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Jump-Diffusion Processes and the Term Structure of Interest Rates

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ABSTRACT

The authors investigate the term structure of interest rates when the underlying state variables and production technologies follow the jump-diffusion processes. Even in some cases where the traditional expectations theory about the term structure is consistent with general equilibrium under diffusion processes, the traditional theory is not consistent under jump-diffusion processes. It is shown that bond prices are strictly higher under jump risks than otherwise and that consumers with logarithmic utility functions will develop hedge portfolios in the presence of jump diffusion.

A NUMBER OF CONTINUOUS-TIME asset-pricing models have been derived under stochastic diffusion processes with continuous sample paths. (See, e.g., Black and Scholes [2], Merton [21], Breeden [3], and Cox, Ingersoll, and Ross [8].) However, recent empirical studies (see, e.g., Brown and Dybvig [4], Feinstone [10], and Jarrow and Rosenfeld [17]) suggest that the underlying processes may follow discontinuous sample paths. Asset-pricing models when the underlying processes have discontinuous sample paths have not been adequately developed although the reasonableness of the topic has been recognized. (See Merton [22].) This difference between the development of the underlying theoretical models and the empirical observations motivates this paper.

In two influential papers, Cox, Ingersoll, and Ross [8, 9] derive a general-equilibrium asset-pricing model assuming diffusion processes and use it to analyze the term structure of interest rates. In this paper, we employ their methodology to derive a model that is driven by jump-diffusion processes. Recently, Ahn and Thompson [1] developed a similar model to examine the effect of regulatory risks on the valuation of public utilities. They found that these "jump risks" were priced even though they were uncorrelated with market factors. This suggests that jump risks cannot be ignored in the pricing of assets. Furthermore, it suggests that jump risks may have important implications for interest rates.

In this paper, we seek to answer the following questions:

1. How is the investment behavior of a consumer affected by jump risks?
2. How do jump risks affect the valuation of an asset?
3. How are interest rates affected by jump processes?
4. Is the traditional expectations theory valid under jump-diffusion processes?

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In Section I, we describe the economy where jump diffusions exist and derive an asset-pricing model. This model will then be used in Section II to analyze the equilibrium interest rate under jump-diffusion processes. Section II also examines the implications for the term structure of interest rates. In Section III, we derive a single-beta asset-pricing model and examine why a consumption Capital Asset Pricing Model (CAPM) would not hold under jump-diffusion processes. In Section IV, we impose strong restrictions on preferences, technologies, and state variables in order to study more easily the analytical structure of the equilibrium interest rate and the investment behavior of consumers. Section V provides a brief summary and conclusions.

I. The Economy

Consider an economy with a single good and a large number of infinitely lived consumers, identical in their endowments and preferences.[1] Consumers have homogeneous beliefs about production technologies and state variables. Each consumer seeks to maximize his or her lifetime expected utility given by

$$\mathrm{E}_0\left[\int_0^\infty \exp(-\rho t)u(C(t), X, t)\, dt\right], \tag{1}$$

where E_0 is a conditional expectation operator based on the consumer's information set at time 0, ρ is the rate of impatience, $C(t)$ denotes the consumption rate at time t, X is a vector of the state variables, and $u(\cdot)$ is a von Neumann-Morgenstern utility function. The utility function u is strictly concave, increasing, and twice differentiable. We assume that $u(0) = 0$ and $u'(0) = \infty$ to ensure interior solutions.

We differ from the models cited above in that we assume that there exist K state variables, determined by a system of the stochastic differential equations[2]

$$dX(t) = \{\mu(X) - M^*(X)\}dt + S(X)dZ + M(X)dY, \tag{2}$$

[1] This economy is similar in scope to that postulated by Cox, Ingersoll, and Ross [8]. The extension to the multi-good case is straightforward. Since we focus on the underlying stochastic processes, we would consider only a single good economy.

[2] Stochastic differential equations used in this paper are discussed in detail in Kushner [18] or Gihman and Skorohod [13]. Let $dx = c(x, t)dt + A(x, t)dZ + B(x, t)dY$. Let $F(x, t)$ be twice continuously differentiable in (x, t). Then by Itô's Lemma,

$$dF = [F_t(x, t) + F_x(x, t)c(x, t) + (½)\mathrm{tr}(F_{xx}(x, t)A(x, t)A(x, t)')]dt$$
$$+ [F_x(x, t)A(x, t)]dZ + [F(x + B(x, t), t) - F(x, t)]dY.$$

The expected value of dF conditioned on $x(t)$ is

$$\mathrm{E}[dF] = L(F)dt = [F_t(x, t) + (½)\mathrm{tr}(F_{xx}(x, t)A(x, t)A(x, t)')$$
$$+ F_x(x, t)c(x, t) + \textstyle\sum_i \pi_i \mathrm{E}_{gi}[F(x + B(x, t)g_i, t) - F(x, t)]]dt,$$

where g_i is the ith jump magnitude and L is a differential operator.

where

$\mu(X)$ is a $K \times 1$ vector of the expected drifts,
$S(X)$ is a $K \times (K+N)$ diffusion matrix such that SS' is positive definite,
Z is a $(K+N) \times 1$ vector of independent Wiener processes,
Y is a $(K+N) \times 1$ vector of independent Poisson processes with a $(K+N) \times 1$ intensity vector $\pi(X)$ (i.e., $\pi(X)$ is a probability vector for components' jump),
$M(X)$ is a $K \times (K+N)$ matrix that translates jumps in Y into the corresponding jumps in X, and
M^* is a $K \times 1$ vector of the expected values of the jump magnitudes of state variables; i.e., the ith element of M^* is $\sum_{j=1}^{K+N} \pi_j \mathrm{E}_{g_j}[M_{ij} g_j]$, where M_{ij} is the (i, j)th element of the matrix M and E_{g_j} is the expectation operator with respect to the jump amplitude g_j.

The jump amplitude of the ith component of Y, g_i, is assumed to be independent of g_j for $j \neq i$. It is also independent of π and Z. Consequently, the underlying processes for the state variables follow discontinuous sample paths.

The economy also has N distinct technologies for production of the single good. The $(N \times 1)$ output vector, dP, is also determined by a system of the stochastic differential equations,

$$dP = I_p\{\alpha(X) - Q^*(X)\}dt + I_pG(X)dZ + I_pQ(X)dY, \tag{3}$$

where

$\alpha(X)$ is an $N \times 1$ vector of the expected rates of return on the output,
$G(X)$ is an $N \times (K+N)$ diffusion matrix such that GG' is positive definite,
$Q(X)$ is an $N \times (K+N)$ matrix,
I_p is an $N \times N$ diagonal matrix with an ith element that is the ith element of P,
Q^* is an $N \times 1$ vector with an ith element that is $\sum_{j}^{K+N} \pi_j \mathrm{E}_{g_j}[Q_{ij} g_j]$, where Q_{ij} is the (i, j)th element of the matrix Q, and
the vectors Z and Y are as previously defined.

