Journal of International Financial Markets, Institutions and Money 11 (2001) 167–197

Journal of INTERNATIONAL FINANCIAL MARKETS, INSTITUTIONS & MONEY

www.elsevier.com/locate/econbase

Estimation for factor models of term structure of interest rates with jumps: the case of the Taiwanese government bond market

Bing-Huei Lin [a,*], Shih-Kuo Yeh [b]

[a] *Department of Business Administration, National Taiwan University of Science and Technology, 43 Keelung Road, Section 4, Taipei 106, Taiwan, ROC*

[b] *National Kaohsiung First University of Science and Technology, Taipei, Taiwan, ROC*

Received 30 March 1999; accepted 11 March 2000

Abstract

This paper examines the Ornstein–Uhlenbeck (O–U) process used by Vasicek, J. Financial Econ. 5 (1977) 177, and a jump-diffusion process used by Baz and Das, J. Fixed Income (Jnue, 1996) 78, for the Taiwanese Government Bond (TGB) term structure of interest rates. We first obtain the TGB term structures by applying the B-spline approximation, and then use the estimated interest rates to estimate parameters for the one-factor and two-factor Vasicek and jump-diffusion models. The results show that both the one-factor and two-factor Vasicek and jump-diffusion models are statistically significant, with the two-factor models fitting better. For two-factor models, compared with the second factor, the first factor exhibits characteristics of stronger mean reversion, higher volatility, and more frequent and significant jumps in the case of the jump-diffusion process. This is because the first factor is more associated with short-term interest rates, and the second factor is associated with both short-term and long-term interest rates. The jump-diffusion model, which can incorporate jump risks, provides more insight in explaining the term structure as well as the pricing of interest rate derivatives. © 2001 Elsevier Science B.V. All rights reserved.

JEL classification: E43; G15

Keywords: Jump-diffusion process; Term structure of interest rates; B-spline approximation

* Corresponding author. Tel.: +886-2-27376748; fax: +886-2-27376744.
E-mail address: lin@ba.ntust.edu.tw (B.-H. Lin).

1042-4431/01/$ - see front matter
PII: S1042-4431(00)00049-4

1. Introduction

Financial variables such as stock prices, foreign exchange rates, and interest rates are conventionally assumed to follow a diffusion process with continuous time paths when pricing financial assets. Despite their attractive statistical properties and computation convenience, more and more empirical evidence has shown that pure diffusion models are not appropriate for these financial variables. For example, Ball and Torous (1983), Jarrow and Rosenfeld (1984), Ball and Torous (1985a,b), Akgiray and Booth (1986), Lin and Yeh (1997), Jorion (1988) all found evidence indicating the presence of jumps in the stock price process. Akgiray and Booth (1988), Tucker and Pond (1988), Park et al. (1993) studied foreign exchange markets and concluded that the jump-diffusion process is more appropriate for foreign exchange rates. In pricing and hedging with financial derivatives, jump-diffusion models are particularly important, since ignoring jumps in financial prices will cause pricing and hedging risks.

The jump-diffusion process is particularly meaningful for interest rates, since the interest rate is an important economic variable, which is, to some extent, controlled by the government as an instrument for its financial policy. Hamilton (1988) investigated US interest rates and found changes in regime for the interest rate process. Das (1994) found movements in interest rates display both continuous and discontinuous jump behavior. Presumably, jumps in interest rates are caused by several market phenomena, such as money market interventions by the Fed, news surprises, shocks in the foreign exchange markets, and so on.

Classical term structure of interest rate models, such as the Vasicek (1977) model, the Cox et al., 1985 model, the Brennan and Schwartz (1978) model, and other extended models all assume that processes of state variables (such as the short-term interest rate, the long-term interest rate, or others), which drive interest rate fluctuations, follow various diffusion processes. Their assumptions are inconsistent with the a priori belief and empirical evidence regarding the existence of discontinuous jumps in interest rates. At a cost of additional complexity, Ahn and Thompson (1988) extended the CIR model by adding a jump component to the square root interest rate process. Using a linearization technique, they obtained closed-form approximations for discount bond prices. Similarly, Baz and Das (1996) extended the Vasicek model by adding a jump component to the Ornstein–Uhlenbeck (O–U) interest rate process, and obtained closed-form approximate solutions for bond prices by the same linearization technique. They also showed that the approximate formula is quite accurate.

Although theoretical derivations for the jump-diffusion term structure models have been developed, the associated empirical work is quite limited. A formal model of the term structure of interest rates is necessary for the valuation of bonds and various interest rate options. More importantly, parameter values or estimates are required for the implementation of a specific model. To price interest rate options, with closed-form solutions or by numerical methods, one must have values of the parameters in the stochastic processes that determine interest rate dynamics.

Hence parameter estimation is a very first step in the application and analysis of interest rate option pricing models.

In this study, besides the classical Vasicek (1977) model, we also investigated a jump-diffusion process, which is a mixture of an O–U process with mean-reverting characteristics used by Vasicek (1977) and a compound Poisson jump process, for interest rates. Closed-form approximate solutions for discount bond prices were derived by Baz and Das (1996). The approximate model is essentially a one-factor term structure model. It has the disadvantage that all bond returns are perfectly correlated, and it may not be adequate to characterize the term structure of interest rates and its changing shape over time. However, the model at least can incorporate jump risks into the term structure model, making the model more complete relative to the pure diffusion Vasicek model. In addition, similar to the simple diffusion Vasicek model, the short-term interest rate can move to negative values under the extended jump-diffusion Vasicek model.

Realizing the drawback of one-factor models, in this study, we extended both the Vasicek (1977) model and the Baz and Das (1996) one-factor jump-diffusion model to a multi-factor model. We then developed a methodology for estimating both the one-factor and two-factor Vasicek and jump-diffusion term structure of interest rate models and completed an empirical study for Taiwanese Government Bond (TGB) interest rates. The state variables (such as the instantaneous short-term interest rate, and other factors) that drive the term structure of interest rates dynamics were not observable, and the observed bond prices are functions of the state variables. Thus, we can use the change of variable technique to obtain the likelihood function in terms of the observed bond prices, in order to conduct a maximum likelihood estimate. The estimation procedure of this study is similar to Chen and Scott (1993), Pearson and Sun (1994).

In the empirical study, for analytical purposes, we first need to obtain the TGB term structure of interest rates (that is the zero-coupon bond yield curve). However, the term structure is not directly observable because most of the TGBs are not zero-coupon bonds. Thus the coupon bond price, which may contain coupon effects, can not provide a good substitute for calculating the term structure of interest rates. Fortunately, a coupon bond is nothing but a portfolio of zero-coupon bonds with maturity consistent with coupon dates. We, thus, can use a curve fitting methodology, called the B-spline approximation suggested by Shea (1984) and successfully generalized and adopted by Steeley (1991), Lin and Paxson (1993), to fit the TGB term structure of interest rates. We use the prices of 45 TGBs to obtain the term structure from January 6, 1996 to August 29, 1998. We then use the estimated weekly interest rates on the 30-day, 180-day, 5- and the 10-year zero-coupon TGBs to estimate parameters in the one-factor and two-factor Vasicek and jump-diffusion models.

Thus, our methodology is a two-stage approach for investigating factor models of the term structure of interest rates. In the first stage, the B-spline approximation is applied to the cross-sectional bond prices to obtain the weekly term structure of interest rates. We then use the longitudinal term structure data obtained in the first stage to estimate the term structure models. This approach is different from that of

De Munnik and Schotman (1994), Sercu and Wu (1997), Ferguson and Raymar (1998), who used cross-sectional data to estimate the term structure models. Although the cross-sectional approach is quite straightforward, it has several limitations. First, the cross-sectional approach cannot easily be applied to multi-factor term structure models since it will be difficult to identify all parameters in its non-linear least squares estimation. Second the cross-sectional approach is inconsistent with inter-temporal equilibrium of the term structure models, in which parameters are assumed to be constant over time. Although one can pool the cross-sectional data and time-series data as done by De Munnik and Schotman (1994), to overcome this problem, problems will still arise since there will be too many parameters to be estimated. With our two-staged methodology, we can investigate the term structure dynamics more freely and flexibly. The first stage is purely statistical and is aimed at fitting the market term structure adequately. We then take the obtained term structures as exogenous to examine the economic meaning of the term structure models. The major disadvantage of the two-stage approach is that it is difficult to assess the estimation error, since efforts can be introduced in either of the two stages.

The empirical results show that both the one-factor and two-factor Vasicek and jump-diffusion models are statistically significant, with the two-factor models fitting better. This is as expected, since one-factor models do not fit the versatile term structure of interest rates very well. For both the two-factor Vasicek and jump-diffusion models, compared with the second factor, the first factor exhibits characteristics of stronger mean-reversion, higher volatility, and more frequent and significant jumps in the case of the jump-diffusion process. This is because the first factor is more associated with short-term interest rates. The second factor is associated with both short- and long-term interest rates, which is as a general factor driving the whole term structure dynamics. There is not a great difference in fitting power of the two-factor Vasicek model and the jump-diffusion model, but the jump-diffusion model, which can incorporate jump risks, provide more insight in explaining the term structure as well as the pricing of interest rate derivatives.

Following the internationalization and liberalization of financial markets, the Taiwanese capital market has become one of the most important markets in the Asia–Pacific area. Although the TGB market is small and not liquid compared with other developed bond markets, it is increasingly important and worthy of study. Moreover, since the assumption of an appropriate stochastic process for the interest rate and the estimation of its associated parameters is of critical importance when pricing and hedging with the term structure of interest rates and interest rate derivatives, the results and the methodology for estimating parameters in the jump-diffusion process have important implications for the area of financial engineering.

The rest of this paper is organized as follows. Section 2 specifies the Vasicek and the jump-diffusion term structure of interest rate models, and extends the one-factor model to a multi-factor model. Section 3 presents the empirical methodology used in this study. Section 4 describes the B-spline approximation methodology for fitting the term structure of interest rates. Section 5 specifies the data, and analyzes the results of parameters estimation and term structure fitting. Section 6 is the summary of the study.

