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A SIMPLE MODEL FOR POPULATION PROJECTIONS APPLIED TO ETHNIC GROUP AND SMALL AREA POPULATIONS.

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CONTENTS

ABSTRACT

A simple population projection model is presented in both matrix and algrbraic forms. The model is used to project the future numbers of persons belonging to the West Indian eyhnic group in Leeds County Borough from 1971 to 1986. In a second application the population of East and West Hunslet ward in Leeds is forecast over a one year period 1971-2. Although the model is initially specified and used in a single region, cohort survival context, multiregional and accounts-generating versions are described.

1. The status of the projection model

In this paper we outline a new model for projecting the population of a single region or for forecasting the future population of a social group in a region. The model has the basic feature of the Leslie closed system model*: the multiplication of a vector of populations by a matrix of age-sex disaggregated survival rates.

It differs from the 'closed system' cohort survival model in incorporating out-migration rates into the survival rate definition and in dealing with in-migration as a separate input. In this respect the new model resembles a single region version of Roger's multiregional cohort survival model. However, the survival rates differ from those in Roger's model being defined to be consistent with those derived from population accounts tables. Births are generated by multiplying a sequentially generated female population at risk by a set of age disaggregated fertility rates. The model incorporates infants born in the period who migrate into the equation that estimates the first age group. In the second section of the paper we outline the model as a set of equations in matrix format. The third second gives details of the contents of the matrices and vectors involved. The equations are expressed in an alternative algebraic representation in section four. Section five shows how the model is related to a number of other models of population growth recently outlined (Rees and Wilson, 1973). The next two sections outline two examples of the use of the model. The model is employed to project the West Indian population of Leeds C.B. in the first example, and to project the whole population of East and West Hunslet ward in Leeds C.B. in the second. We conclude with a discussion of the numerous difficulties involved in applying the model and with a consideration of the possible advantages involved in its use.

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*The population projection models referred to are reviewed in Rees and Wilson (1974).

2. The matrix representation of the model

The projection model consists of four equations expressed in matrix notation. The first calculates the numbers of persons who survive a period of interest within a region and adds to them the number who migrate into the region and survive there. The second works out the populations of women at risk of giving birth in the region in the period of interest. The third equation establishes how many children are born to these women at risk in the region and adds to them the number of children born elsewhere who migrate into the region, most usually with their parents, and survive there. The fourth equation simply combines the numbers surviving the period with those born into the period in final vector of population of the region at the end of the period.

The equations are

$$\underline{w}_S^X(t+T) = \underline{S}^X(t,t+T)\,\underline{w}^X(t) + \underline{I}_S^X(t,t+T) \tag{1}$$

$$\underline{w}^F(t,t+T) = ((\underline{w}_S^F(t+T) - \underline{w}^F(t))0.5) + \underline{w}^F(t) \tag{2}$$

$$\underline{w}_B^X(t+T) = \underline{F}(t,t+T)\,\underline{w}^F(t,t+T)\,\sigma^X(t,t+T)\,s_{01}^X(t,t+T) + \underline{I}_B^X(t,t+T) \tag{3}$$

$$\underline{w}^X(t+T) = \underline{w}_S^X(t+T) + \underline{w}_B^X(t+T) \tag{4}$$

The variables and their subscript labels in the equations have the following meaning.

$\underline{w}_S^X(t+T)$ — A column vector of population of sex X for a region who have survived the period. The survival of the population is denoted by the subscript S. The population refers to point in time t+T where t is the starting point of the period of interest and T is the length in time units of the period. The population is disaggregated by age.

$\underline{S}^X(t,t+T)$ — A matrix of survival rates of persons of sex X over the period t to t+T. The rates refer to the survival from one age group into another.

$\underline{w}^X(t)$ — A column vector of population of sex X for a region at time t, the start of the period of interest. The population is broken down by age.

$\underline{I}_S^X(t,t+T)$ — A column vector of persons of sex X, who migrate into the region and survive (S) there to the end of the period. The surviving in-migrants are disaggregated into age groups.

$\underline{w}^F(t,t+T)$ — A column vector of average female population for the period who constitute an estimate of the population of women at risk of giving birth in the period in the region. In this case the sex superscript X has been given the value of F indicating females. The letter M refers to males.

$\underline{w}_B^X(t,t+T)$ — A column vector of persons of sex X born (B) in the period t to t+T who end the period surviving in the region.

$\underline{F}(t,t+T)$ — A matrix of fertility rates (numbers born divided by numbers at risk) for both sexes for the regions in the period t to t+T. The rates refer to particular age groups of women in the population at risk, It is possible to break down fertility rates by sex of the children born but this is not usually done. We disaggregate births by sex using a sex proportion σ^X.

$\sigma^X(t,t+T)$ — The proportion of births in the region in period t to t+T that are of sex X.

$s_{01}^X(t,t+T)$ — The probability that a person of sex X born in period t to t+T will survive in the region at the end of period. The person is considered as coming from age group 0 and surviving in age group 1.

$\underline{I}_B^X(t,t+T)$ — A column vector of persons of sex X who are born (B) elsewhere in the period t to t+T and who migrate into the region and survive there at the end of the period time t+T.

Equations (1) to (4) look a little elaborate as attached to each term are time point or time period labels. We have done this to emphasise that the values of survival rates, fertility rates or in-migrant numbers may be changed according to the period considered. Thus we can incorporate into the projection model forecasted changes in survival or fertility rates, or in in-migrant numbers. However, the model may be re-stated in a simpler form

$$\underline{w}_{\mathcal{S}}^X(t+T) \quad = \quad \underline{S}^X \underline{w}^X(t) \quad + \quad \underline{I}_{\mathcal{S}}^X \tag{5}$$

$$\hat{\underline{w}}^F \quad = \quad ((\underline{w}_{\mathcal{S}}^F(t+T) \; - \; \underline{w}^F(t))0.5) \quad + \quad \underline{w}^F(t) \tag{6}$$

$$\underline{w}_B^X(t+T) \quad = \quad \underline{F}\,\hat{\underline{w}}^F \sigma^X s_{01}^X \quad + \quad \underline{I}_B^X \tag{7}$$

$$\underline{w}^X(t+T) \quad = \quad \underline{w}_{\mathcal{S}}^X(t+T) \quad + \quad \underline{w}_B^X(t+T) \tag{8}$$

3. The contents of the model matrices and vectors

Laid out in matrix form equations (1) and (5) look as follows;

$$\underset{\substack{\underline{w}_{\mathcal{S}}^X(t+T)\\ R\times 1}}{\begin{bmatrix} 0 \\ w_2^X(t+T) \\ w_3^X(t+T) \\ \vdots \\ w_R^X(t+T) \end{bmatrix}} = \underset{\substack{\underline{S}^X\\ R\times R}}{\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ s_{12}^X & 0 & 0 & \cdots & 0 \\ 0 & s_{23}^X & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots\; s_{R-1R}^X & s_{RR}^X \end{bmatrix}} \times \underset{\substack{\underline{w}^X(t)\\ R\times 1}}{\begin{bmatrix} w_1^X(t) \\ w_2^X(t) \\ w_3^X(t) \\ \vdots \\ w_R^X(t) \end{bmatrix}} + \underset{\substack{\underline{I}_{\mathcal{S}}^X\\ R\times 1}}{\begin{bmatrix} 0 \\ I_{*2}^X \\ I_{*3}^X \\ \vdots \\ I_{*R}^X \end{bmatrix}} \tag{9}$$

The population vectors, $\underline{w}_{\mathcal{S}}^X(t+T)$ and $\underline{w}^X(t)$ have been disaggregated into R age groups. Note that no population is recorded in the first age group of the end of period population vector, nor in the first age group of the in-migrant vector in this particular representation of the matrix contents. The survival rates matrix is structured as in cohort survival models with the principal sub-diagonal occupied by survival rates. These survival rates have the general form s_{r-1r}^X and they measure the number of persons who survive from age group r-1 at the beginning of the period into age group r at the end of the period within the region. Note that survival rates are of this form only if the age group intervals involved are all equal and are equal to the period over which the projection is made. Typical arrangements which satisfy those conditions are;

5 year age groups and a 5 year period
1 year age groups and a 1 year period
10 year age groups and a 10 year period.

Periods of a greater lengths than this run into the problem that some of the women at risk of giving birth may well themselves have been born in the same period.

Although equal age group intervals equal to the length of the period are the most convenient arrangement, sometimes lack of data or small population in the age/sex groups so defined prevents the adoption of such a model structure. This is so when the age group length exceeds the period. One common case is 5 year age groups and a 1 year period. The matrix contents of equations (1) and (5) now differ a little from those specified in equation (9). It is now possible to survive within an age group as well as from one age group to the next. Figure 1, a Lexis diagram, in which age is plotted against time, shows what happens.

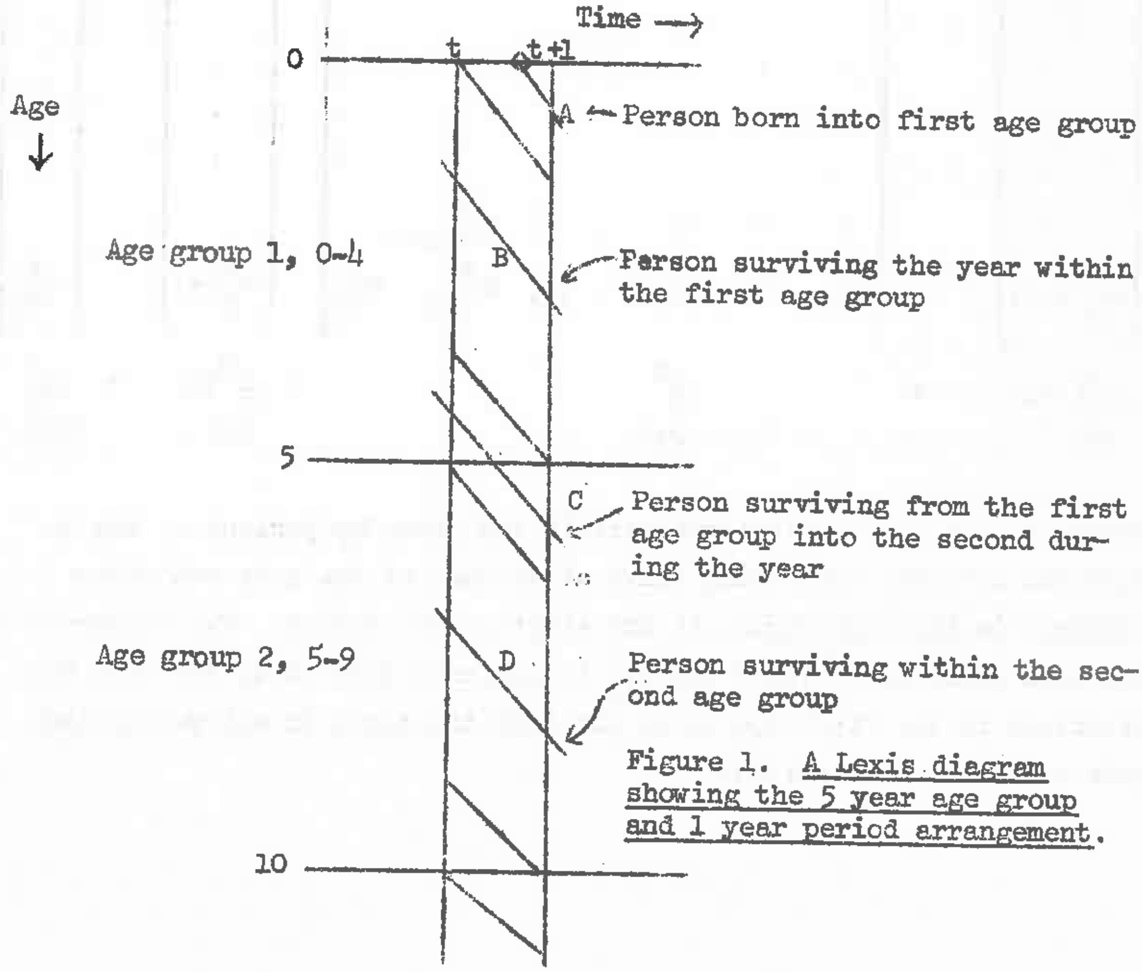


Figure 1. A Lexis diagram showing the 5 year age group and 1 year period arrangement.

Person A is born during the year (t,t+1) and survives into the first five year age group (0-4 years) at the end of the year (t+1). The lifeline labelled B is for a person who survives the year within the first age group. If the distributionof persons within the five year age group at the end of the year is rectangular (that is, even) some $\frac{4}{5}$ ths of the persons in any age group (bar the last) will have survived within the age group and one fifth will have survived from the previous age group. In the case of the 5-9 age group at the end of the year the five year olds, represented by lifeline C, will have survived from the first age group. The six to nine year olds, represented by lifeline D, survive within the second age group. They were aged 5 to 8 years at the start of the year.