All consumers are assumed to be competitive price takers. They have equal access to production technologies, and continuous investment and trading are available without transaction costs. There is a competitive market for instantaneous borrowing and lending at the spot interest rate r. There is also a competitive market for a default-free discount bond with dynamics that are given by

$$dF = F\{\beta_F - e_F^*\}dt + F\sigma_F dZ + Fe_F dY, \tag{4}$$

where β_F is the expected rate of return on the default-free discount bond F, σ_F is a $(K+N) \times 1$ diffusion vector, e_F is a $(K+N) \times 1$ vector, and e_F^* is the expected value of the jump magnitude of F, which is $\sum_{i}^{K+N} \pi_i \mathrm{E}_{g_i}[e_{F_i} g_i]$ where e_{F_i} is the ith element of e_F. The value of β_F will be determined subsequently by Theorem 3.

By Itô's Lemma, the values of σ_F and e_F are given by

$$F\sigma_F = \quad F_W \sigma_W + F_X S \tag{5}$$

$$Fe_F = \begin{bmatrix} F(W + e_{W_1}, X + M_1, t) - F(W, X, t) \\ \vdots \\ F(W + e_{W_{K+N}}, X + M_{K+N}, t) - F(W, X, t) \end{bmatrix}'. \tag{6}$$

In (6), e_{W_i} is the wealth jump change created by only the ith element of Y, and M_i is the ith column of the matrix M.

At each instant, each consumer will allocate his or her wealth among investments in the production technologies, the discount bonds, riskless borrowing or lending, and consumption (C). Let $a(t)$ be the vector of the proportion of wealth invested in each of the N production technologies, and let $b(t)$ be the number of discount bonds in the investor's portfolio. Define $W(t)$ to be the consumer's wealth at time t. Each consumer solves the problem given by

$$\max_{a,b,C} \mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} u(C(t), X, t)\, dt \right]$$

subject to

$$dW(t) = \beta_W dt + \sigma_W dZ + e_W dY$$

and

$$W(0) = W_0, \tag{7}$$

where

$$\beta_W = Wa'(\alpha - r1_N - Q^*) + rW + bF(\beta_F - r - e_F^*) - C, \tag{8}$$

$$\sigma_W = Wa'G + bF\sigma_F, \tag{9}$$

$$e_W = Wa'Q + bFe_F, \tag{10}$$

and 1_N is an $N \times 1$ vector of ones.

We assume that a solution exists and that the control functions, $a(W, X, t)$, $b(W, X, t)$, and $C(W, X, t)$, and the indirect utility function,

$$J(W(t), X(t)) = \max_{\{a,b,C\}} \mathrm{E}_t \left[\int_t^\infty \exp(-\rho(s - t)) u(C(s), x, s)\, ds \right], \tag{11}$$

satisfy the Bellman equation,

$$\begin{aligned} 0 = \max_{\{a,b,C\}} \Big[& u(C, X, t) - \rho J + \beta_W J_W + (1/2)\sigma_W \sigma_W' J_{WW} \\ & + \sigma_W S' J_{WX} + (\mu - M^*) J_X + (1/2) \sum_i \sum_j \mathrm{cov}(X_i, X_j) J_{X_i X_j} \\ & + \textstyle\sum_i^{K+N} \pi_i \mathrm{E}_{g_i}[J(W + e_{W_i} g_i, X + M_i g_i) - J(W, X)] \Big]. \end{aligned} \tag{12}$$

The first-order conditions for the consumer's optimization are

$$u_C - J_W = 0, \tag{13}$$

$$J_W(\alpha - r1_N - Q^*) + G\sigma'_W J_{WW} + GS'J_{WX} + \textstyle\sum_i^{K+N} \pi_i \mathrm{E}_{gi}[J_W(W + e_{W_i}g_i, X + M_ig_i)Q_ig_i] = 0, \quad (14)$$

$$J_W(\beta_F - r - e_F^*) + \sigma_F\sigma'_W J_{WW} + \sigma_F S' J_{WX} + \textstyle\sum_i^{K+N} \pi_i \mathrm{E}_{gi}[J_W(W + e_{W_i}g_i, X + M_ig_i)e_{F_i}g_i] = 0. \quad (15)$$

The additions of the last terms in (14) and (15) mark the differences from the first-order conditions of Cox, Ingersoll, and Ross [8]. It is well known that no trading is Pareto optimal in an identical consumer economy, and so, following Cox, Ingersoll, and Ross [8], we have the equilibrium conditions

$$a^{*\prime}1_N = 1, \quad (16)$$

$$b = 0, \quad (17)$$

where a^* is an $N \times 1$ vector denoting the optimal proportions invested in technologies.

We follow the usual rational-expectations approach assuming that each consumer solves his or her optimization problem taking α, r, and β_F as given. Then α, r, and β_F are determined in the model by the equilibrium conditions.

The first-order condition (14) produces an asset-pricing model that differs from Merton's [21] multi-beta CAPM because of the additional term in (14). Furthermore, even if we assume a time-additive state-independent utility function, like Breeden [3], a single-consumption-beta CAPM cannot be obtained. The movements of the investment opportunities cannot be completely captured by the consumption beta. We will examine in detail the reason why a single-consumption-beta CAPM does not hold in Section III, where we will derive a single-beta asset-pricing model.

II. The Equilibrium Interest Rate

In this section, we investigate the spot interest rate and the equilibrium term structure of interest rates. We note the differences in our model from that of Cox, Ingersoll, and Ross [8, 9]. In what follows, we state without proof[3] a number of theorems and compare the implications of these theorems with those derived by exclusively using diffusions.

THEOREM 1: *The equilibrium spot interest rate is given by*

$$r = a^{*\prime}\alpha - \left(-\frac{J_{WW}}{J_W}\right)\frac{\mathrm{var}\ W}{W} - \textstyle\sum_i^K \left(-\frac{J_{WX_i}}{J_W}\right)\frac{\mathrm{cov}(W, X_i)}{W} + \textstyle\sum_i^{K+N} \pi_i \mathrm{E}_{gi}\left[\left\{\frac{J_W(W + e_{W_i}g_i, X + M_ig_i) - J_W}{J_W}\right\}\frac{e_{W_i}g_i}{W}\right]. \quad (18)$$

The difference between the equilibrium interest rate given above and that of Cox, Ingersoll, and Ross [8] is the addition of the last term in (18). Since the remaining terms are identical, our discussion focuses on that last term. (The specific values of a^* and indirect utility function-related terms (e.g., J_W, J_{WW}, etc.) will, of

[3] Proofs are available from the authors upon request.

course, differ between the two models, but we ignore this fact to concentrate on the structure.) It is a weighted sum of the inner products of proportional changes in marginal utility from jumps and proportional wealth changes from jumps. The last term would be zero only if $J_W(W + e_W g_i, X + M_i g_i)/J_W$ were 1 for all i. This is an unlikely case, so we can rest assured that the additional term would exist in most cases.