2. The jump-diffusion interest rate model

One of the classical term structure of interest rate models is the Vasicek (1977) model. In the Vasicek model, the instantaneous short-term interest rate r is defined by the following diffusion process called the Ornstein–Uhlenbeck (O–U) process

$$\mathrm{d}r(t) = \alpha(\beta - r(t))\mathrm{d}t + \sigma \mathrm{d}W(t), \tag{1}$$

where α is the mean reversion coefficient, β is the long-term mean of the short-term interest rate, t denotes time path, and σ is the instantaneous volatility of the short-term interest rate. $\mathrm{d}W(t)$ is the increment of a standard Wiener process. Let the random variable $r_t \equiv (r(t)|r(0) = r_0)$ denote the level of a short-term interest rate at time t, conditional on the level of a short-term interest rate at the initial time 0, $r(0) = r_0$. Under the O–U process, r_t, is a normal distribution with a mean and variance as follows,

$$E(r_t) = \mathrm{e}^{-\alpha t} r_0 + \beta(1 - \mathrm{e}^{-\alpha t}); \tag{2}$$

$$\mathrm{Var}(r_t) = \frac{\sigma^2}{2\alpha}(1 - \mathrm{e}^{-2\alpha t}). \tag{3}$$

The price of a zero-coupon bond at time t, maturing at time T, $P(r,t,\tau)$ can be determined by a function of the short-term interest rate r and time to maturity $\tau = T - t$. That is,

$$P(r, t, \tau) = \exp(-A(\tau)r(t) + B(\tau)), \tag{4}$$

where

$$A(\tau) = \frac{1 - \mathrm{e}^{-\alpha\tau}}{\alpha};$$

$$B(\tau) = \left(\beta - \frac{\xi\sigma}{\alpha} - \frac{\sigma^2}{2\alpha^2}\right)(A(\tau) - \tau) - \frac{\sigma^2 A^2(\tau)}{4\alpha^3},$$

and ξ is the market price of interest rate risk, which is assumed to be constant.

To incorporate the discontinuous jump behavior in interest rate dynamics, following Baz and Das (1996), the short-term interest rate r is defined by the extended Vasicek jump-diffusion process

$$\mathrm{d}r(t) = \alpha(\beta - r(t))\mathrm{d}t + \sigma \mathrm{d}W(t) + J\mathrm{d}N(t), \tag{5}$$

where α is the mean reversion coefficient; β, the long-term mean of the short-term interest rate; t, the time path; and σ, the instantaneous volatility of the short-term interest rate associated with the diffusion component. $\mathrm{d}W(t)$ is the increment of a standard Wiener process, $N(t)$ represents a Poisson process with intensity rate λ. The probability that only one jump occurs during the instantaneous period $[t,t+$

$\mathrm{d}t]$ is $\lambda\mathrm{d}t$. If one jump occurs during the period $[t,t+\mathrm{d}t]$ then $\mathrm{d}N(t)=1$, while $\mathrm{d}N(t)=0$ represents the absence of a jump during that period. J denotes the magnitude of a jump, which is assumed to be a normal variable with mean equal to θ and S.D. equal to δ. Moreover, $\mathrm{d}W(t)$ is assumed to be independent of $\mathrm{d}N(t)$, which means that the diffusion component and the jump component of the short-term interest rate are independent of each other. Under the process specified in Eq. (5), as shown by Lin and Yeh (1998), r_t is defined as

$$r_t = \mathrm{e}^{-\alpha t}\left(r_0 + \int_0 \mathrm{e}^{\alpha u}\alpha\beta\,\mathrm{d}u + \int_0 \mathrm{e}^{\alpha u}\sigma\,\mathrm{d}W(u) + \sum_{j=1}^{N(t)} \mathrm{e}^{\alpha T_j}J_j\right), \tag{6}$$

where T_j is the time that the j-th jump occurs and $0 < T_1 < T_2 < \ldots < T_{N(t)} < t$, $N(t)$ represents the number of jumps occurring during the period between time 0 and time t. It can be shown that the probability density function for r_t is,

$$f(r_t) = \sum_{n=0}^{\infty}\left[\frac{\mathrm{e}^{-\lambda t}(\lambda t)^n}{n!}\right]\int_0 \int_0 \ldots \int_0 \omega(r_t;\, M,\, S)\frac{1}{t^n}\,\mathrm{d}\tau_1\mathrm{d}\tau_2\ldots\mathrm{d}\tau_n, \tag{7}$$

where $\omega(r_t;\, M,S)$ denotes a normal density function with mean M and standard deviation S, and

$$M = \mathrm{e}^{-\alpha t}r_0 + \beta(1-\mathrm{e}^{-\alpha t}) + \theta\mathrm{e}^{-\alpha t}\sum_{j=1}^{n}\mathrm{e}^{\alpha\tau_j};$$

$$S = \frac{\sigma^2}{2\alpha}(1-\mathrm{e}^{-2\alpha t}) + \delta^2\mathrm{e}^{-2\alpha t}\sum_{j=1}^{n}\mathrm{e}^{2\alpha\tau_j}.$$

The mean and variance of the conditional random variable r_t can be obtained as

$$E(r_t) = \mathrm{e}^{-\alpha t}r_0 + \left(\beta + \frac{\lambda\theta}{\alpha}\right)(1-\mathrm{e}^{-\alpha t}); \tag{8}$$

$$\mathrm{Var}(r_t) = \frac{\sigma^2 + \lambda(\theta^2+\delta^2)}{2\alpha}(1-\mathrm{e}^{-2\alpha t}). \tag{9}$$

Assume that the market price of interest rate diffusion risk is constant and equal to ξ, and the jump risk is diversifiable. Under the jump-diffusion, interest rate process specified in Eq. (5), according to Baz and Das (1996), using a linearization technique, the price of a zero-coupon bond at time t, which matures at time T, $P(r,t,\tau)$ can be determined approximately by a function of the short-term interest rate r and time to maturity $\tau = T - t$. That is,

$$P(r,\, t,\, \tau) = \exp[A(\tau)r(t) + B(\tau)], \tag{10}$$

where

$$A(\tau) = \frac{1-\mathrm{e}^{-\alpha\tau}}{\alpha};$$

$$B(\tau) = \frac{-E\mathrm{e}^{-2\alpha\tau}}{4\alpha^3} + \frac{(\alpha D + E)\mathrm{e}^{-\alpha\tau}}{\alpha^3} + \frac{(2\alpha D + E)\tau}{2\alpha^3} - C;$$

$$C = \frac{D}{\alpha^2} + \frac{3E}{4\alpha^3};$$

$$D = \xi\sigma - \alpha\beta - \theta\lambda;$$

$$E = \sigma^2 + (\delta^2 + \theta^2)\lambda.$$

To ensure that bond prices converge to zero for an arbitrarily large maturity, the additional condition $2\alpha D + E < 0$ is necessary. Under the model in Eq. (10), the short-term interest rate r has a linear relationship with the logarithm of discount bond prices, that is

$$r(t) = \frac{-\log P(r, t, \tau) + B(\tau)}{A(\tau)}. \tag{11}$$

The yield to maturity of a zero-coupon bond expiring $T-t$ periods hence is given by

$$R(r, t, \tau) = \frac{A(\tau)r(t) - B(\tau)}{\tau}. \tag{12}$$

The entire term structure of interest rates can then be defined.

The approximate model is essentially a one-factor term structure model. It has the disadvantage that all bond returns are perfectly correlated, and it may not be adequate to characterize the term structure of interest rates and its changing shape over time. However, the model at least can incorporate jump risks into the term structure model, making the model more complete relative to the pure diffusion Vasicek model. In addition, similar to the simple diffusion Vasicek model, the short-term interest rate can move to negative values under the extended jump-diffusion Vasicek model.

The one-factor models (either the Vasicek model or the jump-diffusion model) can be extended easily to two-factor models with the assumption that the two factors are mutually orthogonal. Take the jump-diffusion model as an example. Assume that the dynamics of the two factors y_1, and y_2, that drive the instantaneous short-term interest rate r, follow the following jump-diffusion processes,

$$\mathrm{d}y_1(t) = \alpha_1(\beta_1 - y_1(t))\mathrm{d}t + \sigma_1 \mathrm{d}W_1(t) + J_1 \mathrm{d}N_1(t); \tag{13}$$

$$\mathrm{d}y_2(t) = \alpha_2(\beta_2 - y_2(t))\mathrm{d}t + \sigma_2 \mathrm{d}W_2(t) + J_2 \mathrm{d}N_2(t), \tag{14}$$

where $r(t) = y_1(t) + y_2(t)$, $\mathrm{d}y_1(t)$ and $\mathrm{d}y_2(t)$ are independent of each other. Under the processes, y_1, and y_2 follow a probability distribution as in Eq. (7). Similar to Eq. (10), the bond price under the two-factor model is

$$P(r, t, \tau) = \exp[-A_1(\tau)y_1(t) - A_2(\tau)y_2(t) + B_1(\tau) + B_2(\tau)], \tag{15}$$

where

$$A_i(\tau) = \frac{1 - \mathrm{e}^{-\alpha_i r}}{\alpha_i};$$

$$B_i(\tau)=\frac{-E_i e^{-2\alpha_i\tau}}{4\alpha_i^3}+\frac{(\alpha_i D_i+E_i)e^{-\alpha_i\tau}}{\alpha_i^3}+\frac{(2\alpha_i D_i+E_i)\tau}{2\alpha_i^3}-C_i;$$

$$C_i=\frac{D_i}{\alpha_i^2}+\frac{3E_i}{4\alpha_i^3};$$

$$D_i=\xi_i\sigma_i-\alpha_i\beta_i-\theta_i\lambda_i;$$

$$E_i=\sigma_i^2+(\delta_i^2+\theta_i^2)\lambda_i;$$

$$i=1,2.$$

The yield to maturity of a zero-coupon bond expiring $\tau=T-t$ periods hence is given by

$$R(r,t,\tau)=\frac{A_1(\tau)y_1(t)+A_2(\tau)y_2(t)-B_1(\tau)-B_2(\tau)}{\tau}. \tag{16}$$

Therefore the entire term structure of interest rates can be defined.

3. Empirical methodology

For the one-factor model, assume that there are $T+1$ observations for the state variable (the instantaneous short-term interest rate), $r(0)$, $r(1)$, $r(2)$, …, $r(T)$. Since the processes specified in Eqs. (1) and (5) are Markovian, the log-likelihood function for the sample of observations is

$$\log L(r;\Theta)\equiv\log L(r_1,r_2,\ldots,r_T;\Theta)=\sum_{i=1}^{T}\log f[r(i)|(i-1)]. \tag{17}$$

where $\Theta=(\alpha,\beta,\sigma,\theta,\delta,\lambda)$ for the jump-diffusion model, and $\Theta=(\alpha,\beta,\sigma)$ for the Vasicek model, which denotes the parameter set to be estimated in the model.

Since the state variable in the model is the instantaneous short-term interest rate, which is unobservable, to develop the maximum likelihood estimator for the parameters of the processes that derive interest rate changes, we develop a likelihood function for the observed discount bond price as a function of the unobservable state variables. According to Eq. (11), the logarithm of the price of a discount bond is a linear function of the state variable, and the change of variable technique can be used to obtain the joint density functions and the log-likelihood function for a sample of observations on the discount bond price.