The equivalent of equation (3) is, under this 5 year age group, 1 year time period arrangement, as follows

$$\begin{bmatrix} w_1^X(t+1) \\ w_2^X(t+1) \\ w_3^X(t+1) \\ \cdot \\ \cdot \\ \cdot \\ \\ w_R^X(t+1) \end{bmatrix} = \begin{bmatrix} s_{11}^X & 0 & 0 & \cdots & & & 0 \\ s_{12}^X & s_{22}^X & 0 & \cdots & & & 0 \\ 0 & s_{23}^X & s_{33}^X & \cdots & & & 0 \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ 0 & 0 & 0 & \cdots & s_{R-1R-1}^X & & \\ 0 & 0 & 0 & \cdots & s_{R-1R}^X & & s_{RR}^X \end{bmatrix} \times \begin{bmatrix} w_1^X(t) \\ w_2^X(t) \\ w_3^X(t) \\ \cdot \\ \cdot \\ \cdot \\ \\ w_R^X(t) \end{bmatrix} + \begin{bmatrix} I_{11}^X \\ I_{*2}^X \\ I_{*3}^X \\ \cdot \\ \cdot \\ \cdot \\ \\ I_{*R}^X \end{bmatrix}$$

$$\underset{R\times 1}{\underline{w}_S^X(t+1)} = \underset{R\times R}{\underline{S}^X} \times \underset{R\times 1}{\underline{w}^X(t)} + \underset{R\times 1}{\underline{I}_S^X} \qquad (10)$$

where s_{rr}^X is the survival rate within age group by persons o sex X. Survival involves both being alive at the end of the year and being resident in the same region at the start of the period. The subscript 1 has been added to $w_{11}^X(t+1)$ and I_{11}^X to indicate that these are only the survivors in the first age group and that the terms do not yet include persons born in the period.

The equations population at risk of giving birth, equations (2) and (6) can be supplied with contents in the following form when five year age groups are employed with either a five year or a one year period.

$$\begin{bmatrix} w_1^F \\ w_2^F \\ w_3^F \\ \hline w_4^F \\ \cdot \\ \cdot \\ \cdot \\ w_{10}^F \\ \hline w_{11}^F \\ \cdot \\ \cdot \\ \cdot \\ w_R^F \end{bmatrix} = \left[\begin{bmatrix} w_{11}^F(t+T) \\ w_2^F(t+T) \\ w_3^F(t+T) \\ \hline w_4^F(t+T) \\ \cdot \\ \cdot \\ \cdot \\ w_{10}^F(t+T) \\ \hline w_{11}^F(t+T) \\ \cdot \\ \cdot \\ \cdot \\ w_R^F(t+T) \end{bmatrix} - \begin{bmatrix} w_1^F(t) \\ w_2^F(t) \\ w_3^F(t) \\ \hline w_4^F(t) \\ \cdot \\ \cdot \\ \cdot \\ w_{10}^F(t) \\ \hline w_{11}^F(t) \\ \cdot \\ \cdot \\ \cdot \\ w_R^F(t) \end{bmatrix} \right] \times \begin{bmatrix} 0.5 \end{bmatrix} + \begin{bmatrix} w_1^F(t) \\ w_2^F(t) \\ w_3^F(t) \\ \hline w_4^F(t) \\ \cdot \\ \cdot \\ \cdot \\ w_{10}^F(t) \\ \hline w_{11}^F(T) \\ \cdot \\ \cdot \\ \cdot \\ w_R^F(t) \end{bmatrix}$$

$$\underset{R\times 1}{\underline{w}^F} = ((\underset{R\times 1}{\underline{w}_S^F(t+T)} - \underset{R\times 1}{\underline{w}^F(t)})\ \underset{1\times 1}{0.5}) + \underset{R\times 1}{\underline{w}^F(t)} \tag{11}$$

This expansion of equations (2) and (6) computes the mid-point or average population of women in the period, and thus accounts for any addition to the population through immigration or subtraction from the population via emigration or death. It corresponds with the population denominator used in the calculation of age specific fertility rates, either, exactly, if the mid-point population is used by the rate denominator, or approximately, if the mid-period population is used.

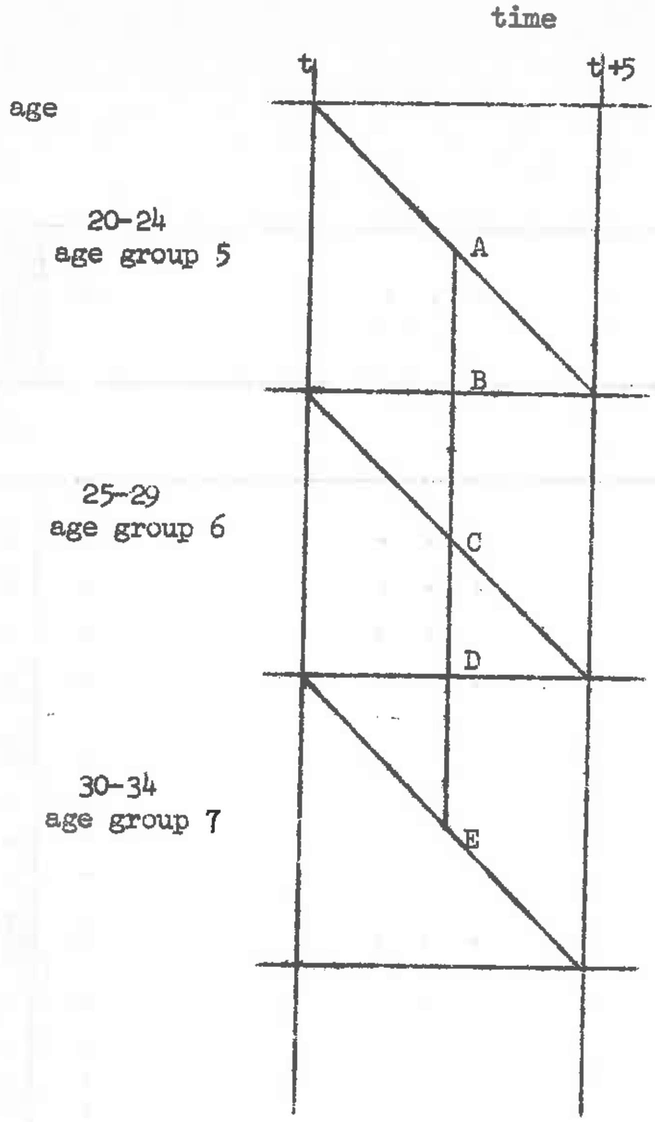
An alternative population at risk should be used if cohort specific fertility rates are used. The population at risk equation then takes the form

$$
\begin{bmatrix} w^F_{12} \\ w^F_{23} \\ \hdashline w^F_{34} \\ \cdot \\ \cdot \\ \cdot \\ w^F_{10,11} \\ \hdashline w^F_{11,12} \\ \cdot \\ \cdot \\ \cdot \\ w^F_{RR} \end{bmatrix}
=
\left[
\begin{bmatrix} w^F_2(t+T) \\ w^F_3(t+T) \\ \hdashline w^F_4(t+T) \\ \cdot \\ \cdot \\ \cdot \\ w^F_{11}(t+T) \\ \hdashline w^F_{12}(t+T) \\ \cdot \\ \cdot \\ \cdot \\ w^F_{R+1}(t+T) \end{bmatrix}
-
\begin{bmatrix} w^F_1(t) \\ w^F_2(t) \\ \hdashline w^F_3(t) \\ \cdot \\ \cdot \\ \cdot \\ w^F_{10}(t) \\ \hdashline w^F_{11}(t) \\ \cdot \\ \cdot \\ \cdot \\ w^F_R(t) \end{bmatrix}
\right]
\times
\begin{bmatrix} 0.5 \end{bmatrix}
+
\begin{bmatrix} w^F_1(t) \\ w^F_2(t) \\ \hdashline w^F_3(t) \\ \cdot \\ \cdot \\ \cdot \\ w^F_{10}(t) \\ \hdashline w^F_{11}(t) \\ \cdot \\ \cdot \\ \cdot \\ w^F_R(t) \end{bmatrix}
$$

$$
\underset{R\times 1}{\underline{w}^F} = ((\underset{R\times 1}{\underline{w}^F_S(t+T)} - \underset{R\times 1}{\underline{w}^F(t)})\ \underset{1\times 1}{0.5}) + \underset{R\times 1}{\underline{w}^F(t)} \qquad (12)
$$

In equation (12) the average population in a cohort that ages from one age group into another is considered to be the population at risk. The terms in the $\underline{w}^F(t+T)$ vector are listed one age group older than those in the $\underline{w}^F(t)$ vector. In both equations (11) and (12) the vectors have two dashed lines between certain terms. The age groups between the two dashed lines are those in which women are at risk of giving birth. In the second cohort case, the $\underline{w}^F(t)$ vector has to be marked off from age group 3 (10-14 yearsold) because these girls will enter the fertile age group 4 (15-19 years old) in part, if the period is less than five years in length and in total if the period is five years or longer.

The two population at risk definitions can be clarified through display on a Lexis diagram.



Populations at risk

Cohort 20-24 25-29: lifelines crossing AC

Age group 25-29: lifelines crossing BD

Cohort 25-29 30-34 lifelines crossing CE

Figure 2. A Lexis diagram showing the two kinds of population at risk

The population at risk of giving birth in age group 25-29, say, consists of the lifelines crossing line BD in Figure 1. Note that the women may be members of either the 20-24 year old cohort or the 25-29 year old cohort (the cohorts being defined at time t). The women of the 25-29 year old cohort (defined at time t) at risk in the five year period are represented by the lifelines crossing line CE. Line ABCDE is located at time $t+\frac{T}{2}$ or $t+\frac{5}{2}$ if the average population in the period is equal to the mid-point.

The population at risk of giving birth vector, calculated in equations (11) or (12) is used together with the appropriate age-specific or cohort rates in equation (3), which can be expanded as follows:

$$
\begin{bmatrix} w_{01}^{X}(t+T) \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & f_4 & f_5 & \cdots & f_{10} & 0 & \cdots & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix}
\times
\begin{bmatrix} w_1^F \\ w_2^F \\ w_3^F \\ \hline w_4^F \\ w_5^F \\ \vdots \\ w_{10}^F \\ \hline w_{11}^F \\ \vdots \\ w_R^F \end{bmatrix}
\times
\begin{bmatrix} \sigma^X \end{bmatrix}
\times
\begin{bmatrix} s_{01}^X \end{bmatrix}
+
\begin{bmatrix} \underline{I}_{01}^X \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

$$
\underset{R\times 1}{\underline{w}_B^X(t+T)} = \underset{R\times R}{\underline{F}} \quad \underset{R\times 1}{\underline{w}^F} \quad \underset{1\times 1}{\sigma^X} \quad \underset{1\times 1}{s_{01}^X} + \underset{R\times 1}{\underline{I}_B^X} \qquad (13)
$$

The fertility rates occupy only the first row of the $\underline{F}$ matrix which is otherwise empty. Fertility rates are entered in the first row only where statistics are conventionally collected, namely, for age group 15-19 through to age group 45-49. Births do occur to women aged 10-14 and aged 50-54, but these are very few in number. They are conventionally counted in with the births to 15-19 year old women and the 45-49 year old women respectively. Because there are zeroes in the non-fertile age group positions in the first row of the $\underline{F}$ matrix, the populations at risk in age groups 1 to 3 and 11 to the last do not come into play and are, in effect, made redundant. Equation (13) is specified for age-specific fertility rates for five year age groups. Slightly different specifications are needed for cohort rates or for rates not specified by five year age groups.

The births produced by multiplying $\underline{F}$ and $\underline{w}^F$ together have to be sexed by multiplying by a sex proportion σ^X which is fairly constant over time and age groups and is represented by a scalar in equation (13). The births of each sex have then to be survived over that fraction of the period remaining after birth by multiplication by an appropriate survival rate, s_{01}^X, which incorporates the risk of dying in situ, out-migrating and surviving, and out-migrating and dying just as did the $\underline{S}$ matrix of survival rates in equation (1). To these survived, sexed births must be added the infants born outside the region in the period who migrate into the region and survive there, the I_{01}^X term.

The final model equation ((4) or (8)) can be expanded thus in the five year equal age group/period case.

$$\begin{bmatrix} w_1^X(t+T) \\ w_2^X(t+T) \\ w_3^X(t+T) \\ \cdot \\ \cdot \\ \cdot \\ w_R^X(t+T) \end{bmatrix} = \begin{bmatrix} 0 \\ w_2^X(t+T) \\ w_3^X(t+T) \\ \cdot \\ \cdot \\ \cdot \\ w_R^X(t+T) \end{bmatrix} + \begin{bmatrix} w_{01}^X(t+T) \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

$$\underset{R\mathrm{x}1}{\underline{w}^X(t+T)} = \underset{R\mathrm{x}1}{\underline{w}_S^X(t+T)} + \underset{R\mathrm{x}1}{\underline{w}_B^X(t+T)} \qquad (14)$$

or thus in the five year age group/one year period case

$$\begin{bmatrix} w_1^X(t+1) \\ w_2^X(t+1) \\ w_3^X(t+1) \\ \cdot \\ \cdot \\ \cdot \\ w_R^X(t+1) \end{bmatrix} = \begin{bmatrix} w_{11}^X(t+1) \\ w_2^X(t+1) \\ w_3^X(t+1) \\ \cdot \\ \cdot \\ \cdot \\ w_R^X(t+1) \end{bmatrix} + \begin{bmatrix} w_{01}^X(t+1) \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

$$\underset{R\times 1}{\underline{w}^X(t+1)} = \underset{R\times 1}{\underline{w}_S^X(t+1)} + \underset{R\times 1}{\underline{w}_B^X(t+1)} \qquad (15)$$

