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VACANCY CHAIN MODELS FOR STRATIFIED HOUSING AND EMPLOYMENT SYSTEMS

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CONTENTS

ABSTRACT

In this paper we formulate deterministic vacancy chain models for examining the relationship between mobility, allocation and demand/supply adjustments in stratified housing and employment systems. Three models are discussed, each differing according to the particular criteria adopted in their derivation, while requiring identical data sets for their implementation. Two are constructed within a mathematical programming framework in which an information theoretical approach is used to generate the vacancy chain models. These are compared conceptually and numerically with the third, the traditional fractional flow (renewal) model which is derived from certain constant flow proportion assumptions.

The motivation for the present study is the construction of a general and computationally flexible framework for the study of mobility and dynamical adjustments in stratified systems in which the macroscopic behaviour is characterised by constraints in forecasting or planning contexts. An indication of the potential of the programming approach is given, and directions for further research are outlined.

1. Introduction

Many socio-economic systems exhibit interdependencies between events accompanying the purchase, exchange or allocation of durable commodities or entities. Thus the possibility of chains of moves in housing or employment systems arises directly from the fact that certain households or individuals associated with a dwelling or a job may contribute at the same time to the demand and supply sides in the allocation or exchange process. At the micro-level the simple consequence that a change of state - a move - is dependent on the existence or creation of a vacancy gives rise to a series of exchange interdependencies or vacancy chains.

Because a vacant dwelling usually has a high opportunity cost associated with it, the administration of the exchange process in the housing market gives rise to a clear perception by participants of the linkage between moves. At the head of the chain a household will move into a new or currently vacant dwelling, releasing a vacancy which triggers subsequent moves. The chain will terminate when a vacancy is filled by a "new entrant", or when a dwelling remains empty. The propagation of vacancies due to transitions in an employment system - which we might take at the labour market or at an organisational level - is less dramatic in terms of the administration of exchange, but no less an essential characteristic of system dynamics.

The phenomenon of vacancy chains has been examined widely in the context of household and occupational mobility (see, for example, White (29), Hua (10), Hyler and Cole (9), Bartholomew (3), LGORU (17)). Several examples may be found in the literature on Manpower Planning at the organisational level, as reviewed by Purkiss (22), and the applications to the [illegible] in [illegible] using the [illegible] model are particularly noteworthy ([illegible] et al. [21]; [illegible] [32]; Bartholomew and Forbes [4]).

The occurrence of exchange interdependencies or vacancy chains are, however, seldom reflected in the examination of stratified housing and employment systems specified at the urban or national level, although exceptions appear in the work of [illegible] [17] and Holm *et al.* [15]. There is little doubt that lack of consideration of the vacancy chain approach in forecasting studies in favour of simple allocation models (as discussed in Section 2) arises from two main reasons: firstly, the usual focus of attention in forecasting at these spatial scales is typically the *association* of supply and demand classes and of potential mismatches in the stratified system, and not on *mobility* per se; secondly, if vacancy chain models were constructed *to serve the same purpose* as that cited, their data requirements would be considered extravagant compared with, for example, the cross sectional approach used to update a base year allocation matrix collected at a single cross section.

Although we shall have cause at a number of points to refer to alternative forecasting strategies, including comparative static, dynamic incremental and vacancy chain analyses, our focus will be on *mobility* and *allocation* characteristics of the stratified system for which the vacancy chain approach is appropriate. By adopting a common accounting framework and exploiting the same forecasting strategy, we shall juxtapose the study of housing and employment systems, and concentrate on common methodological aspects.

The motivation for the current study on vacancy chain models is both methodological and numerical. In the application of conventional models, which we outline in Section 3, it is necessary to invert a matrix or execute a power series solution in order to incorporate 'multiplier effects' which accommodate the interdependencies between moves. This is a straightforward process, common in input-output modelling, and calls for no elaboration at this stage. However, in certain contexts it is necessary to examine the effect of certain constraints on the interaction between supply and demand. These may have a direct significance within a forecasting or planning process, or may be an artefact of the model building procedure. It may be necessary to examine: the effect of credit restrictions or building society/mortgage institution credit allocation policies on housing mobility; the effect of preferential treatment for certain groups of individuals or households, such as first time buyers or school leavers; "bottlenecks" in the demand or supply sides on the allocation and exchange processes; or the effect of assumptions about vacancy and unemployment rates in mobility studies. Examples are not hard to find and we have cited but a few.

An expression of these issues may be made through the imposition of constraints on a "basic" stock-flow model. In such cases a solution strategy which suggests itself is the modification of this basic flow solution to incorporate the constraints through an embedding procedure in which a vacancy chain model is modified by 'outer-loop' iteration. Such an approach is adopted, for example, in land use (Lowry-type) models in which density or industrial mix constraints are imposed (Wilson, 31). However, even when the constraints have a simple form it may not always be a straightforward matter to design a heuristic procedure which is guaranteed to converge [illegible] overall while in more difficult [illegible] of a solution [illegible] may pose severe problems.

The initial motivation then was a search for a systematic procedure for incorporating constraints within vacancy chain models, and for the design of appropriate solution strategies. At a rather more general level our intention is to provide a flexible framework for the examination of a wide and interesting range of questions relating to the mobility and allocation in stratified systems which we would suggest have not been given sufficient attention in the literature. A natural framework within which to express and incorporate constraints is that provided by mathematical programming and in this paper we shall present such an approach to vacancy chain analyses.

The search for an appropriate programming framework then focuses directly on the characteristics and interpretation of the extremal process used to generate the vacancy chain model which is obtained as the first order conditions for the maximisation (or minimisation) of an objective function subject to a set of constraints. The criterion adopted for model formulation in this paper is related to the Entropy Maximising approach (Wilson, 30) and more specifically, to the Minimum Information Adding Approach discussed by Snickars and Weibull ([illegible]5). We wish to stress at the outset that our study does not concern the microscopic analysis of chains of move in stochastic representations, but in the reflection of these interdependencies in deterministic forecasting models relating mobility, allocation and demand/supply adjustments in a time interval (t,t + T). Particular attention will be given to certain multiplier relations which characterize these models.

In Section 2 the stock-flow or continuity conditions which reflect the logic of transitions and interdependencies between moves in stratified housing and employment systems are developed. A distinction is made between 'exchange movers', who may contribute to both supply and demand sides of the allocation process, and new entrants who contribute only to the latter. These equations are interpreted as defining a region or space within which feasible solutions - vacancy chain models - may be found. The three models developed in this paper adopt different criteria for identifying a unique feasible solution. The most widely adopted vacancy chain model, suitably interpreted within the stratified system, is derived in Section 3 from the stock-flow conditions, by imposing certain 'constant flow proportion' assumptions.

In Section 4 two vacancy chain models are derived from mathematical programming principles. The primal and dual programmes are presented and the solution characteristics discussed. The models differ according to the precise nature of the objective function in the information theoretic approach. Because the conventional (fractional flow) and mathematical programming models are derived from different principles, it is not expected that their solutions will be identical. Through numerical examples we investigate the nature and extent of these differences in Section 5.

We return in Section 6 to the essential motivation for the development of the mathematical programming framework and discuss the incorporation of constraints within vacancy chain forecasting models. In a final section possible directions for further research are outlined.

2. Stock-Flow Relations for Stratified Housing and Employment Systems

The appropriate level of market segmentation or stratification in housing or employment systems is, in principle, related to the statistical and/or theoretical significance of heterogeneity on the demand and supply sides in the particular forecasting or policy testing contexts for which a model is designed. In practice, it is often governed by available data and statistical aspects relating to the sparseness of the resultant allocation matrices. In this paper we shall not commit ourselves to specific categories as our emphasis is primarily methodological, and we shall refer in both housing and employment systems to N demand classes or states $\alpha^1 \ldots \alpha^N$, referring to the categorisation of households and individuals respectively, and M supply classes or states $\beta^1 \ldots \beta^M$ referring to dwellings and jobs. The classes α^i and β^j will in general be formed from combinations of the various levels of a number of factors which are deemed appropriate for the stratification procedure. They will therefore appear as lists of attributes, and to avoid repetition of the superscripts we shall simply refer to the arbitrary classes α^i and β^j in terms of the vectors $\underline{\alpha}$ and $\underline{\beta}$. Examples of the various factors which might contribute to the vectors $\underline{\alpha}$ and $\underline{\beta}$ are given below in Table 1.

	Housing System	Employment System
$\underline{\alpha}$	life cycle, income, family size, SEG...	age, sex, occupation, qualifications...
$\underline{\beta}$	tenure, location, size, price, type..	industry, grade, wage, hrs worked...

Table 1. Examples of factors, some of which may contribute to the vectors $\underline{\alpha}$ and $\underline{\beta}$ in Housing and Employments Systems.

In the exploration of dynamical behaviour in stratified systems, the formation of a set of accounts provides a framework within which data may be organized and transitions between states consistently documented. When supplemented by particular assumptions they can provide the basis for forecasting quantities relating to the design of the accounting procedure. The accounts may be derived for categorical data collected according to a variety of designs (eg. multiple cross sections, longitudinal and mixed schemes) and we refer to the work of Hyler and Gale (5) for a discussion of these. Although the latter reference pays specific attention to the housing market its comprehensive considerations may readily be reinterpreted within an employment context. The work of Gleave and Palmer (12) explicitly considers accounting relations for the labour market.

In this paper we shall not be concerned with accounting schemes in their most general form but in their relevance to the development of stock-flow vacancy chain models designed to examine the relationship between demand and supply adjustments in the time interval (t, t + T). T will be taken to be sufficiently long to allow stochastic processes to be ignored. In this time interval T the possible transitions between the two cross sections represented by the aggregations $(\underline{a} : \underline{s})_t$ and $(\underline{a}' : \underline{s}')_{t+T}$ may be organised within an accounting scheme involving the elements

$$N\{(\underline{a} : \underline{s})_t \to (\underline{a}' : \underline{s}')_{t+T}\},$$

which is the number of persons or households of type $\underline{a}$ in state $\underline{s}$ at time t who make transitions to $\underline{a}'$ and $\underline{s}'$ in the interval. In the process of model building a structure is imposed on this data according to the particular assumptions adopted for forecasting. Here we shall be concerned with forecasting flows for the interval (t, t + T) from flow data, embodied in a matrix $\underline{N}$, collected in the interval (t − T, t).

The state of a housing or employment system at some future point in time will in general be dependent on a variety of interacting demographic, social, economic and political processes. In the construction of forecasting models, it is common practice to make simplifying assumptions about the interaction between processes, involving their conditional and quantitative dependence. Usually a two-stage process is adopted in which the evolution of the demand and supply sides is considered prior to the allocation process allowing a matching to take place between $\underline{q}$ and $\underline{g}$ categories. The latter process involves a model which is underpinned by either economic or information theoretic assumptions. Typically, it is assumed that the allocation is a function of the updated social, demographic and economic states on the demand side. We shall adopt this assumption below.

In cross sectional (comparative static) models the allocation involves the whole population in the housing or employment system x_D ($\underline{q}$) and the total stock x_S ($\underline{g}$) which have been updated (forecast) in appropriate models. Dynamic incremental methods, as for example, those adopted in the NBER housing model (Ingram *et al.* (9); Kain *et al.* (10)) differ in that the quantities x_D and x_S refer only to those contributions relevant to the time slice $(t, t + T)$ and are accumulated in demand and supply pools prior to allocation. Thus movers may contribute to both x_D and x_S although information about the state from which they are derived will not be retained, limiting the model's usefulness in the study of interdependencies and mobility. Vacancy chain models differ in this respect, requiring that the states entered and vacated be explicitly considered in the formulation of stocks and flows in the interval T. As in dynamic incremental models mentioned above, *potential* movers are accumulated in mover pools prior to the allocation process.

In the models developed in this paper the dependent variable (which is to be forecast) is taken as $T(\underline{\alpha}; \underline{\beta}', \underline{\beta})$, the number of persons or households in state $\underline{\alpha}$ who move between $\underline{\beta}'$ and $\underline{\beta}$ in the time interval considered. We shall make the common assumption that the potential demand $D(\underline{\alpha})$ corresponding to new entrants and potential movers in the interval may be computed external to the allocation procedure. In Figure 1 we note the contributions to both the demand and supply sides. Particularly important aspects of the contributions to the demand and supply sides are the distinctions between: firstly, new entrants (new households, immigrants, etc.) and internal transfers (exchange movers); and secondly, new stock (new build, and created jobs) and 'second-hand' or existing stock. If we wish to make these distinctions explicit in the notation, we shall refer to membership the classes A_1 and A_2 respectively on the demand side, and to B_1 and B_2 on the supply side. It is the consistent dynamical treatment of internal transfers ($\underline{\alpha} \in A_2$) moving from existing stock ($\underline{\beta} \in B_2$) for which the vacancy chain model is explicitly designed. Where the above distinctions are not made explicit, it should be assumed that an extra attribute is added to the $\underline{\alpha}$ and $\underline{\beta}$ vectors to make the differences implicit. We will on occasion refer to a distinction between so called *primary* and *induced* vacancies. The former will correspond to the members of the B_1 class and to that contribution of B_2 which becomes vacant in $(t, t + T)$ due to household dissolution (deaths, retirement, outmigration, etc.), while the latter will refer to those vacancies induced by the internal movement in the system.

