WORKING PAPER 386

THE ROLE OF ATTRACTIVENESS FUNCTIONS THE DETERMINATION OF EQUILIBRIUM SOLUTIONS TO PRODUCTION-CONSTRAINED SPATIAL INTERACTION MODELS

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1. Introduction

In recent years much attention has focussed on the solution of the equilibrium version of the production-constrained spatial interaction model (eg. Harris and Wilson, 1978, Clarke, 1981, Clarke and Wilson, 1983, Rijk and Vorst, 1982, Harris, 1983).

In particular, a great deal of interest lies in the numerical experiments undertaken with these models that examine how the equilibrium solutions change due to changes in model parameters and data. Indeed a lot of the theoretical developments that have been made have arisen out of the results of numerical work.[1]

However there are times when the numerical results that are obtained do not seem to accord with the prior expectations of the model. The temptation, in this case, is often to reject the results as either spurious or sub-optimal and to try again. In this short paper we shall demonstrate that it is often the case that it is these unexpected results that can force us to rethink the original theory, refining it and thus provide an explanation for the results. We illustrate this with reference to the role of the attractiveness function in the equilibrium models.

2. The equilibrium version of the production-constrained spatial interaction model

The production-constrained spatial interaction model can be written as

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[1] For a discussion of the nature of this theoretical/experimental relationship see Clarke and Wilson (1984).

$$S_{ij} = A_i e_i P_i \, f(W_j) \, e^{-\beta c_{ij}} \tag{1}$$

$$A_i = \left[\sum_j f(W_j) \, e^{-\beta c_{ij}}\right]^{-1} \tag{2}$$

where

S_{ij} = flow of revenue from zone i to zone j

e_i = per capita expenditure on retail goods in i

P_i = Population of i

W_j = size of retail facilities in j

c_{ij} = cost of travel from i

A_i = a balancing factor, given by (2)

β = distance deterrence parameter

The Harris-Wilson (1978) conditions for an equilibrium solution are

$$\sum_i S_{ij} = D_j = k_j W_j, \quad \forall_j \tag{3}$$

where D_j is defined as zonal revenue and k_j is the unit cost of providing retail floorspace in j. In other words an equilibrium will be obtained when revenue balances costs of provision and any excess profit is competed out of the system.

The nature of the attractiveness function can take a number of forms. The two we consider in this paper are

$$\text{(a)} \quad f(W_j) = W_j^{\alpha} \tag{4}$$

$$\text{(b)} \quad f(W_j) = \delta/\left[1 + \theta e(-\varepsilon(W_j + d))\right] \tag{5}$$

Function (a) is the conventional attractiveness factor used by Lakshmanan-Hansen (1964) and others. It implies that attractiveness is not linearly proportional to size but that there may be economies or diseconomies of scale, depending on the value of α. Function (b) is taken from Poston and Wilson (1977) and is clearly a logistic function. In both cases, α, δ, θ, ε and d are model parameters that determine the precise nature of the function. Function (b) is explored in detail in Clarke, Clarke and Wilson (1984).

3. Mechanisms for change in the equilibrium solution

Harris and Wilson (1978) proposed a mechanism for changing equilibrium solutions in the case of functional form (a). These have been subsequently expanded and modified in Wilson and Clarke,(1979),Clarke(1981)and Clarke (1984). Two main conclusions have emerged. As α increases (α, greater than 1) the number of centres (ie. $W_j \neq 0$) decreases, as increasing benefit or attractiveness from size emerges. As β increases the number of centres increases with decreasing average size. The theory also tells us that for $\alpha < 1$, every centre should have some facilities, however small.

The Poston and Wilson (1977) argument relating to function (b) centred around the relationship between benefits of travelling to large centres, and the average cost of travelling to them. The general argument can be illustrated graphically.

In Figure 1 (a, b and c) three different situations are illustrated. In each case a plot of attractiveness or "benefit" against size of facility (u_1) and average cost of travel against size of facility (u_2) is given, along with a plot of $u_1 + u_2$, the net utility associated with facility size. As the aim is to maximise net utility the optimal facility size will occur at the maxima of $u_1 + u_2$. In Figure 1(a) with a low value of β this maximumoccurs at a large positive value of W_j. In Figure 1(b) as β increases a local maxima occurs at the origin, $W_j = 0$ but the global maximumremains at the positive upper value. However in Figure 1(c) β has achieved a sufficiently high value such that the global maximum now exists at the origin. Hence increasing β we would expect to move from a situation of few large centres to many small ones. (For further description of this model see Clarke, Clarke and Wilson 1984).

