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# Some Issues in the Application of Mathematical Programming in Human Geography.*

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At the outset, it is perhaps worthwhile remembering Peter Gould's (1979, p. 142) recent reference to a graduate student's "highly creative dissertation using linear programming in a geographic and behavioural context":

"Of course, it was unacceptable, and had to be completely rewritten to conform to accepted geographic standards".

Contents:

# 1. Introduction.

## 1.1 Aims.

The operational research literature contains many applications of mathematical programming models, which optimise the value of a well-defined objective given a set of well-defined constraints. At the beginning, it is important to remove any verbal confusion arising from the term 'mathematical programme'. It is not computer programming, although computers facilitate, and are necessary for, the large number of calculations found within a mathematical programme; computer packages, especially for linear programmes, are in fact, easily accessible and widely employed (see for instance, Land and Powell (1973), Orchard-Hays (1969), White (1978)). The analytical power and rigour of a mathematical formulation is found in mathematical programmes, and although this often proves extremely useful in complex analyses, geographers must remember that techniques per se cannot say anything about geographical problems.

To demonstrate the relevance of mathematical programming formulations for geographers, this paper emphasises their relationship to model building, and little attention is given to the available solution techniques which are discussed in many text-books, such as Rao (1978). The general line of the argument, in so doing, is not hidden in a mass of excessive detail which could impair comprehension. Moreover, geographers have long appreciated the possibilities of an increased understanding generated by the systematic approach of building (mathematical) models (see, for example, Chorley and Haggett 1967). In addition to insights gained from the computation of an optimal solution, it is suggested that the building of mathematical models can offer assistance in the comprehension of complex inter-relationships found in systems of interest.

Stress is, therefore, placed on presenting a broad overview of the underlying rationale of mathematical programming, and on giving a number of examples to mirror these arguments. Although the examples are generated from topics of personal interest, it is hoped that they will indicate theoretical and practical areas where other geographers can profitable employ mathematical programming formulations (or, at least, see old ideas in a new perspective). The extensive list of references should help the interested reader build up his knowledge of mathematical programming, and it is hoped that the necessary over-simplification and incompleteness, inherent in such an undertaking, is acceptable to the specialists.

## 1.2 Contents.

As the basis of mathematical programmes is optimisation, some discussion of this facet is required at the beginning. Section two, therefore, considers optimisation in general and also contains brief examinations of normative modelling and of how a comparison between ideal and actual spacial patterns can be useful.

Section three acquaints the reader with some important features of mathematical programmes. Linear programmes, the most widely applied, special case of a mathematical programme, are probed in some detail, and this forms the foundation from which to analyse two specific features of non-linear programming: the embedding of one mathematical programme into another, and aspects of structural stability. The latter is one area of analysis which demonstrates that the model building exercise is not completed when the optinal solution is computed; additional insights can also be derived from an examination of the so-called duals, and from the post-optimality procedures of sensitivity analysis and parameterisation. The need to test for local/global optima is also mentioned. Finally, although the issue of structural stability led to a concern with the temporal dimension, further treatment of this topic is given through reference to dynamic programming, control theory, and an optimal evolution argument.

Section four presents a detailed study of location-allocation problems, which have been widely applied in public facility location. Attention focuses on two features - the analytical structure and the behavioural context. Briefly, it can be considered as a generalisation of the location-allocation problem in continuous space by the introduction of a spatial interaction model, which relaxes the unrealistic assumption of travel to the nearest facility. Total travel cost is considered to be an inappropriate indicator of user benefit when consumer choice is incorporated, and an alternative welfare function, consistent with the behavioural model, is discussed. In addition, extensions include placement in hierarchical framework and a means of incorporating elastic demand.

Central place theory is modelled as a mathematical programme in section five, and interesting theoretical insights are described by considering spatial supply and demand interactions. In addition to details of the static representation of this model, directions towards a dynamic formulation are indicated.

# 2. Optimisation.

## 2.1 Optimisation: Some Thoughts.

Simply stated, the rationale underlying mathematical programming is a method of optimising (either maximising or minimising) the value of an objective function subject to a set of specified constraints. (The objective function and constraints must be well-defined, and this is considered in more detail in the next section). At the outset, therefore, the belief that geographers can successfully employ mathematical programming formulations requires some justification of the relevance of the notion of optimality (that is, making the 'best' possible choice from the alternatives available).

Theoretical science, in fact, has long thought that nature operates most efficiently. Its general appeal is illustrated by numerous principles in the natural and social sciences: Fermat's principle of least time; Maupertuis' principle of least action; Hamilton's principle; Smith's 'invisible hand'; Darwin's principle of natural selection and its extensions into 'Social Darwinism'; Zipf's principle of least effort; and so on. In geographical work, the early location theories of Christaller (1933) and Losch (1944) were founded on optimality principles. More recently, Wilson (1970) has applied an entropy-maximising (or an equilibrium statistical mechanics) approach in urban and regional modelling and Keys and Thrift ( have presented a novel analysis of the geography of the manufacturing industry using niche theory's competitive exclusion principle.

Whilst the concept of optimality/efficiency has daily applicability, its ultimate relevance to theoretical and applied geographical studies remains the concern of individual geographers. Undoubtedly, it will be an invalid assumption in certain circumstances. However, some indication of its usefulness is given through a brief consideration of normative modelling and how insights can be gained by comparing optimal and real world patterns. Mathematical programming enables such analyses by formulating the concept of optimality in a manageable, operationable mode.

### 2.2 Normative Modelling.

Whilst it is perhaps dangerous to recognise a clear-cut dichotomy, positive and normative models, interest in normative modelling focuses on what 'should be', essentially relinquishing study of existing spatial patterns. In order to determine what the location configuration should be, it is necessary to have a well-defined criterion and a set of well-defined constraints. Comparing the results of various problems ,formulated using different behavioural assumptions and under dissimilar conditions, would probably be a beneficial exercise.

It is important to remember that these optimal patterns are usually founded on a number of simplified assumptions, such as rational, decision-making under conditions of perfect information. One, useful normative framework which incorporates aspects of risk and uncertainty is game theory, and there are a number of examples of its application in the geographical literature (see, for example, Gould (1963)).

The application of mathematical programming techniques must be recognised as a means of gaining greater comprehension, and, hopefully, this paper offers some illumination on how this can be achieved. In addition to its explanatory power, knowledge of the optimal spatial organisation can often be of prescriptive significance.

In a scramble for 'social relevance' with respect to urgent contemporary problems, however, it is necessary to be aware that the outcomes of mathematical programming formulations should not be thought of as master plans for the development of a more efficient and equitable society. Social and political values underpin these representations, whether it be explicitly stated in the objective function or implicitly derived from not having actually been stated in the objective function. Whilst appreciating the extremes of 'social engineering', with its accompanying conscious management by technocrats, it must be realised that valuable information on the implications of various actions can be gained. Prescriptive statements on facility location, for example, could be made, but the ultimate exposition on whether the ends justify the means is a political decision.

These ideas of spatial efficiency and, therefore, of the employment of mathematical programming models are also directly dependent on the geographical scale of examination; it is highly unlikely that the optimal spatial organisation of a given region viewed from a national perspective coincides with that pattern derived from considering the region in isolation. A fundamental corollary of this point concerns the areas of control of planning authorities, and the determination of the most apposite level(s) at which to employ mathematical programming techniques.

### 2.3 A Comparison Between the Optimal Pattern and Reality.

A discussion of the comparison between the optimal pattern (generated from a mathematical programming model) and reality follows naturally on from some of the issues raised in the preceeding sub-section concerned with normative modelling. For instance, after deriving the optimal, so-called Loschian landscape, Losch (1954, p 363) prescribed a methodology to test the rationality of reality (rather than to attempt to falsify the theory by studying reality).

> "Comparison now has to be drawn no longer to test the theory, but to test reality! Now it must be determined whether reality is rational".

Discrepancies between the actual and optimal patterns can also offer valuable indications on how to improve the models.

Given a well-defined problem, mathematical programming techniques enable us to determine the ideal, or normative, locational pattern; this can be compared with the actual configuration to illustrate the degree of spatial inefficiency. What, for example, are the additional 'costs' incurred by sub-optimally located facilities?

A number of geographers have used the ideal pattern of spatial organisation (usually generated by a linear programming model) as a benchmark in their studies: Chisholm and O'Sullivan (1973) investigated the efficiency of freight flows in the British space-economy; Gould and Sparks (1969) were concerned with the nutritional value of human diets in rural Guatemala; Osayimwese (1974) examined the efficiency of the movement of groundnuts from areas of production to the export ports in Nigeria; Wolpert (1964 ) compared the spatial variations in Swedish agriculture to the configuration that would be generated by so-called, rational, 'Economic Men'; and so on. In addition, a comparison between an ideal and an actual pattern can indicate where improvements to a model can be made. For example, Cox (1965) used a linear programming model to look at the inter-state flows of aluminium in the U.S.A., and tested whether there was a statistically significant difference between the two configurations (that is, whether the variations were due to more than chance). Finding a statistically significant difference, the predicted, ideal flows enabled Cox to derive regression equations from which he computed the regression residuals (underestimated or overestimated flows). Analysis (which, unfortunately, did not include a check for spatial autocorrelation) of the distribution of the regressed residuals, the areas deviating from the normative model's aggregate distance minimisation objective, indicated that

".... the higher positive residual values refer to flows between states separated by a comparatively large distance, .... and the lower negative residual values often refer to flows between states separated by comparatively short distances...."

Interpretation of these features resulted in the conclusion that the linear relationship between transportation costs and distance, as assumed in the model, was invalid and, therefore, indicated where alterations were required.

A number of difficulties accompanying this comparative procedure are important. In addition to problems already mentioned with respect to spatial scale, the temporal dimension plays a significant role in determining what is optimal. For example, Hay's (1977, p 18) description of Osayimwese's (1974) analysis of groundnut flows in Nigeria notes that

".... if the behaviour of the operators is considered they will frequently be responding to day-to-day or week-to-week patterns of demand. Most academic studies will use the aggregate data for much longer periods in which the short term fluctuation have been concealed. For this reason optimal short term behaviour may appear as sub-optimal in the longer term".

Further problems relate to aspects of structural stability and sensitivity, which are considered later. Note also that there may be a number of feasible solutions which are only slightly sub-optimal in comparison to the value of the objective function of the optimal pattern; however, their spatial configuration may be drastically different, which clearly hinders comparison.

## 2.4 Optimisation: Two Warnings.

A pointer to one difficulty related to the application of optimality procedures follows from the implication that the divergence between optimal and actual patterns can be related to the well-defined nature of the mathematical programme; other facets, such as 'intangibles' associated with social and environmental quality, which were excluded from the model, may be appealing explanations of the differences. The ceteris paribus assumption is however, a vital element in the formulation, and this must be remembered. Care must be taken so that the problem does not become tautological; whilst the 'other factors' may be significant, it does not suggest that the actual organisation is the outcome of some optimising behaviour which takes the 'other factors' into account (otherwise, any pattern can be thought of as ideal).

Furthermore, mathematical programming models should not be seen as a panacea for geographers. There is always a danger in this type of paper to exagerate or oversell a specific formulation's usefulness; such an optimising framework can offer new insights. Mathematical programming models should be viewed as one of a number of ways through which to approach a given problem, and, interestingly, such representations can be shown to have equivalences or similarities with other models (Macgill and Wilson, 1979 ). There are, obviously, times when other formulations are more apposite.

## 3. Aspects of Mathematical Programming.

### 3.1 Mathematical Programming : A General Statement.

A mathematical programme involves the optimaisation (maximisation or minimisation) of the value of a specified objective (or criterion) function subject to a number of well-defined constraints. This is represented as a precise mathematical statement (which is formally given in vector notation),

$$\max_{\{\underline{x}\}} Z = f\ (\underline{x}) \tag{3.1.1}$$

subject to

$$g_j(\underline{x}) \leqslant b_j \qquad j = 1, \ldots, m \tag{3.1.2}$$

and to the common non-negativity constraints,

$$\underline{x} \geqslant 0 \tag{3.1.3}$$

Note that although the above equations are optimising the vector of x variables subject to the constraints, no loss of generality occurs by considering maximisation (which is only the reverse of minimisation).