In our estimation, we use the prices of four discount TGBs (30-day, 180-day, 5- and 10-year) to estimate the model. Following the procedure of Chen and Scott (1993), we add measurement errors as additional random variables in the estimation, in order to perform a change of variables from the unobservable state variables to the observed bond prices. The 30-day TGB price is used and modeled without error. The system of equations for the model estimation is

$$\ln P(t,\tau_1)=-A(\tau_1)r(t)+B(\tau_1);$$

$$\ln P(t,\tau_2)=-A(\tau_2)r(t)+B(\tau_2)+e_{1t};$$

$$\ln P(t,\, \tau_3) = -A(\tau_3)r(t) + B(\tau_3) + e_{2t};$$
$$\ln P(t,\, \tau_4) = -A(\tau_4)r(t) + B(\tau_4) + e_{3t}, \tag{18}$$

where $P(t,\tau_i)$ is the price of the discount bond with time to maturity equal to τ_i, and τ_1 is equal to 30 days. e_{1t}, e_{2t}, and e_{3t} are measurement errors. In the estimation, we allow serial correlation and contemporaneous correlation between the measurement errors. The serial correlation is modeled as a first-order autoregressive process

$$e_{jt} = \rho_j e_{j,t-1} + \varepsilon_{jt}; \quad j = 1, 2, 3, \tag{19}$$

where the innovation of the measurement error is assumed to be normally distributed, that is $\varepsilon_{jt} \sim N(0,\sigma^2(\varepsilon_j))$. Thus, measurement errors are assumed to have a joint normal distribution. The log-likelihood function of the estimation then has the following form,

$$\ell(\hat{r},\, \Theta) = \log L(\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_T;\, \Theta) - T\ln|J| - \frac{3T}{2}\log(2\pi) - \frac{T}{2}\log|\Omega|$$
$$-\frac{1}{2}\sum \varepsilon_t'\Omega^{-1}\varepsilon_t, \tag{20}$$

where $\hat{r}$ is the substitute for the unobservable state variable (the instantaneous short-term interest rate). $\hat{r}$ is estimated, according to Eq. (11), by inverting the observed 30-day TGB interest rate, which is modeled without measurement errors. $\varepsilon_t' = (\varepsilon_{1t},\varepsilon_{2t},\varepsilon_{3t})$, and Ω is the covariance matrix for ε_t, which is assumed to be a diagonal matrix with elements $\sigma(\varepsilon_1)$, $\sigma(\varepsilon_2)$, and $\sigma(\varepsilon_3)$ along the diagonal. The elements of the matrix J are functions of $A(\tau_i)$, the coefficients in the linear transformation from r to $\log P(t,\tau_i)$,and the Jacobian of the transformation is $|J^{-1}|$. The likelihood function for the measurement error is conditional on an initial value for e_0. The estimation procedure has been discussed by Lin and Yeh (1998).

For the two-factor models, assume that there are $T+1$ observations for the state variables y_1 (0), y_1 (1), y_1 (2), …, y_1 (T) and y_2 (0), y_2 (1), y_2 (2), …, y_2 (T). Similar to Eq. (17), the log-likelihood function for the sample of the state variable is,

$$\log L(y_i;\, \Theta_i) \equiv \log L(y_{i1}, y_{i2}, \ldots, y_{iT};\, \Theta_i) = \sum_{j=1}^{T} \log f[y_i(j)|y_i(j-1)], \tag{21}$$

where Θ_i ($i = 1, 2$) denotes the parameter set for the i-th factor to be estimated in the model.

Similar to the case of the one-factor model in Eq. (18), we set the system of equations for the two-factor model estimation as,

$$\ln P(t,\, \tau_1) = -A_1(\tau_1)r_1(t) - A_2(\tau_1)r_2(t) + B_1(\tau_1) + B_2(\tau_1);$$
$$\ln P(t,\, \tau_2) = -A_1(\tau_2)r_1(t) - A_2(\tau_2)r_2(t) + B_1(\tau_2) + B_2(\tau_2) + e_{1t};$$
$$\ln P(t,\, \tau_3) = -A_1(\tau_3)r_1(t) - A_2(\tau_3)r_2(t) + B_1(\tau_3) - B_2(\tau_3) + (e_{1t} + e_{2t});$$
$$\ln P(t,\, \tau_4) = -A_1(\tau_4)r_1(t) - A_2(\tau_4)r_2(t) + B_1(\tau_4) + B_2(\tau_4) + e_{2t}, \tag{22}$$

where $P(t,\tau_i)$ is the price of the discount bond with time to maturity equal to τ_i, and τ_1 is equal to 30 days. e_{1t}, and e_{2t}, are measurement errors. The unobservable state variables in this two-factor model are computed by inverting $\ln P(t,\tau_1)$ and $\ln P(t,\tau_2)+\ln P(t,\tau_3)+\ln P(t,\tau_4)$. The measurement errors are also assumed to follow the same distribution as in the one-factor model. The log-likelihood function of the estimation then has the following form

$$\begin{aligned}&\ell(\hat{y}_1,\hat{y}_2;\Theta)\\&=\sum_{i=1}^{2}\log L(\hat{y}_{i1},\hat{y}_{i2},\ldots,\hat{y}_{iT};\Theta_i)-T\ln|J|-T\log(2\pi)-\frac{T}{2}\log|\Omega|\\&-\frac{1}{2}\sum_{j=1}^{T}\varepsilon_t'\Omega^{-1}\varepsilon_t'\end{aligned}\tag{23}$$

where $\Theta=(\Theta_1,\ \Theta_2)$, which is the parameter set to be estimated in the two-factor model. The estimation procedure is similar to the case of the one-factor model as described above.

4. Fitting the term structure of interest rates

To obtain the term structure of interest rates, assume that there are n coupon bonds included in the sample. The price of the coupon bond i, B_i is a linear combination of a series of pure discount bond prices. That is

$$B_i=\sum_{m=1}^{N_i}d_i(\tau_m)P(\tau_m),\tag{24}$$

where $-\tau_m$ is the time when the m-th coupon or principal payment is made, N_i is the number of coupons and principal payments before the maturity date of bond i, $d_i\ (\tau_m)$ is the cash flow paid by bond i at time t_m, and $P(\tau_m)$ is the pure discount bond price with face value \$1 (called the discount function) with maturity at time τ_m Once the discount function, $P(\tau)$ is defined, the spot interest rate (the pure discount bond yield) is also defined by $R(\tau)=-\ln P(\tau)/\tau$. To estimate $P(\tau)$ in Eq. (24), the simplest and most straightforward way is to construct a regression model similar to that adopted by Carleton and Cooper (1976). However, this methodology does not satisfy most researchers because it cannot provide a smooth, continuous yield curve.

An alternative methodology to estimate a continuous yield curve is to use a spline function approximation technique. Specifically, let

$$P(\tau)=\sum_{j=1}^{k}b_jg_j(\tau),\tag{25}$$

where $g_j(\tau)$ is the j-th approximation function, which is dependent on time, and b_js are the coefficients to be estimated, which are applied to the k approximation functions. Combining Eqs. (24) and (25), and introducing an error term[1], results in

$$B_i=\sum_{j=1}^{k}b_j\left(\sum_{m=1}^{N_i}d_i(\tau_m)g_j(\tau_m)\right)+\varepsilon_i.\tag{26}$$

[1] The error term may be caused by transaction costs, coupon effects, market imperfection, and so on.

In this discount fitting model, an additional restriction on the coefficients needs to be imposed to ensure that the estimated discount function will take a value of unity at maturity. That is,

$$P(0) = \sum_{j=1}^{k} b_j g_j(0) = 1.0. \tag{27}$$

Having specified the function $g_j(\tau)$, Eq. (26) is a well-defined linear regression model. The crucial problem is how to choose the function $g_j(\tau)$ and the number of functions k. Unfortunately, there is no economic theory or rule for this purpose. The only rule is empirical. If the model can fit the observed data well, and results in a smooth spot yield curve and a well-behaved forward yield curve, then it is assumed to be an appropriate model.

In choosing spline approximation functions in Eq. (26), we use the B-spline functions to approximate the discount function. The B-spline functions were suggested by Shea (1984) and have been generalized successfully and used by Steeley (1991) to estimate the UK Gilt-edged bond term structure, and Lin and Paxson (1993) to estimate the German government bond term structure. They all concluded that B-spline methodology can approximate the discount function appropriately and result in reliable and smooth spot and forward yield curves. Moreover, after examining various techniques used to term structure fitting, Deacon and Derry (1994) concluded that the B-spline approach is the most preferred methodology for practitioners.

The B-spline function was defined by Powell (1981) as,

$$g_s^p(\tau) = \sum_{i=s}^{s+p+1} \left[\left(\prod_{j=s,\, j\neq i}^{s+p+1} \frac{1}{(\tau_j - \tau_i)} \right) \right] [\max(\tau - \tau_j, 0)]^p, \tag{28}$$

where $g_s^p(\tau)$ is called the s-th ($s = 1, 2, \ldots, p+m$) p-order B-spline function. It is non-zero only if τ is in the interval (τ_s, τ_{s+p+1}), and zero otherwise. m is the number of sub-periods between time zero ($\tau = 0$) and the longest maturity date of the sample bonds (called the approximation space). There are $p+m$ B-spline functions required in this procedure. The two ends of any time interval (τ_s, τ_{s+1}) are called knots. There are $2p+m+1$ knots required within the time horizon. For example, in this article, we set $p = 3$ (cubic B-spline function)[2] and $m = 3$. We then need to specify ten knots and define 6 B-spline functions $g_S(\tau)$. The time to maturity of the sample bonds is between 0 and 15 years. We break it into three sub-periods — from 0 to 5, 5 to 8, and 8 to 15. To have all the six functions well-defined, we need to specify the knots beyond the two ends, 0 and 15 years. We then add -3, -2, -1 and 20, 30, and 40 in the time horizon. As a result, we set $\tau_1, \ldots, \tau_{10}$ equal to $-3, \ldots, 40$[3]. It is these 6 B-spline functions, which construct the

[2] To obtain a smooth forward yield curve, one needs at least a cubic spline function to approximate the discount function. Because the forward interest rate is the first derivative of the discount function, to have a smooth forward yield curve, the discount function must be twice differentiable continuously.

[3] There is no economic theory or rule for this purpose. The method of selection in this article is more or less ad hoc. We will describe this later.

basis for approximating the discount function. Having defined the B-spline functions, we can run Eq. (26) to estimate the coefficients for the discount fitting model.

5. The data and term structure fitting

In the empirical study, we looked at the weekly term structure of TGB interest rates for the period from January 6, 1996 to August 29, 1998. For fitting the term structure, we used the weekly (weekend, normally Saturday, occasionally Friday) price of 42 government bonds offered by the Grand Cathay Security Company. The sample bonds are summarized in Table 1.