Cox, Ingersoll, and Ross [8] have shown that their $J(W, X)$ is strictly increasing and strictly concave in $W(t)$. The same result holds for our model. Consequently, the proportional change in marginal utility has a sign different from that of the proportional change in wealth, thus making the value of the last term in (18) negative.[4] Thus, the jump components have the effect of decreasing the equilibrium interest rate.[5] Economically, the result is reasonable since a hedging service is provided by the discount bond with certain payoff against uncertainty from jumps.

The last term in (18) can be rewritten as

$$\sum_i^{K+N} \pi_i \left[\frac{\mathrm{E}_{g_i}[J_W(W + e_{W_i} g_i, X + M_i g_i) - J_W]\mathrm{E}_{g_i}[e_{W_i} g_i]}{J_W W} + \frac{\mathrm{cov}_{g_i}[J_W(W + e_{W_i} g_i, X + M_i g_i), e_{W_i} g_i]}{J_W W} \right] \tag{19}$$

to provide additional insight. Since J_W is decreasing in W, the covariance term must always be negative. The higher the variance of the distribution of the jump amplitude g_i, the lower the spot equilibrium interest rate, ceteris paribus. This formulation is also easily interpreted. It suggests that the greater the uncertainty about the future wealth, the greater the value of a bond providing certain payoff in the future.

For completeness, we can also represent the interest rate in a manner similar to that of Theorem 1 of Cox, Ingersoll, and Ross [8].

[4] We can demonstrate this in detail as follows:

$$\begin{aligned}
&\mathrm{E}_{g_i}\left[\frac{J_W(W + e_{W_i} g_i, X + M_i g_i) - J_W}{J_W} \times \frac{e_{W_i} g_i}{W}\right] \\
&= \mathrm{E}_{g_i}\left[\frac{J_W(W + e_{W_i} g_i, X + M_i g_i) - J_W(W, X + M_i g_i)}{J_W} \times \frac{e_{W_i} g_i}{W} \right. \\
&\qquad \left. + \frac{J_W(W, X + M_i g_i) - J_W}{J_W} \times \frac{e_{W_i} g_i}{W}\right].
\end{aligned}$$

The first half of the equation has a negative sign since marginal utility of wealth is decreasing in wealth. The second half also does because $J_{WX_i} > 0$ for unfavorable changes (i.e., $e_{W_i} g_i < 0$) and $J_{WX_i} < 0$ for favorable changes, $e_{W_i} g_i > 0$).

[5] Of course, the overall effect of the jump components on the level of the equilibrium interest rate can be found only after obtaining closed-form solutions for a^* and indirect utility function-related terms. It may decrease or increase the equilibrium interest rate.

THEOREM 2: *The equilibrium interest rate is*

$$r = \rho - \frac{L(J_W)}{J_W}, \tag{20}$$

where

$$\begin{aligned} L(F) = {} & \textstyle\sum_i^K (\mu_{X_i} - M^*)\partial F/\partial X_i + (1/2) \sum_i^K \sum_j^K (SS')_{ij}\partial^2 F/\partial X_i \partial X_j \\ & + \beta_W \partial F/\partial W + (1/2)\sigma_W \sigma'_W \partial^2 F/\partial W^2 + \sigma_W S' \partial^2 F/\partial W \partial X \\ & + \textstyle\sum_i^{K+N} \pi_i \mathrm{E}_{gi}[F(W + e_{W_i}, X + M_i) - F(W, X)]. \end{aligned} \tag{21}$$

Theorem 2 is identical to Theorem 1 in Cox, Ingersoll, and Ross [8] except for the difference in the definition of the differential operator L, which, in (21), includes jump-related terms.

The rate of return on discount bonds is given in Theorem 3.

THEOREM 3: *The equilibrium expected rate of return on the default-free discount bond is*

$$\begin{aligned} (\beta_F - r)F = {} & [\phi_W, \phi_{X_1}, \cdots, \phi_{X_K}][F_W, F_{X_1}, \cdots, F_{X_K}]' \\ & + [\pi_1, \pi_2, \cdots, \pi_{K+N}][\phi_1, \phi_2, \cdots, \phi_{K+N}]', \end{aligned} \tag{22}$$

where

$$\phi_W = \left[\left(-\frac{J_{WW}}{J_W}\right)\mathrm{var}\ W + \textstyle\sum_i^K \left(-\frac{J_{WX}}{J_W}\right)\mathrm{cov}(W, X_i)\right],$$

$$\phi_{X_i} = \left[\left(-\frac{J_{WW}}{J_W}\right)\mathrm{cov}(W, X_i) + \textstyle\sum_j^K \left(-\frac{J_{WX_j}}{J_W}\right)\mathrm{cov}(X_i, X_j)\right], \quad \textit{and}$$

$$\begin{aligned} \phi_i = \mathrm{E}_{g_i}\bigg[& -\left\{\frac{J_W(W + e_{W_i}g_i, X + M_i g_i) - J_W}{J_W}\right\} \\ & \times \{F(W + e_{W_i}g_i, X + M_i g_i, t) - F(W, X, t)\}\bigg]. \end{aligned}$$

The difference between Theorem 3 above and Theorem 2 in Cox, Ingersoll, and Ross [8] is the addition of the second half of the right-hand side of the equation. Consequently, we focus on that difference. The second half of equation (22) is a weighted sum of the inner products of the proportional changes in marginal utility from jumps and the jump changes of the discount bond's value. The equilibrium excess expected rate of return on the discount bond is determined by wealth, the K state variables, and the $K + N$ jump component. The greater the covariability between the jump changes of the discount bond's value and the proportional changes in marginal utility, the smaller the equilibrium expected rate of return on the discount bond, ceteris paribus. The underlying intuition is quite simple. The consumer would charge a smaller expected rate of return on the discount bond that provides greater payoff when marginal utility is greater.

Since the payoff of the discount bond is certain, its value is assumed to be independent of wealth, i.e., $F_W = 0$. Consequently, (22) can be rewritten as

$$\beta_F - r = \sum_i^K \frac{F_{X_i}}{F}\left[\left(-\frac{J_{WW}}{J_W}\right)\operatorname{cov}(W, X_i) + \sum_j^K \left(-\frac{J_{WX_j}}{J_W}\right)\operatorname{cov}(X_i, X_j)\right]$$
$$+ \sum_i^{K+N} \pi_i \mathrm{E}_{g_i}\left[-\left\{\frac{J_W(W + e_{W_i}g_i, X + M_i g_i)}{J_W} - 1\right\}\right.$$
$$\left.\times \left\{\frac{F(W + e_{W_i}g_i, X + M_i g_i, t)}{F} - 1\right\}\right]. \tag{23}$$

If $\pi_i = 0$ for all i, then (23) becomes (38) in Cox, Ingersoll, and Ross [7].

Cox, Ingersoll, and Ross [7] define the term premium to be $\beta_F - r$. They also state that the traditional local expectations hypothesis holds if the term premium is zero. They have demonstrated that the local expectations hypothesis does not hold in general. However, they have noted that the local expectations hypothesis holds in three cases. In what follows, we will show that, in two cases among them, the local expectations hypothesis does not hold under the jump-diffusion processes. Only the third case remains.

