MODULE 1

PROBABILITY

PROBLEMS

1. Let $\Omega = \{1, 2, 3, 4\}$. Check which of the following is a sigma-field of subsets of Ω:

(i) $\mathcal{F}_1 = \{\phi, \{1, 2\}, \{3, 4\}\}$;

(ii) $\mathcal{F}_2 = \{\phi, \Omega, \{1\}, \{2, 3, 4\}, \{1, 2\}, \{3, 4\}\}$;

(iii) $\mathcal{F}_3 = \{\phi, \Omega, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{2, 3, 4\}, \{1, 3, 4\}\}$.

2. Show that a class $\mathcal{F}$ of subsets of Ω is a sigma-field of subsets of Ω if, and only if, the following three conditions are satisfied: (i) $\Omega \in \mathcal{F}$; (ii) $A \in \mathcal{F} \Rightarrow A^C = \Omega - A \in \mathcal{F}$; (iii) $A_n \in \mathcal{F}, \mathrm{n} = 1, 2, \cdots \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

3. Let $\{\mathcal{F}_\lambda : \lambda \in \Lambda\}$ be a collection of sigma-fields of subsets of Ω.

(i) Show that $\bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ is a sigma-field;

(ii) Using a counter example show that $\bigcup_{\lambda \in \Lambda} \mathcal{F}_\lambda$ may not be a sigma-field;

(iii) Let $\mathcal{C}$ be a class of subsets of Ω and let $\{\mathcal{F}_\lambda : \lambda \in \Lambda\}$ be a collection of all sigma-fields that contain the class $\mathcal{C}$. Show that $\sigma(\mathcal{C}) = \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$, where $\sigma(\mathcal{C})$ denotes the smallest sigma-field containing the class $\mathcal{C}$ (or the sigma-field generated by class $\mathcal{C}$).

4. Let Ω be an infinite set and let $\mathcal{A} = \{A \subseteq \Omega : A \text{ is finite or } A^C \text{ is finite}\}$.

(i) Show that $\mathcal{A}$ is closed under complements and finite unions;

(ii) Using a counter example show that $\mathcal{A}$ may not be closed under countably infinite unions (and hence $\mathcal{A}$ may not be a sigma-field).

5. (i) Let Ω be an uncountable set and let

$$\mathcal{F} = \{A \subseteq \Omega : A \text{ is countable or } A^C \text{ is countable}\}.$$

(a) Show that $\mathcal{F}$ is a sigma-field;

(b) What can you say about $\mathcal{F}$ when Ω is countable?

(ii) Let Ω be a countable set and let $\mathcal{C} = \{\{\omega\} : \omega \in \Omega\}$. Show that $\sigma(\mathcal{C}) = \mathcal{P}(\Omega)$.

6. Let $\mathcal{F} = \mathcal{P}(\Omega)$ =the power set of $\Omega = \{0, 1, 2, \dots\}$. In each of the following cases, verify if $(\Omega, \mathcal{F}, P)$ is a probability space:

(i) $P(A) = \sum_{x \in A} e^{-\lambda} \lambda^x / x!\,, A \in \mathcal{F}, \lambda > 0$;
(ii) $P(A) = \sum_{x \in A} p(1-p)^x\,, A \in \mathcal{F}, 0 < p < 1$;
(iii) $P(A) = 0$, if A has a finite number of elements, and $P(A) = 1$, if A has infinite number of elements, $A \in \mathcal{F}$.

7. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $A, B, C, D \in \mathcal{F}$. Suppose that $P(A) = 0.6, P(B) = 0.5,\ P(C) = 0.4,\ P(A \cap B) = 0.3,\ P(A \cap C) = 0.2,\ P(B \cap C) = 0.2,\ P(A \cap B \cap C) = 0.1,\ P(B \cap D) = P(C \cap D) = 0, P(A \cap D) = 0.1$ and $P(D) = 0.2$.
Find:
(i) $P(A \cup B \cup C)$ and $P(A^C \cap B^C \cap C^C)$;
(ii) $P((A \cup B) \cap C)$ and $P(A \cup (B \cap C))$;
(iii) $P((A^C \cup B^C) \cap C^C)$ and $P((A^C \cap B^C) \cup C^C)$;
(iv) $P(B \cap C \cap D)$ and $P(A \cap C \cap D)$;
(v) $P(A \cup B \cup D)$ and $P(A \cup B \cup C \cup D)$;
(vi) $P\big((A \cap B) \cup (C \cap D)\big)$.

8. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let A and B be two events (i.e., $A, B \in \mathcal{F}$).
(i) Show that the probability that exactly one of the events A or B will occur is given by $P(A) + P(B) - 2P(A \cap B)$;
(ii) Show that $P(A \cap B) - P(A)P(B) = P(A)P(B^C) - P(A \cap B^C) = P(A^C)P(B) - P(A^C \cap B) = P((A \cup B)^C) - P(A^C)P(B^C)$.

9. Suppose that $n\ (\geq 3)$ persons $P_1, \dots, P_n$ are made to stand in a row at random. Find the probability that there are exactly r persons between P_1 and P_2; here $r \in \{1, 2, \dots, n-2\}$.

10. A point (X, Y) is randomly chosen on the unit square $S = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ (i.e., for any region $R \subseteq S$ for which the area is defined, the probability that (X, Y) lies on R is $\frac{\text{area of } R}{\text{area of } S}$). Find the probability that the distance from (X, Y) to the nearest side does not exceed $\frac{1}{3}$ units.

11. Three numbers a, b and c are chosen at random and with replacement from the set $\{1, 2, \dots, 6\}$. Find the probability that the quadratic equation $ax^2 + bx + c = 0$ will have real root(s).

12. Three numbers are chosen at random from the set $\{1, 2, \dots, 50\}$. Find the probability that the chosen numbers are in
(i) arithmetic progression;

(ii) geometric progression.

13. Consider an empty box in which four balls are to be placed (one-by-one) according to the following scheme. A fair die is cast each time and the number of dots on the upper face is noted. If the upper face shows up 2 or 5 dots then a white ball is placed in the box. Otherwise a black ball is placed in the box. Given that the first ball placed in the box was white find the probability that the box will contain exactly two black balls.

14. Let $\big((0,1], \mathcal{F}, P\big)$ be a probability space such that $\mathcal{F}$ is the smallest sigma-field containing all subintervals of $\Omega = (0,1]$ and $P((a,b]) = b - a$, where $0 \le a < b \le 1$ (such a probability measure is known to exist).

(i) Show that $\{b\} = \bigcap_{n=1}^{\infty} \left(b - \frac{1}{n+1}, b\right], \forall b \in (0,1]$;

(ii) Show that $P(\{b\}) = 0, \forall b \in (0,1]$ and $P\big((0,1]\big) = 1$ (Note that here $P(\{b\}) = 0$ but $\{b\} \neq \phi$ and $P\left((0,1)\right) = 1$ but $(0,1) \neq \Omega$);

(iii) Show that, for any countable set $A \in \mathcal{F}, P(A) = 0$;

(iv) For $n \in \mathbb{N}$, let $A_n = \left(0, \frac{1}{n}\right]$ and $B_n = \left(\frac{1}{2} + \frac{1}{n+2}, 1\right]$. Verify that $A_n \downarrow, B_n \uparrow$, $P(\mathrm{Lim}_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n)$ and $P(\mathrm{Lim}_{n\to\infty} B_n) = \lim_{n\to\infty} P(B_n)$.

15. Consider four coding machines M_1, M_2, M_3 and M_4 producing binary codes 0 and 1. The machine M_1 produces codes 0 and 1 with respective probabilities $\frac{1}{4}$ and $\frac{3}{4}$. The code produced by machine M_k is fed into machine $M_{k+1} (k = 1,2,3)$ which may either leave the received code unchanged or may change it. Suppose that each of the machines M_2, M_3 and M_4 change the received code with probability $\frac{3}{4}$. Given that the machine M_4 has produced code 1, find the conditional probability that the machine M_1 produced code 0.

