Working paper

Relational models of disability: what technique should be used to estimate parameters?

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Abstract

This paper evaluates three different procedures for the estimation of parameters in a Brass relational model of disability. These are ordinary least squares regression (OLS), weighted least squares regression (WLS) and maximum likelihood estimation (ML). This contribution is valuable for two reasons. First, the method for estimating relational model parameters is an area of debate with examples in the literature that recommend each of the three approaches above for the estimation of mortality schedules. Second, the use of relational models to estimate disability schedules represents a new application which requires a solid grounding in terms of the best technique for parameter estimation.

The relational model of disability fitted here is based on Brass' relational model for mortality. The model combines data from two sources, using a schedule of limiting long term illness rates (2001 census) as the standard and adjusting this using two parameters to represent a disability schedule for a particular disability type (Health Survey for England 2000/01). This is useful because even at national level schedules of disability rates from the Health Survey for England show considerable sampling fluctuations particularly at the oldest ages which are smoothed by the model estimates.

Brass relational models are fitted using each of the three procedures for overall disability and locomotor (mobility) disability and for males and females. In addition to a visual comparison of model fit, two tests are used to evaluate the models; ratios between model residual sums of squares and comparison of the model and observed crude rates of disability.

The results show that ML consistently overestimates rates of disability at the oldest ages. ML has a significantly higher residual sum of squares than the OLS and WLS models thus explaining less of the variability in observed disability rates. The OLS and WLS model fitting procedures give very similar and good fits to the observed data.

The conclusion of this paper is to recommend WLS as the procedure for fitting relational models of disability. WLS gives a better fit to the observed data compared to ML and whilst OLS gives a comparable fit this approach is known to be less robust in a statistical sense.

Acknowledgements

I am very grateful to Professor Ian Plewis for his suggestions and advice on the model fitted here and also to Dr Paul Norman for his comments on this paper. The paper is much improved as a result of these inputs. I am grateful to the ESRC who funded this research through a postdoctoral fellowship (award number - PTA-026-27-2543). I take responsibility for any errors in this working paper.

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1. Introduction

1.1 Aim

This paper considers how a Brass relational model of disability should be fitted evaluating the three main approaches in the literature; ordinary least squares, weighted least squares and maximum likelihood. Brass relational models are fitted to estimate schedules of overall disability and mobility disability in England (Health Survey for England) using a schedule of limiting long term illness (census) as the standard. The findings of this working paper inform other research in which specific and more complicated relational models are developed to generate estimates of particular disability types both nationally (Marshall et al. 2012) and for sub-national areas (Marshall 2012).

1.2 Relational models of disability

Estimates and projections of sets of age-specific disability rates distinguishing disability type are essential for planning purposes with sub-national information important for the local provision of specialist services (Siegal 2002). Like other demographic characteristics, such as mortality, fertility and migration, disability rates tend to follow a strong age pattern with low rates across the younger ages and increasing rates through the middle and older ages (see figure 1). As with the estimation of deaths, births and migration, this age pattern can be exploited to generate estimates and projections of populations with a disability through the product of age-specific rates and information on population structure for given points in time.

It is often useful to represent a set of age-specific rates with a model curve that is defined by a smaller number of parameters. For example, fitting a curve to a schedule of rates can overcome the problem of missing rates at particular ages or can smooth fluctuations in rates that stem from sampling variability. If we are interested in projecting changes in schedules of disability rates over time, a salient exercise, especially given debates on population ageing and the implications of a compression or expansion of disability, then we can more easily capture such changes if we are comparing a small number of parameters rather than a large number of age-specific rates. A number of methods have been developed to graduate (or fit curves to) schedules of demographic characteristics. For example, mortality schedules in a particular population can be represented by a mathematical equation with suitably estimated parameter values or through the selection of a lifetable from a set of lifetables on the basis of a suitable mortality index (such as infant mortality).

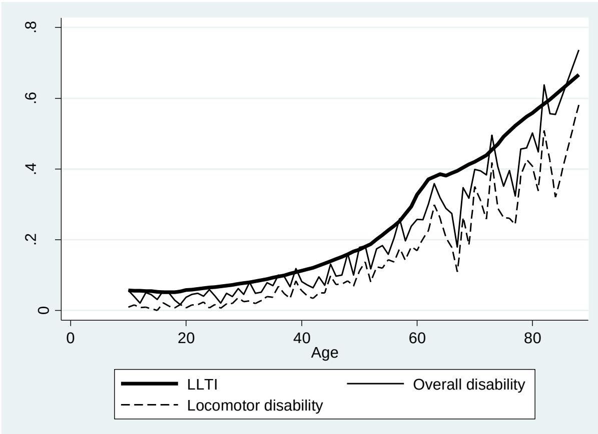
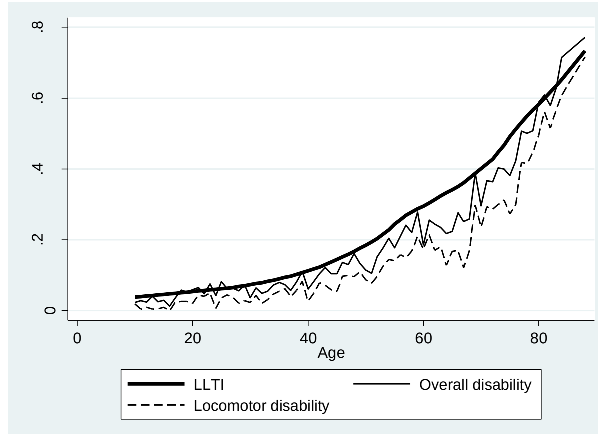
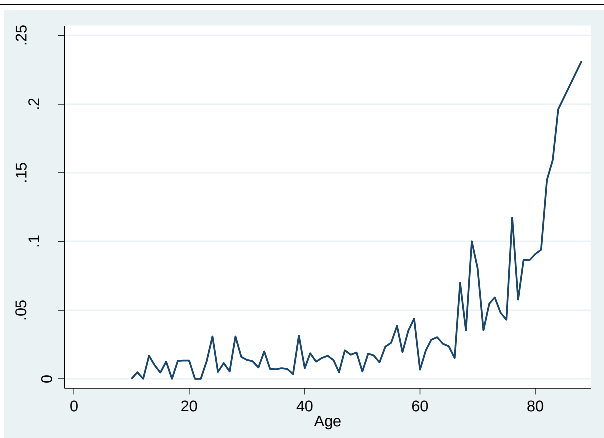
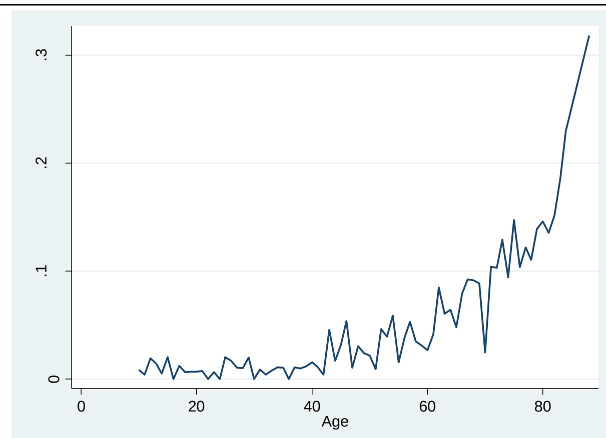
LLTI, overall and locomotor and disability schedules (England: males – 2000/2001)	LLTI, overall and locomotor and disability schedules (England: females – 2000/2001)
	
Sight disability (England: females – 2000/01)	Hearing disability (England: females – 2000/01)
	

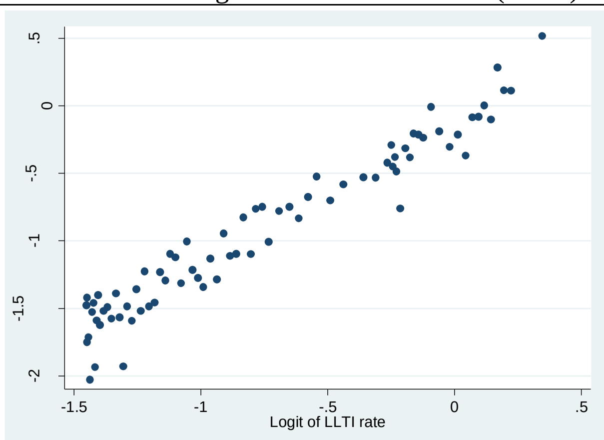
Figure 1: Limiting Long Term Illness (LLTI) and selected disability schedules for England

Source: Health Survey for England (2000/2001) and Census (2001)

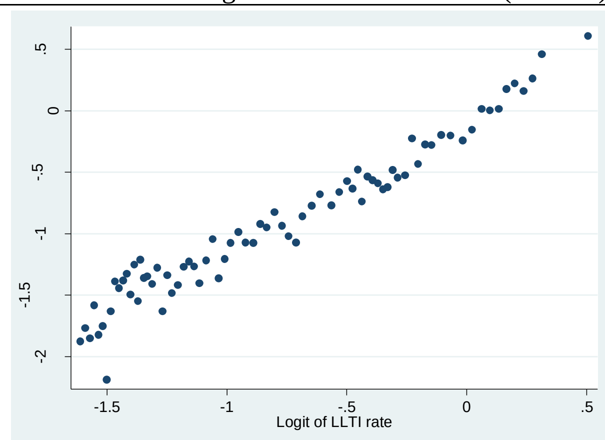
Relational models offer a compromise between the lifetable and mathematical equation approaches involving the adjustment of a 'standard' reliable schedule of rates to derive a schedule of rates in another population. The relational approach involves a 'standard' reliable schedule and a relational rule that adjusts this standard set of age-specific rates to represent a schedule in a population where estimates are either unavailable, unreliable or lacking sufficient age detail. The simplest form of relational model developed by Brass (1971) exploits the tendency for the logit transformation of the proportions surviving to age x in different populations to display a remarkably linear relationship. This allows the relationship between two logit schedules of survivorship probabilities to be expressed by two parameters (intercept and slope) (Newall 1988). More complex relational models involving additional parameters have been developed to improve the model fit particularly at the oldest and youngest ages (Ewbank et al. 1983; Zaba 1979). Relational models have been extended to other characteristics such as fertility (Brass 1981), migration (Zaba 1985; Zaba 1987) and disability (Marshall 2009).