We now present some results relating to both these attractiveness functions where the equilibrium pattern of $\{W_j\}$ does not immediately accord with what we might expect.

4. Numerical experiments, unexpected results and interpretation

The two examples we focus on both relate to changing β values and the resultant equilibrium solutions obtained. In the first case the solutions are obtained from a 30 demand point, 900 supply point spatial system representing the Leeds Metropolitan District (see Clarke,1984,for details).

In the second case we use a hypothetical regular spatial grid system consisting of 729 (27 x 27) demand and supply points. This is shown in Figure 2. In both cases the Euclidean distance between demand and supply points forms the $\{c_{ij}\}$ matrix, e_iP_i is obtained from census data in case 1 and assumed uniform in case 2. The solution procedure used is as follows

Step 1. Set all the W_js equal to some initial starting value.

Step 2. Solve

$$\sum_i S_{ij} = \sum_i \left[e_i P_i f(W_j) \exp(-\beta c_{ij}) \;/\; \sum_j f(W_j) \exp(-\beta c_{ij}) \right]$$

Step 3. Let $D_j = \sum_i S_{ij}$

Step 4. If $D_j = kW_j, \forall_j,$

then one has an optimal solution. If not then set $W_j = (1/k)D_j$ and return to Step 2.

This procedure has proven fairly robust for solving a range of functional forms, $f(W_j)$, but care does need to be taken in the interpretation of results with respect to globally optimal solutions.

Case 1

In this example we examine the first functional form of the attractiveness function, $f(W_j) = W_j^{\alpha}$, and focus on behaviour when $\alpha < 1$. According to Harris and Wilson (1978) every zone should have some facilities present and this proves to be the case. In Table 1 we present three sets of results for successively smaller values of β. It should be pointed out that the zero entries in the matrix of $\{W_j\}$ values are not actually zero but less than 0.5. They are therefore rounded down to zero to provide a clearer presentation of the results.

The interesting observation however is that as β decreases the size of previously large centres diminishes and the size of small centres increases. This is not what we would have anticipated - as β decreases we would normally expect centre size to increase and large centres to grow and small centres to decline. This does happen when $\alpha > 1$ (see Clarke and Wilson, 1983). $\alpha < 1$ provides a special case, which can now be given a theoretical explanation.

First it is worth recapping on the role of β, - it can be argued that for high β, consumers will patronise local facilities. As β decreases the consumers become indifferent to specific locations but highly sensitive to the size of retail facilities when $\alpha > 1$ and consumer scale economies exist. For $\alpha < 1$ these scale economies no longer exist, as $\partial W_j^{\alpha} / \partial W_j < 1$ for $W_j > 1$. This implies that if centres grow their attractiveness grows at a slower rate, this rate diminishing with size (see Figure 3).

Attention can now be focussed on the behaviour of two zones under varying parameter conditions, in particular with β decreasing. We can observe from Figure 4 that for the same unit increase in W_j the smaller centre (W^2) has a substantial advantage in the amount of "attractiveness" (A-A') compared with the larger centre W^1 (B-B'). (Note we have exaggerated the slope of the line to illustrate the argument). Therefore, as β decreases and proximity to demand points is of less importance, zones with smaller amounts of W_j can grow at the expense of larger centres - as Table 1 (a, b and c) demonstrates.

The conclusion to be drawn from this particular example is that $\alpha < 1$ represents a rather unusual, but interesting case where the role of the β parameter is reversed. We could argue that in practice we would not be likely to encounter a situation where $\alpha < 1$ for this particular attractiveness function. An interesting extension of this argument can be found in Harris, Choukroun and Wilson (1981).

Case 2

We again focus on β change but in this case with the second attractiveness function. Figure 5 presents a set of results for the 729 zone system with β variation from 3.5 to 0.5. In the Figure 5.1 we can observe a very widespread distribution of retail facilities and again this accords with our expectation and previous results (Clarke and Wilson, 1983). However as we successively reduce the value of β, some rather unusual spatial structures emerge. This is best illustrated in the case of $\beta = 0.5$. With a more conventional attractiveness function (eg. $f(W_j) = W_j^{\alpha}$, $\alpha > 1$) we would expect to have just a single retail facility located in the central zone. So again an explanation is needed.

