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Discussion of thresholds for the initiation of sediment movement

Pamela S. Naden

School of Geography,
University of Leeds,
Leeds LS2 9JT.

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Contents

Abstract

Thresholds for the initiation of large particle movement based on instantaneous velocity fluctuations are derived for three different particle geometries of uniform grain size. These thresholds are shown to be consistent with the Shields' criterion. The work is extended to cover a further selection of geometries associated with a mixture of two grain sizes, one twice the diameter of the other.

I. Introduction

The purpose of this paper is to bring together some of the diverse strands of argument in the fluid mechanics literature relating to the initiation of particle motion; to describe the appropriate constants to use in a range of well-defined situations and to explain where these constants come from. Four simplifying assumptions have been made. First, the analysis as it stands applies solely to spherical particles but there is no reason why the same framework for analysis cannot be used for other particle shapes. Secondly, the analysis is applicable only to large particle sizes. Thirdly, and notwithstanding the work of Francis (1973) and Halley (1969), the effect of spin on the particles has been ignored. They are considered as essentially sliding along the bed (c.f. Kirkby and Statham's (1975) work on scree slopes). Fourthly, the analysis is performed for a horizontal bed but again this is an assumption which can easily be relaxed.

Further simplification at this stage has not been considered. In particular, the forces which are discussed are the instantaneous forces acting on the particle. These arise from the instantaneous velocity fluctuations and it is the instantaneous velocity which, following Christensen (1965; page 305), is assumed to be normally distributed about the time mean velocity. Here "instantaneous" denotes a time scale of the order of 0.01 seconds in line with McQuivey's (1973a) analysis of energy spectra (page A41).

It only remains to specify the appropriate forces and derive the associated thresholds for simple grain geometries first in uniformly sized material and then for mixed grain sizes. The major aim in this is to ensure that definitions are consistent and constants derived from the literature are relevant.

II Forces acting on a single grain

(i) within the flow

The instantaneous forces acting on a single stationary grain within the flow may be described as a vertical lift force, F_y; a drag force acting in the direction of flow, F_x; and the submerged weight of the particle, G. In fully turbulent flow, assuming a horizontal bed, the instantaneous downstream or horizontal drag force on the particle, F_x is given by Yalin (1977; pages 10-12) as:

$$F_x = C\rho D^2 u^2 \qquad (1)$$

where C is a constant drag coefficient

ρ is the density of water (ρ = 1000 kg/m^3)

D is the diameter of the particle

u is the instantaneous velocity in the downstream (x) direction of the water at the grain centre

The corresponding instantaneous lift force, F_y, may similarly be defined (Yalin 1977; page 144) as:

$$F_y = C\rho D^2 v^2 \qquad (2)$$

where v is the instantaneous velocity in the vertical (y) direction of the water at the grain centre

The resultant instantaneous force on the particle, F, as shown in figure 1a, is, therefore, given as:

$$F = \frac{F_x}{\cos\theta} = \frac{F_y}{\sin\theta} \qquad (3)$$

where θ is the angle between the resultant force and the horizontal

which gives:

$$F_y = F_x \tan\theta = C\tan\theta\, \rho D^2 u^2 \qquad (4)$$

where $C\tan\theta$ might be referred to as a lift coeffient.

At this point, it is necessary to consider briefly the definition of fully turbulent flow around a grain. Yalin (1977; page 13) defines this by:

$$\frac{uD}{\nu} > 10^{3} \qquad (5)$$

where u is the instantaneous water velocity at the grain centre
D is the diameter of the grain
ν is the kinematic viscosity of the water
$\nu = 1.145 \times 10^{-6}\ m^{2}/s$ Francis 1976; page 7)

Thus, for a grain diameter of $D = 0.05m$, $u > 0.023$ m/s and for larger grain diameters, this critical velocity decreases. Therefore, given the observed magnitudes of velocity fluctuations (see figure 2), fully turbulent flow does seem to be an adequate assumption for large grains within a few grain diameters of the bed. Away from the bed and for small grain sizes, however, the vertical force may have to be reassessed. For instance, when considering sand grains going into suspension, Krishnappan (1976) uses the Stokes' equation in the vertical.

Finally, in order to use equations (1) and (2), the constant C must be specified. Data provided by Yalin (1977; figure 1.8, page 12) for the case of a sphere in an "infinite" flow give a value of C of 0.157. Instanteneous forces acting on a stationary spherical particle in fully turbulent flow may, therefore, be summarized as:

$$F_x = 0.157 \rho D^{2} u^{2} \qquad (6)$$

$$F_y = 0.157 \rho D^{2} v^{2} \qquad (7)$$

$$G = (\rho_s - \rho)\, g\, \frac{\pi}{6} D^{3} \qquad (8)$$

where ρ_s is density of particle ($\rho_s = 2650\ kg/m^{3}$ for quartz)
g is acceleration due to gravity ($g = 9.81$ m/s)

(ii) <u>in contact with the bed - threshold I</u>

Forces acting on a stationary particle in contact with the bed are depicted in figure 1b. They are defined as above by equations (6), (7), (8) and a friction force described by:

$$FF = (G - F_y) \tan \phi \qquad (9)$$

where $\tan \phi$ is the coefficient of friction appropriate to the grain geometry. For a sphere on a bed of similar sized uniform spheres, $\phi = 30^{o}$ (see section VI)

and F_y is positive for an upward force; negative for a downward force.

The velocities of flow involved are those measured at the centre of the sphere. Accounting for packing, this represents a distance above the theoretical bed level, defined by Einstein and El-Samni (1949; page 521) as 0.2D below the planar surface, of:

$$y = 0.37D + 0.2D = 0.57D \tag{10}$$

Grain motion will be initiated when there is a net downstream force on the grain. Thus, given the grain geometry, depicted in figure 1b, of a single spherical grain resting on a bed of similar sized uniform grains, the threshold of motion can be defined from equations (6) and (9) as:

$$0.157\rho D^2 u^2 = (G - F_y) \tan \phi \tag{11}$$

where u is measured at y = 0.57D

The formulation of this threshold is identical with that of Christensen and Bush (1971; page 406). However, it corresponds to a very different situation from that quoted in their figure 2 (page 421) and employs different values for the depth and the drag coefficient (page 407).

For saltation to commence, a further criterion must be added to that above. There must be a net upward force on the grain and this threshold can be defined by equations (7) and (8) as:

$$0.157\rho D^2 v^2 = (\rho s - \rho)\, g\, \frac{\Pi}{6} D^3 \tag{12}$$

where v is measured at y = 0.57D.

(iii) <u>within the bed - thresholds II and III</u>

Figures 1c and 1d represent two other grain geometries of uniform spheres for which thresholds must be defined. These essentially refer to grains as part of the bed either unsupported (fig. 1c) or supported (fig. 1d) by a downstream sphere. It is clear that in these two cases the procedure outlined above cannot be used. Forces about the grain centre would then be in terms of the velocity at a negative depth!

The experiments of Einstein and El-Samni (1949) provide the answer to this problem. They measured the lift force on one such hemispherical particle (using hemispheres of diameter 0.069 m) and attempted to describe the pressure difference on the particle as:

$$\Delta p = C_L \frac{\rho}{2} u^2 \tag{13}$$

where Δp is pressure difference
C_L is a lift coefficient
u is an appropriate velocity

They discovered that the lift coefficient was a constant at 0.178 "<u>if and only if</u> the flow velocity was measured at a distance of 0.35D from the theoretical wall" (page 522). They further add that "for every other reference distance, the lift coefficient varies with the flow".

Consequently, the definition of the lift force (equation (7)) for these two cases (taking Christensen and Bush's use of the areal shape factor a such that aD^2 "represents the area of the bed associated with one grain" ([illegible], page 405) and, therefore, for spheres $a = \frac{\Pi}{4}$) becomes:

$$F_y = 0.178 \, \frac{\rho}{2} \cdot \frac{\Pi}{4} \cdot D^2 \, u^2_{0.35D}$$

$$\rightarrow \quad F_y = 0.07 \, \rho \, D^2 \, u^2_{0.35D} \tag{14}$$

where $u_{0.35D}$ is measured at $y = 0.35D$

An appropriate constant for the drag force must now be found and it seems reasonable to attempt to be consistent with the previous analysis of forces acting on a grain within the flow. Thus, substituting equation (4) into (14) gives:

$$C = \frac{0.07}{\tan \theta} \tag{15}$$

where
$$\tan \theta = \frac{F_y}{F_x} \tag{16}$$

Expressing the instantaneous downstream and vertical velocities as:

$$u = \bar{u} + u^1_t \tag{17}$$

and
$$v = \bar{v} + v^1_t \tag{18}$$

where $\bar{u}$ and $\bar{v}$ are time mean velocities at the reference depth
u^1_t and v^1_t are the velocity fluctuations at the reference depth

and substituting into equation (16) gives:

$$\frac{F_y}{F_x} = \frac{(\bar{v} + v^1 t)^2}{(\bar{u} + u^1 t)^2} \tag{19}$$

At the bed, $\bar{u} = 0$ and $\bar{v} = 0$ therefore:

$$\tan\theta = [illegible] \tag{20}$$

This substitution is appropriate despite the fact that Einstein and El-Samni's equation relates the lift force to the velocity at a distance of 0.35D <u>above</u> the bed. Their equation is only a computational device and the force referred to is that operative <u>on</u> the bed. In fact, the rationale for this can easily be seen from the logarithmic flow law:

$$\bar{u} = [illegible]^{\frac{1}{2}} \log_{10}\left(\frac{30.1y}{D}\right) \tag{21}$$

where τ_o is shear stress on the bed

$\left(\frac{\tau_o}{\rho}\right)^{\frac{1}{2}}$ = [illegible] the shear velocity on the bed.

$\bar{u}$ is time mean velocity at depth y

Now $\bar{u} = u_*$ when $\frac{y}{D} = 0.05$. The depth being used by Einstein and El-Samni (1949) is $y = 0.35D$. This leaves a difference of 0.3D which is the definition of the size of roughness elements of the bed. Furthermore, the bed shear stress is composed of the velocity fluctuations <u>at</u> the bed, $\tau_o = \rho u^1 v^1$ and therefore $u_* = \sqrt{(u^1 v^1)}$ (Francis 1976; pages 189-190). Consequently, what is required is the ratio of vertical to horizontal velocity fluctuations.