As relational models were originally developed for mortality they are particularly applicable for the estimation of disability schedules because many disability types follow a similar mortality-like age pattern. This implies that we might model schedules of various disabilities using a single reliable schedule and a relational rule based on those developed for the estimation of mortality. Figure 2 shows that the relationship between logit schedules of rates of LLTI and overall/locomotor disability follows the linear pattern that is essential to the relational approach.

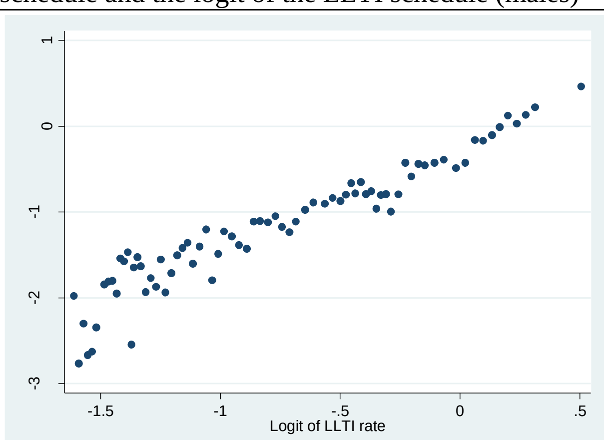
Relationship between the logit of the overall disability schedule and the logit of the LLTI schedule (males)



Relationship between the logit of the overall disability schedule and the logit of the LLTI schedule (females)



Relationship between the logit of the locomotor disability schedule and the logit of the LLTI schedule (males)



Relationship between the logit of the locomotor disability schedule and the logit of the LLTI schedule (females)

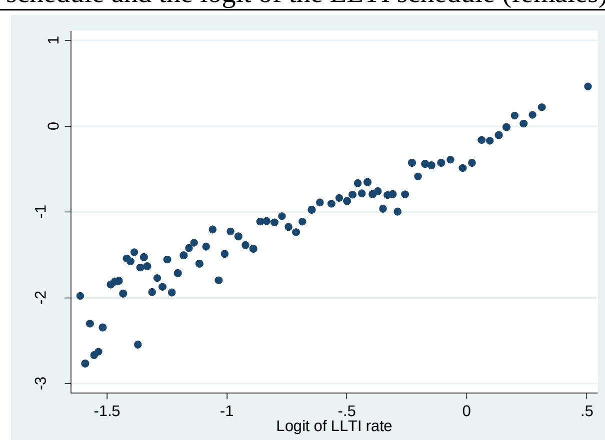
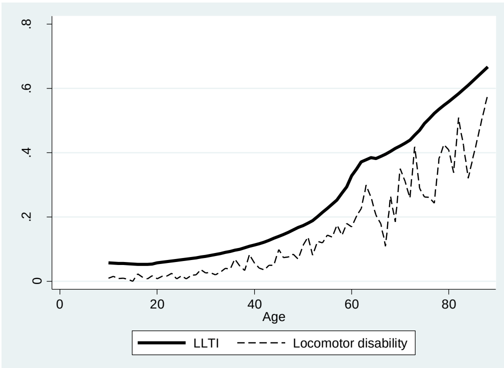


Figure 2: Relationship between the logit schedules of disability rates (overall and locomotor) and LLTI (England; males and females)

Additionally, the nature of data availability in the UK favours the use of relational models of disability. In the UK, schedules of limiting long term illness (LLTI) (see table 1 for the census LLTI question) are available from the census and are reliable even for sub-national areas such as districts (434 districts in the UK, average population =120,000). However, the data on the prevalence of specific disabilities tends to come from specialist disability surveys and even at national level these are subject to sampling variability once divided into age and sex groups. Additionally such survey data are often not available below Government Office Region (there are 9 Government Office regions in England). The problem generated by the disability data availability in the UK requires a method that can smooth fluctuations in national disability schedules and produce local disability schedules where direct estimates may not exist.

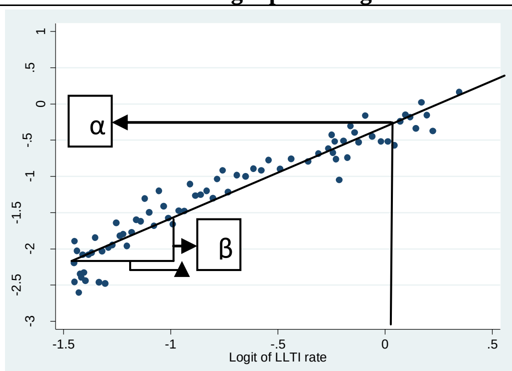
A solution to this problem using a Brass relational model is illustrated in figures 3 and 4 where a locomotor disability schedule is estimated for England (figure 3) and for two districts within England (figure 4). A full algebraic specification of the Brass relational model of disability is given in section 3. In figure 3 the LLTI schedule for England (2001 census) is used as a reliable 'standard' schedule which is adjusted to represent the schedules of a particular disability type, here locomotor disability (Health Survey for England 2000/2001). The relational parameters estimated in figure 3 for England as a whole are then utilized to adjust local LLTI schedules to derive locomotor disability schedules in the districts of Bury and South Bucks. Marshall et al. (2012) demonstrate that relational models are capable of capturing the relationship between LLTI and several disability types although in most cases a more complex model that the Brass approach is needed. Marshall (2012) confirms the validity of relational models for the local estimation of disability schedules.

1. Schedules of rates of LLTI (census) and locomotor disability (HSE) (England – Males)



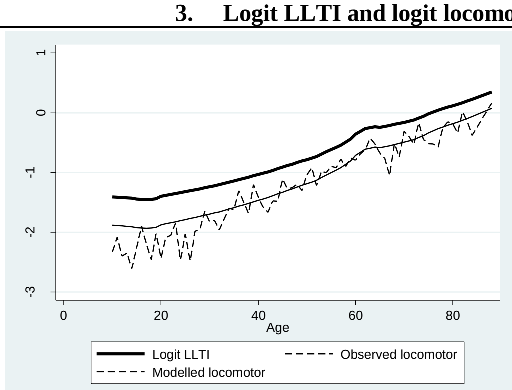
This graph shows age specific rates of LLTI (census) and locomotor disability (HSE 2000/01) which display a similar shape curve but at different levels. The curve of rates for locomotor disability is ragged due to fluctuations that stem from sampling variability.

2. Age specific logit LLTI rates versus age specific logit locomotor disability rates (England – Males)



The plot of the logit transformation LLTI and locomotor disability schedules is remarkably linear. We can represent this linear relationship between the logits with a straight line that is defined by an intercept term (α) of around -0.3 and a slope term (β) of just over 1 (1.1).

3. Logit LLTI and logit locomotor disability schedules (observed and modelled) (England – Males)

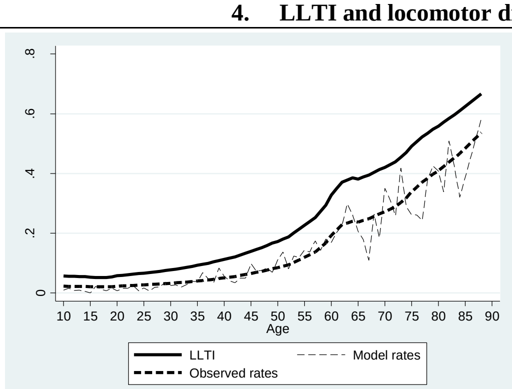


The slope (1.1) and intercept (-0.3) terms can be used to adjust the logit LLTI schedule (downwards) to represent the logit locmotor disability schedule.

The negative intercept term moves the logit LLTI schedule downwards. As the slope term is close to 1 this means little change to the steepness of the logit LLTI schedule is required

There is some evidence here of a poor fit at the youngest ages (10 to 30). Marshall et al. (2012) demonstrate that a more complicated relational model with an additional parameter corrects for this.

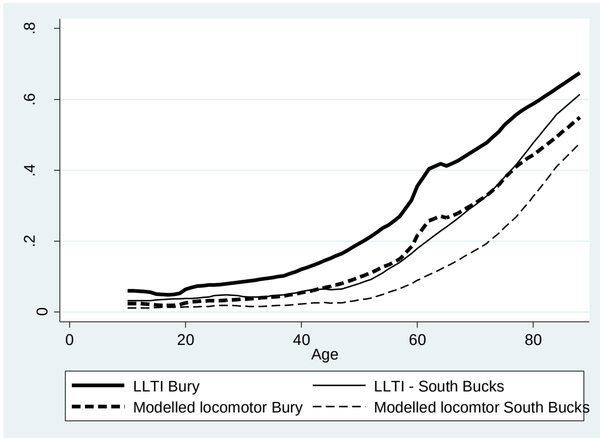
4. LLTI and locomotor disability schedules (observed and modelled) (England – Males)



We can convert the logit rates in the previous graph back to rates by reversing the logit transformation.

Figure 3: Worked example of a Brass relational models of locomotor disability (England)

LLTI schedules and modelled locomotor disability schedules– (Bury and South Bucks - Males)



Apply the slope and intercept parameters that were estimated for England as a whole to logit LLTI rates locally (census) gives locomotor disability schedules for areas that lack direct or reliable survey estimates.

In this graph these calculations are applied to LLTI data in Bury and South Bucks. Bury has higher levels of LLTI than South Bucks and this is reflected in the estimated rates of locomotor disability.

Figure 4: Worked example of the use of a Brass relational model to estimate locomotor disability in Bury and South Bucks districts.