In the first case, the dynamics of aggregate consumption are locally certain and the utility function is state independent. As Breeden [3] argues, the equilibrium excess expected rate of return over the equilibrium interest rate is determined by its instantaneous covariance with aggregate consumption. If aggregate consumption is locally certain, the excess rate of return must be zero. This is true without the jump components. However, under the jump-diffusion processes, even if aggregate consumption is locally certain, the excess rate of return cannot be zero because of the additional term from jumps in (14). In other words, Breeden's CAPM argument does not work in this framework. Consequently, the local expectations hypothesis does not hold for this case.

In the second case, Cox, Ingersoll, and Ross [7] assume that the returns on physical capital are uncorrelated with changes in the state variables and that the utility function is state independent and Bernoulli logarithmic.[6] Without the jump components, the term premium must be zero, as readily shown in (23). However, the term premium cannot be zero, in general, under the jump-diffusion processes since the third term in (23) cannot be zero. Thus, the jump diffusion rules out the local expectations hypothesis in this case.

In the third case, Cox, Ingersoll, and Ross [7] argue that, if the real interest rate is nonstochastic under certain inflation, the local expectations hypothesis holds in terms of the real interest rate even though the nominal interest rate may be stochastic. This follows directly from the constant real interest rate. They state sufficient conditions for the nonstochastic real interest rate under uncertain inflation and note that these conditions require a time-additive iso-

[6] If the instantaneous utility function is the logarithmic type, i.e., $u(C) = \ln C$, then the term $J_{WX} = 0$ where J denotes the indirect utility function. This is true under the jump-diffusion processes as well. According to Merton's [21] terminology, it means zero hedging demand against the unfavorable changes in the investment opportunities from changes in the state variables. The zero value of J_{WX} implies that the consumer is indifferent with respect to the following two choices: choice A has the even chance for $(\underline{W}, \underline{X})$ or $(\bar{W}, \bar{X})$ and choice B has the even chance for $(\underline{W}, \bar{X})$ or $(\bar{W}, \underline{X})$ where $\bar{W} > \underline{W}$ and $\bar{X} > \underline{X}$.

elastic utility function with a constant rate of time impatience and a constant real investment opportunity set (i.e., no state variable). These conditions are still sufficient for delivering the nonstochastic interest rate under the jump-diffusion processes. This can be easily seen from (23). Hence, the local expectations hypothesis holds under the jump-diffusion processes in this special case.

To sum up, we must conclude that the local expectations theory is valid in the presence of jump diffusions only in this special third case.

III. A Single-Beta Asset-Pricing Model

In this section, we derive a single-beta asset-pricing model and examine in detail how Breeden's single-consumption-beta CAPM would not hold.

The equilibrium price of a discount bond can be derived by the methods used by Cox, Ingersoll, and Ross [8].

THEOREM 4: *The equilibrium price of the discount bond satisfies*

$$\begin{aligned}
&(1/2)\operatorname{var} W F_{WW} + \textstyle\sum_i^K \operatorname{cov}(W, X_i) F_{WX_i} + (1/2) \sum_i^K \sum_j^K \operatorname{cov}(X_i, X_j) F_{X_i X_j} \\
&+ \Bigg\{ r(W, X, t) W - C^*(W, X, t) \\
&- \textstyle\sum_i^{K+N} \pi_i \mathrm{E}_{gi}\left[\dfrac{J_W(W + e_{W_i} g_i, X + M_i g_i)}{J_W} e_{W_i} g_i\right]\Bigg\} F_W \\
&+ \textstyle\sum_i^K F_{X_i}\Bigg[\mu_{X_i} - M_i^* - \left(-\dfrac{J_{WW}}{J_W}\right)\operatorname{cov}(W, X_i) \\
&- \textstyle\sum_j^K \left(-\dfrac{J_{WX_j}}{J_W}\right)\operatorname{cov}(X_i, X_j)\Bigg] \\
&+ \textstyle\sum_i^{K+N} \pi_i \mathrm{E}_{gi}\Bigg[\dfrac{J_W(W + e_{W_i} g_i, X + M_i g_i)}{J_W} \\
&\times \{F(W + e_{W_i} g_i, X + M_i g_i, t) - F(W, X, t)\}\Bigg] \\
&+ F_t - r(W, X, t) F = 0,
\end{aligned} \tag{24}$$

where $r(W, X, t)$ *is given from Theorem* 1.

There are several differences between Theorem 4 and Theorem 3 in Cox, Ingersoll, and Ross [8]. First, the equilibrium interest rate that appears in the last term of (24) will be different under jump-diffusion processes. We will show that it will be strictly lower under a logarithmic utility function. Second, additional terms in the coefficients of F_W and F_X appear because of the jump components. The third and major difference comes in the jump risk premia on the discount bond, $\sum_i^{K+N} \pi_i \phi_i$, which were shown in (22). If the probability of a jump is zero for all the processes, i.e., $\pi_i = 0$, then Theorem 4 becomes exactly the same as Theorem 3 of Cox, Ingersoll, and Ross [8]. Consequently, (24) is a generalization of Theorem 3 of Cox, Ingersoll, and Ross [8].

Since (24) holds for any contingent claim as well for a discount bond, the equilibrium price of a contingent claim can be represented in Theorem 5.[7]

THEOREM 5: *Given a boundary condition, $F(W(T), X(T), T) = \theta(W(T), X(T), T)$, the solution to the valuation equation* (24) *is given by*

$$F(W(t), X(t), t) = \mathrm{E}_t\left[\theta(W(T), X(T), T)e^{-\rho(T-t)}\frac{J_W(W(T), X(T), T)}{J_W(W(t), X(t), t)}\right]. \tag{25}$$

A similar representation of the equilibrium price of a contingent claim is obtained by Cox, Ingersoll, and Ross [8] under diffusion processes.

Let $r_{i,t,T}$ denote $\dfrac{\theta(W(T), X(T), T)}{F(W(t), X(t), t)}$. From (13), equation (25) can be rewritten as

$$1 = \mathrm{E}_t[r_{i,t,T}m_{t,T}], \tag{26}$$

where

$$m_{t,T} = e^{-\rho(T-t)}\frac{u_{C_T}}{u_{C_t}}$$

is the intertemporal marginal rate of substitution. Equation (26) is the stochastic Euler equation. This equation holds even in a discrete-time framework. (See Lucas [19].) Using the law of iterated expectation, a covariance expansion, and suppressing t and T subnotations, we have

$$\mathrm{E}(r_i) = r - r\,\mathrm{cov}(r_i, m), \tag{27}$$

where $r = 1/\mathrm{E}(m)$ is the rate of return on the risk-free asset. Applying (27) for m (the intertemporal marginal rate substitution), using $r_m = m/\mathrm{E}(m^2)$, and substituting for $-r$ into (27) gives

$$\mathrm{E}(r_i) = r + \frac{\mathrm{cov}(r_i, r_m)}{\mathrm{var}(r_m)}[\mathrm{E}(r_m) - r], \tag{28}$$

which is an unconditional version of the single-beta asset-pricing model. (See Hansen and Richard [15] for a conditional version.)

