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Jump-Diffusion Interest Rate Process: An Empirical Examination

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1. INTRODUCTION

Conventionally, financial variables such as stock prices, foreign exchange rates, and interest rates are assumed to follow a diffusion process with continuous time paths when pricing financial assets. Despite their attractive statistical properties and computation convenience by which they are unanimously accepted for theoretical derivation, more and more empirical evidence has shown that pure diffusion models are not appropriate for these financial variables. For example, Jarrow and Rosenfeld (1984), Ball and Torous (1985a and 1985b), Akgiray and Booth (1986), Jorion (1988) and Lin and Yeh (1997) all found evidence indicating the presence of jumps in the stock price process. Tucker and Pond (1988), Akgiray and Booth (1988) and Park, Ann and Fujihara (1993) studied foreign exchange markets and concluded that the jump-diffusion process is more appropriate for foreign exchange rates. In pricing and hedging with financial derivatives, jump-diffusion models are particularly important, since ignoring jumps in financial prices will cause pricing and hedging risks.

For interest rates, jump-diffusion processes are particularly meaningful since the interest rate is an important economic

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variable which is, to some extent, controlled by the government as an instrument for its financial policy. Hamilton (1988) investigated US interest rates and found changes in regime for the interest rate process. Das (1994) found movements in interest rates display both continuous and discontinuous jump behavior. Presumably, jumps in interest rates are caused by several market phenomena, such as money market interventions by the Fed, news surprises, and shocks in the foreign exchange markets, and so on.

Classical term structure of interest rate models, such as the Vasicek (1977) model, the Brennan and Schwartz (1979) model, the Cox, Ingersoll and Ross (CIR, 1985) model, the Babbs and Webber (1994) model, the Bakshi and Chen (1997) model, and other extended models for pricing interest rate derivatives, such as the Ho and Lee (1986) model, the Babbs (1990) model, the Hull and White (1990) model, and the Heath, Jarrow and Morton (1992) model, all assume that processes of state variables (such as the short-term interest rate, or the long-term interest rate, or others) which drive interest rate fluctuations follow various diffusion processes. Their assumptions are inconsistent with the *a priori* belief and empirical evidence regarding the existence of discontinuous jumps in interest rates. At a cost of additional complexity, Ahn and Thompson (1988) extended the CIR model by adding a jump component to the square root interest rate process. Using a linearization technique, they obtained closed-form approximations for discount bond prices. Similarly, Baz and Das (1996) extended the Vasicek model by adding a jump component to the O-U interest rate process, and obtained closed-form approximate solutions for bond prices by the same linearization technique. They also showed that the approximate formula is quite accurate.

Although theoretical derivations for the jump-diffusion term structure models have been developed, the associated empirical work has not been done. A formal model of the term structure of interest rates is necessary for the valuation of bonds and various interest rate options. More importantly, parameter values or estimates are required for the implementation of a specific model. To price interest rate options, with closed-form solutions or by numerical methods, one must have values of the parameters in the stochastic processes that determine interest rate dynamics. Hence parameter estimation is a very first step in the application and analysis of interest rate option pricing models.

In this study, we investigated a jump-diffusion process, which is a mixture of an O-U process with mean-reverting characteristics used by Vasicek (1977) and a compound Poisson jump process, for interest rates. Closed-form approximate solutions for discount bond prices were derived by Baz and Das (1996). Essentially the approximate model is a one-factor term structure model. It has the disadvantage that all bond returns are perfectly correlated, and it may not be adequate to characterize the term structure of interest rates and its changing shape over time. However, the model at least can incorporate jump risks into the term structure model, making the model more complete relative to the pure diffusion Vasicek model. In addition, just as the simple diffusion Vasicek model, the short-term interest rate can move to negative values under the extended jump-diffusion Vasicek model. Realizing these limitations, this study did not attempt to propose a 'perfect' model for pricing term structure of interest rates, instead its purpose was to show whether a jump-diffusion model is superior to a simple diffusion model, or the simple diffusion model is superior to the jump-diffusion model.

We developed a methodology for estimating the extended Vasicek jump-diffusion term structure of interest rates model and completed an empirical study for the US money market interest rates. The state variable (the instantaneous short-term interest rate) that drives the term structure of interest rates dynamics was not observable, and the observed bond prices are functions of the state variable. Thus we needed to use the change of variable technique to obtain the likelihood function in terms of the observed bond prices, in order to conduct a maximum likelihood estimate. The estimation procedure of this study is similar to Chen and Scott (1993) and Pearson and Sun (1994).

In the empirical study, we used interest rates on the three-month, six-month, one-year, and three-year Treasury securities to estimate parameters in the Vasicek model and the jump-diffusion model. The estimation was conducted for both weekly and monthly data. The weekly data was from January 2, 1970 to June 20, 1997, and the monthly data was from January 1960 to May 1997. The results showed that, for weekly data the jump-diffusion model was significant. However, the jump-diffusion model was not supported for monthly data, due to the fact that less frequently observed data is not appropriate for detecting the jump component in an interest rate process. In comparison, both models predict an upward sloping term structure when the short-term interest rate is low, and a downward sloping term structure when the short-term interest rate is high. With the same level of short-term interest rate, the jump-diffusion model predicts a lower level of the term structure than the Vasicek model. This is because the term structure predicted by the jump-diffusion model is driven down by additional jump risks. The Vasicek model fits the term structure relatively better when the interest rate is high. When the interest rate is low, the Vasicek model tends to overestimate the term structure. In contrast, the jump-diffusion model fits the term structure reasonably well during periods when the interest rates are low, but underestimates the term structure when interest rates are at high levels. The jump-diffusion model fits the term structure poorly particularly when interest rates are high and the term structure is upward sloping.

Since the assumption of an appropriate stochastic process for the interest rate and the estimation of its associated parameters are of critical importance when pricing and hedging with term structure of interest rates and interest rate derivatives, the results and the methodology for estimating parameters in the jump-diffusion process have important implications for the area of financial engineering.

The rest of this paper is organized as follows: Section 2 specifies the Vasicek and the jump-diffusion term structure of interest rates models. Section 3 presents the empirical methodology used in this study. Section 4 specifies the data, and analyzes the results of parameters estimation and term structure fitting. Section 5 is the summary of the study. Some proofs are in the Appendices.

2. THE JUMP-DIFFUSION INTEREST RATE MODEL

One of the classical term structure of interest rate models is the Vasicek (1977) model. In the Vasicek model, the instantaneous short-term interest rate r is defined by the following diffusion process called the Ornstein-Uhlenbeck (O-U) process:

$$dr(t) = \alpha(\beta - r(t))dt + \sigma dW(t), \tag{1}$$

where α is the mean reversion coefficient, β is the long-term mean of the short-term interest rate, t denotes time path, and σ is the instantaneous volatility of the short-term interest rate. $dW(t)$ is the increment of a standard Wiener process. Let the random variable $r_t \equiv [r(t)|r(0) = r_0]$ denote the level of short-term interest rate at time t, conditional on the level of short-term interest rate at the initial time 0, $r(0) = r_0$. Under the O-U process, r_t follows a normal distribution with mean and variance as follows:

$$E(r_t) = e^{-\alpha t} r_0 + \beta(1 - e^{-\alpha t}); \tag{2}$$

$$\mathrm{Var}(r_t) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}). \tag{3}$$

The price of a zero-coupon bond at time t, maturing at time T, $P(r, t, T)$ can be determined by a function of the short-term interest rate r and time to maturity $T-t$. That is:

$$P(r, t, T) = \exp[-A(t, T)r + B(t, T)], \tag{4}$$

where:

$$A(t, T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha};$$

$$B(t, T) = \left(\beta - \frac{\xi\sigma}{\alpha} - \frac{\sigma^2}{2\alpha^2}\right)(A(t, T) - (T - t)) - \frac{\sigma^2 A^2(t, T)}{4\alpha^3}$$

and ξ is the market price of interest rate risk, which is assumed to be constant.

