Advanced Term Structure Models

Applied Finance Project

Bhavya Aggarwal, Renee Reynolds, Silvi Pandey, Sravya Divi, Upamanyu Pathare

A thesis presented for the degree of

Master of Financial Engineering

Haas School of Business

University of California, Berkeley

03/03/2021

Advanced Term Structure Models

Jump diffusion extensions of classical interest rate models

Abstract

There is growing evidence of the presence of jump risks within the fixed income markets. The crises of 2008, 2013, and 2020 all shocked the bond markets in an instantaneous manner that is not fully captured by the classical interest rate models which assume a more continuous evolution. Our paper extends the classical models such as Vasicek and Cox-Ingersoll-Ross (CIR) to incorporate the effects of jump risks in the market. We explore modern methods to evaluate and calibrate such models and compare their pricing performance against observed market prices for various financial instruments to that of classical models.

Contents

1 Introduction

There exists a considerable body of research focused on the development of continuous, stochastic interest rate processes. However, recent empirical studies suggest that the underlying process might follow discontinuous sample paths. Although research on the effectiveness of discontinuous sample paths is still ongoing, the difference between the outcomes of the theoretical models with continuous paths and jumps motivates this paper.

Recently, the models developed by Ahn and Thompson [AT88] studied the effect of regulatory risk on the valuation of public utilities, where they identified that jump risks are one of the contributing factors in the price even though they are uncorrelated with market factors, implying that jump risks cannot be ignored in asset pricing.

Many studies in the past have tried to model the regime change or shocks in financial variables. There is compelling evidence for jumps both in emerging and developed economies. Specific sources of jumps in interest rates, including economic news and moves by central banks, are put forward in [BW97], [ELP07], and [Joh03]. [Joh03] provides evidence of jumps in US treasury bill rates and highlights the importance of incorporating jumps in pricing interest rate options. Many empirical studies note the importance of jump-diffusion in explaining observed interest rates rather than pure diffusion models.

[BNP07] models traded uncertainty by a Poisson process and extends the HJM framework to derive the forward rate dynamics for jump diffusion under the real world probability measure. [GK03] also characterizes both jumps and diffusion using discretely compounded forward rates evolving continuously or forward rates. They allow for dependence between jump time, jump sizes and interest rates. Using this framework, they price derivative securities like interest rate caps, floors and options on swaps. They argue that the rationale for using simple forwards lies in using observable interest rate data and study the effect of jumps on skew, smile and implied volatilities of interest rate derivatives.

[Zho01b] argues that the CIR model is only able to explain mean persistence, long run mean reversion and volatility in short rate. It fails to capture the volatility persistence and therefore advocates introducing a stochastic volatility factor into diffusion function. Conceptually, they argue, a stochastic volatility term is similar to the jump-diffusion approach. They implement a Multivariate Weighted Non-Linear Least Squares estimator (MWNLS-JD) and the instantaneous interest rate is modeled as a mixture of a square root diffusion process and a Poisson jump process.

The literature has found that jump-diffusion models can be effective across economies. The experiments of [LY99] studied the performance of the jump-Vasicek model as compared to the classical Vasicek model on US Treasury securities. The results showed that the jump-diffusion model fit the term structure relatively well when interest rates were low, but underestimated the term structure when interest rates were high, while the classical Vasicek model without jumps was better for fitting the term structure when interest rates were high and tended to overestimate the term structure when interest rates were low.

Other studies such as [MCO09] and [LY01] applied jump-diffusion models to economies across the globe. In

[MCO09], the jump-Vasicek model was found to be a reasonable representation of Mexican CETES data, while [LY01] studied the application of one-factor and two-factor Vasicek and jump-Vasicek models to Taiwanese Government Bond (TGB) term structure of interest rates. [LY01] finds that all four models–one-factor Vasicek, one-factor jump-diffusion, two-factor Vasicek, and two-factor jump-diffusion–fit the term structure relatively well when interest rates are low, with the two-factor models fitting slightly better than the one-factor models. However, when interest rates are high, the one-factor models, especially the one-factor jump-diffusion model–similar to the results of [LY99]–underestimate the term structure, while the two-factor models continue to fit the term structure well, with the two-factor Vasicek underestimating the shorter-term interest rates and overestimating longer-term interest rates, and two-factor jump-diffusion the opposite. The authors conclude that the two-factor jump-diffusion model, despite neither two-factor model showing clear and significant superiority in performance than the other, has more implications for describing term structure fluctuations as it includes more risk characteristics in term structure behavior, hence greater economic value in pricing interest rate derivatives.

Though we do not study it in this paper, we note that in addition to interest rate derivatives, some research applies jump-diffusion models to credit risk, as done in [Wu14], [Zho97], [Zho01a], and [Tia+14]. For example, [Wu14] uses the jump-diffusion CIR model to derive pricing of the defaultable zero-coupon bond as well as the fair premium of a credit default swap (CDS) in a reduced form model of credit risk.

Another aspect of our models and experiments is calibration. The literature around the calibration of the classical Vasicek and CIR models suggests that calibration methods vary, ranging from simple to more complex. Some of the most commonly implemented simple methods for calibrating the Vasicek and CIR models include least squares, maximum likelihood estimation, and yield curve fitting as used in [Ber16], [Čer12], [TEB16], [OR97], while more complex methods and novel methods include the use of generating functions as in [RM14]. For the calibration of jump-diffusion models, approaches include maximum likelihood estimation, as in [LY99], as well as Monte Carlo methods, as in [PK15].

The rest of this paper is organized as follows: Section 2 defines the models, methodology, and data. Section 3 presents preliminary results for the classical Vasicek and CIR models as well as variance reduction techniques. Section 4 specifies the two calibration approaches used – yield curve fitting and maximum likelihood estimation. Section 5 contains pricing results, with a comparison of the performance of the two aforementioned calibration approaches. Section 6 highlights our final conclusions, and finally, Section 7 outlines future work. The code implemented for the experiments can be found in the Appendices.

1.1 Problem Definition/Statement of Hypotheses

We hypothesize that jump-diffusion processes more accurately model the term structure of interest rates than those without. To test this hypothesis, we price debt instruments, including swaps and bonds, using data obtained from Bloomberg, and compare the accuracy of models with and without jumps.

We proceed in 4 steps:

1. Extend classical interest models to include jump diffusion processes.
2. Develop Monte-Carlo with variance reduction techniques to evaluate the price curve for an arbitrary set of parameters.
3. Fit the model parameters using various regression and root-finding techniques to obtain a price curve that agrees with market rates for Treasury bonds.
4. Compare the prices obtained in the Jump diffusion setup versus prices for the classical models. Use standard measures such as MSE, RMSE, and/or MAE to evaluate the pricing error.

Based on the size and significance of the pricing errors of jump models compared to classical models will enable us to make a conclusion regarding the validity of jump models and the existence of jump risks in fixed income product pricing.

We focus on two classical models, the Vasicek model and the Cox-Ingersoll-Ross (CIR) model. Both these models have a mean-reversion property whereby the instantaneous rate converges to a long-run mean rate. We chose these models for their ubiquity and relative ease of calibration. Unlike related models like the Heath-Jarrow-Morton (HJM), we do not need future/forward rates and unlike the Black-Derman-Toy (BDT) model, we have a mean-reversion feature built-in.

This is not to say that Vasicek and CIR are perfect models; Vasicek has historically been criticized for allowing negative interest rates, however with observations of negative interest rates in many sovereign bond markets, this criticism has inverted onto CIR instead, which disallows negative interest rates completely. Nevertheless, we believe Vasicek and CIR represent a good balance between ease of use and complexity and hence our choice to restrict our analyses to these two models.

2 Experiment Setup

2.1 Model Definition

We begin with two classical interest rate models, Vasicek and Cox-Ingersoll-Ross (CIR):

$$dr(t) = \kappa(t)[\mu_r(t) - r(t)]dt + \sigma(t)dW(t) \qquad \text{(Vasicek)}$$
$$dr(t) = \kappa(t)[\mu_r(t) - r(t)]dt + \sigma(t)\sqrt{r(t)}dW(t) \qquad \text{(CIR)}$$

Both have a mean-reversion tendency with speed of mean reversion controlled by κ. The key difference between the two models is the presence of a $\sqrt{r(t)}$ term in CIR that imposes non-negative interest rates on the model (as $r(t)$ approaches 0, the contribution of the random component shrinks to 0).

For simplicity, we begin by assuming $\kappa(t)$, $\mu_r(t)$, and $\sigma(t)$ to be constants. The exact values of the constants will be calibrated to fit the observed yield curve of Treasury bonds.