In section two, mention was made that one has to be aware that objective functions contain implicit values; the precise statement of the objective function by its very nature also answers other questions. The set of constraints determine the feasible region within which the optimal solution can be found. Occasionally, a given model will have conflicting constraints which cannot be satisfied together. This may raise interesting points with respect to reconstructing a model with a feasible solution, although Charnes and Cooper (1961) suggested a linear programming-type model, termed a goal programming model, which met all the constraints as fully as possible by minimising deviations from the desired levels.

In fact, one of the advantages of using mathematical programmes is the flexibility of model representation which they permit; this clearly has significant theoretical and practical ramifications. Constraints, for instance, can be considered as a straight forward assertion of limits of interrelated effects, a feature portrayed in the covering problems described in section four. Note also the difference between accounting and resource constraints.

Of more practical application are models associated with multiple objectives. In addition to optimising a large weighted objective function, one can treat every objective except one as a constraint, leaving the remaining one as the actual objective function. Comparison and evaluation of the solutions of the various formulations could then be undertaken. It is interesting to note that the planning philosophy adopted by many Swedish geographers, such as Hagerstrand (1975), involves a study of the available choices from given constraints.

Thus, although the building of an unequivocal, exact model is a complex, perhaps impossible, task, mathematical programming allows a search into potential solutions and enables a comparison of results derived from various formulations.

## 3.2 Linear Programming.

### 3.2.1 Linear Programming : An Introduction.

A linear programme is computationally the most straightforward mathematical programme, and it has been widely applied by geographers, for, say, the optimal allocation of resources over space. Since its development during the Second World War, its application has been facilitated by easy accessibility to numerous computer packages, and it is now feasible to formulate and solve linear programmes of a very large dimension.

This subsection will present a description of a linear programming model, and a brief overview of some of its applications in geography. (More detailed discussions are to be found in Greenberg (1978) and in Hay (1977)). This forms a useful foundation on which to consider some features of nonlinear programming.

### 3.2.2 A Linear Programme.

As implied by its name, a linear programming problem has a linear objective function to optimise subject to a set of linear constraints. This is formally exemplified by

$$\max_{\{\underline{x}\}} LP = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

subject to

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n &\leqslant c_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n &\leqslant c_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots\quad &\quad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n &\leqslant c_m \end{aligned} \tag{3.2.1}$$

and

$$\underline{x} \geqslant 0$$

### 3.2.3 Geographical Applications of Linear Programming.

Two general complementary strands of geographical applications of linear programming can be recognised : one illustrating its practical applicability and the other demonstrating its theoretical potential. Before considering these uses, mention will be made of the so-called transportation problem, one of the most well-known linear programmes.

The transportation problem is explicitly geographical, optimising the pattern of commodity flows over space between demand and supply points. It is a capacitated problem, which minimises aggregate transportation costs whilst ensuring that supply meets demand without exceeding the capacity of any supply facility. Usually this is represented as

$$\text{Min T.C.} = \sum_{ij}\sum c_{ij}x_{ij} \qquad (3.2.2)$$

subject to

$$\sum_j x_{ij} \leq s_i \qquad (3.2.3)$$

$$\sum_i x_{ij} \geq d_j \qquad (3.2.4)$$

$$x_{ij} \geq 0 \qquad (3.2.5)$$

where $c_{ij}$ is the transport cost of a unit of the commodity from a supplier at i to a demand point j, and $x_{ij}$ is the volume of the commodity moving from i to j. $s_i$ is the supply capacity at i and $d_j$ is the demand from j. Equation (3.2.5) is the non-negativity constraint.

Similar linear programming formulations have been applied to a number of spatial allocation problems: for example, in the district planning of schools, (Maxfield, 1972; Trifon and Livnat, 1973; Yeates, 1963), of hospitals (Gould and Leinbach, 1966), and in social administration (Massam, 1975).

In fact, the topic of 'regions as a districting problem' (Haggett, Cliff, and Frey, 1977) can be analysed in a mathematical programming framework. It would be interesting for geographers to use non-linear formulations here; electoral districting, for instance, requires the assemblages of Enumeration Districts which satisfy a number of criteria, such as equality of the voting population, spatial contiguity, and social, political and economic homogeneity (Haggett, Cliff, and Frey, 1977). Alternative representations could optimise one criterion subject to constraints on other criteria, and then compare and evaluate the results. (Morrill's (1973) paper applying some of these features is of particular interest).

Other applications include studies of the behavioural aspects of decision making Swedish agriculture (Wolpert, 1964) and in the inter-state flows of aluminium bars in the U.S.A. (Cox, 1965), whilst Morrill and Garrison (1960) considered inter-regional trade patterns of wheat and flour, Dickason and Wheeler (1967) analysed Indian wheat transportation, and Casetti (1966) investigated the pattern of coal and iron ore and steel shipments which minimised the cost of the production and the distribution of steel around the St. Lawrence Seaway. More recently, linear programming techniques were found to be of assistance in the production of a structure plan for the County of Tyne and Wear (Hayton, 1979),and this methodology has been seen to be salient to regional planning in general (Jenkins and Robson, 1974).

The conceptualisation of modern urban and agricultural location theory has followed von Thunen's (1826) well-known analysis of agricultural land-use. Von Thunen's model in fact, has been formulated as a linear programme by both Henderson (1959; 1968) and Stevens (1968), and Day and Tinney (1969) offered a dynamic representation by applying recursive linear programming. Urban (micro-) economic theory has also incorporated the bid-rent approach to rational decision making (see, for example, Alonso (1964)), and some of its operationalisations have been through linear programming models (Stevens, 1961). One such example is the Herbert-Stevens residential location model, and in this field, Senior and Wilson's (1974) examination of linear programming and spatial interaction models is significant as the entropy maximising non-linear model overcomes the problem assiciated with the number of non-zero state variables being restricted to the number of constraints. In these preceeding studies concern focuses on optimising partial, linear models; in contrast, Schlager's (1965) pioneering land-use plan design model was a general, linear programming model.

The spatial price differentiation implicit in von Thunen's land-use analysis raises questions concerning the interplay of spatial supply and demand functions. A non-linear programming formulation for optimal facility size and location is considered in Section Five, whilst Tonn (1978) has presented a linear programming solution to the problem of optimal price-volume combination. Other linear programming spatial equilibrium models include Samuelson's (1952) work on consumers' and producers' surplus and Takayama and Judge's (1972) analysis of inter-industry flows. More recently, mathematical programming models (including non-linear formulations) have been used in a general equilibrium framework to integrate urban and welfare economics - 'New Urban Economics' (see Richardson (1977) for more details), and to explore the introduction of time into spatial economic theory (Macmillan, 1978). Other competitive situations of interest to geographers have been analysed using game theory (Gould, 1963), which is closely associated with optimisation problems. Linear programming models, in fact, have been connected to gaming strategies (Dorfman, Samuelson and Solow, 1958; Karlin, 1959) and this theoretical mould has recently been applied to the location-allocation problem (Beaumont, forthcoming), which is discussed in section four.

### 3.2.4 Linear Programming : Some Comments.

Without doubt, the attraction of linear programming is its relative computational simplicity, and often this leads to the desire to approximate non-linear functions by linearly proportional relationships. Such manipulations, however, are sometimes not appropriate. Cox (1965), for instance, concluded that there is not a direct linear relationship between distance and transportation costs - the 'laws' of diminishing marginalities, therefore, cannot be modelled.

A more technical point is that the variables are implicitly assumed to be of a continuous nature, but geographers are probably more interested in discrete variables. How can we locate half a school? Although integer programming could be utilised, linear programming's computational efficiency would be lost. Care must, however, be taken if 'rounding' of optimal linear programming solutions is employed - the result may not be optimal. It should be noted that the optimal allocation patterns generated by the transportation problem will be integer values if the quantities supplied and demanded are also integer values.

Perhaps a fundemental point for geographers to remember is the incapability of portraying spatial interaction between activities. The simplex solution method for linear programmes is an iterative search procedure which explores all the basic solutions (the vertices of the feasible region which is defined by the interaction of the constraints). This method is founded on the intrinsic property that the total number of non-zero state variables in an optimal solution cannot exceed the total number of constraints. Thus, only a few interaction variables will be non-zero at optimality (if the model is not highly constrained). More realistic spatial interaction models are, therefore, non-linear in nature, such as the gravity model. (Note the work of Evans (1973) on a relationship between the gravity model for trip distribution and transportation problems in linear programming, and Senior and Wilson's (1974) use of entropy maximising residential location models).

There is also a need for non-linear programming to satisfactorily mirror such importatnt locational properties as external economics of scale, which, along with internal scale economics, should be major features of models of urban growth.

## 3.3 Non-Linear Programming.

### 3.3.1 Non-Linear Programming : An Introduction.

Non-linearities engender complexity, and it can be argued that many linear programming formulations are non-linear ones in which the non-linearities have been disregarded or approximated in some way. The balance between mathematical tractability and realism is extremely important and this is mirrored in the advance from linear to non-linear programming models (Williams, 1979). Computational complexities are greatly increased, which often limits the dimensionality of the analysis.

This subsection will mention two topics which are related to the existence of non-linearities : one, how a spatial interaction submodel can be embedded within a mathematical fromework which optimises a given objective function and, two, how the optimisation of non-linear formulations shares something of the structure of bifurcation and catastrophe theory.

### 3.3.2 Embedding One Mathematical Programme Within Another.

The connecting of optimising models can be achieved by embedding one mathematical programme within another to generate a new, single problem. Depending on the specific model, this can be achieved in a number of ways, and, therefore, has significant computational and model-building implications. For example, an alternative objective function which implicitly comprises of the optimal conditions, a direct incorporation of the conditions into the objective function, or a placement of the conditions into the set of constraints may be used.

This procedure is exemplified here by the integration of a spatial interaction model within a mathematical programming framework which maximises consumer surplus. (This is based on the original expositions by Coehlo and Wilson (1977) and by Cochlo, Williams and Wilson (1978). to which the reader is referred for more details). The well-known Huff (1964) and Lakshmanan-Hansen (1965) shopping model is described by

$$S_{ij} = A_i e_i P_i W_j^{\alpha} e^{-\beta c_{ij}} \tag{3.3.1}$$

where

$$A_i = (\sum_j W_j^{\alpha} e^{-\beta c_{ij}})^{-1} \tag{3.3.2}$$

is a balancing factor which ensures that

$$\sum_j S_{ij} = e_i P_i \tag{3.3.3}$$

$S_{ij}$ is the flow of expenditure from residents from zone i to the facilities in zone j; $e_i$ is the per capita expenditure on goods and services by residents on zone i; $P_i$ is the population of zone i; $W_j$ is the size of service facilities in zone j (which is usually taken as a measure of attractiveness); $c_{ij}$ is the travel cost, suitable defined, from zone i to zone j; and $\alpha$ and $\beta$ are parameters.

This equilibrium, singly (production) - constrained model can be derived using entropy maximising principles, (see below for more details), and can be formulated as the following mathematical programme (Wilson, 1970).

$$\underset{\{S_{ij}\}}{\text{Max}}\ S = -\sum_{ij} S_{ij} \log S_{ij} \tag{3.3.4}$$

subject to

$$\sum_j S_{ij} = e_i P_i \tag{3.3.5}$$

$$\sum_{ij} S_{ij} \log W_j = B \tag{3.3.6}$$

$$\sum_{ij} S_{ij} c_{ij} = C \tag{3.3.7}$$

where B and C are aggregate benefits (assumed to be measured on a logarithmic scale) and costs, respectively.

