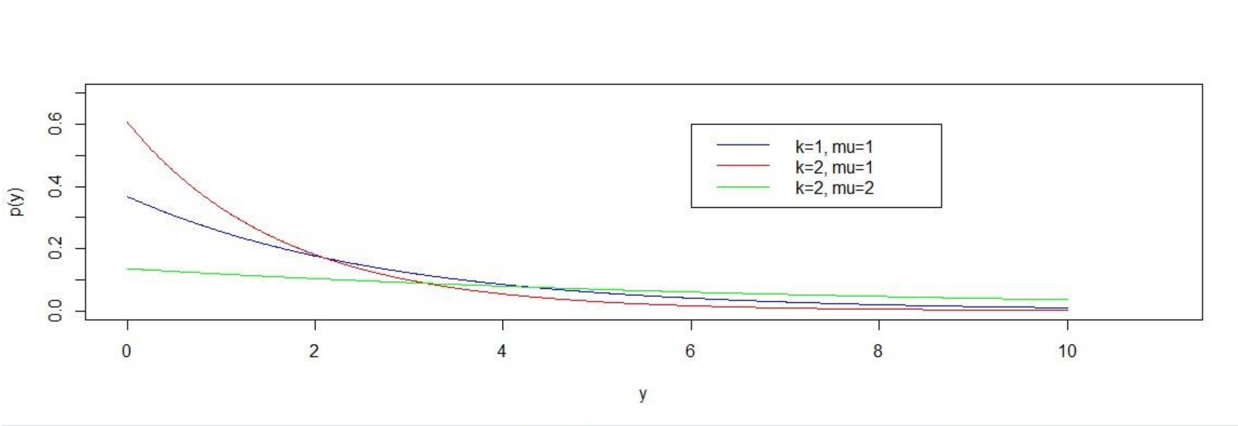
ASSIGNMENT-2

The exponential distribution is a probability distribution for non-negative real numbers. It is often used to model waiting or survival times. The version that we will look at has a probability density function of the form p(y | v) = exp((−e^(−v)*)*y – v),

where y ∈ R+, i.e., y can take on the values of non-negative real numbers. In this form it has one parameter: a log-scale parameter v. If a random variable follows a gamma distribution with log-scale v we say that Y ∼ Exp(v). If Y ∼ Exp(v), then E [Y] = e^v and V [Y] = e^2v

1)

Here is the plotted graph:



Imagine we are given a sample of n observations y = (y1, . . . , yn). Write down the joint probability of this sample of data, under the assumption that it came from an exponential distribution with log-scale parameter v (i.e., write down the likelihood of this data).

Take the negative logarithm of your likelihood expression and write down the negative loglikelihood of the data y under the exponential model with log-scale v.

Question - 2

2.2

$$\rightarrow p(y|v) = \prod_{i=1}^{n} \left(\frac{1}{e^{y_i \times e^{-v}} \times e^{v}} \right)$$

$$= \left(\prod_{i=1}^{n} \left(\frac{1}{e^{y_i \times e^{-v}}} \right) \right) \times \frac{1}{e^{nv}}$$

$$= \frac{1}{e^{nv}} \times \prod_{i=1}^{n} \left(e^{-y_i \times e^{-v}} \right)$$

$$= \frac{1}{e^{nv}} \times \prod_{i=1}^{n} \left(\left(e^{-y_i}\right)^{e^{-v}} \right)$$

$$= \frac{1}{e^{nv}} \times e^{-\sum_{i=1}^{n} y_i \times e^{-v}}$$

let $m = \sum_{i=1}^{n} y_i$

$$= \frac{1}{e^{nv}} \times e^{-me^{-v}}$$

$$\rightarrow p(y|v) = \boxed{e^{(-me^{-v} - nv)}}$$

2.3

$$L(y|v) = -\log(p(y|v))$$

$$= -\log\left(e^{-(me^{-v}+nv)}\right)$$

$$= \boxed{me^{-v} + nv}$$

Derive the maximum likelihood estimator ˆv for v. That is, find the value of v that minimises the negative log-likelihood.

2.4

$\rightarrow L(y|v) = m e^{-v} + nv$

$\Rightarrow \dfrac{\partial L}{\partial v} = 0$

$\Rightarrow \dfrac{\partial L}{\partial v} = -e^{-v} \sum_{i=1}^{n} y_i + n = 0$

$\Rightarrow n = e^{-v} \sum_{i=1}^{n} y_i$

$\rightarrow e^{v} = \dfrac{\sum_{i=1}^{n} y_i}{n}$

$\rightarrow v = \boxed{\log \left| \dfrac{\sum_{i=1}^{n} y_i}{n} \right|} = \boxed{\log \left| \dfrac{m}{n} \right|}$

Determine the approximate bias and variance of the maximum likelihood estimator ˆv of v for the exponential distribution.

2.5

→ Bias :-

$$B(\hat{v}) = E[\hat{v}(y)] - v$$

$$\rightarrow E(\hat{v}_m) = E\left(\log\left(\frac{y_1+y_2+\cdots+y_n}{n}\right)\right)$$

$$= \log\left(E\left(\frac{y_1}{n}\right)\right) + \log\left(E\left(\frac{y_2}{n}\right)\right) + \cdots \log\left(E\left(\frac{y_n}{n}\right)\right)$$

→ As in the question, $E(y) = e^v$.

$$\Rightarrow E(\hat{v}_m) = \log\left(\frac{e^v}{n}\right) + \log\left(\frac{e^v}{n}\right) + \cdots \log\left(\frac{e^v}{n}\right)$$

$$= \frac{v}{n} + \frac{v}{n} + \cdots \frac{v}{n}$$

$$= \frac{nv}{n}$$

$$\rightarrow E(\hat{v}_m) = \boxed{v} \implies B(\hat{v}) = v - v = \boxed{0} \rightarrow \text{Bias.}$$

→ Variance :-

$$\rightarrow V(\hat{v}_m) = V(\hat{v}(\bar{y})) = V\left(\log\left|\frac{y_1+y_2+\cdots y_n}{n}\right|\right)$$

→ Since y_i are independent so variance will become sum of variance.

$$\Rightarrow V(\hat{v}_m) = V\left(\log\left(\frac{y_1}{n}\right) + \log\left(\frac{y_2}{n}\right) + \cdots \log\left(\frac{y_n}{n}\right)\right)$$

$$= \log^2\left(V\left(\frac{y_1}{n}\right)\right) + \log^2\left(V\left(\frac{y_2}{n}\right)\right) + \cdots \log^2\left(V\left(\frac{y_n}{n}\right)\right)$$

$$= \log^2\left(\frac{e^{2v}}{n^2}\right) + \log^2\left(\frac{e^{2v}}{n^2}\right) + \cdots \log^2\left(\frac{e^{2v}}{n^2}\right)$$

$$= 2 \times \frac{2V}{n^2} \times n$$

$$\rightarrow Var(\hat{V}_m) = \boxed{\frac{4V}{n}}$$