Next, we extend the classical model by adding a Jump process. The arrival of the jumps is driven by a Poisson process with a parameterized arrival probability. Furthermore, the jump itself is random with a normally distributed size. This is all we need to do to extend the Vasicek model to include jumps and we obtain the following SDE:

$$dr(t) = \kappa(t)[\mu_r(t) - r(t)]dt + \sigma(t)dW(t) + JdP \qquad \text{(Jump-Vasicek)}$$
$$J = N(\mu, \gamma), P = Poisson(dt * h)$$

In the case of CIR, we have to be more careful with the model definition. Since the Jumps are normally distributed we cannot allow an unconstrained discretely arriving Jump process for a CIR as such a jump could potentially drive the interest rates negative. Zhou (2001) [Zho01b] demonstrated that this is equivalent to imposing the condition $-r(t_-) < J < \infty$. In our model, we go even further and impose a symmetry condition on the normal distribution: $-r(t_-) < J < r(t_-)$. This allows us to preserve symmetry (useful when we proceed to MC with antithetic variates) and use the truncated normal distribution. The CIR model is thus specified as follows:

$$dr(t) = \kappa(t)[\mu_r(t) - r(t)]dt + \sigma(t)\sqrt{r(t)}dW(t) + JdP \qquad \text{(Jump-CIR)}$$
$$J = N_{\text{trunc}}(\mu, \gamma, -r(t_-), r(t_-)), P = Poisson(dt * h)$$

One may be tempted to think that since there is no restriction on the truncated normal based on the value of the Poisson realization, we could have potentially negative rates if we get a larger value (> 1) of the dP process. However, this is not possible in the continuous world (though it remains a concern for discretized approximations) since the probability of drawing n jumps from the Poisson distribution is proportional to $(dt * h)^n$. Thus, as dt shrinks, we can neglect the probability value of all draws greater than 1.

The Jump diffusion terms simulate the "jump risk" that may exist in the market. Jumps arrive randomly and can move the interest rate instantaneously. This attempts to simulate real-world shocks to the yield curve such as a surprise Fed rate decisions (2013 and 2018), commercial paper liquidity crisis during COVID-19, and the credit crunch triggered by bank failures in 2008-09.

It is important to note that for simplicity we assume that these SDEs are in the risk-neutral probability measure (Q). We further note that all the random processes in the model (Poisson, Wiener, and Jump) are completely independent of each other both within and across time periods. When multiple jumps occur, they are all independent of each other and previous jumps. This may or may not hold in the real-world, however such analysis is beyond the scope of the paper.

2.2 Model PDE

2.2.1 Vasicek

We obtain the following PDE for zero coupon bond prices under the Vasicek model:

$$0 = \frac{1}{2}\sigma^2 V_{rr} + \kappa(\mu_r - r)V_r + V_t - rV \qquad \text{(Vasicek-PDE)}$$

We can guess a solution of the form $Z(t, t+T) = exp[A(T) - B(T)r]$ where $A(T)$ and $B(T)$ are given by:

$$B(T) = \frac{1 - e^{\kappa T}}{\kappa}$$
$$A(T) = (B(T) - T))(\mu_r - \frac{\sigma^2}{2\kappa^2}) - \frac{\sigma^2 B(T)^2}{4\kappa}$$

2.2.2 CIR

We have an analogous PDE for the CIR model:

$$0 = \frac{1}{2}\sigma^2 r V_{rr} + \kappa(\mu_r - r)V_r + V_t - rV \qquad \text{(CIR-PDE)}$$

With the guessed solution of the same form as Vasicek, we have the following equations for $A(T)$ and $B(T)$:

$$B(T) = \frac{2(e^{\gamma T} - 1)}{(\gamma + \kappa)(e^{\gamma T} - 1) + 2\gamma}$$
$$A(T) = \frac{2\kappa\mu_r}{\sigma^2} log[\frac{2\gamma e^{(\gamma+\kappa)T/2}}{(\gamma + \kappa)(e^{\gamma T} - 1) + 2\gamma}]$$
$$\gamma = [\kappa^2 + 2\sigma^2]^{1/2}$$

2.3 Methodology

2.3.1 Classical Monte-Carlo

We proceed to discretize the model SDEs described in section 2.1 in preparation for a Monte-Carlo simulation. We obtain the following Finite Difference Equations (FDE(s)) [PKK06]:

$$r_j = r_{j-1} + \kappa[\mu_r - r_{j-1}]\Delta t + \sigma\epsilon_j\sqrt{\Delta t} + JP_{\Delta t} \text{ (Vasicek)}$$

$$r_j = r_{j-1} + \kappa[\mu_r - r_{j-1}]\Delta t + \sigma\epsilon_j\sqrt{r_{j-1}\Delta t} + J^{trunc}P_{\Delta t} \text{ (CIR)}$$

ϵ_j is N(0,1), $P_{\Delta t}$ is Poisson with arrival likelihood $h\Delta t$. For Vasicek, J is N(μ, γ) whereas for CIR J^{trunc} is truncated Normal where the truncation is symmetric and such that r_j is guaranteed to be non-negative. This corresponds to a truncated Normal distribution given by:

$$J^{trunc} \sim N^{trunc}(\mu, \gamma, \pm\frac{(r_{j-1} + \kappa(\mu_r - r_{j-1})\Delta t + \sigma\epsilon_j\sqrt{r_{j-1}\Delta t})}{P_{\Delta t}})$$

The Monte-Carlo simulation will simulate the interest rate over a 20-year period (t = 0, T = 20) with m = 104 time steps per year (i.e. $\Delta t = \frac{1}{104}$ years). We will use 100 simulations (n = 100) for each time step.

For now, we chose these parameters for simplicity and will perform a more detailed analysis/calibration of the parameters later.

A complete interest rate path (consisting of $365(T-t)$ time steps) will be used to discount the bond price to obtain a single sample price curve for the time period from t to T. The different price curves will be averaged out using sample mean to obtain a Classical MC price curve:

$$\frac{1}{n}\sum_{i=1}^{n} exp(-\sum_{j=1}^{m} -r_{ij}\Delta t)$$

2.3.2 Variance Reduction in Monte-Carlo

2.3.2.1 Antithetic Variates

The classical Monte-Carlo simulation provides us with bond price estimates as the average of the bond prices obtained for each spot rate path.

In order to reduce the variance of these estimates, we use antithetical variates. We obtain the antithetic Finite Difference Equation (FDE) in relation to the FDE(s) derived for the classical Monte-Carlo simulation in 2.3.1

The anithetical FDE(s) are

$$r_j = r_{j-1} + \kappa[\mu_r - r_{j-1}]\Delta t - \sigma\epsilon_j\sqrt{\Delta t} - JP_{\Delta t} \text{ (Vasicek)}$$

$$r_j = r_{j-1} + \kappa[\mu_r - r_{j-1}]\Delta t - \sigma\epsilon_j\sqrt{r_{j-1}\Delta t} + J^{trunc\_anti}P_{\Delta t} \text{ (CIR)}$$

ϵ_j is N(0,1) and the same sample drawn for the original spot rate. $P_{\Delta t}$ is Poisson with arrival likelihood $h\Delta t$ and the same sample drawn for the original spot rate. For Vasicek, J is N(μ, γ) and the same sample drawn for the original spot rate whereas for CIR J^{trunc_anti} is truncated Normal where the truncation is symmetric and such that r_j is guaranteed to be non-negative. This corresponds to a truncated Normal distribution given by:

$$J^{trunc\_anti} \sim N^{trunc}(\mu, \gamma, \pm\frac{(r_{j-1} + \kappa[\mu_r - r_{j-1}]\Delta t - \sigma\epsilon_j\sqrt{r_{j-1}\Delta t})}{P_{\Delta t}})$$

The sign of the truncated normal is such that $J^{trunc} * J^{trunc\_anti} < 0$ where J_{trunc} corresponds to the original spot rate.

Similar to the classical Monte-Carlo simulation, we simulate over a 20-year period (t = 0, T = 20) with m = 104 time steps per year (i.e. $\Delta t = \frac{1}{104}$ years). We use 100 simulations (n = 100) for each time step. Effectively, we get 200 paths because for each simulated path, we also obtain the corresponding antithetic path.

A complete interest rate path (consisting of $104(T-t)$ time steps) is used to discount the bond price to obtain a single sample price curve for the time period from t to T. The price for each path and its antithetic path is averaged out followed by the average of all the averaged out prices to obtain the antithetical MC price curve:

$$\frac{1}{n}\sum_{i=1}^{n}\frac{exp(-\sum_{j=1}^{m} -r_{ij}\Delta t) + exp(-\sum_{j=1}^{m} -r_{ij\_anti}\Delta t)}{2}$$

Let the variance of the bond price be V (We use sample variance as an estimator for the true variance). The variance of the above estimate is hence given by:

$$\frac{1}{n^2}\sum_{i=1}^{n}\frac{V + V + 2 * \rho * \sqrt{V}\sqrt{V}}{4}$$

where ρ is the correlation between the bond prices obtained using spot rates and their antithetic spot rates. With antithetic variates, we try to obtain a negative correlation so the variance of the estimate is minimized.

2.3.2.2 Control Variates

Our interest rate models are an extension to existing classical models (i.e without jumps). We expect the bond prices obtained from the classical and the jump models to be correlated with each other. We use the closed form solution described in section 2.2 as a control variate for obtaining the bond price using jump model. We define the following estimator for bond price using jump model:

$$\theta = V_{jump} + c(V_{classical} - Z))$$

where V_{jump} is the bond price using jump model, $V_{classical}$ is the bond price using classical model and Z is the true bond price obtained using the classical model (2.2) ($E^Q(V_{classical}) = Z$). It can be shown that $E(\theta) = E(V_{jump})$, hence θ is an unbiased estimator for bond price using jump model.

