Jump-diffusion CIR model and its applications in credit risk

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Abstract   In this paper, the author discusses the distribution of the jump-diffusion CIR model (JCIR) and its applications in credit risk. Applying the piecewise deterministic Markov process theory and martingale theory, we first obtain the closed forms of the Laplace transforms for the distribution of the jump-diffusion CIR model and its integrated process. Based on the obtained Laplace transforms, we derive the pricing of the defaultable zero-coupon bond and the fair premium of a Credit Default Swap (CDS) in a reduced form model of credit risk. Some numerical calculations are also provided.

Key words   jump-diffusion CIR model; reduced form model of credit risk; Laplace transform; defaultable zero-coupon bond; Credit Default Swap

1   Introduction

As we know, Cox et al. (1985) proposed the classical Cox-Ingersoll-Ross (CIR) process which is defined by an equation of the form

$$\mathrm{d}y_t = \lambda(\eta - y_t)\mathrm{d}t + \theta\sqrt{y_t}\,\mathrm{d}W_t, \tag{1.1}$$

where λ is the rate of mean reversion, η is the long-run level, θ is the volatility coefficient and W_t is a standard Brownian motion.

Compared with the Vasicek process (Vasicek, 1977), although the CIR equation (1.1) does not have a closed-form solution, the CIR process is always positive. If y_t reaches zero, the diffusion term $\mathrm{d}W_t$ disappears and the positive drift term pushes the process in the positive territory. The precise behavior of the CIR process near zero depends on the values of parameters. If $\theta^2 \leq 2\lambda\eta$, the positive drift term will always drive the process y_t away from zero before it will come too close. If $\theta^2 > 2\lambda\eta$, the process y_t will occasionally touch zero and reflect. On the other hand, the CIR process also has the character of mean-reverting. Due to the characters of non-negativity and mean-reverting, the CIR process is better for modelling the interest rate or default intensity in credit risk than the Vasicek model.

Over the recent years, some authors put their attention to the CIR processes and their applications in finance and insurance. We can refer the reader to Chen and Scott (1992), Delbaen (1993), Heston (1993), Berardi (1995), Chou and Lin (2006), Guo (2008), Wu et al. (2009), Ewald and Wang (2010), Trutnau (2011), Song et al. (2012), Bao and Yuan (2013).

In practice, there are primary events such as the governments fiscal and monetary policies, the release of corporate financial reports, some natural disasters and terrorist attacks etc., that will possibly result in some positive jumps in a firm's default intensity process. As time passes, the default intensity process decreases as the firm tries its best to avoid being in bankruptcy after the arrival of a primary event. This decrease will continue until another event occurs, which will result in another positive jump in its intensity processes. In order to describe the appearance of positive jumps in the default intensity process, we consider the jump-diffusion CIR model which has the following structure

$$\mathrm{d}y_t = \lambda(\eta - y_t)\mathrm{d}t + \theta\sqrt{y_t}\,\mathrm{d}W_t + \mathrm{d}J_t, \tag{1.2}$$

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where λ, η, θ and W_t are as in the previous model (1.1). We assume that $\lambda > 0$, $\eta \geq 0$ and $\theta \geq 0$. J_t is a compound Poisson process which is given by

$$J_t = \sum_{j=1}^{M_t} X_j, \tag{1.3}$$

where M_t is a Poisson process with frequency ρ and stands for the total number of jumps up to time t. $\{X_j, j \geq 1\}$ are the jump sizes and assumed to be independent and identically distributed random variables with distribution function $F(x)\,(x > 0)$.

Clearly (1.1) is a special case of (1.2) for $\rho = 0$. In addition, we can find that $\eta = \theta = 0$ would lead to shot noise processes for y_t. It is well known that shot noise models have been applied to diverse areas such as finance, insurance and electronics. Therefore, from an applied point of view, it is very significant to investigate the wider class of jump-diffusion CIR models.

Let $Y_t = \int_0^t y_u \mathrm{d}u$ be the integrated process of y_t. In this work, we will first study the Laplace transforms of the distributions of the processes y_t and Y_t. Then we will discuss the applications of these Laplace transforms in credit risk.

The rest of this article is organized as follows. In Section 2, we obtain the Laplace transforms for jump-diffusion CIR models and their integrated processes. In section 3, based on the result of the previous section, we derive the pricing of the defaultable zero-coupon bond and the fair premium of a Credit Default Swap (CDS) in a reduced form model of credit risk. Some numerical calculations and concluding remarks are presented in Section 4.

2 The Laplace transforms of the distribution of jump-diffusion CIR model

In this section, by applying the piecewise deterministic Markov process theory and martingale theory, we first derive the joint Laplace transform of the distribution of the vector process (y_t, Y_t). Then we obtain the Laplace transforms of the distribution of the jump-diffusion CIR model. The piecewise deterministic Markov process theory was developed by Davis (1984) and has been proved to be a very powerful mathematical tool for examining non-diffusion models. More details on this theory can be found in Davis (1984).