5. The algebraic expression of the model equations

Wilson (1972) has pointed out that it is often useful to express models normally expressed in matrix form, as our model has been, in the form of algebraic equations. This further expresses the structure of the model and helps when a computer program for the model, or a variant of it, is written. For the standard case (equal age group intervals equal to the time period length) our simple projection model may be stated as follows:

for age groups $1 \leq s \leq R$

$$w_s^X(t+T) = s_{s-1}^X w_{s-1}^X(t) + I_{s-1}^X(t,t+T) \qquad (16)$$

for age group s = R, the last

$$w_R^X(t+T) = s_{R-1R}^X w_{R-1}^X(t) + I_{R-1R}^X(t,t+T) + s_{RR}^X w_R^X(t) + I_{RR}^X(t,t+T) \qquad (17)$$

for women at risk of giving birth in the fertile age groups, = α...β

age specific rates for age groups $\alpha \leq s \leq \beta$

$$w_s^F(t,t+T) = ((w_s^F(t+T) - w_s^F(t))0.5) + w_s^F(t) \qquad (18)$$

cohort rates for age groups $\alpha-1 < s < \beta$ at time t

$$w_s^F(t,t+T) = ((w_{s+1}^F(t+T) - w_s^F(t))0.5) + w_s^F(t) \tag{19}$$

<u>for the first age group s = 1</u>

using age specific rates

$$w_1^X(t+T) = s_{01}^X \sigma^X (\Sigma_{s=\alpha}^{\beta} f_s w_s^F(t,t+T)) + I_{01}^X(t,t+T) \tag{20}$$

or using cohort rates

$$w_1^X(t+T) = s_{01}^X \sigma^X (\Sigma_{s=\alpha-1}^{\beta} f_s w_s^F(t,t+T)) + I_{01}^X(t,t+T) \tag{21}$$

For the <u>shorter period case</u> (equal age group intervals larger than the time period length) the model equations can be expressed algebraically as follows:

<u>for age groups 1 < s < R</u>

$$w_s^X(t+T) = s_{s-1s}^X w_{s-1}^X(t) + I_{s-1s}^X(t,t+T) + s_{ss}^X w_s^X(t) + I_{ss}^X(T,t+T) \tag{22}$$

<u>for age group s = R</u>

equation (17) as before

<u>for women at risk of giving birth in the fertile age groups</u>

equation (18) or (19) as before

<u>for the first age group s = 1</u>

using age-specific rates

$$w_1^X(t+T) = s_{01}^X \sigma^X (\Sigma_{s=\alpha}^{\beta} f_s w_s^F(t,t+T)) + I_{01}^X(t,t+T) + s_{11}^X w_1^X(t) + I_{11}^X(t,t+T) \tag{23}$$

or using cohort rates

$$w_1^X(t+T) = s_{01}^X \sigma^X (\Sigma_{s=\alpha-1}^{\beta} f_s w_s^F(t,t+T)) + I_{01}^X(t,t+T) + s_{11}^X w_1^X(t) + I_{11}^X(t,t+T) \qquad (24)$$

Note that we still need four equations in the standard case when these are expressed algebraically but they differ from those in the matrix formulation.

5. The relationship of the model to other demographic work

As presented above our simple model is a single region population projection model. In fact, no locational superscript has been used to indicate the region to which the model equations apply. It is, however, quite easy to convert it into a multi-regional model by adding locational superscripts and by representing in-migration not as a flow, but as a product of an out-migration rate multiplied by the population of the sending region. We do this for the standard case algebraic version of the model for equations (16) through (21):

for age group $1 < s < R$

$$w_s^{iX}(t+T) = s_{s-1s}^{iiX} w_{s-1}^{iX}(t) + \Sigma_{j\neq i} m_{s-1s}^{jiX} w_{s-1}^{jx}(t) \qquad (25)$$

where m_{s-1s}^{jiX} is the rate at which persons of sex X in age group s-1 at time t, migrate from region j to region i and survive there aged s at time t+T. If we identify the variable $M_{s-1s}^{jiX}(t,t+T)$ as a count of the persons who so migrate then

$$m_{s-1s}^{jiX} = M_{s-1s}^{jiX}(t,t+T)/w_{s-1}^{jx}(t) \qquad (26)$$

noting that

$$\Sigma_{j\neq i} M_{s-1s}^{jiX}(t,t+T) = I_{s-1s}^X(t,t+T) \qquad (27)$$

for age group s = R

$$w_R^{iX}(t) = s_{R-1R}^{iiX} w_{R-1}^{iX}(t) + \Sigma_{j\neq i} m_{R-1R}^{jiX} w_{R-1}^{jX}(t) + s_{RR}^{iiX} w_R^{iX}(t) + \Sigma_{j\neq i} m_{RR}^{jiX} w_R^{jX}(t) \quad (28)$$

for women at risk of giving birth

$$W_s^{iF}(t,t+T) = ((w_s^{iF}(t+T) - w_s^{iF}(t))0.5) + w_s^{iF}(t) \quad (29)$$

in the age specifications case, or the cohort rate case

$$W_s^{iF}(T+T) = ((w_{s+1}^{iF}(t+T) - w_s^{iF}(t))0.5) + w_s^{iF}(t) \quad (30)$$

for the first age group

$$w_1^{iX}(t+T) = s_{01}^{iiX} \sigma^{iX} (\Sigma_{s=\alpha}^{\beta} f_s^i W_s^{iF}(t,t+T)) + \Sigma_{j\neq i} m_{01}^{jiX} \left[\sigma^{jX}(\Sigma_{s=\alpha}^{\beta} f_s^j W_s^{jF}(t,t+T))\right] \quad (31)$$

where m_{01}^{jiX} is the rate at which persons of sex x born in region j migrate to region i and survive there in the first age group. If we identify the variable M_{01}^{jiX} with such persons then the rate may be defined as

$$m_{01}^{jiX} = M_{01}^{jiX}(t,t+T)/B_{01}^{j*X}(t,t+T) \quad (32)$$

where B_{01}^{j*X} are the live births that occur of persons of sex X in region i which are modelled as

$$B_{01}^{j*X}(t,t+T) = \sigma^{jX}(\Sigma_{s=\alpha}^{\beta} f_s^j W_s^{jF}(t,t+T) \quad (33)$$

As before the locationally disaggregated migration flows sum to the total in-migration into the region

$$\Sigma_{j\neq i} M_{01}^{jiX} = I_{01}^{X}(t,t+T) \quad (34)$$

Summation of women aged from $s=\alpha-1$ to $s=\beta$ in equation (32) makes it appropriate for use with cohort rates.

A similar ser of multi-regional projection equations can be defined for the shorter period case.

The multiregional version is undoubtedly a more general statement of the model. However, we do not use it in our later examoles (sections 6 and 7) because of the difficulty of defining suitable out-migration rates. In the example of the West Indian population in Leeds (section 6) it is more appropriate to make an estimate of the migration flows of West Indians into Leeds from overseas based on knowledge of work permit quotas, knowledge about dependants yet to immigrate and recent trends in numbers entering the country, than to study and estimate an out migration rate from the rest of the world outside Leeds. Similarly, in Leeds ward example, it is easier and probably more reliable, to estimate the total migrating into the ward than to estimate the out-migration rates of all other wards and the rest of the world to the one ward.

However, the convension of the model into a multi-regional form does enable us to show the rates used are connected with those derived from spatial demographic accounts (Rees and Wilson,1973a, Rees 1973; Wilson and Rees, 1974). In a review of available models of population growth, Rees and Wilson, 1973b, showed that all such models are associated with a set of accounting equations, either explicitly or implicitly(and more usually the latter), and that few sets of accounting equations were fully consistent.

Table 1 shows how an accounts table corresponding to the simple model of population projection presented here might be laid out for age group $1 < s < R$ involving age transitions $s-1, s$. Table 2 is for age group 1 and Table 3 for age group R. The accounts notation is slightly extended from that used in Rees (1973) Wilson and Rees (1973) and Smith and Rees (1974) in that all life states (existance and survival as well as birth and death) are distinguished:

starting state   starting state location   ending state   ending state location

$$K^{\alpha(i)\omega(j)x}_{rs} \quad \text{Sex}$$

starting state age   ending state age

K:	population, persons
starting states:	$\left.\begin{array}{l}\varepsilon \text{ existence (at time t)} \\ \beta \text{ birth in period t to t+T}\end{array}\right\}\alpha$
ending states:	$\left.\begin{array}{l}\sigma \text{ survival at time t+T} \\ \delta \text{ death in period t to t+T}\end{array}\right\}\omega$
locations:	i, j. R rest of the world
starting age group:	r
ending age group:	s
sex:	X

The usual row and column accounting relationships hold((Rees,(1973), Wilson and Rees (1973)) for initial regional populations births in regions regional deaths and final regional populations. For example, the row identity holds for the initial population in a region (from Table 1)

$$K_{s-1s}^{\varepsilon(i)\sigma(i)X} + K_{s-1s}^{\varepsilon(i)\sigma(R)X} + K_{s-1s-1}^{\varepsilon(i)\delta(i)X} + K_{s-1s}^{\varepsilon(i)\delta(i)X} + K_{s-1s-1}^{\varepsilon(i)\delta(R)X} + K_{s-1s}^{\varepsilon(i)\delta(R)X} \equiv K_{s-1*}^{\varepsilon(i)*(*)X} \quad (35)$$

where an asterisk indicates summation over the relevant life state, location or age group. If we know all the terms in the accounts we can define transition rates by dividing any term by the corresponding row total.

$$h_{rs}^{\alpha(i)w(j)X} = K_{rs}^{\alpha(i)\omega(j)X}/K_{r*}^{\alpha(i)*(*)X} \quad (36)$$

These h rates are true probabilities if we are able to supply the exact accounts information to the right hand side of equation (36).

We examine the probabilities of surviving within a reign as derived from the accounts in detail as these correspond with the survival rates incorporated in the simple population projection model. They are defined as:

<u>for s = 1</u>

$$h_{01}^{\beta(i)\sigma(i)X} = K_{01}^{\beta(i)\sigma(i)X}/K_{0*}^{\beta(i)*(*)X} \quad (37)$$

Transition From \ To Age Group			Region of survival at time t+T		Region of death in period t to t+T				TOTALS
			Region i	Region R	Region i		Region R		
			s	s	s-1	s	s-1	s	
Region of existence at time t	Region i	s-1	${}_{s}X_{s-1s}\,{}_{w}X_{s-1}(t)$ ${}_{s}^{ii}X_{s-1s}\,{}_{w}^{i}X_{s-1}(t)$ $K^{\varepsilon(i)\sigma(i)X}_{s-1s}$	$K^{\varepsilon(i)\sigma(R)X}_{s-1\,s}$	$K^{\varepsilon(i)\delta(i)X}_{s-1s}$	$K^{\varepsilon(i)\delta(i)X}_{s-1s}$	$K^{\varepsilon(i)\delta(R)X}_{s-1s-1}$	$K^{\varepsilon(i)\delta(R)X}_{s-1s}$	$w^{X}_{s-1}(t)$ $w^{iX}_{s-1}(t)$ $K^{\varepsilon(i)*(*)X}_{s-1*}$
	Region R	s-1	I^{X}_{s-1s} $m^{RiX}_{s-1s}\,w^{RX}_{s-1}(t)$ $K^{F(R)\sigma(i)X}_{s-1s}$	-	$K^{\varepsilon(R)\delta(i)X}_{s-1s}$	$K^{\varepsilon(R)\delta(i)X}_{s-1s}$	-		-
TOTALS			$w^{X}_{s}(t+T)$ $w^{iX}_{s}(t+T)$ $K^{\varepsilon(*)\sigma(i)}_{s-1s}$	-	$K^{\varepsilon(*)\delta(i)X}_{s-1s-1}$	$K^{\varepsilon(*)\delta(i)X}_{s-1s}$	-		-

Table 1. The correspondence of model and accounts terms (Standard)

For age group 1 < s < R

Transition From \ To		Age Group	Region of survival at time t+T: Region i	Region of survival at time t+T: Region R	Region of death at time t to t+T: Region i	Region of death at time t to t+T: Region R	TOTALS
			1	1	1	1	
Region of Birth	Region i	0	$s_{01}^{X}\sigma^{X\beta}(\Sigma_{s=\alpha} f_s^{F})$ $s_{01}^{iiX}\sigma^{iX\beta}(\Sigma_{s=\alpha} f_s^{i} w_s^{iF})$ $K_{*1}^{\beta(i)\sigma(i)X}$	$K_{*1}^{\beta(i)\sigma(R)X}$	$K_{*1}^{\beta(i)\delta(i)X}$	$K_{*1}^{\beta(i)\delta(R)X}$	$K_{*1}^{\beta(i)*(*)X}$
	Region R	0	I_{01}^{X} $\Sigma_{j\neq i} m_{01}^{jiX}(\sigma^{jX\beta}(\Sigma_{=\alpha} f_s^{j} w_s^{jF}))$ $K_{*1}^{\beta(R)\sigma(i)X}$	-	$K^{\beta(R)\delta(i)X}$	-	-
TOTALS			$w_1^{X}(t+T)$ $w_1^{iX}(t+T)$ $K_{*1}^{\beta(*)\sigma(i)X}$	-	$K_{*1}^{\beta(*)\delta(i)X}$	-	-

Table 2. <u>The correspondence of model and accounts terms (Standard case) for age group S=1</u>

Transition: From / To / Age Group			Region of survival at time t+T: Region i	Region of survival at time t+T: Region R	Region of deaths in period t to t+T: Region i	Region of deaths in period t to t+T: Region R	Totals
			R	R	R	R	
Region of Existence at time t	Region i	R-1	$s^{X}_{R-1R}w^{X}_{R-1}(t)$ $s^{iiX}_{R-1R}w^{iX}_{R-1}(t)$ $K^{\varepsilon(i)\sigma(i)X}_{R-1R}$	$K^{\varepsilon(i)\delta(i)X}_{R-1R}$	$K^{\varepsilon(i)\delta(i)X}_{R-1R}$	$K^{\varepsilon(i)\delta(R)X}_{R-1R}$	$K^{\varepsilon(i)*(*)X}_{R-1*}$
		R	$s^{X}_{RR}w^{X}_{R}(t)$ $s^{iiX}_{RR}w^{iX}_{R}(t)$ $K^{\varepsilon(i)\sigma(i)X}_{RR}$	$K^{\varepsilon(i)\sigma(R)X}_{RR}$	$K^{\varepsilon(i)\delta(i)X}_{RR}$	$K^{\varepsilon(i)\delta(R)X}_{RR}$	$K^{\varepsilon(i)*(*)X}_{R*}$

TABLE 3. The correspondence of model and account terms (standard case)

For age group S = R

TABLE 3. Continued.

