

Recall that a linear operator T on a vector space V of dim n is diagonalizable if \exists a basis consisting of eigenvectors of T .

If $\{v_1, \dots, v_n\}$ are linearly ind. eigenvectors of T , then T is a diagonalizable.

Theorem: Let T be a linear operator on a finite dimensional vector space V of dimension n . If $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of T and suppose v_i is an eigenvector of T corresponding to λ_i for each i . Then $\{v_1, v_2, \dots, v_k\}$ are linearly independent.

Proof: Let a_1, \dots, a_k be s.t.

$$a_1 v_1 + a_2 v_2 + \dots + a_k v_k = 0$$

$$(T - \lambda_k I) v_j = (\lambda_j - \lambda_k) v_j$$

$$\therefore (T - \lambda_k I) (a_1 v_1 + \dots + a_k v_k) = (T - \lambda_k I) 0 = 0$$

$$\Rightarrow a_1 (\lambda_1 - \lambda_k) v_1 + \dots + a_{k-1} (\lambda_{k-1} - \lambda_k) v_{k-1} = 0$$

Consider $(T - \lambda_{k-1} I) v_j = (\lambda_j - \lambda_{k-1}) v_j$

By a similar argument as above, we have

$$a_1 (\lambda_1 - \lambda_k) (\lambda_1 - \lambda_{k-1}) v_1 + \dots + a_{k-2} (\lambda_{k-2} - \lambda_k) (\lambda_{k-2} - \lambda_{k-1}) v_{k-1} = 0$$

After $k-1$ such steps, we have

$$a_1 (\lambda_1 - \lambda_k) (\lambda_1 - \lambda_{k-1}) \dots (\lambda_1 - \lambda_2) v_1 = 0$$

$$\Rightarrow a_1 = 0 \quad \left(\text{since } (\lambda_1 - \lambda_k) \dots (\lambda_1 - \lambda_2) \neq 0 \right)$$

Hence we have

$$a_2 v_2 + \dots + a_k v_k = 0$$

By a similar argument, $a_2 = 0$. & so on.

Therefore

$\{v_1, \dots, v_k\}$ are linearly independent. \longrightarrow

Corollary: If an $n \times n$ matrix A has n distinct eigenvalues then A is diagonalizable.

Example:

Consider

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$$

The characteristic polynomial of A

is given by $f(\lambda) = (1-\lambda)(4-\lambda) + 2$

$$= \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3).$$

Example 1: $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

The characteristic polynomial of A is

$$f(\lambda) = (\lambda - 2)^2$$

Example 2: $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

Characteristic polynomial of A is given by

$$f(\lambda) = (\lambda - 2)^2.$$

Hence 2 is the only eigenvalue of A.

Let $v = (x, y)$ be an eigenvector.

$$L_A v = Av = 2v \quad \text{i.e.} \quad (2x+y, 2y) = 2(x, y)$$

$$\Rightarrow \quad 2x+y=2x \quad \& \quad 2y=2y \quad \Rightarrow \quad y=0.$$

$$\text{i.e.} \quad E_2 \subseteq W = \{(x, y) : y=0\}.$$

$$\dim(W) = 1.$$

$$\Rightarrow \dim(E_2) \leq \dim(W) = 1.$$

$$\Rightarrow \dim(E_2) = 1.$$

Hence there does not exist a basis of V consisting of eigenvectors of T .
Therefore T is not diagonalizable.

Example 3 : Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

The characteristic poly of A is

$$\underline{f(\lambda) = (\lambda-2)^2(\lambda-3)}.$$

Hence 2 and 3 are eigenvalues of A .

Let $v \in E_2$ then $L_A v = 2v$.

$$\begin{pmatrix} 2x \\ 2y+3 \\ 3z \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}.$$

$$\Rightarrow 2y+3 = 2y \quad \text{or} \quad 3=0.$$

$$E_2 \subset W_1 = \{ (x, y, z) : z=0 \}.$$

$v_1 := (1, 0, 0)$, $v_2 := (0, 1, 0)$ is a basis of W ,

$$Av_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ \& hence } v_1 \in E_2.$$

$$Av_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ \& hence } v_2 \in E_2.$$

$$\begin{aligned} E_3 &\subseteq W_2 = \{ (x, y, z) : 2x = 3x, 2y + \beta = 3y, 3z = 3\beta \} \\ &= \{ (x, y, z) : x = 0 \text{ \& } y = \beta \} \\ &= \text{span} \{ (0, 1, 1) \}. \end{aligned}$$

Q note that $v_3 = (0, 1, 1) \in E_3$.