It can be shown that the variance is minimum when:

$$c = -\frac{Cov(V_{classical}, V_{jump})}{Var(V_{classical})}$$

It can also be shown that the minimum variance of θ is:

$$min(Var(\theta)) = Var(V_{jump}) - \frac{Cov(V_{classical}, V_{jump})^2}{Var(V_{classical})}$$

Hence with a higher covariance between $V_{classical}$ and V_{jump}, we obtain a lower variance for the estimator of bond price using jump model (i.e. θ)

2.4 Data Sources & Description

We have used the 'US Treasury Zero-Coupon Yield Curve' for 30 years from QUANDL. The data can be downloaded from the link - https://www.quandl.com/data/FED/SVENY-US-Treasury-Zero-Coupon-Yield-Curve This is how the data looks like

LATEST VALUES

DATE 2021-02-05	SVENY01 0.08	SVENY02 0.10
SVENY03 0.20	SVENY04 0.34	SVENY05 0.50
SVENY06 0.66	SVENY07 0.81	SVENY08 0.96
SVENY09 1.09	SVENY10 1.21	SVENY11 1.31
SVENY12 1.41	SVENY13 1.49	SVENY14 1.56
SVENY15 1.62	SVENY16 1.68	SVENY17 1.73
SVENY18 1.77	SVENY19 1.81	SVENY20 1.85
SVENY21 1.88	SVENY22 1.91	SVENY23 1.94
SVENY24 1.97	SVENY25 2.00	SVENY26 2.03
SVENY27 2.06	SVENY28 2.08	SVENY29 2.11
SVENY30 2.14		

Figure 1: US Treasury Zero Coupon Yield Curve on 5/2/2021

The data itself is obtained by QUANDL from the Federal Reserve Database (FRED), and the yield curve is itself a Nelson-Siegel-Svensson yield curve constructed in accordance with the Federal Reserve's specifications.[GSW06]

These yield curves are an off-the-run Treasury yield curve based on a large set of outstanding Treasury notes and bonds, and are based on a continuous compounding convention. The maturities are from 1 year (SVENY01) to 30 years (SVENY30). Yields are provided in percentage units. We obtain the bond prices from these yields using the below equation :

$$P(\text{Bond with maturity T}) = e^{-yT/100}$$

In addition to the ZCB yield curve, we use Yield to Maturity (YTM) time series for Treasury Bond yields downloaded from the St. Louis Federal Reserve database (FRED). More specifically, we use DGS3MO (Daily, 3-month T-bill yield) to fit model parameters using the MLE approach (discussed later).

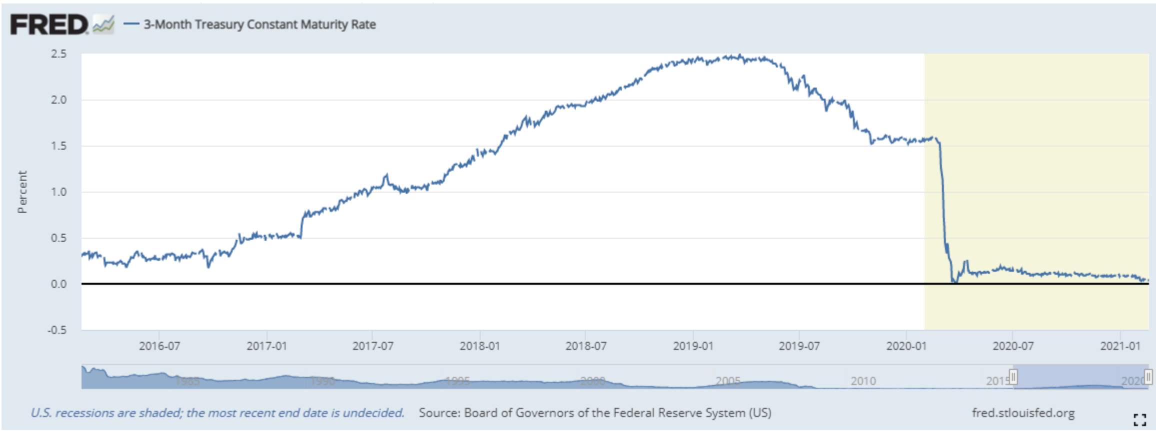


Figure 2: 3 Month Zero-Coupon bond historical yield

3 Preliminary Results

To simplify the initial simulation, we assume constant values for the model parameters such as μ_r, κ, μ, γ and h. We shall calibrate these values exactly later, however for now we just use the following dummy values:

		μ_r	κ	μ	γ	h	r_0	m	n
Vasicek	Classical	0.03	0.5	-	-	-	0.04	-	-
	Jump	0.03	0.5	0	0.01	10	0.04	104	100
CIR	Classical	0.03	0.5	-	-	-	0.04	-	-
	Jump	0.03	0.5	0	0.01	10	0.04	104	100

Table 1: Paremeter Values

3.1 Classical Approach

3.1.1 Vasicek

We run MC simulation for the Vasicek model to obtain the following price curve for the bonds.

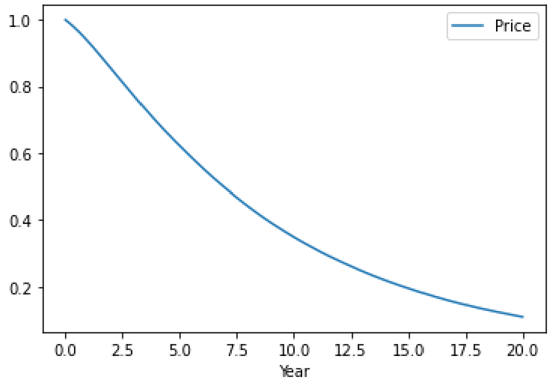


Figure 3: Vasicek Price Curve

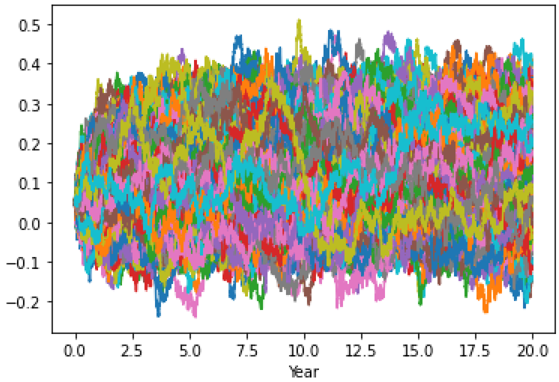


Figure 4: Vasicek Simulated Rates

Figure 3 represents the average prices obtained from the 100 interest rate simulations. Figure 4 plots the simulations. Note that both positive and negative interest rates are observed in the evolution.

3.1.2 CIR

We run MC simulation for the CIR model to obtain the following price curve for the bonds.

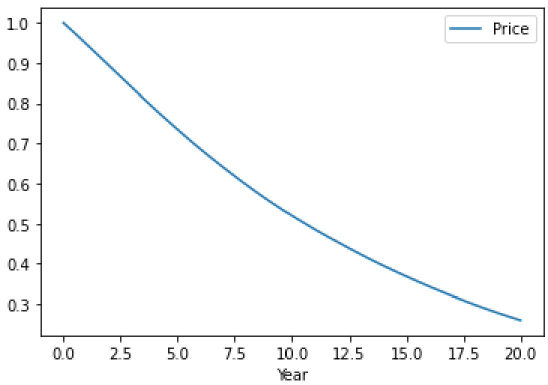


Figure 5: CIR Price Curve

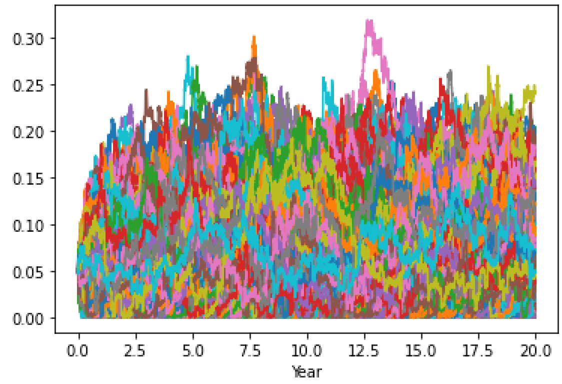


Figure 6: CIR Simulated Rates

Figure 15 represents the average prices obtained from the 100 interest rate simulations. Figure 6 plots the simulations. Note that unlike Vasicek, rates are bounded below by zero. This is because of the truncated normal sampling performed for the CIR simulation.

3.2 Variance Reduction

3.2.1 Antithetic Variates

3.2.1.1 Vasicek

We run antithetic variates based MC simulation for Vasicek. A sample spot rate path along with its antithetic counterpart looks like below

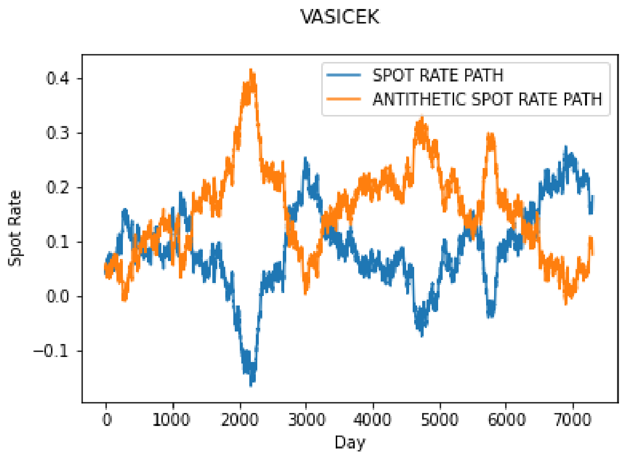


Figure 7: Spot Rate and Antithetic Spot Rate Path - Vasicek

It is evident that the paths are quite negatively correlated.

We also compare the variances of bond price estimates using classical MC and antithetical MC.