An extension to this model is to determine the shop size in zone j, $W_j$, within the model. This problem was studied by Coehlo and Wilson (1976), and requires an additional constraint to limit the total shopping facility size (to, say, W). Thus, the model becomes

$$\underset{\{S_{ij},\, W_j\}}{\text{Max}} \quad S = - \sum_{ij} S_{ij} \log S_{ij} \tag{3.3.8}$$

subject to

$$\sum_j S_{ij} = e_i P_i \tag{3.3.9}$$

$$\sum_{ij} S_{ij} \log S_{ij} = B \tag{3.3.10}$$

$$\sum_{ij} S_{ij} C_{ij} = C \tag{3.3.11}$$

$$\sum_j W_j = W \tag{3.3.12}$$

Now it might be thought desirable to study shopping facility size and location with respect to maximising consumers' welfare, that is, trading-off the benefits of zone j's shopping size ($\frac{\alpha}{\beta} \ln W_j$) and the cost of travel to that zone for a resident of zone i ($c_{ij}$). Following Wilson (1976) this interpretation is generated by re-writing the term, $W_j^{\alpha}$, in equation (3.3.1), as

$$W_j^{\alpha} = \exp(\alpha \log W_j) = \exp\{\, \beta(\tfrac{\alpha}{\beta}) \log W_j \} \tag{3.3.13}$$

which enables equation (3.3.1) to be restated as

$$S_{ij} = A_i e_i P_i \exp\{- \beta(c_{ij} - \tfrac{\alpha}{\beta} \log W_j)\} \tag{3.3.14}$$

The mathematical programme, maximising consumers' benefit with consumers' behaving according to the Huff and Lakshmanan-Hansen shopping model, follows naturally:

$$\underset{\{W_j\}}{\text{Max}} \quad CB = \sum_{ij} S_{ij}\left(\frac{\alpha}{\beta}\log W_j - c_{ij}\right) - \frac{1}{\beta}\sum_{ij} S_{ij}\log S_{ij} \tag{3.3.15}$$

subject to

$$S_{ij} = A_i e_i P_i W_j^{\alpha} e^{-\beta c_{ij}} \tag{3.3.16}$$

$$A_i = \left(\sum_j W^{\alpha} e^{-\beta c_{ij}}\right)^{-1} \tag{3.3.17}$$

$$\sum_j W_j = W \tag{3.3.18}$$

This model, which has non-linearities in both the objective function and constraints, cannot be solved by existing algorithms, and, therefore, an alternative formulation is desirable.

Towards this end, the entropy maximising mathematical programming formulation of the shopping model, described by equarions (3.3.4), (3.3.5), (3.3.6) and (3.3.7), can be implanted into the mathematical programme to maximise consumer's benefit and, in so doing, remove the non-linear constraints of the initial model. Computer programmes are available to solve this equivalent mathematical programme, which has a separable, non-linear objective function and linear constraints,

$$\underset{\{S_{ij},\ W_j\}}{\text{Max}} \quad CS = \sum_{ij} S_{ij}\left(\frac{\alpha}{\beta}\log W_j - c_{ij}\right) - \frac{1}{\beta}\sum_{ij} S_{ij}\log S_{ij} \tag{3.3.19}$$

subject to

$$\sum_j S_{ij} = e_i P_i \tag{3.3.20}$$

and

$$\sum_j W_j = W \tag{3.3.21}$$

The behavioural $\{S_{ij}\}$ mathematical programme is embedded inside the planning $\{W_j\}$ mathematical programme, where the constraints of the shopping programming model, equations (3.3.6) and (3.3.7), have been included in the objective function through the multipliers, $\alpha$ and $\beta$; this maintains the consumers' welfare interpretation. Furthermore, it can be demonstrated that the original shopping model, represented by equations (3.3.1) and (3.3.2), is satisfied (Coehlo, Williams and Wilson, 1978) even though it is not explicitly included.

### 3.3.3 Structural Stability and Non-linear Optimisation.

The concept of stability is of paramount importance with respect to the optimal solution of a mathematical programme. A system's environment may evolve through small, continuous alterations, and bifurcation and catastrophe theory (see, for example, Poston and Stewart (1978), Thom (1975) and Zeeman (1977)) and analogous work from the field of optimal engineering (Thompson and Hunt, 1973) have examined how an equilibrium path abruptly becomes unstable giving qualitativly different situations. Such facts have significant ramifications for non-linear mathematical programming, and in addition, connect static and dynamic representations. (The latter is examined in more detail below). Changes in the control parameters generate new equilibria for the state variables. In fact, it is assumed to be an instantaneous response, and, therefore, the non-optimal pattern arising from a degree of inertia, which would probably exist in reality after such a change, is not found.

Analysis focuses on structural stability: the equilibrium position may become increasingly unstable or even vanish for a value or values of the parameters. Consequently, discontinuous system behaviour exists with the system jumping from one equilibrium to another. The power of the theoretical work is the classification into canonical forms of the possible structural singularities which can arise at discrete critical points in the control parameter space (associated with the disappearance of the stability determinant in a Hessian matrix.). This delimits the potential topological forms of a singularity, and projection of the locus of critical states into the control manifold gives the 'bifurcation set'. An additional point to remember in connection with our present concern with optimality is the fact that many studies using catastrophe theory have been interested in local stability, and therefore, employ the so-called 'delay-rule'. Optimal systems, however, are related to global optima, and, therefore, are associated with the so-called 'Maxwell convention'.

One major implication of this work is an awareness of the possibility of multiple optima. In fact, a system may possess secondary points of bifurcation and compound critical points (where primary and secondary bifurcation points coalesce).

Following the work of Thompson and Hunt (1973), one is able to comprehend that an optimal, 'perfect' system can be inherently unstable because of alterations in its environment. In spite of the complexities of the systems modelled which remain simplifications of the real world, it is unlikely that reality can be 'perfectly' represented by a mathematical programme. Further analysis of perturbations of a so-called 'perfect' mathematical programming model may, therefore, be fruitful. The outcome is dependent initially on the form of singularity in the 'perfect' system, and on the quantity and character of perturbation parameters included. The range of potential topological forms of behaviour which can arise are ascertained using Thom's theorem. Work in engineering has demonstrated that optimal systems are extremely sensitive to minor imperfections (Thompson and Hunt, 1973).

As a system evolves, the size and timing of perturbations is of fundemental importance if they are to affect the system's behaviour; they have no impact away from criticality. These notions relate to recent analyses of chemical and biological systems, which have illustrated how fluctuations are key elements in an evolutionary process (Nicolis and Prigogine, 1977). Fundementally, the evolution of a complex, non-linear system cannot be fully comprehended by examining deterministic equations which describe the system's dynamics. The impact of stochastic fluctuations can force the system to qualitatively different forms of behaviour - 'order through fluctuation'.

This brief exposition has direct pertinence to non-linear mathematical programming, especially to its underlying rationale of optimisation. As attempts are made to model more complex systems, the possible severity of the optimal pattern's structural instability increases, and, in so doing, accerbates the probability of undermining the basic reason for employing the rigorous techniques of mathematical programming. This should not be seen as a desire for a reversal of society's trend to become increasingly complex, but for an awareness of what the use of mathematical programming models entails). Awareness of these features is, obviously, important. As Revelle,Marks & Liebman(1970, p. 695 ) remind us,

> "Are locations set in order to be most efficient with today's population, ten years hence, or twenty years hence or an average of all three?"

In view of the likely changes over such periods, structural stability is a vital concept. Moreover, valuable linkages have been made between static and dynamic representations, although we now know that there could be discord between their optima. (Interestingly, Chisholm (1971, p. 130) emphasises the uncertainty inherent in the temporal dimension, and he advises that, because of this "deadly enemy", "normative theory must be static and not dynamic"). Is it possible to determine a configuration which is optimal for a multitude of siuations?

## 3.4 Duality.

### 3.4.1 Duality : An Introduction.

A mathematical programme, a primal problem, has a symmetrically related dual, mathematical programming problem. The optimal solution of a primal problem coincides with the optimal solution of a dual problem, and this fact often proves of enormous computational assistance in solving non-linear programmes. As the dual of a dual is a primal, the specific delimitation of primal and dual problems is of no consequence.

This sub-section will give a general description of the dual of the general linear programming which was represented above, and also specifically examine the dual of the transportation problem to portray the possible economic interpretations generated by such a formulation. The topic of non-linear duality, which is considered with respect to entropy maximising spatial models, will also be refered to, because of its computational ramifications. The employment of duals as indicators of the optimal solution's sensitivity is mentioned.

### 3.4.2 The Dual of a Linear Programme.

Associated with the linear programming problem, given by equation (3.2.1), there is a dual formulation

$$\min_{\{\underline{y}\}} Z = c_1y_1 + c_2y_2 + \ldots. + c_my_m \qquad (3.4.1)$$

subject to

$$\begin{array}{l} a_{11}y_1 + a_{21}y_2 + \ldots. a_{m1}y_m \geqslant a_1 \\ a_{12}y_1 + a_{22}y_2 + \ldots. a_{m2}y_m \geqslant a_2 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ a_{1n}y_1 + a_{2n}y_2 + \ldots. a_{mn}y_m \geqslant a_n \end{array} \qquad (3.4.2)$$

and

$$\underline{y} \geqslant 0 \tag{3.4.3}$$

A number of features of the primal-dual relationship of a linear programme are worth noting. Both the mathematical programmes are linear programmes, the primal is a maximisation problem and the dual is a minimisation problem in this example. The primal objective function has as many state variables ($x_i$; $i = 1,2, \ldots,n$) as there are constraints in dual formulation (excluding the non-negative constraints) and as many constraints (excluding the non-negativity constraints) as there are state variables ($y_j$; $j = 1,2,\ldots,m$) in the dual formulation. The state variables' coefficients in the primal objective function ($a_i$; $i = 1,2,\ldots,n$) become the right-hand side constraints in the dual programme's constraints, and the right-hand side constraints in the primal formulation's constraints ($c_j$; $j = 1,2,\ldots,m$) are the state variables' coefficients in the dual programme's objective function. The coefficients in the constraint set of the primal programme ($a_{ij}$; $i = 1,2,\ldots,n$ and $j = 1,2,\ldots m$) are, in matrix terminology, transposed to form the coefficients of the constraint set in the dual programme. Finally, in the maximisation problem the constraint inequalities are always 'less than or equal to' statements, whilst in the minimisation problem they are always 'greater than or equal to' statements (noting that, for the simplex method to work, it is necessary to alter the inequalities to equalities through the introduction of 'slack' variables); non-negativity constraints operate on the state variable in both the primal and dual linear programmes.

This rather tedious and prolonged description illustrates the symmetry of the primal-dual relationship, and it is a useful foundation from which to discuss the dual linear programme of the transport problem (For a discussion of the difficulties associated with 'degeneracy' (where more than one dual solution exists) the reader is referred to Baumol (1977)).

### 3.4.3 The Dual of the Transportation Problem.

The linear programme termed the transportation problem, which is portrayed by equations (3.2.2) to (3.2.5), has the associated dual programme

$$\underset{\{\underline{a},\underline{b}\}}{\text{Max}}\ Z = \sum_j d_j a_j - \sum_i s_i b_i \tag{3.4.4}$$

subject to

$$a_j - b_i \leqslant c_{ij}\ , \quad \text{for all i's and j's.} \tag{3.4.5}$$

$$\underline{a} \geqslant 0,\ \underline{b} \geqslant 0 \tag{3.4.6}$$

From the previous general discussion of the primal-dual, linear programme relationship, the derivation of the dual of the transportation problem, represented by equations (3.4.4), (3.4.5) and (3.4.6), is readily comprehended. The dual variables, ($a_j$ and $b_i$), have been introduced, and it should be noted that the negative sign in the dual's objective function arises from the inequality constraint on supply capacity in the primal programme, equation (3.2.3). In this minimisation problem the inequality should be of the form, 'greater than or equal to', therefore, multiplication by -1 is required.

One of the reasons for formulating a dual programme is to assist interpretation, and, therefore, it has significant implications for model building in general. (Chisholm and O'Sullivan (1973), Maxfield (1969), Morrill and Garrison (1960), Senior and Wilson (1974), and Stevens (1961) have all demonstrated that dual formulations can assist explanation). The dual formulation of the transportation problem has explicit economic meaning. The dual variables, $\underline{a}$ and $\underline{b}$ are termed 'shadow prices', and they vary over space. Specifically, $a_j$ is related to the demand at location j and $b_i$ is related to the supply capacity at location i; these facets are associated with the demand and supply constraints in the primal programme. The shadow price of each constraint mirrors the alteration in the aggregate value of the primal objective function resulting from a marginal reinforcement or loosening of the right-hand side of a constriant, that is, a constraints opportunity cost.