Turbulence data has been conveniently collected together by McQuivey (1973b) and several sets of data give values for both the downstream and vertical velocity fluctuations under a variety of flume conditions (Table 1b, page B32; Table 2, pages B34 and B35; Table 3c, pages B48 and B49). These fluctuations are expressed in terms of the root mean squared instantaneous velocity fluctuation i.e. the standard deviation of the velocity. McQuivey's data is plotted in Figure 2 and the relationship between vertical and downstream fluctuations is given with 95% confidence by:

$$\sigma_v = 0.823\ \sigma_u - 0.0045 \pm 0.011 \tag{22}$$

$$R = 0.989 \text{ and } N = 185$$

The intercept of this equation is not significantly different from zero and an equation fitted through the origin yields:

$$\sigma_v = 0.77\ \sigma_u \tag{23}$$

Given the existence of correlated turbulence in a boundary layer (Francis 1976; page 190), this is equivalent to:

$$v^1{}_t = 0.77\ u^1{}_t \tag{24}$$

and substituting into equation (2) gives:

$$\tan\theta = (0.77)^2 = 0.59 \tag{25}$$

Finally, substituting this value of tan θ into equation (18) gives a drag coefficient in the downstream direction of $C = 0.12$. Thus, the instantaneous force due to the flow acting in a downstream direction becomes:

$$F_x = 0.12\ \rho\ D^2 u^2{}_{0.35D} \tag{26}$$

This constant is very different from that deduced to have been used by Christensen and Bush (1971) in their derivation of thresholds but which is said to come from Einstein and El-Samni's (1949) data.

The thresholds at which motion is initiated for grains within the bed can now be formulated. For the case depicted in Figure 1c without a downstream supporting grain, equations (26), (14), (8) and (9) may be combined to give a zero balance of forces in the downstream direction and threshold II as:

$$0.12\ \rho D^2 u^2 = ((\rho_s - \rho)\ g\ \frac{\Pi}{6}\ D^3 - 0.07\ \rho D^2 u^2)\tan\phi \tag{27}$$

where u is measured at $y = 0.35D$.

In the case shown in figure 1d where the grain is one of a continuous bed layer, movement can only occur by saltation (i.e. $F_y > G$) and, therefore, threshold III is:

$$0.07\ \rho L^2 u_?^2 = (\rho_s - \rho)\ g\ \frac{\Pi}{6}\ D^3 \tag{28}$$

where u is measured at $y = 0.35D$.

More detailed threshold determinations such as those given by Yalin (1977; page 76) for other grain geometries will not be considered here.

III Implementation

In order to use the equations defined above, it is necessary to be able to predict both the time mean velocity and the velocity fluctuations at a given height above the bed.

(i) Variation of time mean velocity with depth

The time mean downstream component of water velocity is simply taken from the logarithmic flow law using an appropriate depth value. As described above, for a particle in contact with the bed, $y = 0.57D$ and for a particle within the bed, $y = 0.35D$. Given the mean velocity which occurs at 0.4 of the maximum depth, the time mean velocity at a given depth is found from:

$$\bar{u}_y = \left(\frac{\bar{u}_{0.4\, y_{max}}}{\log_{10}\left(\frac{30.1 \times 0.4\, y_{max}}{D}\right)} \right) . \quad \log_{10}\left(\frac{30.1y}{D}\right) \qquad (29)$$

where $\bar{u}_y$ is time mean velocity at depth y above the theoretical bed

$\bar{u}_{0.4\, y_{max}}$ is the overall time mean velocity of the flow which occurs at depth $0.4\, y_{max}$

y_{max} is maximum depth above the theoretical bed

(ii) Variation of velocity fluctuations with depth

McQuivey (1973b) presents six sets of turbulence data from flume experiments using a bed comprised of lead shot of diameter 0.0044 m (Table 1a, page B31). This seems to be the most relevant data to the particular situations which have been described above. Figure 3 shows the time mean velocity (u) plotted against the relative depth ($^y/_D$) for the six runs. As can be seen from Figure 3, the data fit a semi-logarithmic form. Table I gives the results of a regression analysis of this data in terms of the shear velocity and a constant (m) as shown in the equation below:

$$\bar{u}_y = 5.75\, u_* \log_{10}\left(\frac{m}{D}y\right) \qquad (30)$$

where u_* is the shear velocity

m is a relative depth constant

In Table I, these values of shear velocity are compared with those obtained using the standard logarithmic flow law in which m = 30.1. Given the very small standard error of the estimate, the values of shear velocity produced by the regression analysis are sufficiently different from the standard logarithmic flow law to warrant their use in later analysis.

Figure 4 shows the variation of the standard deviation of the instantaneous velocity (σ_u) with the relative depth $\left({}^{y}/_{D}\right)$. Clearly, this does not present a case for predicting velocity fluctuations directly from the relative depth and various transformations have been tried. Theoretically, it would seem to be appropriate to relate the velocity fluctuations to the shear stress or shear velocity. Figures 5 and 6 show the results of plotting the velocity fluctuations scaled by the shear velocity at the bed against the relative depth. Figure 5 uses the results from the standard logarithmic flow law while Figure 6 uses the results from Table I derived in the regression analysis. An alternative transformation is to scale the velocity fluctuations by the time mean velocity for the given depth. This is shown in Figure 7 and clearly represents a curvilinear relationship. Figure 8 presents the same data in logarithmic form.

It is clear that none of these transformations has eliminated both the variation in slope and the variation in the intercept between the six data sets. Table II presents the regression analyses for Figures 5, 6 and 8. No immediate improvement to these relationships is apparent as the pattern of variation in the regression constants does not mirror the pattern of any of the variables changed between data sets. However, what is required in this paper is just one of these regression lines to act as a simple rule of thumb in the calculations of the thresholds for grain movement. Given that the aim is to predict velocity fluctuations from the depth of flow, priority must be given to the equation with the least standard error in the estimate. As Table II shows, this occurs for the case of the velocity fluctuations scaled by the time mean velocity at the same depth. The fact that the data points in this case diverge slightly with increasing relative depth as shown in Figure 8 need not be unduly worrying given that the emphasis will rest on fluctuations fairly near to the bed.

Velocity fluctuations, therefore, can justifiably be calculated with 95% confidence using the equation:

$$\log_{10}\left(\frac{\sigma_u}{\bar{u}_y}\right) = -0.65\log_{10}\left(\frac{y}{D}\right) - 0.81 \pm 0.16 \qquad (31)$$

$R = -0.96$ and $N = 56$

$$\rightarrow \frac{\sigma_u}{\bar{u}_y} = 0.16\left(\frac{y}{D}\right)^{-0.65} \qquad (32)$$

where the constant 0.16 may vary between 0.11 and 0.23.

IV A worked example

In order to check the feasibility of the thresholds defined above, an example is given for a grain diameter of 0.05 m. A flow with average velocity 2 m/s and maximum depth 1 m is assumed.

For the case of a grain in contact with the bed (threshold I; figure 1b), the height of the centre of the grain above the theoretical bed is given by $y = 0.57D$ (equation (10)).

From the logarithmic flow law expressed in equation (29) the time mean velocity at a depth $y = 0.57D$ is given by:

$$\bar{u}_y = \frac{2}{\log_{10}\left(\frac{30.1 \times 0.4}{0.05}\right)} \log_{10}(30.1 \times 0.57)$$

$$= 1.04 \text{ m/s}$$

From equation (32), the standard deviation of the velocity fluctuations at this depth is given by:

$$\sigma_u = 0.16\,(0.57)^{-0.65} \cdot 1.04$$

$$= 0.24 \text{ m/s}$$

The threshold of motion is defined by equations (11), (7) and (8) for positive velocity fluctuations only as:

$$0.157\,\rho D^2 u^2 = \left((\rho s - \rho)\, g\, \frac{\pi}{6} D^3 - 0.157\,\rho D^2 v^2\right)\tan\phi$$

where $u = \bar{u} + u^1_t$

$v = v^1_t = 0.77\,u^1_t$

$\left(\frac{\rho s - \rho}{\rho}\right) = 1.65$

$\phi = 30^o$

$g = 9.81$ m/s

$$\rightarrow (\bar{u} + u^1{}_t)^2 + (0.77)^2 \tan\phi \, u^1{}_t{}^2 = \left(\frac{\rho s - \rho}{\rho}\right) g \; \frac{\Pi}{6} \, D \, \frac{\tan\phi}{0.157} \qquad (33)$$

$$\rightarrow u^1{}_t{}^2 \left(1 + (0.77)^2 \tan\phi\right) + 2\,\bar{u}\,u^1{}_t + \left(\bar{u}^2 - \left(\frac{\rho s - \rho}{\rho}\right) g \; \frac{\Pi}{6} \, D \, \frac{\tan\phi}{0.157}\right) = 0$$

This is a quadratic equation in $u^1{}_t$ and can be solved for the positive fluctuation by:

$$u^1{}_t = \frac{-\bar{u} + \sqrt{\bar{u}^2 - (1 + (0.77)^2 \tan\phi)\left(\bar{u}^2 - \left(\frac{\rho s - \rho}{\rho}\right) g \, \frac{\Pi}{6} \, D \, \frac{\tan\phi}{0.157}\right)}}{(1 + (0.77)^2 \tan\phi)}$$

$$= \frac{-1.04 + 1.31}{1.34}$$

$$= 0.2 \text{ m/s}$$

The question now to be asked is how likely velocity fluctuations of this order of magnitude and greater are in this example. The required positive velocity fluctuation lies 0.83 standard deviations away from the mean and velocity fluctuations greater than or equal to this can be expected with a probability of 0.2 (from tables of the standard normal distribution).

For the case of grains lying within the bed (thresholds II and III; figures 1c and 1d), the reference height above the theoretical bed must be taken as $y = 0.35D$.

The time mean velocity at this depth is given using equation (29) as:

$$\bar{u}_y = \frac{2}{\log_{10}\left(\frac{30.1 \times 0.4}{0.05}\right)} \cdot \log_{10}(30.1 \times 0.35)$$

$$= 0.86 \text{ m/s}$$

Velocity fluctuations are given by equation (32) as:

$$\sigma_u = 0.16\,(0.35)^{-0.65} \cdot 0.86$$

$$= 0.27 \text{ m/s}$$

Threshold II for a grain within the bed layer but unsupported by a downstream grain is defined by equation (27), again for positive fluctuations only:

$$0.12\ \rho D^2u^2 = \left((\rho s-\rho)\ g\ \frac{\Pi}{6}\ D^3 - 0.07\ \rho D^2u^2\right)\tan\phi$$

where $u = \bar{u} + u^1{}_t$

$$\phi = 30^\circ$$

$$\rightarrow (\bar{u} + u^1{}_t)^2\ (0.12 + 0.07\ \tan\phi) = \left(\frac{\rho s-\rho}{\rho}\right) g\ \frac{\Pi}{6}\ D\ \tan\phi$$

$$\rightarrow u^1{}_t = +\sqrt{\left(\frac{\left(\frac{\rho s-\rho}{\rho}\right) g\ \frac{\Pi}{6}\ D\ \tan\phi}{(0.12 + 0.07\ \tan\phi)}\right)} - \bar{u}$$

$$= +1.23 - 0.86$$

$$= 0.37\ \text{m/s}$$

Thus, the required positive velocity fluctuation at the threshold lies 1.37 standard deviations from the mean and the probability of movement is therefore 0.09.