$\beta = ((1, 0, 0), (0, 1, 0), (0, 1, 1))$ is a basis of \mathbb{R}^3 consisting of eigenvectors of A .

Definition of (algebraic) multiplicity of an eigenvalue:

Let T be a linear operator on a finite dimensional vector space V of dimension n . Let λ_0 be an eigenvalue of T .

Then the (algebraic) multiplicity of λ_0 is the largest

positive integer k s.t. $(\lambda - \lambda_0)^k$ divides the characteristic polynomial $f(\lambda)$ of T .

Theorem: Let T be a linear operator on a finite dim. vector space V . Let λ_0 be an eigenvalue of T . Then $1 \leq \dim(E_{\lambda_0}) \leq$ multiplicity of λ_0 .

Proof: Let $k = \dim(E_{\lambda_0})$

Then $\exists \{v_1, \dots, v_k\}$ which is a basis of E_{λ_0} .

Extending this we get a basis $\beta = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$

$$[T]_{\beta}^{\beta} = \begin{pmatrix} \lambda_0 I_k & C_{k \times (n-k)} \\ 0_{(n-k) \times k} & B_{(n-k) \times (n-k)} \end{pmatrix}$$

Then the characteristic poly of T is given

$$f(\lambda) = \det([T]_{\beta}^{\beta} - \lambda I_n) = \det \begin{pmatrix} (\lambda_0 - \lambda) I_k & C \\ 0 & B - \lambda I_{n-k} \end{pmatrix}$$

$$= \det(\lambda_0 - \lambda) I_k \det(B - \lambda I_{n-k}).$$

$$= (\lambda_0 - \lambda)^k g(\lambda)$$

Hence $(\lambda - \lambda_0)^k$ divides $f(\lambda)$.

$\Rightarrow k \leq \text{multiplicity of } \lambda_0.$

i.e. $\dim(E_{\lambda_0}) \leq \text{ " } \quad \text{---} \quad \blacksquare$

Proposition: Let T be a linear operator on a finite dimensional vector space V . Let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of T and suppose $v_i \in E_{\lambda_i}$ for $i=1, 2, \dots, k$. If $v_1 + v_2 + \dots + v_k = 0$, then $v_i = 0$.

Proof: Exercise.

Theorem: Let T be a linear operator on V & $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of T . Suppose S_j be a linearly

independent set consisting of eigenvectors with eigenvalue λ_j .

Then $S = S_1 \cup S_2 \cup \dots \cup S_k$ is linearly independent.

Proof: Let $S_j = \{v_{j1}, v_{j2}, \dots, v_{jn_j}\}$.

Let a_{ji} be such that

$$\sum_{j=1}^k \sum_{i=1}^{n_j} a_{ji} v_{ji} = 0$$

Define

$$w_j = \sum_{i=1}^{n_j} a_{ji} v_{ji}$$

then $w_j \in E_{\lambda_j}$ (since $w_j \in \text{span}(S_j) \rightarrow S_j \subseteq E_{\lambda_j}$)

$$\text{Also } \sum_{j=1}^k \sum_{i=1}^{n_j} a_{ji} v_{ji} = \sum_{j=1}^k w_j = 0$$

By the previous proposition, we have

$$\Rightarrow \sum_{i=1}^{n_j} a_{ji} v_{ji} = 0 \quad \forall j$$

$$\Rightarrow a_{ji} = 0 \quad \forall j \text{ \& \& } i$$

hence S is linearly independent.

Theorem: Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial of T splits. Then T is diagonalizable iff the $\dim(E_{\lambda_i}) = \text{multiplicity of } \lambda_i$ for each eigenvalue λ_i of T .

Proof: Let $n = \dim(V)$ & $\lambda_1, \dots, \lambda_k$ be ^{distinct} eigenvalues of T .

Assume that T is diagonalizable.

Let β be a basis of V consisting of eigenvectors of T .