We now turn to consider the logic and accounting of moves in the system which are expressed in a set of stock-flow relations.

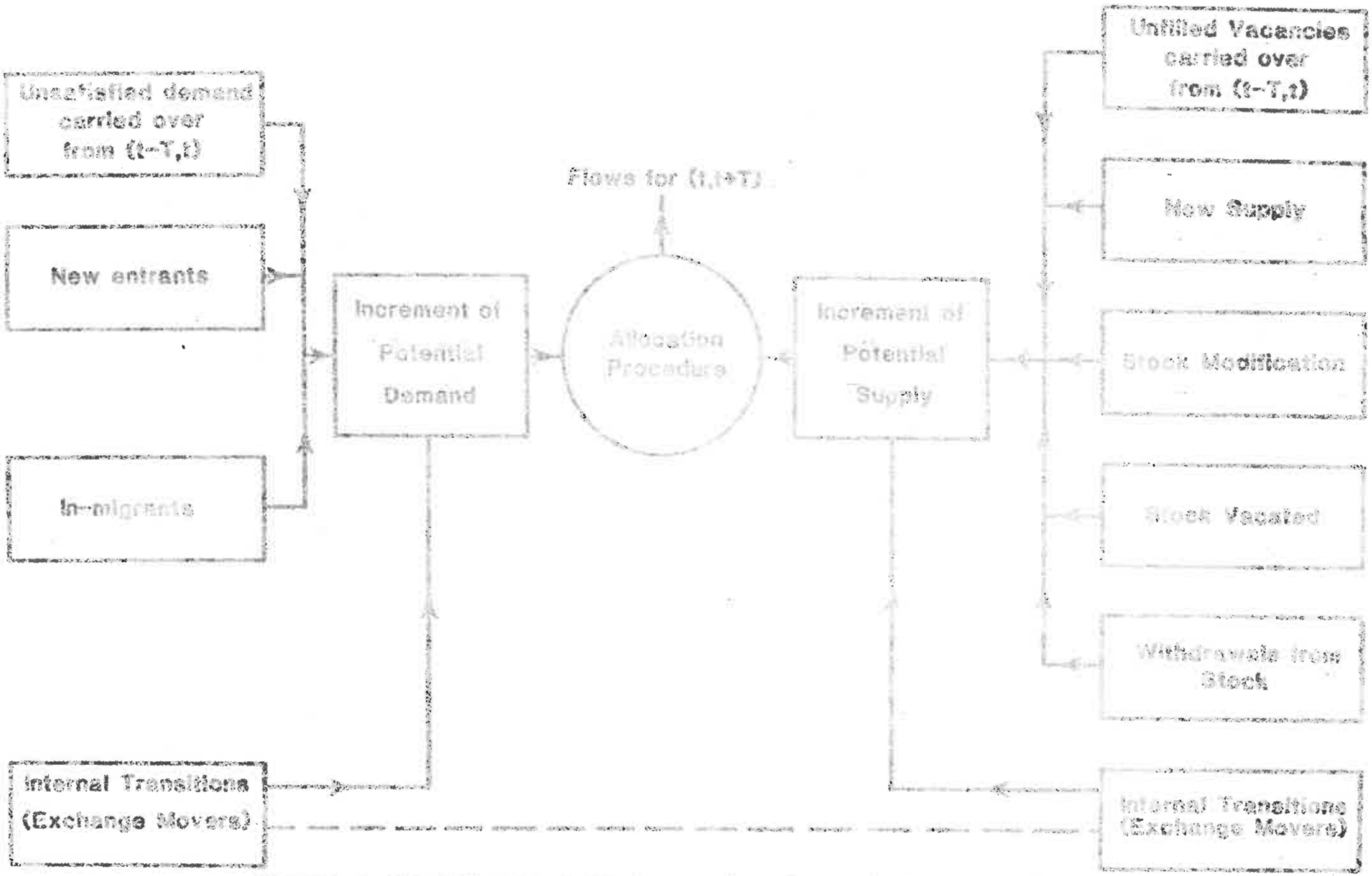


Figure 1. Contributions to demand and supply increments in Housing and Labour Markets in the interval

Stock-Flow [illegible]

In order to reflect the interdependencies between the intrastate movements we start by expressing the fundamental physical restriction underlying the dynamical behaviour of the system, namely

$$(1) \qquad \text{Flow of persons or households into stock state } \underline{\beta} = \text{Vacancies available in state } \underline{\beta}$$

The recognition that vacancies arise from new building etc. and from the internal movement in the system allows us to write

$$(2) \qquad \text{Flow of persons into state } \underline{\beta} = \text{Primary vacancies in state } \underline{\beta} + \text{Vacancies arising from moves out of state } \underline{\beta}$$

In terms of the flow variables we can express this relation for new and 'existing' stock as follows

$$(3) \qquad \sum_{\underline{\alpha} \in A_1} X(\underline{\alpha} : *, \underline{\beta}) + \sum_{\underline{\alpha} \in A_2} \sum_{\underline{\beta}' \in B_2} X(\underline{\alpha} : \underline{\beta}', \underline{\beta}) = S_1(\underline{\beta}) \qquad \underline{\beta} \in B_1$$

$$(4) \qquad \sum_{\underline{\alpha} \in A_1} X(\underline{\alpha} : *, \underline{\beta}) + \sum_{\underline{\alpha} \in A_2} \sum_{\underline{\beta}' \in B_2} X(\underline{\alpha} : \underline{\beta}', \underline{\beta})$$

$$= S_2(\underline{\beta}) + \sum_{\underline{\alpha} \in A_2} \sum_{\underline{\beta}' \in B_1, B_2} X(\underline{\alpha} : \underline{\beta}, \underline{\beta}') \qquad \underline{\beta} \in B_2$$

in which S_1 and S_2 are the primary vacancies in classes B_1 and B_2, respectively. The equations simply state that contributions to the flow into new stock come both from new entrants - the asterisk denoting a dummy supply state - and from internal transfers. The vacancies created by exchange movers who make transitions to either new or existing (second-hand) stock are added to the vacancies $S_2(\beta)$ to form the total supply of vacancies in the set B_2. These are again filled, or partially filled, by new entrants and exchange movers.

The flow elements must also satisfy the following demand restrictions

$$(5) \qquad \sum_{\beta \in B_1, B_2} X(\alpha : *, \beta) \leq D_1(\alpha) \qquad \alpha \in A_1$$

$$(6) \qquad \sum_{\beta \in B_2} \sum_{\beta' \in B_1, B_2} X(\alpha : \beta', \beta) \leq D_2(\alpha) \qquad \alpha \in A_2$$

We shall, as discussed above, treat the total potential demand $D_1(\alpha)$ and $D_2(\alpha)$ created by new entrants and exchange movers as exogenous information derived from other models. Note that these equations are *inequalities*, any slack being interpreted as vacancies in stock and unsatisfied demand, which would, in dynamic models, contribute to the supply and demand pools in the subsequent time interval. One particularly important special case, however, corresponds to the (forecasting) assumption of an excess demand system for which the equations (3) and (4) are expressed as equalities. These represent the conditions in a "supply driven" (SD) system and is that most commonly discussed in the applications of vacancy chain models. An essential feature of the SD-system may be exhibited by summing Equation (3) over all β classes in B_1, and Equation (4) over all β classes in B_2 and then adding the resulting equations. The internal moves are cancelled out and the following relation is obtained

$$(7) \qquad \sum_{\alpha \in A_1} \sum_{\beta \in B_1, B_2} X(\alpha : *, \beta) = \sum_{j=1,2} \sum_{\beta \in B_j} S_j(\beta).$$

This states the well known condition, valid only for the SD-system, that the total number of new entrants drawn into the system by mechanisms involving primary and induced vacancies is equal to the total number of primary vacancies.

Corresponding results may be obtained for excess supply or demand driven (DD) systems for which the equations (5) and (6) are expressed as equalities. A distinction is sometimes made between SD and DD systems in the terminology of stochastic processes where reference is made to Markov (push) and Renewal (pull) systems. Corresponding cases may be found in and close analogies drawn from input-output commodity flow systems.

Clearly there will in general exist mixed systems which are characterised by a disposition of slack variables associated with some classes on both demand and supply sides. We wish to stress that the equations (3) - (6) simply define a feasible region within which stock-flow (vacancy chain) models may be considered to lie. To construct a unique solution it is necessary to inject some assumption which fully determines the system of equations. Three different assumptions are employed in Sections 3 and 4.

Matrix and Graphical Representations

The stock-flow equations (3) - (6) may readily be summarised in Matrix form (see also Byler and Gayle (5)) as the system of linear equations

(8) $$\underline{\underline{F}}\ \underline{X} \leq \underline{S}$$

(9) $$\underline{\underline{G}}\ \underline{X} \leq \underline{D}$$

in which the column vector $\underline{X}$ contains all the components $X(\underline{\alpha} : \underline{\beta}', \underline{\beta})$, $\underline{S}$ contains the primary vacancies in new and existing stock $S_1(\underline{\beta})$ and $S_2(\underline{\beta})$, while $\underline{D}$ contains demand contributions. With the distinction between S_1 and S_2 elements, which we will include in separate vectors $\underline{S}_1$ and $\underline{S}_2$, Equation (8) may be written in the form

$$(10) \qquad \left[\begin{array}{c} \underline{F}_1 \\ \hdashline \underline{F}_2 \end{array}\right] \left[\begin{array}{c} \underline{X} \end{array}\right] = \left[\begin{array}{c} \underline{S}_1 \\ \hline \underline{S}_2 \end{array}\right]$$

Here, the matrix $\underline{F}_1$ contains elements 1 and 0, the former corresponding to contributions of flow into primary (new) vacancies, while the rows of $\underline{F}_2$ contain 1, 0's and -1. The positive contribution corresponds to flows into primary or induced vacancies, while the negative contribution arises from the creation of vacancies due to movement out of a state.

Analogies between these flow systems in Housing and Employment contexts and the multicommodity network flow problems, found in transportation assignment contexts, are evident and relate to the structure of the $\underline{F}$ matrix which in a network context is the node-arc incidence matrix. We may readily develop graphical or network representations of the stock-flow system in which as usual, the structure of the network represents actual or feasible moves. The details of the network depends on whether flows of vacancies and of individuals are treated symmetrically and on the distinctions made in the $\underline{\beta}$ notation - whether for example, reference is made to new and existing stock explicitly or simply to primary and *induced* vacancies as discussed above. In Figure 2 we give a simple example of a network involving 2 α-classes and 2 β-classes in which the latter distinction is made. The flows out of the supply nodes (m_1, m_2)

and (m_1^o, m_2^o) correspond to primary and induced vacancies respectively. Flows into these nodes by new entrants, who are considered to join the system at (n_1, n_2), and by exchange movers enable some or all of the vacancies to be filled. The 'channels' represented by heavy lines correspond to vectors of flow the components of which are associated with each separate $\underline{g}$-class. We have omitted 'intracell' flows, which would appear as 'bubbles' on the internal nodes m_1^o, m_2^o. Their importance will clearly depend on the definition of the $\underline{g}$ classes. In certain cases, it may be necessary to consider them explicitly, when for example, there are restrictions or constraints which involve the *total* number of moves in the system. The present network will be adopted to later comparative tests for the vacancy chain models. Because it corresponds to a simple system the general notation for the flows has been suspended in favour of that indicated, with the superscript denoting the $\underline{g}$-class.

Generalisation to more complex networks which we shall label $(N:M,M)$ corresponding to the elements $\{X(\underline{g} : \beta', \beta)\}$ is straightforward. The stock-flow equations may readily be reproduced by examining flows into and out of the nodes, with allowance for appropriate inequalities. It is clear that the benefits obtained from an improved representation of mobility and internal transitions, which cannot be achieved in a simple allocation system expressed through elements $\{X(\underline{g} : \beta)\}$, must be paid for in the increased complexity of the system equations and their data requirements. We would, however, emphasize that if N and M are large, the resultant matrix $\underset{\sim}{F}$ in Equation (8) and the corresponding $(N:M,M)$ network will typically be exceedingly sparse when expressed in terms of base (observed) flows. This issue will re-emerge in Section 5. We now turn to the generation of the vacancy chain models, in the first instance, unencumbered by additional constraints referred to in the introduction.