1.3 Estimation of parameters in relational models of disability

A key issue is how the parameters of these models of disability schedules should be estimated. This paper evaluates the three main approaches identified in the literature; maximum likelihood, ordinary least squares and weighted least squares. This is a useful exercise as there is some ambiguity as to which approach should be used for mortality estimation and additionally the question has not been explored for relational models of disability. For relational models of mortality, Preston et al. (2000) recommend the OLS approach; Congdon (1993) fits relational models of mortality using WLS, whilst Stewart (2004), after evaluating various methods by which relational models of mortality could be fitted (including OLS and WLS), concludes that ML is most appropriate.

The main weakness of the OLS approach as a model-fitting procedure for relational models is that it fails to accommodate the tendency for rates to be more reliable at certain ages than at others (Brass 1971). Weighted least squares offer a means to accommodate the challenge of varying reliability of rates across the age range. This approach is identical to OLS except that each age has a weight which is multiplied by the residual at that age. Weights are important to ensure that any heteroscedasticity in error structure is accommodated within the loss function. Although weights might lead to a slight loss of fit they give an improved specification of the regression and proper estimates of standard errors of parameters which are important in assessing the extent of overparameterisation (Congdon 1993). Maximum likelihood avoids the issue of differing variances at particular ages and the use of a set of weights to accommodate this by considering each age-specific probability of disability (p) in a distribution as a binomial probability with variances that vary according to p and the sample size. The binomial probability parameter is represented by a smoothed age-specific probability of disability (from a relational model) where the 'successes' are the number of people with disabilities at a particular age and the number of trials is the total population at a particular age (Stewart 2004). An important assumption within this approach is that the probability of individuals having a disability at a particular age should be independent. This assumption is returned to in the discussion.

1.4 Evaluation of model fit

The evaluation of model fit adopted draws on the procedure used by Stewart (2004) in his comparison of estimation strategies for mortality schedules. In addition to a visual examination of model and observed disability age-specific rates, two tests of model fit are undertaken including; the ratio of residual sums of squares between models (F-statistic) and the model crude disability rate (CDisR) compared to the observed CDisR.

The first test involves calculating the residual sum of squares associated with each estimation strategy. The ratio of two sets of residual sum of squares follows an F-distribution with, in this case, 74 degrees of freedom in the numerator and denominator (there are 76 age specific rates and 2 parameters in the models; 76-2=74). For simplicity the F-statistics reported in tables contain the residual sum of squares associated with the ML estimation strategy in the denominator.

The second statistic used to compare the model rates derived from the three estimation strategies is the crude disability rate (CDisR). A good model should produce estimates close to the observed CDisR in the Health Survey for England. However, a caveat should be noted here; Marshall et al. (2012) develop more complex relational models that give a better fit than Brass models for both locomotor and overall disability. Therefore, in this paper, which focuses on the simple Brass relational model, the interest is in the performance of the different estimation strategies relative to each other rather than in absolute terms.

2. Data

2.1 The census of population (UK)

The census has been carried out since 1801, during which time sporadic questions on health and disability have been asked (Charlton 2000). The main advantage of the census as a source of data on disability is its coverage of the total population (all ages, households, and institutions) and the fine geographical detail at which data are reliably available. In 2001 the census included a question on limiting long term illness that records any illnesses, health problems, or disabilities that limit an individual in their daily activities. The question on LLTI (see table 1) features a prompt for elderly people to include problems that are due to old age. This is useful because it is known that the elderly tend to discount some health problems as being a result of ageing (Bajekal, et al. 2003). There is some undercount in the census which is larger in some areas of the country and for certain population groups (Cook 2004). However, these problems are small compared to the uncertainty associated with sample data.

Do you have any long standing illness, health problem or disability which limits your daily activities or the work that you can do? Include problems which are due to old age. (Yes/No)

Table 1: Limiting long term illness question – census 2001.

Source: 2001 Census household questionnaire. Available at

http://www.statistics.gov.uk/census2001/pdfs/H1.pdf

In terms of the general utility of self-reported limiting long term illness, a large body of work supports the validity of self-assessed health (Mitchell 2005) with LLTI found to be most strongly associated with general health perceptions, more serious health conditions (Manor, et al. 2001) and physical limitations rather than with psychological health (Cohen, et al. 1995). There are strong relationships between LLTI and other health outcomes including all cause and cause-specific mortality (Charlton, et al. 1994; Bentham, et al. 1995; Idler & Benyamini 1997) as well as sickness benefits claims from different health conditions (Bambra and Norman 2006; Norman and Bambra 2007).

The census data on LLTI is downloaded from tables ST16 and ST65 which record the population with (and without) LLTI with age and sex detail for the household and institutional populations respectively[1].

2.2 The Health Survey for England

The Health Survey for England (HSE) was set up in 1991 to monitor the health of the private household population in England and the progress towards targets laid out in the *Health of the Nation* strategy. The survey

[1] 2001 census standard tables can be downloaded from the Nomis website -https://www.nomisweb.co.uk/Default.asp

follows a multistage, stratified probability sampling design. The sample size was increased from around 4,000 to 16,000 in 1994 enabling analysis for Health Authority Regions and between socio-economic groups. From 1995 a sample of 4,000 children between the ages of 2 and 15 were included in the sample (Bajekal 2000). Each year of the HSE has a particular focus, with a module measuring disability included in 1995, 2000, 2001, and 2005. In this paper the data on disability in 2000 and 2001 are combined to increase sample sizes and to overlap the data collection date of the 2001 census, a feature that is particularly useful for the models that combine HSE and census data. The 2000 survey focused on disability amongst the elderly with a boosted sample of elderly people including the elderly living in residential and care homes along with a reduced sample of the general population (Bajekal and Prescott 2003). It should be noted that the HSE in 2000 does not give a complete coverage of the disabilities in communal establishments, missing younger people in institutions and covering only care and residential homes (and not, for example, hospitals). However, rates of LLTI are almost identical in the household and total population below the age of 65 in the UK and its constituent countries. This implies that the lack of coverage of the institutional population under the age of 65 in the HSE00/01 is not likely to be a problem in this analysis.

Disability is measured according to five domains: locomotion (mobility), personal care, sight, hearing, and communication. A person is classed as having no disability or a disability at a lower or higher level for each of the five domains based on their answers to questions on ability to perform everyday tasks (see appendix C). The highest score for any of the five types of disabilities is taken as the overall disability score. A score of 1 indicates a lower severity disability, a score of 2 indicates a higher severity disability and a score of 0 indicates no disability. In this paper model rates are produced for overall disability (where a person has one of the five disabilities listed above) and locomotor disability. Severity of disability is not distinguished and so the rates of disability include those with either a higher or lower severity disability. The HSE allows respondents to take into account the use of aids for hearing and sight disabilities, however, for the other domains the use of aids to perform tasks are not allowed. Data are collected using face-to-face computer assisted personal interviewing and the disability module applies to all people aged 10 or over. Proxy answers are not permitted for adults but parents answer for children under the age of 13 (Bajekal and Prescott 2003).

2.3 Data preparation

The analysis of the merged 2000-01 HSE datasets requires the use of two types of weights to ensure that estimates are representative of the target population. First, child weights are needed to compensate for the sample design at these ages which involves limiting the number of children interviewed in each household to two. Second, the HSE in 2000 includes weights to account for the oversampling of the elderly and institutional population. In order to compensate for the lack of older people living in institutions in 2001, the weights associated with people living in institutions in 2000 are doubled.

All models use rates by single year of age up to the age of 84 with an age of 88 to represent all those aged over 84. The use of 88 as the upper age limit is based upon the average age of the population aged over 84 (as calculated using the 2001 census Sample of Anonymised Records[2]). census tabulations of LLTI are only released with quinary age detail (from the age of 20 upwards) and in order to generate single year estimates, these five year rates are smoothed using an Excel based tool developed by Popgroup[3] users specifically for this purpose. The Excel smoothing tool and more information on the smoothing approach are available at: http://www.ccsr.ac.uk/popgroup/about/manuals.html

3. Specification of Brass relational model

The Brass model is defined as below (equations 1 to 5)

Let:

p_{xd} = prevalence of disability d at age x in England (HSE00/01)

p_{xL} = prevalence of LLTI (*l*) at age x in England (census01)

d_x= number of cases of disability at age x (Health Survey for England)

N_x=number of people sampled at age x (Health Survey for England)

First, define, Y_{xd} and Y_{xl} as:

$$Y_{xd} = \frac{1}{2}\ln\left(\frac{p_{xd}}{1-p_{xd}}\right) \qquad [1]$$

$$Y_{xL} = \frac{1}{2}\ln\left(\frac{p_{xL}}{1-p_{xL}}\right) \qquad [2]$$

Then a simple Brass relational model is defined as:

[2] The SARs are samples of individual records from the 2001 (and 1991) Censuses.

[3] For more information on POPGROUP see www.ccsr.ac.uk/popgroup/

$$Y_{xd} = \alpha + \beta Y_{xL} + e_x \qquad [3]$$

Or in its full form, as below:

$$\frac{1}{2}\ln\left(\frac{p_{x.d}}{1-p_{x.d}}\right) = \alpha + \beta\left(\frac{1}{2}\ln\left(\frac{p_{x.L}}{1-p_{x.L}}\right)\right) + e_x \qquad [4]$$

From equation four we can derive the modeled probability of having a particular disability (p_x) by reversing the logit transformation:

$$\hat{p}_{xd} = \frac{e^{(2(\hat{\alpha}+\hat{\beta}Y_{xL}))}}{1+e^{(2(\hat{\alpha}+\hat{\beta}Y_{xL}))}} \qquad [5]$$

4. Specification of fitting procedures

4.1 Ordinary Least Squares

In the ordinary least squares (OLS) approach we calculate values of α and β that minimise the residual sum of squares RS_d (equation 6):

$$RS_d = \sum_{x=10}^{88} (\hat{Y}_{xd} - Y_{xd})^2 \qquad [6]$$

The regression procedure (*regress* in the Stata statistical program) is used to estimate α and β using the OLS approach described above. The summation is from age 10 upwards in equation 6 because disability questions in the Health Survey for England are only asked of those aged over 10.