To incorporate the discontinuous jump behavior in interest rate dynamics, following Baz and Das (1996), the short-term interest rate r is defined by the extended Vasicek jump-diffusion process:

$$dr(t) = \alpha(\beta - r(t))dt + \sigma dW(t) + J dN(t), \tag{5}$$

where α is the mean reversion coefficient, β is the long-term mean of the short-term interest rate, t denotes time path, and σ is the instantaneous volatility of the short-term interest rate associated with the diffusion component. $dW(t)$ is the increment of a standard Wiener process, $N(t)$ represents a Poisson process with intensity rate λ. The probability that only one jump happens during the instantaneous period $[t, t + dt]$ is λdt. If there is one jump during the period $[t,\ t+dt]$ then $dN(t)=1$, and $dN(t)=0$ represents no jump during that period. J denotes the magnitude of a jump, which is assumed to be a normal variable with mean equal to θ and standard deviation equal to δ. Moreover, $dW(t)$ is assumed to be independent of $dN(t)$, which means that the diffusion component and the jump component of the short-term interest rate are independent of each other. Under the process specified in equation (5), as shown in Appendix A, r_t is defined as:

$$r_t = e^{-\alpha t}\left(r_0 + \int_0^t e^{\alpha u}\alpha\beta du + \int_0^t e^{\alpha u}\sigma dW(u) + \sum_{j=1}^{N(t)} e^{\alpha T_j} J_j\right), \tag{6}$$

where T_j is the time that the j-th jump happens and $0 < T_1 < t_2 < \ldots < T_{N(t)} < t$, $N(t)$ represents the number of jumps happening during the period between time 0 and time t. It can be shown in Appendix B that the probability density function for r_t is:

$$f(r_t)=\sum_{n=0}^{\infty}\left[\frac{e^{-\lambda t}(\lambda t)^n}{n!}\right].\int_0^t\int_0^t\ldots\int_0^t \overline{\omega}(r_t;\ M,\ S).\frac{1}{t^n}d\tau_1 d\tau_2\ldots d\tau_n, \tag{7}$$

where $\omega(r_t;\ M,\ S)$ denotes a normal density function with mean M and standard deviation S, and:

$$M = e^{-\alpha t}r_0 + \beta(1-e^{-\alpha t}) + \theta e^{-\alpha t}\sum_{j=1}^{n} e^{\alpha\tau_j};$$

$$S = \frac{\sigma^2}{2\alpha}(1-e^{-2\alpha t}) + \delta^2 e^{-2\alpha t}\sum_{j=1}^{n} e^{2\alpha\tau_j}.$$

Also as shown in Appendix C, the mean and variance of the conditional random variable r_t can be obtained as:

$$E(r_t) = e^{-\alpha t}r_0 + (\beta + \frac{\lambda\theta}{\alpha})(1-e^{-\alpha t}); \tag{8}$$

$$\mathrm{Var}(r_t) = \frac{\sigma^2 + \lambda(\theta^2+\delta^2)}{2\alpha}(1-e^{-2\alpha t}). \tag{9}$$

Assume that the market price of interest rate diffusion risk is constant and equal to ξ, and the jump risk is diversifiable.[1] Under the jump-diffusion interest rate process specified in equation (5), according to Baz and Das (1996), using a linearization technique, the price of a zero-coupon bond at time *t*, which matures at time *T*, $P(r, t, T)$ can be approximately determined by a function of the short-term interest rate *r* and time to maturity *T-t*. That is:

$$P(r, t, T) = \exp[-A(t, T)r + B(t, T)], \tag{10}$$

where:

$$A(t, T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha};$$

$$B(t, T) = \frac{-Ee^{-2\alpha(T-t)}}{4\alpha^3} + \frac{(\alpha D + E)e^{-\alpha(T-t)}}{\alpha^3} + \frac{(2\alpha D + E)(T-t)}{2\alpha^3} - C;$$

$$C = \frac{D}{\alpha^2} + \frac{3E}{4\alpha^3};$$

$$D = \xi\sigma - \alpha\beta - \theta\lambda;$$

$$E = \sigma^2 + (\delta^2 + \theta^2)\lambda.$$

To ensure that bond prices converge to zero for an arbitrarily large maturity, the additional condition $2\alpha D + E < 0$ is necessary. Under the model in equation (10), the short-term interest rate *r* has a linear relationship with the logarithm of discount bond prices, that is:

$$r = \frac{-\log P(r, t, T) + (B(t, T)}{A(t, T)} \tag{11}$$

And the yield to maturity of a zero-coupon bond expiring $T-t$ periods hence is given by:

$$R(r, t, T) = \frac{A(t, T)r - B(t, T)}{T - t} \tag{12}$$

The entire term structure of interest rates then can be defined.

Essentially the approximate model is a one-factor term structure model. It has the disadvantage that all bond returns are perfectly correlated, and it may not be adequate to characterize the term structure of interest rates and its changing shape over time. However, the model at least can incorporate jump risks into the term structure model, making the model more complete relative to the pure diffusion Vasicek model. In addition, just as the simple diffusion Vasicek model, the short-term interest rate can move to negative values under the extended jump-diffusion Vasicek model.

3. THE EMPIRICAL METHODOLOGY

In this section, we specify an empirical methodology for the jump-diffusion model. With appropriate specifications, this methodology is also applicable for the Vasicek model as well. Assume that there are $T+1$ observations for the state variable (the instantaneous short-term interest rate): $r(0)$, $r(1)$, $r(2)$, $\ldots$, $r(T)$. Since the jump-diffusion process specified in equation (5) is Markovian, the conditional likelihood function for the sample is:

$$L(r_1, r_2, \ldots, r_T; \Theta) = f[r(1)|r(0)] \cdot f[r(2)|r(1)] \ldots f[r(T)|r(T-1)], \tag{13}$$

where $\Theta = (\alpha, \beta, \sigma, \theta, \delta, \lambda)$, which denotes the parameter set to be estimated in the model. The log-likelihood function for the sample of observations is:

$$\log L(r; \Theta) \equiv \log L(r_1, r_2, \ldots, r_T; \Theta) = \sum_{i=1}^{T} \log f[r(i)|r(i-1)]. \tag{14}$$

Since the state variable in the model is the instantaneous short-term interest rate which is unobservable, to develop the maximum likelihood estimator for the parameters of the processes that derive interest rate changes, we develop a likelihood function for the observed bond price as functions of the unobservable state variables. The methodology is similar to that of Chen and Scott (1993), and Pearson and Sun (1994) in estimating and testing the CIR model. In the jump-diffusion model, according to equation (11), the logarithm of the price of a discount bond is a linear function of the state variable, and the change of variable technique can be used to obtain the joint density functions and the log-likelihood function for a sample of observations on discount bond price.