The explanation is again relatively straightforward, but very interesting. The shape of the attractiveness function was shown in Figure 1 and repeated in Figure 6. Above a certain point the slope of the curve flattens out towards the upper asymptote. The value at which this asymptote is reached is controlled by the parameter δ in Equation 5. Now for β change a similar argument applies as in Case 1. Above a certain point, q_j, $\delta f(W_j) / \delta W_j$ becomes very small compared with below q_j. Thus in effect the point q_j represents an upper threshold beyond which W_j is unlikely to grow. Hence other zones can pick up the remaining revenue, up to their own threshold value, q_j (q_j will vary by zone). This provides a simple explanation for Figure 5(h). We can also re-run the model for $\beta = 0.5$ but with different values of δ, which either raise or lower the upper threshold. Some examples of this are given in Figure 7. The effect is entirely as expected. For high δ we get just one centre, for low δ we get many centres, and with the highest value of W_j relatively low.

5. Conclusions

It is hoped that the above illustrations show the value of undertaking numerical experiments with these and other types of models. It should also be pointed out that the models often seem to be one step ahead of our explanations. To paraphrase Peter Gould (1981) we should spend more time "letting the equations speak for themselves".

(a)

$\beta = 0.35$

RESULTS FOR LEEDS 30 ZONE SHOPPING MODEL

0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	10.	4.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	1.	307.	128.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	4.	3.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	1.	2.	274.	5.	1.	2.	16.	175.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	1.	0.	0.	0.	1.	18.	7.	4.	13.	47.	55.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	30.	307.	1.	0.	0.	0.	3.	21.	28.	192.	35.	7.	3.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	5.	34.	1.	0.	0.	3.	10.	75.	31.	5.	1.	1.	18.	200.	1.	0.	0.	0.	0.	1.	1.	0.	0.	0.	0.
0.	0.	0.	0.	1.	0.	0.	1.	34.	83.	12.	4.	1.	0.	1.	3.	6.	2.	3.	4.	0.	1.	100.	131.	0.	0.	0.	0.
0.	0.	0.	0.	0.	2.	2.	4.	68.	47.	8.	2.	1.	1.	1.	2.	5.	4.	24.	318.	2.	2.	61.	59.	0.	0.	0.	0.
0.	1.	0.	0.	2.	154.	99.	11.	33.	42.	37.	74.	10.	109.	37.	31.	173.	20.	13.	3.	1.	1.	1.	0.	0.	0.	0.	0.
0.	1.	2.	5.	11.	35.	22.	9.	102.	99.	47.	98.	14.	35.	56.	88.	74.	65.	27.	2.	1.	3.	4.	0.	0.	0.	0.	0.
10.	284.	37.	240.	84.	17.	6.	3.	11.	19.	15.	15.	13.	17.	37.	194.	32.	142.	43.	2.	1.	34.	424.	0.	0.	0.	0.	0.
1.	19.	9.	31.	41.	98.	31.	1.	1.	13.	55.	20.	112.	97.	7.	3.	3.	3.	3.	1.	2.	15.	7.	0.	0.	0.	0.	0.
1.	0.	0.	0.	3.	139.	51.	1.	1.	13.	226.	18.	65.	56.	3.	1.	0.	2.	7.	4.	7.	53.	3.	0.	0.	0.	0.	0.
0.	0.	0.	1.	0.	1.	1.	0.	0.	1.	3.	5.	37.	72.	4.	1.	1.	8.	259.	19.	9.	229.	1.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	15.	107.	4.	3.	5.	6.	14.	1.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	3.	6.	0.	0.	0.	0.	1.	1.	14.	343.	2.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	1.	0.	76.	193.	0.	0.	1.	1.	0.	0.	2.	10.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	1.	1.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	6.	375.	4.	0.	0.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	1.	0.	0.	0.	0.	3.	312.	2.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

(b)

