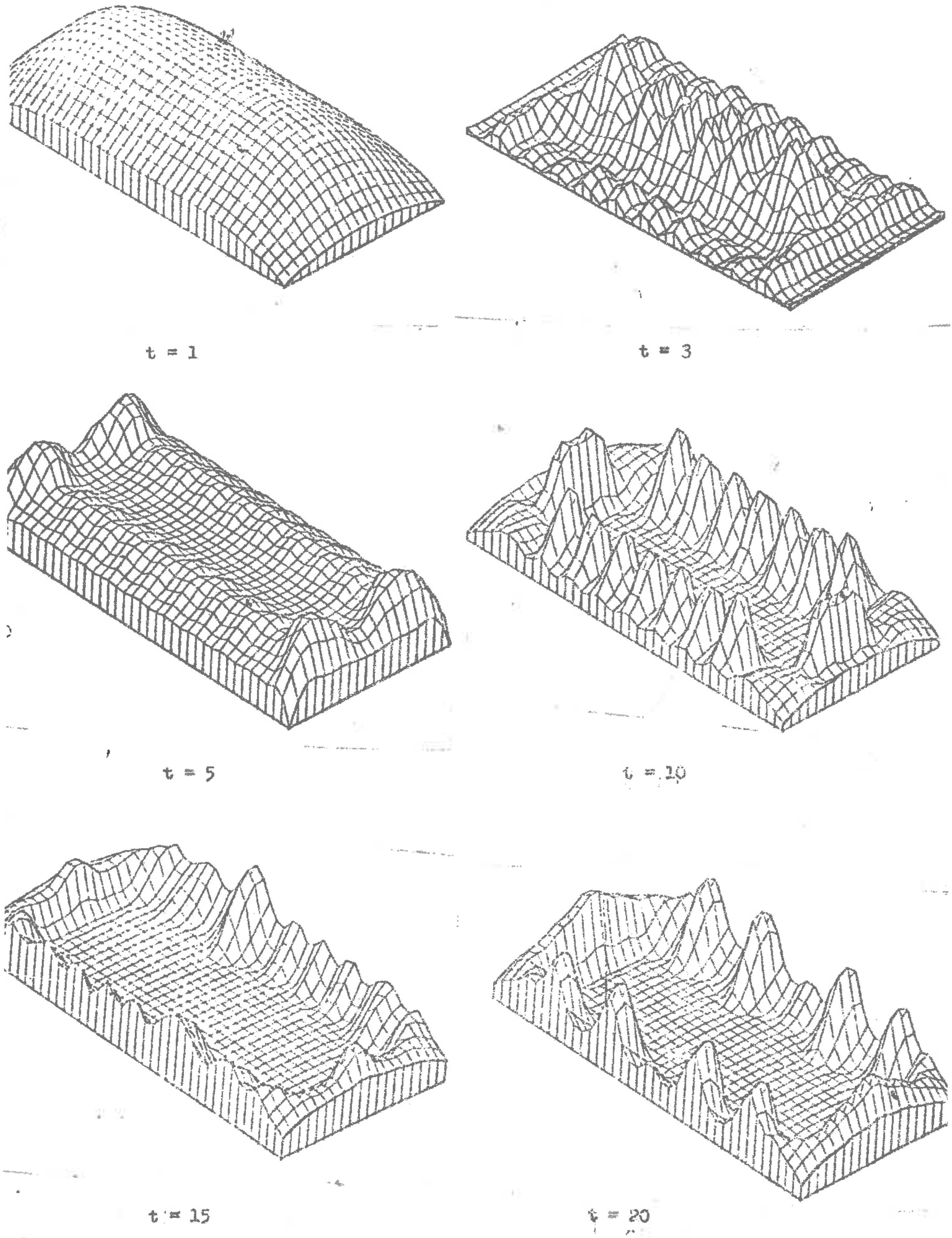


Fig. 5.2

$\sigma = 0.04$ $\beta = 0.5$

$\gamma_2 = 1.2$ $\eta = 1.0$

$\gamma_2 = 1.0$ $a = 1.5$