In our estimation, we use the interest rates on four US Treasury securities (three-month, six-month, one-year, and three-year) to estimate the Vasicek and the jump-diffusion model. Following

Chen and Scott (1993), we add measurement errors as additional random variables in the estimation, in order to perform a change of variables from the unobservable state variables to the observed bond rates. The thirteen-week (three-month) Treasury-bill rate is used and modeled without error because it is one of the most actively traded Treasury securities, and it is frequently used as an indicator of short-term interest rates. The system of equations for the model estimation is:

$$\begin{aligned}
\ln P(t,\ T_1) &= -A(t,\ T_1)r(t) + B(t,\ T_1);\\
\ln P(t,\ T_2) &= -A(t,\ T_2)r(t) + B(t,\ T_2) + e_{1t};\\
\ln P(t,\ T_3) &= -A(t,\ T_3)r(t) + B(t,\ T_3) + e_{2t};\\
\ln P(t,\ T_4) &= -A(t,\ T_4)r(t) + B(t,\ T_4) + e_{3t};
\end{aligned}$$

where $P(t, T_i)$ is the price of the discount bond with time to maturity equal to $T_i - t$, and $T_1 - t$ is equal to 3 months. e_{1t}, e_{2t}, and e_{3t} are measurement errors. In the estimation, we allow serial correlation and contemporaneous correlation between the measurement errors. The serial correlation is modeled as a first-order autoregressive process:

$$e_{jt} = \rho_j e_{j,\,t-1} + \varepsilon_{jt};\ j = 1,\ 2,\ 3, \tag{16}$$

where the innovation of the measurement error is assumed to be normally distributed, that is $\varepsilon_{jt} \sim N(0,\ \sigma^2(\varepsilon_j))$. Thus measurement errors are assumed to have a joint normal distribution. The log-likelihood function of the estimation then has the following form:

$$\begin{aligned}
\ell(\hat{r},\ \Theta) = \log L(\hat{r}_1,\ \hat{r}_2, \ldots, \hat{r}_T;\ \Theta) - T\ \ln|J| - \frac{3T}{2}\log(2\pi)\\
-\frac{T}{2}\log|\Omega| - \frac{1}{2}\sum \varepsilon_t' \Omega^{-1} \varepsilon_t,
\end{aligned} \tag{17}$$

where $\hat{r}$ is the substitute for the unobservable state variable (the instantaneous short-term interest rate). $\hat{r}$ is estimated, according to equation (11), by inverting the observed thirteen-week Treasury-bill rate, which is modeled without measurement errors. $\varepsilon_t' = (\varepsilon_{1t},\ \varepsilon_2 t,\ \varepsilon_{3t})$, and Ω is the covariance matrix for ε_t, which is assumed to be a diagonal matrix with elements $\sigma(\varepsilon_1)$, $\sigma(\varepsilon_2)$, and $\sigma(\varepsilon_3)$ along the diagonal. The elements of the matrix J are functions of $A(t, T_i)$, the coefficients in the linear transformation for r to $\log P(t, T_i)$, and the Jacobian of the transformation is $|J^{-1}|$. The likelihood function for the measurement error is conditional on an initial value for e_0.

In calculating the likelihood function for r_t, since the density function in equation (7) involves multiple integrals, it is very difficult to proceed. To make the estimation possible, we substitute the true density function by an approximate function.[2] That is:

$$f(r_t) = \sum_{n=0}^{\infty} \left[\frac{e^{-\lambda t}(\lambda t)^n}{n!} \right] .\overline{\omega}(r_t;\, M',\, S'), \tag{18}$$

where:

$$M' = e^{-\alpha t} r_0 + \beta(1 - e^{-\alpha t}) + \frac{n}{\alpha t}\theta(1 - e^{-\alpha t});$$

$$S' = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) + \frac{n}{2\alpha t}\delta^2(1 - e^{-2\alpha t}).$$

With this approximate density function, when $\alpha t \to 0$, it converges to the true density function in equation (7). Thus either when α or t is small, the approximate density function will provide an appropriate substitute for the true density function.

4. THE DATA AND EMPIRICAL ANALYSIS

In estimating the term structure of interest rates, we used interest rates on four Treasury securities: the three-month and six-month Treasury-bills and the one-year and three-year Treasury bonds to estimate the term structure of interest rates. Both weekly and monthly data were used. The weekly data was from January 20, 1970 to June 20, 1997, which contained 1,434 observations in total. The monthly data was from January 1960 to May 1997, and contained 449 observations in the series. The data source was from the Aremos USFIN databank. The summary statistics of the sampled interest rates for parameters estimation is in Table 1. During the period from 1970 to 1997, for example, the average weekly three-month interest rate was about 6.83%, with the maximum and minimum interest rate equal to 16.76% and 2.69% respectively. On average, the longer the time to maturity,

Table 1

Summary Statistics of Sampled Interest Rates for Parameter Estimation

Time to Maturity	*Three-month*	*Six-month*	*One-year*	*Three-year*
Weekly Data: January 2, 1970 to June 20, 1997 Number of Observations : 1,434				
	%	%	%	%
Average	6.83	6.98	7.57	8.05
Standard Deviation	2.71	2.63	2.80	2.50
Maximum	16.76	15.76	17.15	16.47
Minimum	2.69	2.82	3.02	4.09
Monthly Data : January 1960 to May 1997 Number of observations : 449				
	%	%	%	%
Average	6.07	6.23	6.72	7.13
Standard Deviation	2.72	2.66	2.86	2.70
Maximum	16.30	15.52	16.72	16.22
Minimum	2.24	2.44	2.81	3.23

the higher the interest rate, revealing that an upward sloping term structure is a normal case.

Figure 1 plots the weekly three-month and three-year interest rates from January 2, 1970 to June 20, 1997. Normally, the three-year interest rate is higher than the three-month interest rate, which reflects an upward sloping term structure. Occasionally, particularly during periods when the interest rate is high, the three-year interest rate is lower than the three-month interest rate, reflecting a negatively sloping term structure. For example, during the periods from 1973 to 1974, from 1979 to 1981, and in 1989, a negatively sloping term structure of interest rates is observed quite often.

Table 2 shows the results of parameters estimated in the Vasicek model. All parameters in the model are statistically significant either for weekly or for monthly data. The parameter α is estimated as 0.3772 and 0.2981 for weekly and monthly data respectively, which are all strongly significant,[3] implying that a strong mean reversion is found in the interest rate dynamics. The estimate of 0.3772 for α implies a mean half life of 1.84 years, and the estimate of 0.2981 implies a mean half life of 2.40 years[4] in

