In $\mathbb{R}^3$, consider $S = \{v_1 = (1,0,1), v_2 = (1,2,0), v_3 = (3,4,1)\}$

$$\text{span}(S) = \{ a_1 v_1 + a_2 v_2 + a_3 v_3 : a_i \in \mathbb{R} \}.$$

$$(3,4,1) = (1,0,1) + 2(1,2,0)$$

$$v_3 = v_1 + 2v_2 \quad \text{———} \quad (*)$$

$$\boxed{v_1 + 2v_2 - v_3 = 0}$$

Let $a_1 v_1 + a_2 v_2 + a_3 v_3 \in span(S)$.

Then $a_1 v_1 + a_2 v_2 + a_3 v_3 = a_1 v_1 + a_2 v_2 + a_3 (v_1 + 2 v_2)$

$= (a_1 + a_3) v_1 + (a_2 + 2a_3) v_2 \in span(\{v_1, v_2\})$.

i.e $span(S) \subseteq span(\{v_1, v_2\})$.

But $span(\{v_1, v_2\}) \subseteq span(S)$

$\Rightarrow span(S) = span(\{v_1, v_2\})$.

Definition of linear dependence

Let S be a subset of a vector space. We say that S is linearly dependent if $\exists$ vectors $v_1, \ldots, v_n \in S$ and scalars $a_1, \ldots, a_n \in \mathbb{R}$, with not all a_i equal to zero, s.t.

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

Notice that $0 = 0v_1 + 0v_2 + \cdots + 0v_n$.

Let $S = \{v_1, \ldots, v_n\}$ be a finite subset of a vector space V. Then S is said to be linearly dependent if $\exists$ $a_1, \ldots, a_n \in \mathbb{R}$, not all equal to zero s.t

$$a_1v_1 + \cdots + a_nv_n = 0.$$

Definition of linear independence

Let $S \subseteq V$ be a subset of a vector space V. We say that S is linearly independent if S is not linearly dependent.

i.e Given $a_1, \dots, a_n \in \mathbb{R}$ & $v_1, \dots, v_n \in S$ s.t.

$$a_1 v_1 + \cdots + a_n v_n = 0, \quad \text{then} \quad a_i = 0 \ \forall i.$$

Examples: Let $S = \{v_1 = (1,2,3), v_2 = (1,0,0)\}$

then $a_1 v_1 + a_2 v_2 = 0$

$$a_1(1,2,3) + a_2(1,0,0) = (0,0,0).$$

Hence
$$a_1 + a_2 = 0$$
$$2a_1 = 0$$
$$3a_1 = 0$$

$\Rightarrow \quad a_1 = 0, \quad a_2 = 0$

i.e. S is linearly independent.

(*) By convention, the empty set is considered to be linearly independent.

<u>Exercise</u>: Let $S \subseteq V$ be a subset containing the zero vector, then S is linearly dependent.

Exercise: Let $S' \subseteq S$ be a subset of a linearly independent set. Then S' is linearly independent.

Exercise: Let $S = \{v\}$ where v is a non-zero vector in V. Prove that S is linearly independent.

Theorem: Let S be a subset of a vector space V.

If S is linearly dependent, then $\exists\ v \in S$ such that $span(S \setminus \{v\}) = span(S)$. $A \Rightarrow B$

Conversely if S is linearly independent, then for any strict subset $S' \subsetneq S$, we have $B \Rightarrow A$

$span(S') \subsetneq span(S)$. $\neg A \Rightarrow \neg B$

Proof: Suppose S is linearly dependent. i.e.

$\exists\ v_1, \ldots, v_n \in S$ (distinct) and $a_1, \ldots, a_n \in \mathbb{R}$ not all equal to zero s.t. $a_1 v_1 + \cdots + a_n v_n = 0$

Assume without loss of generality (after renumbering the indices of a_i's & v_i's if needed), $a_1 \neq 0$

then $v_1 = \left(-\frac{a_2}{a_1}\right) v_2 + \left(\frac{-a_3}{a_1}\right) v_3 + \cdots + \left(\frac{-a_n}{a_1}\right) v_n$. ——— (*)

Let us consider a linear combination in S.