4.2 Weighted least Squares

A model that uses the logit of a proportion as the dependent variable should use weights at each age x for disability d (w_{xd}) based on equation 7 (Congdon 1993):

$$w_{xd} = p_{xd}(1-p_{xd})N_{xd} \qquad [7]$$

Where:

p_{xd} = the prevalence of disability d at age x in England (Health Survey for England)

N_{xd} = the sample size at age x in England in the Health Survey for England

Proportions that are close to 0 or 1 or that are based on small samples are given less weight during the model fitting process. Conversely, proportions that are near 0.5 or that are based on larger sample sizes are given greater weight. For both overall and locomotor disability the largest weights are at the oldest ages because disability proportions close to 0.5 at these ages. Smaller weights are found at the youngest ages as proportions are very close to 0. These 'logit' weights that are recommended by Congdon (1993) contrast with a set of 'population' weights used by Stewart (2004) who bases his WLS weights for relational models of mortality on the person-years lived within each age interval. Figure 5 illustrates the differences between the logit and population weights in the context of relational models of disability. The population weights are the Health Survey for England sample size (N_{xd}) for disability type d, at each individual year of age x. Both sets of weight are centred (divided by their mean value) for comparative purposes. Stewart's population-based weights are shown here as they will be referred to in the discussion section of this paper.

The logit weights specified in equation 7 are used for the WLS models in this paper and seem appropriate to their aim in that they increase with age corresponding well to the observed fluctuations in the logit disability schedule which is least reliable at the youngest ages (see figure 2).

The weights (w_{xd}) are used calculate a weighted residual sum of squares (WRS_d) (equation 8) which is then minimised using stata's regress procedure in the same way as for OLS:

$$WRS_d = \sum_{x=10}^{88} w_{xd} \, (\hat{Y}_{xd} - Y_{xd})^2 \qquad [8]$$

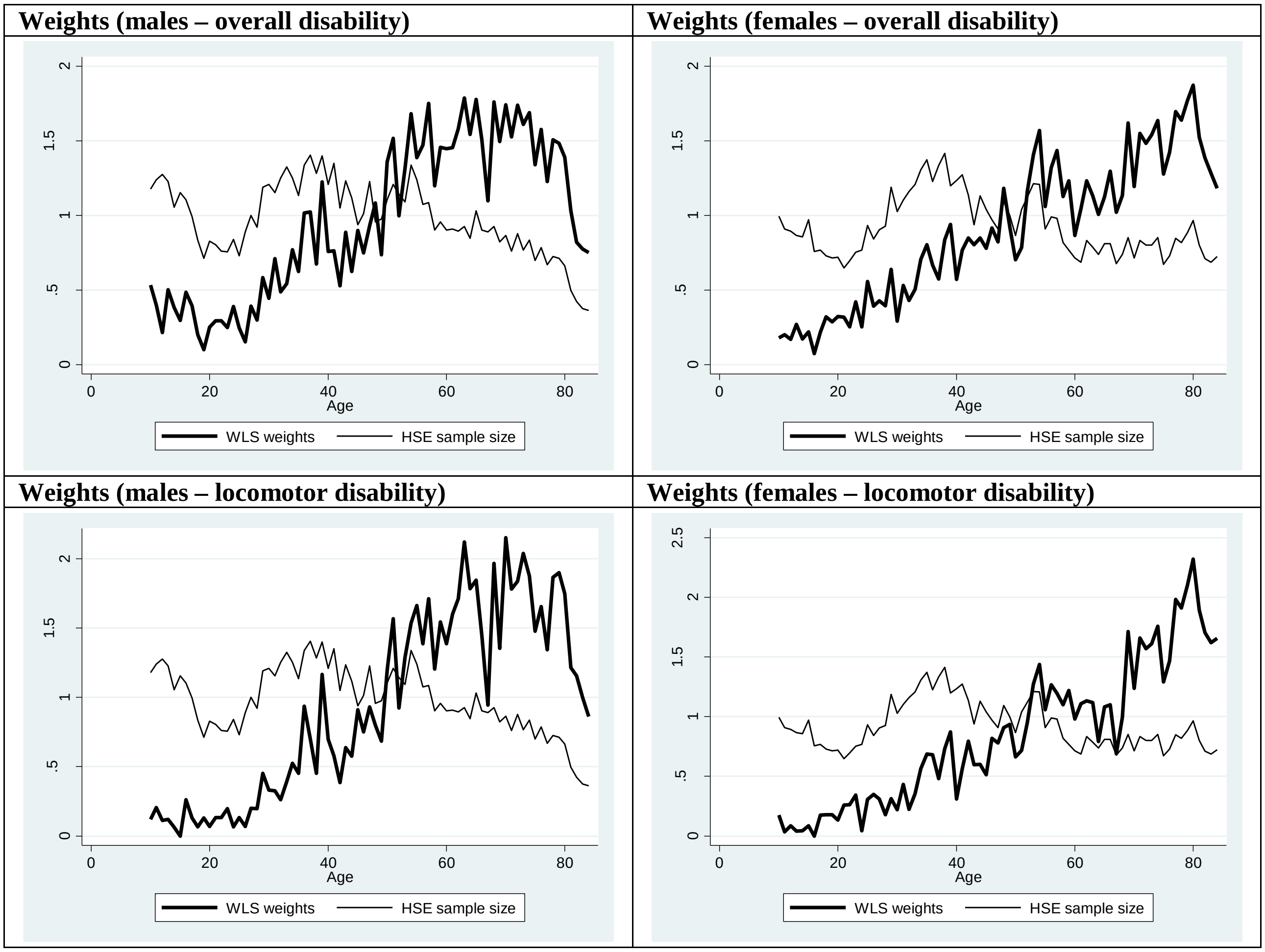
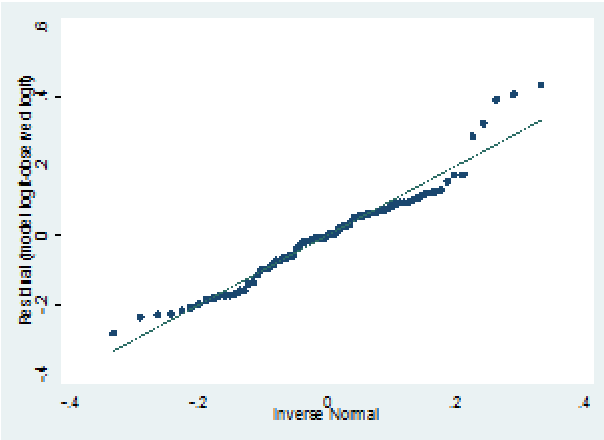


Figure 5: Age specific WLS weights and population based weights (overall and locomotor disability; males and females)

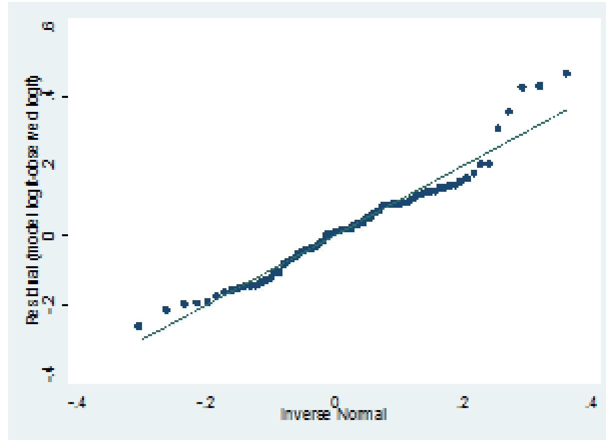
The QQ plots in figures 6 (overall disability) and 7 (locomotor disability) suggest that residuals are normally distributed in both OLS and WLS models

An important assumption of the OLS and WLS models is that residuals are normally distributed. Normal Quantile-Quantile plots for the residuals allow us to test this assumption for our residuals from OLS and WLS models of overall disability (figure 6) and locomotor disability (figure 7). Figures 6 and 7 suggest the assumption of normally distributed residuals is largely met although there is some evidence of heavy tails in the distribution of residuals with an excess of extreme values relative to the normal distribution. The plot for overall disability (females) provides some evidence of an outlier however exclusion of this outlier had almost no effect on the model results. The QQ plots for OLS and WLS are very similar indicating that the use of weights do very little to adjust the distribution of residuals.