The (infinitesimal) generator $\mathcal{A}$ of the unique solution to SDE (1.1), is given by

$$\mathcal{A}f(y) = \lambda(\eta - y)\,\frac{\partial f}{\partial y} + \frac{1}{2}\,\theta^2\, y\,\frac{\partial^2 f}{\partial y^2}, \tag{2.1}$$

where f is an arbitrary twice continuously differentiable function. We assume that y_t is a jump-diffusion CIR model which is a solution of the SDE (1.2). With the aid of piecewise deterministic Markov process theory and using Theorem 5.5 in Davis (1984), one can see that the (infinitesimal) generator of the process (Y_t, y_t, t) acting on a function $f(Y, y, t)$ is given by

$$\begin{aligned}
\mathcal{A}f(Y,\, y,\, t) &= \frac{\partial f}{\partial t} + y\frac{\partial f}{\partial Y} + \lambda(\eta - y)\,\frac{\partial f}{\partial y} + \frac{1}{2}\,\theta^2\, y\,\frac{\partial^2 f}{\partial y^2} \\
&\quad + \rho\left\{\int_0^\infty f(Y,\, y + x,\, t)\mathrm{d}F(x) - f(Y,\, y,\, t)\right\},
\end{aligned} \tag{2.2}$$

where $f : (0, \infty) \times (0, \infty) \times \mathrm{R}^+ \to (0, \infty)$ satisfies:

(1) $f(Y, y, t)$ is bounded on arbitrary finite time intervals;

(2) $f(Y, y, t)$ is differentiable with respect to all t, y, Y;

(3)

$$\left|\int_0^\infty f(Y,\, y + x,\, t)\mathrm{d}F(x) - f(Y,\, y,\, t)\right| < \infty.$$

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For the sake of simplicity in the presentations throughout the rest of this article, we will use the following functions which are given by

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

In order to obtain the joint Laplace transform for the distribution of the vector process (y_t, Y_t), we first present the following lemma.

Lemma 2.1. Assume that m, k are two constants such that $m > 0$ and $k \geq 0$. Then for $0 \leq t < m/\sqrt{\lambda^2 + 2k\theta^2}$,

$$\exp\left\{-A(t)y_t - kY_t + \rho \int_0^t \left[1 - h(A(v))\right]dv + \lambda\eta \int_0^t A(v)dv\right\} \tag{2.3}$$

is a martingale where

$$h(\xi) = \int_0^\infty e^{-\xi x} dF(x), \tag{2.4}$$

$$A(t) = -\frac{\lambda}{\theta^2} - \frac{\sqrt{\lambda^2 + 2k\theta^2}}{\theta^2} \coth\left(\frac{\sqrt{\lambda^2 + 2k\theta^2} t - m}{2}\right). \tag{2.5}$$

Proof. Let

$$f(Y, y, t) = \exp\left\{-A(t)y - kY + R(t)\right\}. \tag{2.6}$$

From Theorem 7.6.1 in Jacobsen (2006), $f(Y, y, t)$ has to satisfy $\mathcal{A}f(Y, y, t) = 0$ for it to be a martingale. Hence by (2.2), it should hold

$$-A'(t)y + R'(t) - ky - \lambda(\eta - y)A(t) + \frac{1}{2}\theta^2 y A^2(t) + \rho\left[h(A(t)) - 1\right] = 0. \tag{2.7}$$

Solving the equation (2.7), we get

$$A(t) = \frac{\left(\sqrt{\lambda^2 + 2k\theta^2} - \lambda\right) + \left(\sqrt{\lambda^2 + 2k\theta^2} + \lambda\right)\exp\left(\sqrt{\lambda^2 + 2k\theta^2} t - m\right)}{\theta^2\left(1 - \exp\left(\sqrt{\lambda^2 + 2k\theta^2} t - m\right)\right)} \tag{2.8}$$

and

$$R(t) = \rho \int_0^t \left[1 - h(A(v))\right]dv + \lambda\eta \int_0^t A(v)dv. \tag{2.9}$$

By the definition of $\coth x$, it holds

$$\coth\left(\frac{\sqrt{\lambda^2 + 2k\theta^2} t - m}{2}\right) = \frac{\exp\left(\sqrt{\lambda^2 + 2k\theta^2} t - m\right) + 1}{\exp\left(\sqrt{\lambda^2 + 2k\theta^2} t - m\right) - 1}. \tag{2.10}$$

Combining (2.8) and (2.10), we get (2.5). Plugging (2.5) and (2.9) into (2.6), (2.3) follows immediately. The proof is completed.

Let $\mathcal{G}_t = \sigma(y_s, 0 \leq s \leq t)$. Now by means of Lemma 2.1, we give the joint Laplace transform of the distribution of the vector process (y_t, Y_t).