Threshold III for a grain within the bed layer which is supported by a downstream grain is given by equation (28), again for a reference distance of $y = 0.35D$ and for positive velocity fluctuations:

$$0.07\ \rho D^2u^2 = (\rho s-\rho)\ g\ \frac{\Pi}{6}\ D^3$$

$$\rightarrow (\bar{u} + u^1{}_t)^2 = \left(\frac{\rho s-\rho}{\rho}\right) g\ \frac{\Pi}{6}\ D\ \frac{1}{0.07} \qquad (34)$$

$$\rightarrow u^1{}_t = +\sqrt{\left(\left(\frac{\rho s-\rho}{\rho}\right) g\ \frac{\Pi}{6}\ D\ \frac{1}{0.07}\right)} - \bar{u}$$

$$\doteq +2.46 - 0.86$$

$$= 1.6\ \text{m/s}$$

In this case the positive fluctuation is 5.9 standard deviations away from the mean and, therefore, lies outside the distribution of instantaneous velocities. Consequently, under the assumed flow conditions and a bed made up of 0.05 m diameter spheres, there will be no motion from a planar bed.

The orders of magnitude of these probabilities for the movement of grains of diameter 0.05 m in a mean flow of 2 m/s and maximum depth 1 m. seem eminently reasonable. For the general case, however, these thresholds must be compared with the Shields' criterion which expresses the value of the <u>mean</u> dimensionless shear stress for which a grain is in <u>incipient</u> motion.

V Comparison with Shields' function

Following Christensen and Bush (1971), incipient motion may be defined by a probability of movement of 0.01. This corresponds to a positive velocity fluctuation 2.33 standard deviations from the mean (normal tables tables - one tailed test). For a grain diameter of 0.05 m this yields the following dimensionless shear stresses.

Threshold I for a grain resting on top of the bed uses a reference height of $y = 0.57D$ and to be in incipient motion requires a velocity fluctuation from equation (32) of:

$$u^1_t = 2.33\,\sigma_u = 2.33 \times 0.16 \times (0.57)^{-0.65}\,\bar{u}$$
$$= 0.54\,\bar{u}$$

Substituting into equation (33) gives the time mean velocity (at a height of $y = 0.57D$) required for incipient motion as:

$$(1.54\,\bar{u})^2 + (0.77)^2 \tan\phi\,(0.54)^2\bar{u}^2 = \left(\frac{\rho s-\rho}{\rho}\right) g\,\frac{\Pi}{6}\,D\,\frac{\tan\phi}{0.157}$$

$$+\bar{u} = +\sqrt{\frac{\frac{(\rho s-\rho)}{\rho}\,g\,\frac{\Pi}{6}\,D\,\frac{\tan\phi}{0.157}}{(1.54)^2 + (0.77)^2\tan\phi\;(0.54)^2}}$$

$$= 0.79 \text{ m/s.}$$

Shields' criterion is given by:

$$\theta_o = \frac{\tau_o}{(\rho s-\rho)\,g\,D} \tag{35}$$

where τ_o is mean bed shear stress.

Combining this with the logarithmic flow law (equation (21)) gives:

$$\theta_o = \left(\frac{\bar{u}}{5.75\log_{10}\left(\frac{30.1\,y}{D}\right)}\right)^2 \cdot \left(\frac{1}{\frac{(\rho s-\rho)}{\rho}\,g\,D}\right) \tag{36}$$

Thus, substituting the value of the time mean flow velocity at a depth of $y = 0.57D$ for which the grain is in incipient motion yields a value of θ_o of 0.015.

Threshold III for a grain which is part of the bed layer relies upon a reference height of $y = 0.35D$ and thus requires a velocity fluctuation for incipient motion from equation (32) of:

$$u^1_t = 2.33\,\sigma_u = 2.33 \times 0.16 \times (0.35)^{-0.65}\,\bar{u}$$
$$= 0.74\,\bar{u}$$

Substituting into equation (34), the time mean velocity at height $y = 0.35D$ necessary for incipient motion is given as:

$$\bar{u} = \frac{1}{1.74} \cdot \sqrt{\left(\frac{\rho s-\rho}{\rho}\right) g \frac{\Pi}{6} D \frac{1}{0.07}}$$
$$= 1.42 \text{ m/s}$$

Substituting this value of the time mean velocity into equation (36) gives the criterion for incipient motion as $\theta_o = 0.07$.

These figures for the critical value of the dimensionless shear stress for incipient motion in the cases where the grain is resting on the bed and where the grain is part of the bed layer can be compared to the flume data of Fenton and Abbott (1977). There is no data available for comparison with the geometrical situation in Figure 1c where the grain is part of the bed but unsupported by a downstream grain. For different degrees of relative protrusion of a single grain above and below the general level of a flume bed made up of hexagonally packed grains of diameter 0.005 to 0.01 m, Fenton and Abbott (1977) measured the dimensionless shear stress at incipient motion and plotted this against the relative protrusion of the grain. In fully turbulent flow, they recommend the use of $\theta_o = 0.1$ for planar surfaces and $\theta_o = 0.01$ for over-riding grains. These values are close to those quoted in the above calculations. Indeed, bearing in mind the difficulty of observing incipient motion and the poorly defined nature of the constant in equation (32), these calculations fit remarkably well on Abbott and Fenton's figure 5 (1977; page 532) extended by Table 3 (page 533) and reproduced here as figure 9.

VI Extension to mixed grain sizes

The extension of these thresholds to cover the case of mixed grain sizes begs two questions. Firstly, what happens to the angle of friction when either small grains override larger grains or large grains override a bed of smaller ones. Figure 10a gives the definition of the angle of friction as the angle between the normal through the point of contact (or the line joining the centre of the overriding grain and the centre of the downstream supporting grain) and the vertical. Thus, the angle of friction (ϕ) is given by:

$$\sin \phi = \frac{R}{R + r} \qquad (37)$$

where r is the radius of the overriding grain
R is the radius of the downstream supporting grain

For uniform grains (i.e. $r = R$), the angle of friction is 30° and is independent of grain size. For different values of the ratio of the radius of overriding grain to that of the bed grain $\left(\frac{r}{R}\right)$, the variation of the friction angle (ϕ) is plotted in Figure 10b. As can be seen from the diagram, for the case where the overriding grain is twice the size of the bed grain $\phi = 19.4^\circ$ and for the case where the overriding grain is half the size of the bed grain $\phi = 42^\circ$. The direction of this variation is consistent with that observed for angular grains in scree experiments by Kirkby and Statham (1975; fig.2, page 352) although the actual values, as expected, are different. The second question which must be broached is that of the set of alternative grain geometries produced in mixed sediments. The question may be split into two parts:

(1) the effect of the downstream geometry, and
(2) the effect of the upstream geometry.

The problem of the downstream geometry (i.e. an <u>adjacent</u> grain of varying size) can be resolved using the angle of friction argument. Figure 11a illustrates the case in point. Again, a friction angle can be defined as the angle between the normal at the point of contact and the vertical. Assuming the grains have the same base level, the friction angle can be defined by:

$$\cos \phi = \frac{r - R}{r + R} \qquad (38)$$

where r is the radius of the grain to be moved
R is the radius of the downstream adjacent grain

For uniform grains (i.e. $r = R$) this angle is 90° and only a lift force greater than its weight will move the grain; as is consistent with equation (28). Figure 11b shows the variation in the friction angle (ϕ) as the grain size ratio between the grain to be moved and the downstream adjacent grain $\left(\frac{r}{R}\right)$ varies. For a grain twice the size of the downstream adjacent grain, the friction angle becomes 70°. Motion will, therefore, be more likely than in the case of uniform grains. For a grain half the size of the downstream adjacent grain, the friction angle is 109°. This essentially means that the smaller grain must be moved upstream in order to round the circumference of the larger grain. As upstream motion is linked with a <u>negative</u> lift force, motion for this grain is impossible. These results are consistent with Yalin's discussion of the surface of a mobile bed (1977; page 78) and Church's definition of normal, underloose and overloose boundaries (1978; page 758).

The problem of the upstream grain geometry revolves around the questions of what reference depth to use in the flow equation and which is the appropriate threshold.

As illustrated in Figure 12a, defining the radius of the sphere to be moved as r and the radius of the upstream sphere as R, the theoretical bed level can be given, following Einstein and El-Samni (1949; page 521), as 0.4R below the top of this upstream grain. The depth (y) which is required is the height of the centre of the sphere to be moved above this theoretical bed level. Assuming both grains are referred to the same base level, this is given by:

$$y = r - 1.6 R \tag{39}$$

This may be quoted in terms of the upstream grain size as:

$$\frac{y}{2R} = \frac{r}{2R} - 0.8 \tag{40}$$

The variation of this depth with the ratio between the radius of the grain to be moved and that of the upstream grain $\left(\frac{r}{R}\right)$ can now be considered.

For a grain size ratio $\left(\frac{r}{R}\right)$ of 1.6, the relative depth becomes zero. However, substitution of this value into the logarithmic flow law (equation (21)) gives a negative velocity. For zero velocity at the bed, the grain size ratio must be 1.67 and this should, therefore, be regarded as the upper limit to the use of thresholds II or III for grains within the bed; these thresholds being based on a depth (y) of 0.7 R. For greater values of the grain size ratio, threshold I for grains in contact with the bed can be used with the appropriate depth (y) as given by equation (40) and shown in Figure 12b.

Finally the criterion for no motion can also be defined in terms of the grain size ratio. This occurs when the top of the grain to be moved drops below the theoretical bed level i.e. at a grain size ratio of 0.8.

VII <u>Further examples</u>

In order to check the feasibility of this approach viz à viz the probability of grain movement, the example calculation given above has been extended to cover a wider selection of grain geometries in mixed sediment. The results are presented in Table III which gives the parameters y, ϕ, D_B (the diameter of particles making up the bed) and D_m (the diameter of the particle to be moved) to be used in the equations and the choice of threshold. Table III also shows the order in which the particles would move as the flow increases. Some of these results are surprising e.g. the ease with which a large particle overrides smaller ones (II) and the reluctance of a small particle to move over a larger one (VI); the ease with which a small particle will move from between two larger ones (I) and yet the difficulty of moving a large particle from between two small ones above which it is protruding (XIV). All these examples raise interesting questions concerning the sedimentary structures observed in gravel bed rivers - the likelihood of such structures being the inverse of the probabilities of movement. Equally, the <u>range</u> of probabilities at which different grains are mobile in mixed sediment, expressed in Table III as 0.16 to 8.1 standard deviations of the given flow, illustrates the importance of varying grain geometry as a control on sediment transport in gravel bed rivers.