Region of Existence at time t	Region R	R-1	I^{X}_{R-1R} $\Sigma_{j\neq i}\, m^{jiX}_{R-1R}\, w^{jX}_{R-1}(t)$ $K^{\varepsilon(R)\sigma(i)X}_{R-1R}$	-	$K^{\varepsilon(R)\delta(i)X}_{R-1R}$	-	-
		R	I^{X}_{RR} $\Sigma_{j\neq i}\, m^{jiX}_{RR}\, w^{jX}_{R}(t)$ $K^{\varepsilon(R)\sigma(i)X}_{RR}$	-	$K^{\varepsilon(R)\delta(i)X}_{RR}$	-	-
TOTALS			$w^{X}_{R}(t+T)$ $w^{iX}_{R}(t+T)$ $K^{\varepsilon(*)\sigma(i)X}_{*R}$	-	$K^{\varepsilon(*)\delta(i)X}_{*R}$	-	-

for 1 < s < R

$$h^{\varepsilon(i)\sigma(i)X}_{s-1s} = K^{\varepsilon(i)\sigma(i)X}_{s-1s} / K^{\varepsilon(i)*(*)X}_{s-1*} \quad (38)$$

and for the age transition R,R

$$h^{\varepsilon(i)\sigma(i)X}_{RR} = K^{\varepsilon(i)\sigma(i)X}_{RR} / K^{\varepsilon(i)*(*)X}_{R*} \quad (39)$$

Now, if the survival rates equal the corresponding accounts based transition probabilities, that is, if

$$s^{iiX}_{01} = h^{\beta(i)\delta(i)X}_{01} \quad (40)$$

$$s^{iiX}_{s-1s} = h^{\varepsilon(i)\sigma(i)X}_{s-1s} \quad (41)$$

and

$$s^{iiX}_{RR} = h^{\varepsilon(i)\sigma(i)X}_{RR} \quad (42)$$

then the simple model for population projection has a correct accounting basis in its survival mechanism. Similar points apply to the way births are modelled and immigrants are recorded.

The only way equations (40), (41) and (42) and other similar correspondences will be fully satisfied will be to construct spatial demographic accounts. This is a fairly demanding prodedure which may be very difficult in many circumstances. So, the survival rates must usually be estimated in a cruder fashion but in a way which ensures that the same kind of rate is being measured.

This point can be illustrated by describing the survival rates used in the first, standard case examples were obtained. No information on survival specific to the Leeds population or the West Indian population in Leeds was to hand. What was available was a population accounts table for a West Yorkshire Study Area (Rees, Smith and King, 1974) and the associated transition rates. The probabilities of survival, and of migration and survival over the five year intercensal period 1961-1966 were calculated and added together to give the probability of survival anywhere*. Three regions were involved:

*These are the accounting equivalents of the life table survival rates.

a West Yorkshire Study Area, the rest of England and Wales and the rest of the world, which we label WY, REW and RTW respectively. The procedure was

$$h_{s-1s}^{\varepsilon(WY)\sigma(*)X} = h_{s-1s}^{\varepsilon(WY)\sigma(WY)X} + h_{s-1s}^{\varepsilon(WY)\sigma(REW)X} + h_{s-1s}^{\varepsilon(WY)\sigma(RTW)X} \qquad (43)$$

For males in age group 20-24 at Census date 1961, for example, the probabilities were

$$h_{56}^{\varepsilon(WY)\sigma(*)M} = 0.99437 = 0.86927 + 0.09117 + 0.03393$$

An estimate was made of the probability of migrating out of Leeds C.B. and surviving the period 1961-66 (the exact equations are in King 1974), and this was subtracted from the West Yorkshire Study Area survival "anywhere" probability to yield an estimated survival rate within Leeds for the 1961-66 period:

$$s_{s-1s}^{LLX} = h_{s-1s}^{\varepsilon(WY)\sigma(*)X} - \Sigma_{j\neq L} h_{s-1s}^{\varepsilon(L)\sigma(j)X} \qquad (45)$$

where L refers to Leeds. For the males in Leeds aged 20-24 at census date 1961 this works out as

$$s_{56}^{LLM} = 0.99437 - 0.26601 = 0.72836 \qquad (46)$$

Note that s_{56}^{LLM} is a good deal lower than $h_{56}^{\varepsilon(WY)\sigma(WY)M}$ because a larger proportion of the population of a small area outmigrate from such an area than do from a larger area. The s_{s-1s}^{LLX} survival rates are used in the West Indian projection model. By the time these ar employed in the model we have been forced to assume that

(1) the probability of suriving anywhere for Leeds inhabitants are the intercensal period 1961-66 is the same as the equivalent probability in the West Yorkshire Study Area.

(2) the age disaggregation of migration out of Leeds is the same as out of the West Yorkshire Study Area.

(3) the survival rates estimated for the whole of the Leeds population apply with no change to the West Indian population, and that

(4) the 1961-66 survival rates so estimated are applicable for the next four intercensal periods (1966-1971, 1971-1976, 1976-1981 and 1981-1986) without change.

Only a formidable demographic information system and much accounting work could allow us to relax these assumptions and to calculate the survival rates for the West Indian population in Leeds more directly. In the absence of such a system and such work we can only state our belief that the true survival rates do not differ significantly from those used in the model.

In this section we have, it is hoped, been able to show that the model of projection described in sections 2 to 4 is a much simplified but consistent version of the projection model associated with spatial demographic accounts. The price of the simplification is usually a set of assumptions such as those set out above.

In the next two sections we show two examples of the model in operation. This should give the reader unversed in abstract mathematics but familiar with numerical computation a better idea of how the model works.

The first example concerns the population of West Indian origin living in Leeds C.B. in 1971. The residential distribution within Leeds of persons born in the West Indies is shown in Figure 3. West Indians are concentrated in inner city districts north of the city centre, particularly in the Chapeltown area, but West Indian families are found in all wards of the city. So the projections of the West Indian population from 1971 to 1986 described in the next section have different implications for the various districts of the city.

The first example involves projection of the population of a small population group for the whole city. The second example examines the likely evolution of the whole population of a small area of the city, East and West Hunslet ward, located to the south of the city centre (Figure 3).

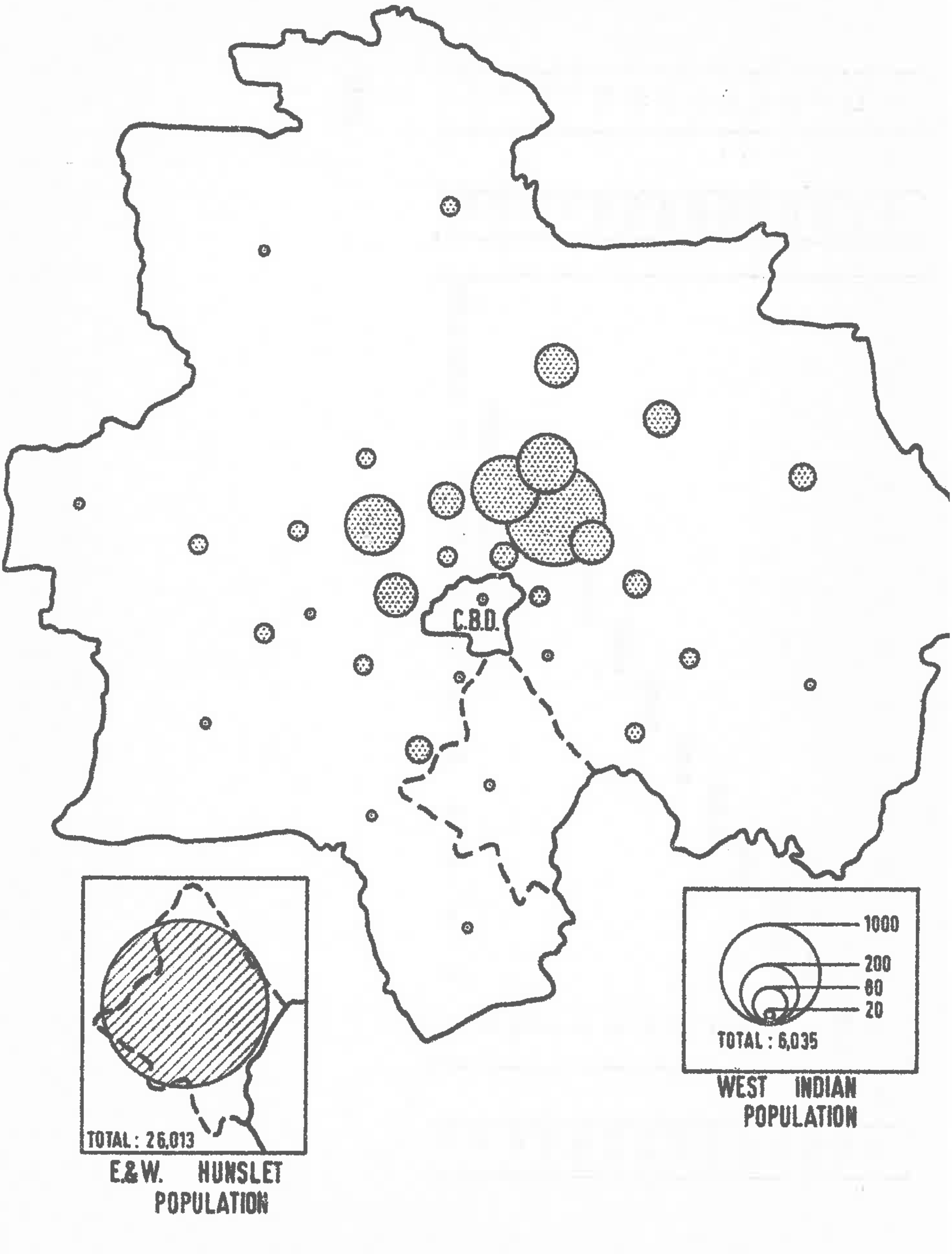


Figure 3 The West Indian and East and West Hunslet populations of Leeds C.B. at census date 1971 (April 25/26)

$$
\begin{bmatrix} 0 \\ 370 \\ 272 \\ 257 \\ 274 \\ 192 \\ 265 \\ 343 \\ 313 \\ 214 \\ 128 \\ 74 \\ 55 \\ 34 \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.861 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0.864 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & 0.830 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & 0.696 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & 0.758 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & 0.757 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0.859 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0.857 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0.917 & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0.908 & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0.896 & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0.875 & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.851 & 0.688
\end{bmatrix}
\begin{bmatrix} 412 \\ 314 \\ 247 \\ 257 \\ 173 \\ 267 \\ 351 \\ 334 \\ 220 \\ 131 \\ 75 \\ 63 \\ 20 \\ 24 \end{bmatrix}
+
\begin{bmatrix} 0 \\ 15 \\ 1 \\ 52 \\ -95 \\ -61 \\ -63 \\ 41 \\ 27 \\ 13 \\ 9 \\ 6 \\ 0 \\ 0 \end{bmatrix}
\tag{47}
$$

The two examples involve populations of similar magnitude - there were some 6,035 West Indians in Leeds and some 26,013 inhabitants of East and West Hunslet at census date 1971.

The East and West Hunslet projection is presented only as an example of what might happen to the population of an Inner City ward if present migration patterns persisted. Of course, they are likely to change quite radically. The modelling and forcasting of migration flows and rates is a field in need of further development.

6. A standard case example: the West Indian population in Leeds 1961-1986

Leeds had in 1971, a small community of ~~just~~ over 6000 persons who were either born in the West Indies or whose parents were born there. They make up about 1.2 per cent of the population of the city. Recent measurement of the fertility rates of West Indian women (King, 1974) indicates that they have been experiencing levels of fertility about twice those of the British born population although the levels are in process of decline. We can examine what implications these fertility levels have for the size of the West Indian community by using the simple model for population projection we have outlined. We show all the model equations for the first projection period 1971-76.

Equation (47) below contains the relevant figures for 1971-76 for West Indian women. It is the "numbers" version of equation (9): where "c.d. 1976" refers to census date in April 1976, "c.d. 1971" refers to census date April 24/25 1971 and "1971-76" refers to the intercensal period between these dates.

We recall that multiplication of two matrices involves the following operation:

$$c_{ij} = \sum_k a_{ik} b_{kj} \qquad (48)$$

where c_{ij} is the element in the ith row and jth column of the product matrix C, a_{ik} is the element in the ith row and kth column of the first multiplicand matrix A and b_{kj} the element in the kth row and jth column of the second multiplicand matrix B. Each element of $\underline{w}_S^F$(c.d.1976) is similarly constructed. For example, the element in the fourth row of $\underline{w}_S^F$(c.d.1976) which refers to age group 15-19, is calculated as follows

$$
\begin{aligned}
w_4^F(\text{c.d.}1976) = & \ (0 \times 412) + (0 \times 314) + (0.830 \times 247) + (0 \times 257) \\
& + (0 \times 173) + (0 \times 267) + (0 \times 351) + (0 \times 334) \\
& + (0 \times 220) + (0 \times 131) + (0 \times 75) + (0 \times 63) \\
& + (0 \times 20) + (0 \times 24) + 52
\end{aligned} \qquad (49)
$$

This reduces because of all the zero rates, to

$$
w_4^F(\text{c.d.}1976) = (0.830)247 + 52 = 257 \qquad (50)
$$

which is the numerical version of equation (16) for age group 4

The survival rates in the subdiagonal do not decline as a simple function of age as do life table survival rates or accounts based survival anywhere rates. They decline with age, then rise, then fall again. This is because the survival rates involve out-migration as well as death. The rate of out-migration is highest in the (3,4), (4,5) and (5,6) age group transitions (Smith and Rees, 1974, Figure 26). The first rows of the $\underline{w}_S^F$(c.d. 1976) vector, the $\underline{S}^F$ matrix and the $\underline{I}_S^F$ vector all contain zeroes because we calculate these terms in later equations, but it is necessary to represent them there conceptually because we need to use all 14 age groups in the $\underline{w}_S^F$(c.d. 1971) vector.

