An elementary row operation is a map which takes an $m \times n$ matrix and gives an $m \times n$ matrix.

Column

<u>Type 1 row operations</u>:

column

Let A be an $m \times n$ matrix. A type 1 row operation

Column Column

exchanges a row i with a row j.

eg: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & 4 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{row 2 \& row 3}]{\text{interchanging}} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 6 & 7 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$

Consider a matrix E obtained by interchang the i^{th} row and j^{th} row of the identity matrix of size m.

Column Column size n

Then the elementary row operation of type 1 is obtained by multiplying E to A from the left.

Exercise: E is invertible.

Such a matrix E is called an elementary matrix of type 1.

Type 2 row operations

In this type of row operation, a row i of the matrix is multiplied by a non-zero scalar c.

eg: $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & 4 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & 4 \\ 5 & 0 & 10 \\ 0 & 0 & 1 \end{pmatrix}$$

multiplying by 5 to row 3

Consider a matrix E obtained by multiplying the i^{th} row by non-zero scalar c of the identity matrix I_m.

Then the type 2 row operation described above is obtained by multiplying EA.

Exercise: Check that E is invertible.

Such matrices are called elementary matrices of type - 2.

Type - 3 row operation

In this operation a scalar times row i is added to row j of the matrix A.

eg $\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & 4 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{added to row 1}]{\text{3 times row 4}} \begin{pmatrix} 1 & 2 & 6 \\ 6 & 7 & 4 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

Suppose E is a matrix obtained by adding c times the i^{th} row to the j^{th} row of the identity matrix I_m, then a type 3 row operation is obtained by multiplication of such a matrix E to A.

Exercise: Check that E is invertible.

Such a matrix E is said to be an elementary matrix of type - 3.

Row - Echelon form of a matrix

A matrix is said to be in its row echelon form if every row is either zero or if every row has the first non-zero entry as 1 and s.t. every entry below is zero.

eg:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$ is in the row echelon form.

$$\begin{pmatrix} 1 & 0 & \pi & e & 2 \\ 0 & 1 & 5 & 2 & 6/7 \\ 0 & 0 & 0 & 1 & 22 \end{pmatrix}$$ is in the row-echelon form.

Consider a system of linear equation

x_0 is a soln to $Ax = b$ $\iff$ x_0 is a soln to $BAx = Bb$ where B is invertible

Let

$E_1, E_2, \ldots, E_n$ be elementary matrices s.t

$E_1 E_2 \ldots E_n A$, we get a row-echelon form.

$$EE_2 \ldots E_n A x = E_1 E_2 \ldots E_n b$$

Recall that if $T: V \to W$ be a linear transformation, then $\text{rank}(T) = \dim(R(T))$.

<u>Proposition</u>: If $T: V \to W$ be a linear transformation between vector spaces each of dimension n. Then T is invertible if and only if $\text{rank}(T) = n$.

<u>Proof</u>: If $\text{rank}(T) = n$, then $R(T) = W \Rightarrow T$ is surjective.

$$\dim(V) = n = \dim(N(T)) + n \Rightarrow N(T) = \{0\}$$

$\Rightarrow$ T is injective.

$\Rightarrow$ T is invertible.

<u>Proposition</u>: Let $T : V \to W$ be a linear transformation. Suppose $S : U \to V$ and $Q : W \to Z$ be invertible linear transformations. Then

$$\text{rank}(T) = \text{rank}(TS) = \text{rank}(QTS) = \text{rank}(QT)$$

Proof: Observe that $R(TS) = TS(\mathcal{U})$.

S - invertible $\Rightarrow$ S - surjective $\Rightarrow$ $S(\mathcal{U}) = V$

$\Rightarrow$ $R(TS) = R(T)$

$\Rightarrow$ $\text{rank}(TS) = \text{rank}(T)$.

$\Rightarrow$ $\text{rank}(QTS) = \text{rank}(QT)$

Since is invertible

$R(QT) = QT(V) = Q(R(T))$.

