Problem 1: Let $V = \{(x, y) : x, y \in \mathbb{R}\}$. Define vector addition in V component-wise and scalar multiplication as follows:

$$a(x, y) = (x, 0) \quad \forall (x, y) \in V \text{ and } a \in \mathbb{R}.$$

Is V a vector space with these operations.

Solution: The vector addition is given by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in V$$

$\forall\ (x_1, y_1)\ \&\ (x_2, y_2) \in V.$

Let $(x, y) \in V$ and $c \in \mathbb{R}$.

Then $c(x, y) = (x, 0) \in V.$

Hence V is closed under vector-addition & scalar multiplication.

<u>Property I</u> : Let (x_1, y_1) and $(x_2, y_2) \in V$

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2) \longrightarrow (*)$$

Notice that $x_1 + x_2 = x_2 + x_1$ & $|||^{ly}$ $y_1 + y_2 = y_2 + y_1 \longrightarrow (**)$

Then $(x_2, y_2) + (x_1, y_1) = (x_2 + x_1, y_2 + y_1)$

$= (x_1 + x_2, y_1 + y_2)$ (By (**)).

$= (x_1, y_1) + (x_2, y_2)$ (by (*))

Property I is satisfied.

<u>Property II</u> Let $(x_1, y_1), (x_2, y_2)$ & $(x_3, y_3) \in V$

$$\big((x_1, y_1) + (x_2, y_2)\big) + (x_3, y_3)$$

$$= (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$$

$$= \big((x_1 + x_2) + x_3, (y_1 + y_2) + y_3\big).$$

$$= \left(x_1 + (x_2 + x_3),\ y_1 + (y_2 + y_3)\right)$$

$$= (x_1, y_1) + (x_2 + x_3,\ y_2 + y_3)$$

$$= (x_1, y_1) + \left((x_2, y_2) + (x_3, y_3)\right)$$

Hence property II is satisfied.

Property III : $(0,0)$ is the zero vector for the vector addition

$$(x, y) + (0,0) = (x+0,\ y+0) = (x, y)$$

Property III is satisfied.

Property IV: Let $(x, y) \in V$.

Then $(-x, -y) \in V$ and

$$(x, y) + (-x, -y) = (x - x, y - y) = (0, 0)$$

Property IV is satisfied.

Property V: $(x, y) \in V$ $\qquad (1 \cdot v = v \quad \forall v \in V)$.

$$1(x, y) = (x, 0) \quad \text{(by definition)}$$

Howevery if $(x, y) = (2, 3)$

$$1(2, 3) = (2, 0) \neq (2, 3)$$

Hence Property I is not satisfied.

Therefore V is not a vector space with these operations.

<u>Property VI</u> $\quad (ab)(x, y) = (x, 0)$

$$a\left(b(x,y)\right) = a(x,0) = (x,0)$$

Hence Property $\underline{\text{VI}}$ is satisfied.

<u>Property $\boxed{\text{VII}}$</u> : $(a+b)(x,y) = (x,0)$

$$a(x,y) + b(x,y) = (x,0) + (x,0) = (2x,0)$$

$$(1+1)(1,2) = 2(1,2) = (1,0)$$

$$1(1,2) + 1(1,2) = (1,0) + (1,0) = (2,0)$$

Property $\overline{\text{VIII}}$ is not satisfied.

Property $\overline{\text{VII}}$: $a\big((x_1, y_1) + (x_2, y_2)\big) = a(x_1+x_2, y_1+y_2)$

$$= (x_1+x_2, 0)$$

$$a(x_1, y_1) + a(x_2, y_2) = (x_1, 0) + (x_2, 0)$$

$$= (x_1+x_2, 0)$$

Property $\overline{\text{VII}}$ is satisfied.

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Problem 2: Prove that the set $W_1 = \{(x,y,z) \in \mathbb{R}^3 : 2x+3y+z=0\}$ is a subspace of $\mathbb{R}^3$, however $W_2 = \{(x,y,z) \in \mathbb{R}^3 : 2x+3y+z=1\}$ is not a subspace of $\mathbb{R}^3$.

Proof: To check that W_1 is a subspace, enough to show that W_1 is closed under the vector addition & scalar multiplication from V.