Let $\beta_i = \beta \cap E_{\lambda_i}$ & $n_i = \# \beta_i$

Let $d_i = \dim(E_{\lambda_i})$ and $m_i = \text{multiplicity of } \lambda_i$

By a previous theorem, $d_i \leq m_i$

& the fact that a linearly ind. set in a vector space of dim d_i has size at most d_i implies that

$$n_i \leq d_i$$

Notice that $\sum n_i = n$ (since β is a basis)

Also $\sum m_i = \deg(f(\lambda)) = n$

$$n = \sum n_i \leq \sum d_i \leq \sum m_i = n$$

$$\Rightarrow \left. \begin{array}{l} \sum (m_i - d_i) = 0 \\ m_i - d_i \geq 0 \quad \forall i \end{array} \right\} *$$

$$(*) \Rightarrow m_i - d_i = 0 \quad \forall i$$

Let us now assume that $d_i = m_i \quad \forall i$

Let β_i be a basis of E_{λ_i}

By the previous theorem $\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_k$ is linearly independent.

$$\# \beta = \sum \# \beta_i = \sum d_i = \sum m_i = n = \dim(V)$$

Hence β is a basis consisting of eigenvectors of T .

Hence T is diagonalizable.

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Definition: Let T be a linear transformation from a vector space V to itself. A subspace W is said to be invariant under T or T -invariant if $Tv \in W \quad \forall v \in W$ (i.e. $T(W) \subseteq W$).

Examples: 1) $\{0\}$ is a T -invariant for every linear operator T .

2) V is T -invariant for every linear operator T .

3) $W = \text{Null}(T)$ is a T -invariant subspace.

4) $W = R(T)$ is a T -invariant subspace.

5) Let v be an eigenvector with eigenvalue λ .

$$\text{Let } W = \text{span}(\{v\})$$

$$T(av) = a\lambda v = \lambda av \in \text{span}(\{v\}).$$

Hence W is T -invariant.

6) E_λ the eigenspace corresponding to λ is T -invariant.

If $T|_W$ is the restriction of T to the subspace W ,
then $T|_W$ is a linear operator on W .

Theorem: Let T be a linear operator on V and W be a T -invariant subspace. Then the characteristic polynomial of $T|_W$ divides the char. poly. of T .
a finite dimensional v.s.

Proof: Let $\alpha = (v_1, \dots, v_k)$ be an ordered basis of W .
and $\beta = (v_1, \dots, v_k, v_{k+1}, \dots, v_n)$ be an ordered basis of V .

$$[T|_W]_{\alpha}^{\alpha} = B \quad \text{a } k \times k \text{ matrix}$$

$$A = [T]_{\beta}^{\beta} = \begin{pmatrix} B & C_{k \times (n-k)} \\ 0_{(n-k) \times k} & D_{(n-k) \times (n-k)} \end{pmatrix}$$

Let $f(\lambda)$ be the char. poly of T & $g(\lambda)$ be the char. poly of $T|_W$. i.e $f(\lambda) = \det(A - \lambda I_n)$ & $g(\lambda) = \det(B - \lambda I_k)$.

$$\begin{aligned}
 f(\lambda) = \det(A - \lambda I_n) &= \det \begin{pmatrix} B - \lambda I_k & C \\ 0 & D - \lambda I_{n-k} \end{pmatrix} \\
 &= \det(B - \lambda I_k) \det(D - \lambda I_{n-k}). \\
 &= g(\lambda) p(\lambda).
 \end{aligned}$$

& hence $g(\lambda) \mid f(\lambda)$. □

Another example of a T-invariant subspace.

Let $v \in V$ and define

$$W = \text{span} \{ v, Tv, T^2v, \dots \}$$

where $T^k v = \underbrace{T(T(T \dots (Tv)))}_{k\text{-times}}$

Exercise: W is invariant under T .

Definition: The subspace W is called the T -cyclic subspace generated by v .

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Theorem: Let W be a T -cyclic subspace generated by a non-zero vector v .
Suppose $\dim(W) = k$. Then $\{v, Tv, \dots, T^{k-1}v\}$ is a basis of W . Moreover, if $a_0v + a_1Tv + \dots + a_{k-1}T^{k-1}v + T^kv = 0$, then the characteristic polynomial of $T|_W$ is given by $g(\lambda) = (-1)^k (a_0 + a_1\lambda + \dots + a_{k-1}\lambda^{k-1} + \lambda^k)$.