The TGB market is small and not as liquid as other developed markets. Table 1 provides the list of the sample bonds used in this paper. The original life of TGBs is from 3 to 15 years, and the issuing size ranges from 10 to 50 billion Taiwan dollars. Bonds issued before 1996 are paying coupons semi-annually. After 1996, the government issued bonds with annual coupon payments. In 1995, the government issued two zero-coupon bonds, with original duration equal to 3 years.

To estimate the term structure of interest rates, we used the B-spline approximation technique as described in Section 4. The first step of the estimation procedure required the identification of the parameters p, m, and the knots as specified in Section 4. In order to obtain a smooth forward yield curve for the discount fitting model, we used cubic ($p = 3$) B-spline functions to approximate the discount functions. To decide m, we used a trial-and-error procedure to compare the average of squared predicting errors (the actual price minus the model price) and the number of significant coefficients estimated with different values of m. Once m was defined, the within-the sample knots could be identified simply by equalizing the number of samples for each sub-period between these knots. We found that when $m = 3$ (the within-the-sample knots were 0, 5, 8, and 15), and the 6 ($k = p + m$) coefficients were all significant at the significance level 0.05. When m increased, although the average of squared predicting errors decreased to a trivial value, the number of significant coefficients estimated did not increase significantly. When $k = 6$, the average standard predicting error made by the model is only about 0.25% of the par value, a quite satisfactory level. As to the out-of-the-sample knots, we found that prediction errors were not changed significantly using different out-of-sample knots. By an ad hoc decision, we set the out-of-sample knots equal to -3, -2, -1, and 20, 30, 40. To incorporate the coupon effects on the bond price, we added two variables to eq. 26. These are the number of coupon payments and the coupon rate of the TGB.

Without losing generality, we report the results of the estimation for the case of July 6, 1996, July 7, 1997, and July 4, 1998. Table 2 shows the results of the estimation. All but two of the coefficients estimated were significant statistically at a significance level of 1%, implying that the estimation is adequate. The fitting error, in terms of percentage of par-value, was 19, 24, and 23 basis points, respectively. In terms of yields, the standard fitting error was 4, 5, and 4 basis points, respectively. The coupon effects were significant for the July 6, 1996 case.

Table 1
Sample Taiwanese government bonds used for estimating the term structure

No.	Bond number	Issuing date	Original life	Maturity date	Coupon (%)	Coupon payments per year	Coupon payment date	Balance (million)
1	00813	02/21/92	4	02/21/96	8.50	2	8/21-2/21	40 000
2	00811	11/22/91	5	11/22/96	9.00	2	5/22-11/22	24 500
3	00814	03/13/92	5	03/13/97	8.50	2	9/13-3/13	40 000
4	00121	06/23/93	4	06/23/97	8.25	2	12/23-6/23	20 000
5	00821	07/24/92	5	07/24/97	8.50	2	1/24-7/24	30 000
6	00822	08/28/92	5	08/28/97	8.50	2	2/28-8/28	25 000
7	00851	10/20/95	3	10/20/98	0	0	–	10 000
8	00852	11/24/95	3	11/24/98	0	0	–	20 000
9	00823	11/27/92	7	11/27/99	8.50	2	5/27-11/27	40 000
10	00824	02/19/93	7	02/19/00	8.50	2	8/19-2/19	40 000
11	00832	12/17/93	7	12/17/00	8.25	2	6/17-12/17	40 000
12	00064	03/25/94	7	03/25/01	8.00	2	9/25-3/25	18 000
13	00842	12/16/94	7	12/16/01	7.60	2	6/16-12/16	30 000
14	A87102	11/21/97	5	11/21/02	6.125	1	11/21	30 000
15	00853	12/22/95	7	12/22/02	7.25	1	12/22	20 000
16	A85104	06/18/96	7	06/18/03	7.00	1	6/18	10 000
17	00831	09/22/93	10	09/22/03	8.75	2	3/22-9/22	30 000
18	A86103	11/19/96	7	11/19/03	6.60	1	11/19	35 000
19	00841	11/18/94	10	11/18/04	7.75	2	5/18-11/18	45 000
20	A87104	03/17/98	7	03/17/05	6.25	1	3/17	30 000
21	A85308	04/26/96	10	04/26/06	7.20	1	4/26	10 000
22	A86101	09/24/96	10	09/24/06	6.90	1	9/24	30 000
23	A86104	12/20/96	10	12/20/06	6.80	1	12/20	50 000
24	A87101	09/2397	10	09/23/07	6.375	1	9/23	30 000
25	00442	06/16/95	15	06/16/10	8.00	2	12/16-6/16	20 000
26	00551	07/21/95	15	07/21/10	7.75	2	1/21-7/21	−10 000
27	00753	09/22/95	15	09/22/10	7.35	2	3/22-9/22	10 000
28	00955	03/22/96	15	03/22/11	7.30	1	3/22	15 000
29	A86310	01/21/97	15	01/21/12	6.90	1	1/21	40 000
30	A87103	12/19/97	15	12/19/12	6.875	1	12/19	30 000
31	A87201	02/20/98	15	02/20/13	6.875	1	2/20	30 000

Table 2
Results of coefficient estimation in term structure fitting for selected dates[a]

Coefficients	07/06/1996		07/05/1997		07/04/1998	
	Coefficient estimated	Standard deviation	Coefficient estimated	Standard deviation	Coefficient estimated	Standard deviation
$b1$	8.3861[b]	0.0304	8.4139[b]	0.0319	8.4929[b]	0.0355
$b2$	9.3663[b]	0.0605	9.3115[b]	0.0643	9.1676[b]	0.0685
$b3$	10.0141[b]	0.0758	10.0498[b]	0.0989	9.8258[b]	0.0854
$b4$	9.0921[b]	0.1288	9.2973[b]	0.1594	8.8746[b]	0.1102
$b5$	7.0417[b]	0.2534	7.0679[b]	0.2302	6.7706[b]	0.1272
$b6$	8.5749[b]	0.3508	8.8452[b]	0.4506	8.8988[b]	0.2865
$b7$	−0.9951[b]	0.4545	−0.8048[b]	0.4105	−0.6648[b]	0.3319
$b8$	0.1031[b]	0.0562	0.0807	0.0606	0.0652	0.0813
Sample Size	21		22		25	
Standard error in price		0.193430		0.244185		0.233258
Standard error in yield		0.000387		0.000528		0.000440

[a] (1) Discount fitting model; $B_i = \Sigma_{j=1}^{6} b_j\left(\Sigma_{m=1}^{N_i} d_i(t_m) g_j(t_m)\right) + b_1\ Payments_i + b_8\ Coupon_i + \varepsilon_i$; (2) $Payments_i = 1$, if bond i pays coupon semi-annually, and $Payments_i = 0$ if bond i pays coupon annually; (3) $Coupon_i$ is bond i's coupon in percentage of par value 100

[b] Statistically significant at a significance level of 5%.

The coefficient for the number of payments was -0.995 1, reflecting that annual payment bonds were traded at higher prices (lower yields). As to the coupon rate level, the coefficient was 0. 103 1, implying higher coupon bonds were traded at higher prices (lower yield). In the case of July 5, 1997 and July 4, 1998, the coupon payments effect was significant, but the coupon rate effect was not.

Thus in general, we can identify the coupon payment effect on the TGB prices. Bonds with semi-annual coupon payments are traded at a higher yield than those with annual coupon payments. Presumably, this is because bonds with annual coupon payments are more recent issues and more actively traded, thus, they are traded at a lower yield. Another reason is that bonds with semi-annual coupon payments are subject to higher reinvestment risks, thus, they are traded at a higher yield. On the other hand, there was no coupon rate effect for most cases, although we found a negative coupon rate effect for some cases. Presumably, this is due to the differences in taxing interest income for individuals and institutions. For individuals, interest income from bond investment is taxed on a cash basis, while for institutions interest income from bond investment is taxed on an accrued basis. Moreover, there are no taxes on capital gains for securities trading in Taiwan. As a result, bonds held by individuals are transferred to the hands of institutions just before the coupon payment date. The individual, who does not receive the interest payment, thus, pay no taxes on the income. The institution, on the other hand, holds the bond for only a few days, and thus, has to pay a small amount of interest income taxes. As a result, bond trading in Taiwan is essentially tax-free, and the bond prices are not subject to tax effects.

Having obtained the continuous term structure of interest rates we calculated discrete yield curves for 30 semi-annually compounded zero-coupon bond yields with times to maturity from 0.5 to 15 years. The results are shown in Fig. 1.

6. Empirical results for term structure models

In estimating the term structure models, we used interest rates for four (the 30-day, 180-day, 5- and 10-year) discount TGBs obtained from the Section 4, to estimate the term structure of interest rates. Weekly data from January 6, 1996 to August 29, 1998, with a total sample of 139 observations, were used. The summary statistics of the sampled interest rates for parameter estimation can be found in Table 3. During the period from 1996 to 1998, for example, the average weekly 10-day interest rate was about 5.6%, with the maximum and minimum interest rate equal to 6.33 and 4.91%, respectively. On average, the longer the time to maturity, the higher is the interest rate, revealing that an upward sloping term structure is a normal case. The standard deviation for the short-term interest rate is in general greater than the long-term interest rate, revealing that short-term interest is more volatile than long-term interest rate. The non-zero skewness and excess kurtosis show that interest rates are not normally distributed.