3. The Constant Proportion Vacancy Chain Model

In this and the following section we discuss the formation of vacancy chain models for SD systems for which

$$(11)\qquad \sum_{\underline{\alpha}\in A_1} X(\underline{\alpha}:*,\underline{\beta}) + \sum_{\underline{\alpha}\in A_2}\sum_{\underline{\beta}'\in B_2} X(\underline{\alpha}:\underline{\beta}',\underline{\beta}) = S_1(\underline{\beta}) \qquad \underline{\beta}\in B_1$$

$$(12)\qquad \sum_{\underline{\alpha}\in A_1} X(\underline{\alpha}:*,\beta) + \sum_{\underline{\alpha}\in A_2}\sum_{\underline{\beta}'\in B_2} X(\underline{\alpha}:\underline{\beta}',\underline{\beta}) \qquad \underline{\beta}\in B_2$$

$$= S_2(\underline{\beta}) + \sum_{\underline{\alpha}\in A_2}\sum_{\underline{\beta}'\in B_1,B_2} X(\underline{\alpha}:\underline{\beta},\underline{\beta}')$$

The essence of the constant proportion vacancy chain model (CPVC) is contained in the assumption that the components of flow maintain a constant proportion of the total flow into a supply node m_β, which in this SD system we may equate to the total vacancies $V(\underline{\beta})$. Introducing the dummy β-class for new entrants we can express the assumption as follows

$$(13)\qquad X(\underline{\alpha}:\underline{\beta}',\underline{\beta}) = p(\underline{\alpha}:\underline{\beta}',\underline{\beta})\;V(\underline{\beta}) \qquad \text{for all } \underline{\alpha},\underline{\beta},\underline{\beta}' \text{ with}$$

$$(14)\qquad V(\underline{\beta}) = \sum_{\underline{\alpha}}\sum_{\underline{\beta}'\in B_2,*} X(\underline{\alpha}:\underline{\beta}',\underline{\beta}),$$

the quantity on the RHS of Equation (14) being the total flow from all sources into node m_β.

The constant of proportionality $p(\underline{\alpha}, \underline{\beta}', \underline{\beta})$ is obtained from the base flows

$$(15) \qquad p(\underline{\alpha} : \underline{\beta}', \underline{\beta}) = \frac{\overline{X}(\underline{\alpha} : \underline{\beta}', \underline{\beta})}{\sum_{\underline{\alpha}} \sum_{\underline{\beta}'} \overline{X}(\underline{\alpha}, \underline{\beta}', \underline{\beta})}$$

the bar over a variable denoting here, as elsewhere, its determination from the base stock-flow data corresponding to the interval (t-T, t). Thus when primary or induced vacancies are created, they are filled by (and shared between) appropriate classes of new entrant and exchange mover in a fixed proportion, which constitutes a forecasting assumption about the transmission of flows. This assumption provides the extra conditions which allows the system of Equations (11) and (12) to be fully determined and solved uniquely.

Firstly, note that

$$(16) \qquad V(\underline{\beta}) = S_1(\underline{\beta}) \qquad \underline{\beta} \varepsilon B_1$$

giving immediately

$$(17) \qquad X(\underline{\alpha} : *, \underline{\beta}) = p(\underline{\alpha} : *, \underline{\beta})\, S_1(\underline{\beta}) \qquad \underline{\alpha} \varepsilon A_1 \; ; \; \underline{\beta} \varepsilon B_1$$

$$(18) \qquad X(\underline{\alpha} : \underline{\beta}', \underline{\beta}) = p(\underline{\alpha} : \underline{\beta}', \underline{\beta})\, S_1(\underline{\beta}) \qquad \underline{\alpha} \varepsilon A_2 \; ; \; \underline{\beta}' \varepsilon B_2 \; ; \; \underline{\beta} \varepsilon B_1$$

From the expression for V(β) and from equation (12) we may write

$$(19)\quad V(\underline{\beta}) = S_2(\underline{\beta}) + \sum_{\underline{\alpha}\in A_1} \sum_{\underline{\beta}'\in B_1} N(\underline{\alpha} : \underline{\beta}, \underline{\beta}') + \sum_{\underline{\alpha}\in A_2} \sum_{\underline{\beta}'\in B_2} N(\underline{\alpha} : \underline{\beta}, \underline{\beta}') \qquad \underline{\beta}\in B_2$$

which with (13) and (18) may be written

$$(20)\quad V(\underline{\beta}) = S_2^{*}(\underline{\beta}) + \sum_{\underline{\beta}'\in B_2} p(\underline{\beta}, \underline{\beta}')\, V(\underline{\beta}') \qquad \underline{\beta}\in B_2$$

The quantities $S_2^{*}(\underline{\beta})$ and $p(\underline{\beta}, \underline{\beta}')$ are defined as follows

$$(21)\quad S_2^{*}(\underline{\beta}) = S_2(\underline{\beta}) + \sum_{\underline{\alpha}\in A_2} \sum_{\underline{\beta}'\in B_1} p(\underline{\alpha} : \underline{\beta}, \underline{\beta}')\, S_1(\underline{\beta}') \qquad \beta\in B_2$$

$$(22)\quad p(\underline{\beta}, \underline{\beta}') = \sum_{\underline{\alpha}\in A_2} p(\underline{\alpha} : \underline{\beta}, \underline{\beta}') \qquad \underline{\beta}\in B_2$$

The system of equations (20) expressing the total induced vacancies in (existing) stock, is now amenable to solution by matrix algebra in the usual way. Defining column vectors $\underline{V}$ and $\underline{S}^{*}$ with components $V(\underline{\beta})$ and $S_2^{*}(\underline{\beta})$ respectively, and a square matrix $\underline{\underline{P}}$ with components $\{p(\underline{\beta}, \underline{\beta}')\}$ we can write (20) in matrix form

$$(23)\quad \underline{V} = \underline{S}_2^{*} + \underline{\underline{P}}\,\underline{V}$$

yielding the solution

$$(24)\quad \underline{V} = \left[\underline{\underline{1}} - \underline{\underline{P}}\right]^{-1} \underline{S}_2^{*}$$

with $\underline{\underline{1}}$ the unit matrix.

We can now summarise the CPVC model as follows

$$(25) \qquad X(\underline{\alpha} : \underline{\beta}', \underline{\beta}) = p(\underline{\alpha} : \underline{\beta}', \underline{\beta}) \; V(\underline{\beta}) \qquad \text{for all } \underline{\alpha}, \underline{\beta}, \underline{\beta}'$$

in which

$$(26) \qquad V(\underline{\beta}) = S_1(\underline{\beta}) \qquad \text{for } \underline{\beta} \, \varepsilon B_1$$

and

$$(27) \qquad V(\underline{\beta}) = \left(\left[\underline{1} - \underline{P} \right]^{-1} \underline{S}_2^* \right)_{\underline{\beta}} \qquad \underline{\beta} \, \varepsilon B_2$$

with $(\underline{Z})_{\underline{\beta}}$ denoting the $\underline{\beta}$-component of any column vector $\underline{Z}$. The close analogies between the system of equations (25) - (27) with the solution of commodity flow input-output models are again evident.

Before progressing to other vacancy chain models derived from mathematical programming, we summarise a few relevant characteristics of the above model:

(i) The solution to Equation (27) may be achieved in the usual way by matrix inversion or by power series expansion

$$(28) \quad X(\underline{\alpha} : \underline{\beta}', \underline{\beta}) = p(\underline{\alpha} : \underline{\beta}', \underline{\beta}) \left[(\underline{S}_2^*)_{\underline{\beta}} + (\underline{P}\,\underline{S}_2^*)_{\underline{\beta}} + (\underline{P}^2 \underline{S}_2^*)_{\underline{\beta}} + \ldots \right] \quad \underline{\beta} \, \varepsilon B_2$$

in which successive terms in the expansion correspond to 'vacancy chains', or paths through the network linked to primary vacancies, of different length.

(ii) for the SD system the components X_λ are linearly dependent on the scale of $\underline{S}$. That is, under the transformation $(\underline{S} \rightarrow k\,\underline{S})$ with k a positive constant, the flows transform $(\underline{X} \rightarrow k\,\underline{X})$.

(iii) a component of flow $X(\underline{\alpha}, \underline{\beta}', \underline{\beta})$ will vanish if the corresponding flow in the base system $\overline{X}(\underline{\alpha}, \underline{\beta}', \underline{\beta})$ vanishes, if the supply vector $\underline{S}$ vanishes, or if the node m_{β} is 'disconnected' from continuous paths to nodes corresponding to positive contributions of primary vacancies.

(iv) the vacancy chain model is characterised by well known multiplier effects (see the discussion in White (27), Murie *et al.* (20) among others). Due to internal adjustments, the total number of moves will in general be greater than the total number of primary vacancies by a factor μ, where the multiplier is given by

(29)
$$\mu = \frac{\sum\limits_{\underline{\alpha}} \sum\limits_{\underline{\beta}, \underline{\beta}'} X(\underline{\alpha}, \underline{\beta}', \underline{\beta})}{\sum\limits_{\underline{\beta}} S(\underline{\beta})}$$

(30)
$$= \frac{\sum\limits_{\beta} V(\underline{\beta})}{\sum\limits_{\underline{\beta}} S(\underline{\beta})}$$

or

$$(31) \qquad \mu = \frac{\sum\limits_{\underline{\beta}\in B_1} g_1(\underline{\beta}) + \sum\limits_{\underline{\beta}\in B_2} \left(\left[I - \underline{P} \right]^{-1} \underline{g}_2^{*} \right)_{\underline{\beta}}}{\sum\limits_{\underline{\beta}\in B_1} g_1(\underline{\beta}) + \sum\limits_{\underline{\beta}\in B_2} g_2(\underline{\beta})}$$

Note that μ is invariant (that is, independent of k) under the transformation $\underline{g} \to k\underline{g}$.

If we define the total mobility of the system in the interval $(t, t + T)$ to be

$$(32) \qquad \mathcal{M}_t = \sum_{\alpha} \sum_{\underline{\beta}, \underline{\beta}'} X(\underline{\alpha}; \underline{\beta}', \underline{\beta})$$

then the expression (24) may be written

$$(33) \qquad \mathcal{M}_t = \mu \left[\sum_{\underline{\beta}\in B_1} g_1(\underline{\beta}) + \sum_{\underline{\beta}\in B_2} g_2(\underline{\beta}) \right]$$

which is useful in rule-of-thumb calculations if the multiplier μ is available from base flow information.

The properties (i) - (iv) will provide interesting points of comparison with the models derived within a mathematical programming framework.

4. Vacancy Chain Models derived within a Mathematical Programming Framework.

The two SD-vacancy chain models presented in this section will appear in the form of necessary conditions for the optimisation of an objective function subject to a set of constraints. The latter will be those expressing the stock-flow conservation relations (Equations 11 and 12) and the demand conditions (5) and (6), while the extremal principle adopted – the criterion underpinning the forecasting approach – will relate to the Entropy Maximising method discussed by Wilson (30), and more specifically to the Minimum Information adding rationale (Kullback (11), Snickars and Weibull (25)). The two models developed here will differ according to the exact nature of the objective functions selected.

Before embarking on their derivation, it is instructive to reconsider briefly the widely used process of bi-proportional adjustment (Bacharach (2)), which is the term expressing the updating (forecasting) of an allocation matrix such as $X(\underline{\alpha} : \underline{\beta})$, adopted in a cross sectional or dynamic incremental model (see Section 2), given base information relating to the allocation $\overline{X}(\underline{\alpha} : \underline{\beta})$ between demand and supply classes. We shall draw parallels between this process and the derivation of vacancy chain models.

