1 PMSM Maximum Torque per Current

The torque of a PMSM is given by:

$$T = \frac{3}{2}p(\psi_p + (L_d - L_q)i_d)i_q.$$

For the case of an SPMSM we have $L_d = L_q$, in case of an IPMSM we have $L_d < L_q$. The given torque equation includes both cases.

To maximize efficiency of the PMSM, the total current $i_s = \sqrt{i_d^2 + i_q^2}$ is to be minimized for a given torque reference T^*. This task can be formulated as an optimization problem with a binding constraint:

$$\begin{aligned}&\min i_s\\&\text{s.t.}\, T = T^*.\end{aligned}$$

It can be solved via the method of Lagrange multipliers:

$$\begin{aligned}\mathcal{L}(i_d, i_q, \lambda) &= i_s^2 + \lambda(T - T^*),\\ &= i_d^2 + i_q^2 + \lambda(\frac{3}{2}p(\psi_p + (L_d - L_q)i_d)i_q - T^*).\end{aligned}$$

$$\begin{aligned}
&& \begin{bmatrix}\frac{\partial}{\partial i_d}\\ \frac{\partial}{\partial i_q}\end{bmatrix}\mathcal{L}(i_d, i_q, \lambda) &= \begin{bmatrix}\frac{\partial}{\partial i_d}\\ \frac{\partial}{\partial i_q}\end{bmatrix}(i_d^2 + i_q^2) + \begin{bmatrix}\frac{\partial}{\partial i_d}\\ \frac{\partial}{\partial i_q}\end{bmatrix}\lambda(\frac{3}{2}p(\psi_p + (L_d - L_q)i_d)i_q - T^*) \overset{!}{=} 0\\
\Leftrightarrow && \begin{bmatrix}2i_d\\ 2i_q\end{bmatrix} &= -\lambda\frac{3}{2}p\begin{bmatrix}(L_d - L_q)i_q\\ \psi_p + (L_d - L_q)i_d\end{bmatrix}\\
\Leftrightarrow && \begin{bmatrix}\frac{2i_d}{(L_d-L_q)i_q}\\ \frac{2i_q}{\psi_p+(L_d-L_q)i_d}\end{bmatrix} &= -\lambda\frac{3}{2}p\begin{bmatrix}1\\ 1\end{bmatrix} && \text{(equate first and second row)}\\
\Leftrightarrow && \frac{2i_d}{(L_d - L_q)i_q} &= \frac{2i_q}{\psi_p + (L_d - L_q)i_d}\\
\Leftrightarrow && \psi_p i_d + (L_d - L_q)i_d^2 &= (L_d - L_q)i_q^2\\
\Leftrightarrow && i_d^2 - i_q^2 + \frac{\psi_p}{L_d - L_q}i_d &= 0
\end{aligned}$$

All operation points that maximize torque per current will satisfy this equation, it is therefore called the MTPC curve (maximum torque per current). In order to determine the maximum achievable torque we assume operation at the nominal current limit $i_s = i_n$.

$$\begin{aligned}&& i_d^2 + i_q^2 &= i_n^2\\ \Leftrightarrow && i_q^2 &= i_n^2 - i_d^2.\end{aligned}$$

$$\begin{aligned}
&& i_d^2 - (i_n^2 - i_d^2) + \frac{\psi_p}{L_d - L_q}i_d &= 0\\
\Leftrightarrow && i_d^2 + \frac{\psi_p}{2(L_d - L_q)}i_d - \frac{i_n^2}{2} &= 0\\
\Leftrightarrow && i_d &= -\frac{\psi_p}{4(L_d - L_q)} \pm \sqrt{\left(\frac{\psi_p}{4(L_d - L_q)}\right)^2 + \frac{i_n^2}{2}}
\end{aligned}$$

Under the assumption of $L_d < L_q$ we can see that the current is minimized if the „minus“ solution for i_d is used. Thus, i_q is given by:

$$i_q = \sqrt{i_n^2 - \left(-\frac{\psi_p}{4(L_d - L_q)} - \sqrt{\left(\frac{\psi_p}{4(L_d - L_q)}\right)^2 + \frac{i_n^2}{2}}\right)^2}$$

Consequently, the maximum achievable torque at nominal current is given by:

$$\begin{aligned}i_d^* &= -\frac{\psi_p}{4(L_d - L_q)} - \sqrt{\left(\frac{\psi_p}{4(L_d - L_q)}\right)^2 + \frac{i_n^2}{2}},\\ i_q^* &= \sqrt{i_n^2 - (i_d^*)^2}.\end{aligned}$$

Analysing these equation for the case of surface mounted magnets ($L_d = L_q$), we get:

$$i_d^* = \lim_{L_d \to L_q} \left(-\frac{\psi_p}{4(L_d - L_q)} - \sqrt{\left(\frac{\psi_p}{4(L_d - L_q)}\right)^2 + \frac{i_n^2}{2}} \right) = 0$$

$$i_q^* = i_n$$

The given optimal currents can then be used to calculate the maximum torque (or nominal torque):

$$T^* = \frac{3}{2} p(\psi_p + (L_d - L_q) i_d^*) i_q^*$$