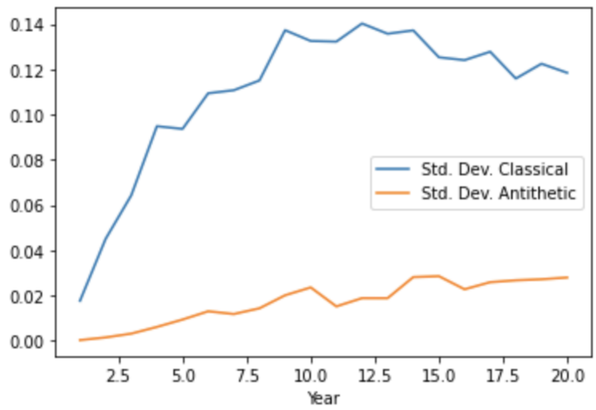


Figure 8: Standard Deviations - Vasicek

As expected, we observe that the variance for antithetic is much lower than that of classical MC.

3.2.1.2 CIR

Likewise, we run MC using antithetic variates for CIR. A sample spot rate path and it's antithetical counterpart looks like below

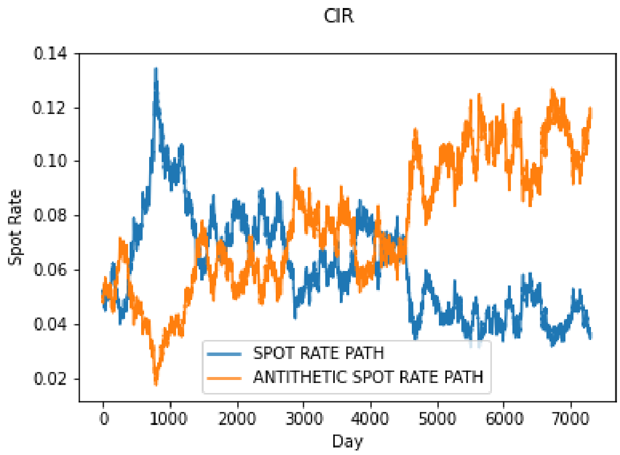


Figure 9: Spot Rate and Antithetic Spot Rate Path - CIR

We observe negative correlation in this case as well. However, the magnitude of jumps are not the same for both the paths.

Theoretically, the variance reduction in CIR should be lesser than that in Vasicek. This is because in CIR, the jump normal is re-sampled for antithetical path and hence can be different from the original spot rate path. We make sure that the two samples have opposite signs in order to maximize the effect of negative correlation on variance reduction.

The variances of bond price estimates using classical MC and antithetical MC looks like the below

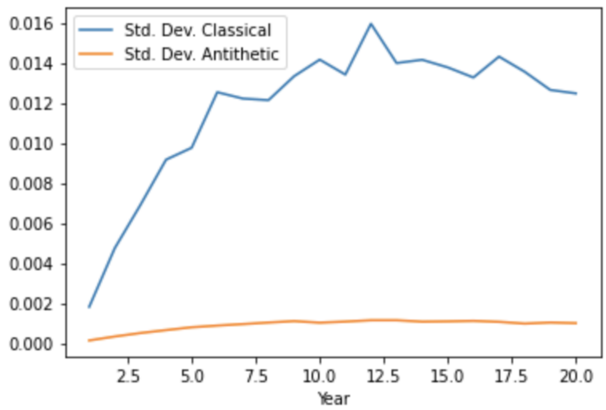


Figure 10: Standard Deviations - CIR

On an average, the classical standard deviation for Vasicek is 13.82 times that of Vasicek. For CIR, this ratio is 12.77.

3.2.2 Control Variates

Our control-variate approach is slightly different from the one learned in MFE 230D. Unlike 230D where we used a known security to price an unknown security, here we are using one data generating process to control for another.

We simulate two different interest rate paths, one for the classical process and one for the jump diffusion process. We then correct the exact price (for classical) by the difference in price between the simulated jump diffusion process and the simulated classical process.

3.2.2.1 Vasicek

We run control variates based MC simulation for Vasicek. With the bond price from classical model as our control variate, we obtain the following plots for the standard deviations for classical MC, antithetic MC and control variates MC.

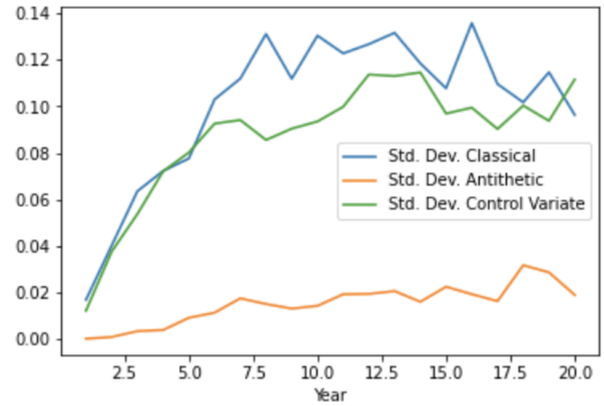


Figure 11: Standard Deviations - Vasicek

3.2.2.2 CIR

Similarly, we obtain the standard deviation of classical, antithetic and control for CIR

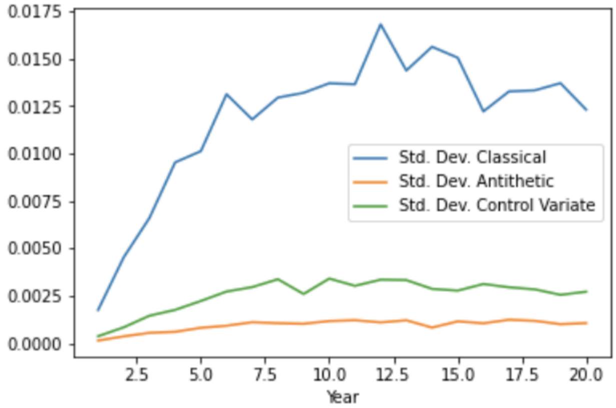
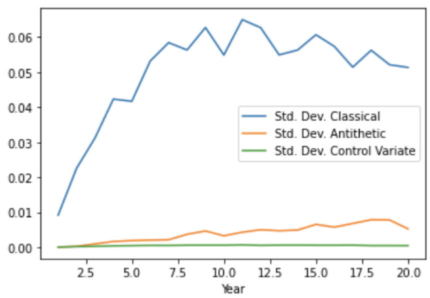


Figure 12: Standard Deviations - CIR

In order to test control variates, we simulate a scenario where the jump is significantly small with $\gamma = 0.00001$ and $h : 0.000001$. As expected, the standard deviation almost becomes 0 for both Vasicek and CIR.

[Vasicek]



[CIR]

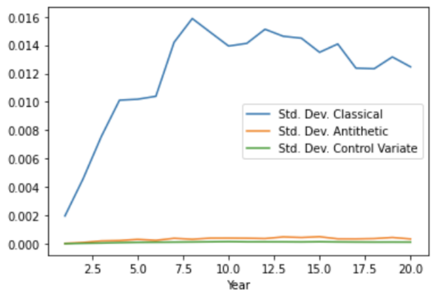


Figure 13: Standard Deviations - Small γ and h

3.2.3 Control-Antithetic Variates

3.2.3.1 Vasicek

We run another version of control variates where we evaluate the prices of bonds using antithetic MC simulation. We observe the following plot for the standard deviations for all 4 methods.

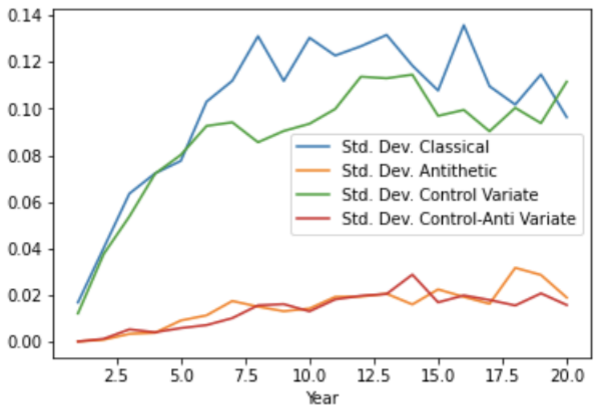


Figure 14: Standard Deviations - Vasicek

3.2.3.2 CIR

Similarly, we obtain the standard deviations for CIR

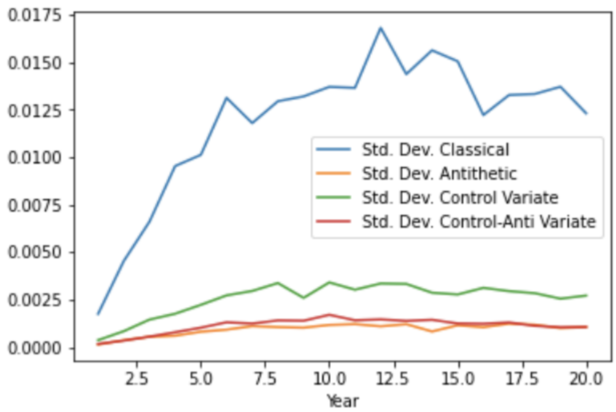


Figure 15: Standard Deviations - CIR

3.2.4 Interpretation

Clearly, antithetic variates and antithetic plus control variates are the most effective for reducing the variance of the estimates.

The below table provides the standard deviations of bond price estimates of maturities ranging 1 to 20 years for all the 4 methods in units of the standard deviations of estimates obtained using the classical MC approach.