Biproportional Adjustment and the Minimum Information Adding Approach

Biproportional updating of a matrix $\overline{X}(\underline{\alpha}, \underline{\beta})$ for revised demand and supply elements $D(\underline{\alpha})$ and $S(\underline{\beta})$ involves the determination of quantities a_i and b_j for each row and column of the allocation matrix such that the updated matrix is 'biproportional' to $\underline{\overline{X}}$ in the following sense

$$(34) \qquad X(\underline{\alpha}^i, \underline{\beta}^j) = a_i b_j \overline{X}(\underline{\alpha}^i, \underline{\beta}^j) \qquad \begin{array}{l} i = 1 \ldots N \\ j = 1 \ldots M \end{array}$$

The adjustment factors or multipliers must be determined in such a way that the following conditions are met

$$(35) \qquad \Sigma_j \; X(\underline{\alpha}^i, \underline{\beta}^j) = D(\underline{\alpha}^i) \qquad i = 1 \ldots N$$

$$(36) \qquad \Sigma_i \; X(\underline{\alpha}^i, \underline{\beta}^j) = S(\underline{\beta}^j) \qquad j = 1 \ldots M$$

Note that here $S(\underline{\beta})$ refers to the *total* vacancies available from primary and induced sources and is assumed to be computed exogenously. The substitution of (34) in (35) and (36) allows the mutual dependence of the vectors $\underline{a}$ and $\underline{b}$ to be expressed

$$(37) \qquad a_i = \frac{D(\underline{\alpha}^i)}{\sum_j b_j \overline{X}(\underline{\alpha}^i, \underline{\beta}^j)} \qquad i = 1 \ldots N$$

$$(38) \qquad b_j = \frac{S(\underline{\beta}^j)}{\sum_i a_i \overline{X}(\underline{\alpha}^i, \underline{\beta}^j)} \qquad j = 1 \ldots M$$

Solution of these equations for $\underline{a}$ and $\underline{b}$ may be achieved by the usual Furness Iterative method, and convergence will be achieved if the system is balanced, with

$$(39) \qquad \sum_{i=1}^{N} D(\underline{\alpha}^i) = \sum_{j=1}^{M} S(\underline{\beta}^j) = X_{**}$$

It has long been recognised (Murchland (19), Bacharach (2)) that the model (34) can be derived from the following optimisation problem

$$(40) \qquad \max_{X} \left[- \sum_{\alpha} \sum_{\beta} X(\alpha, \beta) \ln \frac{X(\alpha, \beta)}{\bar{X}(\alpha, \beta)} \right]$$

subject to

$$(41) \qquad \sum_{\beta} X(\alpha, \beta) = D(\alpha) \qquad \alpha \epsilon A$$

$$(42) \qquad \sum_{\alpha} X(\alpha, \beta) = S(\beta) \qquad \beta \epsilon B$$

in which we have again dropped explicit reference to the superscripts i and j. The relationship between the objective function and that expressed in an Entropy Maximising approach to model generation is evident.

In order to be consistent with the information theoretic formulation (Kullback (11); Wilson (30); Snickars and Weibull (25); Macgill (18)), and provide a theoretic basis for biproportional adjustment, it is appropriate to express the programme not in terms of the quantities $X(\alpha, \beta)$ and $\bar{X}(\alpha, \beta)$ but in terms of the probability elements defined by

$$(43) \qquad p(\alpha, \beta) = \frac{X(\alpha, \beta)}{X_{**}}$$

$$(44) \qquad \bar{p}(\alpha, \beta) = \frac{\bar{X}(\alpha, \beta)}{\bar{X}_{**}}$$

with X_a defined in Equation (39) as the total number involved in the allocation process.

The model generating programme is expressed in terms of the quantities p and $\bar{p}$ as follows

(45) $$\min_{\underline{p}} \sum_{\underline{\alpha}} \sum_{\underline{\beta}} p(\underline{\alpha}, \underline{\beta}) \ln \frac{p(\underline{\alpha}, \underline{\beta})}{\bar{p}(\underline{\alpha}, \underline{\beta})}$$

subject to

(46) $$\sum_{\underline{\beta}} p(\underline{\alpha}, \underline{\beta}) = \frac{D(\underline{\alpha})}{X_a} \qquad \underline{\alpha} \in A$$

(47) $$\sum_{\underline{\alpha}} p(\underline{\alpha}, \underline{\beta}) = \frac{S(\underline{\beta})}{X_a} \qquad \underline{\beta} \in B$$

The objective function has the interpretation of the information added in the transformation from the prior matrix $\{\bar{p}(\underline{\alpha}, \underline{\beta})\}$ to the posterior probability matrix $\{p(\underline{\alpha}, \underline{\beta})\}$, and the programme is interpreted as determining that matrix $\underline{p}$ which minimises the added information to the prior distribution subject to the information contained in the expression of the constraints (46) and (47). An alternative and equivalent way of interpreting this approach is that in the generation of the resultant matrix $\underline{p}$ we make the least assumptions consistent with known conditions.

It may readily be shown that the model derived from the programme (40) - (42) is equivalent to that obtained from the expression in terms of the probabilities (45) - (47). Moreover, because they are concave maximisation or convex minimisation problems subject to linear constraints, a solution is unique if it exists. One further point may be noted at this stage, namely that if marginal changes occur such that the elements X are close to the base values $\overline{X}$ then the approximation

$$(48) \qquad \ln \frac{X}{\overline{X}} \approx \frac{X - \overline{X}}{\overline{X}}$$

reveals that the operation embodied in (40) can be interpreted in terms of a minimum deviation criterion of the form

$$(49) \qquad \underset{\underline{X}}{\mathrm{Min}} \quad \sum_{\underline{\alpha}} \sum_{\underline{\beta}} \quad X(\underline{\alpha}, \underline{\beta}) \quad \frac{(X(\underline{\alpha}, \underline{\beta}) - \overline{X}(\underline{\alpha}, \underline{\beta}))}{\overline{X}(\underline{\alpha}, \underline{\beta})}$$

subject to the constraints (41) and (42). The updated matrix is then that which is 'nearest' to the base allocation in the sense expressed by (49).

Further discussion of these issues may be found in the work of Snickars and Weibull (25) and Macgill (18).

An Entropy Maximising Vacancy Chain Model

We shall generate a vacancy chain model for the system of flows $\{ X(\underline{\alpha} : \underline{\beta}', \underline{\beta}) \}$ by adopting a similar criterion to that used to form the allocation model $\{ X(\underline{\alpha}, \underline{\beta}) \}$ as described above. The generating equations are expressed in terms of the following maximisation problem

$$(50) \qquad \max_{\underline{X}} \quad Z(\underline{X}) = \left[-\sum_{\underline{\alpha}} \sum_{\underline{\beta}, \underline{\beta}'} \quad X(\underline{\alpha} : \underline{\beta}', \underline{\beta}) \quad \ln \frac{X(\underline{\alpha} : \underline{\beta}', \underline{\beta})}{\overline{X}(\underline{\alpha} : \underline{\beta}', \underline{\beta})} \right]$$

subject to the stock-flow equations which we repeat here for convenience

$$\sum_{\underline{\alpha}\in A_1} X(\underline{\alpha} : *, \underline{\beta}) + \sum_{\underline{\alpha}\in A_2} \sum_{\underline{\beta}'\in B_2} X(\underline{\alpha} : \underline{\beta}', \underline{\beta}) = S_1(\underline{\beta}) \quad \underline{\beta}\in B_1$$

$$\sum_{\underline{\alpha}\in A_1} X(\underline{\alpha} : *, \underline{\beta}) + \sum_{\underline{\alpha}\in A_2} \sum_{\underline{\beta}'\in B_2} X(\underline{\alpha} : \underline{\beta}', \underline{\beta})$$

$$= S_2(\underline{\beta}) + \sum_{\underline{\alpha}\in A_2} \sum_{\underline{\beta}'\in B_1, B_2} X(\underline{\alpha} : \underline{\beta}, \underline{\beta}') \quad \underline{\beta}\in B_2$$

Introducing the sets of multipliers $(\xi^1(\underline{\beta}), \underline{\beta}\in B_1 ; \xi^2(\underline{\beta}), \underline{\beta}\in B_2)$ the Lagrangian $\mathcal{L}$ for this programme may be written as follows

$$(51) \quad \mathcal{L}(\underline{X}, \underline{\xi}^1, \underline{\xi}^2) = Z(\underline{X})$$

$$+ \sum_{\underline{\beta}\in B_1} \xi^1(\underline{\beta}) \left[S_1(\underline{\beta}) - \sum_{\underline{\alpha}\in A_1} X(\underline{\alpha} : *, \underline{\beta}) - \sum_{\underline{\alpha}\in A_2} \sum_{\underline{\beta}'\in B_2} X(\underline{\alpha} : \underline{\beta}', \underline{\beta}) \right]$$

$$+ \sum_{\underline{\beta}\in B_2} \xi^2(\underline{\beta}) \left[S_2(\underline{\beta}) + \sum_{\underline{\alpha}\in A_2} \sum_{\underline{\beta}'\in B_1, B_2} X(\underline{\alpha} : \underline{\beta}, \underline{\beta}') \right.$$

$$\left. - \sum_{\underline{\alpha}\in A_1} X(\underline{\alpha} : *, \underline{\beta}) - \sum_{\underline{\alpha}\in A_2} \sum_{\underline{\beta}'\in B_2} X(\underline{\alpha} : \underline{\beta}', \underline{\beta}) \right]$$

in terms of which the first order conditions for a minimum may be written

$$(52) \qquad \frac{\partial \mathcal{L}}{\partial X_\lambda} = 0 \qquad \text{for all } \lambda = (\underline{\alpha} : \underline{\beta}', \underline{\beta}) \text{ combinations}$$

$$\frac{\partial \mathcal{L}}{\partial \xi^i(\underline{\beta})} = 0 \qquad \text{for } \underline{\beta} \in S_i, \quad i=1,2. \tag{53}$$

The conditions (52) generate the stock-flow vacancy chain model

$$X(\underline{\alpha}: *, \underline{\beta}) = \overline{X}(\underline{\alpha}: *, \underline{\beta}) \exp(-\xi^i(\underline{\beta}))\ \underline{\alpha} \in A_i;\ \underline{\beta} \in S_i\ :\ i=1,2 \tag{54}$$

$$X(\underline{\alpha}: \underline{\beta}', \underline{\beta}) = \overline{X}(\underline{\alpha}: \underline{\beta}', \underline{\beta}) \exp(-\xi^i(\underline{\beta}) + \xi^2(\underline{\beta}')) \tag{55}$$

$$\underline{\alpha} \in A_i\ :\ \underline{\beta}' \in S_2;\ \underline{\beta} \in S_i,\ i=1,2.$$

while Equations (53) simply requires that the multipliers $\underline{\xi}$ are such as to satisfy, *at optimality*, the stock-flow conditions (11) and (12).

Note that it is unnecessary to encumber the above programme with the additional restriction

$$\underline{X} \geq 0.$$

In programmes with the above linear constraints and an objective function of the form (50), all flows will be non-negative and will be zero in those cases for which the corresponding base flows vanish.

We note for later reference that the dual of the programme may be written, following Wolfe (33)

$$\min_{(\underline{X},\, \underline{\xi})} \quad \mathcal{L}(\underline{X}, \underline{\xi}) \tag{56}$$

(57) subject to

$$\frac{\partial \mathcal{L}}{\partial X_\lambda} = 0 \qquad \text{for all components } X_\lambda$$

which on substituting yields the following unconstrained problem

$$(58)\quad \min_{\xi^1,\,\xi^2} \Bigg[\sum_{\substack{\underline{\alpha}\in A_1 \\ i=1,2}} \sum_{\underline{\beta}\in B_i} \tilde{X}(\underline{\alpha} : *,\, \underline{\beta}) \exp(-\xi^i(\underline{\beta}))$$

$$+ \sum_{\underline{\alpha}\in A_2} \sum_{\underline{\beta}'\in B_2} \sum_{\substack{\underline{\beta}\in B_i \\ i=1,2}} \tilde{X}(\underline{\alpha} : \underline{\beta}',\, \underline{\beta}) \exp(-\xi^i(\underline{\beta}) + \xi^2(\underline{\beta}'))$$

$$+ \sum_{\substack{\underline{\beta}\in B_i \\ i=1,2}} \xi^i(\underline{\beta})\, S_i(\underline{\beta}) \Bigg]$$

From duality theory we note that $\hat{\xi}$ will solve the dual if and only if the expressions (54) - (55) with $(\hat{\xi}^1, \hat{\xi}^2) = (\xi^1, \xi^2)$ solve the primal programme.

We shall label the model expressed in the Equations (54) - (55) the Entropy Maximising Vacancy Chain (EMVC) model. Let us consider some of its characteristics.

It may readily be checked that the flows derived from the (EMVC) and (CPVC) models are not in general identical. By construction, the proportion of flow into a supply state contributed by a particular component, is for the (CPVC) equal to that value obtained in the base system. In the EMVC model on the other hand these proportions are functions of the Lagrange multipliers also and are thus dependent on the stock conditions. Thus, for example, with $\underline{\alpha}\varepsilon A_2$, $\underline{\beta}$, $\underline{\beta}'\varepsilon B_2$

$$(59)\quad \frac{X(\underline{\alpha}:\underline{\beta}',\underline{\beta})}{\sum\limits_{\underline{\alpha},\underline{\beta}'} X(\underline{\alpha}:\underline{\beta}',\underline{\beta})} = \frac{\overline{X}(\underline{\alpha}:\underline{\beta}',\underline{\beta})}{\theta_1(\underline{\beta}') \sum\limits_{\underline{\alpha}\varepsilon A_1} \overline{X}(\underline{\alpha}:*,\underline{\beta}) + \sum\limits_{\underline{\alpha}\varepsilon A_2} \sum\limits_{\underline{\beta}''\varepsilon B_2} \overline{X}(\underline{\alpha}:\underline{\beta}'',\underline{\beta})\, \theta_2(\underline{\beta}',\underline{\beta}'')}$$

with

$$(60)\qquad \theta_1(\underline{\beta}') = \exp(-t^2(\underline{\beta}'))$$

$$(61)\qquad \theta_2(\underline{\beta}',\underline{\beta}'') = \exp(-t^2(\underline{\beta}') + t^2(\underline{\beta}'')).$$

In the CPVC model the two factors θ_1 and θ_2 would be unity.

