



NANJING UNIVERSITY

ACM-ICPC Codebook 1
Graph Theory

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Contents

1	Shortest Paths	4
1.1	Single-source shortest paths	4
1.1.1	Dijkstra	4
1.1.2	SPFA	5
1.2	All-pairs shortest paths (Floyd-Warshall)	6
2	Spanning Tree	7
2.1	Minimum spanning tree	7
2.1.1	Kruskal's algorithm	7
2.1.2	Prim's algorithm, adjacency matrix representation	8
2.2	Minimum ratio spanning tree	8
2.3	Manhattan distance minimum spanning tree	9
3	Depth-first Search	12
3.1	Strongly connected components, condensation (Tarjan)	12
3.2	Vertex biconnected components (Tarjan)	13
4	Flow Network	15
4.1	Maximum flow, Dinic's algorithm	15
4.2	MCMF, Ford-Fulkerson with SPFA	17
5	Matching	19
5.1	Maximum cardinality bipartite matching, Hungarian	19

1 Shortest Paths

1.1 Single-source shortest paths

1.1.1 Dijkstra

Dijkstra's algorithm with binary heap.

✗ Can't be performed on graphs with negative weights.

Usage:

<code>V</code>	Number of vertices. PLEASE set this value before use!
<code>add_edge(e)</code>	Add edge e to the graph.
<code>dijkstra(src)</code>	Calculate SSSP from src .
<code>d[x]</code>	distance to x
<code>p[x]</code>	last edge to x in SSSP

Time complexity: $O(E \log V)$

```

1  const int INF = 0x7f7f7f7f;
2  const int MAXV = 10005;
3  const int MAXE = 500005;
4  struct edge{
5      int u, v, w;
6  };
7
8  struct graph{
9      int V;
10     vector<edge> adj[MAXV];
11     int d[MAXV];
12     edge* p[MAXV];
13
14     void add_edge(int u, int v, int w){
15         edge e;
16         e.u = u; e.v = v; e.w = w;
17         adj[u].push_back(e);
18     }
19
20     bool done[MAXV];
21     void dijkstra(int src){
22         typedef pair<int,int> pii;
23         priority_queue<pii, vector<pii>, greater<pii> > q;
24
25         fill(d, d + V + 1, INF);
26         d[src] = 0;
27         fill(done, done + V + 1, false);

```

```

28     q.push(make_pair(0, src));
29     while (!q.empty()){
30         int u = q.top().second; q.pop();
31         if (done[u]) continue;
32         done[u] = true;
33         rep (i, adj[u].size()){
34             edge e = adj[u][i];
35             if (d[e.v] > d[u] + e.w){
36                 d[e.v] = d[u] + e.w;
37                 p[e.v] = &adj[u][i];
38                 q.push(make_pair(d[e.v], e.v));
39             }
40         }
41     }
42 }
43 };

```

1.1.2 SPFA

Shortest path faster algorithm. (Improved version of Bellman-Ford algorithm)

This code is used to replace `void dijkstra(int src)`.

✓ Can be performed on graphs with negative weights.

⚠ For some specially constructed graphs, this algorithm is very slow.

Usage:

`spfa(src)` Calculate SSSP from *src*.

Requirement:

1.1.1 Dijkstra

Time complexity: $O(kE)$, for most graphs, $k < 2$

```

1  // ! This procedure is to replace `dijkstra', and cannot be used alone.
2  bool inq[MAXV];
3  void spfa(int src){
4      queue<int> q;
5      fill(d, d + V + 1, INF);
6      d[src] = 0;
7      fill(inq, inq + V + 1, false);
8      q.push(src); inq[src] = true;
9      while (!q.empty()){
10         int u = q.front(); q.pop(); inq[u] = false;
11         rep (i, adj[u].size()){
12             edge e = adj[u][i];
13             if (d[e.v] > d[u] + e.w){

```

```

14         d[e.v] = d[u] + e.w;
15         p[e.v] = &adj[u][i];
16         if (!inq[e.v])
17             q.push(e.v), inq[e.v] = true;
18     }
19 }
20 }
21 }
```

1.2 All-pairs shortest paths (Floyd-Warshall)

Floyd-Warshall algorithm.