Figure 1

Weekly 3-Month and 3-Year Interest Rates from January 2, 1970 to June 20, 1997

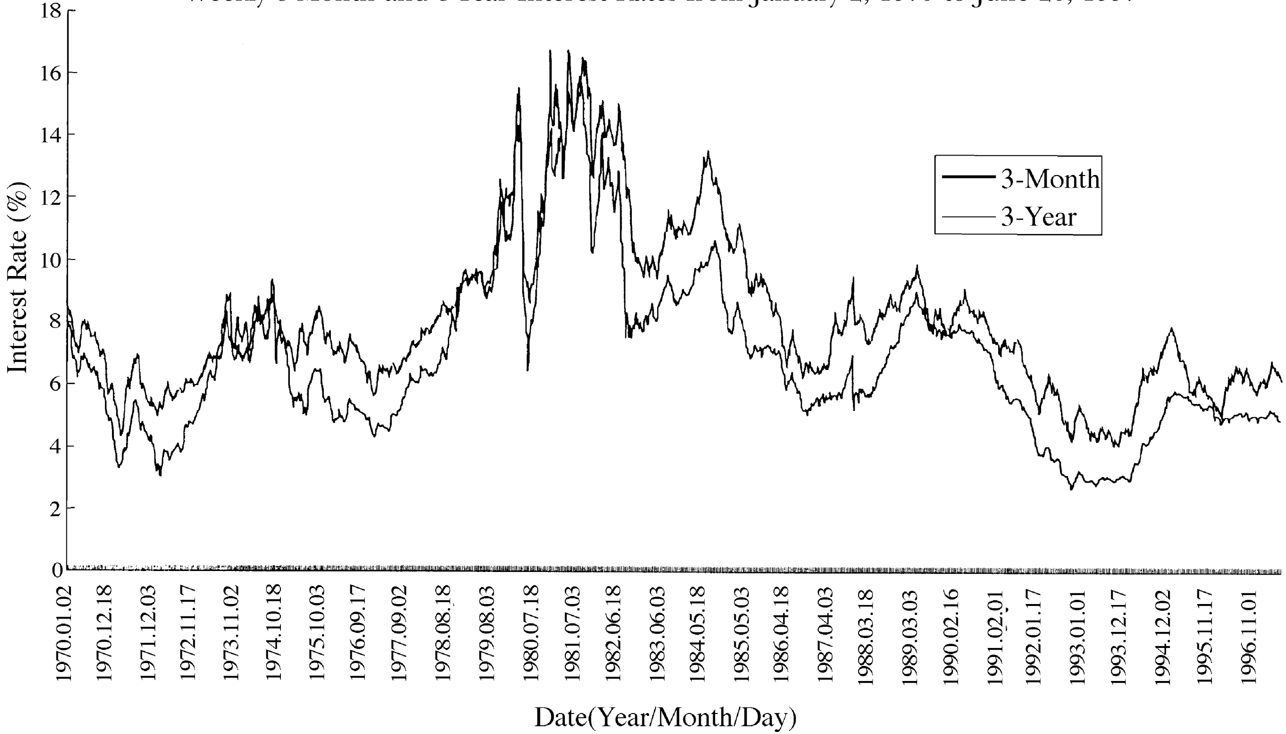


Table 2

Parameters Estimated for the Vasicek Model

	Weekly Data			*Monthly Data*		
Parameter	*Coefficient*	*Std. Error*	*T-test*	*Coefficient*	*Std. Error*	*T-test*
α	0.3772	0.0136	27.79*	0.2981	0.0071	42.01*
β	0.0624	0.0043	14.57*	0.0664	0.0023	28.42*
σ	0.0208	0.0002	84.95*	0.0204	0.0000	241.03*
ξ	−0.8760	0.1299	−6.74*	−0.4684	0.0777	−6.03*
$\sigma(\varepsilon_1)$	0.0004	0.0000	62.24*	0.0007	0.0032	0.20
$\sigma(\varepsilon_2)$	0.0010	0.0000	87.93*	0.0014	0.0000	109.52*
$\sigma(\varepsilon_3)$	0.5026	0.8512	0.59	0.0120	0.0013	9.20*
ρ_1	0.9155	0.0100	91.63*	0.8882	1.0753	0.83
ρ_2	0.9767	0.0038	258.23*	0.9521	0.0035	271.74*
ρ_3	1.4493	0.8302	1.75	0.9554	0.0240	39.87*
Log-Likelihood		52,655.494			14,639.257	

Note:
* Significant at 1%.

the interest rate process. β and σ are estimated as the values that are comparable with the posterior average and standard deviation of the three-month interest rate in Table 1 either for weekly or monthly data. The market price of interest rate risk ξ is estimated as a negative value, which is expected. $\sigma(\varepsilon_1)$, $\sigma(\varepsilon_2)$, and $\sigma(\varepsilon_3)$ are the standard deviation of the innovations in the measurement errors for logarithms of discount bond prices for the six-month, one-year, and the three-year Treasury securities respectively. In the case of weekly data, the standard errors of the model for the annualized yield on six-month, one-year, and three-year Treasury securities were also calculated and are shown in Table 4. The standard errors are 0.00241, 0.00661 and 0.01115 respectively, which are correspondingly equal to 24, 66 and 111 basis points. These measurement errors are economically significant implying that the one-factor model does not fit the term structure very well. This result is similar to that of Chen and Scott (1993).

Table 3 shows the results of parameters estimated in the jump-diffusion model. The parameters of the diffusion part are statistically significant for both weekly and monthly data, whereas the parameters of the jump component (λ, θ, δ) are significant only for the weekly data. The reason why the jump parameters

Table 3

Parameters Estimated for the Jump-Diffusion Model

	Weekly Data			*Monthly Data*		
Parameter	*Coefficient*	*Std. Error*	*T-test*	*Coefficient*	*Std. Error*	*T-test*
α	0.5637	0.0037	156.85**	0.9081	0.0022	416.00*
β	0.0506	0.0006	82.98*	0.0670	0.0004	178.01*
σ	0.0213	0.0001	370.69*	0.0223	0.0000	413.67*
λ	0.3392	0.0069	49.19*	0.0133	0.0492	0.27
θ	−0.0195	0.0014	−14.24*	−0.0282	0.0291	−0.97
δ	0.0183	0.0002	91.35*	0.1609	0.2082	0.77
ξ	−0.1692	0.0346	−4.89*	−0.1679	0.0287	−5.86*
$\sigma(\varepsilon_1)$	0.0000	0.0000	0.58	0.0000	0.0000	0.84
$\sigma(\varepsilon_2)$	0.0000	0.0000	2.71*	0.0000	0.0000	20.95*
$\sigma(\varepsilon_3)$	0.0000	0.0000	32.33*	0.0000	0.0000	169.59*
ρ_1	0.9947	0.0118	84.27*	1.0023	0.0020	487.90*
ρ_2	0.9986	0.0026	376.71*	1.0044	0.0008	1225.69*
ρ_3	1.0024	0.0003	3545.35*	1.0001	0.0001	8287.35*
Log-Likelihood		48,556.020			12,755.979	

Note:
* Significant at 1%.

Table 4

Standard Errors of Term Structure Estimation Based on Weekly Data

Model	*Three-month*	*Six-month*	*One-year*	*Three-year*	*Log-likelihood*
Vasicek	0	0.00241	0.00661	0.01115	52,655.49
Jump-Diffusion	0	0.00266	0.00905	0.01463	48,556.02

are not significant for monthly data is twofold. Theoretically, the jump-diffusion process has finite first and second moments, and the central limit theorem applies. With the increase in the observation time horizon, the conditional distribution of r_t will approach a normal distribution. Thus the jump in the interest rate would be smoothed out in the observed series, resulting in non-significant jump parameters in the model for monthly data. On the other hand, the conditional distribution of r_t used in the empirical study to conduct the maximum likelihood estimate is an approximate one in equation (18), which will converge to the true distribution only when $\alpha t \to 0$. Thus when t is not small, the estimation is bound to be biased.