$b_1 u_1 + b_2 u_2 + \cdots + b_m u_m$ where $u_i \in S$ $\forall i$ (distinct)

If none of the u_i's are v_1, then

$$b_1u_1 + \cdots + b_mu_m \in \text{Span}(S \setminus \{v_1\}).$$

If one of the u_i's is v_1, then WLOG, assume $u_1 = v_1$

then $$b_1u_1 + b_2u_2 + \cdots + b_mu_m = \left(\frac{-a_2}{a_1}\right)v_2 + \cdots + \left(\frac{-a_n}{a_1}\right)v_n + b_2u_2 + \cdots + b_mu_m.$$

$$\in \text{span}(S \setminus \{v_1\}).$$

$$\Rightarrow \quad \text{span}(S) \subseteq \text{span}(S \setminus \{v_1\}).$$

Since $S \setminus \{v_1\} \subseteq S$, we have $\text{span}(S \setminus \{v_1\}) \subseteq \text{span}(S)$.

Hence $\quad \mathrm{Span}(S \setminus \{v_1\}) = \mathrm{span}(S)$.

<u>Converse</u>: If S is linearly independent, then $\mathrm{span}(S') \subsetneq \mathrm{span}(S)$ whenever $S' \subsetneq S$.

Let $S' \subsetneq S$ be a strict subset of S.

i.e. $\exists \; v \in S \setminus S'$.

<u>Claim</u>: $v \in \mathrm{span}(S')$.

If $v \in \text{span}(S')$, then

$$v = a_1 v_1 + \cdots + a_n v_n \quad \text{where } v_1, \ldots, v_n \in S' \text{ and } a_1, \ldots, a_n \in \mathbb{R}.$$

$$\Rightarrow \quad (-1)v + a_1 v_1 + \cdots + a_n v_n = 0$$

Observe that the co-eff. of $v = -1$

& hence $\{v, v_1, v_2, \ldots, v_n\}$ are linearly dependent.

$\Rightarrow$ S is linearly dependent.

This is a contradiction to the linear independence of S.

Therefore our assumption that $v \in span(S')$ is false.

But $v \in span(S)$.

& since $S' \subset S$, $span(S') \subset span(S)$.

Therefore $span(S') \subsetneq span(S)$. ■

Basis:

A subset $S \subseteq V$ is said to be a Basis of V if S is linearly independent and a spanning set of V at the same time.

Example: In $\mathbb{R}^3$ consider $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$.

β is linearly independent for if

$$a(1,0,0) + b(0,1,0) + c(0,0,1) = (0,0,0)$$

then $(a,b,c) = (0,0,0) \Rightarrow a=0, b=0, c=0.$

Check that β spans $\mathbb{R}^3$.

Hence β is a basis of $\mathbb{R}^3$.

Is $S = \{(1,0,0), (0,1,0), (0,0,1), (1,2,3)\}$ a basis of $\mathbb{R}^3$?

NO. (Linearly dependent though S is a spanning set).

Is $S = \{(1,0,0), (0,1,0)\}$. a basis of $\mathbb{R}^3$?

No. S is not a spanning set though S is linearly indep.

$S = \{(1,0,0), (2,0,0)\}$ is not a basis of $\mathbb{R}^3$.

(*) Consider $P_3(\mathbb{R})$.

$\mathcal{B} = \{1, x, x^2, x^3\}$ is a basis of $P_3(\mathbb{R})$.

$\beta = \{1, 1+x, x^2+x, x^3+x^2\}$ is also a basis of $P_3(\mathbb{R})$.

$S = \{1, x, x^2, x^2+x, x^3\}$ is not a basis.

$S = \{1, x+x^2, x^3\}$ is not a basis.

<u>Theorem</u>: Let V be a vector space with a finite basis $\beta := \{v_1, \dots, v_n\}$. Then for every vector $v \in V$, there exist unique scalars $a_1, \dots, a_n$ such that

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

<u>Proof</u>: Let $v \in V$. Since β is a spanning set, $\exists$ $a_1, \ldots, a_n \in \mathbb{R}$ s.t

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Suppose $v = b_1 v_1 + \cdots + b_n v_n$ be any linear combination of $v_1, \ldots, v_n$ equal to v.

Then

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = b_1v_1 + b_2v_2 + \cdots + b_nv_n.$$

$$\Rightarrow \quad (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n = 0$$

But β is linearly independent.

Hence $\quad a_1 - b_1 = 0, \quad a_2 - b_2 = 0, \ldots, \quad a_n - b_n = 0$

$$\Rightarrow \quad a_i = b_i \quad \forall \; i = 1, 2, \ldots, n.$$

Hence every vector in V can be written as an unique linear combination in β. ——— $\blacksquare$.

Example: Let $W = \{ (x, y, z) \in \mathbb{R}^3 : x + y + z = 0 \}$.

Let us try to obtain a basis for W.

Let $v_1 = (1, -1, 0)$. Then $v_1 \in W$.

