MODULE 4

SOME SPECIAL DISCRETE DISTRIBUTIONS AND THEIR PROPERTIES

PROBLEMS

1. Three coins C_1, C_2 and C_3 have probabilities of coming up of a head as $1/2, 1/3,$ and $1/4,$ respectively. A player chooses one of the three coins in such a way that the probability of choosing coin C_1 is $1/3$ and that of choosing coins C_2 and C_3 are $1/4$ and $5/12$, respectively. He flips the chosen coin 4 times. Find the probability of getting: (i) no head; (ii) at least one head; (iii) exactly three heads.

2. Let $X \sim \text{Bin}(n, p)$ and let $k \in \{1, 2, \dots, n\}$. Show that
$$P(X \geq k) = k \binom{n}{k} \int_0^p t^{k-1} (1-t)^{n-k} dt.$$
Hence show that
$$P(X \geq k) \leq \binom{n}{k} p^k.$$

3. The probability of hitting a target, in each shot, is 0.001. Find the approximate probability of hitting the target at least twice in 5000 shots.

4. Let $X \sim \text{Bin}\ (n, p)$, where n is a positive integer and $p \in (0,1)$. Find the mode of the distribution of X.

5. Twenty distinguishable balls are placed at random in six boxes that are labeled as $B_1, \dots, B_6$. Find the probability that boxes with labels B_1, B_2 and B_3 all together contain six balls.

6. Each child in a family is equally likely to be a boy or a girl. Find the minimum number of children the family should have so that the probability of it having at least a boy and at least a girl is at least 0.90.

7. Let n be a positive integer, $r \in \{1,2, \dots, n\}$ and let $p \in (0,1)$. Using probabilistic arguments and also otherwise show that

$$\sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k} = p^r \sum_{k=0}^{n-r} \binom{r+k-1}{k} (1-p)^k.$$

8. A salesman has fishes stored in two boxes B_1 and B_2, each containing N fishes. Each time the salesman needs to sell a fish to a buyer he selects one of the two boxes at random and takes out a fish from the selected box for selling. At some moment the salesman discovers that the box B_1 is empty. Find the probability that at that moment the box B_2 contains exactly k fishes, where $k \in \{0,1,\dots,n\}$.

9. Let $X \sim \mathrm{P}(\lambda)$, where $\lambda > 0$. Find the mode of the distribution of X.

10. There are 180 applicants for a job, out of which only 120 applicants are qualified for the job. Six applicants are selected at random from these 180 applicants. Find the probability that, among the selected candidates, at least two will be qualified for the job. Using an appropriate approximation, find an approximate value of this probability.

11. (i) If $X \sim \mathrm{P}(\lambda)$, find $E\left((2+X)^{-1}\right)$;
    (ii) If $X \sim \mathrm{Ge}(p)$ and r is a positive integer, find $E(\min(X, r))$.

12. (i) If the m.g.f. of a r.v. X is $M_X(t) = (1/3 + 2e^t/3)^5, -\infty, < t < \infty,$ find $P(X \in \{1,3\})$;
    (ii) If the m.g.f. of a r.v. X is $M_X(t) = e^{4(e^t-1)}$, $-\infty < t < \infty$, find $P\left(E(X) - 2\sqrt{\mathrm{Var}(X)} < X < E(X) + 2\sqrt{\mathrm{Var}(X)}\right)$.

13. **(Poisson Approximation to Negative Binomial Distribution)** Show that

$$\lim_{r\to\infty} \binom{r+k-1}{r-1} p_r^r (1-p_r)^k = \frac{e^{-\lambda}\lambda^k}{k!}, \qquad k = 0,1,2,\dots,$$

provided $\lim_{r\to\infty} p_r = 1$ and $\lim_{r\to\infty}\left(r(1-p_r)\right) = \lambda > 0$.

14. If the m.g.f. of a r.v. X is

$$M_X(t) = \frac{2e^t + e^{3t} + 5}{8}, \quad t \in \mathbb{R}.$$

find $P(\{|X - E(X)| \geq \mathrm{Var}(X)\})$.