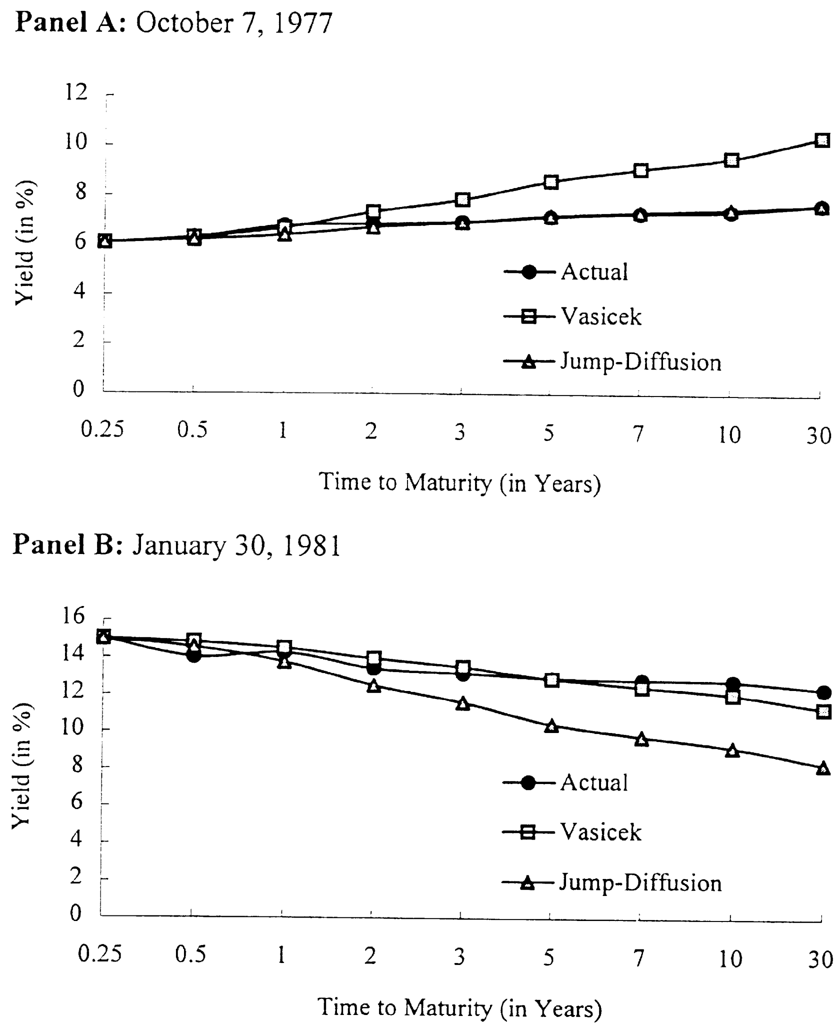
Since the jump-diffusion model is not significant for monthly data, we now concentrate on the results of the weekly data. In the case of weekly data in Table 3, the jump intensity parameter λ is estimated as 0.3392, which implies that on average, a jump happens every three weeks or so.[5] The average jump magnitude θ is estimated as -0.0195, and its standard deviation δ is 0.0183. The negative average on the jump magnitude is understandable, since the market intervention by the government tends to happen when the interest rate is high.

With the estimated jump-diffusion model for weekly data, the standard errors for the annualized yield on six-month, one-year, and three-year Treasury securities were also calculated as 0.00266, 0.00905, and 0.01463 respectively as shown in Table 4. These standard errors are correspondingly equal to 27, 91 and 146 basis points. These measurement errors are even more significant than that from the Vasicek model. Again this is due to the fact that one-factor models, in which all interest rates with different time to maturity fail to specify the flexible term structure well.

Comparing the likelihood function values and standard errors of the Vasicek model and the jump-diffusion model in the case of weekly data as shown in Table 4, we found that overall the jump-diffusion model fit the sampled interest rates even worse than the Vasicek model. For further investigation, we computed estimating errors in predicting term structure of interest rates based on nine different times to maturity on the term structure for the two models. The results are shown in Table 5. In Table 5, we can find that for the sampling period as a whole, the Vasicek model overestimates the term structure by an average of 49.12 basis points, while the jump-diffusion model underestimates the term structure by an average of 58.40 basis points. The standard errors (root mean squared errors) for the two models are 118.99 and 133.35 basis points respectively. Both of the two models fit the term structure poorly, while there are systematic differences. In order to investigate the differences in the way the term structure is fitted by the two models across time, Figure 2 plots the actual and two modeled term structure based on the nine interest rates on the term structure, for nine sampled dates during the sampling period. The nine samples are typical cases for each sub-period during the period from January 2, 1970 to June 20, 1997. From Panels A to I in Figure 2, we discover that

Figure 2

Actual vs. Model Term Structure for Selected Dates

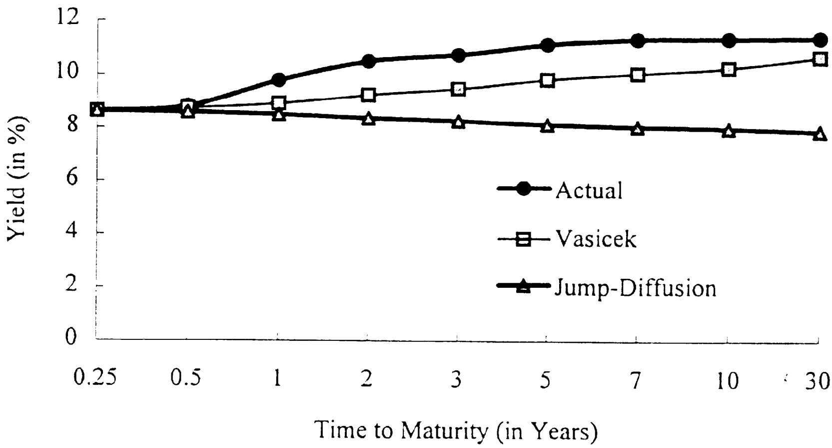


there are systematic differences in term structure fitting done by using the Vasicek model and the jump-diffusion model.

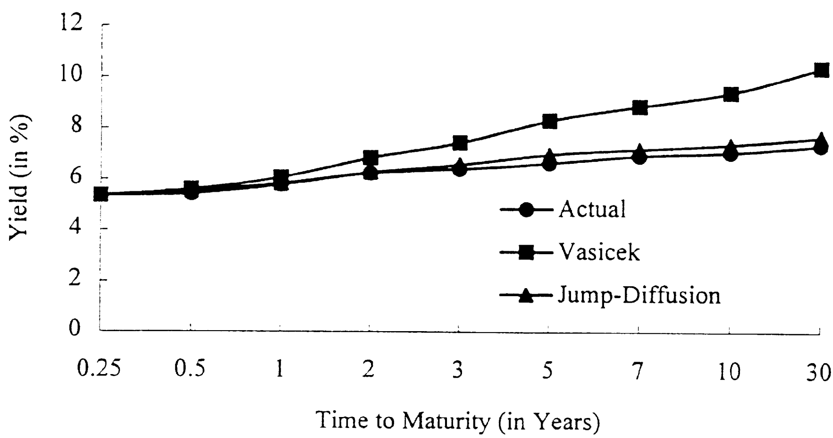
In the first sub-period from January 2, 1970 to December 30, 1977 when interest rates are relatively low, the actual term structure exhibits upward, flat, or sometimes slightly downward sloping in shapes. During this period the jump-diffusion model

Figure 2 (Continued)