Once the population of West Indian women at Census date 1976 has been calculated we can work out the population at risk for use with the cohort fertility rates:

$$\begin{bmatrix} 391 \\ 293 \\ \hline 252 \\ 266 \\ 183 \\ 266 \\ 347 \\ 323 \\ 217 \\ 129 \\ \hline 75 \\ 59 \\ 27 \\ 12 \end{bmatrix} = \left[\begin{bmatrix} 370 \\ 272 \\ \hline 257 \\ 274 \\ 192 \\ 265 \\ 343 \\ 313 \\ 214 \\ 128 \\ \hline 74 \\ 55 \\ 34 \\ 0 \end{bmatrix} - \begin{bmatrix} 412 \\ 314 \\ \hline 247 \\ 257 \\ 173 \\ 267 \\ 351 \\ 334 \\ 220 \\ 131 \\ \hline 75 \\ 63 \\ 20 \\ 24 \end{bmatrix} \right] \times \begin{bmatrix} 0.5 \end{bmatrix} + \begin{bmatrix} 412 \\ 314 \\ \hline 247 \\ 257 \\ 173 \\ 267 \\ 351 \\ 334 \\ 220 \\ 131 \\ \hline 75 \\ 63 \\ 20 \\ 24 \end{bmatrix}$$

The third equation for the West Indian population projections forecasts the number of births to women at risk using cohort fertility rates for five year cohorts derived from a study of the fertility of immigrants in Leeds (King, 1973, 1974a, 1974b). The numbers of females surviving in the first age group at the end of the intercensal period 1971-76 is:

$$
\begin{bmatrix} 448 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0.349 & 0.698 & 0.818 & 0.809 & 0.526 & 0.397 & 0.172 & 0.078 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\times
\begin{bmatrix} 391 \\ 293 \\ 252 \\ 266 \\ 183 \\ 266 \\ 347 \\ 323 \\ 217 \\ 129 \\ 73 \\ 59 \\ 27 \\ 12 \end{bmatrix}
\times
\begin{bmatrix} 0.490 \end{bmatrix}
\times
\begin{bmatrix} 0.907 \end{bmatrix}
+
\begin{bmatrix} 8 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

$$
\underline{w}_B^F(\text{c.d. } 1976) = \underline{F}(1971\text{-}76) \quad \underline{w}^F \quad \sigma^F \quad s_{01}^F + \underline{I}_B^F \qquad (52)
$$

Because cohort fertility rates are used there are non-zero entries in the first row of the F matrix from the third age group to the tenth (10-14 to 45-49). The total fertility rate assumed for West Indian women in the 1971-76 period is calculated in this case by adding up all the individual cohort rates and is some 3.847. Note that we do not need to multiply each rate by five first as they refer to a five year period. This is well above the total fertility of all women either in Leeds or in England and Wales. A sex proportion of 0.490 for females is assumed, and a survival in Leeds rate of 0.907. A very small number of in-migrating children from elsewhere in the period is forecast.

The final step in this example is to combine the survivors vector with the infants vector by adding them together:

$$\begin{bmatrix} 448 \\ \hline 370 \\ 272 \\ 257 \\ 274 \\ 192 \\ 265 \\ 343 \\ 313 \\ 214 \\ 128 \\ 74 \\ 55 \\ 34 \end{bmatrix} = \begin{bmatrix} 0 \\ \hline 370 \\ 272 \\ 257 \\ 274 \\ 192 \\ 265 \\ 343 \\ 313 \\ 214 \\ 128 \\ 74 \\ 55 \\ 34 \end{bmatrix} + \begin{bmatrix} 448 \\ \hline \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

$$\underline{w}^F(\text{c.d. } 1976) = \underline{w}_S^F(\text{c.d. } 1976) + \underline{w}_B^F(\text{c.d. } 1976) \qquad (53)$$

In this standard age group/period case this is merely a tidying up operation. In the shorter period example it has rather more significance.

In order to project the population of West Indian women in Leeds forward to 1986 forecasts have to be made of the contents of the survival rates matrix, the surviving in-migrant vector, the fertility rate schedule, the sex proportion, the survival rate for infants and the number of surviving in-migrants infants. The rates and numbers assumed are set out in Tables 4,5,6,7,8 and 9. The figures are discussed and justified at length in King (1974).

Table 4 The survival rates assumed for West Indian women in Leeds

Age group at start of period No.	Age range	Age group at end of period No.	Age range	Survival rates 1971-76	1976-81	1981-86
1	0- 4	2	5- 9	0.86053	0.86053	0.86053
2	5- 9	3	10-14	0.86415	0.86415	0.86415
3	10-14	4	15-19	0.83019	0.83019	0.83019
4	15-19	5	20-24	0.69573	0.69573	0.69573
5	20-24	6	25-29	0.75777	0.75777	0.75777
6	25-29	7	30-34	0.75697	0.75697	0.75697
7	30-34	8	35-39	0.85894	0.85894	0.85894
8	35-39	9	40-44	0.85562	0.85562	0.85562
9	40-44	10	45-49	0.91669	0.91669	0.91669
10	45-49	11	50-54	0.90820	0.90820	0.90820
11	50-54	12	55-59	0.89633	0.89633	0.89633
12	55-59	13	60-64	0.87484	0.87484	0.87484
13	60-64	14	65&over	0.85106	0.85106	0.85106
14	65&over	14	65&over	0.68814	0.68814	0.68814

Table 5. The surviving in-migrant vectors assumed for West Indian women in Leeds

Age group at start of period No.	Age group	Age group at end of period No.	Age group	In-migrants 1971-76	1976-81	1981-86
1	0- 4	2	5- 9	15	15	15
2	5- 9	3	10-14	1	1	1
3	10-14	4	15-19	52	47	47
4	15-19	5	20-24	95	86	86
5	20-24	6	25-29	61	55	55
6	25-29	7	30-34	63	57	57
7	30-34	8	35-39	41	37	37
8	35-39	9	40-44	27	24	24
9	40-44	10	45-49	13	12	12
10	45-49	11	50-54	9	8	8
11	50-54	12	55-59	6	6	6
12	55-59	13	60-64	0	0	0
13	60-64	14	65&over	0	0	0
14	65&over	14	65&over	0	0	0

Table 6 The fertility rate schedules assumed

Assumption	Cohort Age group at start of period No.	Age group	Age group at end of period No.	Age range	Cohort fertility rates for period : 1971-76	1976-81	1981-86
High Fertility Assumed	3	10-14	4	15-19	0.3488	0.3488	0.3488
	4	15-19	5	20-24	0.6975	0.6975	0.6975
	5	20-24	6	25-29	0.8180	0.8180	0.8180
	6	25-29	7	30-34	0.8090	0.8090	0.8090
	7	30-34	8	35-39	0.5260	0.5260	0.5260
	8	35-39	9	40-44	0.3970	0.3970	0.3970
	9	40-44	10	45-49	0.1720	0.1720	0.1720
	10	45-49	11	50-54	0.0780	0.0780	0.0780
Low Fertility Assumed	3	10-14	4	15-19	0.3488	0.1300	0.1300
	4	15-19	5	20-24	0.6975	0.2600	0.2600
	5	20-24	6	25-29	0.8180	0.7360	0.7360
	6	25-29	7	30-34	0.8090	0.7415	0.7415
	7	30-34	8	35-39	0.5260	0.4080	0.4080
	8	35-39	9	40-44	0.3970	0.1450	0.1450
	9	40-44	10	45-49	0.1720	0.0510	0.0510
	10	45-49	11	50-54	0.0780	0.0025	0.0025

Table 7 The sex proportion

Sex	Period 1971-76	1976-81	1981-86
Female	0.4895	0.4895	0.4895
Male	0.5105	0.5105	0.5105

Table 8. The survival rate for infants

Age group at start of period		Age group at end of period		Period		
No.	Age range	No.	Age group	1971-76	1976-81	1981-86
0	Birth	1	0- 4	0.90709	0.90709	0.90709

Table 9. Numbers of surviving in-migrant infants

Age group at start of period		Age group at end of period		Period		
No.	Age range	No.	Age range	1971-76	1976-81	1981-86
0	Birth	1	0- 4	8	8	8

We have assumed (Table 4) constant survival "in situ" rates over the three projection rates. The death rate component of these rates will undoubtedly decline slightly over time and the migration rate component will fluctuate. But as a first approximation these rates can be assumed constant without too much effect on the future population. The number of surviving in migrants(Table 5) is assumed to decline from 1971-76, when dependants of earlier 1960's migrants will still be arriving, to a lower level in 1976-81 and in 1981-86 when only A and B voucher holders and their dependants will be arriving.

We can in fact compare these in-migration vectors with those of out-migration and survival implied in our model. The rates of out-migration given in King (1974, Table 1) produce in 1971-76 the following numbers of surviving out-migrants which can be compared with corresponding number of in-migrants.

Age transition (see Table 5 for age ranges)	surviving out-migrants	surviving in-migrants	Net surviving in-migrants
1- 2	55	15	-40
2- 3	42	1	-41
3- 4	42	52	10
4- 5	78	95	17
5-6	41	61	20
6-7	64	63	- 1
7-8	47	41	- 6
8- 9	45	27	-18
9-10	15	13	- 2
10-11	9	9	0
11-12	5	6	1
12-13	4	0	- 4
13-14	1	0	- 1
14-14	3	0	- 3
	451	383	-60

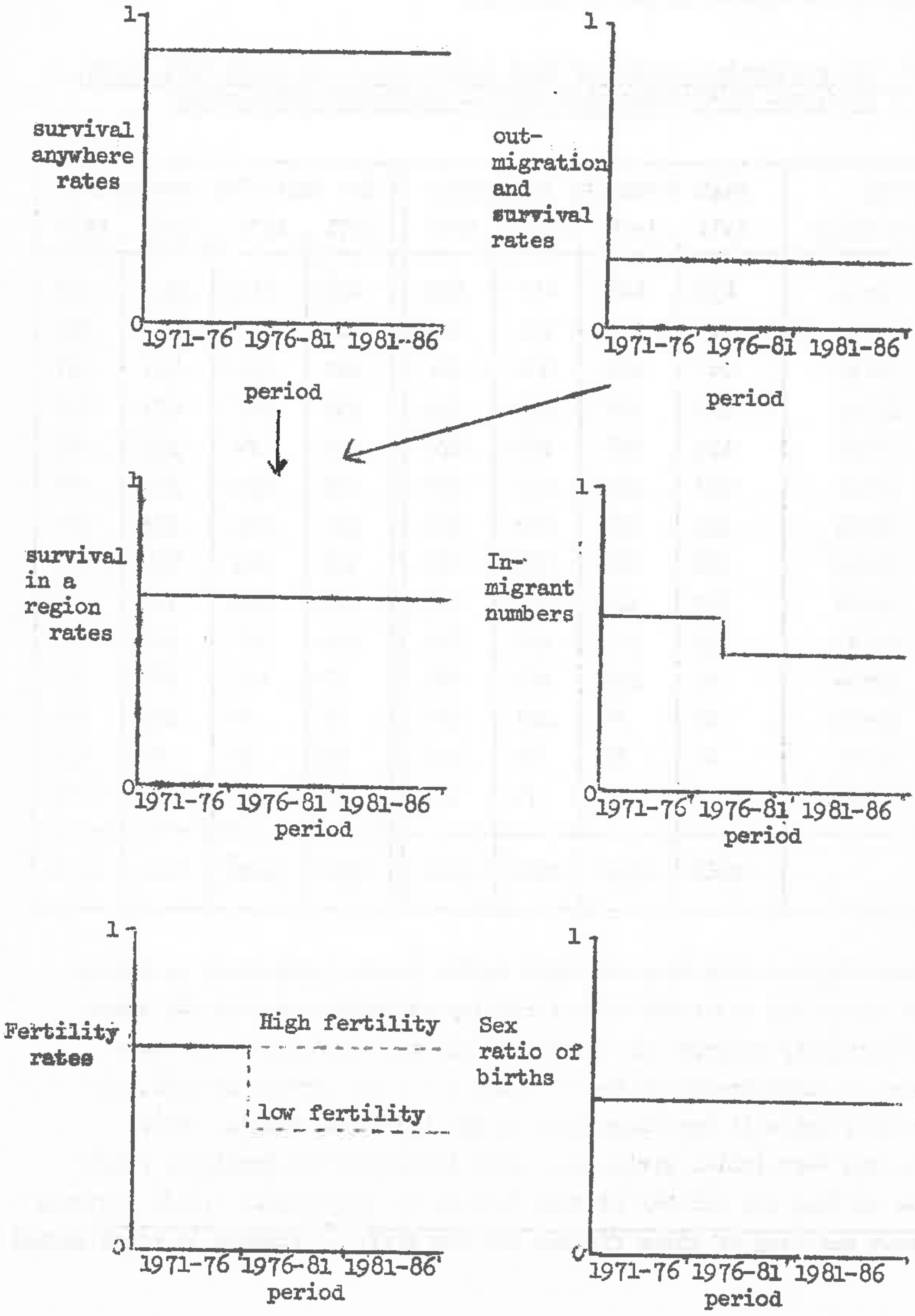
There is a net outflow likely under the assumptions we have adopted though the number of out-migrants may well be lower than those implied here if West Indians fail to move in the same numbers to growing suburban communities around Leeds with the same frequency as the whole population. To the net outflow as a result of migration must be added 29 deaths in the 1971-76 period.