Our objective in this section is to derive a continuous-time limit of (28) under jump-diffusion processes and to examine whether the consumption-beta CAPM holds.

Following Breeden [3], we assume that the instantaneous utility function is state independent. We shrink $T - t$ to dt. By expansion, we have

$$m = e^{-\rho dt}\frac{u'(C_t - dc)}{u'(C_t)} \approx 1 - \rho dt + \textstyle\sum_{n=1}^{\infty}\dfrac{u^{(n+1)}}{u'n!}(dc)^n, \tag{29}$$

where $u^{(n+1)}$ denotes the $(n+1)$-order derivative of u with respect to consumption.

[7] Theorem 5 and subsequent analysis are developed for non-dividend-paying securities with no default risk. This formulation was chosen to simplify the exposition that follows. More complicated contingent claims could be dealt with using the same approach. Nothing fundamental would change.

Assume that consumption and return follow jump-diffusion processes,

$$dc = (\alpha_C - e_C^*)dt + \sigma_C dZ + e_C dY \quad \text{and} \tag{30}$$

$$dr_i = (\alpha_i - e_{r_i}^*)dt + \sigma_{r_i} dZ + e_{r_i} dY, \tag{31}$$

respectively. Note that $(dc)^n$ has the same order of magnitude as dt and cannot be neglected for all n. (See Merton [23].) By a continuous-time limit, we have $r_{i,t,T} = 1 + dr_i$.

Substituting m and r_i into an unconditional version of (26) gives

$$0 = -\rho dt + \sum_{n=1}^{\infty} \frac{u^{(n+1)}}{u' n!} \mathrm{E}(dc)^n + \alpha_i dt + \sum_{n=1}^{\infty} \frac{u^{(n+1)}}{u' n!} \mathrm{E}[(dc)^n dr_i]. \tag{32}$$

Applying (32) for the risk-free asset, we have

$$(\alpha_i - r)dt = -\sum_{n=1}^{\infty} \frac{u^{(n+1)}}{u' n!} \mathrm{E}[(dc)^n dr_i], \tag{33}$$

where

$$\mathrm{E}[(dc)^n dr_i] = \sigma_C \sigma_{r_i}' dt + \sum_{i=1}^{K+N} \pi_i \mathrm{E}_{gi}[\{C(W + e_{W_i} g_i, X + M_i g_i) - C\} e_{r_i} g_i] dt$$

for $n = 1$ and

$$\mathrm{E}[(dc)^n dr_i] = \sum_{i=1}^{K+N} \pi_i \mathrm{E}_{gi}[\{C(W + e_{W_i} g_i, X + M_i g_i) - C\}^n e_{r_i} g_i] dt$$

for $n \geq 2$.

Thus, (33) can be rewritten as

$$\alpha_i - r = -\frac{u''}{u'} \sigma_C \sigma_{r_i}'$$
$$- \sum_{i=1}^{K+N} \pi_i \mathrm{E}_{gi}\left[\sum_{n=1}^{\infty} \frac{u^{(n+1)}}{u' n!} \{C(W + e_{W_i} g_i, X + M_i g_i) - C\}^n e_{r_i} g_i\right]. \tag{34}$$

It can be easily seen that (34) is equivalent to the following asset-pricing model obtained from the first-order condition (14):

$$\alpha - r \times 1_N = \left(-\frac{u''}{u'}\right) G \sigma_C'$$
$$- \sum_{i=1}^{K+N} \pi_i \mathrm{E}_{gi}\left[\frac{J_W(W + e_{W_i} g_i, X + M_i g_i) - J_W}{J_W} Q_i g_i\right]. \tag{35}$$

Applying (33) for a security, m, which has a return perfectly correlated with the intertemporal marginal rate of substitution, we have a single-beta asset-pricing model,

$$\alpha_i - r = \frac{\mathrm{cov}\left(dr_i, \sum_{n=1}^{\infty} \frac{u^{(n+1)}}{u' n!} (dc)^n\right)}{\mathrm{var}\left(\sum_{n=1}^{\infty} \frac{u^{(n+1)}}{u' n!} (dc)^n\right)} [\alpha_m - r]$$
$$= \beta[\alpha_m - r], \tag{36}$$

where α_m is the expected return on the security m and β is a beta measured relative to the security's return.

Under diffusion processes, $(dc)^2 = \sigma_C^2 dt$ and $(dc)^n = o(dt)$ for $n > 2$.[8] This is the reason why consumption is perfectly instantaneously correlated with the marginal rate of substitution.[9] Consequently, consumption can become the benchmark portfolio in (36) so that the consumption CAPM is obtained.[10]

However, under jump-diffusion processes, $(dc)^n = O(dt)$ for $n > 2$. (See Merton [23].) This prohibits perfect correlation between the consumption and the marginal rate of substitution. Consequently, the single-consumption-beta CAPM cannot be obtained.

IV. The Equilibrium Discount Bond Price

In this section, we investigate how jump risks affect the bond price. We also examine the way in which jump risks affect the investment behavior of the consumer.

Since we are interested in finding the equilibrium price of the discount bond, we adopt the case of a logarithmic utility function, as did Cox, Ingersoll, and Ross [9], in order to derive closed-form solutions. We assume that the direct utility function is time additive and state independent. Its form is assumed to be

$$u(C(t), X, t) = \ln C(t). \tag{37}$$

It can be shown that the indirect utility function has the form

$$J(W(t), X(t)) = (1/\rho)\ln W(t) + d(X(t)), \tag{38}$$

where $d(X(t))$ is a solution to the differential equation obtained by substituting $J(W, X)$, C^*, and a^* into (12).

The optimal consumption rate is $\rho W(t)$, and the optimal investment schedule satisfies

$$a^* = (GG')^{-1}(\alpha - r1_N) + \textstyle\sum_i^{K+N} \pi_i \mathrm{E}_{gi}\left[\dfrac{-a^{*\prime} Q_i g_i}{1 + a^{*\prime} Q_i g_i}(GG')^{-1} Q_i g_i\right]. \tag{39}$$

The second term of the right-hand side of (39) shows the hedging behavior of the investor. The demand is greater for a production process with jump outcomes that are more negatively correlated with the market portfolio's jumps. It is well known that, under the diffusion process, the consumer with a logarithmic utility function does not hedge movements in the investment opportunity set. However,

[8] We use the definitions of $O(h)$ and $o(h)$ in the usual sense of $f(h) = O(h)$ if $\lim_{h\to 0} f(h)/h$ is bounded and $f(h) = o(h)$ if $\lim_{h\to 0} f(h)/h = 0$ in our analysis.