Finally, for the case where the particle in incipient motion protrudes 0.5 above a planar surface composed of similarly sized uniform spheres, comparison can again be made with Fenton and Abbott's (1977) results. Following the method set out in section V, the calculated value of θ_o is 0.05. This has been plotted on Figure 9 and shows some measure of agreement with the experimental data. At least, the calculated thresholds occur in the correct order. The wider discrepancy in this last example can be explained by the fact that the experimental conditions are not identical with the specified value of the friction angle as the protruding grain is not directly supported by the grains making up the bed. Thus, it would be easier to move the grain in Abbot and Fenton's (1977) experiment as is shown in Figure 9.

VIII Conclusion

This paper has tried to set out logically and consistently the forces acting on a particle resting on or lying within a particulate bed. Using data on turbulent velocity fluctuations presented by McQuivey (1973b), threshold values for particle movement have been derived and shown to be comparable with the Shields' criterion. The case for mixed grain sizes has been explored using two different grain sizes - one twice the diameter of the other. This has revealed an impressive range of threshold values for different grain geometries - an exercise which has implications for observations of sedimentary structures and bedload transport in gravel bed rivers.

Acknowledgements

I would like to thank Mike McCaig and Mike Kirkby for their helpful comments on the initial draft of this paper.

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Table I : Calculated shear velocities for McQuivey's (1973b) lead shot data

Data set	Number in sample	Best fit equation		Correlation coefficient	Standard error of estimate	Log flow law (m = 30.1)
		u_*	m			u_*
1	10	0.026	30	1.0	0.0049	0.023
2	10	0.046	26	1.0	0.0008	0.039
3	9	0.058	41	1.0	0.0176	0.055
4	9	0.026	28	1.0	0.0033	0.023
5	9	0.046	33	1.0	0.0059	0.043
6	9	0.055	41	1.0	0.0113	0.054

Table II : Regression analysis of velocity fluctuations and relative depth (x) - McQuivey's (1973b) lead shot data

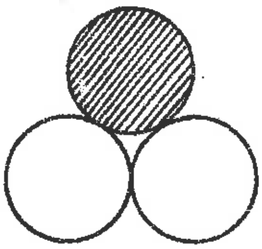
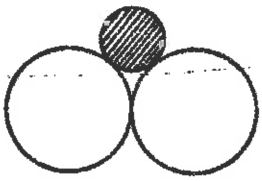
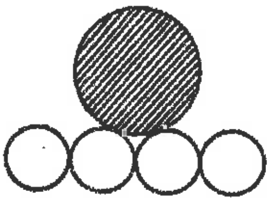
Data set	N	Correlation coefficient	Slope (b)	Intercept (a)	Standard error of estimate	S.E. as % range
A. Velocity fluctuations scaled by shear velocity as given in standard log flow law						
1	10	- 0.99	- 1.16	1.84	0.08	
2	10	- 1.00	- 1.11	1.70	0.06	
3	9	- 0.99	- 1.09	1.49	0.07	
4	9	- 0.98	- 1.09	1.56	0.09	
5	9	- 1.00	- 0.94	1.27	0.04	
6	9	- 0.99	- 1.04	1.21	0.07	
All	56	- 0.91	- 1.11	1.53	0.23	9.6%
B. Velocity fluctuations scaled by shear velocity as given by best-fit line						
1	10	- 0.99	- 1.01	1.60	0.07	
2	10	- 1.00	- 0.95	1.44	0.05	
3	9	- 0.99	- 1.04	1.42	0.06	
4	9	- 0.98	- 0.97	1.39	0.08	
5	9	- 1.00	- 0.87	1.18	0.04	
6	9	- 0.99	- 1.03	1.20	0.07	
All	56	- 0.95	- 1.01	1.38	0.16	8%
C. Velocity fluctuations scaled by time mean velocity at same depth - logged data						
1	10	- 1.00	- 0.61	- 0.71	0.02	
2	10	- 1.00	- 0.63	- 0.75	0.02	
3	9	- 1.00	- 0.62	- 0.82	0.03	
4	9	- 0.99	- 0.65	- 0.78	0.03	
5	9	- 1.00	- 0.66	- 0.87	0.03	
6	9	- 1.00	- 0.72	- 0.91	0.02	
All	56	- 0.96	- 0.65	- 0.81	0.08	5%

Table III : Calculations of the probability of movement of grains in selected geometries; mixed grain sizes

Parameters needed

- y – reference height above theoretical bed level
- D_B – grain size of the bed (as defined by upstream grain)
- D_m – grain size to be moved
- ϕ – friction angle

Thresholds as defined in Section II

Grain geometry (grain to be moved shaded)	Parameters	Calculation	Order of movement
	$D_m = D_B = 0.05$ m $y = 0.57\ D_B$ $\phi = 30^o$ Threshold I (eq. 11)	$\bar{u} = 1.04$ m/s $\sigma_u = 0.24$ m/s $u^1{}_t = 0.20$ m/s i.e. $0.83\ \sigma_u$ $\rightarrow$ prob. = 0.20	III
	$D_m = D_B = 0.1$ m $y = 0.57\ D_B$ $\phi = 30^o$ Threshold I (eq. 11)	$\bar{u} = 1.19$ m/s $\sigma_u = 0.27$ m/s $u^1{}_t = 0.54$ m/s i.e. $2.0\ \sigma_u$ $\rightarrow$ prob. = 0.02	VII
	$D_B = 0.1$ m $D_m = 0.05$ m $y = 0.26\ D_B$ $\phi = 42^o$ Threshold I (eq. 11)	$\bar{u} = 0.86$ m/s $\sigma_u = 0.33$ m/s $u^1{}_t = 0.63$ m/s i.e. $1.9\ \sigma_u$ $\rightarrow$ prob. = 0.03	VI
	$D_B = 0.05$ m $D_m = 0.1$ m $y = 1.11\ D_B$ $\phi = 20^o$ Threshold I (eq. 11)	$\bar{u} = 1.28$ m/s $\sigma_u = 0.19$ m/s $u^1{}_{tt} = 0.12$ m/s i.e. $0.63\ \sigma_u$ $\rightarrow$ prob. = 0.26	II
	$D_B = 0.05$ m $D_m = 0.05$ m $y = 0.35\ D_B$ $\phi = 0^o$ Threshold III (eq.28)	$\bar{u} = 0.86$ m/s $\sigma_u = 0.27$ m/s $u^1{}_t = 1.6$ m/s i.e. $5.9\ \sigma_u$	XIII

able III (continued)

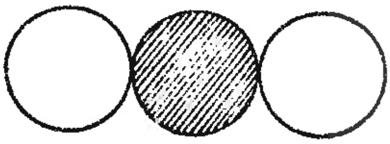
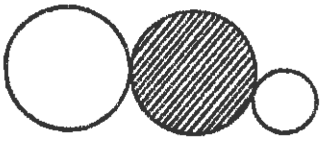
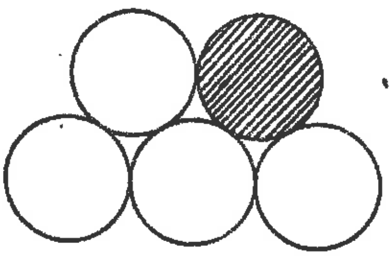
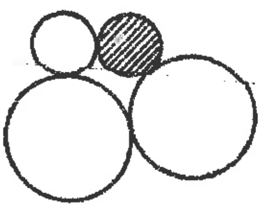
rain geometry	Parameters	Calculation	Order of movement
	D_B = 0.1 m D_m = 0.1 m y = 0.35 D_B ϕ = 0° Threshold III (eq.28)	$\bar{u}$ = 0.98 m/s σ_u = 0.31 m/s $u^1{}_t$ = 2.50 m/s i.e. 8.1 σ_u	XV
	D_B = 0.1 m D_m = 0.1 m y = 0.35 D_B ϕ = 70° Threshold II (eq.27)	$\bar{u}$ = 0.98 m/s σ_u = 0.31 m/s $u^1{}_t$ = 1.75 m/s i.e. 5.6 σ_u	XII
	D_B = 0.05 m D_m = 0.1 m y = 0.2 D_B ϕ = 70° Threshold I (eq. 11)	$\bar{u}$ = 0.65 m/s σ_u = 0.30 m/s $u^1{}_t$ = 2.1 m/s i.e. 7 σ_u	XIV
	D_B = 0.05 m D_m = 0.05 m y = 0.35 D_B ϕ = 30° Threshold II (eq.27)	$\bar{u}$ = 0.86 m/s σ_u = 0.27 m/s $u^1{}_t$ = 0.37 m/s i.e. 1.4 σ_u → prob. = 0.08	IV
	D_B = 0.1 m D_m = 0.1 m y = 0.35 D_B ϕ = 30° Threshold II (eq.27)	$\bar{u}$ = 0.98 m/s σ_u = 0.31 m/s $u^1{}_t$ = 0.77 m/s i.e. 2.5 σ_u → prob. = 0.006	IX
	D_B = 0.05 m D_m = 0.05 m y = 0.35 D_B ϕ = 42° Threshold II (eq.27)	$\bar{u}$ = 0.86 m/s σ_u = 0.27 m/s $u^1{}_t$ = 0.58 m/s i.e. 2.2 σ_u → prob. = 0.01	VIII

Table III (continued)

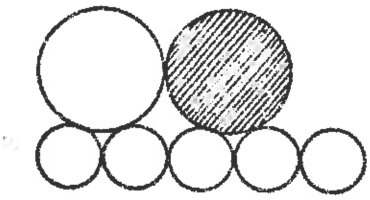
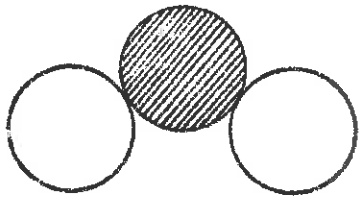
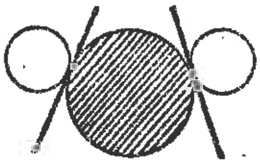
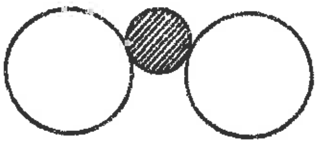
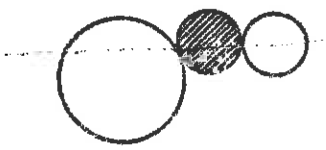
Grain geometry	Parameters	Calculation	Order of movement
	$D_B = 0.1$ m $D_m = 0.1$ m $y = 0.35D_B$ $\phi = 20^\circ$ Threshold II (eq.27)	$\bar{u} = 0.98$ m/s $\sigma_u = 0.31$ m/s $u^1{}_t = 0.48$ m/s i.e. 1.5 σ_u → prob. = 0.07	V
	$D_B = 0.1$ m $D_m = 0.1$ m $y = 0.2\ D_B$ $\phi = 60^\circ$ Threshold I (eq. 11)	$\bar{u} = 0.75$ m/s $\sigma_u = 0.34$ m/s $u^1{}_t = 1.74$ m/s i.e. 5.0 σ_u	XI
	Shaded stone will never move - tangents - at pivot points intersect above the bed		
	$D_B = 0.1$ m $D_m = 0.05$ m $y = 0.35\ D_B$ $\phi = 20^\circ$ Threshold II (eq.27)	$\bar{u} = 0.98$ m/s $\sigma_u = 0.31$ m/s $u^1{}_t = 0.05$ m/s i.e. $0.16\sigma_u$ → prob. = 0.44	I
	$D_B = 0.1$ m $D_m = 0.05$ m $y = 0.35\ D_B$ $\phi = 0^\circ$ Threshold III (eq.28)	$\bar{u} = 0.98$ m/s $\sigma_u = 0.31$ m/s $u^1{}_t = 1.48$ m/s i.e. 4.8 σ_u	X