$\beta = 0.2$

RESULTS FOR LEEDS 30 ZONE SHOPPING MODEL

0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	1.	13.	9.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	1.	8.	142.	91.	4.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	3.	10.	34.	29.	5.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	1.	6.	13.	18.	16.	8.	2.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	1.	10.	76.	30.	22.	24.	35.	43.	2.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	1.	3.	2.	1.	3.	10.	34.	33.	32.	44.	55.	40.	4.	1.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.
0.	1.	0.	14.	73.	11.	4.	4.	8.	21.	37.	45.	81.	50.	26.	12.	4.	1.	0.	0.	0.	0.	0.	1.	0.	0.	0.	0.
0.	0.	1.	14.	51.	19.	8.	8.	13.	34.	42.	38.	27.	21.	18.	27.	38.	7.	3.	2.	1.	3.	3.	2.	0.	0.	0.	0.
0.	0.	1.	5.	15.	15.	12.	13.	30.	38.	29.	23.	17.	15.	17.	23.	24.	15.	15.	14.	8.	11.	52.	38.	1.	1.	0.	0.
1.	0.	1.	4.	11.	19.	21.	22.	42.	45.	29.	22.	18.	18.	21.	28.	31.	30.	52.	94.	23.	25.	69.	34.	1.	0.	0.	0.
0.	1.	4.	10.	24.	72.	48.	37.	46.	50.	48.	51.	35.	41.	57.	72.	104.	60.	55.	43.	31.	29.	33.	4.	0.	0.	0.	0.
1.	9.	19.	34.	49.	62.	51.	38.	63.	66.	56.	65.	45.	56.	72.	91.	90.	86.	64.	38.	35.	47.	33.	4.	0.	1.	0.	0.
9.	94.	67.	136.	104.	57.	40.	30.	37.	47.	48.	48.	48.	53.	60.	105.	68.	89.	64.	34.	41.	98.	150.	8.	0.	0.	0.	0.
3.	25.	31.	54.	63.	72.	40.	32.	22.	44.	71.	55.	80.	76.	41.	32.	39.	32.	33.	31.	40.	58.	34.	3.	0.	0.	0.	0.
0.	3.	4.	9.	18.	52.	41.	14.	14.	39.	104.	48.	63.	59.	31.	21.	30.	28.	37.	33.	39.	35.	18.	1.	0.	0.	0.	0.
0.	0.	1.	2.	4.	10.	13.	10.	10.	17.	24.	27.	43.	51.	27.	19.	20.	34.	78.	34.	25.	49.	3.	0.	0.	0.	0.	0.
0.	1.	0.	0.	1.	2.	3.	9.	10.	8.	7.	7.	20.	34.	32.	26.	31.	23.	18.	6.	2.	1.	0.	1.	0.	0.	0.	0.
0.	0.	0.	0.	1.	2.	6.	17.	23.	9.	3.	4.	6.	9.	12.	33.	97.	10.	3.	1.	0.	0.	1.	0.	0.	0.	0.	0.
0.	0.	0.	0.	1.	3.	8.	43.	59.	10.	4.	3.	3.	3.	4.	11.	14.	2.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	1.	12.	13.	0.	9.	9.	6.	5.	5.	3.	1.	1.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	3.	154.	29.	5.	3.	3.	4.	11.	18.	5.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	1.	19.	8.	1.	1.	1.	3.	23.	182.	13.	1.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	3.	14.	2.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	1.	0.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	1.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

(c)