$S = \{ (1, -1, 0) \}$

Question: Is S a basis?

S is linearly independent.

Claim: S is not a spanning set.

Exercise: Check that $(1,0,-1) \in W$ s.t. $(1,0,-1) \notin \text{span}(S)$.

$$\beta = \{(1,-1,0), (1,0,-1)\}$$

Question: Is β a basis of W?

Claim: β is linearly independent.

Suppose $a(1,-1,0) + b(1,0,-1) = (0,0,0)$

$\Rightarrow \quad \underbrace{-a=0, \quad -b=0} \quad \& \quad a+b=0$

$\Rightarrow \quad a=b=0$

Hence β is linearly indep.

Let $(x, y, z) \in W$ i.e $x+y+z=0$

or $z = -x-y$

We want $a, b \in \mathbb{R}$ s.t

$a(1,-1,0) + b(1,0,-1) = (x, y, -x-y).$

$$(a+b, -a, -b) = (x, y, -x-y).$$

$$\Rightarrow \quad \left.\begin{aligned} a+b &= x \\ -a &= y \\ -b &= -x-y \end{aligned}\right\} \Rightarrow \quad \begin{aligned} a &= -y \\ b &= x+y. \end{aligned}$$

Therefore β is a spanning set & hence a basis.

<u>Theorem</u>: Let V be a vector space and $S \subseteq V$ be a linearly independent set. Let $v \notin S$ be a vector in V.

(i) If $v \in \text{span}(S)$, then $S \cup \{v\}$ is linearly dependent. Also, $\text{span}(S \cup \{v\}) = \text{span}(S)$.

(ii) If $v \notin \text{span}(S)$, then $S \cup \{v\}$ is linearly independent and $\text{span}(S \cup \{v\}) \supsetneq \text{span}(S)$.

Proof: If $v \in \text{span}(S)$, then $\exists\ v_1, \ldots, v_n \in S$ and $a_1, \ldots, a_n \in \mathbb{R}$ s.t.

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

Then $(-1)v + a_1v_1 + \cdots + a_nv_n = 0$

$\Rightarrow$ $S \cup \{v\}$ is linearly dependent.

Let $a_1u_1 + \cdots + a_nu_n$ be an elt. in $\text{span}(S \cup \{v\})$.

Exercise: Check that $a_1u_1 + \cdots + a_nu_n \in \text{span}(S)$.

Therefore, $\text{span}(S) = \text{span}(S \cup \{v\})$. — ■ (proved (i)).

(ii) If $v \notin span(S)$, then $S \cup \{v\}$ is linearly independent & $span(S) \subsetneq span(S \cup \{v\})$.

Suppose $S \cup \{v\}$ is not linearly independent.

then $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n + a v = 0 \longrightarrow (*)$

Suppose $a = 0$

$\Rightarrow a_1 v_1 + \cdots + a_n v_n = 0$

Linear independence of $S \Rightarrow a_i = 0$ which is a contradiction.

Hence $a \neq 0$

Then (*) can be written as

$$v = \left(-a_1/a\right)v_1 + \cdots + \left(-a_n/a\right)v_n$$

$\Rightarrow \quad v \in \text{span}(S)$. which is a contradiction.

Hence our assumption that $S \cup \{v\}$ is linearly dependent is false.

i.e $S \cup \{v\}$ is linearly independent.

Also, $S \subsetneq S \cup \{v\}$.

Then $\quad \text{span}(S) \subsetneq \text{span}(S \cup \{v\})$. $\quad$ —— ■

Let $W = \{ (x, y, z) \in \mathbb{R}^3 : x + y + z = 0 \}$

Recall that $\beta = \{ (1, -1, 0), (1, 0, -1) \}$ is a basis of W. $\beta' = \{ (1, -1, 0), (0, 1, -1) \}$ is also a basis of W.

* Consider $\mathbb{R}^3$

$$\beta = \{ e_1^{(3)} = (1, 0, 0), \; e_2^{(3)} = (0, 1, 0), \; e_3^{(3)} = (0, 0, 1) \}$$

is a basis of $\mathbb{R}^3$.

$\beta' = \{(1,1,0), (0,1,1), (1,0,1)\}$ is also a basis of $\mathbb{R}^3$.

Replacement Theorem:

Let V be a vector space. Suppose S is a spanning set having size n and L, a linearly independent set of size m in V. Then $m \leq n$. Moreover, there exists a subset S' of size $n-m$ such that $S' \cup L$ is a spanning set of V.

Proof: The proof is by induction on m.