The dual programme can be interpreted as the maximisation of system's value-added with respect to shadow prices, subject to the competitive spatial price equilibrium condition that the specific variation in the price of a commodity for any transaction between a supplier located at i and a consumer located at j must not be greater than the transport costs incurred. As Hay (1977, p.17) states

"The constraint ignores any addition to the price made by the producer as a result of pricing policies: but if he raises that price obviously and appreciably an alternative supplier can compete with him".

The employment of a dual, therefore, assists interpretation and has benefits for both the model builder and the policy maker. The significance of a constraint is ascertained, and as many are 'man-made', such as density levels, and minimum development sizes, various policies can be evaluated and modified. If a dual variable is zero at optimality in the dual model, the corresponding constraint in the primal formulation is not binding, and this is often a useful indication of the need to alter the original model.

### 3.4.4 Duality in Non-linear Mathematical Programming with Special Reference to a Spatial Interaction Model.

In addition to a dual's insights into a primal model (as illustrated in the preceding discussion of linear programmes), dual formulations of non-linear mathematical programmes have proved of great computational assistance in work involving spatial interaction. This sub-section will briefly examine duality in non-linear mathematical programming, and exemplify its application by considering in detail, the entropy maximising mathematical programme for spatial interaction.

A number of different formulations of the dual of a non-linear programme have been postulated, conditioned by the form of the model and the aspects of interest in the duality linkage. In the field of transportation, for example, Williams (1976) has employed Rockafellar's (1967) duality theory of separable convex programming. As with the linear programme duality property, the symmetrical relationship that the dual of the dual is the primal is present. In contrast, Wolfe's (1961) duality theory does not contain this feature. It was this formulation for which Balinski and Baumol (1968) offered economic interpretation, and Wilson and Senior (1974) have applied it to analyse entropy maximising models.

Wolfe's (1961) theory can be stated: for a constrained minimisation problem involving a convex primal objective function and a set of concave constraints (all of which are differentiable functions of $\underline{x}$), such as

$$\min_{\{\underline{x}\}} P = f(\underline{x}) \qquad (x_i,\ i = 1,2,\ldots,n) \tag{3.4.7}$$

subject to

$$g_i(\underline{x}) \geqslant 0 \qquad j = 1,2,\ldots,m \tag{3.4.8}$$

the associated dual formulation is

$$\max_{\{\underline{x},\underline{\lambda}\}} D = f(\underline{x}) - \sum_{j=1}^{m} \lambda_j g_j(\underline{x}) \tag{3.4.9}$$

subject to

$$\frac{\partial f(\underline{x})}{\partial \underline{x}} = \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(\underline{x})}{\partial \underline{x}} \tag{3.4.10}$$

$$\underline{\lambda} \geqslant 0 \tag{3.4.11}$$

where equation (3.4.9) is the dual objective function (including the dual variable, $\lambda_j$, associated with the $j^{th}$ constraint in the primal model), which is maximised subject to the constraints, equations (3.4.10) and (3.4.11).

Wolfe (1961) also demonstrates that if the constraints in the primal model are linear, that is, equation (3.4.8) is of the form

$$g_j(\underline{x}) = \underline{A}_j\underline{x} - b_j \tag{3.4.12}$$

where $\underline{A}$ is an m x n matrix which has $\underline{A}_j$ as the $j^{th}$ row. The dual problem can now be rewritten as

$$\underset{\{\underline{x},\underline{\lambda}\}}{\text{Max}}\ D = f(\underline{x}) + \lambda_j(b_j - \underline{A}_j\ \underline{x}) \tag{3.4.13}$$

subject to

$$\frac{\partial f(\underline{x})}{\partial \underline{x}} = \lambda_j \underline{A}_j \tag{3.4.14}$$

and

$$\underline{\lambda} \geqslant 0 \tag{3.4.15}$$

and the primal objective function, $f(\underline{x})$, is found in both the objective function and constraints of the dual. Given constraint (3.4.14), it is possible to restate the objective function (3.4.13) as

$$\underset{\{\underline{x},\underline{\lambda}\}}{\text{Max}}\ D = f(\underline{x}) + \sum_{j=1}^{m} \lambda_j b_j - \underline{x}\ \frac{\partial f(\underline{x})}{\partial \underline{x}} \tag{3.4.16}$$

This transformation proves useful in making the dual programme of the entropy maximising spatial interaction model an unconstrained problem (see equation (3.4.50)).

Given this rather formal presentation of Wolfe's (1961) theory of duality in non-linear programming, an example demonstrating its application is worthwhile. It is proposed to follow Wilson's (1970) entropy maximising mathematical programming formulation of a spatial interaction model (for a full exposition of the approach, the reader is also refered to Gould (1972) and Senior (1979); its derivation will be described as the modelling is relevant to other sections of this paper and it also assists in the interpretation of the dual (for further information see Wilson and Senior, 1974).

Analysis focuses on the spatial pattern of flows, $T_{ij}$, from zone i to zone j, subject to additivity constraints for total flows on each origin zone i,

$$\sum_j T_{ij} = O_i \tag{3.4.17}$$

and on each destination zone j,

$$\sum_i T_{ij} = D_j \tag{3.4.18}$$

A further constraint, a constraint on the aggregate travel budget (C), is also included

$$\sum_{ij} T_{ij} c_{ij} = C \tag{3.4.19}$$

where $c_{ij}$ is the unit cost of a flow from zone i to zone j (the non-negativity constraints ($\underline{T} \geqslant 0$) have been excluded since, for finite values of $\beta$, $T_{ij} > 0$ (Wilson and Senior, 1974)).

Given this structure, there is, obviously, a large number of different flows which meet this spatial pattern. The total number of combinations for a particular distribution is

$$W(T_{ij}) = \frac{T!}{\prod_{ij} T_{ij}!} \tag{3.4.20}$$

where

$$T = \sum_{ij}\sum T_{ij} \tag{3.4.21}$$

It is now possible to formulate a mathematical programming problem to calculate the most probable flow pattern ($T_{ij}$) at the micro-level,

$$\underset{\{\underline{T}\}}{\text{Max}}\ W(T_{ij}) = \frac{T!}{\prod_{ij} T_{ij}!} \tag{3.4.22}$$

subject to constraints operating at the aggregate level,

$$\sum_j T_{ij} = O_i \tag{3.4.23}$$

$$\sum_i T_{ij} = D_j \tag{3.4.24}$$

and

$$\sum_{ij} T_{ij} c_{ij} = C \tag{3.4.25}$$

In fact, for convenience, Wilson (1970) maximised log $W(T_{ij})$ instead of $W(T_{ij})$. (The optimisation problem is unaffected by this transformation, because it involves a monotonic function of $W(T_{ij})$). The objective function can now be written as

$$\underset{\{\underline{T}\}}{\text{Max}} \log W(T_{ij}) = \log T! - \sum_{ij} \log T_{ij}! \tag{3.4.26}$$

"The derivation relies on the use of Sterling's approximation..." (Wilson, 1970, p.5), that is

$$\log T_{ij}! = T_{ij} \log T_{ij} - T_{ij} = T_{ij}(\log T_{ij} - 1) \tag{3.4.27}$$

and the objective function is restated as

$$\underset{\{\underline{T}\}}{\text{Max}} \log W(T_{ij}) = \log T! - \sum_{ij} T_{ij}(\log T_{ij} - 1) \tag{3,4,28}$$

As the term log T! is a constant, it can be taken out of the optimization problem; this leaves the mathematical programme,

$$\underset{\{\underline{T}\}}{\text{Max}} \log W(T_{ij}) = - \sum_{ij} T_{ij}(\log T_{ij} - 1) \tag{3.4.29}$$

subject to

$$\sum_j T_{ij} = O_i \tag{3.4.30}$$

$$\sum_i T_{ij} = D_j \tag{3.4.31}$$

$$\sum_{ij} T_{ij} c_{ij} = C \tag{3.4.32}$$

To demonstrate that this mathematical programme is a spatial interaction model, it is necessary to form the programme's associated Lagrangian function, L,

$$L = - \sum_{ij} T_{ij}(\log T_{ij} - 1) + \sum_i \alpha_i (O_i - \sum_j T_{ij}) + \sum_j \gamma_j (D_j - \sum_i T_{ij}) + \beta(C - \sum_{ij} T_{ij} c_{ij}) \tag{3.4.33}$$

where $\alpha_i$, $\gamma_j$, and $\beta$ are the Lagrangian multipliers (which correspond to the dual variables) related to the constraints (3.4.30) to (3.4.32), respectively. A necessary condition for an optimum, illustrated by the Kuhn-Tucker conditions, is

$$\frac{\partial L}{\partial T_{ij}} = 0 \qquad \text{(note, this arises because, for finite values of } \beta,\ T_{ij} > 0) \tag{3.4.34}$$

and, therefore

$$\frac{\partial L}{\partial T_{ij}} = -\log T_{ij} - \alpha_i - \gamma_j - \beta c_{ij} = 0 \tag{3.4.35}$$

Rewritting this equation gives

$$\log T_{ij} = -\alpha_i - \gamma_j - \beta c_{ij} \tag{3.4.36}$$

or, alternatively,

$$T_{ij} = e^{-\alpha_i - \gamma_j - \beta c_{ij}} \tag{3.4.37}$$

Substitution of this equation in the first constraint ($\sum_j T_{ij} = O_i$) gives

$$O_i = \sum_j e^{-\alpha_i - \gamma_j - \beta c_{ij}} \tag{3.4.38}$$

which, after rearrangement, becomes

$$\frac{O_i}{e^{-\alpha_i}} = (\sum_j e^{-\gamma_j - \beta c_{ij}}) \tag{3.4.39}$$

and, similarly, from the second constraint ($\sum_i T_{ij} = D_j$)

$$D_j = \sum_i e^{-\alpha_i - \gamma_j - \beta c_{ij}}) \tag{3.4.40}$$

which after rearrangement, becomes

$$\frac{D_j}{e^{-\gamma_j}} = (\sum_i e^{-\alpha_i - \beta c_{ij}}) \tag{3.4.41}$$

Now, by letting

$$A_i = \frac{e^{-\alpha_i}}{O_i} \tag{3.4.42}$$

and

$$B_j = \frac{e^{-\gamma_j}}{D_j} \qquad (3.4.43)$$

equation (3.4.37) becomes

$$T_{ij} = A_i B_j O_i D_j e^{-\beta c_{ij}} \qquad (3.4.44)$$

where

$$A_i = (\sum_j B_j D_j e^{-\beta c_{ij}})^{-1} \qquad (3.4.45)$$

and

$$B_j = (\sum_i A_i O_i e^{-\beta c_{ij}})^{-1} \qquad (3.4.46)$$

Equations (3.4.44) to (3.4.46) can be recognised a spatial interaction model, the doubly-constrained gravity model. As the balancing factors, $A_i$ and $B_j$, are functionally interrelared, a common way to solve this spatial interaction model is to fix an initial starting value for $A_i$(or $B_j$) and calculate the value of $B_j$(or $A_i$), and iterate until convergence is reached,(see the computer programmes in Baxter (1976)). An alternative method, obviously, is to solve its mathematical programming version; a third method is to derive and solve the dual programme.

The dual programme is formulated by employing Wolfe's (1961) theorem. Noting that

$$\frac{\partial(-\sum_{ij} T_{ij}(\log T_{ij} - 1))}{\partial T_{ij}}$$

equals $-\log T_{ij}$, the dual objective function can be stated as

$$\underset{\{\underline{T},\underline{\alpha},\underline{\gamma},\beta\}}{\text{Min } D} = \sum_{ij} T_{ij} + \sum_i \alpha_i O_i + \sum_j \gamma_j D_j + \beta C \qquad (3.4.47)$$

This is subject to

$$-\log T_{ij} = \alpha_i + \gamma_j + \beta c_{ij} \tag{3.4.48}$$

The non-negativity constraints on the dual variables are taken out because the primal programme's constraints are equalities. $\alpha_i$, $\gamma_j$ and $\beta$ are the dual variables related to constraints (3.4.30) to (3.4.32), respectively. From equation (3.4.48)

$$T_{ij} = e^{-\alpha_i - \gamma_j - \beta c_{ij}} \tag{3.4.49}$$

this can be placed into the dual objective function to give an unconstrained minimisation problem

$$\underset{\{\underline{\alpha},\underline{\gamma},\beta\}}{\text{Min}} \quad D = \sum_{ij} e^{-\alpha_i - \gamma_j - \beta c_{ij}} + \sum_i \alpha_i O_i + \sum_j \gamma_j D_j + \beta C \tag{3.4.50}$$

The computational advantages of this alternative formulation result from its reduced dimension; mathematical programming models which incorporate spatial interaction features possess a large number of variables and constraints which often prove restrictive in spite of recent advances in computer technology. For example, if a particular problem has n origin zones and m destination zones, the primal mathematical programme considered above would have had (n x m) variables and (n + m + 1) constraints, which is in stark contrast to the dual problem's (m + n + 1) variables. (This feature is particularly well demonstrated in the work on the optimal design of land use plans by Coehlo and Williams (forthcoming)).