		1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
CIR	Antithetic	0.083	0.091	0.068	0.070	0.075	0.067	0.085	0.074	0.081	0.072
	Control-Antithetic	0.092	0.074	0.082	0.081	0.101	0.099	0.104	0.108	0.104	0.123
	Control	0.206	0.184	0.218	0.183	0.219	0.207	0.250	0.260	0.196	0.248
Vasicek	Antithetic	0.011	0.026	0.049	0.076	0.069	0.064	0.130	0.136	0.140	0.134
	Control-Antithetic	0.013	0.034	0.085	0.057	0.076	0.070	0.091	0.119	0.145	0.101
	Control	0.723	0.936	0.846	0.998	1.033	0.899	0.841	0.653	0.808	0.717

Table 2: Standard Deviation (unit of classical std) for 1-10Y bonds

		11Y	12Y	13Y	14Y	15Y	16Y	17Y	18Y	19Y	20Y
CIR	Antithetic	0.085	0.082	0.082	0.074	0.089	0.089	0.075	0.079	0.082	0.075
	Control-Antithetic	0.103	0.086	0.095	0.091	0.082	0.100	0.097	0.083	0.075	0.085
	Control	0.221	0.198	0.231	0.183	0.183	0.255	0.221	0.212	0.185	0.220
Vasicek	Antithetic	0.158	0.168	0.163	0.222	0.146	0.267	0.221	0.139	0.208	0.146
	Control-Antithetic	0.148	0.156	0.156	0.243	0.157	0.147	0.164	0.153	0.181	0.164
	Control	0.813	0.897	0.858	0.967	0.899	0.732	0.823	0.986	0.817	1.157

Table 3: Standard Deviation (unit of classical std) for 11-20Y bonds

Variance reduction is maximum for antithetic MC and control-variate + antithetic MC. Also, the variance reduction is more prominent in Vasicek model as compared to CIR model.

4 Calibration

The Monte-Carlo set up provides us with a framework for pricing yield curves that incorporate jump risk. So far we have used an assumed set of parameters for the model. We now proceed to tune this parameters to obtain yield and price curves that agree with the prevailing market values.

To fit the parameters of the model we propose two broad approaches:

1. Yield Curve Fitting Approach: Attempt to calibrate the model to match the yield curve observed over the past n days.

2. Maximum Likelihood Estimation (MLE) Approach: Attempt to find the optimal parameter set that maximizes the likelihood of observing the given evolution of the spot rate over a time period.

4.1 Yield Curve Fitting (with Solver) Approach

We use the yield curve snapshot from February 5, 2021 as described in the data section.

The algorithm will proceed in the following steps:

1. Start with a guessed set of parameters for κ, μ_r, σ, and γ.*

2. Run a Monte-Carlo estimate of the bond price for each maturity using the assumed parameter set.

3. Compute pricing error vector from the estimate and calculate it's Euclidean norm (l2-norm).

4. Repeat steps 1-3 until a minimum norm is achieved.

*Note: We do not optimize for h, the arrival rate of jumps or the starting rate r_0. This is to reduce the complexity of the minimization problem. The assumed value of h=10 corresponds to roughly 150-200 jump days over a 20 year period. The assumed value of r_0 is 0.001 (0.1%) which is approximately the overnight yield observed in the treasury market. The mean of the jump size, μ, is assumed to be 0 for simplicity (i.e. jumps on average are of size 0). This is a simplification as it makes it much easier to generate antithethic paths for Monte-Carlo.

Furthermore, we add non-zero bounds on each of the parameters, to ensure the algorithm does not waste time searching for negative values which don't make sense from the model's perspective. We also add upper bounds to prevent unrealistic values of the parameters from being optimal ($\gamma < 0.1, \mu_r < 0.2, \sigma < 0.2, \kappa < 1$).

For finding the minima we use Python's *scipy* library with the *scipy.optimize.minimize* function. We use *minimize*'s L-BGFGS (Limited-memory Broyden–Fletcher–Goldfarb–Shanno) algorithm which supports linear constraints on the minimization parameters.

A potential complication that resulted from using an inherently random price calculator like MC was that small deviations in the parameters often caused movements in the value that were within the standard error, thus degrading the quality of the gradient and therefore the minimization problem.
To address this, we set the random seed of the MC calculator to a fixed value before performing step 2. This serves to reduce the random variation arising from different draws and produces more uniform estimates for the solver to minimize.

4.2 Maximum Likelihood Estimation (MLE) Approach

An alternative to fitting the yield curve using a solver-based approach is to fit the evolution of the interest rates. Such a method relies on using a likelihood measure to find the optimal set of model parameters that maximize the likelihood of observing a given interest rate time series.
To proceed, we need an estimate of the transition density for the interest rate time series. We obtain this by discretizating the spot rate model: We assume that the rate evolution from time t to $t + \Delta t$ happens in a single discrete step described by discrete version of the model (similar to the Euler-Maruyama discretization

method used in Monte-Carlo methods). We can then derive transition distributions for the rate at time $r_{t+\Delta t}$ conditioned on the rate at time r_t and given the model parameters.

The Δt used in this approach is determined by the frequency of the available time-series data. For example, with daily data, we would have $\Delta t = \frac{1}{252}$, with weekly it would be $\frac{1}{52}$.

Implicit to the discretization is the assumption that the 3 random processes, dP, dW, and J are all independent of each other.

4.2.1 Vasicek

For Vasicek, we discretize as follows:

$$r_{t+\Delta t} = r_t + \kappa(\mu_r - r_t)\Delta t + \sigma dW_{\Delta t} + J_{\Delta t} dP_{\Delta t}$$

Recall that when we defined the models, we assumed that the rate dynamics were in the risk-neutral Q-measure and not the real-world P-measure. Thus, we need to add a market price of risk λ to construct the P-measure transition distributions. Similar to the work done in 230I, this can be obtained by constructing an adjusted $\hat{\mu_r}$ as follows:

$$\hat{\mu_r} = \mu_r - \frac{\lambda\sigma}{\kappa}$$

and obtain the P-measure discretization:

$$r_{t+\Delta t} = r_t + \kappa(\hat{\mu_r} - r_t)\Delta t + \sigma dW_{\Delta t} + J_{\Delta t} dP_{\Delta t}$$

However, since we are only interested in calibrating the model for pricing, we don't actually care about the actual value of λ or μ_r, only the overall $\hat{\mu_r}$. Since this does not change the estimation problem, going forward, for simplicity, we refer to $\hat{\mu_r}$ as μ_r while noting that this estimated parameter is no longer directly comparable to μ_r estimated in the yield curve fitting method.

Since J ,dW and dP are Normal(μ, γ), Wiener, and Poisson processes respectively, in the discrete world, we can obtain the following distributions:

$$J_{\Delta t} \sim N(\mu, \gamma)$$

$$dW_{\Delta t} \sim N(0, \sqrt{\Delta t})$$

$$dP_{\Delta t} \sim P(h\Delta t)$$

To obtain a distribution for $r_{t+\Delta t}$ we condition the transition density on there being n jumps. Thus for each integer n from $n = 0$ to $n = \infty$, we have $r_{t+\Delta t}$ given by the sum of two normal distributions $J_{\Delta t}$ and $\sigma dW_{\Delta t}$

shifted by the time-step $r_t + \kappa(\mu_r - r_t)\Delta t + n\mu$:

$$p(r_{t+\Delta t}|r_t, \kappa, \mu_r, \mu, \gamma, \sigma, h) = \sum_{n=0}^{n=\infty} p(r_{t+\Delta t}|r_t, dP_{\Delta t} = n, \kappa, \mu_r, \mu, \gamma, \sigma, h)p(dP_{\Delta t} = n|h)$$

with the following expressions for the densities:

$$p(r_{t+\Delta t}|r_t, dP_{\Delta t} = n, \kappa, \mu_r, \mu, \gamma, \sigma, h) = Normal(r_t + \kappa(\mu_r - r_t)\Delta t + n\mu, \sqrt{n(\gamma)^2 + \sigma^2 \Delta t})$$

$$p(dP_{\Delta t} = n|h) = \frac{e^{-h\Delta t}(h\Delta t)^n}{n!} \text{ (Poisson Density/PMF)}$$

It is important to note that the Poisson density contains a factor of $(h\Delta t)^n$. This is of great convenience as it will enable us to ignore higher-order terms of the infinite series when estimating the transition density. Now that we have the transition function we can define the likelihood for a given set of data as follows:

$$L(r_0, ..., r_T) = \prod_{t=0}^{T} p(r_{t+\Delta t}|r_t, \kappa, \mu_r, \mu, \gamma, \sigma, h)$$

For computational convenience we take the natural-log and minimize the negative log likelihood:

$$\kappa, \mu_r, \mu, \gamma, \sigma, h = \underset{\kappa,\mu_r,\mu,\gamma,\sigma,h}{\arg\min} \; - \log L(r_0, ..., r_T)$$

As usual, we will use scipy.optimize.minimize to solve for the log-likelihood function.

4.2.2 CIR

For CIR, we follow a very similar method to obtain a discretized log-likelihood function. The only difference will be in the standard deviation of the normal density (underlined):

$$p(r_{t+\Delta t}|r_t, dP_{\Delta t} = n, \kappa, \mu_r, \mu, \gamma, \sigma, h) = Normal(r_t + \kappa(\mu_r - r_t)\Delta t + n\mu, \underline{\sqrt{n(\gamma)^2 + r_t \sigma^2 \Delta t}})$$

The complication with CIR is that because J is not a true normal but a truncated normal, the exact density is not actually as written above. In fact we have a sum of a normal and truncated normal distributions whose distribution is not analytically known. To avoid this problem, we estimate the distribution using a regular sum of normal, same as in Vasicek. We leave the exact estimation to future papers.