$\beta = 0.05$

RESULTS FOR LEEDS 30 ZONE SHOPPING MODEL

0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	1.	8.	4.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	4.	4.
0.	0.	0.	0.	0.	0.	1.	0.	0.	1.	9.	39.	43.	3.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	1.	0.	0.	1.	3.	12.	31.	26.	8.	2.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	2.	8.	16.	23.	30.	11.	4.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	1.	2.	10.	36.	28.	27.	28.	30.	23.	4.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	1.	0.	1.	2.	3.	2.	5.	12.	28.	33.	35.	40.	39.	27.	8.	2.	1.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	4.	16.	8.	6.	7.	14.	25.	37.	44.	38.	42.	25.	13.	6.	2.	1.	0.	0.	0.	0.	0.	1.	0.	0.	0.
1.	0.	1.	7.	20.	16.	13.	14.	31.	31.	44.	44.	37.	31.	26.	27.	26.	10.	6.	4.	3.	3.	3.	1.	0.	1.	0.	0.
0.	0.	1.	6.	14.	18.	19.	22.	34.	42.	40.	37.	33.	30.	29.	30.	28.	31.	17.	17.	11.	12.	22.	12.	1.	0.	0.	0.
0.	0.	2.	4.	13.	25.	29.	32.	45.	49.	42.	38.	35.	35.	37.	40.	41.	38.	47.	59.	27.	24.	34.	15.	1.	0.	0.	0.
0.	2.	7.	16.	30.	60.	42.	48.	54.	57.	58.	61.	52.	68.	67.	77.	72.	67.	64.	55.	42.	35.	23.	7.	1.	0.	0.	0.
2.	9.	19.	34.	53.	43.	60.	52.	65.	47.	64.	69.	60.	68.	79.	91.	92.	89.	75.	55.	30.	30.	30.	6.	1.	0.	0.	0.
5.	34.	44.	84.	83.	65.	54.	47.	53.	58.	59.	61.	62.	47.	77.	98.	81.	91.	77.	55.	55.	74.	73.	6.	0.	0.	0.	0.
3.	15.	26.	46.	58.	65.	54.	38.	38.	52.	67.	61.	74.	74.	56.	50.	48.	50.	49.	44.	46.	46.	38.	3.	0.	0.	0.	0.
1.	3.	8.	13.	26.	47.	44.	30.	30.	45.	71.	52.	61.	59.	44.	34.	35.	40.	43.	37.	34.	34.	9.	1.	0.	0.	0.	0.
0.	1.	2.	5.	10.	18.	22.	21.	22.	28.	33.	35.	44.	48.	35.	29.	29.	34.	49.	28.	30.	22.	3.	0.	0.	0.	0.	0.
0.	0.	0.	1.	3.	7.	12.	16.	18.	16.	16.	17.	24.	31.	24.	25.	25.	17.	14.	4.	3.	1.	0.	0.	0.	0.	0.	0.
1.	0.	0.	1.	3.	6.	11.	20.	23.	15.	11.	11.	12.	13.	14.	21.	32.	8.	3.	1.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	1.	3.	7.	12.	27.	31.	14.	10.	8.	7.	6.	6.	9.	8.	3.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	2.	12.	15.	11.	11.	11.	9.	9.	8.	5.	3.	1.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	1.	0.	3.	48.	20.	7.	5.	5.	7.	11.	14.	6.	2.	1.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	1.	10.	7.	3.	3.	2.	4.	13.	39.	8.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	1.	0.	0.	0.	0.	0.	0.	1.	3.	3.	2.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

Table 1. Results from equilibrium model, $\alpha < 1$, varying β.

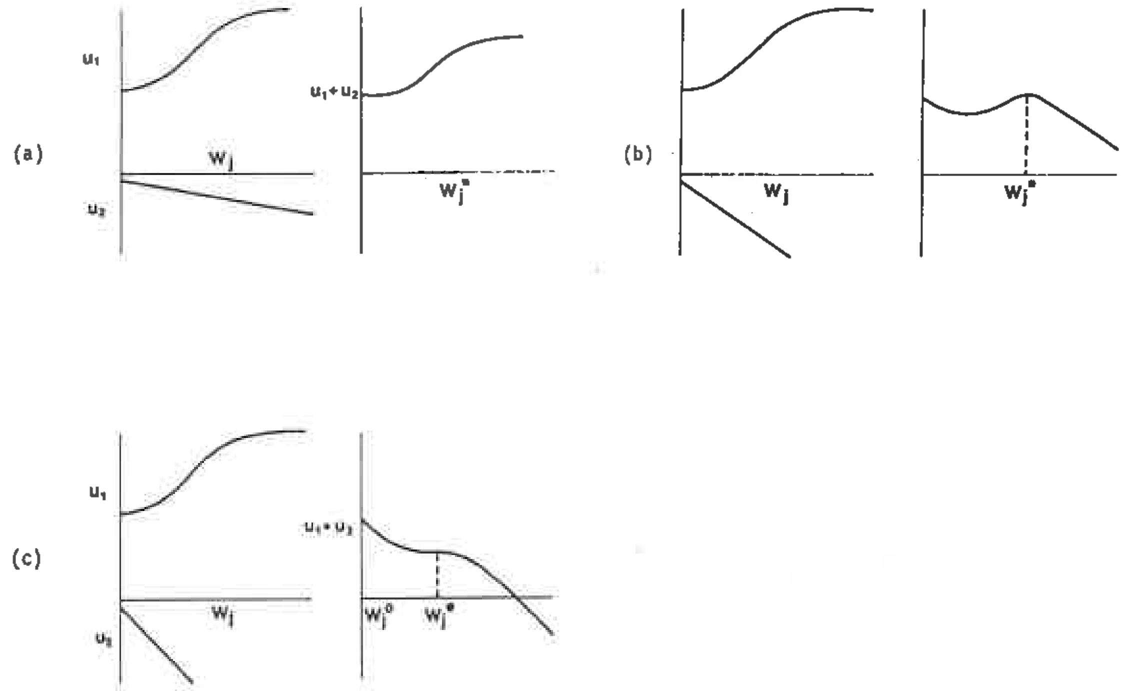
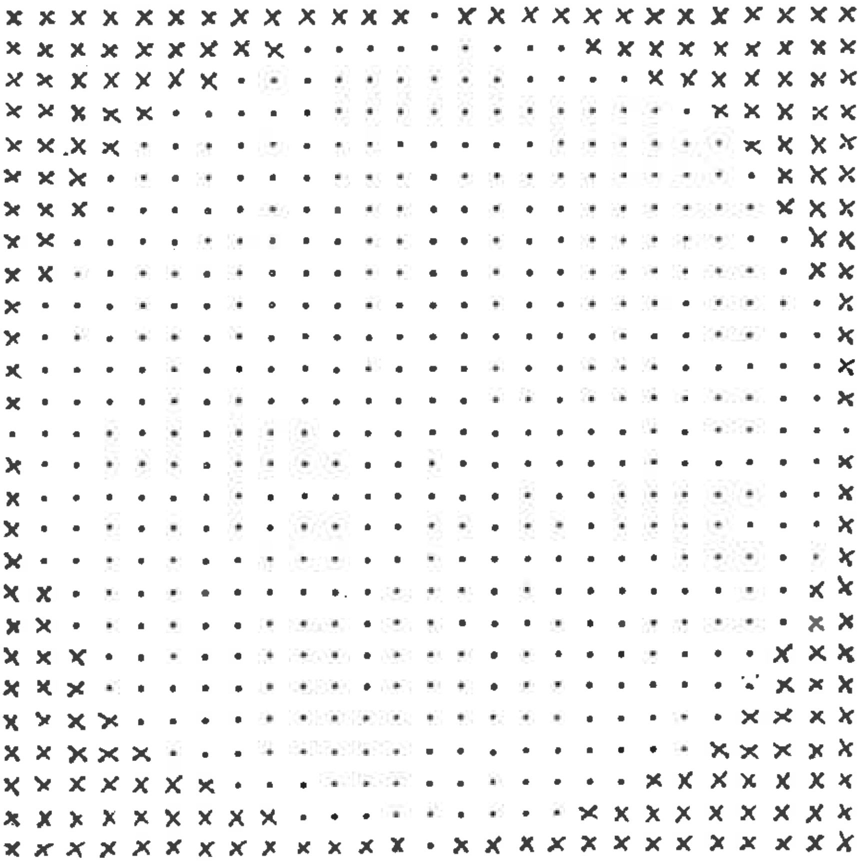