Theorem 2.1. Assume that μ, k are two constants such that $\mu \geq 0$, $k \geq 0$. Then the joint Laplace transform of the distribution of (y_t, Y_t) is given by

$$E\left\{e^{-\mu y_t} e^{-k(Y_t - Y_s)} \big| \mathcal{G}_s\right\}$$

$$= \exp\left(-B_{\mu,k}(s,t)y_s\right)\exp\left(\lambda^2\eta(t-s)/\theta^2\right)\left(C_{\mu,k}(s,t)\right)^{-\frac{2\lambda\eta}{\theta^2}}\exp\left(-\rho\int_0^{t-s}\left[1 - h(B_{\mu,k}(0,u))\right]du\right), \tag{2.11}$$

where

$$B_{\mu,k}(s,t) = \frac{(2k - \lambda\mu) + \mu\sqrt{\lambda^2 + 2k\theta^2}\coth\left(\sqrt{\lambda^2 + 2k\theta^2}\,(t-s)/2\right)}{(\theta^2\mu + \lambda) + \sqrt{\lambda^2 + 2k\theta^2}\coth\left(\sqrt{\lambda^2 + 2k\theta^2}\,(t-s)/2\right)}, \tag{2.12}$$

$$C_{\mu,k}(s,t) = \cosh\left(\sqrt{\lambda^2 + 2k\theta^2}\,(t-s)/2\right)$$
$$+ (\theta^2\mu + \lambda)(\lambda^2 + 2k\theta^2)^{-1/2}\sinh\left(\sqrt{\lambda^2 + 2k\theta^2}\,(t-s)/2\right). \tag{2.13}$$

Proof. By Lemma 2.1, for an arbitrary fixed time t^* $(0 \le s \le t^* < m/\sqrt{\lambda^2 + 2k\theta^2})$, we have

$$E\left\{\exp\left\{-A(t^*)y_{t^*} - kY_{t^*} + \rho\int_0^{t^*}\left[1 - h(A(v))\right]\mathrm{d}v + \lambda\eta\int_0^{t^*}A(v)\mathrm{d}v\right\}\bigg|\mathcal{G}_s\right\}$$
$$= \exp\left\{-A(s)y_s - kY_s + \rho\int_0^s\left[1 - h(A(v))\right]\mathrm{d}v + \lambda\eta\int_0^s A(v)\mathrm{d}v\right\}. \tag{2.14}$$

Then

$$E\left\{\exp\left\{-A(t^*)y_{t^*} - k(Y_{t^*} - Y_s)\right\}\bigg|\mathcal{G}_s\right\}$$
$$= \exp\left(-A(s)y_s\right)\exp\left(-\rho\int_s^{t^*}\left[1 - h(A(v))\right]\mathrm{d}v\right)\exp\left(-\lambda\eta\int_s^{t^*}A(v)\mathrm{d}v\right). \tag{2.15}$$

Set $A(t^*) = \mu \ge 0$. By (2.8), we get

$$m = \sqrt{\lambda^2 + 2k\theta^2}\,t^* - \ln\frac{\mu\theta^2 + \lambda - \sqrt{\lambda^2 + 2k\theta^2}}{\mu\theta^2 + \lambda + \sqrt{\lambda^2 + 2k\theta^2}}. \tag{2.16}$$

Clearly $m > \sqrt{\lambda^2 + 2k\theta^2}\,t^*$, i.e., $t^* < m/\sqrt{\lambda^2 + 2k\theta^2}$. Plugging (2.16) into $A(s)$ and $A(v)$ respectively, by direct computation, we have

$$A(s) = \frac{(\sqrt{\lambda^2 + 2k\theta^2} - \lambda) + (\sqrt{\lambda^2 + 2k\theta^2} + \lambda)\exp(\sqrt{\lambda^2 + 2k\theta^2}s - m)}{\theta^2\left(1 - \exp(\sqrt{\lambda^2 + 2k\theta^2}s - m)\right)}$$
$$= \frac{(2k - \lambda\mu + \mu\sqrt{\lambda^2 + 2k\theta^2}) + (\mu\sqrt{\lambda^2 + 2k\theta^2} + \lambda\mu - 2k)\exp\left(\sqrt{\lambda^2 + 2k\theta^2}(s - t^*)\right)}{(\theta^2\mu + \lambda)\left(1 - \exp\left(\sqrt{\lambda^2 + 2k\theta^2}(s - t^*)\right)\right) + \sqrt{\lambda^2 + 2k\theta^2}\left(1 + \exp\left(\sqrt{\lambda^2 + 2k\theta^2}(s - t^*)\right)\right)}$$
$$= \frac{(2k - \lambda\mu) + \mu\sqrt{\lambda^2 + 2k\theta^2}\coth(\sqrt{\lambda^2 + 2k\theta^2}\,(t^* - s)/2)}{(\theta^2\mu + \lambda) + \sqrt{\lambda^2 + 2k\theta^2}\coth(\sqrt{\lambda^2 + 2k\theta^2}\,(t^* - s)/2)}$$
$$= B_{\mu,k}(s, t^*) \tag{2.17}$$

and

$$A(v) = \frac{(\sqrt{\lambda^2 + 2k\theta^2} - \lambda) + (\sqrt{\lambda^2 + 2k\theta^2} + \lambda)\exp(\sqrt{\lambda^2 + 2k\theta^2}v - m)}{\theta^2\left(1 - \exp(\sqrt{\lambda^2 + 2k\theta^2}v - m)\right)}$$
$$= B_{\mu,k}(v, t^*). \tag{2.18}$$

