MODULE 7

LIMITING DISTRIBUTIONS

PROBLEMS

1. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. $N(\mu, \sigma^2)$ random variables, where $\mu > 0$ and $0 < \sigma^2 < \infty$. Let $Z_n = \sum_{i=1}^{n} X_i$ and let $M_n = \sqrt{n}(\bar{X}_n - \mu)$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$. Show that the sequence $\{Z_n\}_{n\geq 1}$ does not have a limiting distribution, however, the sequence $\{M_n\}_{n\geq 1}$ has a limiting distribution.

2. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables. Let $X_{1:n} = \min\{X_1, \ldots, X_n\}$ and let $Y_n = nX_{1:n}, n = 1, 2, \ldots$. Find the limiting distribution of $\{X_{1:n}\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ when
   (i) $X_1 \sim U(0, \theta), \theta > 0$
   (ii) $X_1 \sim \mathrm{Exp}(\theta), \theta > 0$.

3. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with mean μ and finite variance. Show that
   (i) $\frac{2}{n(n+1)}\sum_{i=1}^{n} i\, X_i \xrightarrow{p} \mu$, as $n \to \infty$;
   (ii) $\frac{6}{n(n+1)(2n+1)}\sum_{i=1}^{n} i^2\, X_i \xrightarrow{p} \mu$, as $n \to \infty$.

4. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables such that the p.m.f. of X_n is given by
$$f_n(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in \left\{-n^{\frac{1}{4}}, n^{\frac{1}{4}}\right\} \\ 0, & \text{otherwise} \end{cases}.$$
   Show that $\bar{X}_n \xrightarrow{p} 0$, as $n \to \infty$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$.

5. Let $X_n \sim NB(n, p_n)$, where $p_n \in (0,1), n = 1, 2, \ldots$ and $\lim_{n\to\infty} n(1 - p_n) = \lambda > 0$. Show that $X_n \xrightarrow{d} X \sim \mathrm{P}(\lambda)$, the Poisson distribution with mean λ.

6. (i) Let $X_n \sim \mathrm{G}\left(n, \frac{1}{n}\right), n = 1, 2, \ldots$. Show that $X_n \xrightarrow{p} 1$, as $n \to \infty$.

   (ii) Let $X_n \sim N\left(\frac{1}{n}, 1 - \frac{1}{n}\right), n = 1, 2, \ldots$. Show that $X_n \xrightarrow{d} Z \sim N(0,1)$, as $n \to \infty$.

7. Consider a random sample of size 80 from the distribution having a p.d.f.
$$f(x) = \begin{cases} \frac{2}{x^3}, & \text{if } x > 1 \\ 0, & \text{otherwise} \end{cases}.$$
Compute, approximately, the probability that not more than 20 of the items of the random sample are greater than $\sqrt{6}$.

8. Let $X_1, X_2, \dots, X_{200}$ be a random sample from $\mathrm{P}(2)$ distribution, and let $Y_{200} = \sum_{i=1}^{200} X_i$. Find, approximately, $P(\{420 \le Y_{200} \le 440\})$.

9. Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables having a common p.d.f.
$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, -\infty < x < \infty.$$
Using the principle of mathematical induction, show that $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \stackrel{d}{=} X_1, \forall n \in \{1, 2, \dots\}$. Hence show that $\{\bar{X}_n\}_{n\ge1}$ does not converge to anything in probability (Note that $E(X_1)$ is not finite and therefore validity of WLLN is not guaranteed).

10. Let $X_n \sim \mathrm{P}(2n), Y_n = \frac{X_n}{n}$ and $Z_n = \frac{{X_n}^2}{n(2n+1)}, n = 1, 2, \dots$.Show that
    (i) $Y_n \xrightarrow{p} 2$ and $Z_n \xrightarrow{p} 1$, as $n \to \infty$;
    (ii) $Y_n^2 + \sqrt{Z_n} \xrightarrow{p} 5$, as $n \to \infty$;
    (iii) $\frac{n^2 Y_n^2 + nY_n}{nY_n + n^2} \xrightarrow{p} 4$, as $n \to \infty$.

11. Let $\bar{X}_n$ be the sample mean based on a random sample of size n from a distribution having mean $\mu \in (-\infty, \infty)$ and variance $\sigma^2 \in (0, \infty)$. Let $Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}, n = 1, 2, \dots$. If $\{Y_n\}_{n\ge1}$ is a sequence of random variables such that $Y_n \xrightarrow{p} 2$, as $n \to \infty$, show that:

    (i) $\frac{2Z_n}{Y_n} \xrightarrow{d} Z \sim N(0,1)$, as $n \to \infty$;
    (ii) $\frac{4Z_n^2}{Y_n^2} \xrightarrow{d} U \sim \chi_1^2$, as $n \to \infty$;
    (iii) $\frac{(2n + Y_n)Z_n}{nY_n + Y_n^2} \xrightarrow{d} Z \sim N(0,1)$, as $n \to \infty$.