16. A student appears in the examinations of four subjects Biology, Chemistry, Physics and Mathematics. Suppose that probabilities of the student clearing examinations in these subjects are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$ respectively. Assuming that the performances of the students in four subjects are independent, find the probability that the student will clear examination(s) of

(i) all the subjects; (ii) no subject; (iii) exactly one subject;

(iv) exactly two subjects; (v) at least one subject.

17. Let A and B be independent events. Show that

$$\max\{P((A \cup B)^c), P(A \cap B), P(A\Delta B)\} \geq \frac{4}{9},$$

where $A\Delta B = (A - B) \cup (B - A)$.

18. For independent events $A_1, \dots, A_n$, show that

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) \leq e^{-\sum_{i=1}^{n} P(A_i)}.$$

19. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $A_1, A_2, \dots$ be a sequence of events $(\text{i.e.}, A_i \in \mathcal{F}, i = 1, 2, \dots)$. Define $B_n = \bigcap_{i=n}^{\infty} A_i$, $C_n = \bigcup_{i=n}^{\infty} A_i, n = 1,2, \dots, D = \bigcup_{n=1}^{\infty} B_n$ and $E = \bigcap_{n=1}^{\infty} C_n$. Show that:

(i) D is the event that all but a finite number of A_ns occur and E is the event that infinitely many A_ns occur;

(ii) $D \subseteq E$;

(iii) $P(E^c) = \lim_{n\to\infty} P(C_n^c) = \lim_{n\to\infty} \lim_{m\to\infty} P(\bigcap_{k=n}^{m} A_k^c)$ and $P(E) = \lim_{n\to\infty} P(C_n)$;

(iv) if $\sum_{n=1}^{\infty} P(A_n) < \infty$ then, with probability one, only finitely many A_ns will occur;

(v) if $A_1, A_2, \dots$ are independent and $\sum_{n=1}^{\infty} P(A_n) < \infty$ then, with probability one, infinitely many $A_n s$ will occur.

20. Let A, B and C be three events such that A and B are negatively (positively) associated and B and C are negatively (positively) associated. Can we conclude that, in general, A and C are negatively (positively) associated?

21. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let A and B two events $(\text{i.e.}, A, B \in \mathcal{F})$. Show that if A and B are positively (negatively) associated then A and B^c are negatively (positively) associated.

22. A locality has n houses numbered $1, \dots, n$ and a terrorist is hiding in one of these houses. Let H_j denote the event that the terrorist is hiding in house numbered $j, j = 1, \dots, n$ and let $P(H_j) = p_j \in (0,1),\ j = 1, \dots, n$. During a search operation, let F_j denote the event that search of the house number j will fail to nab the terrorist there and let $P(F_j|H_j) = r_j \in (0,1),\ j = 1, \dots, n$. For each $i, j \in \{1, \dots, n\}, i \neq j$, show that H_j and F_j are negatively associated but H_i and F_j are positively associated. Interpret these findings.

23. Let A, B and C be three events such that $P(B \cap C) > 0$. Prove or disprove each of the following:

(i) $P(A \cap B|C) = P(A|B \cap C)P(B|C)$;

(ii) $P(A \cap B|C) = P(A|C)P(B|C)$ if A and B are independent events.

24. A *k-out-of-n system* is a system comprising of n components that functions if, and only if, at least k ($k \in \{1,2,\dots,n\}$) of the components function. A 1-out-of-n system is called a *parallel system* and an n-out-of-n system is called a *series system*. Consider n components $C_1, \dots, C_n$ that function independently. At any given time t the probability that the component C_i will be functioning is $p_i(t)$ $\big(\in (0,1)\big)$ and the probability that it will not be functioning at time t is $1 - p_i(t), i = 1, \dots, n$.

(i) Find the probability that a parallel system comprising of components $C_1, \dots, C_n$ will function at time t;

(ii) Find the probability that a series system comprising of components $C_1, \dots, C_n$ will function at time t;

(iii) If $p_i(t) = p(t), i = 1, \dots, n$, find the probability that a k-out-of-n system comprising of components $C_1, \dots, C_n$ will function at time t.