From (2.15), (2.17) and (2.18), we have

$$E\left\{\exp\left\{-\mu y_{t^*} - k(Y_{t^*} - Y_s)\right\}\bigg|\mathcal{G}_s\right\}$$
$$= \exp\left(-B_{\mu,k}(s, t^*)y_s\right)\exp\left(-\rho\int_s^{t^*}\left[1 - h(B_{\mu,k}(v, t^*))\right]\mathrm{d}v\right)\exp\left(-\lambda\eta\int_s^{t^*}B_{\mu,k}(v, t^*)\mathrm{d}v\right). \tag{2.19}$$

Let $u = t^* - v$ in the integral of (2.19), then

$$E\left\{\exp\left\{-\mu y_{t^*} - k(Y_{t^*} - Y_s)\right\}\bigg|\mathcal{G}_s\right\}$$
$$= \exp\left(-B_{\mu,k}(s, t^*)y_s\right)\exp\left(-\rho\int_0^{t^*-s}\left[1 - h(B_{\mu,k}(0, u))\right]\mathrm{d}u\right)\exp\left(-\lambda\eta\int_0^{t^*-s}B_{\mu,k}(0, u)\mathrm{d}u\right). \tag{2.20}$$

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Since t^* is arbitrary, (2.20) remains true for all $0 \le s \le t < m/\sqrt{\lambda^2 + 2k\theta^2}$, then

$$E\left\{\exp\left\{-\mu y_t - k(Y_t - Y_s)\right\} \big| \mathcal{G}_s\right\}$$
$$= \exp\left(-B_{\mu,k}(s,t)y_s\right)\exp\left(-\rho\int_0^{t-s}\left[1 - h(B_{\mu,k}(0,u))\right]du\right)\exp\left(-\lambda\eta\int_0^{t-s}B_{\mu,k}(0,u)du\right). \qquad (2.21)$$

By standard integral calculation, then

$$\exp\left(-\lambda\eta\int_0^{t-s}B_{\mu,k}(0,u)du\right)$$
$$= \exp\left(-\lambda\eta\int_0^{t-s}\frac{(2k - \lambda\mu) + \mu\sqrt{\lambda^2 + 2k\theta^2}\coth(\sqrt{\lambda^2 + 2k\theta^2}\,u/2)}{(\theta^2\mu + \lambda) + \sqrt{\lambda^2 + 2k\theta^2}\coth(\sqrt{\lambda^2 + 2k\theta^2}\,u/2)}du\right)$$
$$= \exp\left(\lambda^2\eta(t - s)/\theta^2\right)\left(\cosh\left(\sqrt{\lambda^2 + 2k\theta^2}\,(t - s)/2\right)\right.$$
$$\left.+(\theta^2\mu + \lambda)(\lambda^2 + 2k\theta^2)^{-1/2}\sinh\left(\sqrt{\lambda^2 + 2k\theta^2}\,(t - s)/2\right)\right)^{-\frac{2\lambda\eta}{\theta^2}}. \qquad (2.22)$$

Plugging (2.22) into (2.21), (2.11) follows immediately. The proof is completed.

Setting $k = 0$ and $\mu = 0$ in (2.11) respectively, we obtain the following corollaries.

Corollary 2.1. Assume that μ, k are two constants such that $\mu \ge 0$, $k \ge 0$. Then the Laplace transforms of the distributions of y_t and Y_t are respectively given by

$$E\left\{e^{-\mu y_t}\big|\mathcal{G}_s\right\}$$
$$= \exp\left(-B_{\mu,0}(s,t)y_s\right)\exp\left(\lambda^2\eta(t - s)/\theta^2\right)\left(C_{\mu,0}(s,t)\right)^{-\frac{2\lambda\eta}{\theta^2}}\exp\left(-\rho\int_0^{t-s}\left[1 - h(B_{\mu,0}(0,u))\right]du\right) \qquad (2.23)$$

and

$$E\left\{e^{-k(Y_t - Y_s)}\big|\mathcal{G}_s\right\}$$
$$= \exp\left(-B_{0,k}(s,t)y_s\right)\exp\left(\lambda^2\eta(t - s)/\theta^2\right)\left(C_{0,k}(s,t)\right)^{-\frac{2\lambda\eta}{\theta^2}}\exp\left(-\rho\int_0^{t-s}\left[1 - h(B_{0,k}(0,u))\right]du\right). \qquad (2.24)$$

To make later calculation somewhat easier, we assume that jumps size in (1.3) follows exponential distribution, i.e., $F(x) = 1 - e^{-\alpha x}$ $(x > 0, \alpha > 0)$. Then from Corollary 2.1, we get the following result.