Panel C: October 7, 1983



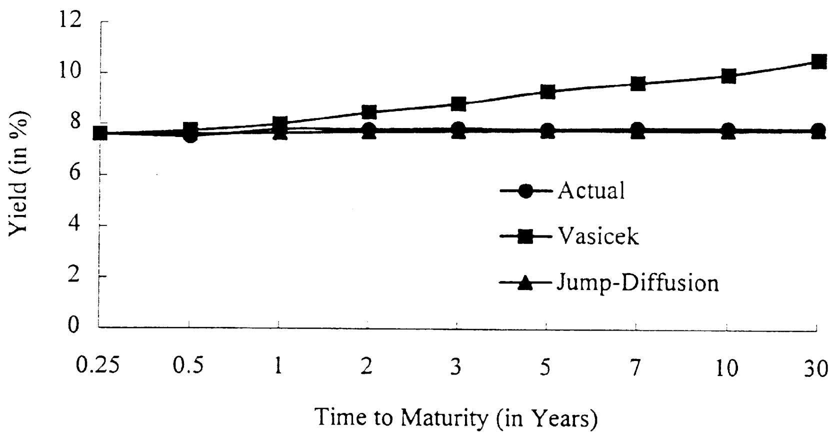
Panel D: January 6, 1987



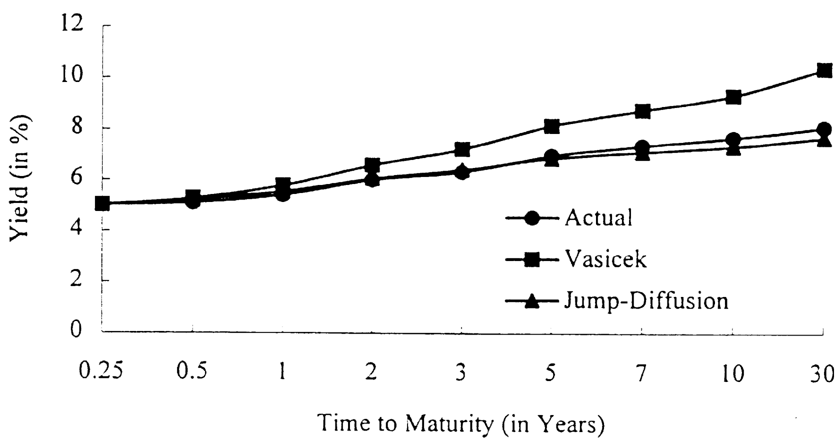
fits the term structure reasonably well, while the Vasicek model seriously overestimates the term structure. Panel A in Figure 2 is a typical case for this period. It is shown in Table 5 that on average the Vasicek model overestimates the term structure by 105.60 basis points in average errors, and the standard error is 116.83 basis points. The jump-diffusion model, on the other hand,

Figure 2 (Continued)

Panel E: October 27, 1989



Panel F: October 25, 1991

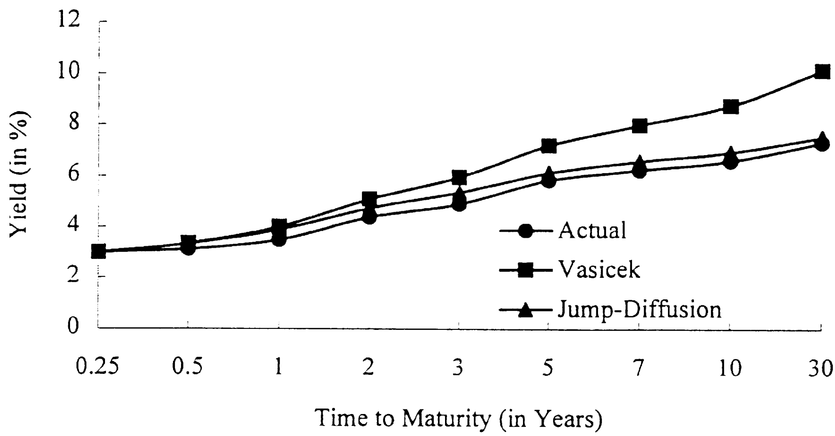


slightly underestimates the term structure by an average of 1.40 basis points. The standard error is 38.50 basis points. Both are much smaller than that of the Vasicek model.

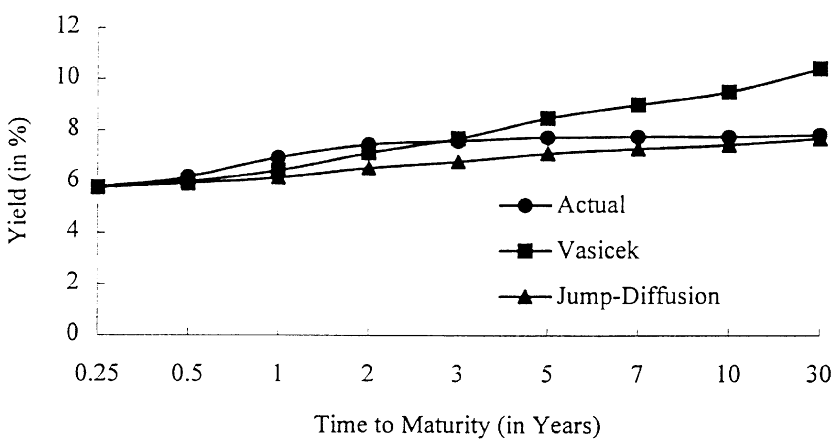
In the second sub-period from January 6, 1978 to December 27, 1985, when interest rates were high and volatile, besides the upward, and flat shapes, the term structure exhibits a downward

Figure 2 (Continued)

Panel G: January 22, 1993



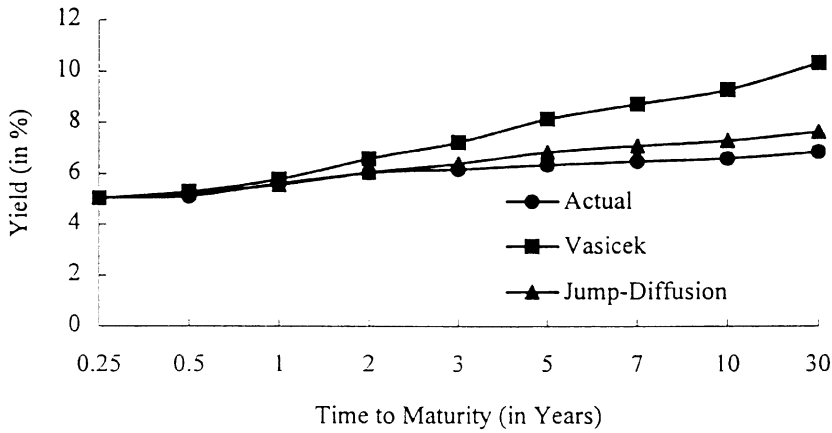
Panel H: January 27, 1995



shape quite often. Panels B and C in Figure 2 are typical examples for that period. During the period, both of the two models underestimate the term structure quite significantly. Particularly, the jump-diffusion model tremendously underestimates the term structure when the interest rate is high and the term structure is upward sloping. Table 5 shows that the

Figure 2 (Continued)

Panel I: January 31, 1997



Vasicek model underestimates the term structure by 59.90 basis points in average errors. The standard error is 117.37 basis points. The jump-diffusion model, on the other hand, even more seriously underestimates the term structure by an average of 195.40 basis points, and with a standard error equal to 221.90 basis points. Supposedly, the underestimation during this period made by both of the models is mostly affected by the estimation of the long-term mean β, which may be actually time dependent. The estimation of β is dominated by the data in the sub-periods 1 and 3, which is probably too low for sub-period 2. This might be the reason why the term structure is so seriously underestimated by the models.

In the third sub-period from January 3, 1986 to June 20, 1997, interest rates were relatively low, and the actual term structure exhibits upward sloping more often than other shapes. In this period, from Table 5 and Panels D to I in Figure 2, it is shown that the jump-diffusion model fits the term structure much better than the Vasicek model. On average the Vasicek model overestimates the term structure by 96.58 basis points, and the standard error is 113.53 basis points. The jump-diffusion model, on the other hand, only slightly overestimates the term structure by an average of 6.07 basis points, and the standard error is 59.50 basis points.