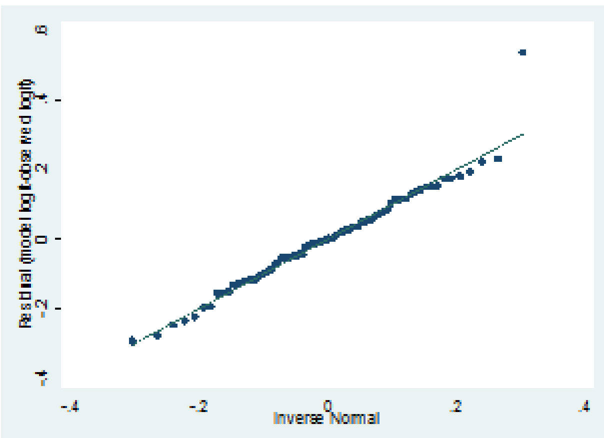
QQ plot – males overall disability (OLS)



QQ plot – males overall disability (WLS)



QQ plot – females overall disability (OLS)



QQ plot – females overall disability (WLS)

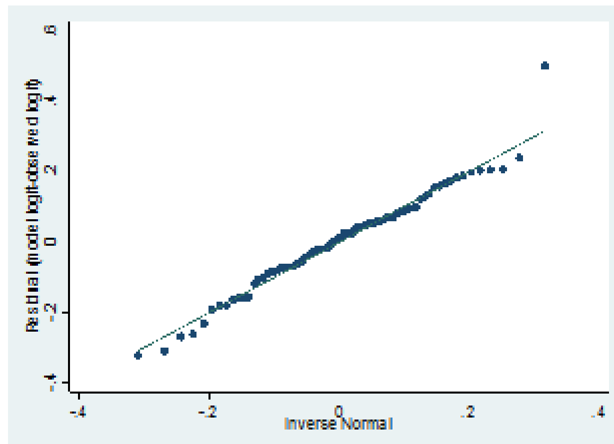


Figure 6: Normal Quantile-Quantile plots for residuals from Brass relational models of overall disability fitted using weighted least squares (WLS) and ordinary least squares (OLS)

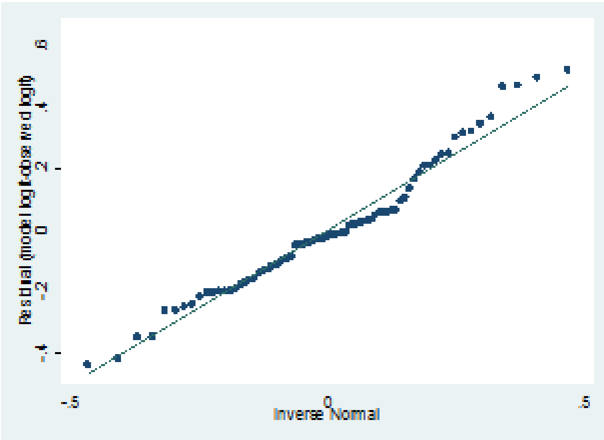
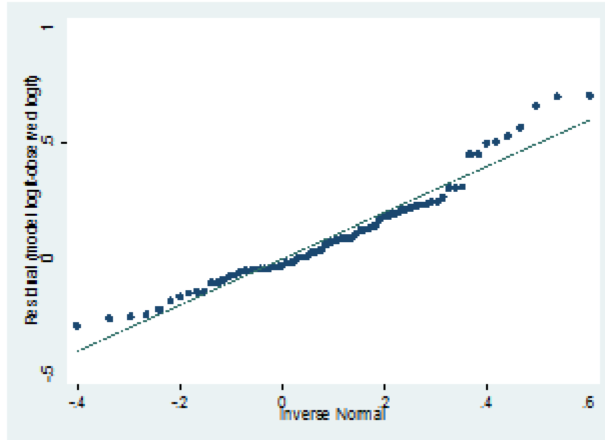
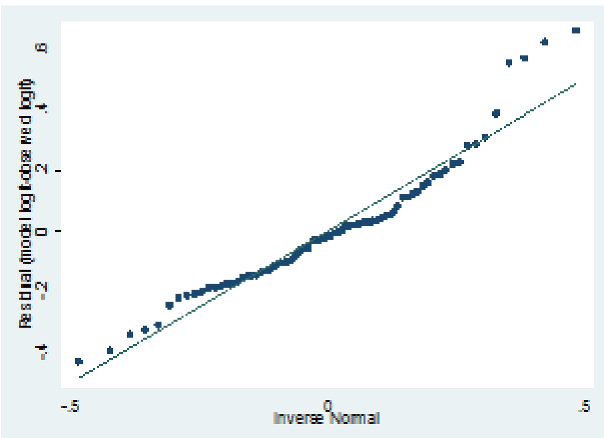
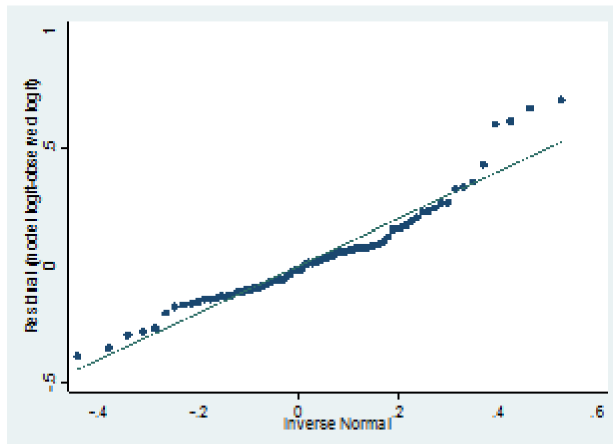
QQ plot – males locomotor disability (OLS)	QQ plot – males locomotor disability (WLS)
	
QQ plot – females locomotor disability (OLS)	QQ plot – females locomotor disability (WLS)
	

Figure 7: Normal Quantile-Quantile plots for residuals from Brass relational models of locomotor disability fitted using weighted least squares (WLS) and ordinary least squares (OLS)

4.3 Maximum likelihood

The maximum likelihood approach involves first specifying a likelihood function. In the case of relational models of disability, the likelihood function (*L*) represents the joint probability of having d_x cases of a particular disability ($pr(D_x=d_x)$) out of a population of n_x people at a particular age x given the smoothed probability of having a disability at this age (estimated using a Brass model). The assumption adopted here is that d_x come from a binomial distribution.

$$L = \prod_{x=10}^{88} f(d_x \mid \hat{p}_{xd}(\alpha,\beta)) \qquad [9]$$

$$L = \prod_{x=10}^{88} (\hat{p}_{xd}(\alpha,\beta))^{d_x}(1-\hat{p}_{xd}(\alpha,\beta))^{n_x-d_x} \qquad [10]$$

Substituting the specification of $\hat{p}_{xd}$ (see equation 5) into equation 10 and taking the logarithm of each side of the equation gives the log-likelihood function in equation 11 below:

$$\ln L = \sum_{x=0}^{88} \left(d_x \ln\left[\frac{e^{(2(\alpha+\beta Y_{Lx}))}}{1+e^{(2(\alpha+\beta Y_{Lx}))}} \right] + (n_x - d_x) \ln\left[1 - \frac{e^{(2(\alpha+\beta Y_{Lx}))}}{1+e^{(2(\alpha+\beta Y_{Lx}))}} \right] \right) \qquad [11]$$

Equation 11 can be simplified to (see appendix A for working):

$$\ln L = \sum_{x=0}^{88} (2\alpha d_x + \beta Y_{Lx} d_x - n_x \ln(1 + (\exp(2(\alpha + \beta Y_{Lx}))))) \qquad [12]$$

Stata's maximum likelihood estimation package (*ml*) is used to maximise the log-likelihood equation in equation 12. This is achieved by setting the two derivatives of the log likelihood function (with respect to α and β) to equal 0 and solving these equations simultaneously. The partial derivatives of the log-likelihood with respect to α and β are shown in equations 13 and 14 respectively (working is shown in appendix B).

$$\frac{d}{d\alpha} \ln(\hat{p}_{xd} \mid d_x) = 2d_x - \frac{2n_x[\exp 2(\alpha + \beta Y_{Lx})]}{\exp(2(\alpha + \beta Y_{Lx}) + 1)} = 0 \qquad [13]$$

$$\frac{d}{d\beta} \ln(\hat{p}_{xd} \mid d_x) = \sum_{x=0}^{88} Y_{Lx} d_x - \frac{n_x Y_{Lx}[\exp 2(\alpha + \beta Y_{Lx})]}{\exp(2(\alpha + \beta Y_{Lx}) + 1)} = 0 \qquad [14]$$

5. Results

Table 2 gives the parameter estimates under each estimation strategy. It is clear that whilst OLS and WLS yield fairly similar estimates for α and β, the solutions under a ML approach are rather different. The α parameter has less influence in ML estimation than in OLS and WLS. For example, the ML estimate of α is not significant for overall disability (males) and unusually, given its role in determining the level of disability, is positive for overall disability (females). The lesser role of α is compensated for by β which is generally further from 1 (where β=1 and α=0 then the LLTI schedule is unaltered) than in the OLS and WLS models.

MEN								
			Coef.	**Std. Err.**	**t**	**P>t**	**95% Confidence interval**	
Overall disability	**OLS**	**α**	-0.13	0.03	-4.58	0	-0.19	-0.07
		β	1.05	0.03	33.59	0	0.98	1.11
	WLS	**α**	-0.11	0.02	-4.9	<0.0000	-0.15	-0.06
		β	1.04	0.03	32.34	<0.0000	0.97	1.10
	ML	**α**	-0.02	0.02	-1.13	0.258	-0.05	0.01
		β	1.15	0.03	44.23	<0.0000	1.10	1.20
Locomotor	**OLS**	**α**	-0.31	0.04	-7.59	<0.0000	-0.39	-0.23
		β	1.26	0.04	28.45	<0.0000	1.17	1.35
	WLS	**α**	-0.31	0.02	-13.47	<0.0000	-0.36	-0.27
		β	1.12	0.04	27.85	<0.0000	1.04	1.19
	ML	**α**	-0.20	0.02	-11.27	<0.0000	-0.24	-0.17
		β	1.29	0.03	41.31	<0.0000	1.23	1.35
WOMEN								
			Coef.	**Std. Err.**	**t**	**P>t**	**95% Confidence interval**	
Overall disability	**OLS**	**α**	-0.11	0.03	-4.21	<0.0000	-0.16	-0.06
		β	1.03	0.03	37.89	<0.0000	0.97	1.08
	WLS	**α**	-0.06	0.02	-3.85	<0.0000	-0.09	-0.03
		β	1.08	0.02	47.13	<0.0000	1.04	1.13
	ML	**α**	0.08	0.01	6.16	<0.0000	0.06	0.11
		β	1.26	0.02	60.43	<0.0000	1.22	1.30
Locomotor	**OLS**	**α**	-0.27	0.04	-6.43	<0.0000	-0.35	-0.18
		β	1.18	0.04	26.91	<0.0000	1.09	1.27
	WLS	**α**	-0.23	0.02	-13.45	<0.0000	-0.26	-0.19
		β	1.18	0.03	42.39	<0.0000	1.12	1.23
	ML	**α**	-0.20	0.02	-11.27	<0.0000	-0.24	-0.17
		β	1.29	0.03	41.31	<0.0000	1.23	1.35

Table 2: Parameter statistics from a Brass relational model of overall and locomotor disability estimated using OLS, WLS and ML

Figure 8 suggests that the ML estimation strategy tends to overestimate rates at the older ages with the possible exception of locomotor disability (females). In contrast both OLS and WLS give similar and good fits to the observed data based on this visual comparison. One reason for the poorer fit of the ML approach could be that associated estimates are distorted by the small population counts at individual years of ages to a greater degree than the other techniques. However, refitting the models using five year age bands gives similar results (see appendix D) and so this does not explain the poor fit of ML. Marshall et al. (2012) demonstrate that a more complicated relational model based on that developed by Ewbank et al. (1983) gives a better fit than the Brass model for overall and locomotor disability. It might be the case the ML approach requires the best fitting relational model in order to generate a good fit. However, this theory can also be discounted as examination of model fit using more complex relational models also reveals similar results to those in figure 8 (see appendix E).