Let (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in W_1$

$$\Rightarrow \quad 2x_1 + 3y_1 + z_1 = 0 \qquad \& \qquad 2x_2 + 3y_2 + z_2 = 0. \longrightarrow (*)$$

Then

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$$

Then

$$2(x_1 + x_2) + 3(y_1 + y_2) + (z_1 + z_2)$$

$$= 2x_1 + 2x_2 + 3y_1 + 3y_2 + z_1 + z_2$$

$$= (2x_1 + 3y_1 + z_1) + (2x_2 + 3y_1 + z_2)$$

$$= 0 + 0 \qquad (\text{by } (*))$$

$$= 0$$

$$\therefore \quad (x_1, y_1, z_1) + (x_2, y_2, z_2) \in W.$$

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Let $(x, y, z) \in W_1$ and $c \in \mathbb{R}$

$$c(x, y, z) = (cx, cy, cz)$$

$$2cx + 3cy + cz = c(2x + 3y + z) = c0 = 0.$$

$\Rightarrow W_1$ is a subspace

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Is W_2 a subspace?

$(x_1, y_1, z_1) \in W_2$ and $(x_2, y_2, z_2) \in W_2$

$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$2(x_1 + x_2) + 3(y_1 + y_2) + z_1 + z_2 =$

$(2x_1 + 3y_1 + z_1) + (2x_2 + 3y_2 + z_2)$

$$1 + 1 = 2$$

Hence W_2 is not closed under vector addition & therefore not a subspace.

<u>Problem 3</u>: Let $S \subset \mathbb{R}$ and $W = \{ f \in \mathcal{F}(S, \mathbb{R}) : f(s_0) = 0 \}$ for a fixed $s_0 \in S$. Then prove that W is a subspace of $\mathcal{F}(S, \mathbb{R})$.

<u>Proof</u>: Recall that $\mathcal{F}(S, \mathbb{R}) = \{ f : S \to \mathbb{R} \}$ with vector addition $(f+g)(s) = f(s) + g(s)$ for $f, g \in \mathcal{F}(S, \mathbb{R})$

$$(cf)(s) = c\,(f(s)) \quad \text{where } f \in \mathcal{F}(S, \mathbb{R}) \ \& \ c \in \mathbb{R}$$

Let $f, g \in W \Rightarrow f(s_0) = 0$ & $g(s_0) = 0$.

Then for $f, g \in W$ and $c \in \mathbb{R}$

$$(f+g)(s_0) = f(s_0) + g(s_0) = 0 + 0 = 0.$$

$$\Rightarrow f+g \in W.$$

$$(cf)(s_0) = c\,f(s_0) = c\,0 = 0$$

$$\Rightarrow cf \in W.$$

W is hence a subspace of $\mathcal{F}(S, \mathbb{R})$. —— ∎.

Problem 4: Check whether the first vector is a linear combination of the other two vectors in the following:

(i) $\{(-2, 2, 2), \ (1, 2, -1), \ (-3, -3, 3)\}$ in $\mathbb{R}^3$

(ii) $\{x^3 - 8x^2 + 4x, \ x^3 - 2x^2 + 3x - 1, \ x^3 - 2x + 3\}$ in $\mathcal{P}_3(\mathbb{R})$.

Solution: Want to check for the existence of $a, b \in \mathbb{R}$ s.t

$$(-2, 2, 2) = a(1, 2, -1) + b(-3, -3, 3)$$
$$= (a - 3b, 2a - 3b, -a + 3b)$$

$$\left.\begin{aligned} a - 3b &= -2 \\ 2a - 3b &= 2 \end{aligned}\right\} \Rightarrow \quad a - 2 = 2$$

$$\Rightarrow a = 4 \Rightarrow 8 - 3b = 2$$

$$\Rightarrow b = 2$$

$$-a + 3b = 2$$

$\Rightarrow a = 4 , b = 2$

Hence $(-2,2,2) = 4(1,2,-1) + 2(-3,-3,3)$

(ii) $\{x^3 - 8x^2 + 4x,\ x^3 - 2x^2 + 3x - 1,\ x^3 - 2x + 3\}$ in $P_3(\mathbb{R})$

Want to check if $\exists\ a, b \in \mathbb{R}$ s.t

$$x^3 - 8x^2 + 4x = a(x^3 - 2x^2 + 3x - 1) + b(x^3 - 2x + 3) \longrightarrow (*)$$

$$= (a+b)x^3 + (-2a)x^2 + (3a-2b)x + (3b-a)$$

$$\Rightarrow \quad \left.\begin{array}{l} a + b = 1 \\ -2a = -8 \end{array}\right\} \Rightarrow a = 4 \Rightarrow b = 1-4 = -3.$$

$$\left.\begin{array}{l} 3a - 2b = 4 \\ 3b - a = 0 \end{array}\right] \longrightarrow 12 + 6 = 18 \neq 4$$

Hence $\nexists$ $a, b \in \mathbb{R}$ s.t $(*)$ is satisfied.