For the Vasicek models, Table 4 shows the results of parameters estimated for the one-factor Vasicek model. All parameters in the model but one, are statistically

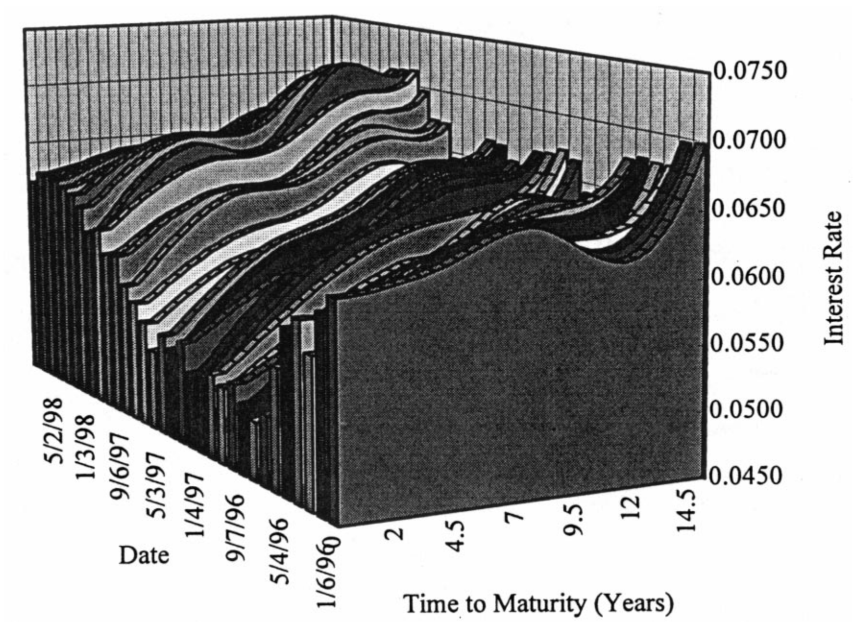

Fig. 1. Term structure of interest rates fitted by B-spline.

significant. The parameter α is estimated as 0.5938[4], which is significant, implying that α mean reversion is found in the interest rate dynamics. The estimate of 0.5938 for α implies a mean half life of 1.1673 years[5] in the interest rate process. β is estimated as 0.0525, which means that the long-term mean of the instantaneous short-term interest rate (the state variable) is approximately 0.0525. σ is the volatility of the interest rate process, which is estimated as 0.0075. Thus, the state

Table 3
Summary statistics of sampled interest rates for parameter estimation

Maturity	10-day	30-day	180-day	5-year	10-year	15-year[a]
Average	0.05601	0.05608	0.05666	0.06114	0.06426	0.06677
Standard deviation	0.00428	0.00426	0.00414	0.00294	0.00269	0.00244
Maximum	0.06330	0.06340	0.06370	0.06690	0.07010	0.07180
Minimum	0.04910	0.04930	0.05030	0.05610	0.05890	0.05980
Skewness	0.14682	0.15087	0.17156	0.29859	0.29555	0.12935
Kurtosis	1.59710	1.59609	1.59796	2.01062	2.74886	3.19975

[a] The estimated 15-year rate may not be reliable due to the lack of long-term sample bonds.

[4] Although weekly data is used for estimation, the parameter is annualized.

[5] The mean half life is the expected time for the process to return halfway to its long-term mean β. The mean half life is calculated as $-\ln(0.5)/\alpha$.

Table 4
Parameters estimated for one-factor Vasicek model

	Coefficient estimated	Standard error	t-test
α	0.5938	0.0096	62.1087[a]
β	0.0525	0.0001	1944.2772[a]
σ	0.0075	0.0001	177.6061[a]
ξ	−0.8767	0.0052	−170.0982[a]
$\sigma(\varepsilon_1)$	0.0029	0.0000	282.8423[a]
$\sigma(\varepsilon_2)$	0.0283	0.0003	82.9262[a]
$\sigma(\varepsilon_3)$	0.0091	0.0001	78.2054[a]
ρ_1	0.8038	0.9506	0.8456
ρ_2	3.7046	0.0985	37.6200[a]
ρ_3	0.8860	0.0299	29.6552[a]

[a] Significant at 1%.

variable (the instantaneous short-term interest rate) is at the same level and is more volatile than the observed interest rates, for example, the standard deviation of the observed 10-day interest is only 0.00428. The market price of interest rate risk ξ is estimated as a negative value, which is expected. $\sigma(\varepsilon_1)$, $\sigma(\varepsilon_2)$, and $\sigma(\varepsilon_3)$ are the standard deviations of the innovations in the measurement errors for logarithms of discount bond prices for the 180-day, 5- and the 10-year TGBs, respectively. The standard errors of the model in terms of discount bond price and annualized yield on the 180-day, 5- and the 10-year TGBs were also calculated. In terms of price, the standard errors are 0.000244, 0.006576, and 0.015569, respectively, which are correspondingly equal to 2, 66, and 156 basis points. In terms of annualized yield, the standard errors are 0.000502, 0.001792, and 0.002981, respectively, which are correspondingly equal to 5, 18, and 30 basis points. These measurement errors are economically significant, implying that the one-factor model does not fit the term structure very well. This result is similar to that of Chen and Scott (1993), who examined the CIR model, and Lin and Yeh (1998), who examined the Vasicek model.

Table 5 shows the results of parameters estimated for the two-factor Vasicek model. All parameters in the model but two are statistically significant. For the first factor, the mean-reversion parameter is estimated as 1.0159, which is stronger than that of the one-factor model. The standard deviation is 0.01, which is also more volatile than that of the one-factor model. For the second factor, the parameter α_2 is estimated as 0.0991, which implies a mean half life of 6.99 years in the state variable process. The volatility parameter σ_2 is 0.0056. Thus, the second factor is less volatile with weaker mean-reversion characteristics than the first factor. The long-term mean of the first factor β_1 is 0.0351, with a value of 0.0210 (β_2) for the second factor. The sum $(\beta_1 + \beta_2)$ is 0.561, which is consistent with the results in the one-factor model, revealing that the estimation is quite reliable. The two-factor model can, thus, discriminate the two distinct factors, which drive the term structure dynamics. $\sigma(\varepsilon_1)$ and $\sigma(\varepsilon_1)$ are the standard deviations of the innovations

Table 5
Parameters estimated for two-factor Vasicek model

	Coefficient estimated	Standard error	t-test
α_1	1.0159	0.0265	38.3034[a]
β_1	0.0351	0.0072	4.8668[a]
σ_1	0.0100	0.0006	15.6399[a]
ξ_1	−2.6823	0.2333	−11.4977[a]
α_2	0.0991	0.0012	85.4414[a]
β_2	0.0210	0.0043	4.9003[a]
σ_2	0.0056	0.0001	67.4288[a]
ξ_2	0.1201	0.2783	0.4316
$\sigma(\xi_1)$	0.0143	0.0002	66.0166[a]
$\sigma(\xi_2)$	0.0007	0.0000	220.8427[a]
ρ_1	−0.1865	2.6429	−0.0706
ρ_2	1.0007	0.0002	6602.8889[a]

[a] Significant at 1%.

in the measurement errors for logarithms of discount bond prices for the 5- and the 10-year TGBs, respectively. The standard errors of the model in terms of discount bond price and annualized yield on 5- and the 10-year TGBs were also calculated. In terms of price, the standard errors are 0.003894 and 0.002933, respectively, which are correspondingly equal to 39 and 29 basis points. In terms of annualized yield, the standard errors are 0.001050 and 0.00055, respectively, which are correspondingly equal to 11 and six basis points. These measurement errors are much less economically significant than that of the one-factor model, implying that the two-factor model is superior to the one-factor model in fitting the term structure of interest rates. This result is also consistent with that of Chen and Scott (1993).

Table 6
Parameters estimated for one-factor jump-diffusion model

	Coefficient estimated	Standard error	t-test
α	1.4619	0.0139	105.2757[a]
β	0.0761	0.0010	77.5038[a]
σ	0.0053	0.0002	28.2780[a]
λ	0.2398	0.1111	2.1585[a]
θ	0.0134	0.0049	2.7346[a]
δ	0.0078	0.0044	1.7646
ξ	−2.8575	0.3616	−7.9027[a]
$\sigma(\varepsilon_1)$	0.0000	0.0000	0.3193
$\sigma(\varepsilon_2)$	0.0000	0.0000	0.1831
$\sigma(\varepsilon 1)$	0.0001	0.0003	0.3538
ρ_1	−0.7678	0.7406	−1.0367
ρ_1	0.8789	0.1827	4.8098[a]
ρ_2	0.8252	0.2051	4.0234[a]

[a] Significant at 1%.

Table 7
Parameters estimated for two-factor jump-diffusion model

	Coefficient estimated	Standard error	t-test
α_1	1.6272	0.0382	42.6184[a]
β_1	0.0533	0.0018	30.3643[a]
σ_1,	0.0055	0.0002	30.4412[a]
λ_1	0.1648	0.0706	2.3344[a]
θ_1	0.0073	0.0029	2.5250[a]
δ_1	0.0093	0.0020	4.6858[a]
ξ_1	−1.7943	0.3059	−5.8655[a]
α_2	0.1120	0.0027	41.3294[a]
β_2	0.0038	0.0001	29.9552[a]
σ_2	0.0004	0.0000	40.6080[a]
λ_2	0.0293	0.0042	6.9687[a]
θ_2	0.0010	0.0001	15.0328[a]
δ_2	0.0009	0.0000	48.5890[a]
ξ_2	−0.2248	0.0340	−6.6133[a]
$\sigma(\varepsilon_1)$	0.0002	0.0013	0.1774
$\sigma(\varepsilon_2)$	0.0000	0.0000	1.1286
ρ_1	0.9139	0.0074	123.3621[a]
ρ_2	0.9336	0.0203	45.9223[a]

[a] Significant at 1%.

For jump-diffusion models, Table 6 shows the results of parameters estimated for the one-factor jump-diffusion model. The parameter α is estimated as 1.4619, which is strongly significant, and implies a mean half life of 0.4741 years in the interest rate process. β is estimated as 0.0761. σ is the volatility of the diffusion part in the interest rate process, and it is estimated as 0.0053. The jump intensity parameter λ is estimated as 0.2398, which implies that on average, jumps happen every 4 weeks or so. The average jump magnitude θ is estimated as 0.0134, and its standard deviation δ is 0.0078. The standard errors for the 180-day, 5- and the 10-year discount TGBs, in terms of price, are 0.000528, 0.0098, and 0.02489, respectively, which are correspondingly equal to 5, 98, and 249 basis points. In terms of annualized yield, the standard errors are 0.001084, 0.002677, and 0.004703, respectively, which are correspondingly equal to 11, 27, and 47 basis points. These measurement errors are even more significant than that of the one-factor Vasicek model, implying that the one-factor jump-diffusion model cannot fit the term structure very well.

Table 7 shows the results of parameters estimated for the two-factor jump-diffusion model. For the first factor, the result is quite similar to that in the one-factor model, except that the long-term mean of the factor is 0.0533, which is lower than that in the one-factor jump-diffusion model. For the second factor, the parameter α_2 is estimated as 0.1120, which implies a mean half life of 6.19 years in the state variable process. The long-term mean β_2 is estimated as 0.0038, and σ_2 is 0.0004. Thus, the second factor is at a lower level and is much less volatile with weaker mean-reversion characteristics than the first factor. The result is similar to the two-factor Vasicek model. Moreover, the jump intensity parameter for the second factor λ_2 is estimated as 0.0293, which implies that on average, a jump happens every 34 weeks or so. The average jump magnitude θ_2 is estimated as 0.0010, and the standard deviation, δ_2 is 0.0009. These values are much smaller than those of the first factor. Thus, the second factor is less volatile, has weaker mean-reversion, and jumps less frequently and less significantly in magnitude than the first factor. In terms of price, the standard errors for the 5- and 10-year TGBs are 0.003846 and 0.002527, respectively, which are correspondingly equal to 38 and 25 basis points. In terms of annualized yield, the standard errors are 0.001002 and 0.000568, respectively, which are correspondingly equal to ten and six basis points. These measurement errors are much less economically significant than that of the one-factor model, implying that the two-factor model is superior to the one-factor model in fitting the term structure of interest rates. The result is also comparable to that of the two-factor Vasicek model.