✓ Can be performed on graphs with negative weights. To detect negative cycle, one can inspect the diagonal, and the presence of a negative number indicates that the corresponding vertex lies on some negative cycle.

△ **Self-loops** and **multiple edges** must be specially judged.

△ If the weights of edges might exceed $\text{LLONG_MAX} / 2$, the line (*) should be added.

Usage:

<code>init()</code>	Initialize the distances of the edges from 0 to V.
<code>floyd()</code>	Calculate APSP.
<code>d[i][j]</code>	distance from i to j

Time complexity: $O(V^3)$

```

1  const LL INF = LLONG_MAX / 2;
2  const int MAXV = 1005;
3  int V;
4  LL d[MAXV][MAXV];
5
6  void init(){
7      Rep (i, V){
8          Rep (j, V) d[i][j] = INF;
9          d[i][i] = 0;
10     }
11 }
12
13 void floyd(){
14     Rep (k, V)
15         Rep (i, V)
16             Rep (j, V)
17                 // ! (*) if (d[i][k] < INF && d[k][j] < INF)
18                     d[i][j] = min(d[i][j], d[i][k] + d[k][j]);
19 }
```

2 Spanning Tree

2.1 Minimum spanning tree

2.1.1 Kruskal's algorithm

Usage:

<code>n, m</code>	The number of vertices and edges, resp.
<code>edges[]</code>	Edges of the graph, numbered from 0.
<code>kruskal()</code>	Run Kruskal's algorithm.

Time complexity: $O(E \log E)$

```

1  const int MAXV = 100005;
2  const int MAXE = 300005;
3
4  int n, m;
5  struct edge{
6      int u, v, w;
7      bool operator < (const edge& e) const {
8          return w < e.w;
9      }
10 } edges[MAXE];
11
12 int p[MAXV];
13 void init(int num){
14     for (int i=1; i<=num; i++) p[i] = i;
15 }
16
17 int parent(int x){
18     if (p[x] == x) return x;
19     return p[x] = parent(p[x]);
20 }
21
22 bool unite(int u, int v){
23     u = parent(u); v = parent(v);
24     p[u] = v; return u != v;
25 }
26
27 void kruskal(){
28     init(n);
29     sort(edges, edges + m);
30     int curn = 1;
31     for (int i = 0; curn < n; i++){
32         if (unite(edges[i].u, edges[i].v)){
33             // choose the i-th edge

```

```

34         currn++;
35     }
36 }
37 }

```

2.1.2 Prim's algorithm, adjacency matrix representation

Calculate minimum spanning tree. The result is represented as a tree rootes at `src`.

Usage:

`adj[i][j]` Adjacency matrix, indexed from 1.
`prim(src)` Run Prim's algoirhtm from `src`.

Time complexity: $O(V^2)$

```

1  const int MAXN = 108;
2
3  int n;
4  LL adj[MAXN][MAXN]; // indexed from 1
5
6  int prev[MAXN]; // note that, prev[src] = 0
7  bool done[MAXN];
8  LL key[MAXN]; // key[v] = adj[prev[v]][v] for v != src when done
9  void prim(int src){
10     Rep (i, n)
11         key[i] = LLONG_MAX, done[i] = false;
12     key[src] = 0, prev[src] = 0;
13     rep (cnt, n){
14         LL u, k = LLONG_MAX;
15         Rep (i, n)
16             if (!done[i] && key[i] < k)
17                 u = i, k = key[i];
18         done[u] = true;
19         Rep (v, n)
20             if (!done[v] && adj[u][v] < key[v])
21                 prev[v] = u, key[v] = adj[u][v];
22     }
23 }

```

2.2 Minimum ratio spanning tree

Minimize $\frac{\sum_{e \in ST} cost[e]}{\sum_{e \in ST} dist[e]}$ where ST is a spanning tree.

Usage:

First, build the edges of the graph as the structure shows; then, implement a usual MST algorithm; finally, call `solve()` to get the answer.

```

1  double k;
2  struct edge{
3      int u, v;
4      double cost, dist;
5      double w(return cost - dist * w);
6      bool operator < (const edge& rhs) const {
7          return w() < rhs.w();
8      }
9  };
10
11 double mst(){
12     // return sum(dist[e])/sum(cost[e]) for all e in mst
13 }
14
15 double solve(){
16     k = 1e5; // initial k estimate
17     double nxt;
18     while (fabs((nxt = mst()) - k)) > 1e-8){ // admissible error
19         k = nxt;
20     }
21     return k;
22 }

```

2.3 Manhattan distance minimum spanning tree

Usgae:

`add_point(x, y)` Add point (x, y) .
`Manhattan_MST()` Calculate Manhattan distance minimum spanning tree.

