

Let V & W be vector spaces and suppose $S: V \rightarrow W$ and $T: V \rightarrow W$ are linear transformations. We define

$S+T: V \rightarrow W$ to be

$$(S+T)v := Sv + Tv \quad \forall v \in V.$$

Proposition: With S & T as above, $S+T$ is a linear transformation.

Proof: For $v_1, v_2 \in V$,

$$\begin{aligned}(S+T)(v_1+v_2) &= S(v_1+v_2) + T(v_1+v_2) \\&= Sv_1 + Sv_2 + Tv_1 + Tv_2 \\&= Sv_1 + Tv_1 + Sv_2 + Tv_2 \\&= (S+T)v_1 + (S+T)v_2.\end{aligned}$$

Check that $(S+T)(av) = a(S+T)v.$

Let $T: V \rightarrow W$ be a linear transformation between vector spaces. Let $a \in \mathbb{R}$. Then we define

$$(aT): V \rightarrow W \text{ to be}$$
$$(aT)(v) := a(Tv)$$

Exercise: Check that (aT) is a linear transformation.

Example: $S: V \rightarrow V$ be $sv = a \cdot v$

and $T : V \rightarrow V$ be $Tv = a_2v$

then $(S+T)v = (a_1+a_2)v$

Let the space of all linear transformations between V & W be denoted by $L(V, W)$. Then $L(V, W)$ is a subset of $\mathcal{F}(V, W)$.

Proposition: The set $\mathcal{L}(V, W)$ is a subspace of $\mathcal{F}(V, W)$.

Proof: We just checked that $\mathcal{L}(V, W)$ is closed under addition & scalar multiplication.

Let $S: V \rightarrow W$ and $T: U \rightarrow V$ be linear transformations between vector spaces. Then we define the product (or composition) of linear transformations $ST: U \rightarrow W$ to be

$$(ST)u := S(Tu) \quad \forall u \in U.$$

Exercise: Prove that ST is a linear transformation.

Proposition:

Let V & W be finite dimensional vector spaces & suppose $\beta = (v_1, \dots, v_m)$ and $\gamma = (w_1, \dots, w_n)$. Let $S: V \rightarrow W$ and $T: V \rightarrow W$ be linear transformations.

$$[S+T]_{\beta}^{\gamma} = [S]_{\beta}^{\gamma} + [T]_{\beta}^{\gamma}$$

Proof: Let $[S]_{\beta}^r = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & & a_{nm} \end{pmatrix}$. Then $Sv_j = a_{1j}w_1 + a_{2j}w_2 + \dots + a_{nj}w_n$

$$\& [T]_{\beta}^r = \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix}$$

We know that $Tv_j = b_{1j}w_1 + \dots + b_{nj}w_n$.

$$\begin{aligned} (S+T)v_j &= (a_{1j}w_1 + \dots + a_{nj}w_n) + (b_{1j}w_1 + \dots + b_{nj}w_n) \\ &= (a_{1j} + b_{1j})w_1 + \dots + (a_{nj} + b_{nj})w_n \end{aligned}$$

$$\begin{pmatrix} a_{11} + b_{11} & \dots & a_{1m} + b_{1m} \\ a_{21} + b_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} + b_{n1} & & a_{nm} + b_{nm} \end{pmatrix} \\
 = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix} = [S]_{\beta}^{\gamma} + [T]_{\beta}^{\gamma}.$$

Exercise: Let $T: V \rightarrow W$ be a linear transformation as in the proposition above & suppose $a \in \mathbb{R}$.

$$\text{Then } [aT]_{\beta}^r = a [T]_{\beta}^r.$$

Proposition: Let U, V & W be finite dimensional vector spaces and suppose $\alpha = (u_1, \dots, u_n)$, $\beta = (v_1, \dots, v_m)$ & $\gamma = (w_1, \dots, w_n)$ be ordered bases of U, V & W respectively. Then, if $S: V \rightarrow W$ and $T: U \rightarrow V$ are linear transformations, we have

$$[ST]_{\alpha}^{\gamma} = [S]_{\beta}^{\gamma} [T]_{\alpha}^{\beta}$$

Example: Let $L : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ be define

$$L(x_1, x_2, \dots) := (x_2, x_3, \dots)$$

$$\& R(x_1, x_2, \dots) := (0, x_1, x_2, \dots)$$

$$\text{Then } LR(x_1, x_2, \dots) = (x_1, x_2, \dots)$$

$$\& RL(x_1, x_2, \dots) = (0, x_2, x_3, \dots)$$

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Proof of proposition: $\alpha = (u_1, \dots, u_e)$, $\beta = (v_1, \dots, v_m)$ and $\gamma = (w_1, \dots, w_n)$ be the ordered bases of U, V & W resp.