Corollary 2.2. Assume that μ, k are two constants such that $\mu \ge 0$, $k \ge 0$ and that $F(x) = 1 - e^{-\alpha x}$ $(x > 0, \alpha > 0)$. Then the Laplace transforms of the distributions of y_t and Y_t are respectively given by

$$E\left\{e^{-\mu y_t}\big|\mathcal{G}_s\right\}$$
$$= \exp\left(-B_{\mu,0}(s,t)y_s + \lambda^2\eta(t - s)/\theta^2\right)\left(C_{\mu,0}(s,t)\right)^{-\frac{2\lambda\eta}{\theta^2}}\left(\frac{2\lambda(\alpha + \mu)\exp(\lambda(t - s))}{\alpha(\theta^2\mu + 2\lambda)\exp(\lambda(t - s)) + (2\lambda\mu - \alpha\theta^2\mu)}\right)^{-\frac{2\rho}{2\lambda - \alpha\theta^2}}$$
$$\qquad (2.25)$$

and

$$E\left\{e^{-k(Y_t - Y_s)}\big|\mathcal{G}_s\right\}$$
$$= \exp\left(-B_{0,k}(s,t)y_s\right)\exp\left\{M_1(k)\,(t - s) + M_2(k)\ln D_k(s,t) - M_3\ln C_{0,k}(s,t)\right\}, \qquad (2.26)$$

where

$$D_k(s,t) = \cosh\left(\sqrt{\lambda^2 + 2k\theta^2}\,(t - s)/2\right) + \frac{\alpha\lambda + 2k}{\alpha\sqrt{\lambda^2 + 2k\theta^2}}\sinh\left(\sqrt{\lambda^2 + 2k\theta^2}\,(t - s)/2\right),$$

$$M_1(k) = \frac{\lambda^2\eta}{\theta^2} - \frac{2k\rho}{\alpha\left(\sqrt{\lambda^2 + 2k\theta^2} + \lambda\right) + 2k} - \frac{\alpha\rho\sqrt{\lambda^2 + 2k\theta^2}}{2k + 2\alpha\lambda - \alpha^2\theta^2},$$

$$M_2(k) = \frac{2\alpha\rho}{2k + 2\alpha\lambda - \alpha^2\theta^2}, \quad M_3 = \frac{2\lambda\eta}{\theta^2}.$$

3 Applications in credit risk

In this section, based on the results of previous section, we derive the pricing of the defaultable zero-coupon bond and the fair premium of a Credit Default Swap (CDS) in a reduced form model of credit risk. Reduced form models of credit risk were pioneered by Artzner and Delbaen (1995). For the literature on the reduced form model, we can refer to Jarrow and Turnbull (1995), Duffie and Singleton (1999), Bai, Hu, and Ye (2007), Liang and Wang (2012), Su and Wang (2013). In some literature on the reduced form model of credit risk, the default arrival time for the firm is defined as the first jump time of the Cox process. Due to some primary events which will possibly result in some positive jumps in a firm's default intensity process, we employ the jump-diffusion CIR model to describe the firm's default intensity.

We first state the definition of Cox process. Many alternative definitions of a Cox process can be found in the previous literature. We adopted the one used by Brémaud (1981).

Definition 3.1. Let $\{\Omega, \mathcal{F}, \mathrm{P}\}$ be a probability space with information structure given by $\{\mathcal{G}_t, t \in [0, T]\}$. Let N_t be a point process adapted to $\{\mathcal{G}_t, t \in [0, T]\}$. Let y_t be a nonnegative process adapted to $\{\mathcal{G}_t, t \in [0, T]\}$ such that

$$\int_0^t y_u \mathrm{d}u < \infty \quad \text{a.s. (no explosions).}$$

If for all $u \in \mathbb{R}$ and $0 \le t_1 \le t_2$,

$$E\left\{ \mathrm{e}^{iu(N_{t_2} - N_{t_1})} \big| \mathcal{G}_{t_2} \right\} = \exp\left\{ (\mathrm{e}^{iu} - 1) \int_{t_1}^{t_2} y_u \mathrm{d}u \right\}, \tag{3.1}$$

where $\mathcal{G}_t = \sigma(y_u, u \le t)$, then N_t is called a Cox process with intensity y_t.

From this definition, we can consider a Cox process as a two-step randomisation procedure. N_t is a Poisson process conditional to y_t and y_t is used to generate N_t by acting as its intensity. Therefore, a Cox process is also called a doubly stochastic Poisson process.