Time complexity: $O(n \log n)$, but constant factor may be large.

```

1  int V = 0;
2  struct pt{int id, x, y;};
3  typedef vector<pt>::iterator vit;
4  vector<pt> pts;
5
6  struct edge{
7      int u, v, w;
8      bool operator < (const edge& e) const {return w < e.w;}
9  };
10 vector<edge> edges;
11
12 struct BIT{

```

```

13 inline int lowbit(int x) {return x&-x;}
14 int N;
15 vector<int> tr;
16 vector<int> minv;
17
18 BIT(int n){
19     tr.resize(N = n + 5);
20     minv.resize(N);
21     fill(range(tr), INT_MAX);
22 }
23
24 int prefmin(int n, int& x){
25     LL ans = INT_MAX;
26     int v = 0;
27     while (n){
28         if (tr[n] < ans) ans = tr[n]; v = minv[n];
29         n -= lowbit(n);
30     }
31     x = ans;
32     return v;
33 }
34
35 void insert(int n, int v, int x){
36     while (n < N){
37         if (tr[n] > x) tr[n] = x, minv[n] = v;
38         n += lowbit(n);
39     }
40 }
41 };
42
43 struct CMP{
44     inline bool operator()(const pt& lhs, const pt& rhs){
45         if (lhs.x == rhs.x) return lhs.y > rhs.y;
46         return lhs.x > rhs.x;
47     }
48 } cmp;
49
50 const int DIFF = 1020; // ! DIFF > max(x_i, y_i); discretize when necessary
51 void make_edge(){
52     sort(range(pts), cmp);
53     BIT bit(DIFF * 2);
54     for (vit it = pts.begin(); it != pts.end(); it++){
55         int vxy;
56         int v = bit.prefmin(it->x - it->y + DIFF, vxy);
57         if (v) edges.push_back(edge{it->id, v, vxy - it->x - it->y});
58         bit.insert(it->x - it->y + DIFF, it->id, it->x + it->y);
59     }

```

```

60 }
61
62 struct UFS{
63     int p[10005];
64     void init(int num){
65         for (int i=1; i<=num; i++) p[i] = i;
66     }
67
68     int parent(int x){
69         if (p[x] == x) return x;
70         return p[x] = parent(p[x]);
71     }
72
73     bool unite(int u, int v){
74         u = parent(u); v = parent(v);
75         p[u] = v; return u != v;
76     }
77 } ufs;
78
79 void kruskal(){
80     ufs.init(V);
81     sort(range(edges));
82     int curn = 1;
83     for (int i = 0; curn < V; i++){
84         if (ufs.unite(edges[i].u, edges[i].v)){
85             // choose the i-th edge
86             curn++;
87         }
88     }
89 }
90
91 inline void add_point(int x, int y){pts.push_back(pt {++V, x, y});}
92
93 void Manhattan_MST(){
94     make_edge();
95     for (vit it = pts.begin(); it != pts.end(); it++) swap(it->x, it->y);
96     make_edge();
97     for (vit it = pts.begin(); it != pts.end(); it++) it->x = -it->x;
98     make_edge();
99     for (vit it = pts.begin(); it != pts.end(); it++) swap(it->x, it->y);
100    make_edge();
101    // restore original coordinates
102    // for (vit it = pts.begin(); it != pts.end(); it++) it->y = -it->y;
103    kruskal();
104 }

```

3 Depth-first Search

3.1 Strongly connected components, condensation (Tarjan)

Find strongly connected components and compute the component graph.

△ The component graph may contain **multiple edges**.

Usage:

<code>V</code>	number of vertices
<code>scc[i]</code>	the SCC that i belongs to, numbered from 1.
<code>sccn</code>	number of SCCs
<code>find_scc()</code>	Find all SCCs.
<code>contract()</code>	Compute component graph.