Fig. 2 (Panel A to K) plot the actual and estimated term structure using the four models for selected dates. When interest rates are at lower level (from mid 1996 to mid 1997, Panel B to G), all four models predict a positively sloped term structure and fit the actual term structure relatively well, with the two-factor (both the Vasicek and jump-diffusion) models fitting more satisfactorily. During the period when interest rates are at relatively higher level (Panel A, and H to K), the one-factor model predicts a flat or even negatively sloped term structure, which is quite far from the actual term structure. In particular, the one-factor jump-diffusion model underestimates terribly the term structure when interest rates are at a relatively high level. The two-factor models are still able to fit the actual term structure relatively well. Comparing the two-factor models, the two-factor Vasicek model tends to underestimate the shorter-term interest rates and overestimate the longer-term interest rates. On the other hand, the two-factor jump-diffusion model

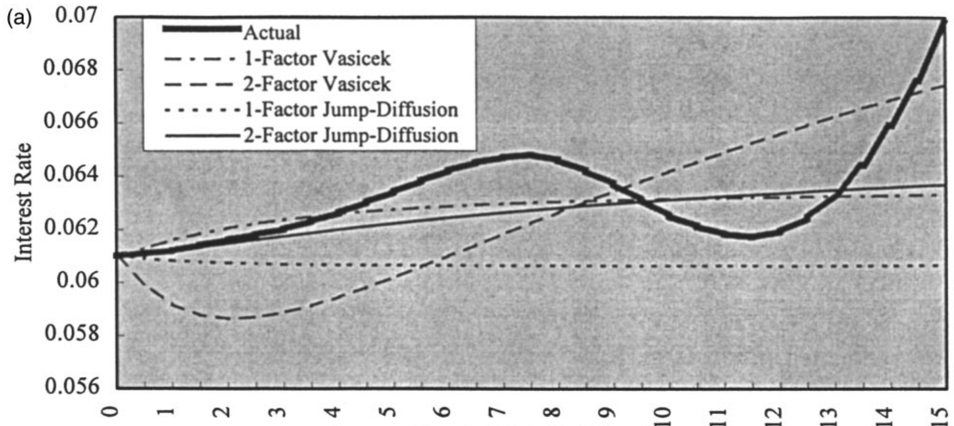

Fig. 2. Actual vs. estimated term structure of interest rates.

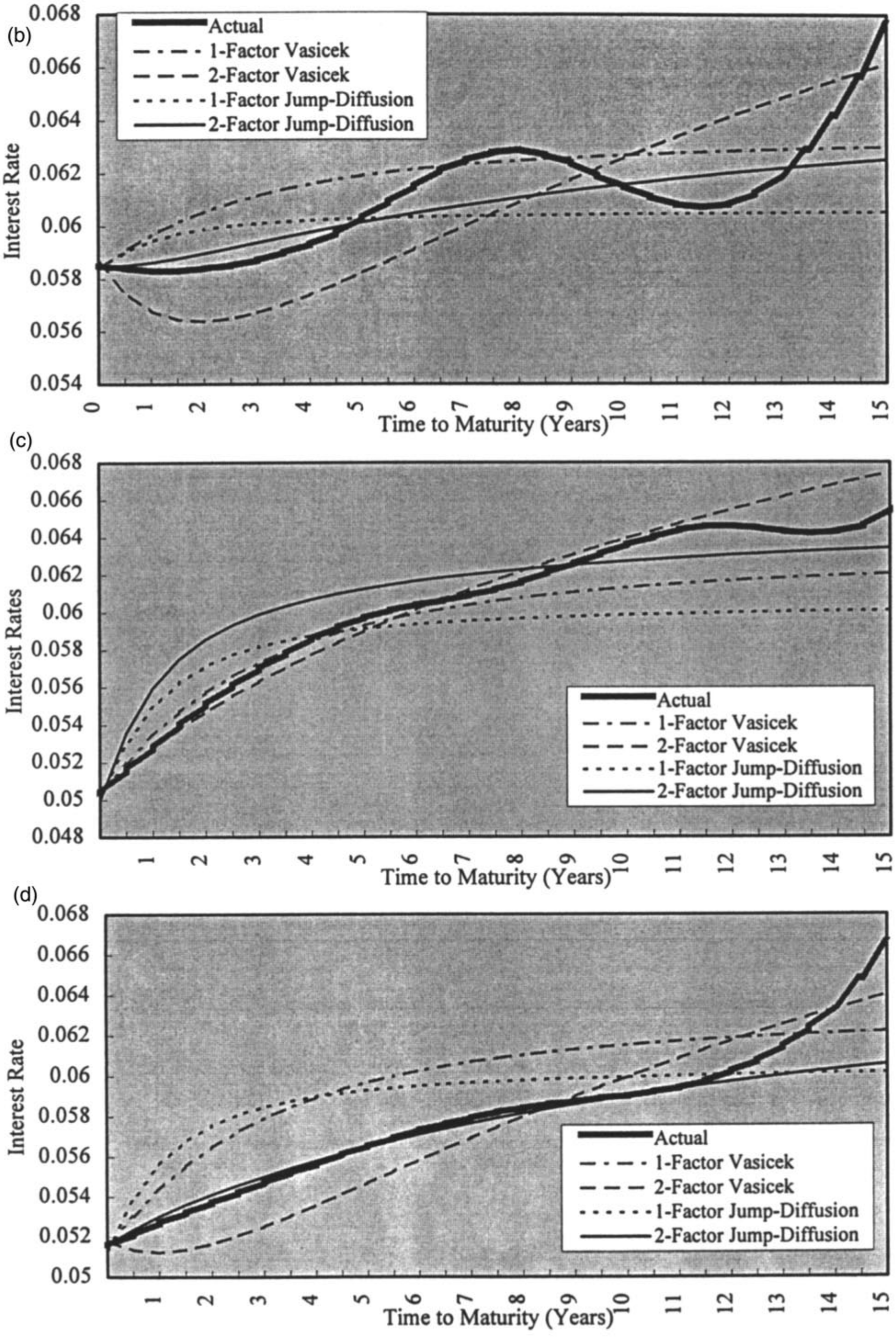

Fig. 2. (*Continued*)

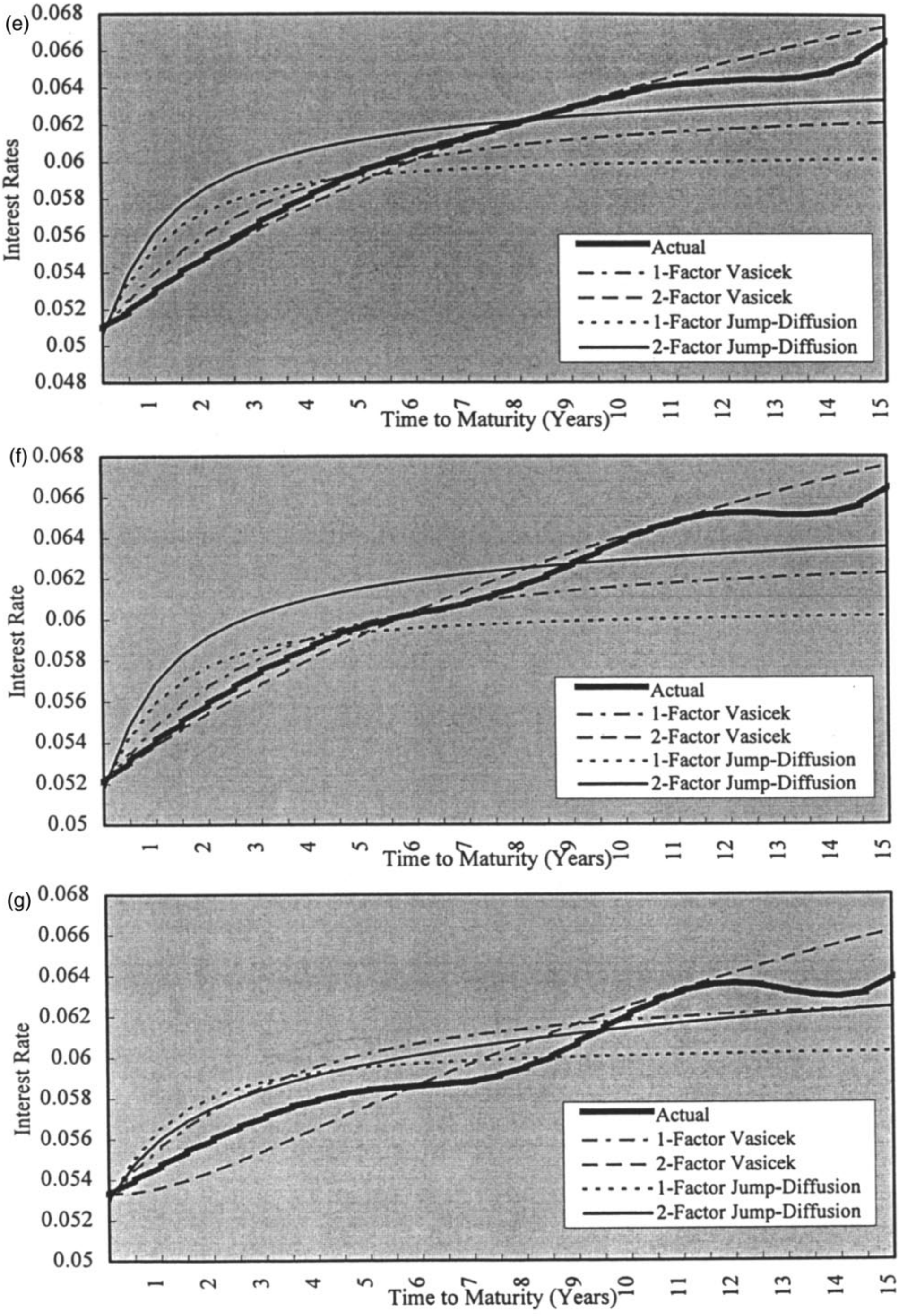

Fig. 2. (*Continued*)