If $m=0$, then L is empty & clearly $m \leq n$. Moreover if $S'=S$, then S' has $n-m$ elts & $S' \cup L = S$ is a spanning set.

Let $m > 0$. Assume that the theorem has been proved upto $m-1$.

Let $L = \{v_1, v_2, \ldots, v_m\}$

Then $\hat{L} = \{v_1, v_2, \ldots, v_{m-1}\}$ is a linearly independent set.

The induction hypothesis tells us that

$m-1 \leq n$ and that $\exists$ a subset $\hat{S}'$ of size $n-m+1$

s.t $\hat{L} \cup \hat{S}'$ is a spanning set of V.

Let $\tilde{S}' = \{w_1, w_2, \ldots, w_{n-m+1}\}$

Since $\tilde{S}' \cup \tilde{L}$ spans V, we have

$$v_m = a_1 v_1 + \cdots + a_{m-1} v_{m-1} + b_1 w_1 + \cdots + b_{n-m+1} w_{n-m+1}. \qquad \text{———} (*)$$

Suppose $m-1 = n \Rightarrow n-m+1 = 0$

Then $\tilde{S}'$ is empty.

i.e. $v_m = a_1 v_1 + \cdots + a_{m-1} v_{m-1}$

$$\Rightarrow \quad a_1 v_1 + \cdots + a_{m-1} v_{m+1} + (-1) v_m = 0$$

which contradicts the fact that L is linearly independent

Therefore $\quad m-1 < n \quad$ or $\quad m \leq n.$

In $(*)$ if $b_1, b_2, \ldots, b_{n-m+1}$ are all 0, then

$$v_m = a_1 v_1 + \cdots + a_{m-1} v_{m-1}$$

$\Rightarrow$ L is linearly dependent which is a contradiction.

Hence at least one of $b_1, \ldots, b_{n-m+1}$ is non-zero.

Assume WLOG that $b_1 \neq 0$

Then $w_1 = \left(-\frac{1}{b_1}\right) v_m + \left(-\frac{a_1}{b_1}\right) v_1 + \cdots + \left(-\frac{a_{m-1}}{b_1}\right) v_1$

$+ \left(-\frac{b_2}{b_1}\right) w_2 + \cdots + \left(-\frac{b_{n-m+1}}{b_1}\right) w_{n-m+1}.$

i.e w_1 is a linear comb. of $v_1, \ldots, v_m, w_2, \ldots, w_{n-m+1}$

Let $S' = \{w_2, \ldots, w_{n-m+1}\}.$

$$\operatorname{span}(S' \cup L) = \operatorname{span}(\tilde{S}' \cup L). \longrightarrow (**)$$

But $\tilde{S}' \cup \tilde{L} \subset \tilde{S}' \cup L$

Thus $\operatorname{span}(\tilde{S}' \cup \tilde{L}) \subset \operatorname{span}(\tilde{S}' \cup L)$.

$\Rightarrow$ $\tilde{S}' \cup L$ is a spanning set.

Therefore $S' \cup L$ is a spanning set by $(**)$.

Hence we have proved the result.

Let V be a vector space with a basis β containing d elts

Corollary: Any subset of V of size less than d cannot be a spanning set.

Proof: Suppose S be a subset of size d' which is less than d and s.t S is a spanning set.

The apply the replacement thm. to $L = \beta$ and S

We get $d' \geq d$. But this is a contradiction to the assumption that $d' < d$.

Therefore S cannot be spanning set. — $\blacksquare$

Hence any spanning set has size at least d.

Corollary: Any subset of V of size greater than d cannot be a linearly independent set.

Proof: Suppose L is a subset of size $d' > d$ and s.t L is linearly independent. Suppose $S = \beta$.

Replacement theorem gives

$d' \leq d$. a contradiction.

Hence L cannot be linearly independent.

Suppose L is infinite. Apply the same argument above to L' a subset of size $d+1$. Then L' is linearly dependent. $\Rightarrow$ L is linearly dependent.

Corollary: Any other basis of V should have d elts.

Proof: Let β' be another basis of size d'.

Then the first corollary tells us that

$$d' \geq d.$$

Second corollary forces $d' \leq d$.

$$\Rightarrow \quad d' = d.$$

Definition of dimension:

Let V be a vector space with a basis of size d. Then d is called the dimension of V. Suppose V is a vector space which does not contain a finite basis. Then V is called infinite dimensional.

Examples 1) $\dim(\mathbb{R}^n) = n$.