Time complexity: $O(V + E)$

```

1  const int MAXV = 100005;
2
3  struct graph{
4      vector<int> adj[MAXV];
5      stack<int> s;
6      int V; // number of vertices
7      int pre[MAXV], lnk[MAXV], scc[MAXV];
8      int time, sccn;
9
10     void add_edge(int u, int v){
11         adj[u].push_back(v);
12     }
13
14     void dfs(int u){
15         pre[u] = lnk[u] = ++time;
16         s.push(u);
17         rep (i, adj[u].size()){
18             int v = adj[u][i];
19             if (!pre[v]){
20                 dfs(v);
21                 lnk[u] = min(lnk[u], lnk[v]);
22             } else if (!scc[v]){
23                 lnk[u] = min(lnk[u], pre[v]);
24             }
25         }
26         if (lnk[u] == pre[u]){
27             sccn++;
28             int x;
29             do {
30                 x = s.top(); s.pop();

```

```

31         scc[x] = sccn;
32     } while (x != u);
33 }
34 }
35
36 void find_scc(){
37     time = sccn = 0;
38     memset(scc, 0, sizeof(scc));
39     memset(pre, 0, sizeof(pre));
40     Rep (i, V){
41         if (!pre[i]) dfs(i);
42     }
43 }
44
45 vector<int> adjc[MAXV];
46 void contract(){
47     Rep (i, V)
48         rep (j, adj[i].size()){
49             if (scc[i] != scc[adj[i][j]])
50                 adjc[scc[i]].push_back(scc[adj[i][j]]);
51         }
52 }
53 };

```

3.2 Vertex biconnected components (Tarjan)

Find all vertex biconnected components (blocks).

Usage:

<code>add_edge(u, v)</code>	Add edge (u, v) to the graph. Vertices are numbered from 0.
<code>find_bcc(n)</code>	Find all BCCs in the graph containing n vertices $[0, n)$.
<code>bcc_cnt</code>	number of BCCs
<code>bccno[]</code>	the id of BCC a vertex belongs to. The id is numbering from 0. If the vertex is a cut vertex, the value is undefined.
<code>iscut[]</code>	whether a vertex is a cut vertex
<code>bcc[]</code>	the vertex set of a BCC
<code>bcce[]</code>	the edge set of a BCC

Time complexity: $O(V + E)$

```

1  const int MAXN = 100005;
2  struct graph {
3      int pre[MAXN], iscut[MAXN], bccno[MAXN], dfs_clock, bcc_cnt;
4      vector<int> adj[MAXN], bcc[MAXN];
5      set<pair<int, int>> bcce[MAXN];

```

```

6
7     stack<pair<int, int>> s;
8
9     void add_edge(int u, int v) {
10         adj[u].push_back(v);
11         adj[v].push_back(u);
12     }
13
14     int dfs(int u, int fa) {
15         int lowu = pre[u] = ++dfs_clock;
16         int child = 0;
17         for (int v : adj[u]) {
18             if (!pre[v]) {
19                 s.push({u, v});
20                 child++;
21                 int lowv = dfs(v, u);
22                 lowu = min(lowu, lowv);
23                 if (lowv >= pre[u]) {
24                     iscut[u] = 1;
25                     bcc[bcc_cnt].clear();
26                     bcce[bcc_cnt].clear();
27                     while (1) {
28                         int xu, xv;
29                         tie(xu, xv) = s.top(); s.pop();
30                         bcce[bcc_cnt].insert({min(xu, xv), max(xu, xv)});
31                         if (bccno[xu] != bcc_cnt) {
32                             bcc[bcc_cnt].push_back(xu);
33                             bccno[xu] = bcc_cnt;
34                         }
35                         if (bccno[xv] != bcc_cnt) {
36                             bcc[bcc_cnt].push_back(xv);
37                             bccno[xv] = bcc_cnt;
38                         }
39                         if (xu == u && xv == v) break;
40                     }
41                     bcc_cnt++;
42                 }
43             } else if (pre[v] < pre[u] && v != fa) {
44                 s.push({u, v});
45                 lowu = min(lowu, pre[v]);
46             }
47         }
48         if (fa < 0 && child == 1) iscut[u] = 0;
49         return lowu;
50     }
51
52     void find_bcc(int n) {

```

```

53     memset(pre, 0, sizeof pre);
54     memset(iscut, 0, sizeof iscut);
55     memset(bccno, -1, sizeof bccno);
56     dfs_clock = bcc_cnt = 0;
57     rep (i, n) if (!pre[i]) dfs(i, -1);
58 }
59 };

```

4 Flow Network

4.1 Maximum flow, Dinic's algorithm

✓ Can be performed on networks with parallel and antiparallel edges.