Most critical for this particular projection are the fertility rates assumed. Two sets of assumptions are made. In the first set (Table 6) the high fertility rates of 1971 are assumed to continue in 1971-76, 1976-81 and 1981-86. In the second set, (Table 6) the fertility rates are assumed to decline to a lower level in 1976-81 and 1981-86, to a level characteristic of the whole Leeds population in 1971. There is a strong evidence for such a decline occuring (King, 1974) and our guess would be that fertility among West Indian women is more likely to be nearer the second set of assumptions than the first.

The sex proportion for births and survival rate for infants are assumed constant over the three projection periods.

Figure 4 shows in simplified form the nature of the assumptions made in projecting the population of West Indian women. The projections

Figure 4. A diagramatic respresentation of the assumptions made in projecting the population of West Indian women



are conditional
on these assumptions proving correct. In the projections we do not foretell the future but rather say what it will be like if certain conditions hold.

If we run the model not only for the period 1971-76 (as we have in equations (47), (51), (52) and (53)) but also for 1976-81 and 1981-86 for both the high and the low fertility rate schedules we obtain the population stock figures given in Table 10.

Table 10 The population stocks of West Indian women in Leeds 1971, 1976, 1981, and 1986 under high and low fertility assumptions

Age group No.	Age range	High fertility assumption 1971	1976	1981	1986	Low fertility assumption 1971	1976	1981	1986
1	0- 4	412	448	447	453	412	448	283	290
2	5- 9	314	370	401	400	314	370	401	259
3	10-14	247	272	320	347	247	272	320	347
4	15-19	257	257	273	313	257	257	273	313
5	20-24	173	274	265	276	173	274	265	276
6	25-29	267	192	263	256	267	192	263	256
7	30-34	351	265	203	256	351	265	203	256
8	35-39	334	343	264	211	334	343	264	211
9	40-44	220	313	317	250	220	313	317	250
10	45-49	131	214	298	303	131	214	298	303
11	50-54	75	128	203	279	75	128	203	279
12	55-59	63	74	121	188	63	74	121	188
13	60-64	20	55	64	106	20	55	64	106
14	65&over	24	34	70	103	24	34	70	103
		2888	3239	3509	3741	2888	3239	3345	3437

These figures show that the West Indian female population is likely to increase by 853 under the high fertility assumption and by 549 under the low fertility assumption: Note that all the differences in these increases is concentrated in the 0- 4 and 5 - 9 age groups in 1986, the members of which will have been born in the 1976-1986 decade. There might be 853 West Indian girls aged under 10 in 1986 if fertility rates continue as they are and 549 if they fall in the way recent trends indicate*.

*The exact matching of these figures and the difference figures is coincidental.

A comprehensive view of the projected population in each age group for both women and men of West Indian origin is given in Figure 5 in the form of population pyramids for 1971, 1976, 1981, and 1986. They show that the population changes from one in 1971 whose age/sex structure is dominated by the differences between age groups in immigration propensity to a more normal form in 1986 either approaching a fairly even distribution of persons in the age groups prior to theoutset of heavy mortality under the low fertility assumptions or a youthful one in the high fertility case. The high fertility histogram has the pyramidal form below age 40 in 1986 characteristic of a fairly rapidly growing population.

7. A shorter period case example: the population of East and West Hunslet ward 1971-1972

The simple population projection model presented in this paper can be applied on an annual basis if the shorter period version is used. We look at the likely evolution of the population of an inner city ward, Hunslet, in Leeds over the year 1971-72. The base population of the ward was obtained from the Census 1971 Ward and Parish library. The survival rates were calculated by subtracting out-migration rates from survival anywhere rates. The survival anywhere rates of the West Riding demographic accounts of 1961-66 (Smith and Rees, 1974) were used after correction for subsequent changes in the probability of dying obtained from nationalwork (O.P.C.S., 1973). The out-migration rates which were subtracted from the survival anywhere rates to yield the ward survival rates were estimated from the results of an interview survey of electors on List B of the electoral register.* The in-migration vector was also obtained from this survey. The fertility rates were those for the West Yorkshire conurbation as a whole in 1971 adjusted to a ward basis by multiplying the city rates by the ratio of children ever born to women in Leeds as a whole. The male/female split used was that for the city as a whole.

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*Our thanks for putting together the necessary survey and census data and for working out the initial 1971-72 model run are due to Susan Smedley, David Landsborough and Susan Bonthrone, students at Leeds University. The work was done during a field week at the Department of Geography in March 1973.

Schedules for the rest of the decade were prepared for survival anywhere rates, fertility rates and male/female split. However,these were not used for a projection as it was not possible to specify reasonable guesses as to what the out-migration or in-migration vectors for the ward population were likely to be for the rest of the decade. These would undoubtedly change substantially in the period as parts of the ward underwent demolition of housing (causing more out-migrants than normal) and as other parts saw the completion of new housing. An effective population projection for East and West Hunslet would require the calculation of the out-migrations and inmigrations consequent on the city's redevelopment and housing programme. This has been done in other contexts (Clarke, 1974; Howell, 1974).

The numerical versions of the matrix equations for East and West Hunslet are presented here for the census year 1971-72. Equations (54) and (55) survive the male and female populations of East and West Hunslet forward from census date 1971 to census date 1972. In equation (56) the female population at risk of giving birth is worked out. This population at risk vector is used to work out surviving male births (equation(57)) and surviving female births (equation (58)). These infants aged under one year are then added to the survivors aged one to four to make up the new total in the first, 0- 4 year old age group.

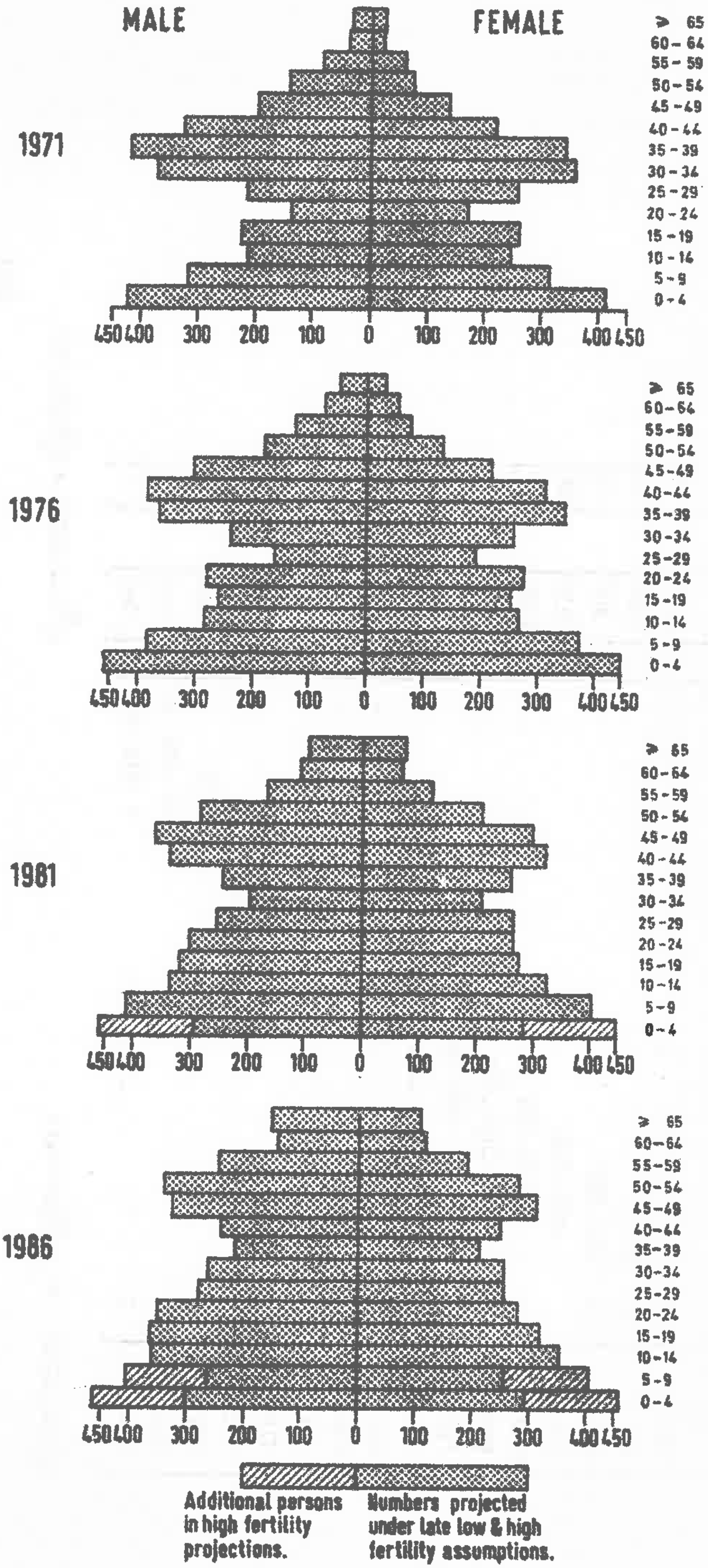

Figure 5 Age/sex pyramids for the projected population of West Indians in Leeds

The survivors' equations

1. Males

$$\begin{bmatrix} 800 \\ 986 \\ 921 \\ 959 \\ 1309 \\ 898 \\ 716 \\ 632 \\ 611 \\ 635 \\ 591 \\ 718 \\ 703 \\ 601 \\ 425 \\ 370 \end{bmatrix} = \left[\begin{array}{cccccccccccccccc}
.671 & 0 & & & \cdots & & & & & & & & & & & 0 \\
.168 & .688 & & & & & & & & & & & & & & \\
0 & 172 & .688 & & & & & & & & & & & & & \\
 & & .172 & .731 & & & & & & & & & & & & \\
 & & & .183 & .737 & & & & & & & & & & & \\
 & & & & .185 & .753 & & & & & & & & & & \\
\cdot & & & & & .189 & .723 & & & & & & & & & \cdot \\
\cdot & & & & & & .181 & .715 & & & & & & & & \cdot \\
\cdot & & & & & & & .180 & .688 & & & & & & & \cdot \\
\cdot & & & & & & & & .172 & .679 & & & & & & \cdot \\
 & & & & & & & & & .171 & .662 & & & & & \\
 & & & & & & & & & & .166 & .679 & & & & \\
 & & & & & & & & & & & .171 & .664 & & & \\
 & & & & & & & & & & & & .167 & .675 & & \\
 & & & & & & & & & & & & & .170 & .689 & 0 \\
0 & & & & \cdots & & & & & & & & & 0 & .172 & .837
\end{array}\right] \begin{bmatrix} 1139 \\ 1057 \\ 986 \\ 937 \\ 1320 \\ 733 \\ 710 \\ 641 \\ 682 \\ 727 \\ 683 \\ 860 \\ 824 \\ 669 \\ 425 \\ 346 \end{bmatrix} + \begin{bmatrix} 35 \\ 68 \\ 61 \\ 104 \\ 164 \\ 102 \\ 64 \\ 45 \\ 27 \\ 24 \\ 15 \\ 21 \\ 9 \\ 12 \\ 19 \\ 7 \end{bmatrix}$$

$$\underline{w}_{\underline{S}}^{M}(\text{c.d.}1972) = \underline{S}^{M}(1971\text{-}72) \quad \underline{w}^{M}(\text{c.d.}1971) + \underline{I}_{\underline{S}}^{M}(1971\text{-}72)$$

(54)

The survivors' equations

2. Females

$$
\begin{bmatrix} 753 \\ 933 \\ 846 \\ 942 \\ 1189 \\ 858 \\ 621 \\ 598 \\ 582 \\ 606 \\ 608 \\ 757 \\ 785 \\ 725 \\ 615 \\ 830 \end{bmatrix}
=
\begin{bmatrix}
.668 & 0 & \cdots & & & & & & & & & & & & & 0 \\
.468 & .683 & & & & & & & & & & & & & & \\
0 & .171 & .683 & & & & & & & & & & & & & \\
 & & .171 & .734 & & & & & & & & & & & & \\
 & & & .184 & .731 & & & & & & & & & & & \\
 & & & & .183 & .755 & & & & & & & & & & \\
\vdots & & & & & .189 & .708 & & & & & & & & & \vdots \\
 & & & & & & .178 & .716 & & & & & & & & \\
 & & & & & & & .180 & .694 & & & & & & & \\
 & & & & & & & & .174 & .692 & & & & & & \\
 & & & & & & & & & .173 & .685 & & & & & \\
 & & & & & & & & & & .172 & .681 & & & & \\
 & & & & & & & & & & & .171 & .670 & & & \\
 & & & & & & & & & & & & .168 & .671 & & \\
 & & & & & & & & & & & & & .168 & .670 & 0 \\
0 & & \cdots & & & & & & & & & & & 0 & .168 & .832
\end{bmatrix}
\begin{bmatrix} 1079 \\ 1010 \\ 906 \\ 918 \\ 1202 \\ 722 \\ 638 \\ 611 \\ 641 \\ 667 \\ 683 \\ 909 \\ 923 \\ 825 \\ 698 \\ 842 \end{bmatrix}
+
\begin{bmatrix} 32 \\ 62 \\ 55 \\ 114 \\ 142 \\ 93 \\ 33 \\ 47 \\ 27 \\ 33 \\ 25 \\ 21 \\ 11 \\ 16 \\ 9 \\ 12 \end{bmatrix}
$$

$$\underline{w}_S^F(\text{c.d.}1972) = \underline{S}^F(1971\text{-}72) \quad \underline{w}^F(\text{c.d.}1971) + \underline{I}_S^F(1971\text{-}72 \qquad (55)$$

The population at risk equation

1. males and 2. females

$$\begin{bmatrix} \\ 930 \\ 1196 \\ 790 \\ 630 \\ 604 \\ 611 \\ 636 \\ \\ \end{bmatrix} = \left[\begin{bmatrix} \\ 942 \\ 1189 \\ 858 \\ 621 \\ 598 \\ 582 \\ 606 \\ \\ \end{bmatrix} - \begin{bmatrix} \\ 918 \\ 1202 \\ 722 \\ 638 \\ 611 \\ 641 \\ 667 \\ \\ \end{bmatrix} \right] \begin{bmatrix} 0.5 \end{bmatrix} + \begin{bmatrix} \\ 918 \\ 1202 \\ 722 \\ 638 \\ 611 \\ 641 \\ 667 \\ \\ \end{bmatrix}$$

$$\underline{w}^F = ((\underline{w}^F_S(c.d.1972) - \underline{w}^F(c.d.1971))0.5) + \underline{w}^F(c.d.\ 1971) \qquad (56)$$

Only the relevant portions (the fertile age groups) of the vectors are displayed.