In addition to the computational advantages, interpretation of the primal problem is enhanced by an examination of the dual variables. It has already been mentioned that the dual variable of a particular constraint mirrors the marginal effect on the value of the objective function of an alteration in the right-hand side of the constraint. Thus, in their relation to the zones of origin and destination, the dual variables, $\alpha_i$ and $\gamma_j$, portray gravity-type potential terms. Such information could prove useful with respect to attempts to equalise/increase accessibility to facilities from particular areas.

### 3.4.5 Duality: Final Comments.

The significance of duality, both for economic interpretation and for computational reasons, have been described. It can, in fact, be argued that such formulations should be an integral part of the building of mathematical programmes (and should not be regarded as an extra exercise to be undertaken occasionally). It should be pointed out, that the information on opportunity costs is limited to a restricted range of the optimal solution. This raises a fundamental question with regard to the stability of the optimal pattern; analysis may be unsound if the optimal solution is unstable.

The dual variables present information about the right-hand side coefficients of the primal's constraints; additional information on the importance of other coefficients in a mathematical programming model would be of interest and this leads onto a brief consideration of parametric analysis on the primal solution.

### 3.5 Parametric and Sensitivity Analysis.

In addition to information about the optimal pattern and the marginal effect of specific constraint values, it can be appreciated that knowledge of the influence of alterations in the coefficients of the objective function and of the constraints would be valuable.

Reference has already been made to the significance of the stability of a an optimal solution. (Poston and Stewart (1978) refer to a 'constraint catastrophe', which can occur when optimal points are generated by constraints rather than the disappearance of derivations. Similarly, a parameter change can result in the solution of a mathematical programme, particularly a linear programme, jumping from one vertex of the feasible region to another.) The degree of sensitivity of the solution to parameter changes must be thought of as part of the process of building a mathematical programme; comparison and evaluation of alternative model formulations and analysis of the results of changes in coefficients is important. With modelling problems arising from the quality of available data, the sensitivity of a solution to changes in coefficients is essential. Furthermore, it links up with the use of mathematical programmes to examine potential impacts of various policies (see Bonsall, et al.'s (1977) detailed study of this issue in transport modelling). These notions of sensitivity are also related to the morphology applied in a mathematical programme; what is the effect on the optimal solution of variations in zonal size and configuration? (Openshaw (1977), for example, has attempted to incorporate scale and aggregation problems into a problem optimising the pattern of zones subject to a number of constraints)

Thus, it can be suggested that the optimal solution of a mathematical programme must be complemented by additional information on its sensitivity to parameter change. Without this knowledge, one must question the value of an optimal solution.

## 3.6 The Temporal Dimension.

### 3.6.1 Some Introductory Comments.

Up to this point, the mathematical programming formulations have not been related to the temporal dimension. The models generate optimal, static, spatial patterns in which all the activity takes place at one instance. This abstraction does not facilitate the modelling of process, the force of change. The combination of optimisation procedures and time raises the issue of optimal processes, a feature which has direct significance to ideas of system contr l. In addition, knowledge of important processes assists in the comprehension of a system.

Bearing in mind these introductory remarks, this sub-section will introduce some features of dynamic programming and optimal control problems. The desire for an explicit representation of process by differential (or difference) equations is mentioned, and a discussion of their associated difficulties leads onto the outline of two alternative arguments.

### 3.6.2 Dynamic Programming and Optimal Control Problems.

The mathematical technique, dynamic programming, was developed by Bellman (1957); it is an optimisation procedure, which is especially apposite for problems comprising of time dependence because optimal values of variables are selected sequentially rather than simultaneously. This feature of multi-stage decision-making has been considered in the geographical literature in connection with regional development (Anuchin, 1973), transportation systems, network construction problems, and water resource management (MacKinnon, 1969). A problem is decomposed into a sequence of individual problems; Bellman's "principle of optimality", for example, states that

"An optimal policy (or set of decisions) has the property that whatever the initial state and initial decision are, the remaining dicisions must constitute an optimal policy with regard to the state resulting from the first decision". (from Rao, 1978, p. 481)

Unfortunately, the dimensionality curse falls on dynamic programming formulations, although their structure is useful for optimisation over time. An alternative conceptualisation of system dynamics involves the incorporatio of differential (or difference) equations; these types of equations are employed in optimal control theory to describe the impact of the control variables on the system's state variables. (Keys (1979), however, has recentl demonstrated that the formulation and solution of these problems does not require such equations).

In addition to deterministic models, stochastic system representations have been estimated from time-series data (Bennett, 1978a) (It should be noted that optimal control models are also significant with respect to contr in spatially divided systems, such as regional economies (Bennett, 1978b)) A final remark, which is especially significant in view of the importance placed upon solution sensitivity in model building, draws attention to Keys (1979) research into the sensitivity of solutions of optimal control problem he has demonstrated that the adjoint variables (the parallel variables of th dual variables) are also related to the sensitivity issue.

### 3.6.3 Equilibrium Point Dynamics.

It is simple to demand a dynamic representation; it is formidable to respond to this desire. Unfortunately, insufficient knowledge of a system's behaviour often precludes attempts to portray it through differential (or difference) equations. Static equilibrium states can, however, be incorporated into dynamic structures.

Following Macgill and Wilson's (1979) argument, based on Bellman's (19 work, a static equilibrium state of a system,

$$\underline{A}\ \underline{x} = \underline{b} \tag{3.6.1}$$

can be embedded within a differential equation describing the state variables, $\underline{x}$, as

$$\frac{d\underline{x}}{dt} = (\underline{A}\ \underline{x} - \underline{b}).\quad F(\underline{x}) \tag{3.6.2}$$

where $F(\underline{x})$ is some function of $\underline{x}$. As Macgill and Wilson note, additional, unknown equilibrium states may exist (that is, the solutions of $F(\underline{x}) = 0$), and a dynamic model may have identical equilibrium states to those of a static model (that is, when $\underline{A}\ \underline{x} = \underline{b}$).

### 3.6.4 Optimal Evolution.

The notion that many of the processes that geographers study are underpinned by optimality principles has been sketched in section two, and it can be linked to the desire for dynamic location models. A system of differential (or difference) equations, given the initial conditions, is sufficient for a dynamical representation. However, the information required for such an explicit formulation is often not present; it is possible, instead, to use a line of argument (which has been extensively employed in biology, see, for example, E.O. Wilson (1975)) that is a short -cut to an evolutionary form. Care, however, must be taken not to mistake it for a dynamical representation.

The optimality thesis, founded on general efficiency principles, offers the final structure of the evolution and the path towards it. In other words, the basic tenet contends that the configuration will evolve to form the optimal pattern, and, therefore, distance from equilibrium/optimality is seen as the major drive behind a system's alteration. Fundamentally, optimality is viewed as a temporal concept, that is, analysis extends beyond efficient, steady states and encompasses efficiency with respect to change over time.

An interesting research topic, for instance, would be to consider evolution in a central place system. The operationalisation of existing notions through mathematical programming formulations could present original contributions to this area of interest (section five exemplifies this assertion). Vance (1970, p. 8), in his individual treatment of the geography of wholesaling, criticised the theoretical framework in which

> "The prime assumption... is an evolution to a final state of order, with the concomitant notion that any departure from that state of order represents a clearly preliminary situation which will soon be "improved" so as to conform to the theoretical mechanism"

It can, however, be argued that model building using mathematical programmes, which incorporates aspects of stability, sensitivity, stochastic fluctuations and so on, is not necessarily rigidly deterministic (and, therefore, unrealistic).

It should be noted here that the 'dynamic' model, outlined in section five below, applies the Lagrangian formulation of the particular mathematical programming model. The evolution path uses Lagrangian partial derivatives, which portray the most efficient trajectory towards equilibrium. Some comments on the topic of structural stability are also given.

This line of argument does require careful thought, and there are problems which raise both theoretical and empirical questions. In this respect, it is important to reiterate that 'optimal evolution' must be regarded as a means to gain insights in the formulation of a dynamical system. It would be only too easy to view them as plausible evolutions _per se_, describing the events in such a way that implies the systems' components actually brought about purposeful alteration. Empirical investigations are, therefore, needed to ascertain whether general, optimising behaviour exists. Without verification, such representations could be unwarranted. Furthermore, and related to the above consideration of objective function definition, various statements could be propounded for specific contexts. Maximising producers' profit, minimising producers' cost, minimising aggregate distance travelled to a facility by a consumer, and so on, for example, may all be salient in a particular situation. Once again analysis and comparison of the different objectives should be theoretically rewarding.

Attention to the stability of the equilibrium state to which a system evolves is necessery, because, in the real world, changes are occurring which affect the system. This feature relates to the previous examinations of the *ceteris paribus* assumption and of criticality, and bears upon the underlying rationale of system efficiency and adaptability.

# 4. Location-Allocation Problems.

## 4.1 Location-Allocation Problems : An Introduction.

The way mathematical programming models can marry practical and analytical aspects of a problem is perhaps best demonstrated by the so-called location-allocation problems, which jointly optimise the location of facilities and the allocation of consumers to them. Such analyses now form an extensive literature on private and public sector facility location (for further information see, for example, Freestone (1977), Hodgart (1978), Lea (1973) and ReVelle, Marks and Liebman (1970)), and, in addition, a number of computer algorithms are available (Rushton, Goodchild, and Ostresh, 1973).

Common features of mathematical programmes are to be found in the various formulations. Public facility location problems, for example, often contain some social welfare function which is to be maximised subject to a set of constraints (which include investment budget limits). The p- median problem, introduced by Hakimi (1964), which optimally locates p facilities in order to minimise the sum of the weighted travel distances, has been computed as an integer linear programme by ReVelle and Swain (1970).

Early work concentrated on facets of spatial efficiency (such as the minimisation of the aggregate distance consumers travel to a facility), but more recently, concern has focused on the conflict between welfare and efficiency criteria in the location of facilities (Dear, 1974; Massam, 1975; McAllister, 1976; McGrew and Monroe, 1975; Morrill, 1974; Morrill and Symons, 1977; and Symons, 1971). Numerous formulations can be analysed and compared in an attempt to examine trade-offs between efficiency and equity: for example, a location-allocation model enabled Hotelling's (1929) well-known, ice-cream selling problem to be revisited in two dimensions, and it was found that optimal, competitive and welfare patterns are not congruent (Beaumont, forthcoming).

The vast majority of location-allocation models allocate people to the nearest facility, and this does not mirror the observed behaviour of consumers. As a consequence of this point, this section incorporates a spatial interaction model of consumer demand (which can be compared to Hodgson's (1978) application of an entropy-maximising interaction model to allocate patrons to facilities); a suitable heuristic solution procedure is also outlined. The line of the argument in Section 4.2 begins with the common multi-facility location problem to minimise aggregate travel (which results in movement to the nearest facility), and it is then suggested that, when a spatial interaction model is included, total travel cost is an inappropriate criterion for user benefit. An alternative welfare function, locational surplus, which is demonstrated to be a generalisation of the multi-facility location problem, is then discussed. In Section 4.3, an alternative formulation, to generate optimal facility patterns - accessibility maximising ; is described. A brief look at covering-type problems, in Section 4.4, suggests a way of determining the number of facilities required and linkages with central place theory demonstrates one area where further research is possible.

## 4.2 Spatial Interaction Models and the Location-Allocation Problem.

As stated above, this sub-section emphasises two aspects of the location-allocation problem - the analytical structure and the behavioural context (more details can be found in Beaumont (forthcoming)).'