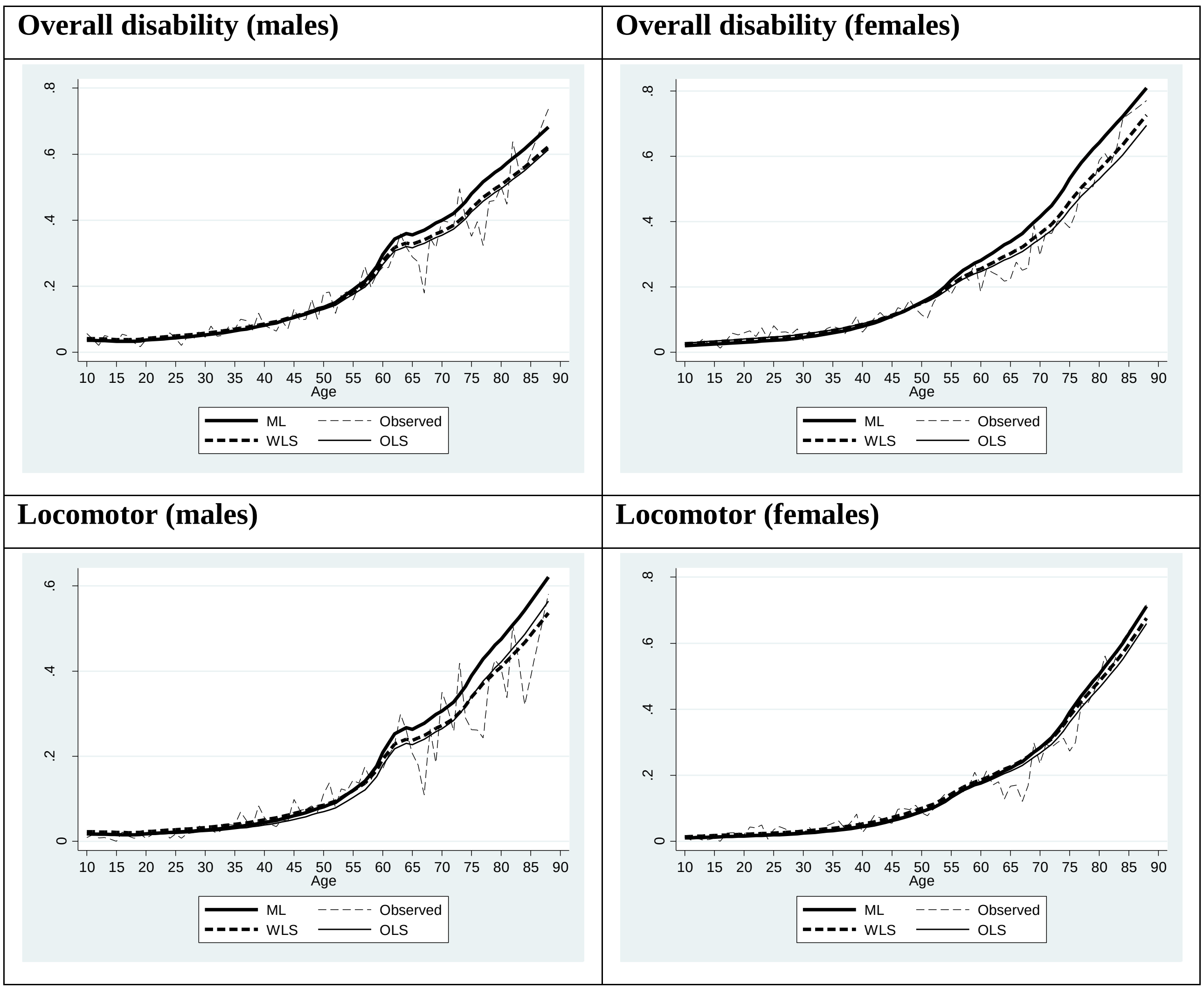


Figure 8: Brass relational model schedules of overall and locomotor disability estimated using OLS, WLS and ML

The results from the two tests of model fit under each estimation strategy are shown in table 3. The F-statistics indicate that the distributions generated with the OLS and WLS have smaller variances around the observed disability distribution compared to the ML procedures estimates. These differences in residual sums of squares are statistically significant with the exception of locomotor disability (females) and suggest that under this measure, OLS and WLS generally give a better fit than ML. The differences between OLS and WLS in terms of residual sums of squares do not reach significance.

Finally, the observed crude rates of disability (overall disability males - 0.18, overall disability females - 0.20; locomotor males - 0.13 locomotor female -0.15) are more closely reproduced by OLS and WLS than by ML which tends to overestimate as might be expected given the plots in figure 8. The performance of the OLS and WLS models is worst for females (overall and locomotor disability) with crude rates falling outside confidence intervals for HSE observed estimates suggesting the need for a more complex relational model in these situations (see Marshall et al. (2012) for such models).

MALES			
		F	Crude disability rate
Overall disability	OLS	0.63	0.18
	WLS	0.65	0.18
	ML	1.00	0.19
Locomotor	OLS	0.61	0.12
	WLS	0.55	0.13
	ML	1.00	0.14
FEMALES			
		F	Crude disability rate
Overall disability	OLS	0.32	0.22
	WLS	0.34	0.23
	ML	1.00	0.25
Locomotor	OLS	0.80	0.18
	WLS	0.93	0.19
	ML	1.00	0.19

Table 3: F statistic and crude disability rate from relational models of overall and locomotor disability estimated using OLS, WLS and ML

6. Discussion and conclusions

The results above suggests that relational models fitted using maximum likelihood do not give as good fit to the observed data compared to weighted or ordinary least squares fitting procedures. Visual examination of model and observed schedules and of model and observed crude disability rates reveal that model rates estimated using ML consistently overestimate the observed rates at the oldest ages. In contrast OLS and WLS provide a similar and good fits to the observed data across the age range. The residual sums of squares (RSS) are significantly lower for models fitted using OLS and WLS compared to ML.

These differences in model rates stem from the different parameter estimates that are returned by the three estimation strategies. If we consider the parameter estimates given their role in adjusting LLTI schedules it is clear that parameter estimates from OLS and WLS are more plausible than from WLS. We would expect α to be negative because age-specific rates of overall and locomotor disability are generally lower than for LLTI (see figure 1) and a negative estimate of α has the effect of moving the LLTI curve downwards. However, under the ML estimation procedure, α is positive for overall disability (females) and is not significantly different to zero for overall disability (males). The ML estimate of α for locomotor disability is lower than for the OLS or WLS estimation procedures for both males and females.

The poorer fit of relational models of disability using ML contradicts research by Stewart (2004) which favoured ML for the estimation of relational models of mortality rather than OLS and WLS. Stewart proposed that the main weakness of least squares regression stemmed from the dependence of their variances on the sum of squared errors of the smoothed logits. An interesting question then is why do such different conclusions appear to hold for relational models of mortality and disability? One explanation might stem from the weighting scheme used in Stewart's paper which uses person-years lived in a particular age interval. This is an unusual choice of weights because the advice on modelling the logit of a proportion in the literature (see Congdon 1993; p252) recommends weights based on equation 15 (below). The use such weights might alter the findings within Stewart's paper.

$$w_{xd} = p_{xd}(1 - p_{xd})N_{xd} \qquad [15]$$

A second question raised by this paper is why the ML procedure gives such a poor fit to observed disability schedules at the oldest ages? One explanation might lies in the differential emphasis given to particular ages when the ML estimation procedure is undertaken. The ML approach has similarities with a WLS weighting scheme based on the population based weights recommended by Stewart (2004). The ML procedure involves modelling the number of people with disabilities at a particular age given the total population at that age. Thus, the ML model attaches most importance to ages where populations are greatest. Figure 5 demonstrates that the main difference between the logit and population weights are that the population weights are higher at the younger ages (10-50) and lower at the older ages (50+). This means that the ML model automatically attaches less importance to model fit at the older ages which might explains the overestimation at the oldest ages. The logit weights seem more reasonable than the population based weights given the distribution of the logit schedules (see figure 2).

Another explanation for the poorer fit of the ML model could be linked to its assumption of independence of probabilities of disability amongst individuals at a particular age. This assumption is unlikely to hold because the Health Survey for England follows a sampling design involving clustering where postcodes are selected as the primary sampling unit. It is highly probable that people from the same postcode may well share similar propensities to developing disability given the evidence of strong clustering of this and other health-related characteristics. This lack of independence may be the cause of the poor fit of the ML model which relies on independence of binomial events (individuals having a disability or not).

In conclusion, the key finding of this paper is to demonstrate that ML does not offer a better method for fitting Brass relational models of disability than procedures based on least squares. The results suggest that there is very little to choose between OLS and WLS models. However given that the literature strongly cautions against the use of OLS because of the potential for statistical instability associated with models fitted in this way (Congdon 1993), it seems that WLS is a best choice for the estimation of relational models of disability.