Figure 1 : Forces acting on a single grain

Figure 1a: Within the flow

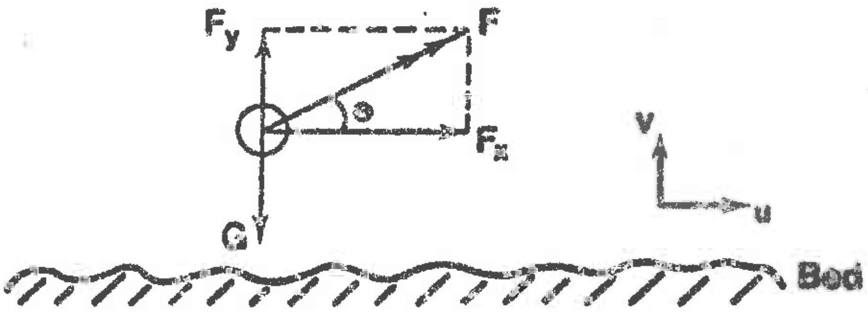


Figure 1b: In contact with the bed

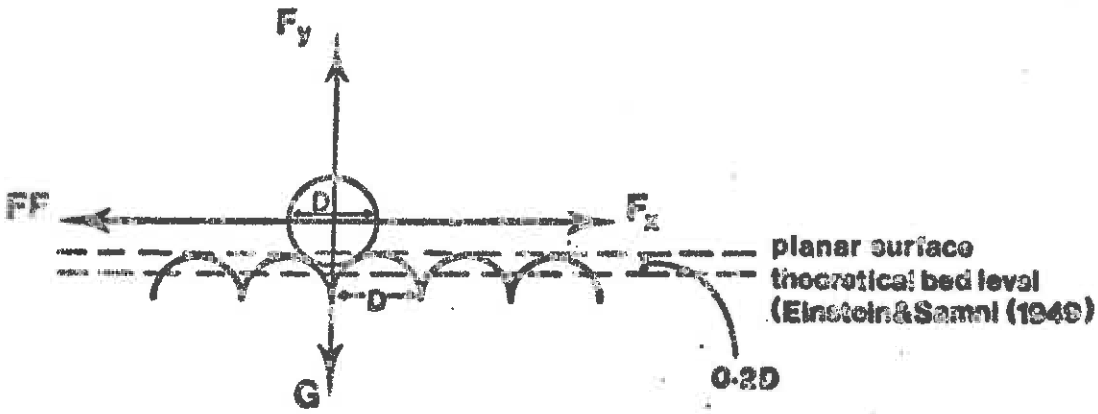


Figure 1c: Grain as part of bed layer but unsupported by downstream grain

Figure 1d: Grain within bed layer

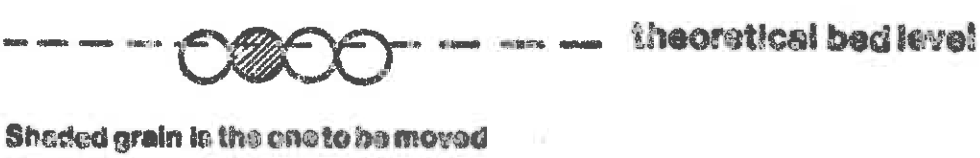


Figure 2 : Vertical velocity fluctuations as a function of downstream velocity fluctuations - McQuivey (1973b) - all data available

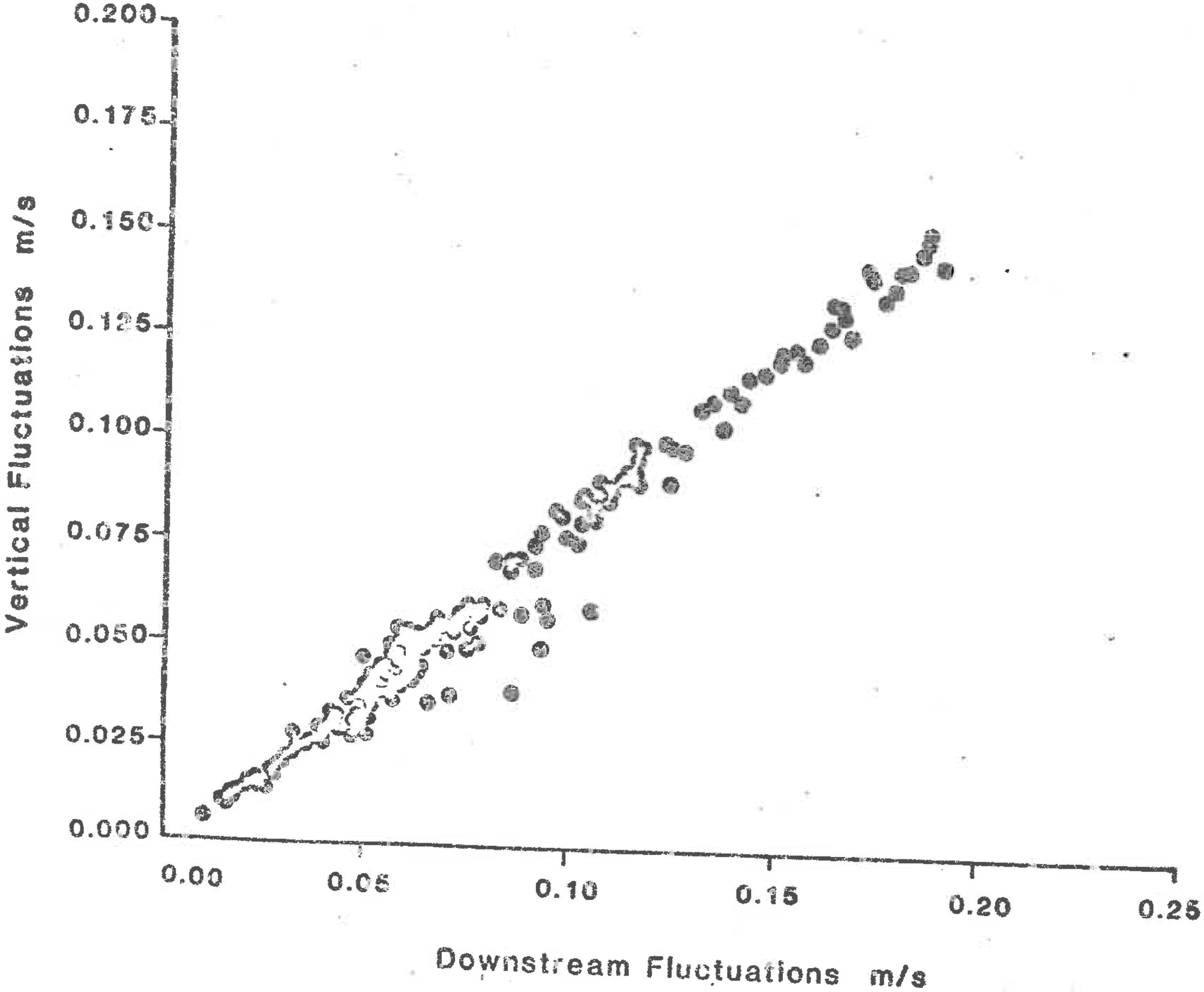


Figure 5 : Time mean velocity as a function of relative depth - McQuivey's (1973b) lead shot data - 6 data sets

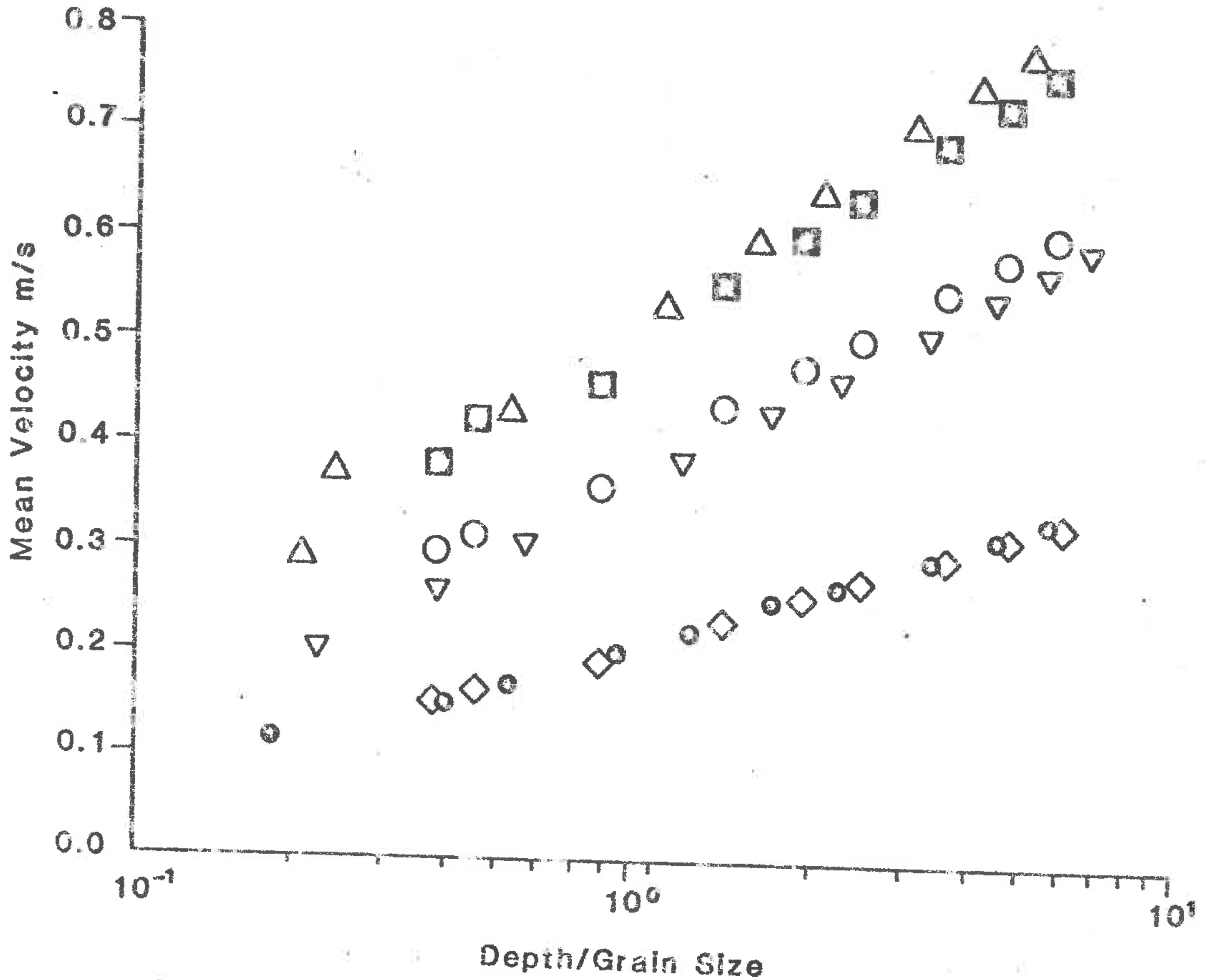


Figure 4 : Velocity fluctuations as a function of relative depth - McQuivey's (1973b) lead shot data - 6 data sets