[9] $$\rho_{m,C} = \frac{\mathrm{cov}(m, dc)}{\sqrt{\mathrm{var}(m)\mathrm{var}(dc)}} = \frac{\mathrm{cov}\left(\dfrac{u''}{u'}dc, dc\right)}{\sqrt{\mathrm{var}\left(\dfrac{u''}{u'}dc\right)\mathrm{var}(dc)}} = \frac{\dfrac{u''}{u'}\mathrm{var}(dc)}{\left|\dfrac{u''}{u'}\right|\mathrm{var}(dc)} = -1.$$

[10] (33) becomes $\alpha_i - r = -\dfrac{u''}{u'}\sigma_C\sigma_{r_i}'$, which delivers $\alpha_i - r = \dfrac{\sigma_C\sigma_{r_i}'}{\sigma_C^2}(\alpha_C - r)$.

under jump-diffusion processes, even the consumer with the logarithmic utility function would hedge against technological change from jumps.

The underlying intuition for this hedging behavior is not difficult to follow. From (39), jump risks are correlated with the market portfolio. In fact, they are systematic risks so that they cannot be neglected in pricing. Consequently, asset demands are affected by jump risks. We can see in (33) that $(dc)^n dr_i$ includes jump risks and that it cannot be neglected (i.e., $(dc)^n dr_i = O(dt)$ for all n).

Since one of our purposes is to find the effect of jump risks on the equilibrium price of the discount bond, we will focus on the equilibrium valuation equation. As mentioned earlier, since the value of the discount bond does not depend on W, i.e., its principal at maturity is one regardless of the level of W, the values of F_W, F_{WW}, and F_{WX} are zero. Consequently, the valuation equation (24) reduces to

$$\begin{aligned}&(\tfrac{1}{2}) \textstyle\sum_i^K \sum_j^K \operatorname{cov}(X_i, X_j) F_{X_i X_j} + \sum_i^K F_{X_i}[\mu_{X_i} - M_i^* - a^{*\prime} GS_i] \\ &+ \textstyle\sum_i^{K+N} \pi_i \mathrm{E}_{g_i}\left[\dfrac{1}{1 + a^{*\prime} Q_i g_i} \{F(W + e_{W_i} g_i, X + M_i g_i, t) - F(W, X, t)\}\right] \\ &+ F_t - r(W, X, t)F = 0. \qquad (40)\end{aligned}$$

Equation (40) is the "jump-diffusion" analog of equation (12) of Cox, Ingersoll, and Ross [9]. Following Merton [20], we assume that the jump amplitude g_i is one with probability one for all i.[11] The valuation equation (40) can then be further reduced to

$$\begin{aligned}&(\tfrac{1}{2}) \textstyle\sum_i^K \sum_j^K \operatorname{cov}(X_i, X_j) F_{X_i X_j} + \sum_i^K F_{X_i}[\mu_{X_i} - M_i^* - a^{*\prime} GS_i] \\ &+ \textstyle\sum_i^{K+N} \pi_i \left[\dfrac{1}{1 + a^{*\prime} Q_i} \{F(W + e_{W_i}, X + M_i, t) - F(W, X, t)\}\right] \\ &+ F_t - rF = 0. \qquad (41)\end{aligned}$$

We will use equation (41) for valuing the discount bond in the remainder of the paper, where two examples that further specify the technological change are used to investigate the implied term structure of interest rates.

Example 1: This is a very simple case with no state variable, i.e., $K = 0$, and only one technology, i.e., $N = 1$. Consequently, the equilibrium valuation equation for the discount bond is

$$\frac{1}{1 + Q} [F(W(1 + Q), t, T) - F(W, t, T)] + F_t - rF = 0 \qquad (42)$$

with the boundary condition $F(W, T, T) = 1$, where T is the maturity date. The

[11] The jump amplitude is assumed to be one with probability one in order to have explicit solutions. However, if we do not make this assumption, we conjecture that the solution for a bond price will have the form of the expected value of a bond price given the Poisson distribution of jumps, similar to the solution for option prices in Merton [22]. The equilibrium valuation equation (41) is similar to (14) in Merton [22]. Merton's equation is derived by the arbitrage method in the partial-equilibrium setting. On the other hand, our equation is derived in the general-equilibrium setting. Cox and Ross [6] derive the formula for call options under the jump process with a similar restriction that the jump magnitude k is nonrandom in order to form the exact hedge.

solution of equation (42) is

$$F(W, t, T) = \exp(-r(T - t)), \tag{43}$$

where $r = \alpha - \sigma^2 - \pi Q^2/1 + Q$, σ^2 is GG', and π is the intensity parameter. Since J_W is positive, $1 + Q$ must be positive. Thus, the interest rate is strictly lower than under the diffusion process. The bond price is an increasing convex function of the probability of a jump.

In this example, we have a flat term structure. The yield to maturity, $y(T, t)$, is simply r. The local expectations hypothesis must hold. It is easy to interpret the result by reference to equation (23). Since there is no state variable, $F_X = 0$. Furthermore, since the value of the discount bond does not depend on the wealth, the change in the value caused by the jump component is zero. Consequently, the right-hand side of equation (23) is zero regardless of the proportional change in marginal utility.

If there are state variables, the local expectations hypothesis does not hold even though we assume that the production processes are uncorrelated with the state variables, as mentioned earlier. This follows from the fact that changes in the value of the discount bond caused by the jump component are not zero in general.

Example 2: In this example, we analyze the case analogous to the one examined by Cox, Ingersoll, and Ross [9]. There are N production technologies. We impose a further restriction on Q in equation (3):

$$Q(X) = [0, q], \tag{44}$$

where 0 is an $N \times N$ matrix of zeros and q is an $N \times 1$ vector with dynamics governed by the following stochastic differential equation:[12]

$$dX = [\epsilon - \eta X]dt + \sqrt{X}SdZ + mXdY, \tag{45}$$

where ϵ and η are positive constants, S is an $(N + 1) \times 1$ constant diffusion vector and m is an $(N + 1) \times 1$ vector with elements that are zero except for the last negative constant element. We additionally assumed that $\alpha = \hat{\alpha}X$, $GG' = \Omega X$, $GS' = \Sigma X$, and $\pi(X) = \hat{\pi}X$, where the elements of $\hat{\alpha}$, Ω, Σ, and $\hat{\pi}$ are constant and $\pi(X)$ denotes the intensity of the last element of Y. It can be shown that the equilibrium interest rate is

$$r(X) = \left(a^{*\prime}\hat{\alpha} - a^{*\prime}\Omega a^* - \frac{\hat{\pi}(a^{*\prime}q)^2}{1 + a^{*\prime}q}\right)X, \tag{46}$$

where a^* solves the equation (39) for this example. By using equation (39), we can show that

$$r(X) = (1_N'\Omega^{-1}1_N)^{-1}(\alpha'\Omega^{-1}1_N - 1 - Pq'\Omega^{-1}1_N)X = \hat{r}X, \tag{47}$$

where P is $\hat{\pi}(a^{*\prime}q)/(1 + a^{*\prime}q)$ and the elements of $\Omega^{-1}1_N$ are assumed to be positive. The value of $r(X)$ in (47) is assumed to be positive. Since $1 + a^{*\prime}q$ is positive and $a^{*\prime}q$ is negative, the interest rate is strictly lower than under the

[12] The stochastic differential equation (45) is exactly the same as in Cox, Ingersoll, and Ross [9] except for the addition of the jump terms.

diffusion process.[13] The higher the probability rate of a jump, the lower the interest rate.