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Appendix A: Derivation of log-likelihood function

In this appendix the log-likelihood function function is simplified. We start with the log-likelihood function in equation 12 (see below):

$$\ln(\hat{p}_{xd} \mid d_x) = \sum_{x=0}^{88} d_x \ln(\left[\frac{e^{(2(\alpha+\beta Y_{Lx}))}}{1+e^{(2(\alpha+\beta Y_{Lx}))}}\right] + (n_x - d_x)\ln\left[1 - \frac{e^{(2(\alpha+\beta Y_{Lx}))}}{1+e^{(2(\alpha+\beta Y_{Lx}))}}\right] \quad \text{[A1]}$$

Let $a = \exp(2(\alpha + \beta Y_{Lx}))$ [A2]

Then substituting *a* into equation A1 we have:

$$\sum_{x=0}^{88} d_x \ln\left(\frac{a}{1+a}\right) + (n_x - d_x)\ln\left[1 - \left(\frac{a}{1+a}\right)\right] \quad \text{[A3]}$$

Simplify second term:

$$\sum_{x=0}^{88} d_x \ln\left(\frac{a}{1+a}\right) + (n_x - d_x)\ln\left[\left(\frac{1}{1+a}\right)\right] \quad \text{[A4]}$$

Using logarithmic identities:

$$\sum_{x=0}^{88} d_x [\ln(a) - \ln(1+a)] + (n_x - d_x)[\ln(1) - \ln(1+a)] \quad \text{[A5]}$$

Multiplying out:

$$\sum_{x=0}^{88} d_x \ln(a) - d_x \ln(1+a) + n_x \ln(1) - d_x \ln(1) - n_x \ln(1+a) + d_x \ln(1+a) \quad \text{[A6]}$$

We can ignore the constant ln(1) terms (as they don't vary with respect to our parameters α and β) and collect together terms:

$$\sum_{x=0}^{88} d_x \ln(a) - n_x \ln(1+a) \quad \text{[A7]}$$

We can now put the full specification of 'a' (see equation A2) back into equation A7

$$\sum_{x=0}^{88} d_x \ln(\exp(2(\alpha + \beta Y_{Lx}))) - n_x \ln(1 + (\exp(2(\alpha + \beta Y_{Lx})))) \quad \text{[A8]}$$

The ln and the exp in the first term of equation A8 cancel:

$$\sum_{x=0}^{88} d_x (2(\alpha + \beta Y_{Lx})) - n_x \ln(1 + (\exp(2(\alpha + \beta Y_{Lx}))))$$ [A9]

Multiply out the first term of A9:

$$\sum_{x=0}^{88} 2\alpha d_x + \beta Y_{Lx} d_x - n_x \ln(1 + (\exp(2(\alpha + \beta Y_{Lx}))))$$ [A10]

Appendix B: Differentiation of log-likelihood function

In this appendix the log-likelihood is differentiated with respect to α and β. The log-likelihood equation is defined below:

$$\ln(\hat{p}_{xd} \mid d_x) = \sum_{x=0}^{88} (2\alpha d_x + \beta Y_{Lx} d_x - n_x \ln(1 + (\exp(2(\alpha + \beta Y_{Lx})))))$$

[B1]

First differentiate with respect to α:

$$\frac{d}{d\alpha} \sum_{x=0}^{88} 2\alpha d_x + \beta Y_{Lx} d_x - n_x \ln(1 + (\exp(2(\alpha + \beta Y_{Lx}))))$$

[B2]

We can differentiate each term separately:

$$\sum_{x=0}^{88} \frac{d}{da} 2\alpha d_x + \frac{d}{da} \beta Y_{Lx} d_x - \frac{d}{da} n_x \ln(1 + (\exp(2(\alpha + \beta Y_{Lx}))))$$

[B3]

The first two terms are differentiated easily:

$$\sum_{x=0}^{88} 2 d_x + 0 - \frac{d}{da} n_x \ln(1 + (\exp(2(\alpha + \beta Y_{Lx}))))$$

[B4]

In order to differentiate the second term we must use the chain rule twice:

First let:

$$u = \sum_{x=0}^{88} \exp(2(\alpha + \beta Y_{Lx}) + 1)$$

[B5]

Then:

$$\frac{d}{da} \ln \sum_{x=0}^{88} \exp(2(\alpha + \beta Y_{Lx}) + 1) = \frac{d \ln(u)}{du} \frac{du}{da}$$

[B6]

$$\frac{d\ln(u)}{du} = \frac{1}{u} \qquad \text{[B7]}$$

$$\frac{du}{da} = \sum_{x=0}^{88} \frac{du}{da}\left[\exp(2(\alpha + \beta Y_{Lx}) + 1)\right] \qquad \text{[B8]}$$

Differentiate each term separately:

$$\frac{du}{da} = \sum_{x=0}^{88} \frac{du}{da}[\exp 2(\alpha + \beta Y_{Lx})] + \frac{du}{da}(1) \qquad \text{[B9]}$$

$$\frac{du}{da} = \sum_{x=0}^{88} 2[\exp 2(\alpha + \beta Y_{Lx})] \qquad \text{[B10]}$$

Plugging equations B10 and B7 into equation B6 we have:

$$2d_x - \frac{2n_x[\exp 2(\alpha + \beta Y_{Lx})]}{\exp(2(\alpha + \beta Y_{Lx}) + 1)} \qquad \text{[B11]}$$

The working for the solution of:

$$\frac{d}{d\beta} \sum_{x=0}^{88} 2\alpha d_x + \beta Y_{Lx} d_x - n_x \ln(1 + (\exp(2(\alpha + \beta Y_{Lx})))) \qquad \text{[B12]}$$

Is as above for $\frac{d}{d\alpha}$ except that the first term is equal to 0 (rather than $2d_x$) and the second term is equal to $Y_{Lx} d_x$ rather than 0. Finally rather than working out $\frac{du}{da}$ as in equation B10 we work out $\frac{du}{db}$ as below:

$$\frac{du}{d\beta} = \sum_{x=0}^{88} \frac{du}{d\beta}\left[\exp(2(\alpha + \beta Y_{Lx}) + 1)\right] \qquad \text{[B13]}$$

$$\frac{du}{d\beta} = \sum_{x=0}^{88} \frac{du}{d\beta}[\exp 2(\alpha + \beta Y_{Lx})] + \frac{du}{d\beta}(1) \qquad \text{[B14]}$$

$$\frac{du}{d\beta} = \sum_{x=0}^{88} Y_{Lx}[\exp 2(\alpha + \beta Y_{Lx})]$$ [B15]

Thus:

$$\frac{d}{d\beta}\sum_{x=0}^{88} 2\alpha d_x + \beta Y_{Lx} d_x - n_x \ln(1 + (\exp(2(\alpha + \beta Y_{Lx})))) = \sum_{x=0}^{88} Y_{Lx} d_x - \frac{n_x Y_{Lx}[\exp 2(\alpha + \beta Y_{Lx})]}{\exp(2(\alpha + \beta Y_{Lx}) + 1)}$$ [B16]

Appendix C: Health survey for England - disability module

Disability Type	Survey Question	Response	Disability score
Locomotor	What is the furthest you can walk on your own without stopping and without discomfort?	Only a few steps More than a few steps but less than 200m More than 200m	2 1 0
	Can you walk up and down a flight of 12 stairs without resting?	Not at all Only if hold on and take rests Yes	2 1 0
	Can you, when standing, bend down and pick up a shoe from the floor?	No Yes	1 0
Personal care	Can you get in and out of bed on your own?	Only with someone to help With some difficulty Without difficulty	2 1 0
	Can you get in and out of a chair on your own?	Only with someone to help With some difficulty Without difficulty	2 1 0
	Can you dress and undress yourself on you own?	Only with someone to help With some difficulty Without difficulty	2 1 0
	Can you wash your face and hands on your own?	Only with someone to help With some difficulty Without difficulty	2 1 0
	Can you feed yourself, including cutting up food?	Only with someone to help With some difficulty Without difficulty	2 1 0
	Can you get to and use the toilet on your own?	Only with someone to help	2

Disability Type	Survey Question	Response	Disability score
		With some difficulty Without difficulty	1 0
Seeing	Can you see well enough to recognise a friend at a distance of four metres (across the road)? If no can you see well enough to recognise a friend at a distance of one metre (at arms length)	Cannot recognise a friend at 1m Can recognise a friend at 1m but not at 4m Can recognise a friend at 4m	2 1 0
Hearing	Is your hearing good enough to follow a TV programme at a volume others find acceptable? If not, can you follow a TV programme with volume turned up?	Cannot follow a TV programme even with the volume turned up Can follow a TV programme with the volume turned up Can follow a TV programme at normal volume	2 1 0
Communication	Can you speak without difficulty?	Yes No	1 0
	Do you have problems communicating with other people?	Difficulty communicating with close relatives Difficulty communicating with other people No communication problem	2 1 0

Disability Scores in the Health Survey for England (2001).