Figure 1. Graphical representation of changing equilibrium solution due to increasing a



X indicates exogenous zone

Figure 2. 729 zone spatial system

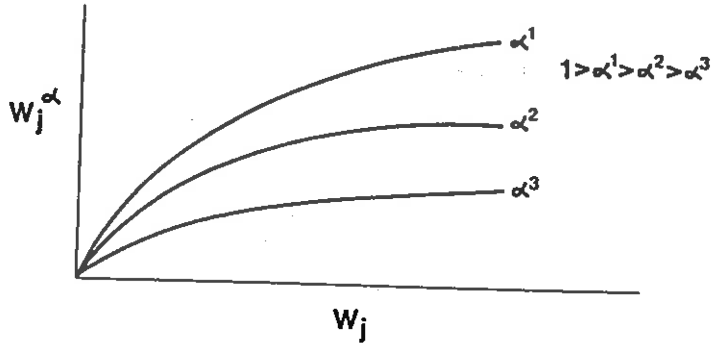


Figure 3. W_j^{α} - W_j plots for $\alpha < 1$

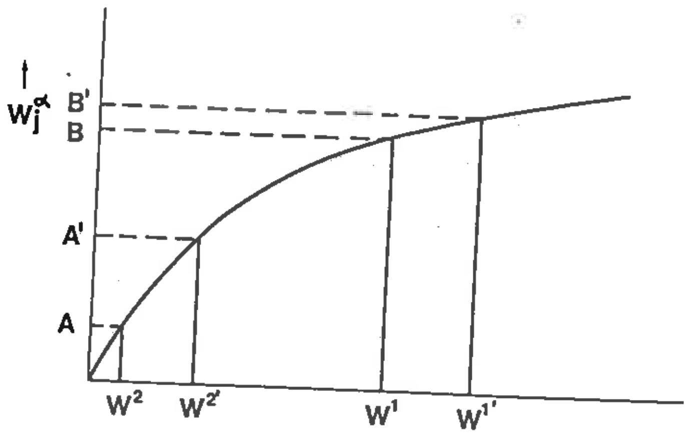


Figure 4. Comparative change in attractiveness for large and small centres, $\alpha < 1$