In other words, the programming model is not consistent with the assumption that vacancies created in a supply state are filled by flow elements in a constant proportion determined in the base system. This is not particularly surprising as an alternative criterion, jointly involving all the flows in the system and their base values, has been adopted for model generation.

The EMVC model embodies a multiplier relation of the following form

$$(62)\qquad V = \frac{\sum\limits_{\underline{\alpha}\varepsilon A_1} \sum\limits_{\underline{\beta}\varepsilon B_1,B_2} X(\underline{\alpha}:*,\underline{\beta}) + \sum\limits_{\underline{\alpha}\varepsilon A_2} \sum\limits_{\underline{\beta}'\varepsilon B_2} \sum\limits_{\underline{\beta}\varepsilon B_1,B_2} X(\underline{\alpha},\underline{\beta}',\underline{\beta})}{\sum\limits_{\underline{\beta}\varepsilon B_1} \theta_1(\underline{\beta}) + \sum\limits_{\underline{\beta}\varepsilon B_2} \theta_2(\underline{\beta})}$$

or

$$(63)\qquad \mu = \frac{\left[\displaystyle\sum_{\underline{\alpha}\in A_1}\ \sum_{\substack{\underline{\beta}\in B_i\\ i=1,2}} \bar{X}(\underline{\alpha} : *, \underline{\beta})\exp(-\xi^i(\underline{\beta})) + \sum_{\underline{\alpha}\in A_2}\ \sum_{\underline{\beta}'\in B_2}\ \sum_{\substack{\underline{\beta}\in B_i\\ i=1,2}} \bar{X}(\underline{\alpha} : \underline{\beta}', \underline{\beta})\exp(-\xi^i(\underline{\beta}) + \xi^2(\underline{\beta}'))\right]}{\displaystyle\sum_{\underline{\beta}\in B_1} S_1(\underline{\beta}) + \sum_{\underline{\beta}\in B_2} S_2(\underline{\beta})}$$

Because the equations are not invariant under the transformation $\{\underline{S} \to k\underline{S}\}$, $\{\underline{X} \to k\underline{X}\}$ the multiplier will in general be a function of the supply conditions. Concommitantly, when the supply elements are transformed the flows will not be scaled by the factor k. This we consider to be a rather unsatisfactory characteristic of the EMVC model, applied to the SD system, the numerical significant of which we shall address in the following section. This feature does, however, encourage an enquiry of whether an expression of the MIA approach in terms of proportions will result in a numerically distinct model which is not tarnished by this characteristic.

<u>The Scaled Entropy Maximising Vacancy Chain (SEMVC) Model</u>

The required flow proportions to generate a model analogous to (45) - (47) are defined as follows

$$(64)\qquad p_\lambda = \frac{X_\lambda}{X_s} \qquad \text{for all flow components } \lambda,$$

with X_* denoting the total flows (transitions) in the system. The $\bar{p}_\lambda$ corresponding to base proportions are defined in a similar way in terms of $\bar{X}_\lambda$, $\bar{X}_*$.

An immediate problem presents itself. While the quantity $\bar{p}_\lambda$ may be derived from the base year information, it can be seen that the denominator of p_λ, X_* - the total number of transitions - is itself an endogenous variable, and prevents any progress on the same basis as before. However, instead of defining proportions in terms of X_* - an unknown - we shall specify corresponding quantities in terms of S_*, the total number of primary vacancies, defined by

$$(65) \qquad S_* = \sum_{\substack{\underline{\beta}\in B_i \\ i=1,2}} S_i(\underline{\beta})$$

and thus attempt to free the resultant multiplier μ from its dependence on a scaling factor k under the transformation $\{\underline{S} \to k\underline{S}\}$.

The modified Entropy Maximising Model (SEMVC) is now written as before but in terms of quantities expressed as fractions of S_*. That is

$$(66) \qquad \max_{\underline{x}} \left[- \sum_{\lambda} x_\lambda \ \mathrm{Ln} \ \frac{x_\lambda}{\bar{x}_\lambda} \right]$$

subject to

$$(67) \qquad \sum_{\underline{\alpha}\in A_1} x(\underline{\alpha} : *, \underline{\beta}) + \sum_{\underline{\alpha}\in A_2} \sum_{\underline{\beta}'\in B_2} x(\underline{\alpha} : \underline{\beta}', \underline{\beta}) = s_1(\underline{\beta}) \qquad \underline{\beta}\in B_1$$

$$(68)\qquad \sum_{\underline{\alpha}\in A_1} x(\underline{\alpha}: *, \underline{\beta}) + \sum_{\underline{\alpha}\in A_2} \sum_{\underline{\beta}'\in B_2} x(\underline{\alpha}: \underline{\beta}', \underline{\beta})$$

$$= s_2(\underline{\beta}) + \sum_{\underline{\alpha}\in A_2} \sum_{\underline{\beta}'\in B_1, B_2} x(\underline{\alpha}: \underline{\beta}, \underline{\beta}') \qquad \underline{\beta}\in B_2$$

in which

$$(69)\qquad \bar{x}_\lambda = \frac{\bar{X}_\lambda}{\bar{S}_s} \qquad \text{for all components } \lambda$$

and

$$(70)\qquad s_i(\underline{\beta}) = \frac{S_i(\underline{\beta})}{S_s} \qquad \text{for all } \underline{\beta},\ i=1,2.$$

The resultant transitions are similarly determined from the solution

$$(71)\qquad X_\lambda = S_s\ x_\lambda \qquad \text{for all components } \lambda, \text{ and are given}$$

by

$$(72)\qquad X(\underline{\alpha}: *, \underline{\beta}) = \frac{S_s}{\bar{S}_s} \cdot \bar{X}(\underline{\alpha}: *, \underline{\beta}) \exp(-\zeta^i(\underline{\beta})) \qquad \underline{\alpha}\in A_1;\ \underline{\beta}\in S_i,\ i=1,2.$$

$$(73)\qquad X(\underline{\alpha}: \underline{\beta}', \underline{\beta}) = \frac{S_s}{\bar{S}_s} \cdot \bar{X}(\underline{\alpha}: \underline{\beta}', \underline{\beta}) \exp(-\zeta^i(\underline{\beta}) + \zeta^2(\underline{\beta}'))$$

$$\underline{\alpha}\in A_2 : \underline{\beta}'\in S_2,\ \underline{\beta}\in S_i,\ i=1,2.$$

in which we have distinguished between the duals ξ and ζ, because the two models (EMVC) and (SEMVC) will in general result in different flow vectors. By construction, the flow components in SEMVC will now scale by a factor k under the transformation $\{\underline{S} \rightarrow k\underline{S}\}$, leaving a multiplier μ independent of k.

Before turning to numerical issues we would remark on two points. Firstly, we are not *strictly speaking* at liberty to characterise the formation of the model in terms of the MIA approach because the quantities x_λ are not interpretable as probabilities, normalised to unity. From a purely *theoretical* viewpoint the procedure in transforming the EMVC model must be regarded to be somewhat ad hoc.

Secondly, although the above programming models may appear rather complicated their formation is in fact very straightforward, and indeed the general solution may be written down immediately if reference is made to an appropriately structured diagram of the network representation. The following simple rules are involved:

(i) Label all nodes and form a directed graph uniquely defining all links (as the flow elements).

(ii) Assign a dual variable (or lagrange multiplier) to each node. The elements $\xi^1(\underline{\beta})$ $\underline{\beta} = 1\ldots M$ are assigned to nodes representing new stock while $\xi^2(\underline{\beta})$ $\underline{\beta} = 1\ldots M$ label those representing "existing" or "second hand" states.

(iii) Write $X(\underline{a}:\ldots) = \overline{X}(\underline{a}:\ldots) \exp(\psi_{mm'})$

where m and m' are the two nodes defining the particular arc representing the flow $X(\underline{a}:\ldots)$.

(iv) For a flow into a node m' designate a negative contribution from the corresponding dual variable at the node. Similarly for a flow out of m designate a positive contribution to ψ.

Thus in Equation (54) flows by new entrants into new or existing stock simply "pick up" a negative contribution from the dual at the appropriate node that is

$$(75) \qquad \psi_{mm'} = -\xi^{i}(\beta_{m'}) \qquad \beta_{m'} \varepsilon B_i \qquad i=1,2$$

while in Equation (55), positive and negative expressions contribute from the two nodes β_m and $\beta_{m'}$ such that

$$(76) \qquad \psi_{mm'} = -\xi^{i}(\beta_{m'}) + \xi^{i}(\beta_m) \qquad \beta_{m'} \varepsilon B_i \quad i=1,2.$$

With appropriate modifications to the quantities X and S a similar procedure may be adopted for the SEMVC model.

Furthermore, a straightforward method of forming the dual programme - adopted in numerical work - is obtained if it is written as follows

$$(77) \qquad \min_{\xi} \left[\sum_{\substack{\text{all links} \\ \lambda}} X_\lambda(\xi) + \sum_{\substack{\text{all nodes} \\ m}} \xi_m S_m \right]$$

subject to

$$(78) \qquad X_\lambda = \bar{X}_\lambda \exp(\psi_{mm'})$$

in which λ includes the states β_m and $\beta_{m'}$.

By writing down the expression (78) as outlined above and substituting these into (77) the dual programme may be immediately expressed as an unconstrained (convex) minimisation problem.

5. Numerical Comparisons of the SD-Vacancy Chain Models

In this paper we have presented three models of vacancy chains - CPVC, EMVC and SEMVC - and it is natural to enquire how and to what extent they differ. As we have stressed throughout, different criteria have been adopted to generate them, although the transition (flow) matrix $\underset{\sim}{Y}$ satisfies the same fundamental stock-flow relations in each case. It has not been our intention to put forward an SD-model which is conceptually superior to the CPVC model, our prime motivation being the presentation of a framework within which an SD-model may be generated identical or similar to the latter expression, and which may accommodate generalisations of the type discussed in Section 1. For this reason, a large numerical deviation from the CPVC model in SD systems must be considered an embarassment in terms of the acceptability of the particular programming approach adopted.

This series of numerical tests has been conducted firstly, and primarily, to examine the difference of the two programming models (EMVC and SEMVC) from each other and from the reference CPVC model; and secondly to support the claim that the programming approach is flexible and will offer no fundamental difficulties in numerical implementation.