Table 5

Estimating Errors for Term Structure of Interest Rates

		Vasicek Model		*Jump-Diffusion Model*	
Sub-Period	*Number of Observations*	*Average Errors*	*Root Mean Squared Errors*	*Average Errors*	*Root Mean Squared Errors*
Jan. 2, 1970– Jun. 20, 1997	1,434	49.12	118.99	−58.40	133.35
Jan. 2, 1970– Dec. 30, 1977	418	105.60	116.83	−1.40	38.50
Jan. 6, 1978– Dec. 27, 1985	417	−59.90	117.37	−195.40	221.90
Jan. 3, 1986– Jun. 20, 1997	599	96.58	113.50	6.07	59.50

Notes:
[1] Numbers in the table are in basis points.
[2] Errors are calculated as the average of the actual and model interest rates of the 3-month, 6-month, 1-year, 2-year, 3-year, 5-year, 7-year, 10-year, and the 30-year Treasury securities.

In summary, over-all the Vasicek model tends to overestimate the term structure, whereas the jump-diffusion model tends to underestimate the term structure. Both models predict a positively sloping term structure when interest rates are low, while they predict a negatively sloping term structure when interest rates are high. With the same level of short-term interest rate, the jump-diffusion model predicts a lower level of term structure than the Vasicek model. This is because the term structure predicted by the jump-diffusion model is driven down by additional jump risks. The Vasicek model fits the term structure relatively better when interest rates are high. When interest rates are low, the Vasicek model tends to overestimate the term structure. In contrast the jump-diffusion model fits the term structure reasonably well during periods when interest rates are low, but underestimates the term structure when interest rates are at high levels. The jump-diffusion model fits the term structure poorly particularly when interest rates are high and the term structure is upward sloping.

As expected, one-factor models, in which all interest rates with different maturities are perfectly correlated, cannot fit the real-world versatile term structure of interest rates very well. Either the Vasicek or the jump-diffusion model can fit the term structure reasonably well for the periods when interest rates are at certain levels, and the term structure exhibits certain shapes. However, the jump-diffusion model, with jump risks incorporated in it, has more implications than the Vasicek model. If these one-factor models have to be used, one needs to modify the parameters more frequently, in order to include more new information about the term structure changes in the models. One can also allow some parameters in the model to be time-dependent in order to catch more information on the term structure dynamics.

5. SUMMARY

Financial variables such as stock prices, foreign exchange rates, and interest rates are always assumed to follow a diffusion process with continuous time paths when pricing financial assets. Despite their attractive statistical properties and computation convenience, more and more empirical evidence shows that pure diffusion models are not appropriate for these financial variables. For interest rates, jump-diffusion processes are particularly meaningful since the interest rate is an important economic variable which is, to some extent, controlled by the government as an instrument for its financial policy.

Although theoretical derivations for the jump-diffusion term structure models have been developed, the associated empirical work has not been found. In this study, we investigated a jump-diffusion process, which is a mixture of an O-U process with mean-reverting characteristics used by Vasicek (1977) and a compound Poisson jump process, for interest rates. Closed-form approximate solutions for discount bond prices were derived by Baz and Das (1996). Essentially the approximate model is a one-factor term structure model. It has the disadvantage that all bond returns are perfectly correlated, and it may not be adequate to characterize the term structure of interest rates and its changing shape over time. However, the model can at least incorporate jump risks into the term structure model, making the model more complete relative to the pure diffusion Vasicek model. In addition, just as the simple diffusion Vasicek model, the short-term interest rate can move to negative values under the extended jump-diffusion Vasicek model. Realizing these limitations, this study did not attempt to propose a 'perfect' model for pricing term structure of interest rates, instead its purpose was to show whether a jump-diffusion model is superior to a simple diffusion model, or the simple diffusion model is superior to the jump-diffusion model.

We developed a methodology for estimating the extended Vasicek jump-diffusion term structure of interest rates model and completed an empirical study for US money market interest rates. The state variable (the instantaneous short-term interest rate) that drives the term structure of interest rates dynamics was not observable, and the observed bond prices are functions of the state variable. Thus we needed to use the change of variable technique to obtain the likelihood function in terms of the observed bond prices, in order to conduct a maximum likelihood estimate. The estimation procedure of this study is similar to Chen and Scott (1993) and Pearson and Sun (1994).

In the empirical study, we used yields of three-month, six-month, one-year, and three-year Treasury securities to estimate parameters in the Vasicek model and the jump-diffusion model. The estimation was conducted for both weekly and monthly data. The weekly data was from January 2, 1970 to June 20, 1997, and the monthly data was from January 1960 to May 1997. The results showed that, for weekly data the jump-diffusion model was significant. The jump-diffusion model was not supported for monthly data, due to the fact that less frequently observed data is not appropriate for detecting the jump component in an interest rate process. In comparison, both models predict a positively sloped term structure when interest rates are low, and a negatively sloped term structure when interest rates are high. With the same level of short-term interest rate, the jump-diffusion model predicts a lower level of term structure than the Vasicek model. This is because the term structure predicted by the jump-diffusion model is driven down by additional jump risks. The Vasicek model fits the term structure relatively well when interest rates are high. When interest rates are low, the Vasicek model tends to overestimate the term structure. On the other hand, the jump-diffusion model fits the term structure reasonably well during periods when interest rates are low, whereas it underestimates the term structure when interest rates are at high levels. The jump-diffusion model fits the term structure poorly, particularly when interest rates are high and the term structure is positively sloped.

As expected, one-factor models do not fit the versatile term structure of interest rates very well. Either the Vasicek or the jump-diffusion model can only fit the term structure reasonably well for the periods when interest rates are at certain levels, and the term structure exhibits certain shapes. If these one-factor models are used, one needs to modify the parameters more frequently, in order to include more new information about the term structure changes in the models. One can also allow some parameters to be time-dependent in the model, in order to catch more information on the term structure dynamics. Since the assumption of an appropriate stochastic process for the interest rate and the estimation of its associated parameters are of critical importance when pricing and hedging with term structure of interest rates and interest rate derivatives, the results and methodology for estimating parameters in the jump-diffusion process have important implications for the area of financial engineering.

APPENDIX A

Following Arnold (1974), we first assume:

$$r_t = \phi(t)(r_0 + \int_0^t \phi(u)^{-1}\alpha\beta du + \int_0^t \phi(u)^{-1}\sigma dW + \sum_{j=1}^{N(t)} \phi(T_j)^{-1}J_j), \tag{A1}$$

where $\phi(t) = \exp(-\int_0^t \alpha du) = e^{-\alpha t}$.

Next we let:

$$R_t = r_0 + \int_0^t \phi(u)^{-1}\alpha\beta du + \int_0^t \phi(u)^{-1}\sigma dW + \sum_{j=1}^{N(t)} \phi(T_j)^{-1}J_j. \tag{A2}$$

The differential form for equation (A2) is:

$$dR_t = \phi(u)^{-1}(\alpha\beta dt + \sigma dW + JdN).$$

According to equation (A1) $r_t = \phi(t)\, R_t$, thus:

$$\begin{aligned} dr_t &= \phi'(t)R_t dt + \phi(t)dR_t, \\ &= -\alpha\phi(t)R_t dt + \alpha\beta dt + \sigma dW + JdN, \\ &= -ar_t dt + abdt + \sigma dW + JdN, \\ &= \alpha(\beta_r - r_t)dt + \sigma dW + JdN. \end{aligned} \tag{A3}$$