The births equations

1. Males

$$
\begin{bmatrix} 301 \\ \hdashline 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & 0 & .147 & .196 & .185 & .084 & .041 & .010 & .000 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} \\ \hdashline 930 \\ 1196 \\ 790 \\ 630 \\ 604 \\ 611 \\ 636 \\ \hdashline \\ \\ \end{bmatrix}
\begin{bmatrix} .51409 \end{bmatrix}
\begin{bmatrix} .9435 \end{bmatrix}
+
\begin{bmatrix} 9 \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix}
$$

$$
\underline{w}_B^M(\text{c.d.}1972) = \underline{F}(1971\text{-}72) \quad \underline{w}^F \quad \sigma^M \quad s_{01}^M(1971\text{-}72) + \underline{I}_B^M(1971\text{-}72 \qquad (57)
$$

2. Females

$$\begin{bmatrix} 284 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & .147 & .196 & .185 & .084 & .041 & .010 & .000 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & 0 & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \\ \hline 930 \\ 1196 \\ 790 \\ 630 \\ 604 \\ 611 \\ 636 \\ \hline \\ \end{bmatrix} \begin{bmatrix} .48591 \end{bmatrix} \begin{bmatrix} .9460 \end{bmatrix} + \begin{bmatrix} 8 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underline{w}_B^F(\text{c.d. } 1972) \quad = \quad \underline{F}(1971\text{-}72) \qquad \underline{w}^F \quad \sigma^F \quad S_{01}^F(1971\text{-}2) + \underline{I}_B^F(1971\text{-}72) \tag{58}$$

<u>The final population equations</u>

<u>1. Males</u>

$$
\begin{bmatrix} 1101 \\ 986 \\ 921 \\ 959 \\ 1309 \\ 898 \\ 716 \\ 632 \\ 611 \\ 635 \\ 591 \\ 718 \\ 703 \\ 601 \\ 425 \\ 370 \end{bmatrix}
=
\begin{bmatrix} 800 \\ 986 \\ 921 \\ 959 \\ 1309 \\ 898 \\ 716 \\ 632 \\ 611 \\ 635 \\ 591 \\ 718 \\ 703 \\ 601 \\ 425 \\ 370 \end{bmatrix}
+
\left[\begin{array}{c} 301 \\ \hline 0 \\ \\ \\ \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \\ \\ \\ \\ \\ \\ 0 \end{array}\right]
$$

$$\underline{w}^{M}(\text{c.d.}1972) = \underline{w}_{S}^{M}(\text{c.d. }1972) + \underline{w}_{B}^{M}(\text{c.d.}1972) \qquad (59)$$

2. Females

$$\begin{bmatrix} 1037 \\ 933 \\ 846 \\ 942 \\ 1189 \\ 858 \\ 621 \\ 598 \\ 582 \\ 606 \\ 608 \\ 757 \\ 785 \\ 725 \\ 615 \\ 830 \end{bmatrix} = \begin{bmatrix} 753 \\ 933 \\ 846 \\ 942 \\ 1189 \\ 858 \\ 621 \\ 598 \\ 582 \\ 606 \\ 608 \\ 757 \\ 785 \\ 725 \\ 615 \\ 830 \end{bmatrix} + \begin{bmatrix} 284 \\ \hline \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

$$\underline{w}^F(c.d.1972) = \underline{w}^F_S(c.d.1972) + \underline{w}^F_B(c.d.1972) \qquad (60)$$

The model equations above and in the previous section of the paper have been set out in normal cohort survival form . However, the model results can be represented in the form of an accounts table. This is possible because the rates were defined in a fashion consistent with population accounts. The accounts table for the male inhabitants of East and West Hunslet is set out in Figure 6. To derive the out-migrant flows the out-migration rates calculated from the interview survey which were subtracted from the survival anywhere rates, were used. To work out the deaths figures the death rates corresponding to the survival anywhere rates were applied to the out-migrant and in-migrant flows summed by initial age group using an equation form adopted from the account based model (Wilson and Rees, 1974):

	Age grp	0-4 1	5-9 2	10-14 3	15-19 4	20-24 5	25-29 6	30-34 7	35-39 8	40-44 9	45-49 10	50-54 11	55-59 12	60-64 13	65-69 14	70-74 15	75+ 16	Sub-total
SURVIVAL IN E&W HUNSLET c.d. 1972																		
Existence in E&W Hunslet c.d. 1971	0-4 1	765	191															956
	5-9 2		727	182														909
	10-14 3			678	170													848
	15-19 4				685	172												857
	20-24 5					973	244											1217
	25-29 6						552	139										691
	30-34 7							513	128									641
	35-39 8								459	115								574
	40-44 9									469	117							586
	45-49 10										494	124						618
	50-54 11											452	113					565
	55-59 12												584	147				731
	60-64 13													547	138			685
	65-69 14														451	113		564
	70-74 15															293	73	366
	75+ 16																290	290
Subtotal		765	918	860	855	1145	796	652	587	584	611	576	697	694	589	406	363	11098
Existence in Elsewhere c.d. 1971	0-4 1	35	14															49
	5-9 2		54	12														66
	10-14 3			49	21													70
	15-19 4				83	49												132
	20-24 5					115	33											148
	25-29 6						69	15										84
	30-34 7							49	7									56
	35-39 8								38	8								46
	40-44 9									19	6							25
	45-49 10										18	2						20
	50-54 11											13	0					13
	55-59 12												21	3				24
	60-64 13													6	2			8
	65-69 14														10	4		14
	70-74 15															15	1	16
	75+ 16																6	6
Subtotal		35	68	61	104	164	182	64	49	27	24	15	21	9	12	19	7	777
Subtotal survivors		800	986	921	959	1309	898	716	632	611	635	591	718	703	601	425	370	11875
Birth in 1971-72 in E&W Hunslet and age of mother at time of birth	15-19 4	66																66
	20-24 5	114																114
	25-29 6	71																71
	30-34 7	26																26
	35-39 8	12																12
	40-44 9	3																3
	45-49 10	0																0
Subtotal		292																292
Birth in 1971-72 Elsewhere and age of mother at time of birth	15-19 4	2																2
	20-24 5	3																3
	25-29 6	2																2
	30-34 7	1																1
	35-39 8	1																1
	40-44 9	0																0
	45-49 10	0																0
Subtotal		9																9
Subtotal B-S		301																301
Grand total in age group		1101	986	921	959	1309	898	716	672	611	635	591	718	703	601	425	370	12176

	Age grp	0-4 1	5-9 2	10-14 3	15-19 4	20-24 5	25-29 6	30-34 7	35-39 8	40-44 9	45-49 10	50-54 11	55-59 12	60-64 13	65-69 14	70-74 15	75+ 16	Sub-total	Sub-total survivors
SURVIVAL IN ELSEWHERE c.d. 1972																			
Existence in E&W Hunslet c.d. 1971	0-4 1	146	36															182	1138
	5-9 2		118	30														148	1057
	10-14 3			110	28													138	986
	15-19 4				64	16												80	937
	20-24 5					82	20											102	1319
	25-29 6						34	8										42	733
	30-34 7							54	13									67	708
	35-39 8								53	13								66	640
	40-44 9									74	18							92	678
	45-49 10										84	20						104	722
	50-54 11											87	22					109	674
	55-59 12												89	22				111	842
	60-64 13													87	21			108	793
	65-69 14														53	13		66	630
	70-74 15															19	5	24	390
	75+ 16																5	5	295
Subtotal		146	154	140	92	98	54	62	66	87	102	107	111	109	74	32	10	1444	12542
Birth in 1971-72 in E&W Hunslet and age of mother at time of birth	15-19 4	3																3	69
	20-24 5	5																5	119
	25-29 6	3																3	74
	30-34 7	1																1	27
	35-39 8	0																0	12
	40-44 9	0																0	3
	45-49 10	0																0	0
Subtotal		12																12	304

	Age grp	0-4 1	5-9 2	10-14 3	15-19 4	20-24 5	25-29 6	30-34 7	35-39 8	40-44 9	45-49 10	50-54 11	55-59 12	60-64 13	65-69 14	70-74 15	75+ 16	Sub-total
DEATH IN 1971-72 IN E&W HUNSLET																		
Existence in E&W Hunslet c.d. 1971	0-4 1	1	0															1
	5-9 2		0	0														0
	10-14 3			0	0													0
	15-19 4				0	0												0
	20-24 5					1	0											1
	25-29 6						0	0										0
	30-34 7							2	0									2
	35-39 8								1	0								1
	40-44 9									2	1							3
	45-49 10										4	0						4
	50-54 11											6	1					7
	55-59 12												14	2				16
	60-64 13													24	3			27
	65-69 14														31	4		35
	70-74 15															30	3	33
	75+ 16																50	50
Subtotal		1	0	0	0	1	0	2	1	2	5	6	15	26	34	34	53	180
Existence in Elsewhere c.d. 1971	0-4 1	0	0															0
	5-9 2		0	0														0
	10-14 3			0	0													0
	15-19 4				0	0												0
	20-24 5					0	0											0
	25-29 6						0	0										0
	30-34 7							0	0									0
	35-39 8								0	0								0
	40-44 9									0	0							0
	45-49 10										0	0						0
	50-54 11											0	0					0
	55-59 12												1	0				1
	60-64 13													0	0			0
	65-69 14														1	0		1
	70-74 15															1	0	1
	75+ 16																1	1
Subtotal		0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	4
Subtotal survivors		1	0	0	0	1	0	2	1	2	5	6	16	26	35	35	54	184
Birth in 1971-72 in E&W Hunslet and age of mother at time of birth	15-19 4	1																1
	20-24 5	2																2
	25-29 6	1																1
	30-34 7	0																0
	35-39 8	0																0
	40-44 9	0																0
	45-49 10	0																0
Subtotal		4																4
Birth in 1971-72 Elsewhere and age of mother at time of birth	15-19 4	0																0
	20-24 5	0																0
	25-29 6	0																0
	30-34 7	0																0
	35-39 8	0																0
	40-44 9	0																0
	45-49 10	0																0
Subtotal		0																0
Subtotal B-S		4																4
Grand total in age group		5	0	0	0	1	0	2	1	2	5	6	16	26	35	35	54	188

	Age grp	0-4 1	5-9 2	10-14 3	15-19 4	20-24 5	25-29 6	30-34 7	35-39 8	40-44 9	45-49 10	50-54 11	55-59 12	60-64 13	65-69 14	70-74 15	75+ 16	Sub-total	Sub total deaths	Total population
DEATH IN 1971-72 ELSEWHERE																				
Existence in E&W Hunslet c.d. 1971	0-4 1	0	0															0	1	1139
	5-9 2		0	0														0	0	1057
	10-14 3			0	0													0	0	986
	15-19 4				0	0												0	0	937
	20-24 5					0	0											0	1	1320
	25-29 6						0	0										0	0	733
	30-34 7							0	0									0	2	710
	35-39 8								0	0								0	1	641
	40-44 9									1	0							1	4	682
	45-49 10										1	0						1	5	727
	50-54 11											2	0					2	9	683
	55-59 12												2	0				2	18	860
	60-64 13													4	0			4	31	824
	65-69 14														3	1		4	39	669
	70-74 15															2	0	2	35	425
	75+ 16																1	1	51	346
Subtotal		0	0	0	0	0	0	0	0	1	1	2	2	4	3	3	1	17	197	12739
																		Sub-total	Sub total births deaths	Total births
Birth in 1971-72 in E&W Hunslet and age of mother at time of birth	15-19 4	0																0	1	70
	20-24 5	0																0	2	120
	25-29 6	0																0	1	75
	30-34 7	0																0	0	27
	35-39 8	0																0	0	13
	40-44 9	0																0	0	3
	45-49 10	0																0	0	0
Subtotal		0																0	4	308

Fig 6 An aggregate table for the ...

$$\text{non-surviving migrants in an age transition} = \left[\frac{\text{death rate for initial age group}}{\text{1- death rate for initial age group}} \right] \begin{array}{l}\text{surviving}\\ \text{migrants}\\ \text{summed}\\ \text{by initial}\\ \text{age group}\end{array}$$

$$\text{X age transition allocation factor} \qquad (61)$$

The rates and calculations are set out in Table 11. Nine tenths of the deaths to persons in an age group s, were estimated as taking place in age groups, and one tenth in age group s+1. Rounding and other minor adjustments were made to ensure whole and consistent numbers in the table.