In the following we assume that y_t is a jump-diffusion CIR model satisfying (1.2) and $y_0 = 0$. Denote $\tau = \inf\{t \ge 0, N_t = 1 \, | \, N_0 = 0\}$, where N_t is a Cox process with intensity y_t defined as (1.2). Then from (3.1), we have

$$P\left(N_t - N_s = k \, \big| \, \mathcal{G}_t \right) = \frac{1}{k!} \left(\int_s^t y_u \mathrm{d}u \right)^k \exp\left(-\int_s^t y_u \mathrm{d}u \right). \tag{3.2}$$

Let $\mathcal{H}_t := \sigma(\{\tau \le s\}, s \le t)$, i.e. the σ-algebra generated by τ up to time t and $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$. By the definition of τ and (3.2), the conditional distributions of τ are given by

$$P\left(\tau > t \, \big| \, \mathcal{G}_t \right) = P\left(N_t - N_0 = 0 \, \big| \, \mathcal{G}_t \right) = \mathrm{e}^{-Y_t}. \tag{3.3}$$

Now we present the survival probability of a firm which has a default intensity process y_t.

Theorem 3.1. Let y_t be a jump-diffusion CIR model satisfying $y_0 = 0$, and N_t be a Cox process with intensity y_t. Then the survival probability is given by

$$
\begin{aligned}
P\left(\tau > t \right) &= \exp\left\{ \left(\frac{\lambda^2 \eta}{\theta^2} - \rho \right) t - \frac{2\lambda \eta}{\theta^2} \ln\left[C_{0,1}(0, t) \right] + \rho \int_0^t h(B_{0,1}(0, u)) \mathrm{d}u \right\} \\
&=: \exp\{\Phi(t)\}.
\end{aligned} \tag{3.4}
$$

Proof. By the definition of the default arrival time τ and (2.24), we have

$$
\begin{aligned}
& P\left(\tau > t \, \big| \, \mathcal{G}_s \right) \\
=\ & P\left(N_t - N_s = 0 \, \big| \, \mathcal{G}_s \right) = E\left\{ \mathrm{e}^{-(Y_t - Y_s)} \big| \mathcal{G}_s \right\} \\
=\ & \exp\left(-B_{0,1}(s, t) y_s \right) \exp\left(\lambda^2 \eta (t - s)/\theta^2 \right) \left(C_{0,1}(s, t) \right)^{-\frac{2\lambda \eta}{\theta^2}} \exp\left(-\rho \int_0^{t-s} \left[1 - h(B_{0,1}(0, u)) \right] \mathrm{d}u \right). \tag{3.5}
\end{aligned}
$$

Note that $y_0 = 0$ and $P\left(\tau > t \right) = P\left(\tau > t \, \big| \, \mathcal{G}_0 \right)$, we can easily get (3.4). The proof is completed.

Remark 3.1. Taking the derivative in (3.4), we can obtain the default probability density as

$$P\left(\tau \in \mathrm{d}t \right) = -\exp\{\Phi(t)\} \partial_t \Phi(t) \, \mathrm{d}t. \tag{3.6}$$

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If we assume that $F(x) = 1 - e^{-\alpha x}$ $(x > 0, \alpha > 0)$, we can get the following corollary.

Corollary 3.1. Let y_t be a jump-diffusion CIR model satisfying $y_0 = 0$, and N_t be a Cox process with intensity y_t. Assume that $F(x) = 1 - e^{-\alpha x}$ $(x > 0, \alpha > 0)$. Then the survival probability is given by

$$
\begin{aligned}
P(\tau > t) &= \exp\Big\{ M_1(1)\, t + M_2(1) \ln D_1(0,\, t) - M_3 \ln C_{0,1}(0, t) \Big\} \\
&=: \exp\{\Psi(t)\}.
\end{aligned}
\tag{3.7}
$$

By some similar arguments as in the proof of Theorem 3.1, we can prove this corollary. Here we omit the details.

Next we will derive the pricing of the defaultable zero-coupon bond and the fair premium of a CDS. In recent years, the rapid expansion of market for credit derivatives has led to a growing interest in investigation of the pricing of the defaultable zero-coupon bond and the fair premium of CDS. A CDS is in fact a contract agreement between protection buyer and seller. Assume that firm A issues a defaultable zero-coupon bond and investor B holds the bond. Then B faces the credit risk arising from default of firm A. In order to protect from this credit risk, B buys a CDS contract which requires B to pay periodic premium to party C (CDS protection seller). In exchange, C will compensate B for his loss in the event of default of the bond.

The following definition of the price process of CDS can be found in Crépey et al. (2009).

Definition 3.2. The model price process of a CDS is given by $P_t = E\{p_T(t)\}$, where $p_T(t)$ corresponds to the CDS cumulative discounted cash flows on the time interval $(t, T]$ and satisfies

$$
\beta(t)p_T(t) = (1 - R)\beta(\tau)I_{(t < \tau < T)} - \kappa \int_t^{\tau \wedge T} \beta(v)\mathrm{d}v.
\tag{3.8}
$$

In equation (3.8), τ is the default arrival time of firm A, $\beta(t) = e^{-\int_0^t r_u \mathrm{d}u}$ is the discount factor. Here we assume that the market interest rate r_t is a deterministic function of the time and that the recovery rate is R. (3.8) describes the change trend of cash flow for investor B. The first term on the right-hand side of (3.8) corresponds to the present value of the investor B's loss $(1 - R)$ resulted by the default of firm A. The second term on the right-hand side of (3.8) corresponds to the present value of the premiums which B pays to C.