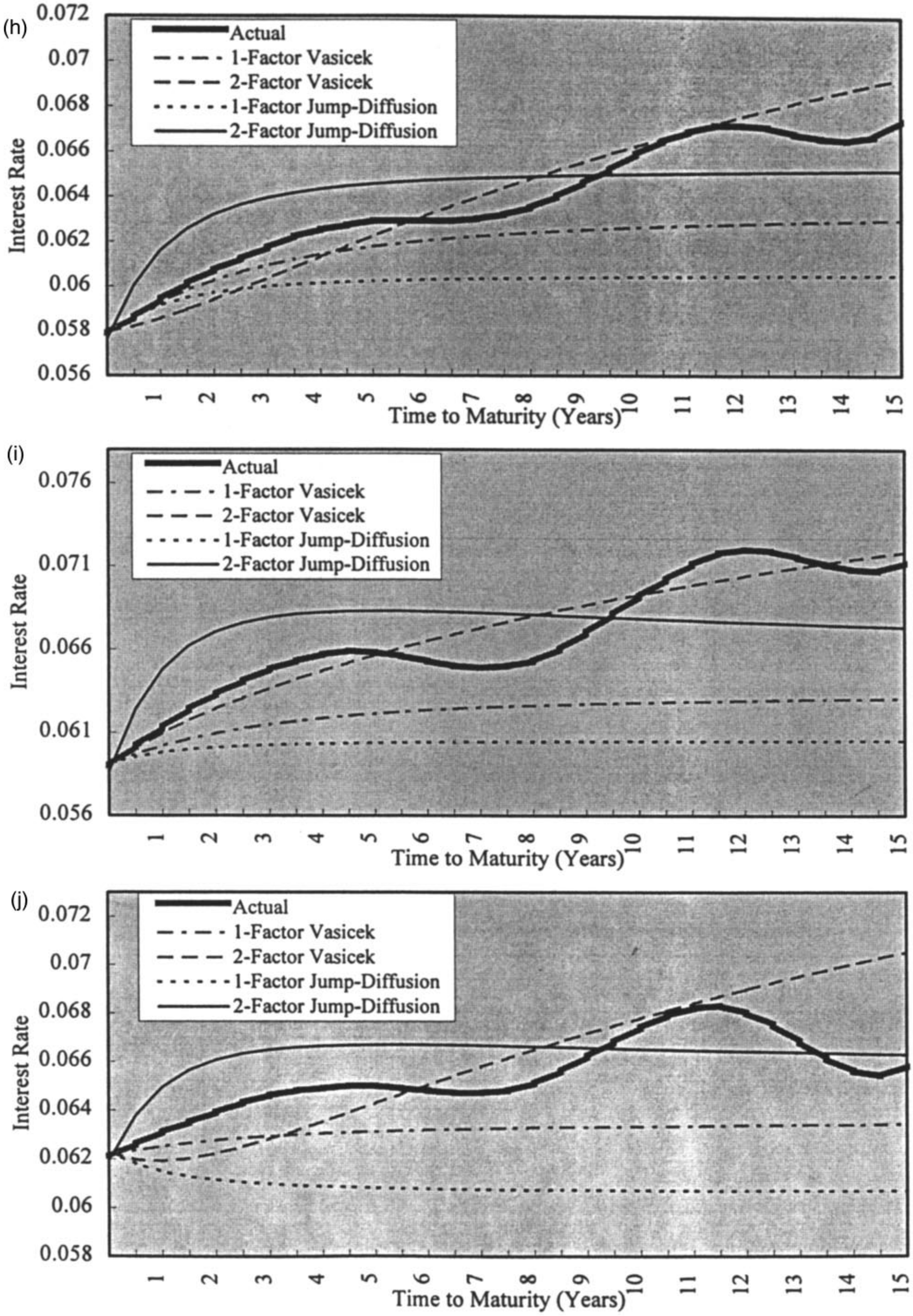

Fig. 2. (*Continued*)

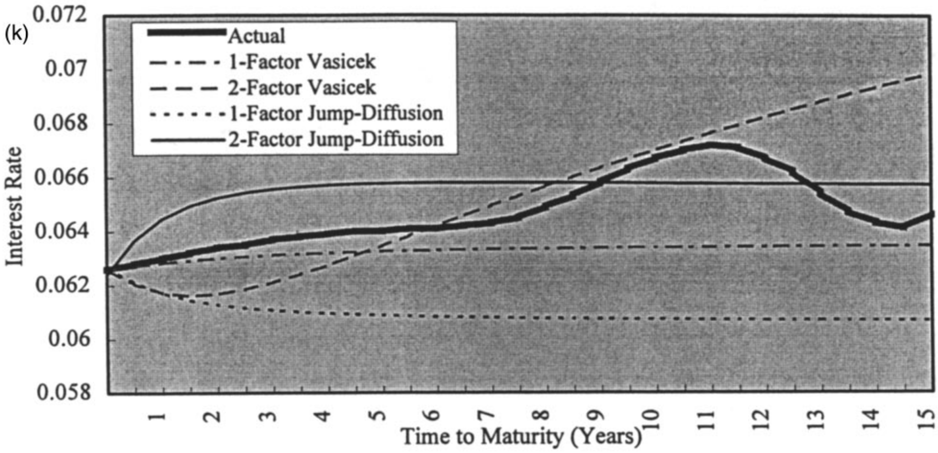

Fig. 2. (*Continued*)

tends to overestimate the shorter-term interest rates and underestimate the longer-term interest rates. Both of the two-factor models are unable to fit the flexible actual term structure very well. In particular, the models are unable to fit the longer-term end of the term structure. Presumably, this is because the actual term structure itself is estimated from the coupon bond prices. Due to the paucity of long-term bond data, the estimation of the term structure in the long-term end is bound to be unreliable.

Tables 8–11 calculate the pricing errors for the four models. Consistent with Fig. 2, the pricing error for the one-factor Vasicek model in Table 8 and the one-factor jump-diffusion model in Table 10 are quite significant. Overall, the one-factor models underestimate the term structure seriously. On average, the one-factor Vasicek model underestimates the term structure by ten basis points and the

Table 8
Errors of term structure estimation (one-factor Vasicek model)

	In price		In yield	
	Average error	Standard error	Average error	Standard error
Sub-period				
During 1996	−11.12	88.13	−0.84	18.18
During 1997	−71.41	133.98	12.54	25.39
During 1998	−106.15	168.72	22.48	33.08
Time to maturity				
0 to 5 Years	4.59	39.61	−2.44	14.29
5 to 10 Years	−34.07	96.35	6.86	19.85
10 to 15 Years	−155.76	206.29	28.18	37.55
All	−57.60	129.57	10.04	25.30

Table 9
Errors of term structure estimation (two-factor Vasicek model)

	In price		In yield	
	Average error	Standard error	Average error	Standard error
Sub-period				
During 1996	6.30	62.67	−1.18	13.52
During 1997	16.58	47.17	−1.86	9.96
During 1998	36.86	94.92	−4.58	19.24
Time to maturity				
0–5 Years	−20.64	29.51	8.70	11.66
5–10 Years	13.69	45.31	−2.67	9.84
10–15 Years	68.18	107.36	−1.23	19.39
All	17.84	67.60	−1.41	14.95

standard error is 25 basis points in yield. The model slightly overestimates the short-term interest rates by an average of two basis points. The one-factor jump-diffusion model underestimates the term structure even more seriously, by an average of 24 basis point and a standard error of 39 basis points in yield. Tables 9 and 10 shows the pricing error of the two-factor Vasicek model. On average, the two-factor Vasicek model overestimates slightly the term structure by 1.4 basis points and the standard error is about 15 basis points, while it underestimates the shorter-term interest rates by an average of 8.7 basis points and a standard error of 11.66 basis points in yield. On the other hand, Table 11 shows that the pricing error of the two-factor jump-diffusion model is −2.06 basis points on average, and the standard error is 12 basis points in yield. This model overestimates the shorter-term interest rates and underestimates the longer-term interest rates by an average of 5.55 basis points with a standard error of 21.35 basis points in yield. Overall, the

Table 10
Errors of term structure estimation (one-factor jump-diffusion model)

	In price		In yield	
	Average error	Standard error	Average error	Standard error
Sub-period				
During 1996	−70.86	143.18	9.77	27.78
During 1997	−134.51	207.26	24.06	38.74
During 1998	−202.39	263.14	43.22	51.89
Time to maturity				
0–5 Years	−6.91	58.17	0.85	21.70
5–10 Years	−116.87	162.50	24.03	33.21
10–15 Years	−283.75	318.27	50.28	56.76
All	−127.79	202.92	23.54	39.09

Table 11
Errors of term structure estimation (two-factor jump-diffusion model)

	In price		In yield	
	Average error	Standard error	Average error	Standard error
Sub-period				
During 1996	−18.21	64.60	0.40	13.35
During 1997	−8.02	53.12	−3.28	9.79
During 1998	12.23	74.58	−6.04	18.24
Time to maturity				
0–5 Years	23.16	33.85	−5.04	9.10
5–10 Years	14.27	46.72	−5.82	12.78
10–15 Years	−65.61	97.86	5.55	21.35
All	−6.73	57.89	−2.06	12.17

pricing errors of the two-factor models are considerably lower than those of the one-factor models.

The empirical evidence does not provide a clear determination of which two-factor model is superior. However, from a theoretical point of view, the jump-diffusion model, although much more complicated than the Vasicek model, has more implications for describing term structure fluctuations. It includes more risk characteristics in term structure behavior, hence has greater economic value in the pricing of interest rate derivatives.

For an explanation of the factors, Figs. 3 and 4 plot the short-term (10 day) interest rate versus the estimated two factors and the sum of the two factors (which are associated with the instantaneous short-term interest rate) for the Vasicek model and the jump-diffusion model. From Figs. 3 and 4, one can find that the sum

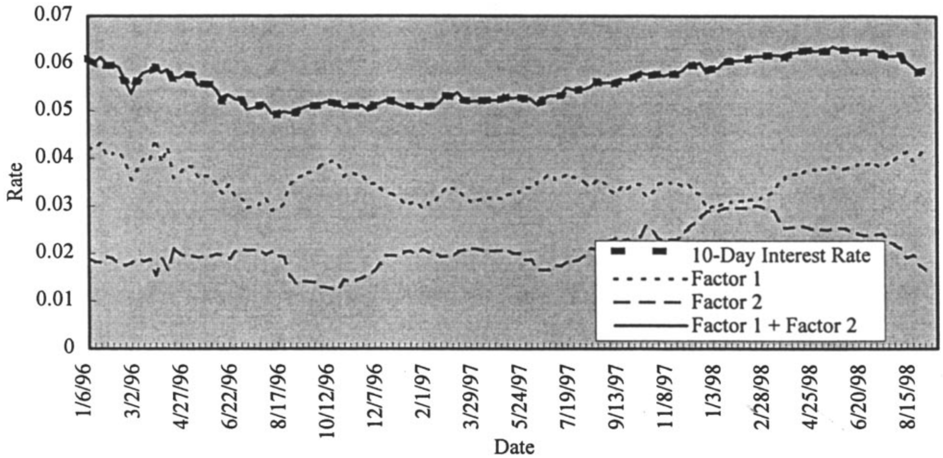

Fig. 3. Factor analysis (two-factor Vasicek model).