Since Q is invertible $\dim(Q(R(T))) = \dim(R(QT))$

//

$$\dim(Q(R(T))) = \dim(R(T)) \quad \text{(by dimension thm.)}$$

$$\operatorname{rank}(QT) = \operatorname{rank}(T).$$

Let A be an $m \times n$ matrix. Then

$L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transf. corresponding to A

$$L_A x := Ax$$

We define the rank of the matrix A to be the rank(L_A).

eg: Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$R(L_A) = \left\{ \begin{pmatrix} x_1 \\ x_3 \\ x_3 \\ 0 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$$

Therefore $\operatorname{rank}(A) = 3$.

Proposition: Let A be an $m \times n$ matrix, B be an invertible $m \times m$ matrix & C an $n \times n$ invertible matrix. Then $\operatorname{rank}(A) = \operatorname{rank}(BA) = \operatorname{rank}(AC) = \operatorname{rank}(BAC)$.

Proof: By defn, $\text{rank}(A) = \text{rank}(L_A)$.

& $\text{rank}(BA) = \text{rank}(L_{BA}) = \text{rank}(L_B L_A)$

$= \text{rank}(L_A)$ (since L_B is an inv. lin. trans.)

$= \text{rank}(A)$.

$\blacksquare$

Suppose A is an $m \times n$ matrix

$\text{rank}(A) = \text{rank}(L_A)$. $\quad L_A : \mathbb{R}^n \to \mathbb{R}^m$

Let $e_1, \ldots, e_n$ be the standard basis.

$L_A e_1$ is the first column of A

$L_A e_j$ is the j^{th} column of A.

Consider the span of the columns of A.

Since $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$, we have $\{L_A e_1, \ldots, L_A e_1\}$ is a spanning set of $R(L_A)$.

<u>Proposition</u>: Let A be an $m \times n$ matrix in its row echelon form. Then the rank of A is equal to the number of

non-zero rows of A.

Proof: If needed, after multiplying the required elementary matrices, we obtain a matrix with the first k-rows non-zero

Then the column space (Span of the columns of A) $\subseteq U = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} : x_i \in \mathbb{R} \right\}$

Since $\dim(U) = k$

we have $\operatorname{rank}(A) \leq k$

So enough to show that $\operatorname{rank}(A) \geq k$

Let $v = \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ be an elt. in U.

Let C_1 be the column of A containing the first non-zero entry of row k. Define

$v_1 = v - x_k C_1$ has zero in all rows below $k-1$. $v_1 = \begin{pmatrix} \vdots \\ y_{k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Suppose y_{k-1} is the entry in the $(k-1)^{th}$ row of v_1

Let C_2 be the column containing the first non-zero entry of row $k-1$

define $v_2 = v_1 - y_{k-1} C_k$.

Follow this procedure to obtain 0 after k-steps.

$\Rightarrow$ $v \in$ span of columns of A.

$\Rightarrow$ $U \subseteq$ Column space of A.

$\Rightarrow$ $k \leq \dim(R(A)) = \text{rank}(A)$. —— $\blacksquare$

Let A be arbitrary $m \times n$ matrix.

Let $E_1, \ldots, E_k$ be elementary matrices s.t

$E_1 E_2 \ldots E_k A$ is in its row echelon form.

<u>Theorem</u>: Let A be an $m \times n$ matrix of rank r. Then after finitely many row & column operations, we get a matrix of the type

$$\begin{pmatrix} I_r & O_{r\times n-r} \\ O_{m-r\times r} & O_{m-r\times n-r} \end{pmatrix}$$ where $O_{k\times l}$ is the zero matrix of size $k\times l$.

<u>Proof</u>: Let A be reduced to its row-echelon form A' s.t the first r rows are non-zero.

$$\Rightarrow \quad A' = \begin{pmatrix} * \\ \hline 0 \end{pmatrix} \begin{matrix} r \text{ rows} \\ \} \, m-r \text{ rows.} \end{matrix}$$

Consider row 1 of A'. Add the relevant multiple of the column containing the first non-zero entry in the first row to subsequent columns to obtain zeroes in the first row. (Column operations of type 3).

Repeat the same process to all subsequent rows.

Now apply column operations of type 1 to obtain a matrix of type required. ——— $\blacksquare$.