Usage:

add_edge(*u*, *v*, *c*) Add an edge from *u* to *v* with capacity *c*.
max_flow(*s*, *t*) Compute maximum flow from *s* to *t*.

Time complexity: For general graph, $O(V^2E)$; for network with unit capacities, $O(\min\{V^{2/3}, E^{1/2}\}E)$; for bipartite network, $O(\sqrt{VE})$.

```

1 struct edge{
2     int from, to;
3     LL cap, flow;
4 };
5
6 const int MAXN = 1005;
7 struct Dinic {
8     int n, m, s, t;
9     vector<edge> edges;
10    vector<int> G[MAXN];
11    bool vis[MAXN];
12    int d[MAXN];
13    int cur[MAXN];
14
15    void add_edge(int from, int to, LL cap) {
16        edges.push_back(edge{from, to, cap, 0});
17        edges.push_back(edge{to, from, 0, 0});
18        m = edges.size();
19        G[from].push_back(m-2);
20        G[to].push_back(m-1);
21    }
22
23    bool bfs() {

```

```

24     memset(vis, 0, sizeof(vis));
25     queue<int> q;
26     q.push(s);
27     vis[s] = 1;
28     d[s] = 0;
29     while (!q.empty()) {
30         int x = q.front(); q.pop();
31         for (int i = 0; i < G[x].size(); i++) {
32             edge& e = edges[G[x][i]];
33             if (!vis[e.to] && e.cap > e.flow) {
34                 vis[e.to] = 1;
35                 d[e.to] = d[x] + 1;
36                 q.push(e.to);
37             }
38         }
39     }
40     return vis[t];
41 }
42
43 LL dfs(int x, LL a) {
44     if (x == t || a == 0) return a;
45     LL flow = 0, f;
46     for (int& i = cur[x]; i < G[x].size(); i++) {
47         edge& e = edges[G[x][i]];
48         if (d[x] + 1 == d[e.to] && (f = dfs(e.to, min(a, e.cap - e.flow))) > 0)
49             {
50                 e.flow += f;
51                 edges[G[x][i]^1].flow -= f;
52                 flow += f;
53                 a -= f;
54                 if (a == 0) break;
55             }
56     }
57     return flow;
58 }
59
60 LL max_flow(int s, int t) {
61     this->s = s; this->t = t;
62     LL flow = 0;
63     while (bfs()) {
64         memset(cur, 0, sizeof(cur));
65         flow += dfs(s, LLONG_MAX);
66     }
67     return flow;
68 }
69
70 vector<int> min_cut() { // call this after maxflow

```



```

70     vector<int> ans;
71     for (int i = 0; i < edges.size(); i++) {
72         edge& e = edges[i];
73         if(vis[e.from] && !vis[e.to] && e.cap > 0) ans.push_back(i);
74     }
75     return ans;
76 }
77 };

```

4.2 MCMF, Ford-Fulkerson with SPFA

✓ Can be performed on networks with parallel and antiparallel edges.

Usage:

<code>add_edge(u, v, c)</code>	Add an edge from u to v with capacity c .
<code>min_cost(s, t, &cost)</code>	Compute MCMF from s to t . Return the maximum flow, and set <code>cost</code> as the minimum cost.
<code>min_cost(s, t, f, &cost)</code>	Compute minimum cost flow from s to t with capacity f . Return whether such flow exists, and set <code>cost</code> as the minimum cost, if exists.