Source: Disability report: Health Survey for England 2001 (Bajekal and Prescott 2003)

Appendix D – Relational models fitted to schedules of rates at five year age bands

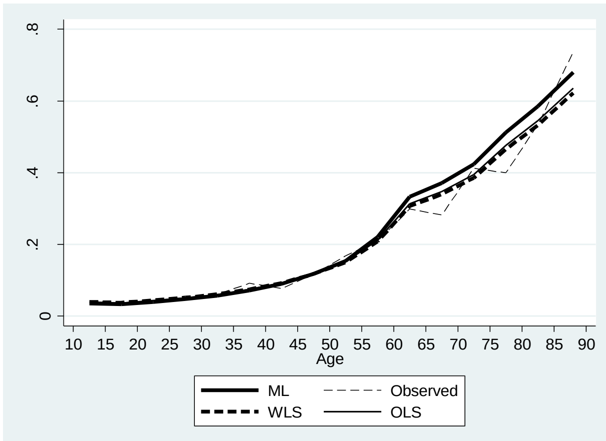
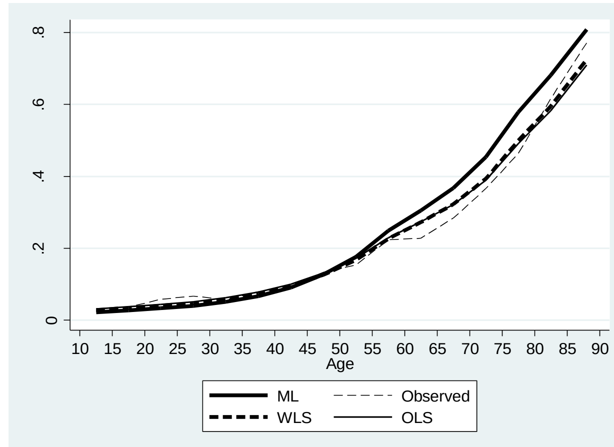
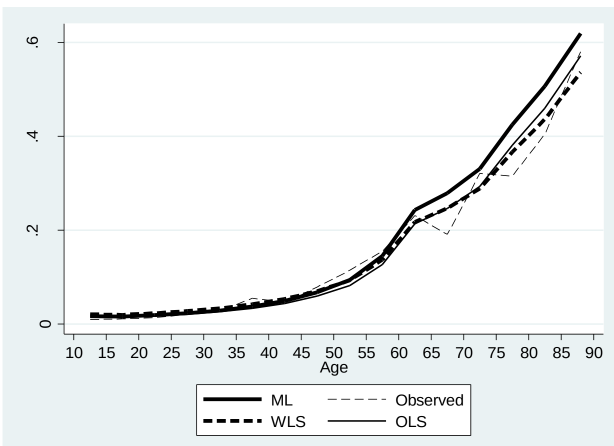
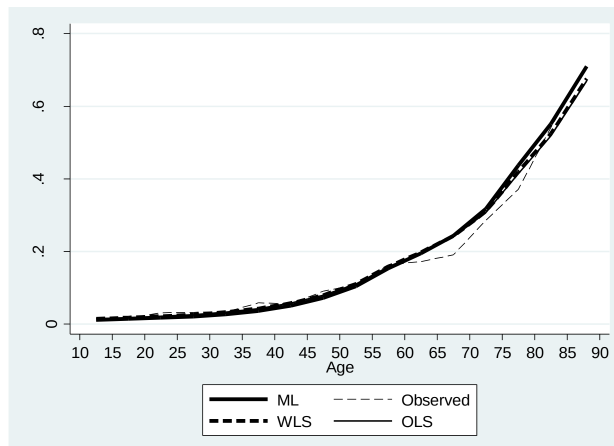
Overall disability (males)	Overall disability (females)
	
Locomotor disability (males)	**Locomotor disability (females)**
	

Figure D1: Brass relational model schedules of overall and locomotor disability estimated using OLS, WLS and ML (five year rates)

Appendix E – More complex relational models

The Brass relational model has been extended by Ewbank et al. (1983) who develop a relational function involving two additional parameters giving greater flexibility and a better model fit at the oldest and younger ages. The additional parameters have little effect at the middle ages and the Ewbank model reduces to a Brass model when these additional parameters tend to zero.

Marshall et al. (2012) develop a Reduced Ewbank model with three parameters which gives a better fit than the Brass model for most disability types. Here we fit this Reduced Ewbank model using maximum likelihood and weighted least squares.

The Ewbank model is defined as below:

First, define function T (which comprises the two additional parameters λ and κ) as:

$$T(p_{xL};\lambda)=\frac{\left(\frac{p_{xL}}{1-p_{xL}}\right)^{\lambda}-1}{2\lambda} \quad \text{if } p_{xL}\geq 0.5 \qquad \text{[E1]}$$

$$T(p_{xl};\kappa)=\frac{1-\left(\frac{1-p_{xL}}{p_{xL}}\right)^{\kappa}}{2\kappa} \quad \text{if } p_{xL}<0.5 \qquad \text{[E2]}$$

Let:

ω =1 if $p_{xL} \geq$0.5 and 0 otherwise

χ =1 if p_{xL}<0.5 and 0 otherwise

Then:

$$\frac{1}{2}\left(\log_e\left(\frac{p_{xd}}{1-p_{xd}}\right)\right)=\left(\alpha\ +\beta\ \ T(p_{xL}:\lambda\ )\right)+e_x \qquad \text{if } p_{xL}\ \geq 0.5 \qquad [E3]$$

$$\frac{1}{2}\left(\log_e\left(\frac{p_{xd}}{1-p_{xd}}\right)\right)=\left(\alpha\ +\beta\ \ T(p_{xL}:\kappa\ )\right)+e_x \qquad \text{if } p_{xL}\ < 0.5 \qquad [E4]$$

Log likelihood

For the calculation of the log-likelihood, the same working applies as for the Brass log likelihood except that we replace Y_{xL} with the function T (defined above). The final likelihood function is then:

$$\sum_{x=0}^{88} 2\alpha d_x + \beta T(P_{xL}:\lambda)d_x - n_x \ln(1+(\exp(2(\alpha+\beta T(P_{xL}:\lambda))))) \qquad \text{if } p_{xL}\ \geq 0.5 \qquad [E5]$$

$$\sum_{x=0}^{88} 2\alpha d_x + \beta T(P_{xL}:\kappa)d_x - n_x \ln(1+(\exp(2(\alpha+\beta T(P_{xL}:\kappa))))) \qquad \text{if } p_{xL}\ < 0.5 \qquad [E6]$$

The Reduced Ewbank model maximum likelihood model drops one of the additional parameters replacing either equation D3 or D4 with the Brass model (see equation 4 in the main text). The log likelihood function for the Reduced Ewbank model, comprises one of the above two equations (D5 or D6) and the log likelihood for the Brass model (see below) depending on which parameter is dropped.

$$\sum_{x=0}^{88} 2\alpha d_x + \beta Y_{Lx} d_x - n_x \ln(1+(\exp(2(\alpha+\beta Y_{Lx})))) \qquad [E7]$$

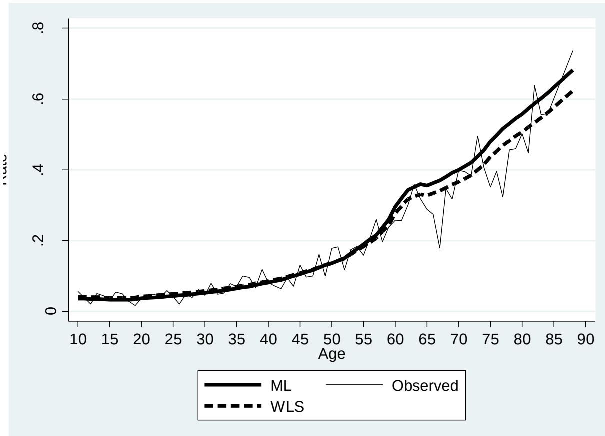
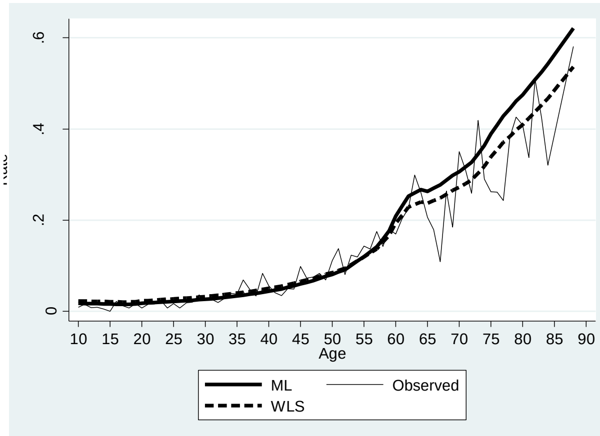
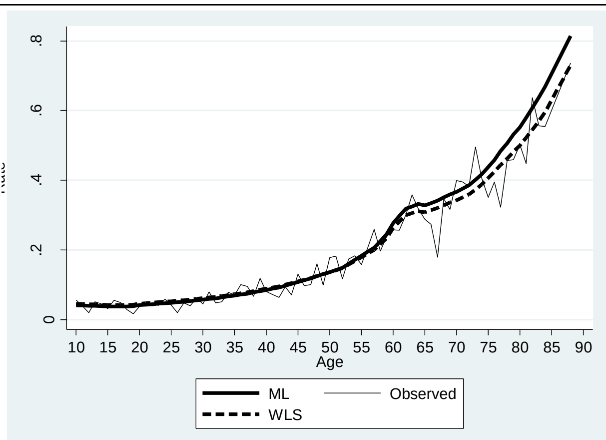
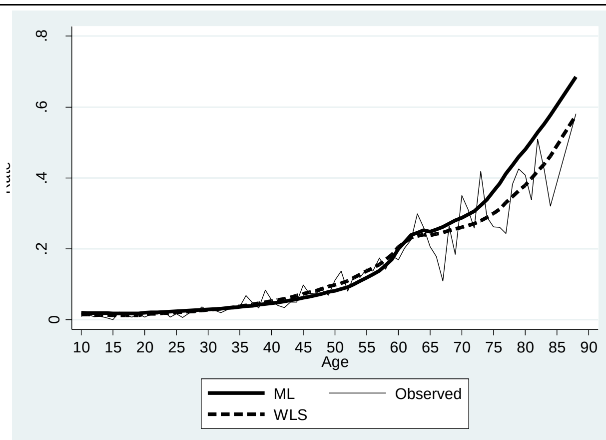
Brass – Overall disability (males)	Brass – Locomotor disability (males)
	
Reduced Ewbank (l) – Overall disability (males)	**Reduced Ewbank (l) – Locomotor disability (males)***
	

Figure E1: Graphs comparing model rates from Reduced Ewbank relational models fitted using weighted least squares and maximum likelihood

***Different model types are selected for locomotor disability (males). In ML a reduced Ewbank with l selected whilst in WLS a reduced Ewbank with k was selected. The graph here is a reduced Ewbank with l for both ML and WLS.**