4.2.3 Pricing Procedure

Since the MLE approach does not attempt to fit the present day yield curve, we have to be careful about the type of maturities we try to price. If we use the historical evolution of the short-term rate (ex. 3 month Treasury yield), we will likely see very small values for parameters like μ_r, since as of February 2021, the 3 month yield is close to 0.

Thus, we will refrain from directly pricing longer maturity bonds and instruments using a model calibrated using the MLE estimates. We will restrict our attention to bonds/derivatives with maturities no greater than 3 years.

4.3 Conclusion

We can summarize the two approaches used as follows:

1. Yield Curve Fitting Approach
    - Advantages: Simple to use, can use standard routines available in Python, Excel, and other common financial tools. We use Python's scipy.optimize library. Can exactly match observed yield curve at any given day.
    - Disadvantages: Algorithm may take too long or may not converge at all. Existence of multiple solutions could force the user to choose between several compelling solutions with little theoretical basis for doing so.
2. Maximum Likelihood Estimation (MLE) approach
    - Advantages: Will provide consistent, efficient estimator of the model parameters.
    - Disadvantages: Contingent upon assumptions about model parameter distributions. May not match prices for other maturities on the yield curve. Distribution for Jump-CIR is approximate only (see Future Work section for suggested improvements).

5 Pricing

With our model properly setup and calibrated, we proceed to price common fixed-income derivatives on the financial markets.

Our models are calibrated using the following datasets

- Yield Curve Fitting Approach - Treasury bond prices of maturities ranging 1 to 30 years. We used prices on 5.2.2021.
- MLE Approach - Treasury bond yields (YTM) of maturity - 3M, over last 6 months.

We use the calibrated models for pricing out-of-sample treasury bonds and treasury swaps. We use the following metrics for evaluating the accuracy of our models

- Yield Curve Fitting Approach -
    - Out-of-sample bond price estimation - MAE of estimated prices and true prices.

– Treasury swap rate estimation - MAE of estimated swap rates and true swap rates

- MLE Approach -

  – Out-of-sample bond price estimation - MAE of estimated prices and true prices

  – Treasury swap rate estimation - MAE of estimated swap rates and true swap rates

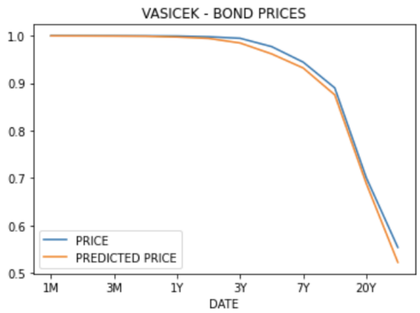
5.1 Yield Curve Fitting Approach

	Calibration Date	κ	μ_r	σ	μ	γ	h
Vasicek	5.2.2021	0.094	0.036	0.01	-	-	-
CIR	5.2.2021	0.076	0.044	0.117	-	-	-
Jump Vasicek	5.2.2021	0.272	0.030	0.01	0	0.01	10
Jump CIR	5.2.2021	0.00329184	0.001	0.04480743	0	0.11385074	10

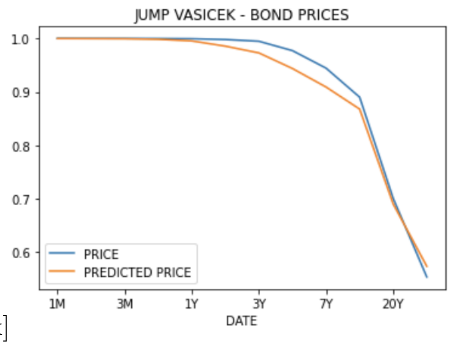
Table 4: Calibrated Model Parameters

5.1.1 Vasicek Vs Jump Vasicek

We obtain the following out-of sample bond price estimations



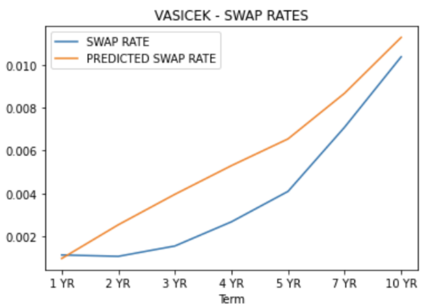
[Vasicek]



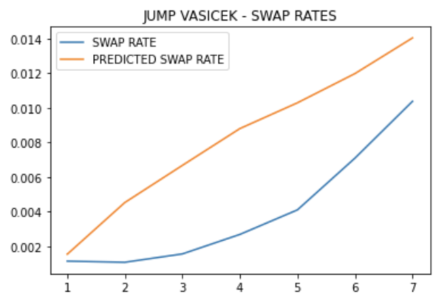
[Jump Vasicek]

Figure 16: Estimated Bond Prices (5.2.2021)

We obtain the following swap rate estimations



[Vasicek]

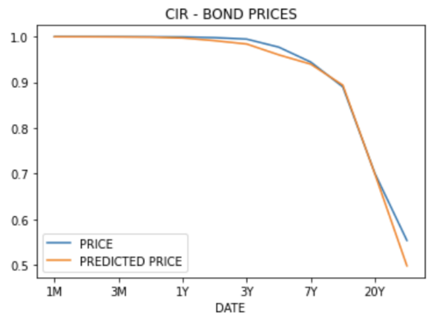


[Jump Vasicek]

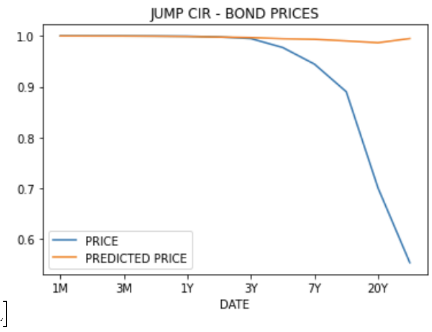
Figure 17: Estimated Swap Rates (5.2.2021)

5.1.2 CIR Vs Jump CIR

We obtain the following out-of sample bond price estimations



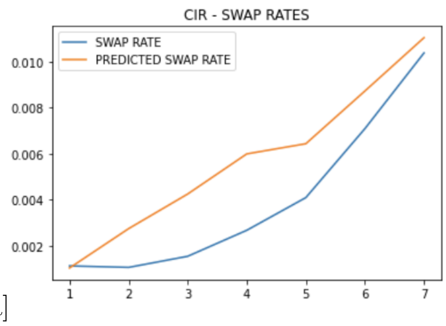
[CIR]



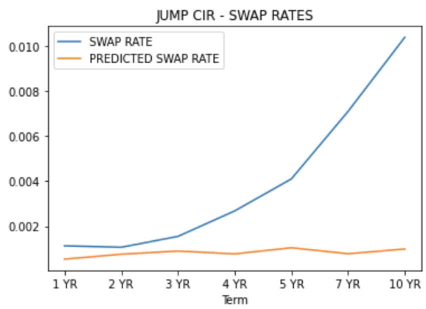
[Jump CIR]

Figure 18: Estimated Bond Prices (5.2.2021)

We obtain the following swap rate estimations



[CIR]



[Jump CIR]

Figure 19: Estimated Swap Rates (5.2.2021)

5.1.3 Pricing Errors

	1M	**2M**	**3M**	**6M**	**1Y**	**2Y**	**3Y**	**5Y**	**7Y**	**10Y**
Vasicek	5.29e-08	2.03e-07	6.84e-07	2.94e-06	2.52e-05	1.78e-04	4.04e-04	1.73e-03	5.17e-03	8.97e-03
J Vasicek	1.33e-06	4.95e-06	3.72e-05	3.84e-05	4.15e-04	2.20e-03	4.80e-03	1.49e-02	5.22e-02	5.71e-02
CIR	2.56e-05	6.61e-05	1.11e-04	3.43e-04	1.57e-03	4.83e-03	9.25e-03	1.87e-02	2.61e-02	6.43e-02
J CIR	3.92e-06	1.21e-05	2.29e-05	7.27e-05	2.86e-04	1.10e-03	1.92e-03	4.56e-03	8.47e-03	1.03e-02

Table 5: Standard Deviation - Bond Price Estimates

	1Y	**2Y**	**3Y**	**4Y**	**5Y**	**7Y**	**10Y**
Vasicek	2.83e-06	2.53e-05	1.18e-04	2.18e-04	3.16e-04	5.20e-04	8.063e-04
J Vasicek	5.11e-05	3.47e-04	1.16e-03	2.09e-03	2.52e-03	4.97e-03	6.48e-03
CIR	0.0003731	0.00108407	0.00231791	0.00293035	0.00338673	0.00502137	0.00539948
J CIR	8.25e-05	2.75e-04	5.25e-04	6.12e-04	9.35e-04	7.36e-04	1.64e-03

Table 6: Standard Deviation - Swap Rate Estimates

	1M	**2M**	**3M**	**6M**	**1Y**	**2Y**	**3Y**	**5Y**	**7Y**	**10Y**
Vasicek	0.000174	0.000264	0.000391	0.000754	0.002150	0.006125	0.010884	0.015354	0.010606	0.00717
J Vasicek	0.000215	0.000364	0.000572	0.001327	0.004030	0.011976	0.021515	0.034538	0.028748	0.03607
CIR	0.000175	0.000272	0.000393	0.000733	0.002475	0.006300	0.010889	0.013007	0.010093	0.01669
J CIR	0.000146	0.000192	0.000259	0.000313	0.000542	0.000062	0.002274	0.017876	0.048792	0.10214