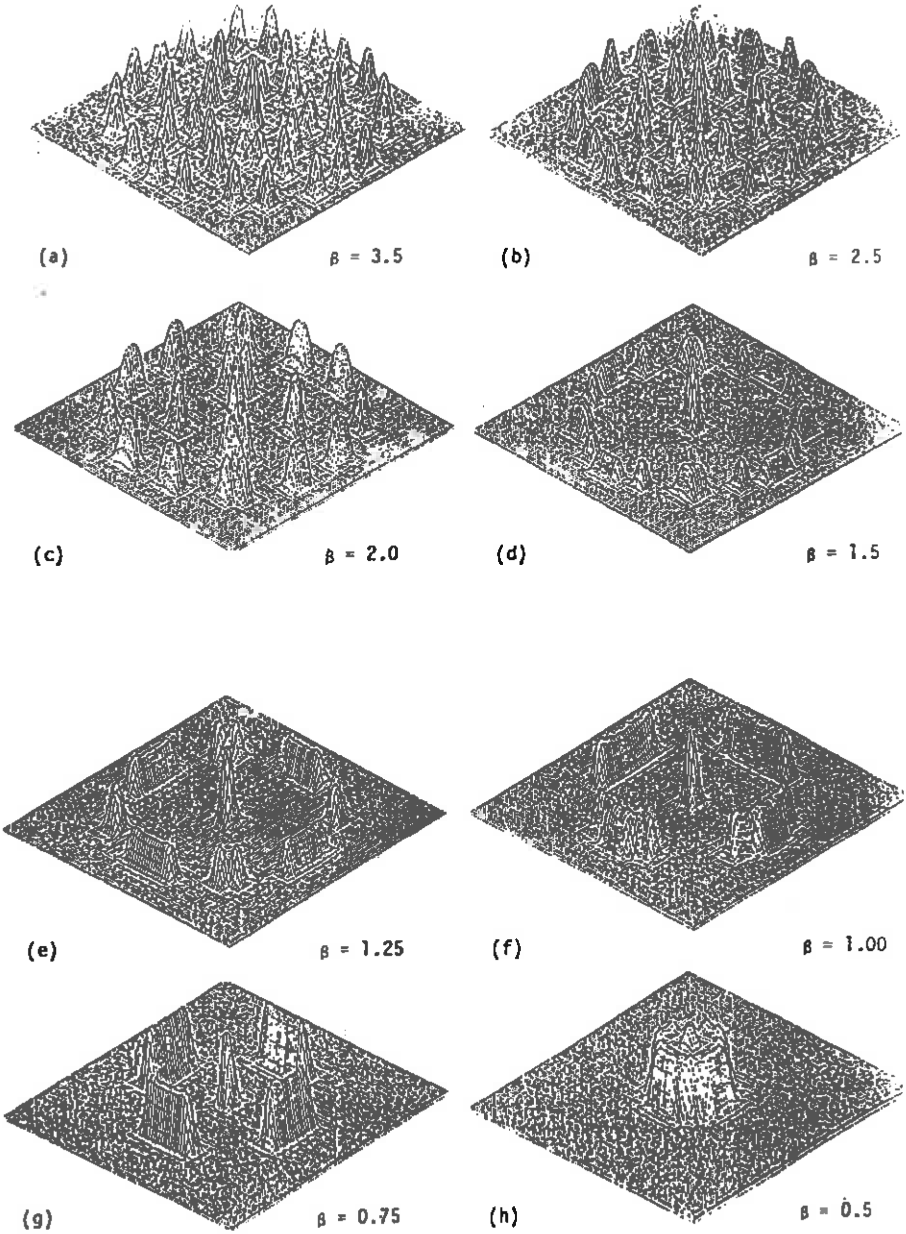


Figure 5. Decreasing values of β, δ = 10.