We have proved that A is an $m \times n$ matrix of rank r, then $\exists$ elementary matrices $E_1, \dots, E_k$ of size m & elementary matrices $F_1, \dots, F_\ell$ of size n s.t.

$$E_1 \dots E_k A F_1 \dots F_\ell = \begin{pmatrix} I_r & O_{r \times n-r} \\ O_{m-r \times r} & O_{m-r \times n-r} \end{pmatrix}$$

$$\Rightarrow \quad A = \underbrace{E_k^{-1} \cdots E_1^{-1}}_{B} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \underbrace{F_\ell^{-1} \cdots F_1^{-1}}_{C} .$$

Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$. Then transpose of A

is given by $\begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}$

Notice that $(AB)^t = B^t A^t$

If $AA^{-1} = I$, then $(A^{-1})^t A^t = I$

$$\Rightarrow \quad (A^t)^{-1} = (A^{-1})^t.$$

<u>Theorem</u>: If an $m \times n$ matrix A has rank r, then so does its transpose.

Proof: By the above theorem,

$$A = B \begin{pmatrix} I_r & O_{r\times n-r} \\ O_{m-r\times r} & O_{m-r\times n-r} \end{pmatrix} C$$

$$A^t = C^t \begin{pmatrix} I_r & O_{r\times m-r} \\ O_{n-r\times r} & O_{n-r\times m-r} \end{pmatrix} B^t .$$

$$\Rightarrow \quad \text{rank}(A^t) = \text{rank}(\quad '' \quad) = r = \text{rank}(A).$$

Theorem: Let $T: V \longrightarrow W$ be a linear transformation. Suppose α & β are finite ordered bases of V & W. Then

$$\operatorname{rank}(T) = \operatorname{rank}\left([T]_\alpha^\beta\right).$$

Proof: Let $\phi_\alpha : V \longrightarrow \mathbb{R}^n$ (where $n = \dim V$) be defined by $\phi_\alpha(v) = [v]^\alpha$

Check that ϕ_α is an isomorphism.

$|||^{ly}$ define $\phi_\beta : W \longrightarrow \mathbb{R}^m$ where $\phi_\beta(w) = [w]^\beta$.

then $[Tv]^\beta = [T]_\alpha^\beta [v]^\alpha$

Rewriting $\phi_\beta(Tv) = L_{[T]_\alpha^\beta} \phi_\alpha(v) \quad \forall \; v \in V$

$\Rightarrow \quad \phi_\beta T = L_{[T]_\alpha^\beta} \phi_\alpha$

$\Rightarrow \quad T = \phi_\beta^{-1} L_{[T]_\alpha^\beta} \phi_\alpha$

$$\mathrm{rank}(T) = \mathrm{rank}\left(L_{[T]_\alpha^\beta}\right) = \mathrm{rank}\left([T]_\alpha^\beta\right).$$

Example: $T : P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ given by

$$Tf = f - xf'$$

Check that T is a linear transformation.

Fix $\beta = \{1, x, x^2, x^3\}$

$$[T]_{\beta}^{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$\Rightarrow \quad \text{rank}(T) = 3.$$

Theorem: Let A be an $n \times n$ matrix. Then A is invertible if and only if A can be written as a product of elementary matrices.

Proof: If A is a product of elementary matrices, since the product of invertible matrices is invertible, we have that A is invertible.

Suppose A is an invertible matrices. Then the linear transformation L_A is invertible.

Recall that $L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ (where A is $n \times n$).

$\Rightarrow \quad \text{rank}(L_A) = n.$

$\Rightarrow \quad \text{rank}(A) = n$

By a theorem proved earlier, we have elementary matrices $E_1, \ldots, E_k$ and $F_1, \ldots, F_\ell$ s.t

$$A = E_1 \ldots E_k I_n F_1 \ldots F_\ell.$$

$$= E_1 \ldots E_k F_1 \ldots F_\ell.$$

$\blacksquare$

Suppose A be an invertible matrix, Then

$$A = E_1 E_2 \ldots E_k.$$

$$E_k^{-1} \cdots E_2^{-1} E_1^{-1} A = I$$

$$E_k^{-1} \cdots E_1^{-1} I = A^{-1}$$

<u>Example:</u> $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\begin{pmatrix}1&2\\3&4\end{pmatrix} \quad \Big| \quad \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\begin{pmatrix}1&0\\-3&0\end{pmatrix} \quad E_1 : R_2 \longrightarrow R_2 - 3R_1 \qquad \begin{pmatrix}1&2\\0&-2\end{pmatrix} \quad \Big| \quad \begin{pmatrix}1&0\\-3&1\end{pmatrix}$$