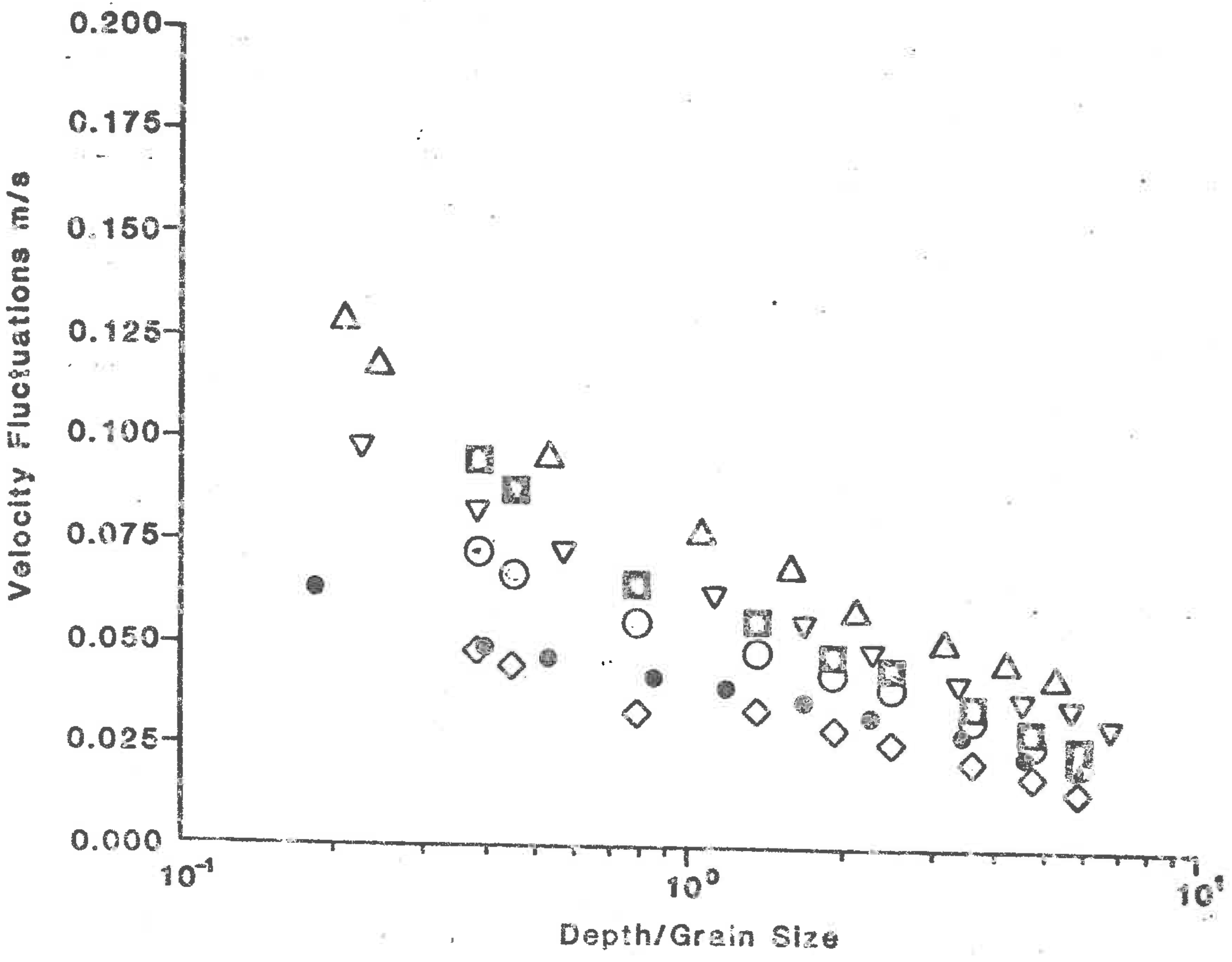


Figure 5 : Velocity fluctuations scaled by shear velocity (calculated from standard logarithmic flow law) as a function of relative depth - McQuivey's (1973b) lead shot data - 6 data sets

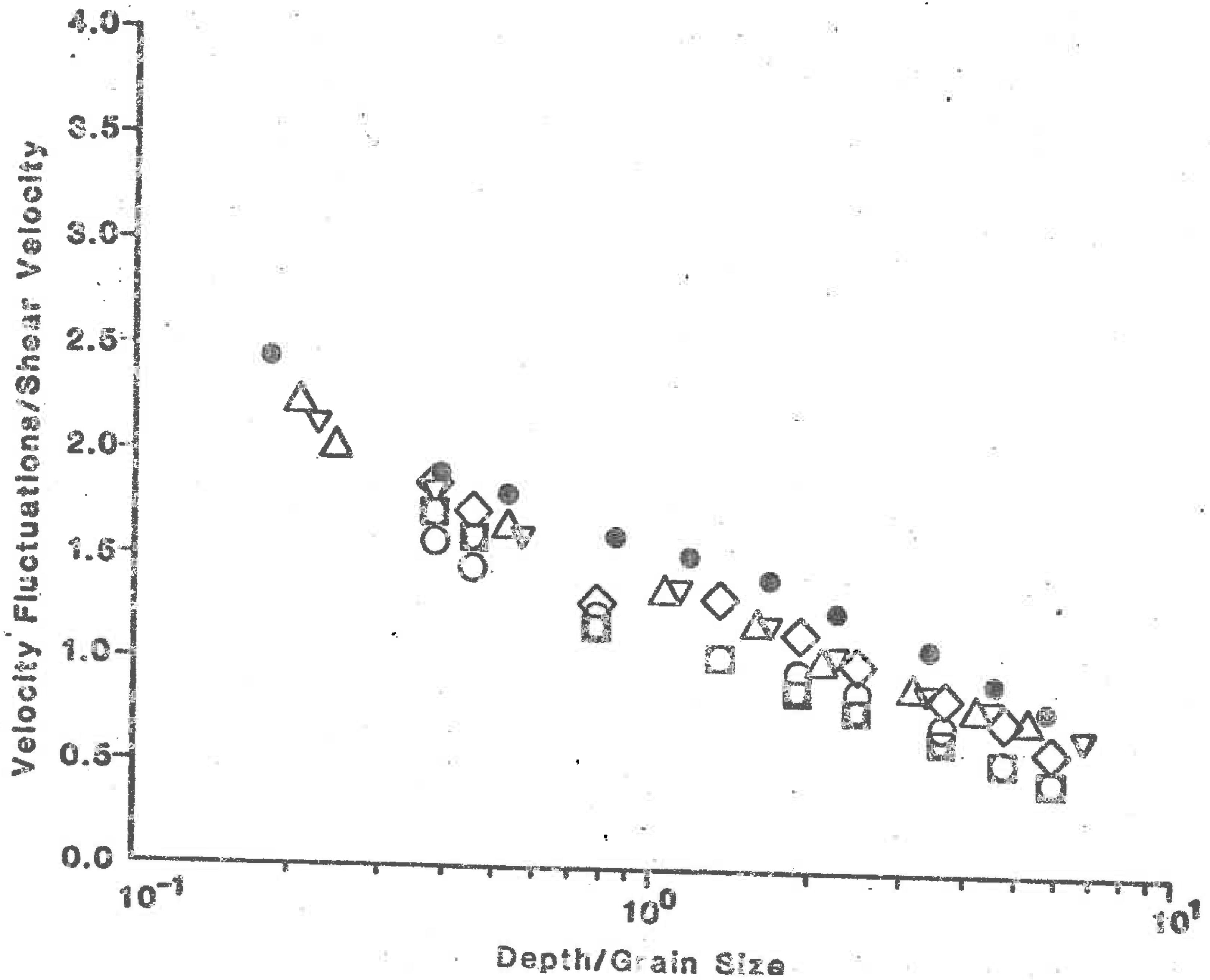
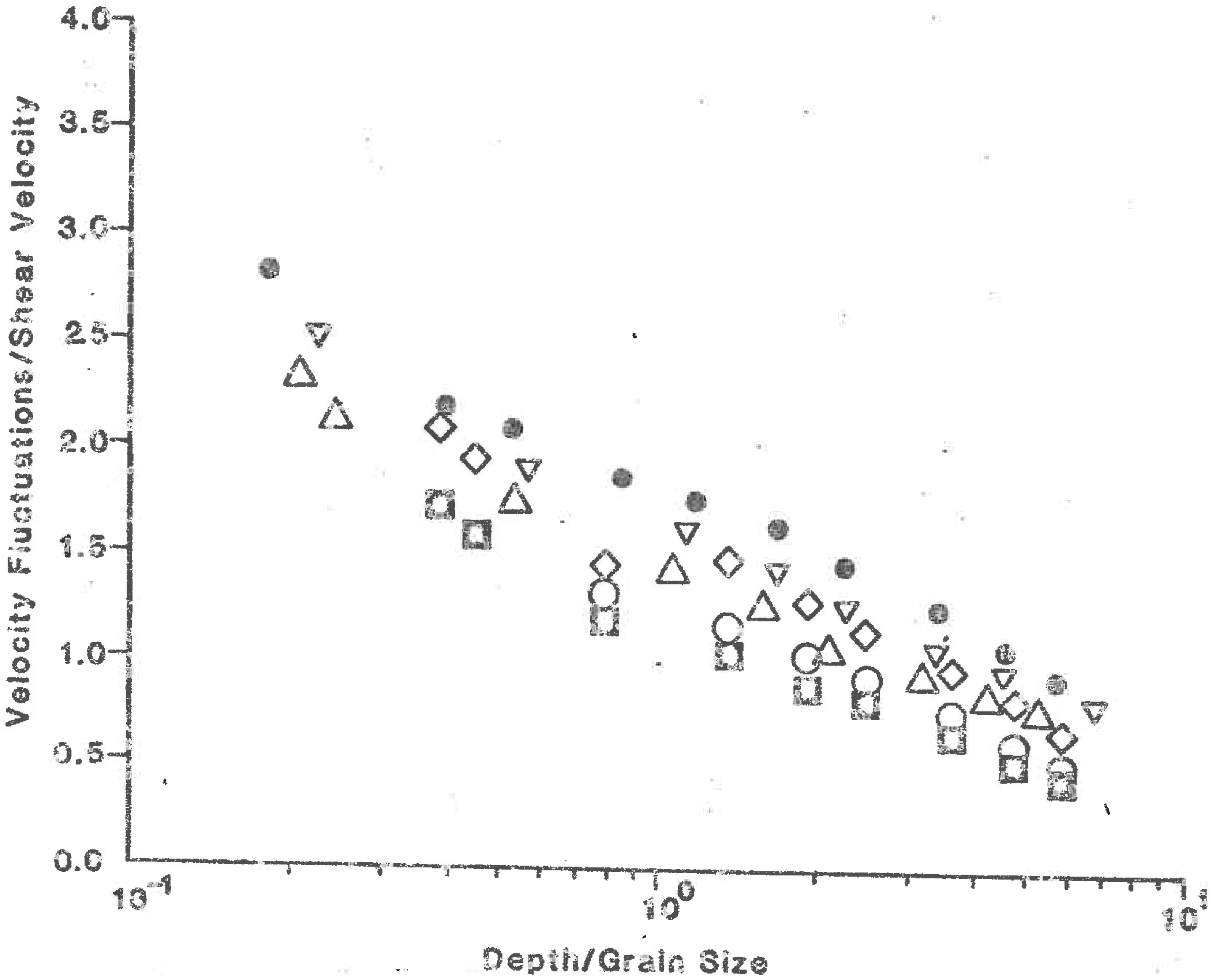