Simply stated, the elementary and fundamental form of the location-allocation problem aims to optimally locate a set of m supply facilities in order to minimise the aggregate transport costs contracted in providing a set of n demand points. The problem is uncapacitatied, and, therefore, at optimality, it is assumed that each demand location will be served by only one supply facility. This continuous space, location-allocation problem may be formally expressed in terms of the optimisation problem

$$\underset{\{\underline{s}\}}{\text{Min}}\ C\ (\underline{d};\underline{s}) \tag{4.2.1}$$

where $\underline{d}$ and $\underline{s}$ are the vectors denoting the location of n demand points, $(x_i,y_i)$, and of m supply points, $(x_j,y_j)$ respectively. This equals

$$\underset{\{\underline{s},\underline{\lambda}\}}{\text{Min}}\quad C = \sum_{i=1}^{n}\ \sum_{j=1}^{m}\ O_i\ \lambda_{ij}\ c_{ij} \tag{4.2.2}$$

subject to

$$\sum_{j=1}^{m}\ \lambda_{ij} = 1\ , \qquad i = 1,\ 2,\ \ldots,\ n \tag{4.2.3}$$

$$\lambda_{ij} = \{ \begin{matrix} 1 \\ 0 \end{matrix} \tag{4.2.4}$$

where $O_i$ is the quantity demanded to location $(x_i,\ y_i)$; and the generalised transport cost between the demand point i and the supply point j is represented by $c_{ij}(i = 1,\ 2,\ \ldots,\ n;\ j = 1,\ 2,\ \ldots,\ m)$ which will be taken as the Euclidean metric,

$$c_{ij} = \{(x_i - x_j)^2 + (y_i - y_j)^2\}^{\frac{1}{2}} \tag{4.3.5}$$

$\lambda_{ij}$ is a binary variable having the value 1 of the demand point i is allocated to supply facility j, 0, otherwise. Constraint (4.2.3) ensures the all-or-nothing allocation procedure.

Both exact and heuristic algorithms are available to solve the above problem. For consistency with later models, a heuristic solution is documented here. It involves iterating through the following equations

$$x_j = \frac{\sum_{i=1}^{n}\ O_i\lambda_{ij}x_i/c_{ij}}{\sum_{i=1}^{n}\ O_i\lambda_{ij}/c_{ij}} \tag{4.2.6}$$

$$y_j = \frac{\sum_{i=1}^{n} O_i \lambda_{ij} y_i / c_{ij}}{\sum_{i=1}^{n} O_i \lambda_{ij} / c_{ij}} \tag{4.2.7}$$

as $c_{ij}$ is a function of both $x_j$ and $y_j$. Equations (4.2.6) and (4.2.7) are produced by differentiating equation (4.2.2) with respect to both $x_j$ and $y_j$, and setting the derivatives equal to zero (that is, $\partial c/\partial x_j = 0$, and $\partial c/\partial y_j = 0$).

Reference, however, has been made to the fact that the assumption of movement to the nearest supply facility can be criticised for not reflecting the observed behaviour of consumers. Williams (1977), for instance, considers nine possible interpretations of observed trip dispersion. Utility can be derived by patronising a more distant outlet. This behavioural assumption can be described by $P_{ij}$, representing the probability of patronising a supply facility at j for an individual at i,

$$P_{ij} = \frac{A_j e^{-\beta c_{ij}}}{\sum_{j=1}^{m} A_j e^{-\beta c_{ij}}}, \qquad \begin{array}{l} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, m \end{array} \tag{4.2.8}$$

where $A_j$ is the exponential transform of the mean utility at j, and $\beta$ is an elasticity/dispersion parameter which depicts a trade-off between transport costs and qualitative spatial choice attributes related different supply locations and goods (Williams, 1977). It takes into account consumers' perception of all non-transportation influences.

"It is in fact related to the variability of perceived costs about the mean value due to the presence of other non-measured factors". (Williams and Senior, 1977, pp. 203-204)

When consumers travel to the nearest facility (as in the common multi-facility problem), the dispersion parameter, $\beta$, is infinite. As choices, tastes, and other non-transportation factors in consumer behaviour increase in importance, $\beta$ tends towards zero, and transport costs are relatively less crucial in determining their behaviour; transport costs are no longer a contributor to consumer choice when β equals zero.

At first sight the natural generalisation of the aggregate transport cost criterion would be given by

$$\underset{\{\underline{s}\}}{\text{Min}}\ Z = \sum_{i=1}^{n} \sum_{j=1}^{m} O_i P_{ij} c_{ij} \tag{4.2.9}$$

and it is noted that it is an unconstrainted problem because the all-or-nothing nature of the problem (portrayed in the nearest facility allocation) has been relaxed. In order to solve this model, differentiation of equation (4.2.9) with respect to $x_j$ and $y_j$ gives

$$\frac{\partial Z}{\partial x_j} = \sum_{i=1}^{n} O_i P_{ij}(1 - \beta(c_{ij} - \sum_{j'=1}^{m} P_{ij'} c_{ij'})) \frac{(x_i - x_j)}{c_{ij}} \tag{4.2.10}$$

and

$$\frac{\partial Z}{\partial y_j} = \sum_{i=1}^{n} O_i P_{ij}(1 - \beta(c_{ij} - \sum_{j'=1}^{m} P_{ij'} c_{ij'})) \frac{(y_i - y_j)}{c_{ij}} \tag{4.2.11}$$

By equating these partial derivatives to zero, the optimal supply locations can be expressed as

$$x_j = \frac{\sum_{i=1}^{n} O_i \frac{P_{ij} x_i}{c_{ij}} (1 - \beta(c_{ij} - \sum_{j'=1}^{m} P_{ij'} c_{ij'}))}{\sum_{i=1}^{n} O_i \frac{P_{ij}}{c_{ij}} (1 - \beta(c_{ij} - \sum_{j'=1}^{m} P_{ij'} c_{ij'}))} \tag{4.2.12}$$

and

$$y_j = \frac{\sum_{i=1}^{n} O_i \frac{P_{ij} y_i}{c_{ij}} (1 - \beta(c_{ij} - \sum_{j'=1}^{m} P_{ij'} c_{ij'}))}{\sum_{i=1}^{n} O_i \frac{P_{ij}}{c_{ij}} (1 - \beta(c_{ij} - \sum_{j'=1}^{m} P_{ij'} c_{ij'}))} \tag{4.2.13}$$

in which it is noted that $P_{ij}$, through its dependence on $c_{ij}(j = 1, 2, \ldots, m)$, is a function of all supply co-ordinates.

The next step in the argument is to question whether the narrow emphasis on the total transport cost criterion is sensible here. Neuberger

(1971), Quarmby and Neuberger (1969) and Williams (1976), for instance, all demonstrate that for transport and land-use evaluation aggregate travel cost (or time) is, in general, an unsuitable benefit criterion when consumers' choice is included. (It is only apposite when there is no differentiation of supply facilities and consumers, and when it is assumed that everyone patronises the nearest facility). More fundementally, transport cost is untenable with respect to the theoretical foundation of the location decision-making process, because it disregards the foundation of spatial choice behind trip dispersion.

Neuberger (1971) suggested that a suitably defined locational surplus function is an appropriate criteria for recording user benefit. In addition, and most importantly, Neuberger explicitly recognised the structural equifinality of models in specific utility maximising and entropy maximising systems. The underlying theoretical framework of any model must, therefore, be made explicit, and also it must be realised that is is not substitutable. Here, random utility theory, forming the basis of a causally interpretable gravity-type spatial interaction model, appears particularly apposite. Unlike the usual foundations of gravity models, its incorporation of the utility generated from residing in zone i and patronising a facility in zone j is consistent with the common idea of travel as a derived demand. (Excellent discussions of this systematic representation of the micro-level decision process are given by Cochrane (1975), McFadden (1974), and Williams (1976; 1977)). Simply, random utility theory of choice assumes that each individual picks that alternative which provides him the greatest <u>net</u> benefit and that

"the trip distribution patterns reflect the overall probability of particular trips being chosen on this basis". (Cochrane (1975, p.1))

In order to derive the locational surplus welfare criterion, the change in benefit perceived by consumers when the configuration of supply points is changed from an initial state $\underline{s}^o = (x_1^o, y_1^o, \ldots, x_m^o, y_m^o)$, with the corresponding cost matrix $\underline{c}^o = (c_{11}^o, \ldots, c_{nm}^o)$, to a final state $\underline{s}' = (x_1^1, y_1^1, \ldots, x_m^1, y_m^1)$ is considered. This alteration in benefit, $\Delta LS$, is given by the path integral

$$\Delta LS = - \sum_{i=1}^{n} \sum_{j=1}^{m} \int_P d\underline{c}\, P_{ij}(\underline{c}) \tag{4.2.14}$$

where P is some path in cost-space between the initial and final states, $\underline{s}^o$ and $\underline{s}^1$ (Williams and Senior, 1977). As the function P satisfies integrability conditions (Williams, 1976), the integral is independent of path, and the change in locational surplus, $\Delta LS$, is given by

$$\Delta LS = \frac{1}{\beta} \sum_{i=1}^{n} O_i \log \frac{\sum_{j=1}^{m} A_j e^{-\beta c_{ij}}}{\sum_{j=1}^{m} A_j e^{-\beta c_{ij}^o}} \tag{4.2.15}$$

The denominator term in equation (4.2.15) can be seen as constant in the context of optimising the supply locations, and the location problem can be expressed as

$$\max_{\{\underline{s}\}} LS(\underline{d}, \underline{s}) \tag{4.2.16}$$

For the spatial interaction model (equation (4.2.8)), the maximisation of locational surplus, LS, can be written as

$$\max_{\{\underline{s}\}} LS = \frac{1}{\beta} \sum_{i=1}^{n} O_i \log \sum_{j=1}^{m} A_j e^{-\beta c_{ij}} \tag{4.2.17}$$

Here the optimisation is involved with locating m supply facilities (each one assumed to be of an adequate capacity to accommodate any demand that may be placed upon it) to maximise locational surplus for n exogenous locations of (inelastic) demand. This problem can be solved by employing a similar algorithm to those already discussed; by differentiating equation (4.2.17) with respect to $x_j$ and $y_j$ and setting them equal to zero ( that is $\partial LS/\partial x_j = 0$, and $\partial LS/\partial y_j = 0$), it can be easily demonstrated that

$$x_j = \frac{\sum_{i=1}^{n} O_i P_{ij} x_j / c_{ij}}{\sum_{i=1}^{n} O_i P_{ij} / c_{ij}} \tag{4.2.18}$$

and

$$y_j = \frac{\sum_{j=1}^{n} O_i P_{ij} y_i / c_{ij}}{\sum_{i=1}^{n} O_i P_{ij} / c_{ij}} \tag{4.2.19}$$

One interesting feature of this model is that the common location-allocation problem of minimising aggregate transport costs (which is described in equations (4.2.2) to (4.2.4) is a limiting case of the locational surplus maximisation model (see Beaumont (forthcoming) for more details). Thus, by varying $\beta$, the nearest centre hypothesis can be thought as merely one member of a family of more general models. This is illustrated by examining the limit of the locational surplus model, $(LS(\beta))$, as $\beta$ tends to infinity

$$\lim_{\beta\to\infty} LS(\beta) = \lim_{\beta\to\infty} \left(\frac{1}{\beta} \sum_{i=1}^{n} O_i \log \sum_{j=1}^{m} A_j e^{-\beta c_{ij}}\right) \tag{4.2.20}$$

$$= \lim_{\beta\to\infty} \left(-\sum_{i=1}^{n} O_i c_{ij*} + \frac{1}{\beta} \sum_{i=1}^{n} O_i \log A_{j*} + \frac{1}{\beta} \sum_{i=1}^{n} O_i \log\left(1 + \sum_{j\neq j*} \frac{A_j}{A_{j*}} e^{-\beta(c_{ij} - c_{ij*})}\right)\right) \tag{4.2.21}$$

where

$$c_{ij*} = \min_{\{\underline{s}\}}(c_{ij}) \tag{4.2.22}$$

Therefore,

$$LS(\infty) = -\sum_{i=1}^{n} \sum_{j=1}^{m} O_i \lambda_{ij} c_{ij} \tag{4.2.23}$$

because the limit of the second and third terms in equation (4.2.21) are zero. The optimal solutions generated by the locational surplus model will, obviously, digress from the more usual aggregate distance location-allocation problem, and it follows that evaluation and comparison of the results will be useful. (see Beaumont (forthcoming) for such a comparative examination).