Proof: Let j be the largest positive integer such that $\{v, Tv, \dots, T^{j-1}v\}$ is a linearly independent set.

Then $T^j v \in \text{span} \{v, \dots, T^{j-1} v\}$.

Claim: $T^l v \in \text{span} \{v, \dots, T^{j-1} v\}$ for all $l \geq j$.

Let us assume that the claim is proved for upto $l-1$.
(Base case is $l=j$).

$$T^l(v) = T(T^{l-1}v)$$

But $T^{l-1}v \in \text{span} \{v, \dots, T^{j-1}v\}$.

$$\Rightarrow T^{l-1}v = b_0 v + \dots + b_{j-1} T^{j-1}v$$

$$\begin{aligned} \Rightarrow T^l v &= (b_0 T v + \dots + b_{j-2} T^{j-1} v) + b_{j-1} T^j v \\ &\in \text{span} \{v, T v, \dots, T^{j-1} v\}. \end{aligned}$$

i.e. $T^l v \in \text{span}(v, \dots, T^{j-1} v) \quad \forall l.$

$$\Rightarrow W \subset \text{span}(v, \dots, T^{j-1} v) \subseteq W$$

$$\text{span}(v, \dots, T^{j-1} v) = W$$

Hence $\{v, \dots, T^{j-1} v\}$ is a basis of W .

Since $\dim(W) = k$, we have $j = k$.

$$\text{Let } -T^k v = a_0 v + a_1 T v + \dots + a_{k-1} T^{k-1} v.$$

$$\text{Then for } \beta = (v, T v, \dots, T^{k-1} v),$$

$$[T|_W]_{\beta}^{\beta} = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & & \vdots & -a_1 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & 0 & -a_{k-1} \\ & & & 1 & \end{pmatrix}$$

Then the characteristic polynomial $g(\lambda)$ is given by

$$\det([T|_W]_{\beta}^{\beta} - \lambda I_k) = \det \begin{pmatrix} -\lambda & 0 & \dots & 0 & -a_0 \\ 1 & -\lambda & & \vdots & \vdots \\ 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & -\lambda & -a_{k-1} \\ & & & 1 & -a_{k-1}-\lambda \end{pmatrix}$$

$$= -\lambda \det \begin{pmatrix} -\lambda & 0 & \dots & -a_1 \\ 1 & -\lambda & & \vdots \\ 0 & 1 & & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \dots & -a_{k-2} \\ & & & -a_{k-1} - \lambda \end{pmatrix} + (-1)^{k+1} (-a_0) \det \begin{pmatrix} 1-\lambda & & & 0 \\ 1-\lambda & & & \\ & \ddots & & \\ 0 & & & 1-\lambda \end{pmatrix}$$

$$= -\lambda \left((-1)^{k-1} (a_1 + a_2 \lambda + \dots + a_{k-1} \lambda^{k-2} + \lambda^{k-1}) \right) + (-1)^k a_0$$

$$= (-1)^k (a_0 + a_1 \lambda + \dots + a_{k-1} \lambda^{k-1} + \lambda^k)$$

Given a polynomial $p(\lambda) = a_0 + a_1\lambda + \dots + a_n\lambda^n$
we define $p(T) := a_0I + a_1T + \dots + a_nT^n$.

Thus $p(T)v = a_0v + a_1Tv + \dots + a_nT^n v$.

Exercise: $p(T)q(T)v = q(T)p(T)v \quad \forall v \in V$
2 polynomials p, q .

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Cayley - Hamilton Theorem :

Let V be a finite dim. vector space & T be a linear operator on V with characteristic polynomial $f(\lambda)$. Then $f(T)$ is the zero operator.

Proof: Enough to show that given a vector $v \in V$,
 $f(T)v = 0$.

Let W be the T -cyclic subspace generated by v .

Then W is invariant under T

$\⊃ k$ suppose $a_0 v + a_1 T v + \dots + a_{k-1} T^{k-1} v + T^k v = 0 \longrightarrow (*)$

then $g(\lambda) = (-1)^k (a_0 + a_1 \lambda + \dots + \lambda^k)$ is the char. poly of $T|_W$.