By using Itô's Lemma, the dynamics of the interest rate can be given by

$$dr = \hat{r}[\epsilon - \eta X]dt + \hat{r}\sqrt{X}SdZ + e_r dY, \tag{48}$$

where $e_r = r(X + mX) - r(X)$. By introducing a new one-dimensional diffusion process, dz, and jump process, dy, equation (48) can be rewritten as

$$dr = k(\theta - r)dt + \sigma\sqrt{r}dz + \delta dy, \tag{49}$$

where $\hat{r}[\epsilon - \eta X] \equiv k(\theta - r)$, $\hat{r}\sqrt{X}SdZ = \sigma\sqrt{r}dz$, and $e_r dY = \delta dy$. The value of δ is assumed to be a negative constant, and the intensity of y is assumed to be πr. The value of δ is assumed to be such that a positive interest rate will always result.

Detailed explanations about the behavior of the interest rate in the case of the diffusion component can be found in Cox, Ingersoll, and Ross [9]. In fact, this diffusion component is a diffusion limit of a discrete branching process.[14] Thus, equation (49) may be regarded as a mixed structure of a Poisson process and a limit of a branching process. The conditional expected value and variance of $r(s)$ given the current interest rate, $r(t)$, are given by[15]

$$\mathrm{E}(r(S) \mid r(t)) = (\theta k/\underline{k}) + [r(t) - (\theta k/\underline{k})]e^{-\underline{k}(s-t)}$$

and

$$\begin{aligned} \mathrm{var}(r(s) \mid r(t)) &= (\theta k\underline{\sigma}^2/2\underline{k}^2)(1 - e^{-\underline{k}(s-t)})^2 \\ &\qquad + r(t)(\underline{\sigma}^2/\underline{k})[1 - e^{-\underline{k}(s-t)}]e^{\underline{k}(s-t)}, \\ \underline{k} &= k - \pi\delta \mathrm{E}[g], \\ \underline{\sigma}^2 &= \sigma^2 + \pi\delta^2\mathrm{E}[g^2], \end{aligned} \tag{50}$$

where g is defined as the jump amplitude of the independent Poisson process. It is clear from (49) and (50) that the interest rate will have the mean-reverting property as in Cox, Ingersoll, and Ross [9]. Thus, the mean-reverting property is not affected by jump risks. It can be shown that, even though the interest rate has a discontinuous sample path, it has the steady-state distribution as in Cox, Ingersoll, and Ross. The steady-state mean and variance are $\theta k/\underline{k}$ and $\theta k\underline{\sigma}^2/2\underline{k}^2$, respectively.

[13] Following Cox, Ingersoll, and Ross, we assume that every production process is active. Thus, $a^{*\prime}q$ is negative. Since $\mathrm{sign}(dr/d\pi) = \mathrm{sign}(q'\Omega^{-1}1_N)$, $\mathrm{sign}(q'\Omega^{-1}1_N) < 0$ is sufficient for demonstrating that the interest rate is strictly lower than under the diffusion process. The negativity of $q'\Omega^{-1}1_N$ follows from $q < 0$ and $\Omega^{-1}1_N > 0$.

[14] A detailed discussion about approximations of the branching process by a diffusion process is provided by Cox and Miller [5, p. 235] and Feller [11]. This approximation is called a "square root" process with mean-reverting drift.

[15] The moment-generating function is given by

$$\mathrm{E}[e^{-zr}] = M(z, s) = \exp\left[\frac{-r(t)kze^{-\underline{k}s}}{\underline{k} + (1/2)\underline{\sigma}^2 z(1 - e^{-\underline{k}s})}\right]\left[\frac{\underline{k}}{\underline{k} + (1/2)\underline{\sigma}^2 z(1 - e^{-\underline{k}s})}\right]^{(2\theta k/\underline{\sigma}^2)}.$$

The density function of r can be obtained by using the inversion integral.

The equilibrium valuation equation for the price of the discount bond (41) becomes

$$(½)\mathrm{var}(X)F_{XX} + (\epsilon - \eta X - a^*GSX)F_X + F_t$$
$$+ \pi(X)\left[\frac{1}{1 + a^{*\prime}q}\{F(W + e_{W_{N+1}}, X + m, t) - F(W, X, t)\}\right] - rF = 0. \quad (51)$$

We can write (51) in terms of the spot interest rate, r, since the state variable and the interest rate are in a one-to-one correspondence as shown in (47).[16] The valuation equation (51) can then be rewritten as

$$(½)\sigma^2 rF_{rr} + k(\theta - r)F_r + F_t$$
$$+ \pi r\left[\frac{1}{1 + a^{*\prime}q}\{F(r + \delta, t, T) - F(r, t, T)\}\right] - \lambda rF_r - rF = 0, \quad (52)$$

with the boundary condition $F(r, T, T) = 1$ where $r = \hat{r}X$. By separating the fourth term into two terms, we can rewrite (52) as

$$(½)\sigma^2 rF_{rr} + k(\theta - r)F_r + F_t + \pi r\{F(r + \delta, t, T) - F(r, t, T)\}$$
$$+ \pi r\left[\frac{-a^{*\prime}q}{1 + a^{*\prime}q}\{F(r + \delta, t, T) - F(r, t, T)\}\right] - \lambda rF_r - rF = 0. \quad (53)$$

The first four terms in (53) are, from Itô's Lemma, the expected price change for the bond, i.e., E[dF]. From equation (5), E[dF] is defined as $F\beta_F$. Consequently, the expected rate of return on the bond is given by

$$r + \frac{\lambda rF_r}{F} + \pi r\left[\frac{a^{*\prime}q}{1 + a^{*\prime}q}\left\{\frac{F(r + \delta, t, T) - F(r, t, T)}{F(r, t, T)}\right\}\right]. \quad (54)$$

The return premium on the bond is determined by two factors. The first factor, λ, is the covariance of changes in the interest rate and changes in optimally invested wealth. This was discussed by Cox, Ingersoll, and Ross [9]. However, the second factor is new. It is nothing but the second term of the right-hand side in (23), i.e., the term related to the jump risk premium. The sign of $1 + a^{*\prime}q$ is positive from $J_W > 0$. The term $a^{*\prime}q$ is the proportional jump change in optimally invested wealth caused by the jump of the interest rate; it is negative. The next term, $F(r + \delta, t, T) - F(r, t, T)$, is the jump change in the bond price caused by the jump of the interest rate. Since the bond price has a negative relationship with the interest rate and δ is negative, $F(r + \delta, r, T)$ is strictly greater than $F(r, t, T)$. Since the jump changes in both optimally invested wealth and the bond have the different signs, the jump risk premium for the discount bond will be negative. This means that the investors would appreciate the hedging service of the bond against the uncertain jump changes of the underlying technologies.

[16] Cox, Ingersoll, and Ross [9] state two sufficient conditions for bond prices to depend only on the spot interest rate. First, the consumers have constant relative risk aversion; uncertainty in technology can be described by a single variable; and the interest rate is a monotonic function of this variable. Second, change in technology is nonstochastic, and the interest rate is a monotonic function of wealth. Our example is the first case of Cox, Ingersoll, and Ross.