Problem 5: Let S_1, S_2 be subsets of a vector space V.

Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$.

Give an example when (i) $\text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$

(ii) $\text{span}(S_1 \cap S_2) \subsetneq \text{span}(S_1) \cap \text{span}(S_2)$.

Solution: Let $v \in \text{span}(S_1 \cap S_2)$

i.e. $\exists$ $v_1, \dots, v_k \in S_1 \cap S_2$ and $a_1, \dots, a_k \in \mathbb{R}$

s.t $v = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$.

$$v_1, \ldots, v_k \in S_1 \cap S_2 \implies v_1, \ldots, v_k \in S_1$$

$$\implies a_1 v_1 + \cdots + a_k v_k \in \operatorname{span}(S_1)$$

By a similar argument $a_1 v_1 + \cdots + a_k v_k \in \operatorname{span}(S_2)$

$$\implies a_1 v_1 + \cdots + a_k v_k \in \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$$

$$\implies v \in \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$$

Hence $\operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$.

Example: (i) In $\mathbb{R}^3$ consider

$$S_1 = \{(1,0,0), (0,1,0), (1,1,0)\}$$

$$S_2 = \{(0,1,0), (1,1,0), (1,2,0)\}.$$

$$S_1 \cap S_2 = \{(0,1,0), (1,1,0)\}$$

$$\text{span}(S_1 \cap S_2) = \{(x, y, z) \in \mathbb{R}^3 : z = 0\} = W$$

$$\text{span}(S_1) = W = \text{span}(S_2).$$

(ii) In $\mathbb{R}^2$, let

$$S_1 = \{(1,0), (0,1)\}$$

$$S_2 = \{(1,1), (1,-1)\}.$$

$$\text{span}(S_1) = \text{span}(S_2) = \mathbb{R}^2$$

$$S_1 \cap S_2 = \phi \quad \& \quad \text{span}(S_1 \cap S_2) = \{0\}.$$

Problem 6: Check whether the following set

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

is linearly dependent in $M_{3\times 2}(\mathbb{R})$

Solution: To check whether S is linearly dependent, we want to check for the existence of $a_1, a_2, \ldots, a_5 \in \mathbb{R}$

s.t $$a_1 \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} + a_4 \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} + a_5 \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a_1+a_4 & a_1+a_5 \\ a_2+a_4 & a_2+a_5 \\ a_3+a_4 & a_3+a_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

i.e.

$$a_1+a_4=0 \qquad a_1+a_5=0$$
$$a_2+a_4=0 \qquad a_2+a_5=0$$
$$a_3+a_4=0 \qquad a_3+a_5=0$$

$\Rightarrow$ $a_1=a_2=a_3$ $\qquad$ $\Rightarrow$ $a_1=a_2=a_3$

$a_4=-a_1$ $\qquad$ & $a_5=-a_1$

If $a_1=1$, then

$$\begin{pmatrix}1&1\\0&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\1&1\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&0\\1&1\end{pmatrix}+(-1)\begin{pmatrix}1&0\\1&0\\1&0\end{pmatrix}+(-1)\begin{pmatrix}0&1\\0&1\\0&1\end{pmatrix}$$

$$=\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}.$$

Hence S is linearly dependent.

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Problem 7: Let S be a finite set of non-zero polynomials in $P(\mathbb{R})$ s.t. no two polynomials have the same degree. Then prove that S is linearly independent.

<u>Proof:</u> Let P_1 be the polynomial in S with least degree. Let $\deg(P_1) = d_1$

Let P_2 be the poly in S with the least degree $> d_1$

$$\deg(P_2) = d_2$$

After finitely many steps, we have

$S = \{P_1, P_2, \dots, P_k\}$ s.t $\deg(P_i) = d_i$

and $d_1 < d_2 < d_3 < \cdots < d_k$.