### 4.3 An Alternative Formulation : Log Accessibility.

This sub-section will briefly describe an alternative mathematical programming model which has been employed by Leonardi (1978) in his analysis of multi-facility optimum location and size. The spatial interaction benefits, described by equation (4.2.17), were defined as locational surplus, although, given the conventional Hansen-type (1959) definition of accessibility ($\sum_j W_j e^{-\beta c_{ij}}$), it is possible to use a different interpretation, accessibility benefits (see, for example, Leonardi (1978), and Williams and Senior (1978)), which is clearly directly relevant to contemporary issues, such as rural transport and accessibility (see Moseley's (1979) recent book).

Leonardi (1978) applied a so-called log-accessibility objective function subject to a constraint on total facility size. Following the notation of this paper, this is formally stated as

$$\underset{\{W_j\}}{\text{Max}} \quad Z = \sum_{i=1}^{n} O_i \log \sum_{j=1}^{m} W_j e^{-\beta c_{ij}} \tag{4.3.1}$$

subject to

$$\sum_j W_j = W \tag{4.3.2}$$

This constrained problem can be translated into an unconstrained Lagrangian expression, L

$$L = \sum_{i=1}^{n} O_i \log \sum_{j=1}^{m} W_j e^{-\beta c_{ij}} + \lambda(W - \sum_j W_j) \tag{4.3.3}$$

where $\lambda$ is the Lagrangian multiplier associated with constraint (4.3.2). The necessary optimality conditions can be ascertained by differentiating the Lagrangian with respect to $W_j$ and $\lambda$, by equating them to zero (that is, $\partial L/\partial W_j = 0$ and $\partial L/\partial \lambda = 0$). Now

$$\frac{\partial L}{\partial W_j} = \sum_{i=1}^{n} \frac{O_i e^{-\beta c_{ij}}}{\sum_k W_k e^{-\beta c_{ik}}} - \lambda \tag{4.3.4}$$

which equals

$$\frac{1}{W_j} \sum_{i=1}^{n} \frac{O_i W_j e^{-\beta c_{ij}}}{\sum_k W_k \; e^{-\beta c_{ik}}} - \lambda \tag{4.3.5}$$

From equations (3.3.1) and (3.3.2), and by disregarding the $\alpha$ parameter, it is possible to rewrite the first term as

$$\frac{\sum_i S_{ij}}{W_j} \tag{4.3.6}$$

where it is noted that $O_i$, the level of demand from zone i, is equivalent to $e_i P_i$ in the shopping model. It is therefore possible to write the optimality condition as

$$\frac{\sum_i S_{ij}}{W_j} - \lambda = 0 \qquad \text{(for } W_j > 0\text{)} \tag{4.3.7}$$

or as

$$\sum_i S_{ij} = \lambda W_j \tag{4.3.8}$$

which is the balancing condition, equating the flow of revenue accruing to zone j to the supply of shops, applied by Harris and Wilson (1978). It is interesting to note that this result occurs even though Leonardi did not explicitly utilise the shopping model. The constraint, equation (4.3.2), is picked up by

$$\frac{\partial L}{\partial \lambda} = W - \sum_j W_j = 0 \tag{4.3.9}$$

## 4.4 The Location, Size and Number of Facilities.

In the previous sub-sections, the models were concerned with the optimal location and size of an exogenously given number of facilities. This sub-section briefly examines the topic of covering-type problems as a way of endogenously determining the number of facilities required (for more details see Beaumont, 1979)

The optimal spatial configuration of facilities is, obviously, dependent on their total number in the system. A significant planning implication (especially as it has been observed that the utility ofmmany public services is dependent upon the distance travelled to receive the service (see, Smolensky, Burton, and Tideman(1970)) is that, in some circumstances, a large number of small facilities may be the answer. With respect to the advantages derived from internal economics of scale, ReVelle and Church (1977) have examined, under a budget constraint, their inverse relationship with the number of facilities. Whilst appreciating this trade-off, it is important to remember that, when considering the cost reductions arising from internal scale economics, an additional welfare cost should also be included associated with the lower levels of service utilisation when there is a smaller number of larger facilities.

When examining central place theory in a location-allocation framework, Rushton (1973) applied the threshold of a good or service (that is, the total effective demand required to support a particular function) to determine the number of facilities. The simple idea to calculate the number of facilities involved dividing the total demand by the threshold level. The method is, however, invalid if demand is not uniformly distributed, because, when consumers travel to the nearest facility, some facilities will, in fact, not necessarily meet the original threshold level applied in the calculation. Such a situation is also likely when the travel to the nearest facility assumption is relaxed, although the difficulty could be overcome by unrealistically assuming that the threshold was both the minimum and maximum size of a facility.

In a consideration of the number of facilities, one potentially interesting avenue to pursue is the development of the so-called covering problems. White and Case (1974), for instance, discussed the total cover problem, which minimises the total number of facilities required to completely meet the demands of a set of consumers. Formally, this can be represented as a zero-one programming problem,

$$\text{Min } Z = \sum_{j=1}^{m} x_j \qquad (4.4.1)$$

subject to

$$\sum_{j=1}^{m} a_{ij} x_j \geq 1 \qquad i = 1, \ldots, n; \; j = 1, \ldots, m \qquad (4.4.2)$$

$$x_j = \begin{cases} 1 \\ 0 \end{cases} \qquad j = 1, \ldots, m \qquad (4.4.3)$$

where $x_j$ is set equal to one if a facility is assigned to a location j, and to zero, otherwise. Similarly, the covering coefficients, $a_{ij}$, have a value of one if consumers at point i are covered by a facility at point j.

It is interesting to note that the determination of the minimum number of facilities required for the total coverage of demand has direct salience to Christaller's (1933, p. 72) well-known 'marketing principle', 'the primary and chief law of the distribution of central places', which organises a central place system so that '<u>all</u> parts of a region are supplied from the minimum possible number of functioning central places' (emphasis added).

A useful extension would be to incorporate the concept of a range of a good or service (that is, the maximum ('economic') distance a consumer would be willing to travel to purchase a good or service), which actually determines the spatial configuration of a central place system, by developing models using the so-called location-set covering framework. This approach is an outgrowth of Toregas, et al 's (1971) application of a maximum distance (or time) constraint as a factor behind the spatial pattern of emergency service facilities. At least one supply facility must be located within s distance (or time) units of each demand location (where s equals the maximum distance (or time) constraint). For each demand point i, let $N_i$ be the set of possible facility locations which satisfy the maximum distance constraint

$$d_{ij} \leqslant s \qquad (4.4.4)$$

where $d_{ij}$ is the distance between a demand point at i and a facility at j. Thus, the set $N_i$, for each n demand points, can be formally defined as

$$N_i = \left\{ j \mid d_{ij} \leq s \right\} , \qquad i = 1, \ldots, n \qquad (4.4.5)$$

The mathematical programming model, employing the previous definitions, can be stated as

$$\text{Min } Z = \sum_{j=1}^{m} x_j \qquad (4.4.6)$$

subject to

$$\sum_{j \epsilon N_i} x_j \geqslant 1 \quad , \qquad i = 1, \ldots, n \qquad (4.4.7)$$

and

$$x_j = \begin{cases} 1 \\ 0 \end{cases} \qquad j = 1, \ldots, m \qquad (4.4.8)$$

These ideas on linking concepts of central place theory within the framework of location-allocation problems, is, as McNulty and Rushton (1978) have outlined, a topic in which potentially beneficial research can be undertaken. Alternative formulations could be compared and evaluated in order to gain new theoretical insights and to generate operational models. For example, an extension of the location-set covering problem, the maximal covering location problem, which

"... seeks the maximum population which can be served within a stated service distance or time given a limited number of facilities". (Church and ReVelle (1974, p. 103)),

may be useful with respect to the locational organisation of a number of different services, particularly those working with limited budgets.

### 4.5 Location-Allocation Problems : Further Suggestions.

The example given in this section offers some indication of the wide-ranging analytical and practical value of such formulation. They clearly portray many geographical facets, such as optimum location, and spatial interaction, which are to be found in applications of mathematical programming in geography.

There has been some demonstration on how common geographical concepts, such as those related to central place theory, can be included in location-allocation models. Further extensions and reformulations should enable other features of location theory to be incorporated. For example, although emphasis in this paper has been on the general problem of multi-facility location, other problems, such as incremental facility location and facility reorganisation can be analysed (Hodgart, 1978). In addition, the location system's morphology need not be restricted to analysis on a plane; much research, in fact, has been concerned with optimal location patterns for given networks, and therefore it mirrors existing transport links (MacKinnon and Barber, 1977). More specifically, it is possible to replace the discrete demand locations by a continuous space representation (such as, Rushton's (1973) use of a spatial trend surface function in which the density of demand is a function of the locational coordinates), and the usual modelling of inelastic demand can be changed to elastic demand formulations in which the level of demand is inversely related to the distance to the distance to the facility (see, for example, Beaumont (1979), and Abernathy and Hershey (1972)). It is also possible to incorporate the temporal dimension, and analysis of dynamic location-allocation systems has been undertaken by Scott (1971; 1975) and Sheppard (1974). There is, therefore, a rich variety of locational analyses which can be tackled within the structure of location-allocation problems.

# 5. A New Way At Looking At An Old Problem.

## 5.1 Introduction.

In this section, a mathematical programming formulation of central place theory (at the intra-urban level) is presented (see Beaumont and Clarke (1979) for more details). Emphasis is centred on the solution characteristics (the Kuhn-Tucker optimality conditions) and interpretation demonstrates a consistency between familiar theoretical concepts, such as threshold of a good or service, and their underlying economic rationale. This minimum size constraint is determined endogenously within a competitive framework, and is therefore, theoretically more satisfying than the use of an exogenous value as in the Lowry-type urban models.

Firstly, a few introductory comments consider the use of mathematical programming to examine central place theory, and then a detailed explanation of the model is given. Some remarks are made about the desirability of a comprehensive structure in which central place theory can be placed.

## 5.2 Modelling Using A Mathematical Programming Representation.

Since the seminal studies of Christaller (1933) and Losch (1944), central place theory has always held the special attention of geographers; indeed, Bunge (1962) described central place theory as "geography's finest intellectual product". Yet doubts continue to be expressed as to the value of the theory as an explanatory tool (see, for example, White's (1978) review of Beavon's (1977) recent book).

In some ways a basic problem is finding an adequate language to express the theoretical notions of central place theory, which in its original formulation is an optimisation problem. Early studies often applied statistical techniques to group centres and to analyse point patterns.

Recently, there has been a growing literature on the geometry of central place systems. In addition, mathematical programming formulations can be developed which are consistent with the theoretical foundations of central place systems (Beaumont and Clarke, 1979; Puryear, 1975; Wilson, 1978). Analyses use a spatial interaction and activity representation of a central place system, and a spatial zoning system (as opposed to a continuous spatial approach); centres' locations are then described by descrete spatial units which make up the area under study. The spacing and size of facilities can, therefore, be readily investigated.

### 5.3 A Mathematical Programming Model Illustrating Some Central Place Concepts.

The mathematical programming model discussed below describes competition within a single sector, that of services, and it is proposed that the size and distribution of facilities can be ascertained for a profit-maximising type behaviour. At the outset, two points should be remembered. Firstly, for simplicity, only one type of service is considered although the model can be easily disaggregated to represent hierarchies. Secondly, alternative formulations can and should be attempted.