We first state the pricing of the defaultable zero-coupon bond and the fair premium of CDS based on the conclusion of Theorem 3.1.

Theorem 3.2. Let $B(0, T)$ be the present value of the defaultable zero-coupon bond at time 0 paying 1 at time T and κ be the fair premium of CDS. Then the following statements hold:

(1) The formula for calculating the value of $B(0, T)$ is given by

$$
B(0, T) = e^{-\int_0^T r_u \mathrm{d}u} \exp\{\Phi(T)\} - R \int_0^T e^{-\int_0^t r_u \mathrm{d}u} \exp\{\Phi(t)\}\, \partial_t \Phi(t)\, \mathrm{d}t.
\tag{3.9}
$$

(2) The pricing of CDS at time t is given by

$$
P_t = I_{(\tau > t)} E\left\{ \int_t^T \big((1 - R)y_v - \kappa\big) e^{-\int_t^v (r_u + y_u)\mathrm{d}u} \mathrm{d}v \,\Big|\, \mathcal{G}_t \right\}.
\tag{3.10}
$$

(3) The fair premium of CDS is given by

$$
\kappa = \frac{-(1 - R)\int_0^T e^{-\int_0^v r_u \mathrm{d}u} \exp\{\Phi(v)\}\, \partial_t \Phi(v)\, \mathrm{d}v}{\int_0^T e^{-\int_0^v r_u \mathrm{d}u} \exp\{\Phi(v)\}\mathrm{d}v}.
\tag{3.11}
$$

Proof. (1) By (3.4), (3.6) and the definition of the defaultable zero-coupon bond, we obtain immediately

$$
\begin{aligned}
B(0, T) &= e^{-\int_0^T r_u \mathrm{d}u} P(\tau > T) + R \int_0^T e^{-\int_0^t r_u \mathrm{d}u}\mathrm{d}P(\tau \leq t) \\
&= e^{-\int_0^T r_u \mathrm{d}u} \exp\{\Phi(T)\} - R \int_0^T e^{-\int_0^t r_u \mathrm{d}u} \exp\{\Phi(t)\}\, \partial_t \Phi(t)\, \mathrm{d}t.
\end{aligned}
$$

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(2) By Definition 3.2, we get

$$
\begin{aligned}
P_t &= E\{p_T(t)|\mathcal{F}_t\} \\
&= E\left\{(1-R)\beta(\tau)\beta^{-1}(t)I_{(t<\tau<T)} - \kappa\int_t^T \beta(v)\beta^{-1}(t)I_{(\tau>v)}\mathrm{d}v\Big|\mathcal{F}_t\right\} \\
&= (1-R)E\left\{e^{-\int_t^\tau r_u\mathrm{d}u}I_{(t<\tau<T)}\Big|\mathcal{F}_t\right\} - \kappa\int_t^T e^{-\int_t^v r_u\mathrm{d}u}E\{I_{(\tau>v)}|\mathcal{F}_t\}\mathrm{d}v \\
&=: P_t' - P_t''.
\end{aligned}
\tag{3.12}
$$

By Theorem 9.23 in McNeil et al. (2005), we have

$$
P_t' = I_{(\tau>t)}(1-R)E\left\{\int_t^T y_v e^{-\int_t^v(r_u+y_u)\mathrm{d}u}\mathrm{d}v\Big|\mathcal{G}_t\right\}.
\tag{3.13}
$$

By Lemma 7.4.1.1 in Jeanblanc et al. (2009) (taking $X\equiv 1$ and $T=v$) and (3.3), for $v\geq t$, we have

$$
\begin{aligned}
E\{I_{(\tau>v)}|\mathcal{F}_t\} &= I_{(\tau>t)}\frac{E\{I_{(\tau>v)}|\mathcal{G}_t\}}{E\{I_{(\tau>t)}|\mathcal{G}_t\}} = I_{(\tau>t)}\frac{E\{I_{(\tau>v)}|\mathcal{G}_t\}}{P(\tau>t|\mathcal{G}_t)} \\
&= I_{(\tau>t)}e^{\int_0^t y_u\mathrm{d}u}E\{E\{I_{(\tau>v)}|\mathcal{G}_v\}|\mathcal{G}_t\} \\
&= I_{(\tau>t)}e^{\int_0^t y_u\mathrm{d}u}E\{P(\tau>v|\mathcal{G}_v)|\mathcal{G}_t\} \\
&= I_{(\tau>t)}e^{\int_0^t y_u\mathrm{d}u}E\{e^{-\int_0^v y_u\mathrm{d}u}|\mathcal{G}_t\} \\
&= I_{(\tau>t)}E\{e^{-\int_t^v y_u\mathrm{d}u}|\mathcal{G}_t\}.
\end{aligned}
\tag{3.14}
$$

Plugging (3.14) into P_t'', we get

$$
P_t'' = I_{(\tau>t)}\,\kappa\,E\left\{\int_t^T e^{-\int_t^v(r_u+y_u)\mathrm{d}u}\mathrm{d}v\Big|\mathcal{G}_t\right\}.
\tag{3.15}
$$

Then (3.10) follows by (3.12), (3.13) and (3.15).

