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<u>A Theoretical Basis for Diffusion In A Central Place System.</u>

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The aim of this note is to take issue with Hudson's (1969) claim that pure hierarchical diffusion cannot cause the empirically observed s-shaped or logistic growth pattern (Brown and Cox, 1971). Moreover, Alves (1974) subsequent rejection of this assertion is shown to be founded upon a false premise. It is, however, demonstrated that a logistic growth pattern is possible under certain conditions in a hierarchical system with diffusion; these conditions are a special case of a general model of hierarchical diffusion which is developed.

Emphasis is placed upon a representation of hierarchical diffusion, the spatial pattern of centres in the diffusion process, which is obviously of fundamental significance, being deliberately omitted in this account. This neglect is seen as essential in the pursuit of a dynamic location theory, particularly as evidence indicates that in the initial stages of a diffusion process the hierarchical impact is relatively more significant and contagious diffusion becomes increasingly important after the elemental configuration is established (Gould, 1969).

Following both Hudson and Alves, a pure hierarchical diffusion system is one in which each centre informs all K centres ($K > 1$) of the next lowest order in each time period. Having assumed that acceptance on contact is certain, that is the acceptance probability equals one, Alves, following Hudson, is incorrect in arguing that the number of cumulative adopters at time t, f(t), is

$$f(t) = K^t \tag{1}$$

f(t), as given by equation (1), is the number of *new* adopters at time t. The number of *cumulative* adopters at time t, S(t), is, in fact, given by

$$S(t) = \sum_{t=0}^{t} f(t). \tag{2}$$

It is, therefore, suggested that Alves' subsequent proof that pure hierarchical diffusion, incorporating a self-limiting mechanism, is consistent with a logistic curve of cumulative adopters is invalid. It is possible to demonstrate this notion, however, and, in so doing, raise a number of issues worthy of more detailed empirical analysis.

Now assume that $S(n)$ is the cumulative number of centres which have adopted a message up to and including the n'th time period. $S(n) - S(n - 1)$ is therefore the number of new adoptions in the n'th time period. If in each time period the message passes down the hierarchy by one order, a centre of order n contacts K centres of order $n + 1$. We define $S(0) = 1$, that is a message starts at time zero in the single centre of order 1, and $g(n)$ is a monotonic function giving the probability of a centre of order n adopting a message after being contacted by a centre of order $n - 1$. Thus, $S(1) = g(1)K + S(0)$ or, more generally, $S(n) = g(n)K + S(n - 1)$.

The cumulative number of centres which have adopted the message after N time periods, $S(N)$, is given by either:

$$S(N) = S(1) + \sum_{n=2}^{N} [S(n - 1) - S(n - 2)] g(n)K, \; N \geq 2 \quad (3)$$

or

$$S(N) = S(N - 1) + [S(N - 1) - S(N - 2)] g(N)K, \; N \geq 2. \quad (4)$$

Here equation (3) is the summation of the number of adopter centres in each order and equation (4) is the addition to $S(N - 1)$ of the number of adopter centres of order N. After rearrangement, equation (4) becomes,

$$S(N) - S(N - 1) = [S(N - 1) - S(N - 2)] g(N)K, \; N \geq 2. \quad (5)$$

This relationship can be rewritten for the right-hand side of equation (5) repeatedly to obtain,

$$S(N) - S(N-1) = [S(N-2) - S(N-3)]\, g(N)g(N-1)K^2$$

$$= [S(N-3) - S(N-4)]\, g(n)g(N-1)g(N-2)K^3$$

$$= [S(1) - S(0)]\, K^{N-1} \prod_{n=2}^{N} g(n), \; N \geqslant 2. \quad (6)$$

Since $S(0) = 1$ and $S(1) = S(0) + Kg(1)$ then $S(1) - S(0) = Kg(1)$ and $S(N) - S(N-1)$, the number of new adoptions in the N'th time period, is given by

$$S(N) - S(N-1) = Kg(1)\, K^{N-1} \prod_{n=2}^{N} g(N)$$

$$= K^N \prod_{n=1}^{N} g(n)$$

$$= \prod_{n=1}^{N} Kg(n), \; N \geqslant 1. \quad (7)$$

To demonstrate that pure hierarchical diffusion can produce a logistic curve of cumulative adopters two conditions must be satisfied: the presence of limiting behaviour and of a point of inflexion. In relation to the asymptotic advance to a limit it must be shown that $S(N)$ converges to a limit as N approaches infinity. From equation (7) it can be readily appreciated that this condition is attained if,

$$Kg(n) < 1, \; n \geqslant N_{crit} \text{ for some integer } N_{crit} \geqslant 1. \quad (8)$$