$$\text{Let } [T]_{\alpha}^{\beta} = \begin{pmatrix} a_{11} & \dots & a_{1l} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{ml} \end{pmatrix}$$

$$\text{Then } T u_i = \sum_{j=1}^m a_{ji} v_j$$

$$[S]_{\beta}^{\gamma} = \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix}$$

$$\text{Then } S v_j = \sum_{k=1}^n b_{kj} w_k$$

$$(ST)u_i = S(Tu_i) = S\left(\sum_{j=1}^m a_{ji}v_j\right)$$

$$= \sum_{j=1}^m a_{ji}(Sv_j) = \sum_{j=1}^m a_{ji}\left(\sum_{k=1}^n b_{kj}w_k\right)$$

$$= \sum_{k=1}^n \left(\sum_{j=1}^m b_{kj}a_{ji}\right) w_k$$

$$\text{Let } c_{ki} = \sum_{j=1}^m b_{kj}a_{ji}$$

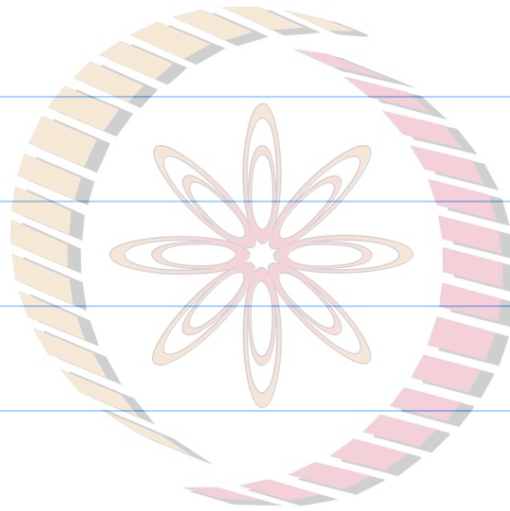
$$(ST)u_i = \sum_{k=1}^n c_{ki} w_k$$

Hence

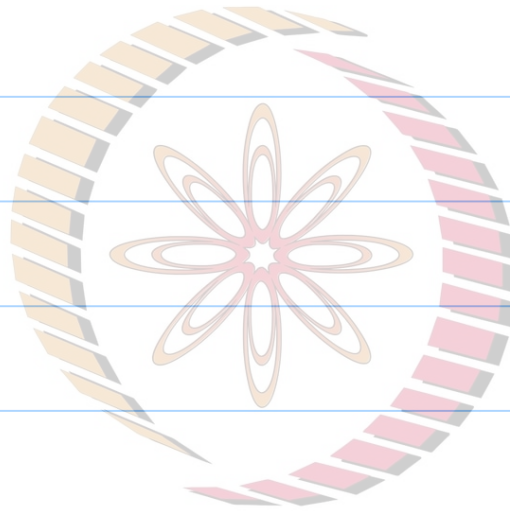
$$[ST]_{\alpha}^{\gamma} = \begin{pmatrix} c_{11} & \cdots & c_{1\ell} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{n\ell} \end{pmatrix}$$

Check that

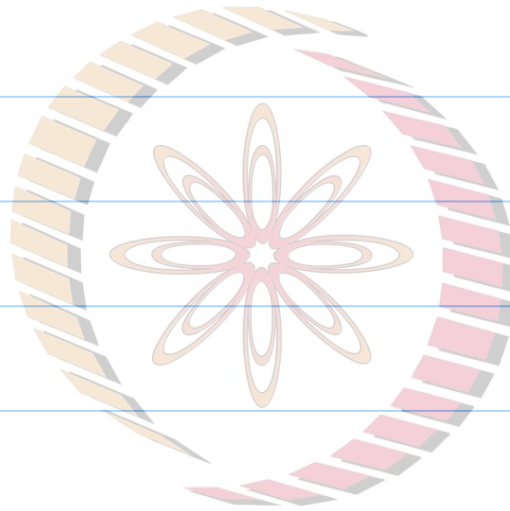
$$\begin{pmatrix} b_{11} & \cdots & b_{m1} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1\ell} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{m\ell} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1\ell} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{n\ell} \end{pmatrix} \quad \square$$