Suppose $\underbrace{a_1 p_1 + a_2 p_2 + \cdots}_{\substack{\parallel \\ q}} + a_k p_k = 0 \quad \text{———} \quad (*)$

Co-eff. of $x^{d_k} = a_k b_k$

where b_k = co-eff. of x^{d_k} in $p_k(x)$.

$\Rightarrow \quad a_k b_k = 0 \quad \Rightarrow \quad a_k = 0$ (since $b_k = 0$ implies that $\deg(p_k) \neq d_k$).

$\Rightarrow \quad a_1 p_1 + a_2 p_2 + \cdots + a_{k-1} p_{k-1} = 0$

By a similar argument $a_{k-1} = 0$

By repeating the above process, we have $a_k = a_{k-1} = \cdots = a_1 = 0$

$\Rightarrow S$ is linearly independent.

<u>Problem 8</u>: Determine whether the following subsets are bases for the subspace $W = \{ax^2 + bx + c \in P_2(\mathbb{R}) : 2a + b + c = 0\}$ in $P_2(\mathbb{R})$.

(i) $S = \{x^2-2,\ x^2+2x-4\}$

(ii) $S = \{3x^2-2x-4,\ x^2-x-1,\ -2x^2+x+3\}$

(iii) $S = \{x^2-2,\ -2x^2+x+3\}$.

<u>Solution:</u> (i) $S = \{x^2-2,\ x^2+2x-4\}$

Let $a, b \in \mathbb{R}$ s.t $a(x^2-2) + b(x^2+2x-4) = 0$

$\Rightarrow (a+b)x^2 + 2bx + (-2a-4b) = 0$

$\Rightarrow \quad 2b = 0 \Rightarrow b = 0$

$a+b = 0 \Rightarrow a = 0$

Hence S is linearly independent

$$\Rightarrow \quad \dim.(W) \geq 2$$

If S is not a spanning set, then

$\exists \quad p(x) \in W$ s.t $p \notin \operatorname{span}(S)$

$S \cup \{p\}$ is linearly ind.

$$\Rightarrow \quad \dim(W) = 3 = \dim(P_2(\mathbb{R})).$$

$\Rightarrow \quad W = P_2(\mathbb{R})$ but this is a contradiction

Hence S is a spaning set.

$\Rightarrow$ S is a basis of W.

$\Rightarrow$ $\dim(W) = 2$.

(ii) S cannot be linearly independent, the size of any linearly independent subset of W cannot be greater than $\dim(W)$.

$\therefore$ S is not a basis.

(iii) By a corollary to the replacement theorem, to check whether S is a basis, it is enough to check for linear independence since S has $\dim(W) = 2$ elts.

$$S = \{x^2 - 2, \ -2x^2 + x + 3\}.$$

If $a(x^2 - 2) + b(-2x^2 + x + 3) = 0$

$$\Rightarrow (a - 2b)x^2 + bx + (3b - 2a) = 0$$

$$a - 2b = 0, \quad b = 0, \quad 3b - 2a = 0$$

$\Rightarrow \quad a=0, \; b=0.$

Hence S is a basis of W.

Problem 9: For a fixed scalar $c \in \mathbb{R}$, let $W = \{ p \in \mathcal{P}_n(\mathbb{R}) : p(c) = 0 \}$ be a subspace of $\mathcal{P}_n(\mathbb{R})$. Determine the dimension of W.

Solution: We know that $p(c) = 0$

iff $\quad p(x) = (x-c)\,q(x)$

Claim: $\quad S = \{ x-c,\ x(x-c),\ x^2(x-c),\ \ldots,\ x^{n-1}(x-c) \}$ is a basis of W.

That S is linearly independent in W follows from problem 7.

$\Rightarrow \quad \dim(W) \geq n.$

If $\quad \dim(W) > n$

$$\Rightarrow \quad \dim(W) = n+1 = \dim(\mathcal{P}_n(\mathbb{R})).$$

$$\Rightarrow \quad W = \mathcal{P}_n(\mathbb{R}).$$

This is a contradiction since $1(c) = 1 \neq 0$

$$\Rightarrow \quad \dim(W) = n. \qquad \text{———} \qquad \blacksquare.$$

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