$\gamma_3 = 1.0$

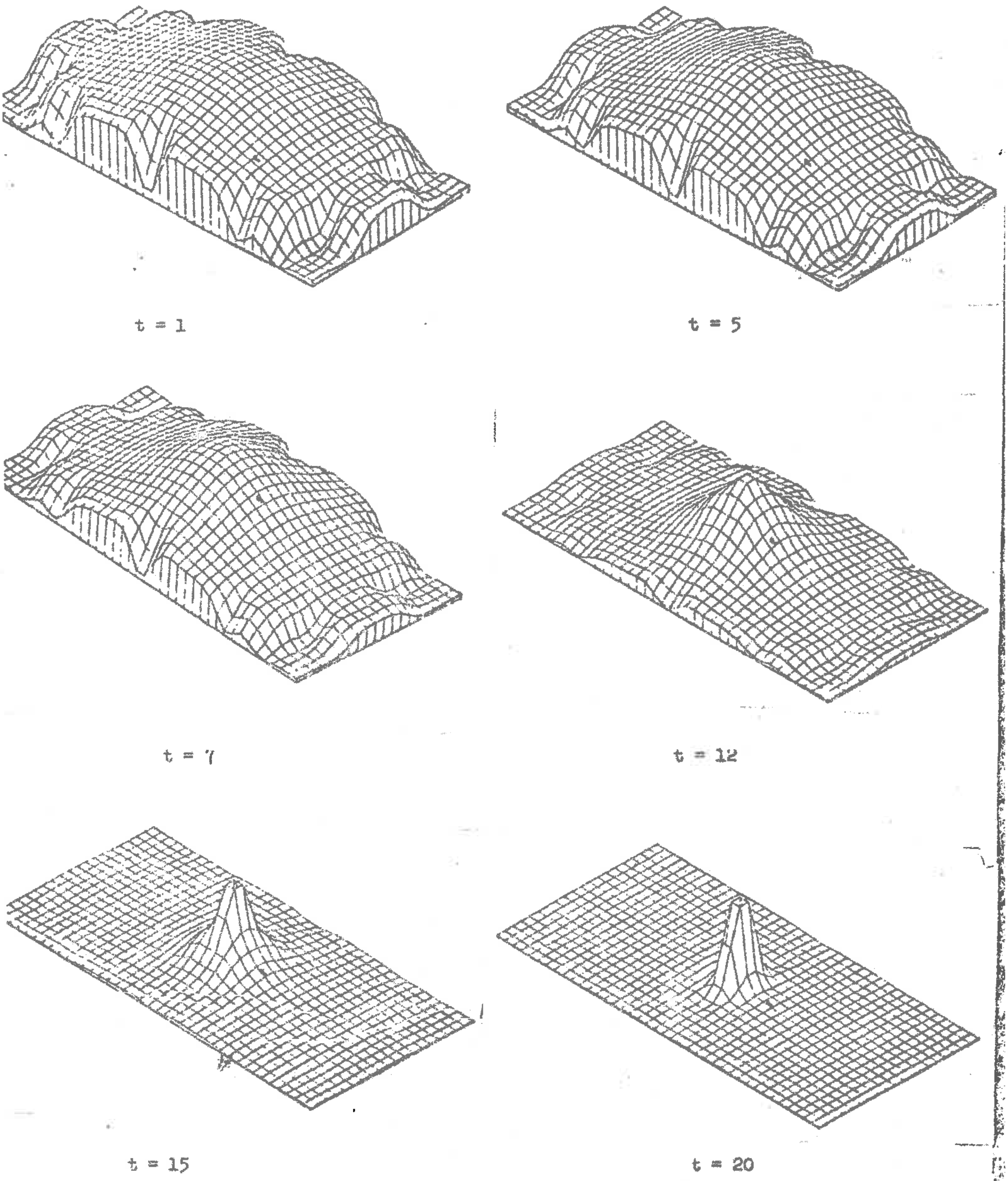
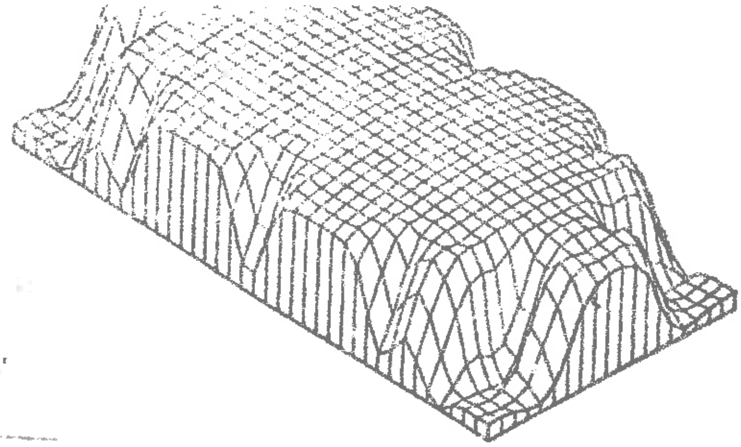
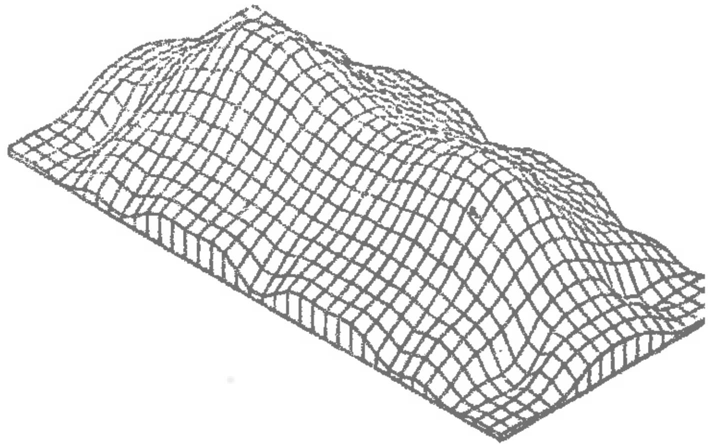