Formally, attention here is focused on maximising total net profit (T.N.P.), subject to land constraints.

$$\max_{\{W_j\}} \text{T.N.P.} = \sum_j (\sum_i S_{ij} - W_j B_j) \tag{5.3.1}$$

subject to the land constraints,

$$W_j \leq \mu L_j \tag{5.3.2}$$

and to the non-negativity constraints

$$W_j \geqslant 0 \qquad (5.3.3)$$

Here $W_j$ is the size of service facilities in zone j (which is usually taken as a measure of attractiveness); $L_j$ is the maximum amount of land available for services in zone j;

, and μ is a suitable defined coefficient. $B_j$ is the cost of a unit of space in zone j (and could be linked to the competitive determination of the price of space through bid rents (Beaumont and Clarke, 1979)). $S_{ij}$ is the flow of expenditure from residents in zone i to the facilities in zone j, and it can be determined by using the Huff (1964) and Lakshmanan-Hansen (1965) shopping model, which is presented again for convenience,

$$S_{ij} = A_i e_i P_i W_j e^{-\beta c_{ij}} \qquad (5.3.4)$$

where

$$A_i = (\sum_j W_j e^{-\beta c_{ij}})^{-1} \qquad (5.3.5)$$

which ensures that

$$\sum_j S_{ij} = e_i P_i \qquad (5.3.6)$$

given that $e_i$ is the per capita expenditure on goods and services by residents of zone i; $P_i$ is the population of zone i; $c_{ij}$ is the travel cost, suitably defined, from zone i to zone j; and β is a travel impedance parameter.

An intuitive interpretation would be that sevice provision only occurs when profit is accruing to producers. (Producers will obviously not offer a service even to make an optimal pattern if the profit is negative). This model demonstrates a usual property of a profit maximising situation - marginal cost equals marginal revenue - and also gives insight into the threshold at which services are provided. (That is, whether at optimality, there is sufficient total effective demand to support a facility in a particular zone). These features are illustrated by examination of the Kuhn-Tucker (1951) conditions necessary for a solution to the model described by equations (5.3.1) to (5.3.3).

This constrained problem is reformulated as a new, unconstrained problem, the associated Lagrangian, L; explicit dependencies between variables in the objective function and constraints are extracted by the introduction of additional variables, the so-called Lagrangian multipliers (which have interpretations similar to duals). The Lagrangian can be formally stated as

$$L = \text{T.N.P.} + \sum_j \lambda_j (\mu L_j - W_j) \qquad (5.3.7)$$

where $\underline{\lambda}$ are the Lagrangian multipliers associated with constraint (5.3.2); such association facilitates interpretation later.

The Kuhn-Tucker conditions for maximisation are

$$\frac{\partial L}{\partial W_j} \leqslant 0 \qquad (5.3.8)$$

$$W_j \frac{\partial L}{\partial W_j} = 0 \qquad (5.3.9)$$

and $$\frac{\partial L}{\partial \lambda_j} \geqslant 0 \qquad (5.3.10)$$

$$\lambda_j \cdot \frac{\partial L}{\partial \lambda_j} = 0 \qquad (5.3.11)$$

which gives rise to the constraint (5.3.2). Also we must have

$$W_j \geqslant 0 \tag{5.3.12}$$

Employing economic rationale, provision of services increases as long as marginal revenue is greater than marginal cost. Facility size is in competitive, profit-maximising equilibruim when marginal cost equals marginal revenue. Analysis here is focused on two situations: when the service is not provided from zone j (because the threshold conditions have not been met), that is $W_j = 0$, and when the service is provided from zone j (because the threshold conditions have been met), that is $W_j \neq 0$.

Firstly, case 1, when $W_j = 0$. We know from equations (5.3.8) and (5.3.9 ), that

$$\frac{\partial L}{\partial W_j} \leqslant 0 \tag{5.3.13}$$

Here, we address

$$\frac{\partial L}{\partial W_j} < 0 \tag{5.3.14}$$

as this is the non-trivial situation; the equality part is examined in case 2 where it is of significance for interpretation. Differentiating the Legrangian equation (5.3.7) with respect to $W_j$, gives

$$\frac{\partial L}{\partial W_j} = \sum_i \left[1 - \frac{S_{ij}}{e_i P_i}\right] \frac{S_{ij}}{W_j} - B_j - \lambda_j \tag{5.3.15}$$

which, from equation (5.3.14), must be less than zero. That is,

$$\left[\sum_i \left( 1 - \frac{S_{ij}}{e_i P_i} \right)\right] \frac{S_{ij}}{W_j} < B_j + \lambda_j \tag{5.3.16}$$

It is the interpretation of this equation which is of particular interest with respect to case 1, when there is no service provision from zone j ($W_j = 0$). The first term on the left-hand side represents the marginal revenue for zone j

On the right-hand side of the equation, $B_j$ is the cost of a unit of space in zone j and $\lambda_j$ is the Lagrangian multiplier associated with constraint (5.3.2) which ensures that the land occupied by service facilities in zone j does not exceed the amount available. Thus, equation (5.3.16) demonstrates that a potential facility in zone j cannot supply any service because the cost of a unit will not be covered by its revenue. In this situation, the subsidy value, which will be at least sufficient to enable the aggregate demand to be satisfied, is not enough to permit a facility to locate in zone j.

An important corollary of this analysis is that if for instance, planners deemed a particular service provision level was desirable, the necessary financial assistance is known. In addition to its applicability from a welfare standpoint, it also has an explicit spatial representation.

Furthermore, to satisfy equations (5.3.10) and (5.3.11), we must have

$$\frac{\partial L}{\partial \lambda_j} = \mu L_j - W_j \geqslant 0 \qquad (5.3.17)$$

which means that total service floorspace must not exceed that available in each zone j.

Secondly, case 2, when $W_j \neq 0$. We know, from equations (5.3.8) and (5.3.9), that

$$\frac{\partial L}{\partial W_j} = 0 \tag{5.3.18}$$

which from equation (5.3.15) can be written as,

$$\sum_i \left[1 - \frac{S_{ij}}{e_i P_i}\right] \frac{S_{ij}}{W_j} - B_j - \lambda_j = 0 \tag{5.3.19}$$

After rearrangement it becomes

$$\sum_i \left[1 - \frac{S_{ij}}{e_i P_i}\right] \frac{S_{ij}}{W_j} = B_j + \lambda_j \tag{5.3.20}$$

Thus, conditions for service provision from zone j, ($W_j \neq 0$), are that the marginal revenue equals the marginal costs, that is, when a supplier is at the profit maximising state. The spatially varying terms, such as $B_j$ and $\lambda_j$ are especially interesting as they determine at the profit-maximising position whether excess or normal profits accrue. This raises new theoretical issues with respect to 'marginal hierarchical' goods and service and 'non-marginal hierarchical' goods and services in a development of a disaggregate model, and also to the analysis of the emergence or disappearance of facilities in particular locations using normal profits as a benchmark. A final point is that spatial differences in $\lambda_j$, which may be reflected in price differentials over space, results in consumers not necessarily having to travel to their nearest supply facility.

## 5.4 Possible Extensions to the Model.

Reference to possibilities of extending the model by disaggregation and of incorporating this partial analysis into a more of comprehensive framework of the whole urban system has already been made; these facets are examined in detail in an earlier paper (Beaumont and Clarke, 1979) which the interested reader should consult.

One additional extension would be an analysis of the evolution of facility location and size. In this direction, it may be useful to complement current research on dynamical formulations, such as the employment of difference and differential equations by White (1977) and by Wilson (1978), by utilising the optimising argument described above (Of course, if it is thought that the dynamic process is fully comprehended and that differential (or difference) equations can be written, the optimality thesis is redundant). For example, using the model outlined for services in sub-section (5.3), the Lagrangian partial derivatives are given in equation (5.3.15). These can be thought of as mirroring efficient paths towards the optimal pattern of facility size and location. The evolution of the system's state variables, shopping size ($W_j$), could be portrayed by

$$\dot{W}_j = \frac{\partial L}{\partial W_j} \tag{5.4.1}$$

which relates to the marginal rate of profit at a particular facility size (and a more sophisticated analysis could write similar equations for the Lagrangian multipliers). It is interesting to note that the Lagrangian multipliers, which have a physical interpretation, play an important role in the system's evolution. The phenomena of the evolutionary structural instability of this system could be examined in a catastrophe theory framework; the Lagrangian multipliers could be employed as control parameters (although as Wilson (1979) indicates many of the other variables will change through time and any of these could cause qualitatively different behaviour if the system is near criticality).

Whilst it is appreciated that this is only a preliminary sketch of a research area which deserves and requires much more extensive examination, sufficient facets have been outlined to demonstrate the potential of pursuing future analyses along this avenue.

## 6. Discussion: Some Concluding Remarks.

This wide-ranging description of mathematical programming has illustrated the availability of new conceptual models to geographers. It is suggested that some geographers should take up the challenge that mathematical programming formulations present to geography, particularly at a time when there is 'little new theoretical work to replenish the subject's conceptual stock' (Cooke and Robson, 1976, p. 89). Geography has not obtained, if it is ever possible, an exhaustive selection of available methods of analysis. Additional tools are always welcome. Moreover, although mathematical programming is apposite for practical and theoretical exercises, much is, obviously, to be gained by evaluation and comparison with other formulations; a comprehension of the powers and limitations of mathematical programming models is essential.

A number of the potential advantages gained through an application of mathematical programming in geography have been examined. The associated analytical rigour will assist explanation and enhance the opportunities for generalising; for example, the sub-section on spatial interaction models and the location-allocation problem demonstrated that the common multi-facility lacation problem was a special case of a more general model. The ability to operationalise location analyses through mathematical programmes indicates one use as an aid to decision-makers assessing the implication of various courses of action. In addition to an ability to present assistance for specific problems, mathematical programming should facilitate understanding of the problems and be instrumental in encouraging theoretical development. No one should be under the impression that mathematical programmes will formulate new hypotheses, rather, they present a means of probing suggested relationships and hypotheses. Exploration of alternative formulations,by varying the objective function or the constraints, has been mentioned a number of times in this paper, and it is an important part of the model building process. The possibility of obtaining new, theoretical insights from the restructuring of established research topics, such as central place systems, has been recognised. With regard to comparing the optimal solutions of various models, one interesting starting point would be the different objectives and constraints employed by Christaller (1933) and by Losch (1944) in their development of central place systems. Furthermore, mathematical programmes permit the integration of disparate theories with the hope of later broadening the theory and displaying additional research areas (Beaumont and Clarke, 1979). Additional indication on the inadequacies of current theory will also be forthcoming, and, in so doing, they will suggest new research topics.

Emphasis has been placed on the interpretation of mathematical programming models and their solutions at the expense of discussing the solution techniques in general. The problem must be fitted to the method, the method should not be allowed to dictate the geography; the problem must be structured as accurately and as completely as possible. Mathematical expediency must not be permitted to delimit the formulation of the model (but, even if this was tolerated, it would most likely produce a misleading solution or prove to be insoluble). Continued research in, for example, non-linear programming, is required to overcome technical difficulties which arise in attempting to solve some representation.

An important feature of mathematical programming models is its ability to enable both qualitative and quantitative interpretation. Whilst honouring both the dangers of Andreski-type 'scorcery' and the evidence of significant qualitative theorising, such as Darwin's theory of evolution, future developments in geography will be improved by their parallel consideration. With problems, such as an 'energy crisis', forming part of an uncertain future, linkages between methodology and philosophy can be mirrored in mathematical programming's ability to examine available future alternatives. It is important not to view the discussion as mere methodology; it is impossible to divorce the ends from the means. Aspects of structural instability, optimal control, sensitivity of optimal spatial patterns and so on all possess extra significance when viewed in this perspective.

One difficulty in the operationalisation of mathematical programming models, which is particularly important with respect to prescriptive applications, is data - its scarcity or its poor quality. Although model variables are rarely inherently unobservable, one must reinforce Cox's (1965, p.235) early statement with regard to the application of linear programming:

"It would be salutary indeed if recent methodological advances in geography only served to emphasize the significance of classical geographic investigation".

In fact, one important outgrowth of model-building has been the greater appreciation of the need for improved monitoring of the system. Although recent developments in computer technology have resulted in the feasibility of large, accessible information systems, high costs of data collection remain a problem.

Finally, in conclusion, it is obvious that mathematical programming models, like all models, are dependent on their underlying rationale and assumptions; fundamentally, is the notion of optimisation appropriate for geographical systems? Nevertheless, mathematical programming presents a conceptual framework to examine many different forms of geographical system; it is, perhaps, too early to make definitive statements concerning their relevance. As more phenomena are described in terms of mathematical programming models, problems will clearly arise, but significant progress may be made in comprehending real world processes. Whatever future awaits mathematical programmes in geography, it is hoped that this paper has introduced the basic issues to geographers, and will allow them to make their own evaluation of its potential.

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