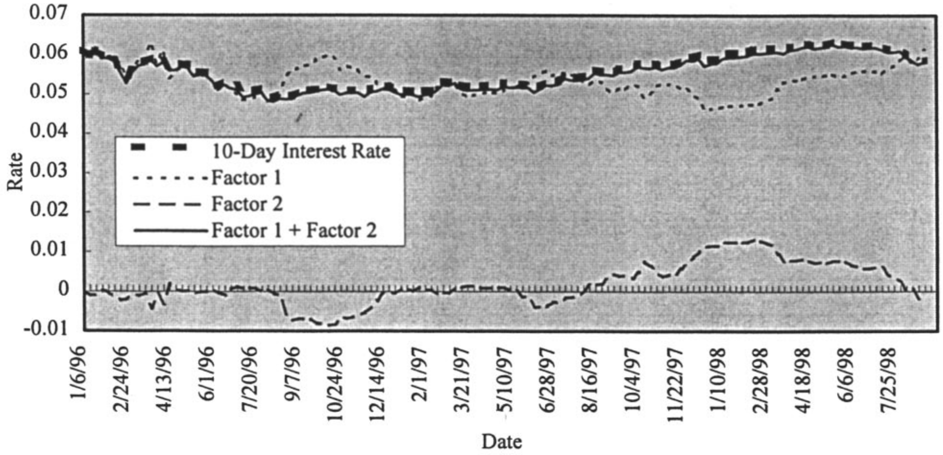

Fig. 4. Factor analysis (two-factor jump-diffusion model).

of the two factors is consistent with the short-term interest rate for either the Vasicek model or the jump-diffusion model.

Moreover, we also ran a regression analysis to investigate the relationship between the observed interest rates and the estimated factors for both the two-factor Vasicek and jump-diffusion models. We used the 10-day, 3-, 7- and 14-year TGB interest rates as the dependent variable. The results are shown in Tables 12 and 13. From Table 12, all coefficients estimated are statistically significant at the level of 1% for the 10-day, 3- and 7-year cases, which implies that the two factors are correlated with the observed interest rates. Comparing the coefficients of the two factors, the coefficients of the second factor remain equally high for models using the 10-day and 3-year interest rates as the dependent variable, and moderate for the model using the 7-year interest rate as the dependent variable. The coefficients of the first factor are lower when using the longer-term interest rate as the dependent variable. This implies that the first factor is more associated with the shorter-term interest rates, while the second factor is associated with the general term structure of interest rates. In the case of the model using the 14-year interest rate as the dependent variable, the explanation power is lower for either the Vasicek model or the jump-diffusion model, while the second factor still has explanation power to some extent. The lower explanation power for the 14-year interest rate may be due to the term structure fitting problem. All TGBs used for the term structure fitting have time to maturity lower than 15 years, and this lack of longer maturity bonds makes the estimated long-term interest rates unreliable.

7. Summary

In this paper, we investigate the O–U process used by Vasicek (1977) and a jump-diffusion process, which is a mixture of an O–U process and a compound

Table 12
Interest rate factor analysis (two-factor Vasicek model)

Dependent variable	Regression coefficient						Adjusted R-squared
	Intercept		Factor 1		Factor 2		
	Coefficient	Standard error	Coefficient	Standard error	Coefficient	Standard error	
10-day rate	0.00108	0.00005	0.96953	0.00115	0.99664	0.00096	0.999896
3-year rate	0.02312	0.00048	0.50440	0.01076	0.89790	0.00901	0.986442
7-year rate	0.04083	0.00093	0.25782	0.02075	0.57439	0.01738	0.887633
14-year rate	0.05918	0.001714	−0.07127	0.04828	0.41283	0.03207	0.607841

Table 13
Interest rate factor analysis (two-factor jump-diffusion model)

Dependent variable	Regression coefficient						Adjusted R-squared
	Intercept		Factor 1		Factor 2		
	Coefficient	Standard error	Coefficient	Standard error	Coefficient	Standard error	
10-day rate	0.00378	0.00006	0.94649	0.00120	0.99635	0.00088	0.999896
3-year rate	0.03373	0.00061	0.45938	0.01120	0.80745	0.00821	0.986442
7-year rate	0.04907	0.00117	0.22475	0.02160	0.50068	0.01583	0.887633
14-year rate	0.07078	0.00216	−0.06191	0.03986	0.29715	0.02920	0.607841

Poisson jump process, used by Baz and Das (1996), for the term structure of interest rates. We develop a methodology for estimating both the one-factor and two-factor Vasicek and jump-diffusion term structure models, and complete an empirical study for the TGB interest rates. Since the state variables that drive the term structure of interest rates dynamics are not observable, we use the change of variable technique to obtain the likelihood function in terms of the observed bond prices, and conduct a maximum likelihood estimate.

We use a two-stage approach to investigate the term structure models. The first stage involves application of the B-spline approximation to cross-sectional bond prices to obtain weekly term structures. This is a purely statistical procedure with the goal of fitting the market term structure adequately. We then use the longitudinal term structure data obtained in the first stage as exogenous to estimate the term structure models. The major disadvantage of the two-stage approach is that it is difficult to assess the estimation error, since errors can arise in either of the two stages. The research sample contained weekly prices of 45 TGBs from January 6, 1996 to August 29, 1998. Having obtained the term structure by applying the B-spline approximation, we then use the estimated weekly interest rates on the 30-day, 180-day, 5- and the 10-year zero coupon TGBs to estimate parameters for the one-factor and two-factor Vasicek and jump-diffusion models.

The results show that both the one-factor and two-factor Vasicek and jump-diffusion model are statistically significant, with the two-factor models fitting better. This is as expected, since one-factor models do not fit the versatile term structure of interest rates very well. For both the two-factor Vasicek and jump-diffusion models, compared with the second factor, the first factor exhibits characteristics of stronger mean reversion, higher volatility, and more frequent and significant jumps in the case of the jump-diffusion process. The first factor is more often associated with shorter-term interest rates, and the second factor is associated with both short-term and longer-term interest rates. There is not a great difference in the fitting power of the two-factor Vasicek model and the jump-diffusion model, but the jump-diffusion model, which can incorporate jump risks, provides more insight in explaining the term structure as well as the pricing of interest rate derivatives.

Since the assumption of an appropriate stochastic process for the interest rate and the estimation of its associated parameters are of critical importance when pricing and hedging with the term structure of interest rates and interest rate derivatives, the results and the methodology for estimating parameters in the jump-diffusion process have important implications in the area of financial engineering.

References

Ahn, C., Thompson, H., 1988. Jump-diffasion processes and the term structure of interest rates. J. Finance 43, 155–174.

Akgiray, V., Booth, G., 1986. Stock price processes with discontinuous time paths: an empirical examination. Financial Rev. 21, 163–184.

Akgiray, V., Booth, G., 1988. Mixed diffusion-jump process modeling of exchange rate movements. Rev. Econ. Stat. 70, 631–637.

Ball, C.A., Torous, W.N., 1983. A simplified jump process for common stock returns, J. Financial Quant. Anal., March, 53–63.

Ball, C.A., Torous, W.N., 1985a. On jumps in stock returns. J. Financial Quant. Anal. 10, 337–351.

Ball, C.A., Torous, W.N., 1985b. On jump in common stock prices and their impact on call pricing. J. Finance 40, 155–173.

Baz, J., Das, S.R., 1996. Analytical approximations of the term structure for jump-diffusion processes: a numerical analysis. J. Fixed Income, June, 78–86.

Brennan, M.J., Schwartz, E.J., 1978. A continuous time approach to the pricing of bonds. J. Banking Finance 3, 133–155.

Carleton, W.T., Cooper, I.A., 1976. Estimation and uses of the term structure of interest rates. J. Finance 31, 1067–1083.

Chen, R.R., Scott, L., 1993. Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates. J. Fixed Income 14-3, 1.

Cox, J.C., Ingersoll, J.E., Ross, S.A., 1985. A theory of the term structure of interest rates. Econometrica 51, 385–408.

Das, S.R., 1994. Poisson–Gaussian processes and the bond markets. Working paper # 95034, Harvard Business School.

Deacon, M., Derry, A., 1994. Estimating the term structure of interest rates. Working paper No. 24, The Bank of England.

De Munnik, J.F.J., Schotman, P.C., 1994. Cross-sectional versus time series estimation of term structure models: empirical results for the Dutch bond market. J. Banking Finance 18, 997–1025.

Ferguson, R., Raymar, S., 1998. A comparative analysis of several popular term structure estimation models. J. Fixed Income, March, 17–33.

Hamilton, J., 1988. Rational expectations econometrics analysis of changes in regime: an investigation of the term structure of interest rates. J. Econ. Dynamics Contr. 12, 385–423.

Jarrow, R.A., Rosenfeld, E.R., 1984. Jump risks and the intertemporal capital asset pricing model. J. Business, July, 337–351.

Jorion, P., 1988. On jump processes in the foreign exchange and stock markets. Rev. Financial Studies 1 (4), 427–445.

Lin, B.H., Paxson, D.A., 1993. Valuing the 'new-issue' quality option in Bund futures. Rev. Futures Markets 12 (2), 349–388.

Lin, B.H., Yeh, S.K., 1997. An empirical study on jump-diffusion process with heteroskedasticity in Taiwan stock market. Conference paper, APFA Annual meeting, Tokyo, 1998.

Lin, B.H., Yeh, S.K., 1998. Jump-diffusion interest rate process: an empirical examination. Forthcoming, J. Business Finance Accounting.

Park, A., Ann, C.M., Fujihara, R., 1993. Optimal hedged portfolios: the case of jump-diffusion risks. J. Int. Money Finance 12, 493–510.

Pearson, N.D., Sun, T.S., 1994. Exploiting the conditional density in estimating the term structure: an application to the Cox, Ingersoll and Ross model. J. Finance 49 (4), 1279–1304.

Powell, M.J.D., 1981. Approximation, Theory and Methods. Cambridge University Press, Cambridge.

Sercu, P., Wu, X., 1997. The information content in bond model residuals: an empirical study on the Belgian bond market. J. Banking Finance 21, 685720.

Shea, G., 1984. Pitfalls in smoothing interest rate term structure data: equilibrium models and spline approximation. J. Financial Quant. Anal. 19, 253–269.

Steeley, J.M., 1991. Estimating the Gilt-edged term structure: basis splines and confidence intervals. J. Business Finance Accounting 18 (4), 513–529.

Tucker, A.L., Pond, I., 1988. The probability distribution of foreign exchange price changes: tests of candidate processes. Rev. Econ. Stat. 70, 638–647.

Vasicek, O., 1977. An equilibrium characterization of the term structure of interest rates. J. Financial Econ. 5, 177–188.