Fig. 4.2

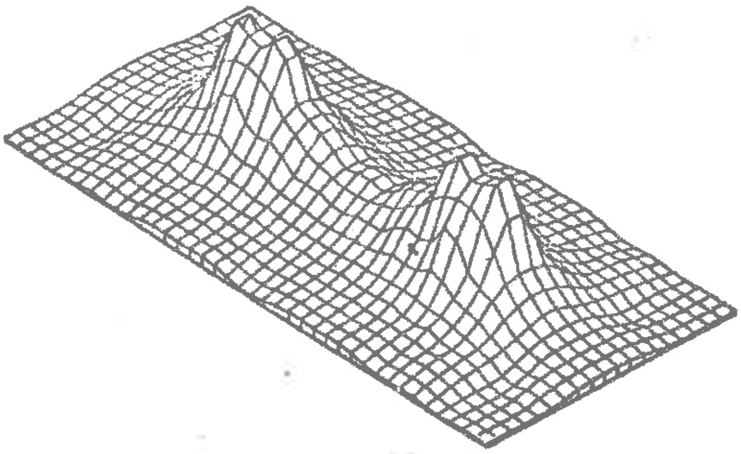
$\beta = 0.1$
$\alpha = 1.1$
$t = 0.04$



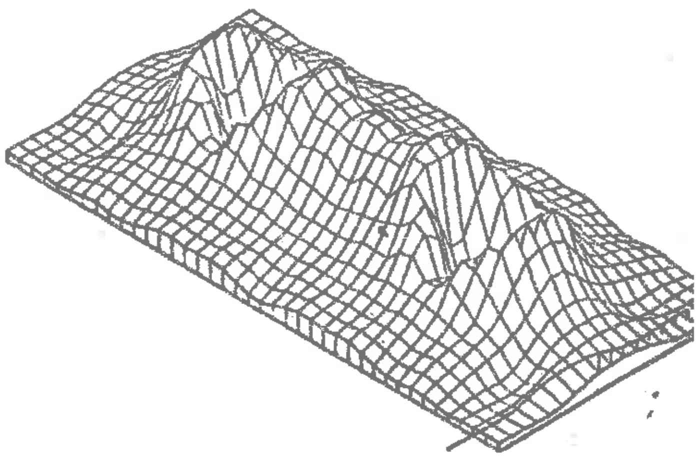
$t = 1$



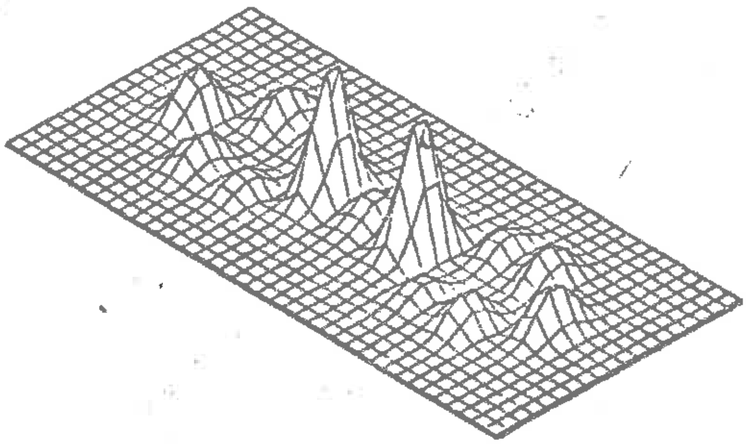
$t = 5$



$t = 10$



$t = 15$



$t = 20$

Fig. 4.3

$\beta = 0.5$
$\alpha = 1.5$ $\quad \varepsilon = 0.06$