MODULE 3

FUNCTION OF A RANDOM VARIABLE AND ITS DISTIRBUTION

PROBLEMS

1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X: \Omega \to \mathbb{R}$, and $h: R_X \to \mathbb{R}$ be given functions, where $R_X = \{X(\omega): \omega \in \Omega\}$. Assuming that there exists a set $S \subseteq \mathbb{R}$ such that $S \notin \mathcal{B}_1$, using appropriate examples, show that:
   (i) X may not be a random variable;
   (ii) if X is a random variable then $h(X)$ may not be a random variable.

2. Let the random variable X have the p.m.f.

$$f_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & \text{if } x \in \{0,1,\dots,n\} \\ 0, & \text{otherwise} \end{cases};$$

   here n is a positive integer and $p \in (0,1)$. Find the p.m.f.s of random variables $Y_1 = X^2$ and $Y_2 = \sqrt{X}$.

3. Let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} e^{-1}, & \text{if } x = 0 \\ \dfrac{e^{-1}}{2(|x|!)}, & \text{if } x \in \{\pm 1, \pm 2, \dots\}. \\ 0, & \text{otherwise} \end{cases}$$

   Find the p.m.f. and the distribution function of $Y = |X|$.

4. Let X be a random variable with

$$P(\{X=-2\}) = \frac{1}{21}, P(\{X=1\}) = \frac{2}{21}, P(\{X=0\}) = \frac{1}{7},$$
$$P(\{X=1\}) = \frac{4}{21}, P(\{X=2\}) = \frac{5}{21}, P(\{X=3\}) = \frac{2}{7}$$

   Find the p.m.f. and distribution function of $Y = X^2$.

5. Let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^x, & \text{if } x \in \{0,1,2,\dots\} \\ 0, & \text{otherwise} \end{cases}.$$

Find the distribution function of $Y = X/(X+1)$ and hence determine the p.m.f. of Y.

6. Let the random variable X have the p.d.f. $f_X(\cdot)$. Find the distribution function and hence the p.d.f.s (provided they exist) of $X^+ = \max(X, 0)$, $X^- = \max(-X, 0)$, $Y_1 = |X|$ and $Y_2 = X^2$ in each of the following cases:

(i) $f_X(x) = \begin{cases} \frac{1+x}{2}, & \text{if} -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$; (ii) $f_X(x) = \begin{cases} \frac{1}{3}, & \text{if} -2 < x < -1 \\ \frac{x^2}{65}, & \text{if} -1 < x < 4 \\ \frac{2x}{27}, & \text{if } 4 < x < 5 \\ 0, & \text{otherwise} \end{cases}$.

7. Let the random variable X have the p.d.f.

$$f(x) = \begin{cases} 4x^3, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Find the p.d.f. and the distribution function of the random variable $Y = -2 \ln X^4$.

8. Let the random variable X have the p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{2}, & \text{if} -2 < x < -1 \\ \frac{1}{6}, & \text{if } 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}.$$

Find the p.d.f. of (i) $Y_1 = |X|$; (ii) $Y_2 = X^2$.

9. Let X be a random variable with p.d.f.

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise}' \end{cases}$$

and let $Y = \sin X$.

(i) Find the distribution of Y and hence find its p.d.f.;

(ii) Find the p.d.f. of Y directly (i.e., without finding the distribution function).

10. Let X be a random variable with p.d.f.

$$f_X(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Find the p.d.f.s of the following random variables: (i) $Y_1 = \sqrt{X}$; (ii) $Y_2 = X^2$; (iii) $Y_3 = 2X + 3$; (iv) $Y_4 = -\ln X$.

11. Let X be a random variable with p.d.f. f_X given in Problem 10 and let $Y = \min(X, 1/2)$.
    (i) Is X of continuous type?
    (ii) Examine whether or not X is of discrete or of absolutely continuous type.

12. Let the random variable X have the p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 < x \leq 1 \\ \frac{1}{2x^2}, & \text{if } x > 1 \\ 0, & \text{otherwise} \end{cases},$$

and let $Y = 1/X$.

(i) Find the distribution function of Y and hence find its p.d.f.;

(ii) Find the p.d.f. of Y directly (i.e., without finding, the distribution function).

13. Let the random variable X have the p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases},$$

where $\theta > 0$. Let $Y = (X - \theta)^2$.
(i) Find the distribution function of Y and hence find its p.d.f.;
(ii) Find the p.d.f. of Y directly (i.e., without finding the distribution function).

14. Let the random variable X have the p.d.f.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty.$$

Find the p.m.f./p.d.f. of $Y = g(X)$, where

$$g(x) = \begin{cases} -1, & \text{if } x < 0 \\ \frac{1}{2}, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}.$$

15. Let the random variable X have the p.d.f.

$$f_X(x) = \begin{cases} \frac{3}{8}(x+1)^2, & \text{if } -1 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

and let $Y = 1 - X^2$.