Time complexity: $O(|f|kE)$.

```

1 struct edge{
2     int from, to;
3     int cap, flow;
4     LL cost;
5 };
6
7 const LL INF = LLONG_MAX / 2;
8 const int MAXN = 5005;
9 struct MCMF {
10     int s, t, n, m;
11     vector<edge> edges;
12     vector<int> G[MAXN];
13     bool inq[MAXN]; // queue
14     LL d[MAXN];    // distance
15     int p[MAXN];   // previous
16     int a[MAXN];   // improvement
17
18     void add_edge(int from, int to, int cap, LL cost) {
19         edges.push_back(edge{from, to, cap, 0, cost});
20         edges.push_back(edge{to, from, 0, 0, -cost});
21         m = edges.size();
22         G[from].push_back(m-2);

```

```

23     G[to].push_back(m-1);
24 }
25
26 bool spfa(){
27     queue<int> q;
28     fill(d, d + MAXN, INF); d[s] = 0;
29     memset(inq, 0, sizeof(inq));
30     q.push(s); inq[s] = true;
31     p[s] = 0; a[s] = INT_MAX;
32     while (!q.empty()){
33         int u = q.front(); q.pop(); inq[u] = false;
34         rep (i, G[u].size()){
35             edge& e = edges[G[u][i]];
36             if (e.cap > e.flow && d[e.to] > d[u] + e.cost){
37                 d[e.to] = d[u] + e.cost;
38                 p[e.to] = G[u][i];
39                 a[e.to] = min(a[u], e.cap - e.flow);
40                 if (!inq[e.to]) q.push(e.to), inq[e.to] = true;
41             }
42         }
43     }
44     return d[t] != INF;
45 }
46
47 void augment(){
48     int u = t;
49     while (u != s){
50         edges[p[u]].flow += a[t];
51         edges[p[u]^1].flow -= a[t];
52         u = edges[p[u]].from;
53     }
54 }
55
56 #ifdef GIVEN_FLOW
57 bool min_cost(int s, int t, int f, LL& cost) {
58     this->s = s; this->t = t;
59     int flow = 0;
60     cost = 0;
61     while (spfa()) {
62         augment();
63         if (flow + a[t] >= f){
64             cost += (f - flow) * a[t]; flow = f;
65             return true;
66         } else {
67             flow += a[t]; cost += a[t] * d[t];
68         }
69     }

```

```

70         return false;
71     }
72 #else
73     int min_cost(int s, int t, LL& cost) {
74         this->s = s; this->t = t;
75         int flow = 0;
76         cost = 0;
77         while (spfa()) {
78             augment();
79             flow += a[t]; cost += a[t] * d[t];
80         }
81         return flow;
82     }
83 #endif
84 };

```

5 Matching

5.1 Maximum cardinality bipartite matching, Hungarian

Usage:

<code>init(nx, ny)</code>	Initialize the algorithm with two parts containing <code>nx</code> , <code>ny</code> vertices, respectively. The vertices are numbered from 0.
<code>add(a, b)</code>	Add an edge from <i>a</i> to <i>b</i> .
<code>match()</code>	Compute and return maximum cardinality bipartite matching..
<code>mx[], my[]</code>	The index of the other vertex if matched; otherwise -1 .

Time complexity: $O(VE)$.

```

1 #include <bits/stdc++.h>
2 using namespace std;
3
4 #define rep(i, n) for (int i = 0; i < (n); i++)
5 #define Rep(i, n) for (int i = 1; i <= (n); i++)
6 #define range(x) (x).begin(), (x).end()
7 typedef long long LL;
8
9 struct Hungarian{
10     int nx, ny;
11     vector<int> mx, my;
12     vector<vector<int> > e;
13     vector<bool> mark;
14

```

```
15 void init(int nx, int ny){
16     this->nx = nx;
17     this->ny = ny;
18     mx.resize(nx); my.resize(ny);
19     e.clear(); e.resize(nx);
20     mark.resize(nx);
21 }
22
23 inline void add(int a, int b){
24     e[a].push_back(b);
25 }
26
27 bool augment(int i){
28     if (!mark[i]) {
29         mark[i] = true;
30         for (int j : e[i]){
31             if (my[j] == -1 || augment(my[j])){
32                 mx[i] = j; my[j] = i;
33                 return true;
34             }
35         }
36     }
37     return false;
38 }
39
40 int match(){
41     int ret = 0;
42     fill(range(mx), -1);
43     fill(range(my), -1);
44     rep (i, nx){
45         fill(range(mark), false);
46         if (augment(i)) ret++;
47     }
48     return ret;
49 }
50 };
```