Table 7: Absolute Error between true bond prices and the estimates

	1Y	**2Y**	**3Y**	**4Y**	**5Y**	**7Y**	**10Y**
Vasicek	0.000166	0.001476	0.002398	0.002580	0.002449	0.001489	0.000896
J Vasicek	0.000398	0.003456	0.005192	0.005863	0.006099	0.005479	0.003236
CIR	0.000133	0.001395	0.002581	0.002511	0.002785	0.001909	0.001766
J CIR	0.000590	0.000309	0.000648	0.001904	0.003049	0.006307	0.009379

Table 8: Absolute Error between true swap rates and the estimates

		Vasicek	J Vasicek	CIR	J CIR
Bond Price Estimates	Maturity $\leq$ 1Y	0.00074	0.0013	0.00072	0.00029
	Maturity $>$ 1Y	0.01086	0.0318	0.01189	0.1475
Swap Rate Estimates	Maturity $\leq$ 3Y	0.00134	0.00302	0.0014	0.00051
	Maturity $>$ 3Y	0.00158	0.00532	0.00201	0.0063

Table 9: Mean Absolute Error

	Bond Price Estimates	Swap Rate Estimates
Vasicek	0.00663	0.001638
J Vasicek	0.01484	0.004225
CIR	0.006486	0.00194
J CIR	0.0739	0.00316

Table 10: Overall Mean Absolute Error

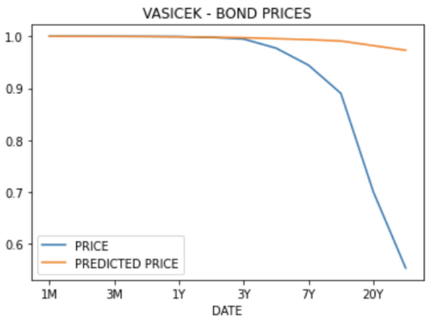
5.2 Maximum Likelihood

	Calibration Time Series	κ	μ_r	σ	μ	γ	h
Vasicek	3-Month T-Rate	32.18	0.0009042	0.00146	-	-	-
CIR	3-Month T-Rate	0.4635	0.0	0.0576	-	-	-
Jump Vasicek	3-Month T-Rate	0.8078	0.0	0.001446	0	0.0001628	4.885
Jump CIR	3-Month T-Rate	0.4782	0.0	0.04627	0	0.0001648	10.03

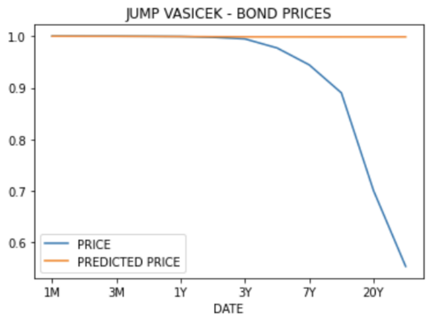
Table 11: Calibrated Model Parameters

5.2.1 Vasicek Vs Jump Vasicek

We obtain the following out-of sample bond price estimations



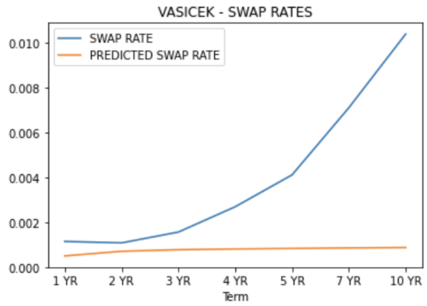
[Vasicek]



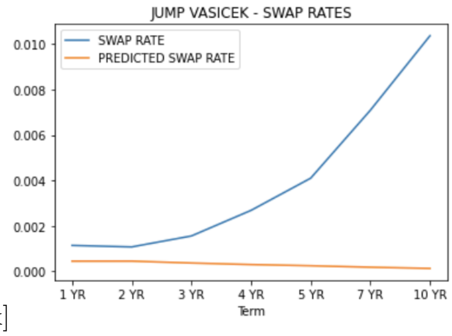
[Jump Vasicek]

Figure 20: Estimated Bond Prices (5.2.2021)

We obtain the following swap rate estimations



[Vasicek]

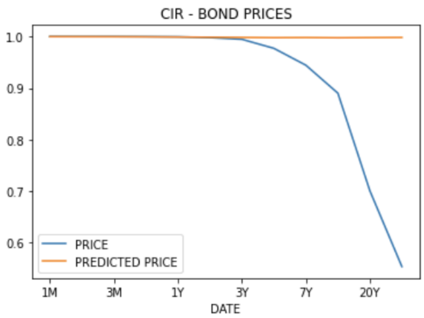


[Jump Vasicek]

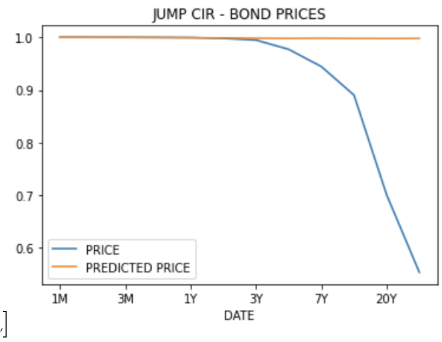
Figure 21: Estimated Swap Rates (5.2.2021)

5.2.2 CIR Vs Jump CIR

We obtain the following out-of sample bond price estimations



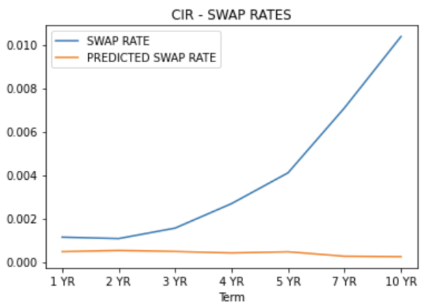
[CIR]



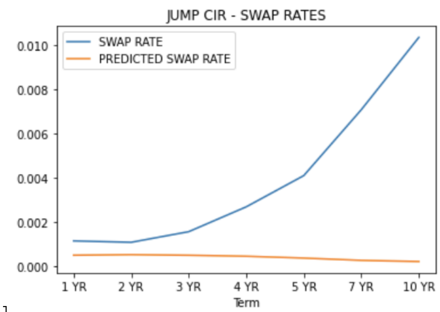
[Jump CIR]

Figure 22: Estimated Bond Prices (5.2.2021)

We obtain the following swap rate estimations



[CIR]



[Jump CIR]

Figure 23: Estimated Swap Rates (5.2.2021)

5.2.3 Pricing Errors

	1M	2M	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y
Vasicek	2.01e-10	3.04e-10	3.85e-10	5.74e-10	1.92e-09	2.91e-09	3.83e-09	6.38e-09	1.01e-08	1.35e-08
J Vasicek	1.49e-09	4.06e-09	9.55e-09	5.55e-08	3.63e-07	2.02e-06	3.66e-06	7.94e-06	1.21e-05	1.58e-05
CIR	5.11e-06	1.66e-05	2.96e-05	1.22e-04	3.04e-04	8.94e-04	1.42e-03	2.00e-03	2.57e-03	2.71e-03
J CIR	4.00e-06	8.83e-06	2.32e-05	6.78e-05	2.51e-04	6.61e-04	1.23e-03	1.47e-03	1.62e-03	1.99e-03

Table 12: Standard Deviation - Bond Price Estimates

	1Y	2Y	3Y	4Y	5Y	7Y	10Y
Vasicek	6.39e-10	9.95e-10	9.46e-10	1.54e-09	1.52e-09	1.54e-09	1.04e-09
J Vasicek	5.42e-08	4.55e-07	6.27e-07	1.03e-06	1.08e-06	1.26e-06	1.67e-06
CIR	9.78e-05	2.75e-04	3.41e-04	4.07e-04	4.66e-04	2.56e-04	2.34e-04
J CIR	7.55e-05	2.15e-04	2.88e-04	2.60e-04	2.35e-04	3.20e-04	2.41e-04

Table 13: Standard Deviation - Swap Rate Estimates

	1M	2M	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y
Vasicek	0.000134	0.000173	0.000231	0.000275	0.000432	0.000174	0.002520	0.018287	0.049238	0.10084
J Vasicek	0.000138	0.000175	0.000226	0.000227	0.000200	0.001014	0.004133	0.021624	0.054362	0.10865
CIR	0.000142	0.000182	0.000236	0.000269	0.000274	0.000579	0.003771	0.020805	0.053473	0.10741
J CIR	0.000142	0.000181	0.000242	0.000270	0.000306	0.000816	0.003617	0.021078	0.053473	0.10773

Table 14: Absolute Error between true bond prices and the estimates

	1Y	2Y	3Y	4Y	5Y	7Y	10Y
Vasicek	0.000645	0.000373	0.000786	0.001879	0.003277	0.006245	0.009510
J Vasicek	0.000693	0.000626	0.001191	0.002388	0.003857	0.006915	0.010250
CIR	0.000656	0.000510	0.001078	0.002221	0.003657	0.006804	0.010164
J CIR	0.000653	0.000534	0.001064	0.002270	0.003768	0.006744	0.010140

Table 15: Absolute Error between true swap rates and the estimates

		Vasicek	Jump Vasicek	CIR	Jump CIR
Bond Price Estimates	Maturity $\leq$ 1Y	0.00024	0.00019	0.00023	0.00022
	Maturity > 1Y	0.14531	0.15535	0.1545	0.1546
Swap Rate Estimates	Maturity $\leq$ 3Y	0.00060	0.00083	0.00074	0.00077
	Maturity > 3Y	0.00634	0.00701	0.0069	0.0069

Table 16: Mean Absolute Error

	Bond Price Estimates	Swap Rate Estimates
Vasicek	0.072	0.0032
J Vasicek	0.077	0.0037
CIR	0.077	0.0036
J CIR	0.077	0.0036

Table 17: Mean Absolute Error

5.3 Conclusions

We see that the errors for the models incorporating jump processes are slightly lower for shorter maturity instruments and bonds. The advantage seems to vanish for longer term securities. This is in line with the findings of [LY01], among others, who noted that jump models in the Taiwanese market do not demonstrate a clear superiority over classical models, but nevertheless hold economic value because of the additional risk parameters they incorporate.