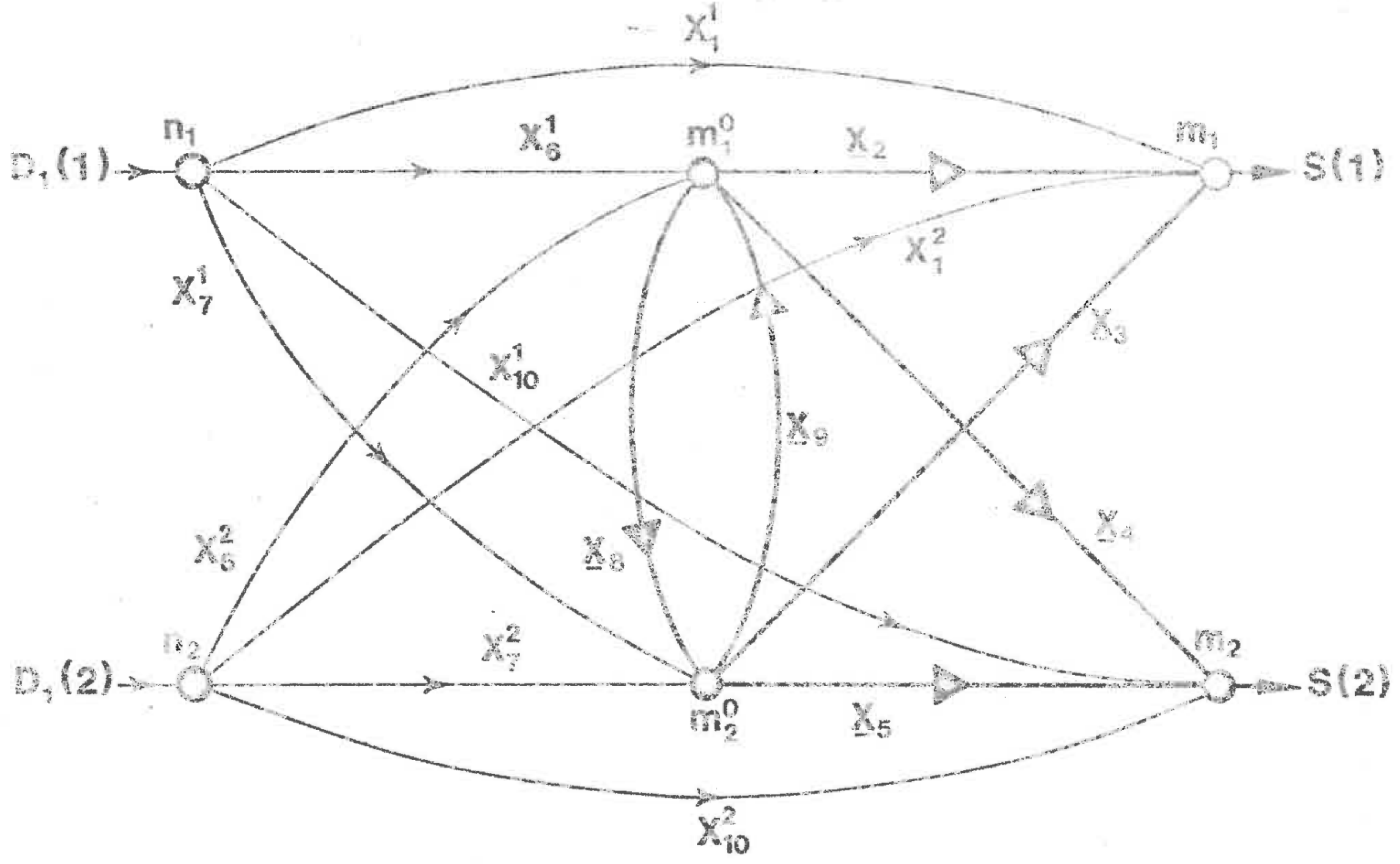
Because any numerical differences will in general be context dependent it would be of value to derive analytic expressions for the deviations. While the results (59) indicate that there will be differences, this is of little value in numerical assessment. For this purpose we have therefore resorted to a series of tests on straightforward but non-trivial BD-contexts. These have involved a set of ten numerical analyses on - to use the (N:M,R) notation adopted in Section 2 - (1 : 2 , 2), (2 : 2 , 2), (2 : 3 , 3) networks. The ten particular networks which were considered have furthermore been examined under different forecasting (supply) conditions. Here we present the results for two such tests on a single (2 : 2 , 2) network which are illustrative of the results obtained from the wider series. Recall that the network notation refers to two person types and two supply states. For this calculation we shall assume that new stock alone contributes to the primary vacancies. A pictorial representation of the network is given in Figure 2 . Because this is a relatively simple network we have suspended the general ($\underline{a}$, $\underline{E}$, $\underline{g}$) notation in favour of that shown in the Figure.

Results have been obtained from the three models in two tests which we shall label TI and TII. They correspond to the following different forecasting (supply) conditions

$$\text{TI} : (\tilde{S}(1) = 100; \ \tilde{S}(2) = 120) \ : \ (S(1) = 120; \ S(2) = 75)$$

$$\text{TII} : (\tilde{S}(1) = 100; \ \tilde{S}(2) = 120) \ : \ (S(1) = 200; \ S(2) = 150).$$

The programme for generating the EMVC model together with its dual, and their solution are shown in Table 2. For the formation of the



$\underline{X}_\sigma$ denotes a vector of flows with components (X_σ^1, X_σ^2)

Figure 2. The network representation of interstate transitions (flows) in time increment.(see also Section 5)

SEMVC model a simple modification is made in accordance with Equations (69) and (70), that is:

(79) $$S(1) \rightarrow s(1) = \frac{S(1)}{S(1) + S(2)}$$

(80) $$S(2) \rightarrow s(2) = \frac{S(2)}{S(1) + S(2)}$$

(81) $$\bar{X}_{\lambda}^{\alpha} \rightarrow \bar{x}_{\lambda}^{\alpha} = \frac{\bar{X}_{\lambda}^{\alpha}}{\bar{S}(1) + \bar{S}(2)} \qquad \lambda = 1 \ldots 10 \; : \; \alpha = 1,2.$$

The programme is then solved in terms of the variable x and the solution of the SEMVC model is obtained as follows

(82) $$X_{\lambda}^{\alpha} = x_{\lambda}^{\alpha} \,.\, (S(1) + S(2)) \qquad \lambda = 1 \ldots 10 \; : \; \alpha = 1,2.$$

The sets of results corresponding to TI and TII are shown in Tables 3a and 3b respectively, with the same base data adopted for both. A comparison of the EMVC and CPVC model results in TI indicates that agreement is reasonably good, with discrepancies typically less than 10%. The results are, however, significantly worse for flows X_8 and X_9 (on the "internal loop" in the diagram), where deviations of the order of 15 to 20 % are found. Some explanation of this may be found if we write down the solutions for these components in the EMVC and SEMVC models, as follows:

Primal Program

$$\max_{\underline{X}} \; - \sum_{\alpha=1,2} \; \sum_{\lambda=1,10} X_\lambda^\alpha \; \mathrm{Ln} \frac{X_\lambda^\alpha}{\bar{X}_\lambda^\alpha}$$

subject to

$$\sum_{\alpha=1,2} (X_1^\alpha + X_2^\alpha + X_3^\alpha) = S(1)$$

$$\sum_{\alpha=1,2} (X_4^\alpha + X_5^\alpha + X_{10}^\alpha) = S(2)$$

$$\sum_{\alpha=1,2} (X_2^\alpha + X_4^\alpha + X_8^\alpha - X_6^\alpha - X_9^\alpha) = 0$$

$$\sum_{\alpha=1,2} (X_3^\alpha + X_5^\alpha + X_9^\alpha - X_7^\alpha - X_8^\alpha) = 0$$

Dual Program

$$\min_{\underline{v}} \Big[\sum_{\alpha=1,2} (\bar{X}_1^\alpha \exp(-v_1) + \bar{X}_2^\alpha \exp(-v_1 + v_3) + \bar{X}_3^\alpha \exp(-v_1 + v_4)$$

$$+ \bar{X}_4^\alpha \exp(-v_2 + v_3) + \bar{X}_5^\alpha \exp(-v_2 + v_4) + \bar{X}_6^\alpha \exp(-v_3)$$

$$+ \bar{X}_7^\alpha \exp(-v_4) + \bar{X}_8^\alpha \exp(-v_4 + v_3) + \bar{X}_9^\alpha \exp(-v_3 + v_4)$$

$$+ \bar{X}_{10}^\alpha \exp(v_2)) + v_1 S(1) + v_2 S(2) \Big]$$

Solution

$$X_1^\alpha = \bar{X}_1^\alpha \exp(-v_1) \qquad X_4^\alpha = \bar{X}_4^\alpha \exp(-v_2 + v_3) \qquad X_7^\alpha = \bar{X}_7^\alpha \exp(-v_4)$$

$$X_2^\alpha = \bar{X}_2^\alpha \exp(-v_1 + v_3) \qquad X_5^\alpha = \bar{X}_5^\alpha \exp(-v_2 + v_4) \qquad X_8^\alpha = \bar{X}_8^\alpha \exp(-v_4 + v_3) \qquad X_{10}^\alpha = \bar{X}_{10}^\alpha \exp(-v_2)$$

$$X_3^\alpha = \bar{X}_3^\alpha \exp(-v_1 + v_4) \qquad X_6^\alpha = \bar{X}_6^\alpha \exp(-v_3) \qquad X_9^\alpha = \bar{X}_9^\alpha \exp(-v_3 + v_4) \qquad \alpha = 1,2$$

Table 2. The Primal and Dual Programs and formal solution of the Entropy Maximising Vacancy Chain Model.

Table 3. The data for and the numerical solutions of the CPVC, EMVC and SEMVC models.

Base Data

$\bar{X}_\lambda^\alpha$	$\bar{X}_1^\alpha$	$\bar{X}_2^\alpha$	$\bar{X}_3^\alpha$	$\bar{X}_4^\alpha$	$\bar{X}_5^\alpha$	$\bar{X}_6^\alpha$	$\bar{X}_7^\alpha$	$\bar{X}_8^\alpha$	$\bar{X}_9^\alpha$	$\bar{X}_{10}^\alpha$
$\alpha=1$	40	15	12	20	30	18	30	16	25	25
$\alpha=2$	20	5	5	10	20	22	50	14	15	15

$\bar{S}(1) = 100$ $\bar{S}(2) = 120$

Table 3a Results for test TI

TEST I: $S(1) = 120$ $S(2) = 75$						
Model	CPVC		EMVC		SEMV	
	$\alpha=1$	$\alpha=2$	$\alpha=1$	$\alpha=2$	$\alpha=1$	$\alpha=2$
X_1^α	48	24	15.3	22.7	16.9	23.5
X_2^α	10	6	16.4	6.5	18.4	6.1
X_3^α	14.4	9.6	15.7	10.4	15.0	10.0
X_4^α	12.5	6.2	13.0	6.5	12.6	6.3
X_5^α	13.8	12.5	19.7	13.1	19.2	12.8
X_6^α	15	18.4	15.7	19.2	15.2	18.6
X_7^α	24.2	40.4	26.0	43.4	25.0	41.7
X_8^α	12.9	11.3	15.9	13.9	14.0	12.2
X_9^α	20.9	12.5	25.2	15.1	22.5	13.5
X_{10}^α	15.6	9.4	14.2	8.5	15.2	9.0
Multi-plier μ	1.80		1.89		1.83	

Table 3b Results for test TII

TEST II: $S(1) = 200$ $S(2) = 150$						
Model	CPVC		EMVC		SEMV	
	$\alpha=1$	$\alpha=2$	$\alpha=1$	$\alpha=2$	$\alpha=1$	$\alpha=2$
X_1^α	80	40	88.6	44.3	78.7	39.3
X_2^α	30	10	24.7	8.2	30.5	10.2
X_3^α	24	16	20.5	13.7	24.7	16.4
X_4^α	25	12.4	22.1	11.1	25.2	12.6
X_5^α	37.5	25	34.4	22.9	38.2	25.5
X_6^α	27.4	33.5	24.2	29.6	27.7	33.8
X_7^α	44.6	74.4	38.9	64.9	45.7	76.1
X_8^α	23.7	20.8	15.5	13.5	25.2	22.1
X_9^α	38.1	22.9	25.9	15.5	40.2	24.0
X_{10}^α	31.2	18.8	37.2	22.3	30.4	18.3
Multi-plier μ	1.81		1.65		1.84	

$$\text{EMVC}: \quad \begin{aligned} X_8^{\alpha} &= \bar{X}_8^{\alpha} \exp(-v_4 + v_3) \\ X_9^{\alpha} &= \bar{X}_9 \exp(-v_3 + v_4) \end{aligned} \qquad \alpha = 1,2$$

$$\text{SEMVC}: \quad \begin{aligned} X_8^{\alpha} &= \frac{S_*}{\bar{S}_*} \bar{X}_8^{\alpha} \exp(-\hat{v}_4 + \hat{v}_3) \\ X_9^{\alpha} &= \frac{S_*}{\bar{S}_*} \bar{X}_9^{\alpha} \exp(-\hat{v}_3 + \hat{v}_4) \end{aligned} \qquad \alpha = 1,2$$

for which we obtain:

$$\text{EMVC}: \quad X_8^{\alpha} X_9^{\alpha} = \bar{X}_8^{\alpha} \bar{X}_9^{\alpha} \qquad \alpha = 1,2$$

$$\text{SEMVC}: \quad X_8^{\alpha} X_9^{\alpha} = \frac{S_*^2}{\bar{S}_*^2} \bar{X}_8^{\alpha} \bar{X}_9^{\alpha} \qquad \alpha = 1,2.$$

For the EMVC model the product of flows on 'internal loops' is a constant and is 'locked into' its base value. One flow can therefore only increase at the expense of the other, even if all supply components are doubled![+] As all flows are mutually dependent in the programming models such behaviour will tend to distort the results for the whole system. This feature is not present in the SEMVC model because the flows have, by construction, been linked to external conditions in the SD-system through the relationships (79) - (84). It is not surprising then that agreement between the CPVC and SEMVC results are considerably closer both for individual flow components and for the multiplier μ.

+ The same would have been true if intra-state transitions (diagramatically represented by "bubbles" at nodes) had been included. In the EMVC model these flows would simply have been equated to their base values, while in the SEMVC model they would have been scaled by $S_*/\bar{S}_*$.

This behaviour is even more apparent in tests TII in which $S(1)$ is doubled and $S(2)$ increased by 25%. For the EMVC model a general deterioration is noted in all components and in X_8 and X_9 in particular where the discrepancy has leapt to as much as 50%. The multiplier μ (EMVC) also shows a significant, and expected, dependence on $\underline{S}$ (see Section 4). The results for the SEMVC model are, in contrast, very close to the CPVC model reference set, the deviation in component values being typically 2 - 3% rising to 4 - 6% on the internal loop. To within round off errors the multipliers μ(CPVC) and μ(SEMVC) retain their constant values in the different tests.