(3) Note that

$$
P(\tau>v) = P(\tau>v|\mathcal{G}_0) = E\left\{e^{-\int_0^v y_u\mathrm{d}u}\Big|\mathcal{G}_0\right\}
$$

and

$$
-\frac{\partial}{\partial v}P(\tau>v) = E\left\{y_v e^{-\int_0^v y_u\mathrm{d}u}\Big|\mathcal{G}_0\right\}.
$$

Then (3.11) follows by setting $P_0 = 0$ in (3.10). The proof is completed.

For the sake of the numerical calculations in next section, we present the following corollary based on the conclusion of Corollary 3.1.

Corollary 3.2. Let $B(0,T)$ be the present value of the defaultable zero-coupon bond at time 0 paying 1 at time T and κ be the fair premium of CDS. Then the following statements hold:

(1) The formula for calculating the value of $B(0,T)$ is given by

$$
B(0,T) = e^{-\int_0^T r_u\mathrm{d}u}\exp\{\Psi(T)\} - R\int_0^T e^{-\int_0^t r_u\mathrm{d}u}\exp\{\Psi(t)\}\,\partial_t\Psi(t)\,\mathrm{d}t.
\tag{3.16}
$$

(2) The fair premium of CDS is given by

$$
\kappa = \frac{-(1-R)\int_0^T e^{-\int_0^v r_u\mathrm{d}u}\exp\{\Psi(v)\}\,\partial_t\Psi(v)\,\mathrm{d}v}{\int_0^T e^{-\int_0^v r_u\mathrm{d}u}\exp\{\Psi(v)\}\mathrm{d}v}.
\tag{3.17}
$$

4   Numerical results and conclusions

In this section, using the conclusions of Corollary 3.2, let us illustrate the price calculations of the defaultable zero-coupon bond and the fair premium of CDS. We also analyse the dynamic relationships between $B(0, T)$, κ and the maturity date T respectively.

Example 4.1. The parameter values used to calculate the pricing of the defaultable zero-coupon bond and the fair premium of CDS are

$$\lambda = 0.1, \ \eta = 0, \ \theta = 0.2, \ \alpha = 15, \ \rho = 1, \ R = 0.4, \ r_t = 0.05.$$

Note that Corollary 3.2 is based on Corollary 3.1. The expressions of $\exp\{\Psi(T)\}$, $\exp\{\Psi(t)\}$, $\partial_t\Psi(t)$, $\exp\{\Psi(v)\}$ and $\partial_t\Psi(v)$ in Corollary 3.2 can be obtained from the conclusion of Corollary 3.1. Therefore, after substituting the above parameter values into $\exp\{\Psi(T)\}$, $\exp\{\Psi(t)\}$, $\partial_t\Psi(t)$, $\exp\{\Psi(v)\}$ and $\partial_t\Psi(v)$, one can obtain the numbers showed in the following Table 4.1 and 4.2 by means of the conclusions of Corollary 3.2 and MATLAB software.

Table 4.1. The dynamic relationship between $B(0, T)$ and T

T	2	4	6	8	10
$B(0, T)$	0.7613	0.5861	0.4733	0.4058	0.3669

Table 4.2. The dynamic relationship between κ and T

T	2	4	6	8	10
κ	0.0950	0.1040	0.1093	0.1123	0.1140

Table 4.1 shows that the dynamic relationship between the pricing of the defaultable zero-coupon bond and the maturity date T. Table 4.2 shows that the dynamic relationship between the fair premium of CDS and the maturity date T.

We can find from Table 4.1 that the price of the defaultable zero-coupon bond is monotonically decreasing function respect to the maturity date T. However, it is indicated from Table 4.2 that the price of the fair premium of CDS is monotonically increasing function respect to the maturity date T. The reason for this monotonically increasing trend is that the ruin probability of firm A increases with prolonged maturity date T.

In this paper, for the sake of simplifying calculation, we assume that the jump sizes are exponentially distributed. It is of interest and challenging to employ other heavy-tailed distributions for the jump sizes, such as Pareto distribution, Gumbel distribution and Fréchet distribution. However, since it is unlikely for us to obtain explicit expressions for the joint Laplace transform of the distribution of the vector process (y_t, Y_t), numerical methods need to be used to calculate the price of the defaultable zero-coupon and the fair premium of CDS.

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