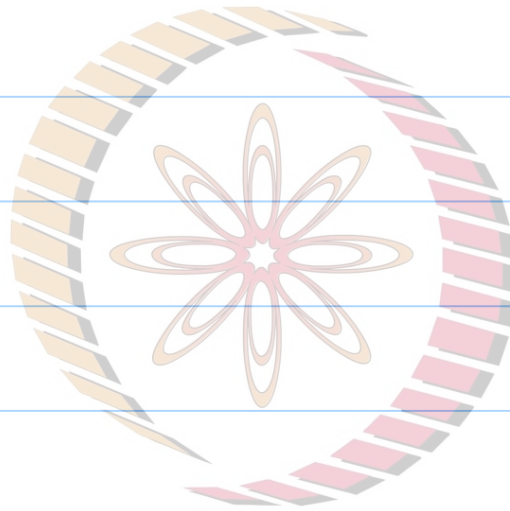
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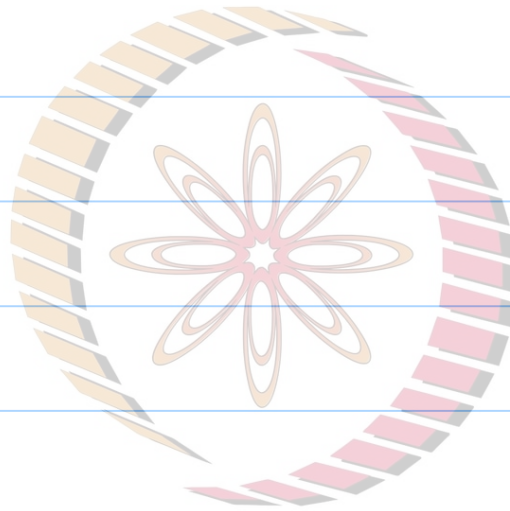
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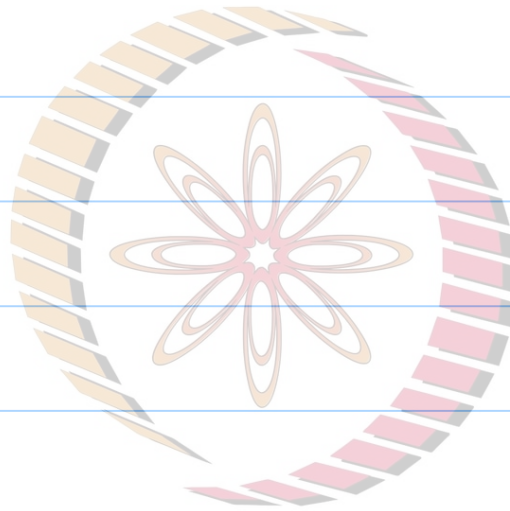
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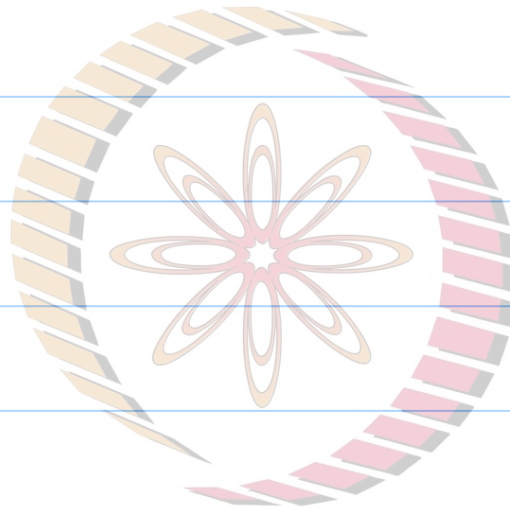
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