These general features of the comparison between the three models are typical of the whole series of tests on the ten networks. It was found that for $S(1)$ and $S(2)$ rather close to their base values $\overline{\underline{S}}$, the EMVC model gave reasonable agreement to the CPVC reference system. The discrepancy increased however the greater the deviation of $\underline{S}$ from $\overline{\underline{S}}$, with the distorting influence of internal loops being particularly apparent. In all cases considered, however, the agreement between the flow components and multiplier μ for the SEMVC and the CPVC models was found to be impressive. This leads us to conclude, that for SD-systems the SEMVC model, derived from a mathematical programme, closely approximates the CPVC model computed on constant proportion assumptions.

On the issue of numerical solution we would add the following points. Firstly, the problem of solving the unconstrained dual offers no difficulties in principle or practice. Several computer codes are available for this procedure, and also for the modifications introduced in Section 6. It is unnecessary to contemplate solving the much larger and constrained primal programme. Secondly, experience with a similar numerical approach for solving Entropy Maximising allocation models (Champernowne *et al.* (6), to which we refer for further computational details) has revealed that there is no difficulty in extending this method to very large systems. For the SD-problems defined above there will be no more than 2N dual variables.

6. Towards More Complex Models: the Incorporation of Constraints

We now turn to the issue of incorporating constraint information in the formulation and solution of the vacancy chain models. The general problem is as follows: a model - say the CPVC - has been constructed and its predictions found to violate one or more constraints which reflect forecasting assumptions and/or policy considerations. It is therefore required to modify the solution to be consistent with these constraints. In developing the programming approach we are not suggesting that it is impossible to modify the CPVC solution, indeed in some cases appropriate measures may readily present themselves. For many interesting cases however the supplementation of a CPVC model by constraints may present severe problems in algorithm design and numerical solution. Because the programming framework allows the addition of constraints in a straightforward manner, the existence and characteristics of the model solution together with algorithms to provide it may be readily examined within this approach.

We develop the methodology in two parts according to the complexity of the constraints. In the first case we confront those planning and forecasting contexts which involve the addition of bounds and inequalities to be superimposed on the vacancy chain model, while in the second we consider the incorporation of general linear constraints. After presenting the characteristics of the model we briefly discuss application contexts.

Bounds and Inequalities on Flow Variables

In the models presented in Sections 3-5 we were concerned with excess demand or ED- systems in which the equality constraints in the stock-flow equations were imposed on the model solution. We shall now formulate a model which is characterised by general inequalities on both the demand and supply conditions expressed in Equations (3) – (6). In addition we shall impose a set of lower and upper bounds on a subset F of the flow components X_λ, $\lambda \varepsilon F$. That is

$$(83) \qquad X_\lambda \leq X_\lambda^U \qquad \lambda \varepsilon F^U$$

$$(84) \qquad X_\lambda \geq X_\lambda^L \qquad \lambda \varepsilon F^L$$

This general formulation will be used later to discuss a set of special cases and various planning and forecasting contexts.

Adopting the general model building approach of Section 4, the Lagrangian for this problem may be written

$$(85) \qquad \mathcal{L}(\underline{x}, \underline{\xi}^1, \underline{\xi}^2, \underline{\rho}^1, \underline{\rho}^2, \underline{\phi}^U, \underline{\phi}^L) = Z(\underline{x})$$

$$+ \sum_{\underline{\beta}\varepsilon B_1} \xi^1(\underline{\beta}) \left[s_1(\underline{\beta}) - \sum_{\underline{\alpha}\varepsilon A_1} x(\underline{\alpha} : *, \underline{\beta}) - \sum_{\underline{\alpha}\varepsilon A_2} \sum_{\underline{\beta}'\varepsilon B_2} x(\underline{\alpha} : \underline{\beta}', \underline{\beta}) \right]$$

$$+ \sum_{\underline{\beta}\varepsilon B_2} \xi^2(\underline{\beta}) \left[s_2(\underline{\beta}) + \sum_{\underline{\alpha}\varepsilon A_2} \sum_{\underline{\beta}'\varepsilon B_1, B_2} x(\underline{\alpha} : \underline{\beta}, \underline{\beta}') \right.$$

$$\left. - \sum_{\underline{\alpha}\varepsilon A_1} x(\underline{\alpha} : *, \underline{\beta}) - \sum_{\underline{\alpha}\varepsilon A_2} \sum_{\underline{\beta}'\varepsilon B_2} x(\underline{\alpha} : \underline{\beta}', \underline{\beta}) \right]$$

$$+ \sum_{\underline{\alpha}\varepsilon A_1} \rho^1(\underline{\alpha}) \left[d_1(\underline{\alpha}) - \sum_{\underline{\beta}\varepsilon B_1, B_2} x(\underline{\alpha} : *, \underline{\beta}) \right]$$

$$+ \sum_{\underline{a} \in A_2} \rho^2(\underline{a}) \left[d_2(\underline{a}) - \sum_{\underline{\beta} \in B_1 \cup B_2} \sum_{\underline{\beta}' \in B_2} x(\underline{a}; \underline{\beta}', \underline{\beta}) \right]$$

$$+ \sum_{\lambda \in F^U} \phi_\lambda^U \left[x_\lambda^U - x_\lambda \right] - \sum_{\lambda \in F^L} \phi_\lambda^L \left[x_\lambda^L - x_\lambda \right]$$

in which the multipliers $\underline{\xi}^1$, $\underline{\xi}^2$, $\underline{\rho}^1$, $\underline{\rho}^2$, $\underline{\phi}^U$, $\underline{\phi}^L$ are associated with the constraint sets (3), (4), (5), (6), (83) and (84) respectively. In terms of the Lagrangian $\mathcal{L}$, the optimality conditions are written as follows

(86) $$\frac{\partial \mathcal{L}}{\partial x_\lambda} = 0 \qquad \text{for all components } \lambda$$

(87) $$\xi^i(\underline{\beta}) . \frac{\partial \mathcal{L}}{\partial \xi^i(\underline{\beta})} = 0; \quad \frac{\partial \mathcal{L}}{\partial \xi^i(\underline{\beta})} \geqslant 0; \qquad \underline{\beta} \in B_i,\ i = 1,2$$

(88) $$\rho^j(\underline{a}) . \frac{\partial \mathcal{L}}{\partial \rho^j(\underline{a})} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \rho^j(\underline{a})} \geqslant 0 \qquad \underline{a} \in A_j,\ j = 1,2$$

(89) $$\phi_\lambda^U \quad \frac{\partial \mathcal{L}}{\partial \phi_\lambda^U} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \phi_\lambda^U} \geqslant 0 \qquad \lambda \in F^U$$

(90) $$\phi_\lambda^L \quad \frac{\partial \mathcal{L}}{\partial \phi_\lambda^L} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \phi_\lambda^L} \geqslant 0 \qquad \lambda \in F^L$$

together with the following bounds on the multipliers

(91) $$\underline{\xi} \geqslant 0; \quad \underline{\rho} \geqslant 0; \quad \underline{\phi}^U \geqslant 0; \quad \underline{\phi}^L \geqslant 0.$$

The complementary slackness conditions in Equations (87) - (90) provide the usual requirement that a multiplier will vanish if the corresponding constraint is slack, and will take up positive values only if the constraint binds. The expression (86) generates the following relationship between $\underline{x}$ and the multipliers

$$(92) \quad x(\underline{\alpha} : *, \underline{\beta}) = \overline{x}(\underline{\alpha} : *, \underline{\beta}) \exp \left[-\varepsilon^{i}(\underline{\beta}) - \rho^{1}(\underline{\alpha}) - \phi^{U}(\underline{\alpha} : *, \underline{\beta}) + \phi^{L}(\underline{\alpha} : *, \underline{\beta}) \right]$$

$$\underline{\alpha} \varepsilon A_1 \quad \underline{\beta} \varepsilon B_i, \ i = 1,2.$$

$$(93) \quad x(\underline{\alpha} : \underline{\beta}', \underline{\beta}) = \overline{x}(\underline{\alpha} : \underline{\beta}', \underline{\beta}) \exp \left[-\varepsilon^{i}(\underline{\beta}) + \varepsilon^{2}(\underline{\beta}') - \rho^{2}(\underline{\alpha}) \right.$$

$$\left. - \phi^{U}(\underline{\alpha} : \underline{\beta}', \underline{\beta}) + \phi^{L}(\underline{\alpha} : \underline{\beta}', \underline{\beta}) \right] \ \underline{\alpha} \varepsilon A_2 ; \underline{\beta}' \varepsilon B_2 \ \underline{\beta} \varepsilon B_i,$$

$$i = 1,2.$$

In order to provide a numerical solution it is again appropriate to manipulate the dual programme which may be expressed as follows:

$$(94) \min_{\underline{\varepsilon}, \underline{\varepsilon}, \underline{\rho}} Z^{*} = \left[\sum_{\underline{\alpha} \varepsilon A_1} \sum_{\substack{\underline{\beta} \varepsilon B_i \\ i = 1,2}} \overline{x}(\underline{\alpha} : *, \underline{\beta}) \exp \left(-\varepsilon^{i}(\underline{\beta}) - \rho^{1}(\underline{\alpha}) - \phi^{U}(\underline{\alpha} : *, \underline{\beta}) \right. \right.$$

$$\left. + \phi^{L}(\underline{\alpha} : *, \underline{\beta}) \right)$$

$$+ \sum_{\underline{\alpha} \varepsilon A_2} \sum_{\underline{\beta}' \varepsilon B_2} \sum_{\substack{\underline{\beta} \varepsilon B_i \\ i = 1,2}} \overline{x}(\underline{\alpha} : \underline{\beta}', \underline{\beta}) \exp \left(-\varepsilon^{i}(\underline{\beta}) + \varepsilon^{2}(\underline{\beta}') - \rho^{2}(\underline{\alpha}) \right.$$

$$\left. - \phi^{U}(\underline{\alpha} : \underline{\beta}', \underline{\beta}) + \phi^{L}(\underline{\alpha} : \underline{\beta}', \underline{\beta}) \right)$$

$$+ \sum_{i=1}^{2} \sum_{\underline{\beta} \in B_i} \xi^i(\underline{\beta})\, s_i(\underline{\beta}) + \sum_{j=1}^{2} \sum_{\underline{\alpha} \in A_j} \rho^j(\underline{\alpha})\, d_j(\underline{\alpha}) + \sum_{\lambda \in F^U} \phi^U_\lambda\, x^U_\lambda - \sum_{\lambda \in F^L} \phi^L_\lambda\, x^L_\lambda \Bigg]$$

subject to

(95) $\quad \underline{\xi} \geqslant 0; \quad \underline{\rho} \geqslant 0; \quad \underline{\phi}^U \geqslant 0; \quad \underline{\phi}^L \geqslant 0.$

Thus the effect of the inequality constraints and the additional bounds on the variables is to generate a multivariate minimisation problem subject only to the non-negativity conditions on the dual variables.

Again the dual programme and the formal model solution may be written down with ease if reference is made to the corresponding network representing the problem, as we discussed in Section 4. If demand is considered to 'enter' the network through demand 'nodes', $n^j_{\underline{\alpha}}$, then quantities $-\rho^j(\underline{\alpha})$ are added to the exponent of the appropriate flow variable. Furthermore, $-\phi^U_\lambda$ and $+\phi^L_\lambda$ are added if the link flow is constrained by upper and lower bounds respectively. In shorthand the dual becomes

(96)
$$\min_{\{\text{dual variables}\}} \Bigg[\sum_{\substack{\text{all links}\\ \lambda}} x_\lambda(\underline{\xi},\underline{\rho},\underline{\phi}) + \sum_{\substack{\text{all supply}\\ \text{nodes}}} \xi_n\, s_n + \sum_{\substack{\text{all demand}\\ \text{nodes}}} \rho_m\, d_m + \sum_{\lambda \in F^U} \phi_\lambda\, x^U_\lambda - \sum_{\lambda \in F^L} \phi_\lambda\, x^L_\lambda \Bigg]$$

subject to non-negativity constraints on all dual variables.

We have expressed the above programmes in scaled form, that is by dividing the variables $\bar{T}$, S, D, X^U, X^L by a quantity X_* in order to endow the resultant model with appropriate properties as discussed above. In order to extract the flows X_λ we must therefore write

$$(97) \qquad X_\lambda = X_* x_\lambda \qquad \text{for all } \lambda$$

The appropriate scaling factor X_* will in general vary with context. Where there is reason to believe that the system is characterised by excess demand and the mobility is limited by available supply (together with any additional constraints, such as credit restrictions in the housing market) then the appropriate value for X_* is S_* as before. Although we will concentrate on SD-vacancy chain models in this paper, it is well to consider those demand driven (Markov) systems characterised by 'flexible' supply. These are particularly relevant to planning contexts in which it is sought to manipulate the supply stock to meet projected demand. Examples may be found in housing systems in which 'short-fall' analysis is an important aspect of the strategic planning process; and in some manpower systems in which promotion policies operate (see a discussion of the Kent model, [32]). In this case, all the dual variables $\underline{\mu}$ and $\underline{\phi}$ are put equal to zero, and the unconstrained minimisation problem is solved as before, in this case for the set of variables $\underline{\rho}$. The appropriate demand driven model is then extracted from Equations (92)-(93) and (97) with X_* equal to D_* the total projected demand.