The population accounts show an estimate of how the male population of East and West Hunslet fared over the year from April 25/26 1971 to April 25/26 1972. In the submatrix in the upper left corner are contained the person who survive within the ward over the year. The numbers of these persons were implicitly calculated in equation (54) in the $\underline{S}^{M}\underline{w}^{M}$ multiplication. Some 87 per cent of the initial population survive in the ward over the year. Some 1444 persons out-migrate and survive in other wards in Leeds or outside the city. The rates of out-migration by age transition are given in Table 11. Overall some 11 per cent of the males move out and survive over the year. Compared with the losses to the population sustained as the result of migration, only small numbers are lost through death while residing in the ward (180) or as a result of death while residing elsewhere (17), though the division between the two is very approximate. Overall some 1.5 per cent of the initial population died in the year.

Table 11 Rates involved in generating the accounts table for males in East and West Hunslet

Age transition: Age range		Age group numbers		Rate of survival anywhere (1)	Rate of out-migration & survival (2)	Rate of survival within ward (3) = ((1)-(2))	Death probability by initial age group (4)
Birth	0- 4	0	1	.9835	.0400	.9435	.0165
0- 4	0- 4	1	1	.7993	.1280	.6713	.0009
0- 4	5- 9	1	2	.1998	.0320	.1678	
5- 9	5- 9	2	2	.7997	.1120	.6877	.0004
5- 9	10-14	2	3	.1999	.0280	.1719	
10-14	10-14	3	3	.7996	.1120	.6876	.0005
10-14	15-19	3	4	.1999	.0280	.1719	
15-19	15-19	4	4	.7993	.0680	.7313	.0009
15-19	20-24	4	5	.1998	.0170	.1828	
20-24	20-24	5	5	.7993	.0620	.7373	.0009
20-24	25-29	5	6	.1998	.0150	.1848	
25-29	25-29	6	6	.7992	.0460	.7532	.0010
25-29	30-34	6	7	.1998	.0110	.1888	
30-34	30-34	7	7	.7989	.0760	.7229	.0014
30-34	35-39	7	8	.1997	.0190	.1807	
35-39	35-39	8	8	.7982	.0830	.7512	.0023
35-39	40-44	8	9	.1995	.0200	.1795	
40-44	40-44	9	9	.7969	.1090	.6879	.0039
40-44	45-49	9	10	.1992	.0270	.1722	
45-49	45-49	10	10	.7942	.1150	.6792	.0072
45-49	50-54	10	11	.1986	.0280	.1706	
50-54	50-54	11	11	.7901	.1281	.6620	.0124
50-54	55-59	11	12	.1905	.0320	.1655	
55-59	55-59	12	12	.7822	.1030	.6792	.0223
55-59	60-64	12	13	.1955	.0250	.1705	
60-64	60-64	13	13	.7704	.1060	.6644	.0370
60-64	65-69	13	14	.1926	.0260	.1666	
65-69	65-69	14	14	.7535	.0790	.6745	.0580
65-69	70-74	14	15	.1885	.0190	.1695	
70-74	70-74	15	15	.7336	.0450	.6886	.0830
70-74	75+	15	16	.1834	.0110	.1724	
75+	75+	16	16	.8510	.0145	.8365	.1490

In the accounts table both out-migration and in-migration flows are displayed and a comparison can be made thus:

Age group	Gain through in-migration & survival	Loss through out-migration and survival	Net balance
0- 4	49	182	-133
5- 9	66	148	- 82
10-14	70	138	- 68
15-19	132	80	52
20-24	148	102	46
25-29	84	42	42
30-34	56	67	- 11
35-39	46	66	- 20
40-44	25	92	- 67
45-49	20	104	- 84
50-54	13	109	- 96
55-59	24	111	- 87
60-64	8	108	-100
65-69	14	66	- 52
70-74	16	24	- 8
75+	6	5	1
Total	777	1444	-667

There is a substantial population loss through migration: the area is undergoing a process of urban redevelopment. The slum dwellings built in the last century are being removed and are being replaced by council owned flat and terrace dwellings. In the most mobile age groups, from 15-19 to 25-29, however, there is a net in-migration of 140 persons. This pattern of net movement is characteristic of inner city areas (or of Greater London as a whole) which provide accommodation suitable for small, young households (singles or couples) but not for families with children. This is still true after redevelopment in many cases. One large high rise block of council flats in the ward (the Hunslet Grange Flats) houses a considerable number of University and Polytechnic students, for example. Families prefer other kinds of council accommodation.

The future course of migration movements out of and into the ward, is, in fact, dependent on the turnover of the population in housing stock that "survives" the period under consideration, and on the out-migrations and in-migrations occasioned by the compulsory purchase, demolition, contruction and occupation sequence of redevelopment. Local authority housing and planning departments are in a position to monitor the planned charges and methods have been developed to incorporate this knowledge in population forecasts (Clarke, 1974). A model making this connection would be very appropriate in the East and West Hunslet ward case.

8. Conclusion

In the first half of the paper a simple model for population projection was described. It contained a number of novel features. These included a sequential treatment of fertility that made it possible to multiply conventional fertility rates by an estimate of the population at risk rather than the initial population. Rates were defined to be consistent with population accounts. Infant within region survival and migration and survival were properly treated (c.f Gilje and Campbell, 1973; Heatherington, 1974). It was shown that the model could be applied to a variety of situations through careful definition of the contents of the matrices involved in the model. The matrix model could usefully be re-expressed in terms of algebraic equations and there were a separate set of these for the two cases treated - the standard and the shorter period cases.

In the second half of the paper the model was applied in two situations. The discussion here suggested that it was often useful to pick out the out-migration flows as well as the in-migration movements, and that it was instructive to examine the whole population accounts implied by the model. We can restate the model more formally to take into account these new demands. The out-migration component can be explicitly represented thus in the first equation:

$$\underline{w}_{S}^{X}(t+T) = \underline{S}^{X}(A)\,\underline{w}^{X}(t) - \underline{O}^{X}\underline{w}^{X}(t) + \underline{I}_{S}^{X} \qquad (61)$$

where $\underline{S}^X(A)$ is a matrix of "survival anywhere" rates for persons of sex X and $\underline{O}^X$ is a matrix of out-migration and survival rates. Table 11 shows the values of these rates for the East and West Hunslet example. They are arranged within the matrices in the same way as the survival within ward rates shown in equations (54) and (55). The equation for the births into first age group becomes:

$$\begin{aligned} \underline{w}^X_B(t+T) &= \underline{F}\,\underline{w}^F\,\sigma^X\,s^X_{01}(A) \\ &\quad - \underline{F}\,\underline{w}^F\,\sigma^X\,o^X_{01} \\ &\quad + \underline{I}^X_B \end{aligned} \qquad (62)$$

where o^X_{01} is the rate of out migration and survival of infants of sex X born in the period.

The population accounts associated with the model can be generated by the following set of equations . The survivors' submatrices are generated thus:

$$\begin{bmatrix}\text{submatrix of}\\ \text{within region}\\ \text{region}\end{bmatrix} = \underline{WS}^X = \underline{\tilde{w}}^X(t)\,\underline{S}^{X'} \qquad (63)$$

$$\begin{bmatrix}\text{submatrix of}\\ \text{out-migrants}\\ \text{who survive}\end{bmatrix} = \underline{OS}^X = \underline{\tilde{w}}^X(t)\,\underline{O}^{X'} \qquad (64)$$

$$\begin{bmatrix}\text{submatrix of}\\ \text{in-migrants}\\ \text{who survive}\end{bmatrix} = \underline{IS}^X = \begin{matrix}\text{the } \underline{I}^X_S \text{ vector broken down}\\ \text{into its compenent (age-}\\ \text{transition) parts and}\\ \text{arranged in matrix form}\end{matrix} \qquad (65)$$

where $\underline{\tilde{w}}^X(t)$ is a matrix with the $\underline{w}^X(t)$ or initial population values along the principal diagonal. By arranging the initial populations in each age group in this way, the product matrix has the dimensions of the accounts submatrix. Note that we must transpose and postmu ply by the $\underline{S}^X$ and $\underline{O}^X$ matrices.

$$\begin{bmatrix} \text{submatrix of persons dying} \\ \text{within a region} \end{bmatrix} = \underline{WD}^X = \underline{\widehat{WSI}}^X \; \underline{D}^X \qquad (66)$$

$$\begin{bmatrix} \text{submatrix of persons out-} \\ \text{migrating and dying} \end{bmatrix} = \underline{OD}^X = \underline{\widehat{OSI}}^X \; \underline{D}^X \qquad (67)$$

$$\begin{bmatrix} \text{submatrix of persons in-} \\ \text{migrating and dying} \end{bmatrix} = \underline{ID}^X = \underline{\widehat{ISI}}^X \; \underline{D}^X \qquad (68)$$

The $\underline{\widehat{WSI}}^X$, $\underline{\widehat{OSI}}^X$, and $\underline{\widehat{ISI}}^X$ matrices are ones in which the row sums of the $\underline{WS}^X$, $\underline{OS}^X$ and $\underline{IS}^X$ matrices have been arranged in the principal diagonal. These matrices are multiplied by a matrix of deaths rates of the form:

$$\underline{D}^X = \begin{bmatrix} c_{11}d^X_{1*} & c_{12}d^X_{1*} & 0 & \cdots & & 0 \\ 0 & c_{22}d^X_{2*} & c_{23}d^X_{2*} & & & \\ 0 & 0 & c_{33}d^*_{3*} & & & \\ \cdot & & \cdot & & & \cdot \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ & & & & c_{R-1R-1}d^X_{R-1*} & c_{R-1R}d^X_{R-1*} \\ 0 & & \cdots & & 0 & 1.0d^X_{R*} \end{bmatrix} \qquad (69)$$

where the d^X_{r*} are the adjusted death rates for persons aged r at the start of the period. If q^X_{r*} is the probability of persons of sex X aged r at time t dying in the period t to t+T then:

$$d^X_{r*} = \frac{q^X_{r*}}{1-q^X_{r*}} \qquad (70)$$

This is multiplied by a co-efficient c_{rr} to give a death rate for the diagonal transitions and c_{rr+1} for the off-diagonal, where

$$c_{rr} + c_{rr+1} = 1.0 \tag{71}$$

This enables us to assign deaths to persons in age group r to age group at death. The values were guessed to be 0.9 for c_{rr} and 0.1 for c_{rr+1} in constructing the East and West Hunslet accounts tables. We assume in these calculations that the out-migrants experience the region of origin's death rate schedule, and that the in-migrants experience the region of destination's death rate schedule, that is, the same schedule.

Similar procedures can be formally defined to construct the birth and survival, and birth and death submatrices.

In conclusion, some general points can be made. An attempt has been made to show how the the cohort survival model in which migration is recognised can be extended in a number of ways that make it more consistent and flexible. We hope that the utility of making the model consistent with a set of population accounts has been demonstrated, and that we have shown that care must be taken in using the model to think very carefully about the schedules of forecast rates to be employed in the model. The examples presented here have not been ideal but we were able to demonstrate in the West Indian case the considerable differences in projected population likely to occur between assuming that present trends in fertility will continue and assuming that they will probably decline. Similar differences would also have shown up if we had been able to forecast changes in migration rates in both examples.

References

Clarke, M.P. (1974) Constrained cohort techniques for small zones. Paper presented at the Population Forcasting Seminar, Summer Annual Meeting, P.T.R.C., University of Warwick July, 1974.

Gilje, E.K. and Campbell, R.A. (1973) A new model for projecting the population of the Greater London Boroughs. Paper presented at the Population Projections Seminar, New Year Meeting, P.T.R.C, University of Sussex, January, 1973. (Research Memorandum No. 408 from the Department of Planning and Transportation, G.L.C.).

Heathrington, M.F. (1974) Two notes concerning migration assumptions recently used in projecting the populations of the Greater London Boroughs. Paper presented at the Population Forcasting Seminar, Summer Annual Meeting, P.T.R.C., University of Warwick, July, 1974.

Howell, J. (1974) The role of population forecasts and demographic analysis in new town planning. Paper presented at the Population Forcasting Seminar, Summer Annual Meeting, P.T.R.C., University of Warwick, July, 1974.

King, J.R. (1973) The social, economic and location characteristics of ethnic births in Leeds C.B. in 1971. Working Paper 47, Department of Geography, University of Leeds.

King, J.R. (1974a) The fertility of immigrants in Leeds, Working Paper 58, Department of Geography, University of Leeds.

King, J.R. (1974b). Immigrant fertility trends and population growth in Leeds. Environment and Planning A, 6, forthcoming. (Working Paper 75, Department of geography, University of Leeds.)

O.P.C.S. (1972) Population projections No. 2, 1971-2021. H.M.S.O., London.

Rees, P.H. (1973) A revised notation for spatial demographic accounts and models. Environment and Planning, 5, pp. 147-155

Rees, P.H.; Smith, A.P.; and King, J.R. (1974). Demography, Chapter 3 in the empirical development of urban and regional models S.S.R.C. Final Report in preparation. A.G. Wilson, P.H. Rees and C.M. Leigh.

Rees, P.H. and Wilson, A.G. (1973) Accounts and models for spatial demographic analysis. I: aggregate population. Environment and Planning 5, pp. 61-90.

Rees, P.H. and Wilson, A.G. (1974) A comparison of available models of population growth. Working Paper 54, Department of Geography, University of Leeds.

Smith, A.P.; and Rees P.H. (1974) Methods of constructing spatial demographic accounts for British Regions. Working Paper 59, Department of Geography, Unversity of Leeds.

Wilson, A.G. (1974) Urban and regional model in geography and planning John Wiley, London, Chapter 11, "Towards comprehensive models".

Wilson, A.G. and Rees, P.H. (1974a) Accounts and models for spatial demographic analysis II: age/sex disaggregated populations. Environment and Planning, 6,pp. 101-116.

Wilson, A.G. and Rees, P.H. (1974b) Population statistics and spatial demographic accounts. The Statistician, forthcoming (Working Paper 53, Department of Geography, University of Leeds).