Then $\prod_{n=1}^{N} Kg(n) \to 0$ as $N \to \infty$. It is noted that as $g(n)$ is the probability of adoption after contact its values range between 0 and 1; thus, this condition is not satisfied if $g(n) = 1 \; \forall \; n$ (as in Hudson's exposition) and a logistic curve is not possible. It would, however, seem possible that $g(n)$ decreases as n increases, at least after a few time generations, perhaps until $Kg(n) < 1$. With respect to the presence of a point of inflexion it is usual to consider differential equations and demand that for $S(n)$ to have a point of inflexion,

$$\frac{d^2S}{dn^2} = 0, \qquad \text{and } \frac{ds}{dn} = 0 \quad (9)$$

Here it is appropriate to use the equivalent difference equations discussed in many texts on analysis, for example, Lang (1968).

$$\nabla S = S(n) - S(n - 1) \quad \text{equivalent to } \frac{dS}{dn} \tag{10}$$

and

$$\nabla^2 S = [S(n + 1) - S(n)] - [S(n) - S(n - 1)] \quad \text{equivalent to } \frac{d^2S}{dn^2}. \tag{11}$$

Considering equation (11), by applying equation (7), evaluating after m time periods, $m \geq 1$, and rearranging it is possible to write

$$\begin{aligned} \nabla S &= K^m \prod_{n=1}^{m} g(n) \\ \nabla^2 S &= [Kg(m + 1) - 1] \left[\prod_{n=1}^{m} Kg(n)\right]. \end{aligned} \tag{12}$$

Given that, by definition $Kg(n) > 0$, it is possible to identify five different cases. (If $Kg(n) = 0$ then, since K is a constant, $g(n) = 0$, there is no adoption at level n and the diffusion process can be considered to have ended).

(i) $\underline{Kg(n) \in]0, 1[\; \forall n}$. Using (12) it can be shown that

$$\nabla S < 1 \quad \text{and} \quad \nabla^2 S < 0. \tag{13}$$

Using the equivalent theory to that of differential equations with respect to the interpretation of first and second derivatives, the function $S(n)$ decreases with n and since $Kg(n) < 1$ it approaches a limit.

(ii) $\underline{Kg(n) \in]1, +\infty[\; \forall n}$. Here (12) indicates

$$\nabla S > 1 \quad \text{and} \quad \nabla^2 S > 0 \tag{14}$$

and unrestricted growth occurs. A special case of this situation is where $g(n) = 1$ as studied by Hudson (1969).

(iii) <u>$Kg(n) = 1 \ \forall n$.</u> Here

$$\nabla S = 1 \quad \text{and } \nabla^2 S = 0 \tag{15}$$

and $S(n)$ is a linear function which does not approach a limit.

The fourth and fifth cases can be recognised as combinations of the first two.

(iv) <u>Let there be an integer m for which, $Kg(n) \varepsilon \,]0, 1[$, $n < m$, and $Kg(n) \varepsilon \,]1, +\infty[$, $n \geq m$.</u>

Now for $n = 1, 2, \ldots, m - 1$ the function is of type (i), ie. decreasing, and for $n = m, m + 1, \ldots$ it is of type (ii), ie. increasing. If $S(n)$ were expressed as a continuous formation of n, a value of n would exist where $\frac{d^2S}{dn^2} = 0$, a point of inflexion. As $S(n)$ is a discrete function the existence of an integer n where $\nabla^2 S = 0$ is not guaranteed. That such a point exists between the integers $n - 1$ and m cannot be denied however. Since this case is similar to (ii) as $n \to \infty$ it is increasing and does not approach a limit.

(v) <u>Let there be an integer m for which $Kg(n) \varepsilon \,]1, +\infty[$ $n < m$ and $Kg(n) \varepsilon \,]0, 1[$ $n \geq m$.</u>

This is similar to (iv) where $S(n)$ increases for $n = 1, 2, \ldots, m - 1$ and decreases for $n = m, m + 1, \ldots$ As $S(n)$ is similar to (i) as $n \to \infty$ it is decreasing and approaching a limit as $n \to \infty$ with a point of inflexion between $S(m - 1)$ and $S(m)$.

Each of the above cases corresponds to certain forms of the function $g(n)$ as K is a constant greater then 1. It can be seen that case (v) gives rise to the logistic curve which has been derived in empirical studies. In order to derive this curve the assumption is that $g(n) \varepsilon \,]\frac{1}{K}, \infty[$, $n < m$ and $g(n) \varepsilon \,]0, \frac{1}{K}]$ $n \geq m$.

To satisfy this condition $g(n)$ must be a monotonic decreasing function of n and $g(n) < \frac{1}{K}$ for some n. The probability of acceptance decreases as the messsge progresses down the hierarchy, a not unrealistic condition.

Within this analytic framework several areas for empirical investigation could be developed; these would typically relate to differing values of K and various forms of the function $g(n)$.

In conclusion we restate the main finding of the analysis, that a pure hierarchical diffusion process can lead to logistic growth of cumulative adopters. This is derived as a special case of a general model of hierarchical diffusion in a central place system. Hence a theoretical foundation for the observed patterns of growth can be found in the purely hierarchical diffusion process.

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