Figure 6 : Velocity fluctuations scaled by shear velocity (as given by best-fit regression lines) as a function of relative depth - McQuivey's (1973b) lead shot data - 6 data sets



<u>Figure 7</u> : <u>Velocity fluctuations scaled by time mean velocity at the given depth as a function of relative depth - McQuivey's (1973b) lead shot data - 6 data sets</u>

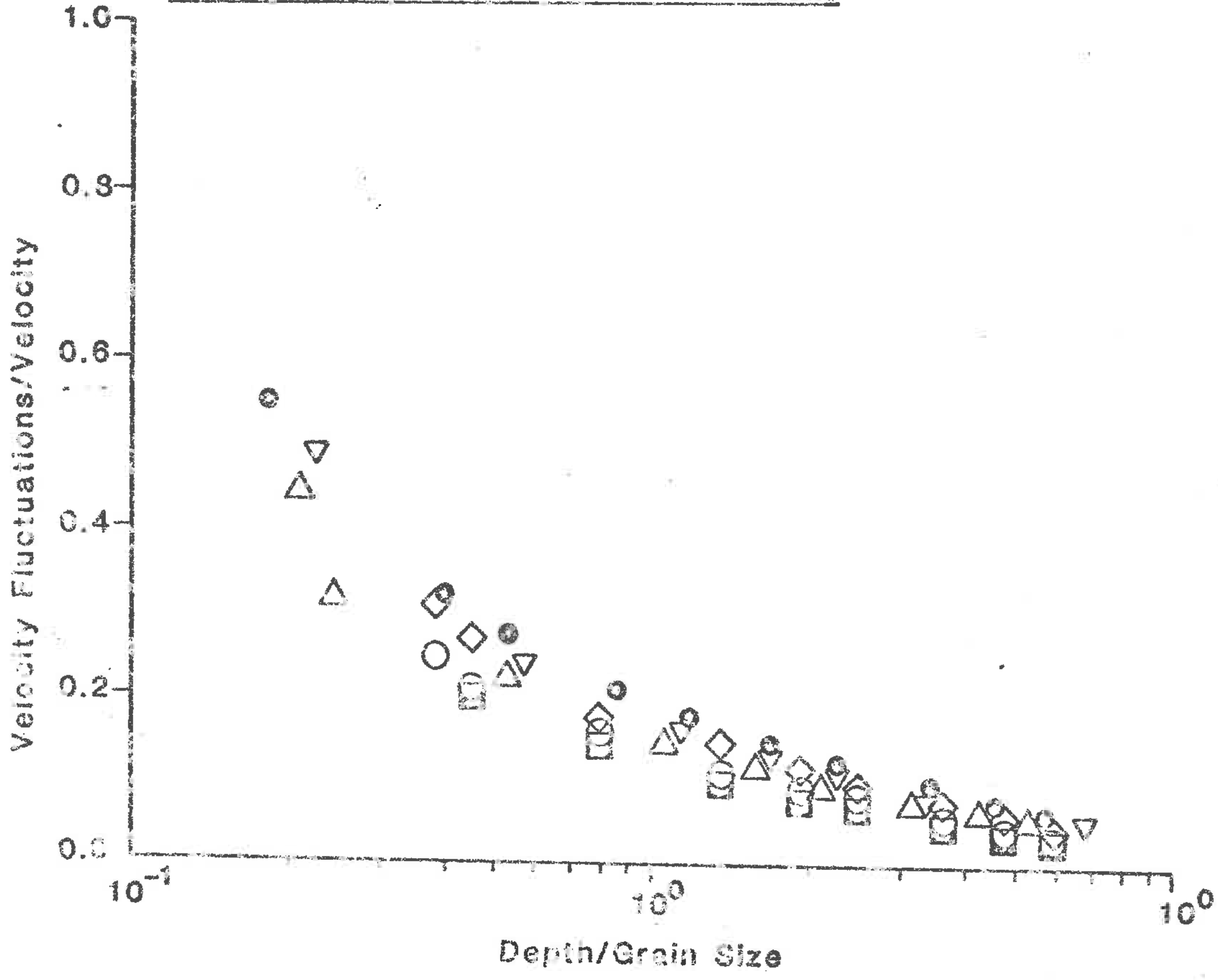


Figure 8 : Figure 7 redrawn as a logarithmic plot

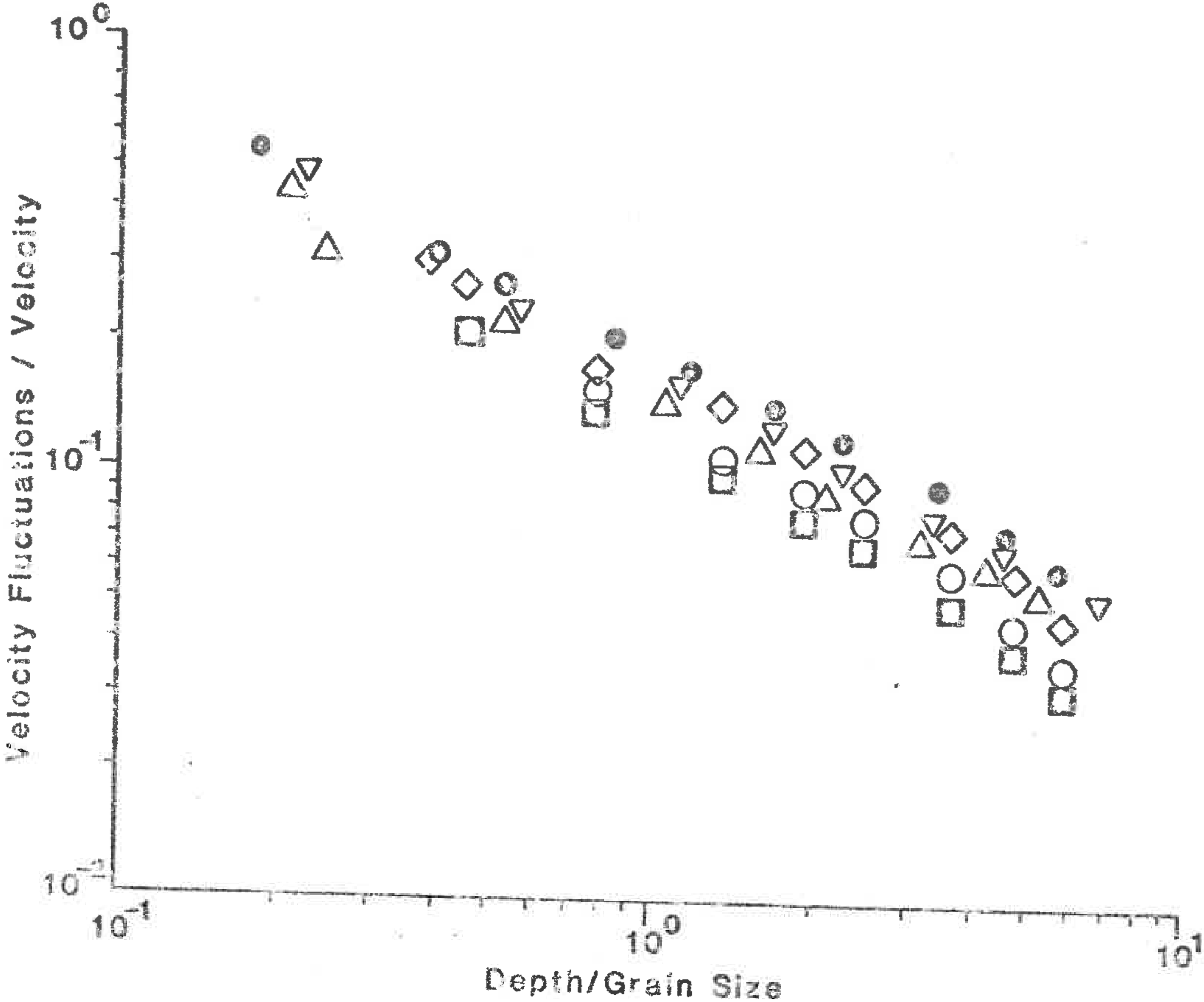
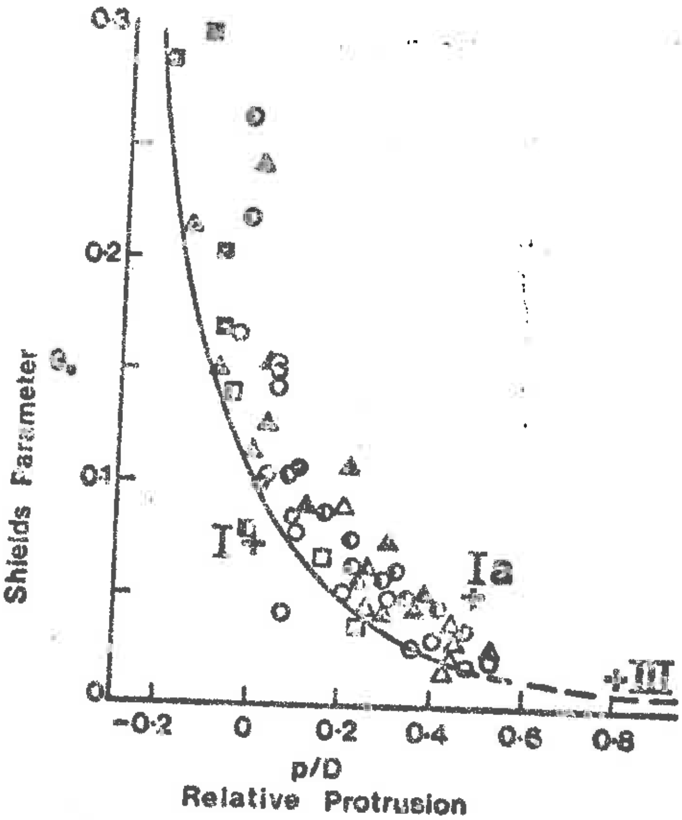