Consider $\beta = \{ e_1 = (1,0,\ldots,0), e_2 := (0,1,0,\ldots), \ldots, e_n = (0,\ldots,0,1)\}$

Then β is a basis of $\mathbb{R}^n$ called the standard basis of $\mathbb{R}^n$.

2) $P_n(\mathbb{R})$. Then $\beta = \{1, x, x^2, \ldots, x^n\}$ is a basis of $P_n(\mathbb{R})$

Hence $\dim(P_n(\mathbb{R})) = n+1$.

3) The zero vector space is the only vector space of dimension 0.

4) Consider $\mathcal{P}(\mathbb{R})$, vector space of all polynomials.

Suppose $\dim(\mathcal{P}(\mathbb{R})) = d$.

Consider $L = \{1, x, x^2, \ldots, x^d\}$. L is a linearly ind. set of size $d+1$. A contradiction to the first corollary.

Hence $\mathcal{P}(\mathbb{R})$ is infinite dimensional.

Proposition: Let V be a vector with a basis β consisting of d elements. Then any spanning set of size d is a basis of V.

Proof: Suppose S be spanning set of V of size d.

If S is not linearly independent, then by a theorem proved earlier, $\exists\ v \in S$ s.t $\text{span}(S \setminus \{v\}) = \text{span}(S)$.

i.e $\text{span}(S \setminus \{v\}) = V$.

Then $S \setminus \{v\}$ is a spanning set.

i.e. $\exists$ a set of size $d-1$ which is a spanning set.

which is a contradiction to the first corollary of the replacement theorem.

Hence S must be linearly independent.

Therefore S is a basis —— $\blacksquare$

Proposition: Let V be a vector space with a basis β containing d elements. Then every linearly independent set of size d is a basis.

Proof: Let L be a linearly independent set of size d.

Suppose L is not a spanning set.

Let $v \in V$ s.t. $v \notin \text{span}(L)$.

Then $L' = L \cup \{v\}$ is a linearly independent set.

But L' has cardinality $d+1$. which is a contradiction to a corollary to the replacement theorem.

Hence L is a spanning set. $\Rightarrow$ L is a basis — $\blacksquare$

Proposition: Let V be a finite dimensional vector space. Then any linearly independent set is contained in a basis.

Proof: Let β be a basis of size d & L be a linearly independent set of size d'.

Then $d' \leq d$.

By applying replacement theorem to L & $S = \beta$, $\exists$ a subset S' of β of size $d - d'$ s.t

$S' \cup L$ spans V.

But $S' \cup L$ has size d which is a spanning set.

$\Rightarrow$ $S' \cup L$ is a basis.

Hence L is contained in a basis ——— $\blacksquare$

<u>Proposition</u>: Let V be a finite dimensional vector space. Then every spanning set contains a basis.

Proof: Let S be a spanning set of V where $\dim(V) = d$.

Let d' is the largest integer there exists a linearly independent set of size d' contained in S.

Clearly $d' \leq d$

Suppose $d' < d$.

Then $\exists$ a set L of size d' & L is linearly independent. Let $v \in S$ s.t $v \notin \text{span}(L)$. Then $L \cup \{v\}$ is a linearly independent subset of V of size $d'+1$ which is a contradiction to our assumption

Hence $d' = d$.

i.e. L is a basis (Contained in S). — $\blacksquare$

Theorem: Let V be a finite dimensional vector space. If W is a subspace of V, then $\dim(W) \leq \dim(V)$. Moreover if $\dim(W) = \dim(V)$, then $W = V$.

Proof: Let $\dim(V) = d$

If W is the zero subspace, then $\dim(W) = 0 \leq \dim(V)$ with equality if $\dim(V) = 0$ i.e. V is the zero subspace.

Let v_1 be a non-zero vector in W.

If $\text{span}(\{v_1\}) = W$, then $\dim(W) = 1 \leq \dim(V)$

Assume $\{v_1\}$ does not spans W.

Suppose $v_2 \in W \setminus \text{span}(\{v_1\})$

Then $\{v_1, v_2\}$ is linearly ind.

If $\text{span}\{v_1, v_2\} = W$, then again $\dim(W) \leq \dim(V)$.

Assume $\{w_1, w_2\}$ does not span W.

Repeat the above algorithm.

After d steps, we would have obtained a subset $\{w_1, \ldots, w_d\}$ which is linearly ind. (& hence a basis)

Hence $\{w_1, \ldots, w_d\}$ spans V. Since $W \subseteq V$

$\Rightarrow$ $\{w_1, \ldots, w_d\}$ spans W.

$\Rightarrow$ $W = V$. ■