$$E_2 : R_1 \longrightarrow R_1 + R_2 \qquad \begin{pmatrix}1&0\\0&-2\end{pmatrix} \quad \Big| \quad \begin{pmatrix}-2&1\\-3&1\end{pmatrix}$$

$$E_3 : R_2 \longrightarrow \frac{-1}{2} R_2 \qquad \begin{pmatrix}1&0\\0&1\end{pmatrix} \quad \Big| \quad \begin{pmatrix}-2&1\\3/2&-1/2\end{pmatrix} \checkmark$$

Let A be a 1×1 matrix. i.e. $A = (a)$

then $\det(A) := a$.

Let A be a 2×2 matrix. say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then define $\det(A) := ad - bc$.

Let A be an $n \times n$ matrix. Then given a row i and column j, we denote by $\hat{A}_{ij}$, the minor of A w.r.t row i & column j.

$$\text{Suppose } A = \begin{array}{cc} & \text{row } j \\ & \downarrow \\ \text{row } i \rightarrow & \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \end{array}$$

Then the minor $\tilde{A}_{ij}$ is an $(n-1)\times(n-1)$ matrix obtained by removing the i^{th} row and j^{th} column.

eg: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Then $\tilde{A}_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$

$$\tilde{A}_{23} = \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix} \quad \text{etc.}$$

<u>Definition of a determinant</u> :

Let A be an $n \times n$ matrix. Pick a row i. Then define

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\tilde{A}_{ij}).$$

This expression is called the cofactor expansion along row i.

eg: Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

Pick row 1. $\tilde{A}_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$; $\tilde{A}_{12} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$; $\tilde{A}_{13} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$

$$\det(A) = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}).$$

eg: (*) Let $A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ \vdots & & & \\ \vdots & & B & \\ \vdots & & & \end{pmatrix}$. Then

$$\det(A) = a_{11} \det(B).$$

(*) $A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 \cdots & 0 \\ \vdots & & \ddots & 0 \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$, then $\det(A) = a_{11} a_{22} \cdots a_{nn}$.

Suppose $v_1, \ldots, v_n$ be n vectors in $\mathbb{R}^n$.
Let $A := (v_1, \ldots, v_n)$, then we denote by
$\det(v_1, \ldots, v_n) := \det(A)$.

Properties of the determinant

Property 1: Let A be $n \times n$ matrix. Suppose

B is obtained by interchanging two rows of A.

Then $\det(B) = -\det(A)$.

Corollary: If two rows of a matrix (say i & j) are the same, then $\det(A) = 0$.

The B obtained by interchanging row i & row j is the same A.

$$\det(A) = -\det(A) \Rightarrow \det(A) = 0.$$

Exercise: (i) $\det(I_n) = 1$.

(ii) If E is an elementary matrix of type 1, then $\det(E) = -1$.

$$\det(EA) = -\det(A) = \det(E)\det(A).$$

Property 2: Let B be a matrix obtained from A by multiplying the i^{th} row by a scalar (say c).

Then $\det(B) = c\det(A)$.

Corollary: If A is a matrix with a row zero, then $\det(A) = 0$.

For $c = 0$, multiplying the zero row by c, we get $B = A$.

$$\det(A) = c\det(A) = 0.$$

Exercise: Check that if E is an elementary matrix of type 2 obtained by multiplying the i^{th} row by a scalar c, then

$$\det(E) = c.$$

$$\det(EA) = c\det(A) = \det(E)\det(A).$$

Property 3: Let B be a matrix obtained from A by an elementary row operation of type 3.

Then $\det(B) = \det(A)$.

Suppose B is obtained by replacing row j by c(row i) + row(j). Consider the defn of determinant along this row.

$$\det(B) = (c\,a_{i1} + a_{j1})\det(\tilde{A}_{j1}) - (c a_{i2} + a_{j2})\det(\tilde{A}_{j2}) + \cdots$$

$$= c\left(\cancel{a_{i1}\det\tilde{A}_{j1} - a_{i2}\det(\tilde{A}_{j2}) + \cdots} \to 0\right) + \det(A)$$

Hence elementary matrices of type 3 has determinant 1.

$$\det(EA) = \det(E)\det(A).$$

Lemma: If E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.