12. Let $X_1, X_2, \dots$ be a sequence of i.i.d. $U(0,1)$ random variables. Let $G_n = (X_1 X_2 \cdots X_n)^{\frac{1}{n}}, n = 1, 2, \dots$ be the sequence of geometric means. Show that, as $n \to \infty$,

(i) $G_n \xrightarrow{p} \frac{1}{e}$;

(ii) $n^b\left(G_n^2 - \frac{1}{e^2}\right) \xrightarrow{d} N(0, \sigma^2)$, for some $b > 0$ and $\sigma^2 > 0$. Find the values of b and σ^2.

13. Let $\{(X_{1n},\ X_{2n})\}_{n\geq 1}$ be a sequence of i.i.d. bivariate random vectors such that $E(X_{11}) = \mu_1 \in \mathbb{R}, E(X_{21}) = \mu_2 \in \mathbb{R}, \mathrm{Var}(X_{11}) = \sigma_1^2 > 0$, $\mathrm{Var}(X_{21}) = \sigma_2^2 > 0$, and $\mathrm{Corr}(X_{11}, X_{21}) = \rho \in (-1, 1)$. Let $\bar{X}_{1n} = \frac{1}{n}\sum_{i=1}^{n} X_{1i}, \bar{X}_{2n} = \frac{1}{n}\sum_{i=1}^{n} X_{2i}$, $C_n = \frac{1}{n-1}\sum_{i=1}^{n}(X_{1i} - \bar{X}_{1n})(X_{2i} - \bar{X}_{2n}), S_{1n}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_{1i} - \bar{X}_{1n})^2, S_{2n}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_{2i} - \bar{X}_{2n})^2$ and $R_n = \frac{C_n}{S_{1n}S_{2n}}, n = 2, 3, \ldots$. Show that, as $n \to \infty$,

   (i) $C_n \xrightarrow{p} \rho\sigma_1\sigma_2$ and $R_n \xrightarrow{p} \rho$;

   (ii) $\sqrt{n}(C_n - \rho\sigma_1\sigma_2) \xrightarrow{d} N(0, (\theta - \rho^2)\sigma_1^2\sigma_2^2)$,

   where $$\theta = \frac{E\left((X_{11}-\mu_1)^2(X_{21}-\mu_2)^2\right)}{\sigma_1^2\sigma_2^2}.$$

14.

   (i) Let $X_n \sim \mathrm{Bin}(n, p_n)$, where $p_n \in (0,1), n = 1, 2, \ldots$ and $\lim_{n\to\infty} p_n = p \in (0,1)$. Show that
   $$\frac{X_n - np_n}{\sqrt{np_n(1-p_n)}} \xrightarrow{d} Z \sim N(0,1), \text{as } n \to \infty;$$

   (ii) Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables of absolutely continuous type. Let $F(\cdot)$ and $f(\cdot)$, respectively, denote the d.f. and the p.d.f. of X_1 and let θ be the median of F $\left(\text{i.e.}, F(\theta) = \frac{1}{2}\right)$. Suppose that $f(\theta) > 0$. Let $M_n = X_{n+1:2n+1}, n = 1, 2, \ldots$, be the middle observation (called the sample median) based on random sample $X_1, X_2, \ldots, X_{2n+1}$. Show that, as $n \to \infty$,

   (a) $\sqrt{n}(M_n - \theta) \xrightarrow{d} N\left(0,\ \frac{1}{4f^2(\theta)}\right)$;

   (b) $M_n \xrightarrow{p} \theta$.

15. Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that, for real constants μ and $\sigma > 0, \sqrt{n}(X_n - \mu) \xrightarrow{d} N(0, \sigma^2)$, as $n \to \infty$. Find the limiting distributions of

   (i) $S_n = \sqrt{n}(X_n^2 - \mu^2), n = 1, 2, \ldots$,

   (ii) $T_n = n(X_n - \mu)^2, n = 1, 2, \ldots$,

   (iii) $U_n = \sqrt{n}(\ln X_n - \ln \mu), n = 1, 2, \ldots$, where $\mu > 0$.

16. Let $X_n \sim \text{Bin}(n, p), n = 1, 2, \ldots$. Find the limiting distribution of $Z_n = \sqrt{n}\left(\frac{X}{n}\left(1 - \frac{X}{n}\right) - p(1 - p)\right), n = 1, 2, \ldots$. Find the limiting distribution (non degenerate) of a normalized version of $Y_n = \frac{X}{n}\left(1 - \frac{X}{n}\right)$ when $p = \frac{1}{2}$.