More typically in short-term analyses in both labour and housing systems will be the assumption of excess demand with the possibility of demand "bottlenecks" associated with certain $\underline{s}$-components. By examining the unconstrained case with all $\underline{p}$ variables equal to zero, such bottlenecks may easily be identified. The incorporation of limiting demand components may then readily be effected by requiring that the appropriate dual attains a non-zero (and non-negative) value.

The imposition of bounds and inequality constraints on the flow variables may be used in certain circumstances to investigate the effects of preferential policies or reflect goals of a planning department (even though the relevant policy instruments are not themselves explicitly considered). Thus in public housing systems it is usual to reflect need of entry in the form of priority categories which might be incorporated into the $\underline{g}$-classification. Constraints may then be imposed to ensure the entry of maximum or minimum proportions of numbers in each class. In a similar way in the labour market bounds may be used to investigate different conditions relating to the entry into the labour market or training schemes by school leavers, and of early retirement schemes. The effect of these bounds on both the total mobility and on the mobility of different $\underline{g}$ groups (measured by the number of moves) would provide useful information on the effect of policy measures on the dynamics of the system.

In the above model formulation the slack variables associated with the demand and supply sides were determined through the optimisation process and were functions of the quantities $\underline{X}$, $\underline{S}$, and $\underline{D}$. It may be considered desirable to incorporate in the model building procedure information of the slack variables which is derived from a previous time period. For example, in the labour market information on vacancies and unemployment, and in the housing system counts of vacant dwellings, can be used to play the same role as the quantity X in the model building procedure.

Thus, if the slack variables in the demand and supply constraints are contained in the vectors $\underline{U}$ and $\underline{V}$ with components U_n; n=1...N, V_m; m=1...M, then the quantities

$$\sum_n U_n \ln \frac{U_n}{\bar{U}_n} \quad \text{and} \quad \sum_m V_m \ln \frac{V_m}{\bar{V}_m}$$

may be added to the objective function and $\underline{U}$ and $\underline{V}$ determined endogenously. An analogous treatment of slack variables in allocation models may be found in the work of Snickars and Weibull (25). In scaled models the quantities V and U would be divided by $X_{..}$ in the usual way.

The Incorporation of Linear Constraints

We turn finally to those cases in which the constraints are expressible by the relation

(98) $$f(\underline{X}) \leq 0$$

and we shall concentrate on those linear functions of the form

(99) $$\sum_{\lambda \in G_q} c_\lambda^q X_\lambda \leq C_q \qquad q=1\ldots Q.$$

The Q linear constraints are formed from combinations of flow variables X_λ belonging to the Q subsets G_q ; q=1...Q, which may be associated with some or all of the various categories making up the $\underline{\alpha}$ and $\underline{\beta}$ vectors.

We shall consider the effect of these constraints on excess demand (ED) systems in which case we must now entertain the possibility of slack variables appearing in Equations (3) and (4). The dual programme for this problem may be written in the following form

$$(100) \quad \min_{\underline{\xi},\, \underline{\gamma}} \; Z^{*}(\underline{\xi}, \underline{\gamma}) = \sum_{\underline{\alpha} \in A_1} \; \sum_{\substack{\underline{\beta} \in B_i \\ i=1,2}} \overline{X}(\underline{\alpha} : *, \underline{\beta}) \exp\left[-\xi^{i}(\underline{\beta}) - \sum_{q=1}^{Q} \gamma_q \, c_{\lambda}^{q} \, \delta(\lambda \mid \underline{\alpha}, *, \underline{\beta}) \right]$$

$$+ \sum_{\underline{\alpha} \in A_2} \; \sum_{\underline{\beta}' \in B_2} \; \sum_{\substack{\underline{\beta} \in B_i \\ i=1,2}} \overline{X}(\underline{\alpha} : \underline{\beta}', \underline{\beta}) \exp\left[-\xi^{i}(\underline{\beta}) + \xi^{2}(\underline{\beta}') - \sum_{q=1}^{Q} \gamma_q \, c_{\lambda}^{q} \, \delta(\lambda \mid \underline{\alpha} : \underline{\beta}', \underline{\beta}) \right]$$

$$+ \sum_{i=1,2} \; \sum_{\underline{\beta} \in B_i} \xi^{i}(\underline{\beta}) \, S_i(\underline{\beta}) + \sum_{q=1}^{Q} \gamma_q \, C_q$$

subject to

$$(101) \qquad \underline{\gamma} \geq 0 \qquad \underline{\xi} \geq 0$$

The delta terms defined by

$$(102) \qquad \delta(\lambda \mid w) = 1 \text{ if } \lambda = w$$
$$\qquad\qquad = 0 \text{ otherwise}$$

allow contributions to the flow vectors to be made from the dual variables γ_q, when λ is equal to the triplet of vectors w which takes the form $(\underline{\alpha} : \underline{\beta}', \underline{\beta})$ and $(\underline{\alpha} : *, \underline{\beta})$. The formal solution of the programme is now expressed

$$(103) \quad X(\underline{\alpha}: *,\underline{\beta}) = \frac{S_*}{\bar{S}_*}\ \bar{X}(\underline{\alpha}: *,\underline{\beta}) \exp\left[-\xi^1(\underline{\beta}) - \sum_{q=1}^{Q} \gamma_q\, c_\lambda^q\, \delta(\lambda \mid \underline{\alpha}: *,\underline{\beta})\right]$$

$$\underline{\alpha} \in A_1;\quad \underline{\beta} \in B_i,\quad i=1,2.$$

$$(104) \quad X(\underline{\alpha}: \underline{\beta}',\underline{\beta}) = \frac{S_*}{\bar{S}_*}\ \bar{X}(\underline{\alpha}: \underline{\beta}',\underline{\beta}) \exp\left[-\xi^1(\underline{\beta}) + \xi^2(\underline{\beta}') - \sum_{q=1}^{Q} \gamma_q\, c_\lambda^q\, \delta(\lambda \mid \underline{\alpha}: \underline{\beta}',\underline{\beta})\right]$$

$$\underline{\alpha} \in A_2;\quad \underline{\beta} \in B_i,\quad i=1,2;\quad \underline{\beta}' \in B_2$$

The dual programme once again is simply a multivariate convex minimisation problem subject to non-negativity bounds on the dual variables $\underline{\gamma}$ and $\underline{\xi}$. Its solution does not pose any difficulties. For saturated systems in which the constraints (99) are made to "bite", the problem is even more straightforward as the variables $\underline{\gamma}$ are now unbounded.

If the symbol ℓ is used to represent the conditions in the Q constraints (99) as specified, for example, by the values of the coefficients $\underline{c}$ and C, then of particular interest in the investigation of the mobility relationship

$$(105) \qquad \frac{\mu(\ell)}{\mu(\infty)} = \frac{\sum_\lambda X_\lambda(\ell)}{\sum_\lambda X_\lambda(\infty)}$$

in which ∞ is used to indicate the absence of the set (99). Solution of the dual programme (100) - (101) provides a very straightforward means of investigating this expression.

We conclude this section with a brief reference to an application context which we take from the housing system. In their recent paper in Urban Studies (7) Curry and Thomas note the lack of detailed study given to the effects of lending (credit allocation) policies by Building Societies or Financial Institutions on mobility in the housing system. The models developed in this paper may address precisely this problem and we sketch an approach to this problem drawing on the above model which incorporates linear constraints. This model framework provides a generalisation of that presented by Curry and Thomas (7).

In a housing system a large majority of the moves are made possible with the aid of finance from a Building Society, Bank or/and Insurance Company and it is well known that a modification to the scale of borrowing either through the amount of money available or through the cost of credit will affect the mobility of all or some groups of household. Likewise a modification in the conditions or policy of lending (whether these are explicit or implicit) will similarly affect group and total mobility. Credit allocation policies may relate to a number of the attributes of the highly heterogeneous group of households and stock of dwellings. For example, specific policy considerations may involve the following dimensions of variability : the distinction between first time and exchange buyers; different income ranges; the price, age and type of dwelling among others. That is, policy considerations at the micro-level will result in a certain allocation of the total amount of credit between different types of individuals and dwellings. This allocation may be expressed in a set of equation of the form (99) in which C_g is the total money available to the subgroup g (of households and/or dwellings) and $C_{\underline{\alpha}}^{\beta} = C(\underline{\alpha} : \beta, \beta')$ is the average amount required by a household of type $\underline{\alpha}$ moving between dwellings of type $\underline{\beta}$ and $\underline{\beta}'$.

For example, $\frac{C_1}{C_1 + C_2}$ may be the total proportion of funds available for first-time buyers, requiring that the following condition be imposed on the resultant flow vector

$$(106) \qquad \sum_{\underline{\alpha} \in A_1} \; \sum_{\underline{\beta} \in B_1, B_2} X(\underline{\alpha} : *, \underline{\beta}) \; C(\underline{\alpha} : *, \underline{\beta}) = C_1,$$

in which $C(\underline{\alpha} : *, \underline{\beta})$ is the mean amount of credit required by a new entrant buying a dwelling of type $\underline{\beta}$.+ Other linear constraints may be used to reflect other characteristics of the policy allocating funds to different $\underline{\alpha}$ and $\underline{\beta}$ classes.

The programming framework developed above and specifically the model expressed by Equations (103) - (104) may now readily be adopted to investigate the effect of a specific policy or any change from the existing strategy which is reflected in the base flow vector $\overline{\underline{X}}$. Note that in the Entropy Maximising approach it is not of prime concern exactly what policy is operative in the time slice(t-T, t) - the quantity $\overline{\underline{X}}$ simply reflects information which must be updated according to the *known* conditions which hold in the time slice (t, t+T).

The model may be embellished in many ways to explore any potential conflicts between mobility and equity issues, most obviously by the explicit consideration of asset accumulation and recycling of credit in which the quantity C - the total amount of credit available - is expressed as a

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+ Any significant differences in the available deposit may be accommodated in the market segmentation of $\underline{\alpha}$.

function of the flow vector itself. These are, however, outside the scope of this paper and will be discussed elsewhere.

7. Directions for Further Research

We have examined the characteristics of alternative vacancy chain models and have proposed a mathematical programming framework for model generation in order to analyse forecasting and planning contexts which involve the specification of constraint relations. It is suggested that the approach will facilitate both the theoretical and practical examination of an important class of problems which have hitherto attracted little formal attention.

The study of vacancy chain representations leaves a number of practical issues to be resolved, notably the determination of appropriate levels of market segmentation, and the formation of the resultant base flow matrix possibly with the aid of matrix "infilling" techniques (Feinberg (8)) obviating the need for a comprehensive mover survey. We conclude, however, with comments on what we consider to be a significant and fruitful *theoretical* area of investigation, namely the development of an alternative basis for the vacancy chain model derived from economic principles. The models discussed in this paper are forecasting devices in which the matrices $\underline{X}$ are updated according to the revised set of conditions reflected in the demand and supply increments, and the *additional* constraints governing macroscopic system characteristics. The approach to model building is, as we discussed in Section 4, analogous to the information theoretic formulation of simple matching or allocation models. As such, it does not provide (nor does it seek to provide) a 'behavioural' explanation for the allocation process or a detailed interface with policy other than may be achieved through the direct imposition of constraints. For allocation models the incorporation of economic principles has provided a significant step to enhancing the understanding of exchange processes, although we would not claim that the corresponding models have been developed to a high degree of realism, particularly in the case of labour markets.

It is natural to enquire whether, for the housing system in particular, economic principles may be used to underpin vacancy chain/allocation models in the tradition of the work by Herbert and Stevens (14), Harris (13), Wilson (31), Senior (23), Kain et al. (10), Ames (1), Williams and Senior (28), Rothenberg (22) and others. In particular can the economic analysis of dispersion, competition and price dynamics be extended to the vacancy chain approach in a straightforward manner, and furthermore does an extremal principle exist which will allow the economic approach to be embedded within a mathematical programming framework? For simple allocation models it has been found that both Entropy Maximising and Utility Maximising approaches may be used to generate models with a similar analytic structure (see, for example, Williams and Wilson (29)). Can these extensions be provided for vacancy chain models, and, if so, do there exist shadow prices analogous to $\underline{r}$ embedded in a programming approach, which may be used as a basis for policy analysis, evaluation and design. We hope to report on these issues shortly.

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