Proposition: Let $T: V \to W$ be a linear transformation and let $\{v_1, \ldots, v_n\}$ be a spanning set of V. Then $\operatorname{span}\{Tv_1, \ldots, Tv_n\} = R(T)$.

Proof: Let $w \in R(T) \implies \exists\, v \in V$ s.t. $Tv = w$

Since $\{v_1, \ldots, v_n\}$ is a spanning set of V, $\exists\, a_1, \ldots, a_n \in \mathbb{R}$

$$v = a_1 v_1 + \cdots + a_n v_n.$$

$$w = Tv = T(a_1 v_1 + \cdots + a_n v_n) = a_1 Tv_1 + \cdots + a_n Tv_n$$

$$\in \operatorname{span}\{Tv_1, \ldots, Tv_n\}$$

Hence $\quad R(T) \subseteq \operatorname{span}\{Tv_1, \dots, Tv_n\} \subseteq R(T)$

$$\Rightarrow \quad R(T) = \operatorname{span}\{Tv_1, \dots, Tv_n\} \quad \text{———} \quad \blacksquare$$

Proposition: Let $T: V \longrightarrow W$ be an injective linear transformation. Suppose $\{v_1, \dots, v_n\}$ be a linearly independent set in V. Then the set $\{Tv_1, \dots, Tv_n\}$ is linearly independent.

Proof: Let $\quad a_1 Tv_1 + \dots + a_n Tv_n = 0$

$$\Rightarrow \quad T(a_1 v_1 + \dots + a_n v_n) = 0$$

$$\Rightarrow \quad a_1 v_1 + \cdots + a_n v_n \in \mathrm{Null}(T) = \{0\}$$

$$\Rightarrow \quad a_1 v_1 + \cdots + a_n v_n = 0$$

$\Rightarrow \quad a_i = 0 \ \forall i$ since $\{v_1, \ldots, v_n\}$ is linearly ind.

$\Rightarrow \quad \{Tv_1, \ldots, Tv_n\}$ is linearly independent.

<u>Theorem:</u> Let V & W be finite dimensional vector spaces and let $T : V \to W$ be a bijective linear transformation. Then $\dim(V) = \dim(W)$.

<u>Proof:</u> Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V

Since β is a spanning set, by the proposition above,

$\{Tv_1, \ldots, Tv_n\}$ spans $R(T)$.

Since T is surjective, we have $R(T) = W$.

$\Rightarrow$ $\{Tv_1, \ldots, Tv_n\}$ spans W.

By the proposition above, since T is injective & β is linearly independent, we have $\{Tv_1, \ldots, Tv_n\}$ is linearly independent & hence a basis.

Therefore $\dim(W) = n = \dim(V)$.

Conversely, if $\{Tv_1, \ldots, Tv_n\}$ is a basis of W, then $\{v_1, \ldots, v_n\}$ is a basis of V.

Theorem: Let V be a finite dimensional vector space and let $\{v_1, \ldots, v_n\}$ be a basis of V. Let $\{w_1, \ldots, w_n\}$ be a subset of W (a vector space). Then there exists a unique linear transformation $T: V \longrightarrow W$ s.t $Tv_j = w_j$ for $j = 1, \ldots, n$.

Uniqueness

Proof: If S and T be two linear transformations

s.t $\quad Sv_j = w_j = Tv_j \quad$ for $j = 1, 2, \ldots, n$

Let $v \in V. \Rightarrow \exists\ a_1, \ldots, a_n$ s.t

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Then $\quad Sv = S(a_1 v_1 + \cdots + a_n v_n) = a_1 S v_1 + \cdots + a_n S v_n$

$$= a_1 w_1 + \cdots + a_n w_n$$

|||ly $\quad Tv = a_1 T v_1 + \cdots + a_n T v_n$

$$= a_1 w_1 + \cdots + a_n w_n = Sv$$

Existence:

Let $v \in V$ and $a_1, \ldots, a_n \in \mathbb{R}$ be s.t

$$v = a_1 v_1 + \cdots + a_n v_n.$$

We define $Tv := a_1 w_1 + \cdots + a_n w_n$.

Well-definedness follows from the fact that every vector of V can be written as a unique linear combination of basis vectors.

Let $v, v' \in V$ and $v = a_1 v_1 + \cdots + a_n v_n$ &
$v' = b_1 v_1 + \cdots + b_n v_n$

Goal: $T(v+v') = Tv + Tv'$

$$v+v' = (a_1+b_1)v_1 + \cdots + (a_n+b_n)v_n.$$

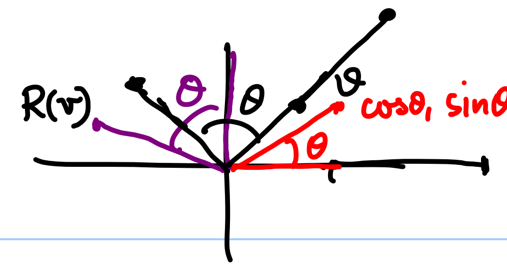
$$\Rightarrow T(v+v') = (a_1+b_1)w_1 + \cdots + (a_n+b_n)w_n.$$

$$Tv = a_1w_1 + \cdots + a_nw_n$$

$|||^{ly}$ $Tv' = b_1w_1 + \cdots + b_nw_n$

$$\therefore Tv + Tv' = (a_1+b_1)w_1 + \cdots + (a_n+b_n)w_n = T(v+v')$$

Check that $T(cv) = cTv \ \forall\, v \in V \ \& \ c \in \mathbb{R}$. —— ∎



Example: We know that the rotation by θ maps $(1,0)$ to $(\cos\theta, \sin\theta)$ &

$(0,1) \longrightarrow (-\sin\theta, \cos\theta)$.

Our rotation can be defined

$$R(x,y) = x R(1,0) + y R(0,1)$$

$$= (x\cos\theta - y\sin\theta, \; x\sin\theta + y\cos\theta)$$

$$R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$