APPENDIX B

Let $X(t)$ be $e^{-\alpha t}(r_0 + \int_0^t e^{\alpha u}\alpha\beta du + \int_0^t e^{\alpha u}\sigma dW(u))$, and $Y(t)$ be $e^{-\alpha t}\sum_{j=1}^{N(t)} e^{\alpha T_j}J_j$, then $r_t = X(t) + Y(t)$. $X(t)$ is independent with $Y(t)$, the distribution of $X(t)$ is a Normal distribution with a mean $E[X(t)] = e^{-\alpha t}r_0 + \beta(1 - e^{-\alpha t})$ and a variance:

$$\mathrm{Var}[X(t)] = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}).$$

The probability density function for r_t can be derived as:

$$\Pr(r_t \le r) = \Pr(X(t) + Y(t) \le r)$$

$$= \sum_{n=0}^{\infty} \Pr[X(t) + Y(t) \le r | N(t) = n] \cdot \Pr[N(t) = n]$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{-\lambda t}(\lambda t)^n}{n!}\right] \cdot \Pr[X(t) + Y(t) \le r | N(t) = n]. \tag{A4}$$

Moreover:

$$\Pr[X(t) + Y(t) \le r | N(t) = n]$$

$$= \Pr[X(t) + e^{-\alpha t}\sum_{j=1}^{n} e^{\alpha T_j}J_j \le r]$$

$$= \int_0^t \int_0^t \ldots \int_0^t \Pr[X(t) + e^{-\alpha T_j}\sum_{j=1}^{n} e^{\alpha T_j}J_j \le r | T_1 = \tau_2, T_2 = \tau_2, \ldots, T_n = \tau_n]$$

$$.f_{T_1}(\tau_1).f_{T_2}(\tau_2)\ldots f_{T_n}(\tau_n)d\tau_1 d\tau_2 \ldots d\tau_n$$

$$= \int_0^t \int_0^t \ldots \int_0^t \Pr[X(t) + e^{-\alpha t}\sum_{j=1}^{n} e^{\alpha\tau_j}J_j \le r].\frac{1}{t^n}d\tau_1 d\tau_2 \ldots d\tau_n. \tag{A5}$$

J_j is assumed to be Normally distributed with mean θ and standard deviation δ. It is also assumed to be independent with $X(t)$. Moreover, as specified above, $X(t)$ is also Normally distributed. Thus under the condition that τ_j is known, the variable:

$$X(t) + e^{-\alpha t}\sum_{j=1}^{n} e^{\alpha\tau_j}J_j$$

is also Normally distributed. Thus equation (A5) can be written as:

$$\Pr[X(t) + Y(t) \le r | N(t) = n] = \int_0^t\int_0^t \ldots \int_0^t \Omega(M,\ S).\frac{1}{t^n}\,d\tau_1\,d\tau_2\ldots d\tau_n, \tag{A6}$$

where $\Omega(M,\ S)$ denotes the cumulative density function for a Normal distribution with mean M and variance S and:

$$M = e^{-\alpha t}r_0 + \beta(1 - e^{-\alpha t}) + \theta e^{-\alpha t}\sum_{j=1}^{n} e^{\alpha\tau_j}, \tag{A7}$$

$$S = \frac{\sigma_2}{2\alpha}(1 - e^{-2\alpha t}) + \delta^2 e^{-2\alpha t}\sum_{j=1}^{n} e^{2\alpha\tau_j}. \tag{A8}$$

It follows that equation (A4) can be expressed as:

$$\Pr[r_t \le r] = \sum_{n=0}^{\infty}\left[\frac{e^{-\lambda t}(\lambda t)^n}{n!}\right].\int_0^t\int_0^t \ldots \int_0^t \Omega(r_t;\ M,\ S).\frac{1}{t^n}\,d\tau_1\,d\tau_2\ldots d\tau_n. \tag{A9}$$

The probability density function for r_t is:

$$f(r_t) = \sum_{n=0}^{\infty}\left[\frac{e^{-\lambda t}(\lambda t)^n}{n!}\right].\int_o^t\int_0^t \ldots \int_0^t \overline{\omega}(r_t; M, S).\frac{1}{t^n}\,d\tau_1\,d\tau_2\ldots d\tau_n, \tag{A10}$$

where $\omega(r_t; M, S)$ represents the probability density function for a Normal distribution.

APPENDIX C

According to Ross (1993), the conditional mean and variance for r_t can be derived by:

$$E(r_r) = E\{E[r_t|N(t)=n]\} = E\{E(E[r_t|N(t) = n|T_j = \tau_j])\}. \quad \text{(A11)}$$

$$\begin{aligned} \mathrm{Var}(r_t) &= E\{\mathrm{Var}[r_t|N(t) = n]\} + \mathrm{Var}\{E[r_t|N(t) = n]\}, \\ &= E\{E(\mathrm{Var}[r_t|N(t) = n|T_j = \tau_j])\} \\ &+ E\{\mathrm{Var}(E[r_t|N(t) = n|T_j = \tau_j])\}. \qquad \text{(A12)} \\ &+ \mathrm{Var}\{E(E[r_t|N(t) = n|T_j = \tau_j])\}. \end{aligned}$$

Based on equations (A11) and (A12) and the density function for r_t in equation (A10), the result is:

$$E(r_t) = e^{-\alpha t} r_0 + (\beta + \frac{\lambda\theta}{\alpha})(1 - e^{-\alpha t}), \qquad \text{(A13)}$$

$$\mathrm{Var}(r_t) = \frac{\sigma_2 + \lambda(\theta^2 + \delta^2)}{2\alpha}(1 - e^{-2\alpha t}). \qquad \text{(A14)}$$

NOTES

1 In the case of systematic jump risk, the bond price is given by equation (10), with λ replaced by $\lambda*$, the risk-neutral jump intensity rate, and

$$\lambda^* = \lambda\left(1 - \frac{\phi_j \delta^2}{(\theta^2 + \delta^2)B^2/2 - \theta B)}\right),$$

where θ_j is the market price of interest rate jump risk, which is assumed to be proportional to the variance of the jump increment. In empirical estimation, one cannot identify whether λ or λ^* is estimated. Fortunately this is not relevant in pricing term structure derivatives.

2 To obtain the approximate conditional density function for the short-term interest rate r_t, consider the last term of M in equation (7), $\sum_{j=1}^{n} e^{\alpha T_j}$, where T_j denotes the time when the j-th jump happens, which is an independent uniform distribution conditional on the number of jumps n in the interval $(0, t)$ is known. Assuming jumps in interest rate spread equally over the time interval $(0, t)$, then the value of $\sum_{j=1}^{n} e^{\alpha T_j}$ is given by:

$$E\left[\sum_{j=1}^{n} e^{\alpha T_j}\right] = \sum_{j=1}^{n} E(e^{\alpha T_j}) = n \cdot E(e^{\alpha T}) = n \cdot \int_0^t e^{\alpha T}\frac{1}{t}\,dT = \frac{n}{\alpha t}(e^{\alpha t} - 1).$$

Similarly, for the last term of S in equation (7):

$$E\left[\sum_{j=1}^{n} e^{2\alpha T_j}\right] = \frac{n}{2\alpha t}(e^{2\alpha t} - 1).$$

Substituting these into equation (7), results in equation (18), the approximate density function.

3 A formal statistical test for the hypothesis $\alpha = 0$ is not simple because the parameter value lies on the boundary of the parameter space. The problem of testing the hypothesis $\alpha = 0$ with a t-statistic or a likelihood ratio test is

similar to the problem of testing for unit roots in time series, and the test statistics do not have the familiar large-sample asymptotic distributions.

4 The mean half life is the expected time for the process to return halfway to its long-term mean β. The mean half life is calculated as $-\ln(0.5)/\alpha$.

5 The jump time follows an exponential distribution with a mean of $1/\lambda$.

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