It can be shown that the price of the bond is

$$F(r, t, T) = A(t, T)\exp(-B(t, T)r), \tag{55}$$

where

$$A(t, T) \approx \left[\frac{2\gamma \exp(k + \lambda + \gamma - \bar{\pi}\delta)(T - t)}{(k + \lambda + \gamma - \bar{\pi}\delta)(\exp\{\gamma(T - t)\} - 1) + 2\gamma}\right]^{2k\theta/(\sigma^2 + \bar{\pi}\delta^2)},$$

$$B(t, T) \approx \frac{2(\exp\{\gamma(T - t)\} - 1)}{(k + \lambda + \gamma - \bar{\pi}\delta)(\exp\{\gamma(T - t)\} - 1) + 2\gamma},$$

$$\gamma = ((k + \lambda - \bar{\pi}\delta)^2 + 2\sigma^2 + 2\bar{\pi}\delta^2)^{1/2},$$

$$\bar{\pi} = \pi/(1 + a^{*'}q).$$

The solution for $B(t, T)$ is based on the second-order approximation, i.e., $\exp(-dB(t, T)) \approx 1 - dB(t, T) + (1/2)d^2B^2(t, T)$. The accuracy of this approximation must be high only if the constant d is close to zero. The relationship between $A(t, T)$ and $B(t, T)$ is given by $A = h \exp(k\theta \int Bdt)$, where h is a constant. Consequently, the approximation of $A(t, T)$ depends on the accuracy of $B(t, T)$.

If the probability of a jump is zero, then (55) coincides with the bond price obtained by Cox, Ingersoll, and Ross [9] and Richard [24].[17]

The bond price is an increasing convex function of the probability rate of a jump. The underlying intuition is simple. Since δ is negative, the increase in the probability rate of a jump would cause the instantaneous reduction in the interest rate, ceteris paribus, from equation (49). Since the bond price is a decreasing function of the interest rate, the bond price would increase.

[17] Recently, Brown and Dybvig [4] tested this special case ($\pi = 0$) by employing the maximum-likelihood estimation procedure. Gibbons and Ramaswamy [12] tested it by employing the Generalized Method of Moments procedure developed by Hansen [14]. Their empirical methodologies can be straightforwardly applied to our (55). For example, following Gibbons and Ramaswamy [12], we can obtain two orthogonality conditions:

$$\mathrm{E}\left[\frac{1}{F(r, t, T)} - \frac{1}{A(t, T)}\exp(B(t, T)r(t))\right] = 0 \quad \text{and}$$

$$\mathrm{E}\left[\frac{1}{F(r, t, T)F(r, s, T)} - \frac{1}{A(t, T)}\frac{1}{A(s, T)}\exp(B(t, T)r(t) + B(s, T)r(s))\right] = 0.$$

The consistent estimator of parameters ($b = (\gamma, \lambda, k, \text{etc.})$) can be obtained by minimizing the quadratic form

$$g_T(b)'\, W_T g_T(b),$$

where g_T is the method-of-moments estimator of orthogonality conditions and W_T is a weighting matrix. (See Hansen and Singleton [16].) The minimized value multiplied by the number of time-series observations is distributed asymptotically as a chi-square with k degrees of freedom, where k is the number of orthogonality conditions minus the number of parameters. This value can be used to test the overidentifying restrictions implied by the model. One interesting empirical study would be to conduct a nested-hypothesis testing such that we can find which specification is more compatible with data among diffusion processes and jump-diffusion processes.

The dynamics of the bond price based on (55) are given by

$$dF = r[(1 - \lambda B(t, T))F - \bar{\pi}\{F(r + \delta, t, T) - F(r, t, T)\}]dt - B(t, T)F\sigma\sqrt{r}dz + \{F(r + \delta, t, T) - F(r, t, T)\}dy. \tag{56}$$

Even if the zero interest rate is reached, the returns on the bond are still uncertain because of the jump risks. This is different from the case of diffusions. Similarly, the bond returns are not certain even though t approaches the maturity date, T. The yield to maturity of the bond, $R(r, t, T)$, is represented by

$$R(r, t, T) = \frac{B(t, T)r - \ln A(t, T)}{(T - t)}. \tag{57}$$

It can be easily shown that

$$R(r, T, T) = r \tag{58}$$

and

$$R(r, t, \infty) \approx \frac{2k\theta}{\gamma + k + \lambda - \bar{\pi}\delta}. \tag{59}$$

The effect of the probability of a jump on (58) and (59) can be easily obtained. The increase in the probability of a jump decreases the spot interest rate, r, as seen in (47). The increase in the probability of a jump decreases $R(r, t, \infty)$. This is apparent from the fact that the yield to maturity is a decreasing convex function of the bond price. The forward interest rate is given by

$$f(r, t, T) = -\frac{F_T}{F} = rB_T - \frac{A_T}{A}. \tag{60}$$

It is easily seen that $f(r, t, t) = f(r, T, T) = r$ and $f(r, t, \infty) = R(r, t, \infty)$. It also can be easily verified that the term premium, $f(r, t, T) - \mathrm{E}(r(T) \mid R(t))$, is nonzero.

V. Conclusion

This paper has investigated the effect of jump components of the underlying processes on the term structure of interest rates. In order to highlight the effects of jumps, we extended the model of Cox, Ingersoll, and Ross [8, 9] to have state variables with jump-diffusion processes. The equilibrium interest rate and the term structure are endogenously determined by the specifications of preference, technology, and the state variables.

We can find several general conclusions that arise from jump-diffusion processes. First, we find that Merton's multi-beta CAPM does not hold in general. Furthermore, Breeden's single consumption beta does not hold. In other words, the discontinuous movements of the investment opportunities cannot be completely captured by a single consumption beta. Second, the equilibrium interest rate under jump-diffusion processes is strictly lower than the one under diffusion processes, ceteris paribus. This follows from the fact that the discount bond provides a hedging service against additional jump risks since the bond delivers a certain payoff at the maturity date. Third, Cox, Ingersoll, and Ross [7] provide cases where the traditional expectations theory on the term structure of interest rates is consistent with general equilibrium models—for example, the case where aggregate consumption is locally riskless or the case where the output is uncorrelated with the state variables under the logarithmic utility. We find that the traditional expectations theory is not consistent with the equilibrium models even in the above cases under the jump-diffusion processes. This is because the term premium, defined as the difference between the expected rate of return on the discount bond and the spot interest rate, is additionally affected by the jump risk premia. Fourth, it is well known that the consumers with logarithmic utility would not hedge movements in the investment opportunities under the diffusion processes. In other words, the covariability of an asset payoff with technological changes is not priced. However, the covariability with technological jump changes is priced even in the case of the logarithmic utility function under the jump-diffusion processes. The consumers with logarithmic utility functions would appreciate the hedging service of an asset against the uncertain jump changes of the underlying technologies.

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