$(*)$ can be rewritten as $g(T)v = 0$.

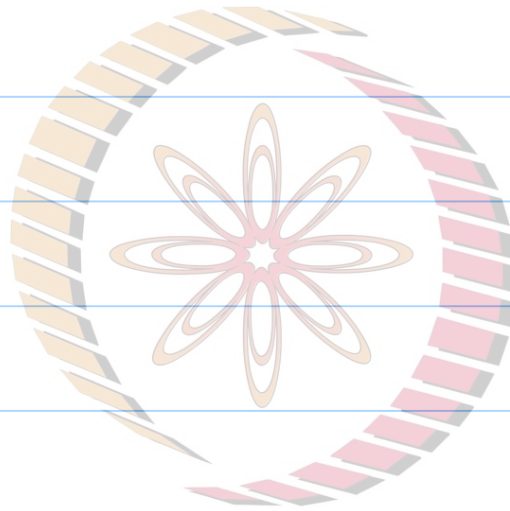
Let $f(\lambda)$ be the char. poly of T . By a theorem above,
 \exists a polynomial $p(\lambda)$ s.t

$$f(\lambda) = p(\lambda) g(\lambda)$$

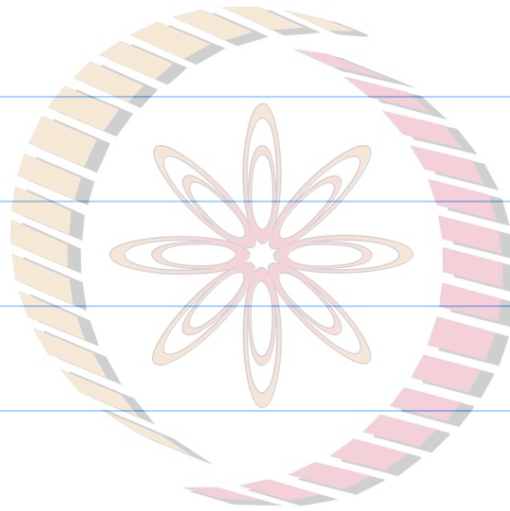
Hence

$$\begin{aligned} f(T)v &= p(T)(g(T)v) \\ &= p(T)0 \\ &= 0 \end{aligned}$$

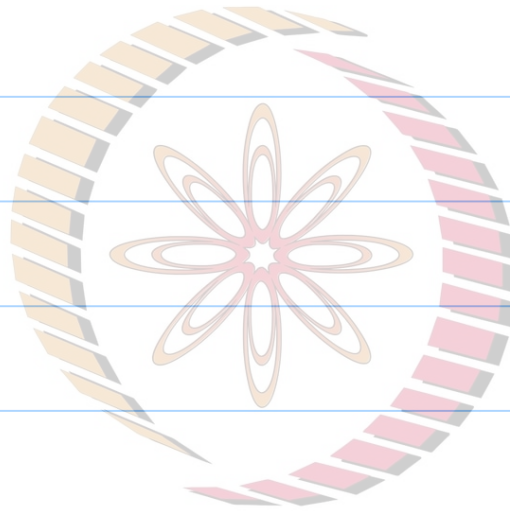
Hence $f(T)$ is the zero operator. — ■



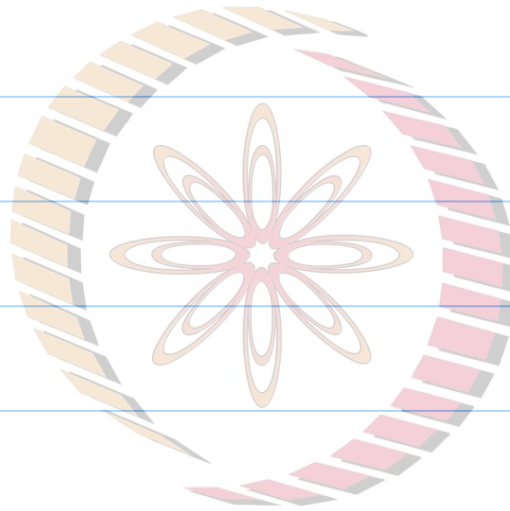
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