6 Final Conclusions

We have developed an extensive and extensible framework for Jump diffusion models. We learned that Monte-Carlo is an effective pricing tool, with fast, low variance, and good quality estimates.
We developed two main calibration tools, one based on yield curve fitting and the other on MLE over the historical short-rate. The former has the advantage of matching observed yield curves exactly while the latter provides a sound statistical basis for the derived parameter values.

In line with related research we find that jump models appear to do a better job at reducing the pricing error of common Treasury derivatives such as swaps. Performance gain is more significant for shorter maturity instruments than longer maturity instruments.

While this is not overwhelming evidence showing the superiority of Jump models over their classical counterparts, we believe that we didn't necessarily need to have strong evidence for the existence of jumps to justify the use of Jump models. The Jump extensions used in these models are relatively simple, painless to evaluate and justify, and a very natural enrichment of the classical models. They capture a source of risk that is perhaps not fully encapsulated by the continuously evolving Wiener random process. Although left unexplored in this paper, the availability of the Jump parameters could provide a theoretical backing for more advanced risk-management, VaR, and "what-if?" analysis for fixed-income instruments.

7 Future Work

This project sets the stage for a lot of potential future research on Jump diffusion models.

7.1 Tree-Based Evaluators

Conventional models such as Vasicek and CIR support an easy approximation through recombining trees. This is chiefly possible because there is only one source of randomness in these models (the Wiener process, dW); thus as the finite time jump for each tree node goes to 0, we can ignore all higher-order terms and discretize the rate movement into finite recombining possibilities in the future.

Unfortunately, a naive application of such an approach for jump models does not work. Since the jumps arrive randomly and are normally distributed, there is no restriction on where the rate may be at time $t + dt$.

Perhaps additional simplifying assumptions might help alleviate this problem somewhat. One could change the distribution of the jump to a discrete (and finite) distribution and distribution of the arrival process to some kind of Bernoulli process which either happens or doesn't (i.e. no multiple jumps).

It may then be possible to create a tree where each branch would be conditioned on whether or not a jump happened, and the size of the jump if it did. It is unlikely that such a tree would recombine and the computational complexity of evaluating it may make it completely infeasible.

If an effective tree based evaluator could be engineered, it would greatly increase the precision of the estimates as well as enable us to evaluate path-dependent derivatives such as barrier callable bonds.

7.2 Closed-Form Solutions

We do not know if a closed-form solution actually exists. If a closed form solution could be found it would be of obvious advantage to pricing jump-based instruments. However, even if one does not exist, perhaps a close approximation could be found which would enable us to reduce error through MC simulations by the Control Variate method. Alternatively a closely related model with a closed form solution could also be used.

7.3 Exact Distributions for Jump CIR calibration

Recall that the MLE calibration for Jump-CIR used approximated the sum of a normal and truncated-normal distribution with a normal distribution. Perhaps a better approximate distribution or a numerical approximation of the actual distribution of the sum could be used instead.

7.4 Improved Algorithms

Currently, the MC algorithms are a little slow for high values of m or n. Perhaps a refactor and rewrite of the existing codebase could improve the performance and reliability.

8 Bibliography/References

References

[AT88] Chang Mo Ahn and Howard E. Thompson. "Jump-Diffusion Processes and the Term Structure of Interest Rates". In: *The Journal of Finance* (1988), pp. 155–174. URL: https://www.jstor.org/stable/2328329.

[BW97] S. Babbs and N. Webber. "Term Structure Modeling under Alternative Official Regimes". In: *Mathematics of Derivatives Securities* (1997).

[OR97] Ludger Overbeck and Tobias Rydén. "Estimation in the Cox-Ingersoll-Ross Model". In: *Cambridge University Press* (1997), pp. 430–461. URL: https://www.jstor.org/stable/3532742.

[Zho97] Chunsheng Zhou. "A Jump-Diffusion Approach to Modeling Credit Risk and Valuing Defaultable Securities". In: *Board of Governors of the Federal Reserve System (U.S.)* (1997).

[LY99] Bing-Huei Lin and Shih-Kuo Yeh. "Jump-Diffusion Interest Rate Process: An Empirical Examination". In: *Journal of Business Finance & Accounting* (1999), pp. 967–995. URL: https://doi.org/10.1111/1468-5957.00282.

[LY01] Bing-Huei Lin and Shih-Kuo Yeh. "Estimation for factor models of term structure of interest rates with jumps: the case of the Taiwanese government bond market". In: *Journal of International Financial Markets, Institutions and Money* (2001), pp. 167–197. URL: https://doi.org/10.1016/S1042-4431(00)00049-4.

[Zho01a] Chunsheng Zhou. "The term structure of credit spreads with jump risk". In: *Journal of Banking & Finance* (2001), pp. 2015–2040. URL: https://doi.org/10.1016/S0378-4266(00)00168-0.

[Zho01b] Hao Zhou. "Jump Diffusion Term Structure and Ito Conditional Moment Generator". In: *Federal Reserve Publications* (2001). URL: https://www.federalreserve.gov/pubs/feds/2001/200128/200128pap.pdf.

[GK03] PAUL GLASSERMAN and S. G. KOU. "THE TERM STRUCTURE OF SIMPLE FORWARD RATES WITH JUMP RISK". In: *Mathematical Finance* (2003), pp. 383–410.

[Joh03] Michael Johannes. "The Statistical and Economic Role of Jumps in Continuous-Time Interest Rate Models". In: *The Journal of Finance* (2003), pp. 227–260. URL: https://doi.org/10.1111/j.1540-6321.2004.00632.x.

[GSW06] R. Gurkaynak, B. Sack, and J. Wright. "The U.S. Treasury Yield Curve: 1961 to the Present". In: *Federal Reserve Board* (2006). URL: https://www.federalreserve.gov/pubs/feds/2006/200628/200628pap.pdf.

[PKK06] Kisoeb Park, Moonseong Kim, and Seki Kim. "Bond Pricing with Jumps and Monte Carlo Simulation". In: *Springer-Verlag Berlin Heidelberg* (2006), pp. 30–37. URL: https://link.springer.com/content/pdf/10.1007%5C%2F11758501_9.pdf.

[BNP07] Nicola Bruti-Liberati, Christina Nikitopoulos-Sklibosios, and Eckhard Platen. "Pricing under the Real-World Probability Measure for Jump-Diffusion Term Structure Models". In: *QUANTITATIVE FINANCE RESEARCH CENTRE* (2007), p. 198.

[ELP07] Lina El-Jahel, Hans Lindberg, and William Perraudin. "Interest Rate Distributions, Yield Curve Modelling and Monetary Policy". In: *Mathematics of Derivatives Securities* (2007).

[MCO09] José Antonio Núñez Mora, José Luis de la Cruz, and Elizabeth Ortega. "Continuous Time Models of Interest Rate with Jumps: Testing the Mexican Data (1998-2006)". In: *Equilibrio Económico* (2009), pp. 123–140.

[Čer12] Jakub Černy. "Calibration of Interest Rates". In: *WDS'12 Proceedings of Contributed Papers: Part I - Mathematics and Computer Sciences* (2012). URL: https://ssrn.com/abstract=2302509.

[RM14] Marianito R. Rodrigo and Rogemar S. Mamon. "An alternative approach to the calibration of the Vasicek and CIR interest rate models via generating functions". In: *Quantitative Finance* (2014), pp. 1961–1970. URL: https://doi.org/10.1080/14697688.2013.765062.

[Tia+14] Lihui Tian et al. "Pricing Vulnerable Options with Correlated Credit Risk Under Jump-Diffusion Processes". In: *The Journal of Futures Markets* (2014), pp. 957–979. URL: https://doi.org/10.1002/fut.21629.

[Wu14] Yongfeng Wu. "Jump-diffusion CIR model and its applications in credit risk". In: *Hacettepe Journal of Mathematics and Statistics* (2014), pp. 1095–1106. URL: https://doi.org/10.15672/HJMS.201447455.

[PK15] Kisoeb Park and Seki Kim. "On Interest Rate Option Pricing with Jump Processes". In: *International Journal of Engineering and Applied Sciences (IJEAS)* (2015). URL: https://www.ijeas.org/download_data/IJEAS0207030.pdf.

[Ber16] Victor Bernal. "Calibration of the Vasicek Model: An Step by Step Guide". In: (2016).

[TEB16] Emmanuel Thompson, Gideon Mensah Engmann, and Austin Butorac. "Short-Term Interest Rate Model: Calibration of the Vasicek Process to Ghana's Treasury Rate". In: *Journal of Finance and Investment Analysis* (2016), pp. 71–79.

9 Appendices

We have released all our code under the GNU GPL v3.0 License to the following GitHub repository: Advanced Term Structures. The repository contains all the codes and machinery required to do the analysis in the paper, as well as result notebooks summarizing key findings.