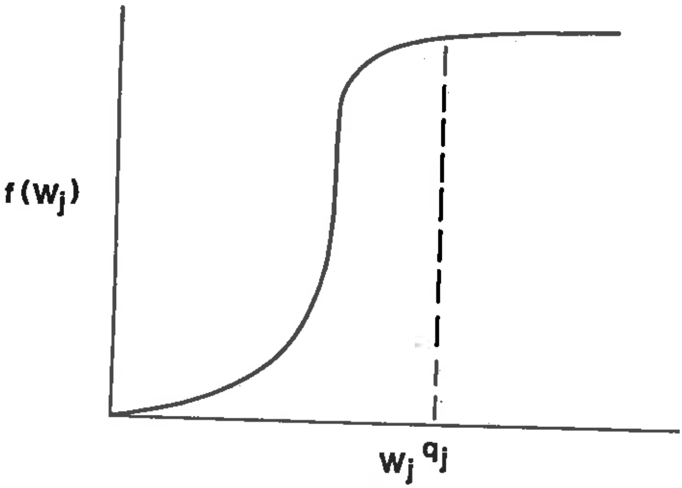
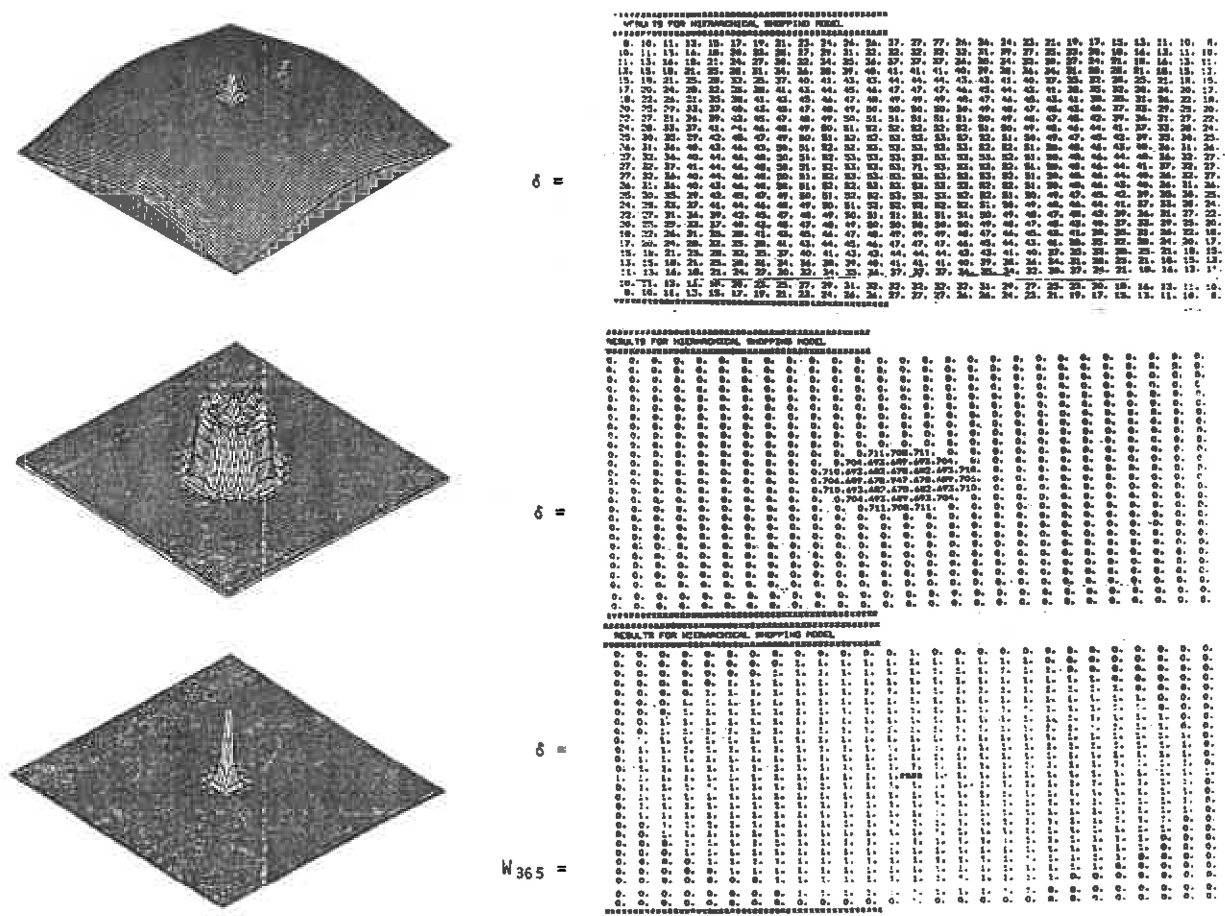


Figure 6. Logistic function and the upper threshold point q_j

Figure 7. Equilibrium model, Case 2, δ variation



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