(i) Find the distribution function of Y and hence find its p.d.f.;

(ii) Find the p.d.f. of Y directly (i.e., without finding the distribution function).

16. Let the random variable X have the p.d.f.

$$f_X(x) = \begin{cases} 6x(1-x), & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

and let $Y = X^2(3 - 2X)$.

(i) Find the distribution function of Y and hence find its p.d.f.;

(ii) Find the p.d.f. of Y directly (i.e., without finding the distribution function).

17. Let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} \frac{1}{n}, & \text{if } x \in \{1, 2, \dots, n\} \\ 0, & \text{otherwise} \end{cases},$$

where $n\ (\geq 2)$ is an integer, Find the mean and variance of X.

18. In three independent tosses of a fair coin, let X denote the number of tails appearing. Let $Y = X^2$ and $Z = 2X^2 + 1$. Find the means and variances of random variables Y and Z.

19. (i) From a box containing N identical tickets, numbered, $1, 2, \dots, N, n\ (\leq N)$ tickets are drawn with replacement. Let $X =$ largest number drawn. Find $E(X)$.

(ii) Find the expected number of throws of a fair die required to obtain a 6.

20. (i) Let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} \frac{c}{x^{2+r}}, & \text{if } x \in \{1, 2, \dots, \} \\ 0, & \text{otherwise} \end{cases},$$

where $c^{-1} = \sum_{n=1}^{\infty} n^{-2-r}$ and $r \geq 0$ is an integer. For what values of $j \in \{0,1,2, \dots\}, E(X^j)$ is finite?

(ii) Find a distribution for which no moment exists.

21. Let X be a random variable with p.d.f.

$$f_X(x) = \begin{cases} \frac{x}{2}, & \text{if } 0 < x \leq 1 \\ \frac{1}{2}, & \text{if } 1 < x \leq 2 \\ \frac{3-x}{2}, & \text{if } 2 < x < 3 \\ 0, & \text{otherwise} \end{cases}.$$

Find the expected value of $Y = X^2 - 5X + 3$.

22. Let $E\left(|X|^\beta\right) < \infty$, for some $\beta > 0$. Show that $E(|X|^\alpha) < \infty, \forall \alpha \in (0, \beta]$.

23. Let X be random variable of absolutely continuous type with p.d.f. $f_X(x)$ that is symmetric about μ ($\in \mathbb{R}$), i.e., $f_X(x + \mu) = f_X(\mu - x), \forall x \in (-\infty, \infty)$. If $E(|X|^3)$ is finite, then show that $E(X^3) = 3\mu E(X^2) - 2\mu^3$.

24. If X is an absolutely continuous type random variable with median m, then show that $E(|X - m|) \leq E(|X - c|), \forall c \in (-\infty, \infty)$.

25. (i) Let X be a random variable with finite expectation. Show that $\lim_{x \to -\infty} xF_X(x) = \lim_{x \to \infty}\left[x\left(1 - F_X(x)\right)\right] = 0$, where F_X is the distribution function of X.
(ii) Let X be a random variable with $\lim_{x \to \infty}[x^\alpha P(|X| > x)] = 0$, for some $\alpha > 0$. Show that $E\left(|X|^\beta\right) < \infty, \forall \beta \in (0, \alpha)$. What about $E(|X|^\alpha)$?

26. (i) Let X be a non-negative random variable $(\text{i.e.}, P(\{X \geq 0\}) = 1)$ of absolutely continuous type and let h be a real-valued function defined on $(0, \infty)$ such that $h(x) \geq 0, \forall x \geq 0$. Define $\psi: [0, \infty) \to \mathbb{R}$ by $\psi(x) = \int_0^x h(t)dt, x \geq 0$. Show that
$$E\left(\psi(X)\right) = \int_0^\infty h(y)P(X > y)dy.$$

(ii) Let α be a positive real number. Under the assumptions of (i), show that

$$E(X^\alpha) = \alpha \int_0^\infty x^{\alpha-1}P(X > x)dx.$$

(iii) Let $F(0) = G(0) = 0$ and let $F(t) \geq G(t), \forall t > 0$, where F and G are distribution functions of absolutely continuous type non-negative random variables X and Y, respectively. Show that $E(X^k) \leq E(Y^k), \forall k > 0$, provided the expectations are finite.

27. Consider a target comprising of three concentric circles of radii $1/\sqrt{3}, 1$ and $\sqrt{3}$ feet. Shots within the inner circle earn 4 points, within the next ring 3 points and within the third ring 2 points. Shots outside the target do not earn any point. Let X

denote the distance (in feet) of the hit from the centre and suppose that X has the p.d.f.

$$f_X(x) = \begin{cases} \dfrac{2}{\pi(1+x^2)}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Find the expected score in a single shot.