Figure 9 : Shields' parameter plotted against relative protrusion of grain in incipient motion - data from Fenton and Abbott (1977) for fully turbulent flow



Results from experimental series B: □ grain 1 △ grain 2 ○ grain 3 Unshaded, $200 < R_* < 350$, half shaded, $350 < R_* < 550$, shaded, $550 < R_* < 830$

+I Theoretical Threshold

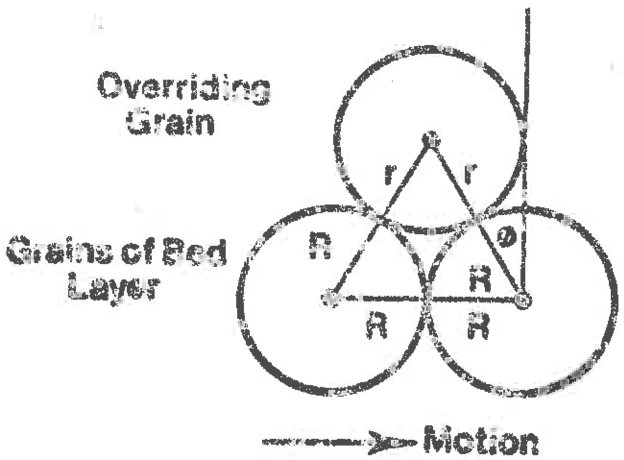
+III Theoretical Threshold

+Ia Threshold I with grain protruding by half its diameter (see table III.)

R_{E*} is grain Reynolds no. ($R_{E*} = \frac{u_* d}{\nu}$

Figure 10 : Friction angles for overriding grains

Figure 10a: Definition of friction angle (ϕ)



r is radius of overriding grain

R is radius of grains making up the bed

$$\sin \phi = \frac{R}{R + r}$$

Figure 10b: Variation of friction angle with grain size ratio $\left(\frac{r}{R}\right)$

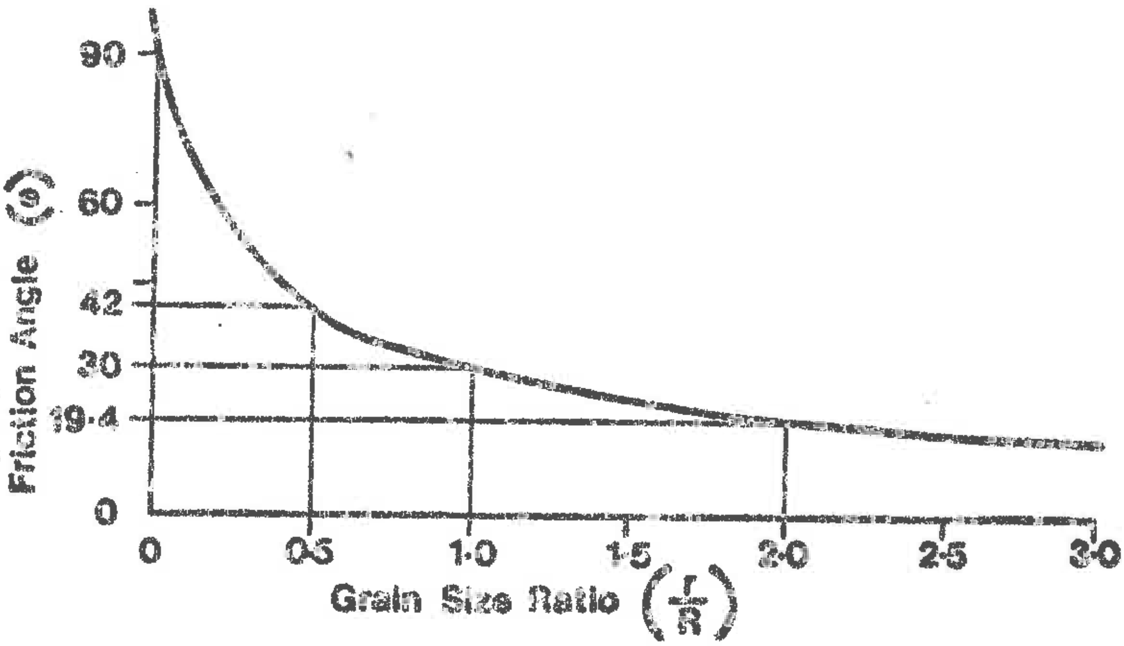
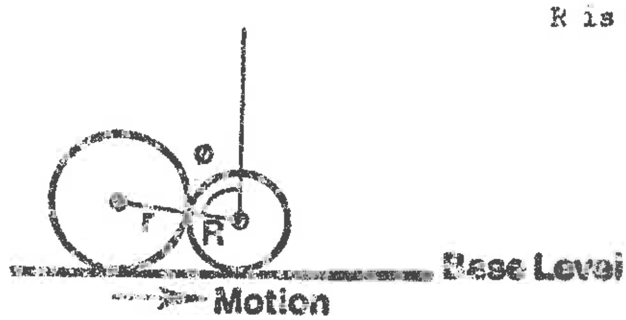


Figure 11 : Friction angles for adjacent grains - the question of downstream grain geometry

Figure 11a: Definition of friction angle (ϕ)

r is radius of grain to be moved

R is radius of downstream adjacent grain



Assuming both grains have same base level

$$\cos \phi = \frac{r - R}{r + R}$$

Figure 11b: Variation of friction angle with grain size ratio $\left(\frac{r}{R}\right)$

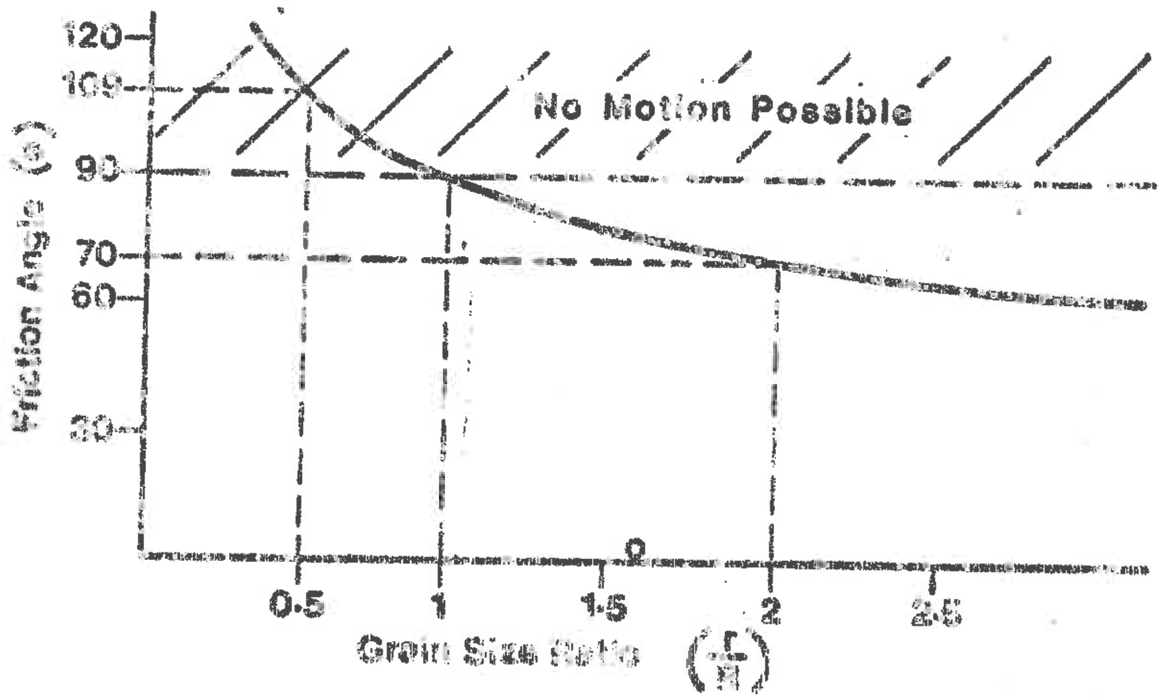
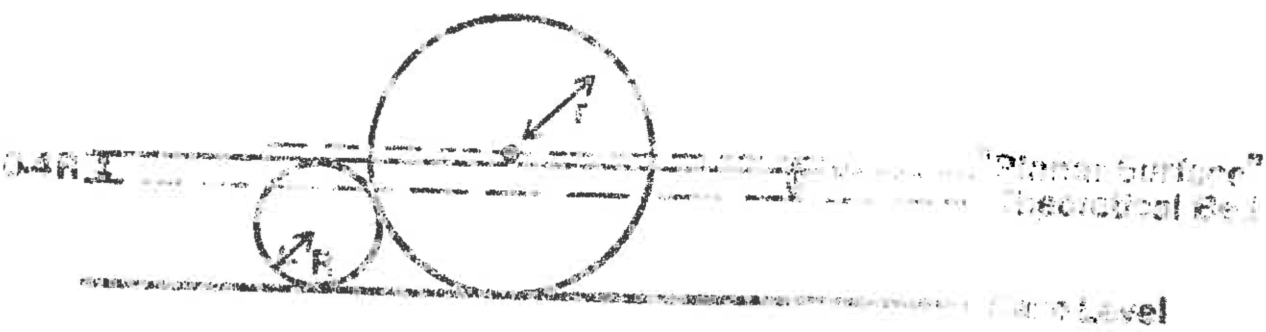


Figure 12: Reference depths and thresholds as a function of upstream grain geometry

Figure 12a: Definition of reference depth with a given upstream grain



r is radius of sphere to be moved
R is radius of upstream sphere

Assuming both grains are referred to the same base level,

$$y = r - 2R + 0.4R$$
$$\therefore y = r - 1.6R$$

Figure 12b: Reference depth and threshold to be used with a given grain size ratio