Theorem: Let A be an $n \times n$ matrix. Then A is invertible iff $\det(A) \neq 0$.

Proof: Suppose A is invertible, then

$$A = E_1 E_2 \dots E_k I$$

$$\det(A) = \det(E_1)\det(E_2)\cdots\det(E_k)\det(I)$$
$$\neq 0$$

Suppose $\det(A) \neq 0$.

We know that $A = E_1 \cdots E_k \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} F_1 \cdots F_\ell$.

$$\Rightarrow E_k^{-1} \cdots E_1^{-1} A F_\ell^{-1} \cdots F_1^{-1} = \begin{pmatrix} I_r & 0_{r\times n-r} \\ 0_{n-r\times r} & 0_{n-r\times n-r} \end{pmatrix} =: A'$$

We know $\det(A') \neq 0$ only if $n = r$.

$$\det\left(E_k^{-1}\cdots E_1^{-1} A F_\ell^{-1}\cdots F_1^{-1}\right) = \det(E_k^{-1})\cdots \det(E_1^{-1})\det(A)\det(F_\ell^{-1})\cdots\det(F_1^{-1}).$$

$\neq 0$

$\Rightarrow \quad r = n.$

i.e $\quad \text{rank}(A) = n.$

$\Rightarrow \quad L_A$ is surjective.

$\Rightarrow \quad L_A$ is Injective

$\Rightarrow \quad L_A$ is invertible

$\Rightarrow \quad A$ is invertible. $\blacksquare$

Lemma: Let $v_1, \ldots, v_n$ be column vectors in $\mathbb{R}^n$. Then $v_1, \ldots, v_n$ is linearly dep. iff $\det(v_1, \ldots, v_n) = 0$.

Proof: $\det(v_1, \ldots, v_n) = 0$

$\iff A = (v_1, \ldots, v_n)$ is not invertible.

$\iff L_A$ is not invertible.

$\iff \operatorname{null}(L_A) \neq 0$.

$\Longleftrightarrow \quad \exists\ (a_1, \ldots, a_n)$ s.t $\quad L_A \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = 0$

$\Longleftrightarrow \quad \exists\ a_1, \ldots, a_n$ s.t $\quad a_1 v_1 + \cdots + a_n v_n = 0$

$\Longleftrightarrow \quad v_1, \ldots, v_n$ are linearly dependent.

<u>Theorem</u>: Let A and B be $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B).$$

<u>Proof</u>: Suppose A is not invertible. $\Rightarrow \det(A) = 0$

$\Rightarrow$ R.H.S $= 0$

A is not invertible $\Rightarrow$ L_A is not invertible

$L_{AB} = L_A L_B \Rightarrow L_{AB}$ is not invertible.

$\Rightarrow \det(AB) = 0$.

If B is not invertible $\left(\Rightarrow \text{R.H.S} = 0 \right)$

Then L_B is not invertible lin. trans.

$L_{AB} = L_A L_B$ is not invertible.

$\Rightarrow \det(AB) = 0$.

Suppose A and B are invertible.

Then $A = E_1 \dots E_k$ and $B = F_1 \dots F_\ell$ where E_i and F_j are elementary matrices.

$$AB = E_1 \dots E_k F_1 \dots F_\ell$$

$$\det(AB) = \underbrace{\left(\det(E_1) \dots \det(E_k)\right)}_{\det(A)} \underbrace{\left(\det(F_1) \dots \det(F_\ell)\right)}_{\det(B)}$$

$\blacksquare$

Theorem: Let A be an $n \times n$ matrix. Then

$$\det(A^t) = \det(A).$$

Proof: Exercise: Check the above when A is non-invertible.

A - invertible $\Rightarrow$ $A = E_1 \dots E_k$

$$A^t = E_k^t \cdots E_1^t$$

$$\det(A^t) = \det(E_k^t) \dots \det(E_1^t).$$

Exercise : Check that $\det(E_k^t) = \det(E_k)$.

Hence

$$\det(A^t) = \det(E_k)\cdots\det(E_1)$$

$$= \det(A) \qquad \blacksquare$$