28. (i) Find the moments of the random variable that has the m.g.f. $M(t) = (1-t)^{-3}, t < 1$.

(ii) Let the random variable X have the m.g.f.

$$M(t) = \frac{e^{-t}}{8} + \frac{e^{t}}{4} + \frac{e^{2t}}{8} + \frac{e^{3t}}{2}.$$

Find the distribution function of X and find $P(\{X^2 = 1\})$.

(iii) If the m.g.f. of a random variable X is

$$M(t) = \frac{e^{t} - e^{-2t}}{3t}, \text{for } t \neq 0,$$

find the p.d.f. of $Y = X^2$.

29. Let X be a random variable with m.g.f $M(t), -h < t < h$. Prove that
$P(X \geq a) \leq e^{-at} M(t), 0 < t < h$ and $P(X \leq a) \leq e^{-at} M(t), -h < t < 0$.

30. (i) Let X be a random variable such that $P(X \leq 0) = 0$ and let $\mu = E(X)$ is finite. Show that $P(X \geq 2\mu) \leq 0.5$.
(ii) If X is a random variable such that $E(X) = 3$ and $E(X^2) = 13$, then determine a lower bound for $P(-2 < X < 8)$.

31. Let the random variable X have the p.m.f.

$$f_X(x) = \begin{cases} \dfrac{1}{8}, & \text{if } x \in \{-1,1\} \\ \dfrac{6}{8}, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}.$$

Using this p.m.f., show that the bound for Chebyshev's inequality cannot be improved (without additional assumptions).

32. For any random variable X having the mean μ and finite second moment, show that $E((X-\mu)^2) \le E((X-c)^2), \forall c \in \mathbb{R}$.

33. Let X be a random variable with p.m.f./p.d.f. $f_X(x)$ that is symmetric about $\mu(\in \mathbb{R})$, i.e., $f_X(x+\mu) = f_X(\mu - x), \forall x \in (-\infty, \infty)$.
(i) If q_1, m and q_3 are respectively the lower quartile, the median and the upper quartile of the distribution of X then show that $m = \frac{q_1+q_3}{2}$;
(ii) If $E(X)$ is finite then show that $E(X) = \mu = m = \frac{q_1+q_3}{2}$.

34. For any values of $\mu \in \mathbb{R}$ and $\sigma > 0$, show that the kurtosis of $N(\mu, \sigma^2)$ distribution is $\beta_2 = 3$..

35. For $\mu \in \mathbb{R}$ and $\lambda > 0$, let $X_{\mu,\lambda}$ be a random variable having the p.d.f.

$$f_{\mu,\lambda}(x) = \begin{cases} \dfrac{1}{\lambda} \mathrm{e}^{-\frac{x-\mu}{\lambda}}, & \text{if } x \ge \mu \\ 0, & \text{otherwise} \end{cases}.$$

(i) Find $C_r(\mu,\lambda) = E((X-\mu)^r), r \in \{1,2,\ldots\}$ and $\mu'_r(\mu,\lambda) = E\left(X^r_{\mu,\lambda}\right), r \in \{1,2\}$;

(ii) For $p \in (0,1)$, find the p-th quantile $\xi_p \equiv \xi_p(\mu,\lambda)$ of the distribution of $X_{\mu,\lambda}$ $(F_{\mu,\lambda}\left(\xi_p\right), = p$, where $F_{\mu,\lambda}(\cdot)$ is the distribution function of $X_{\mu,\lambda}$).

(iii) Find the lower quartile $q_1(\mu,\lambda)$, the median $m(\mu,\lambda)$ and the upper quartile $q_3(\mu,\lambda)$ of the distribution of $X_{\mu,\lambda}$;

(iv) Find the mode $m_0(\mu,\lambda)$of the distribution of $X_{\mu,\lambda}$;

(v) Find the standard deviation $\sigma(\mu,\lambda)$, the mean deviation about median $\mathrm{MD}\big(m(\mu,\lambda)\big)$, the inter-quartile range $\mathrm{IQR}(\mu,\lambda)$, the quartile deviation (or semi-inter-quartile range) $\mathrm{QD}(\mu,\lambda)$, the coefficient of quartile deviation $\mathrm{CQD}(\mu,\lambda)$ and the coefficient of variation $\mathrm{CV}(\mu,\lambda)$ of the distribution of $X_{\mu,\lambda}$;

(vi) Find the coefficient of skewness $\beta_1(\mu,\lambda)$ and the Yule coefficient of skewness $\beta_2(\mu,\lambda)$ of the distribution of $X_{\mu,\lambda}$;

(vii) Find the excess kurtosis $\gamma_2(\mu,\lambda)$ of the distribution of $X_{\mu,\lambda}$;

(viii) Based on values of measures of skwness and the kurtosis of the distribution of $X_{\mu,\lambda}$, comment on the shape of $f_{\mu,\lambda}$.