

Problem 1: Let $V = \{(x, y) : x, y \in \mathbb{R}\}$. Define vector addition in V component-wise and scalar multiplication as follows:

$$a(x, y) = (x, 0) \quad \forall (x, y) \in V \text{ and } a \in \mathbb{R}.$$

Is V a vector space with these operations.

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Solution: The vector addition is given by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in V$$

$\forall (x_1, y_1) \& (x_2, y_2) \in V.$

Let $(x, y) \in V$ and $c \in \mathbb{R}.$

Then $c(x, y) = (x, 0) \in V.$

Hence V is closed under vector-addition & scalar multiplication.

Property I : Let (x_1, y_1) and $(x_2, y_2) \in V$

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \longrightarrow (*)$$

Notice that $x_1 + x_2 = x_2 + x_1$ & $y_1 + y_2 = y_2 + y_1$
 $\longrightarrow (**)$

Then $(x_2, y_2) + (x_1, y_1) = (x_2 + x_1, y_2 + y_1)$

$$= (x_1 + x_2, y_1 + y_2) \quad (\text{By } (*)).$$

$$= (x_1, y_1) + (x_2, y_2) \quad (\text{by } *)$$

Property I is satisfied.

Property II Let $(x_1, y_1), (x_2, y_2) \& (x_3, y_3) \in V$

$$\begin{aligned} & ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) \\ &= (x_1 + x_2, y_1 + y_2) + (x_3, y_3) \\ &= ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3). \end{aligned}$$

$$\begin{aligned}
 &= (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3)) \\
 &= (x_1, y_1) + (x_2 + x_3, y_2 + y_3) \\
 &= (x_1, y_1) + ((x_2, y_2) + (x_3, y_3))
 \end{aligned}$$

Hence property II is satisfied.

Property III : $(0, 0)$ is the zero vector for the vector addition

$$(x, y) + (0, 0) = (x+0, y+0) = (x, y)$$

Property III is satisfied.

Property IV : Let $(x, y) \in V$.

Then $(-x, -y) \in V$ and

$$(x, y) + (-x, -y) = (x-x, y-y) = (0, 0)$$

Property IV is satisfied.

Property V :

$$(x, y) \in V$$

$$(1. v = v \quad \forall v \in V).$$

$$1(x, y) = (x, 0) \quad (\text{by definition}).$$

However if $(x, y) = (2, 3)$

$$1(2, 3) = (2, 0) \neq (2, 3)$$

Hence Property II is not satisfied.

Therefore V is not a vector space with these operations.

Property VI $(ab)(x, y) = (x, 0)$

$$a(b(x, y)) = a(x, 0) = (x, 0)$$

Hence Property VII is satisfied.

Property VIII : $(a+b)(x, y) = (x, 0)$

$$a(x, y) + b(x, y) = (x, 0) + (x, 0) = (2x, 0)$$

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$$(1+1)(1, 2) = 2(1, 2) = (1, 0)$$

$$1(1, 2) + 1(1, 2) = (1, 0) + (1, 0) = (2, 0)$$

Property VII is not satisfied.

Property VII : $a((x_1, y_1) + (x_2, y_2)) = a(x_1 + x_2, y_1 + y_2)$

$$\begin{aligned} a(x_1, y_1) + a(x_2, y_2) &= (x_1, 0) + (x_2, 0) \\ &= (x_1 + x_2, 0) \end{aligned}$$

Property VII is satisfied.



Problem 2: Prove that the set $W_1 = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y + z = 0\}$
is a subspace of \mathbb{R}^3 , however $W_2 = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y + z = 1\}$
is not a subspace of \mathbb{R}^3 .

Proof: To check that W_1 is a subspace, enough to
show that W_1 is closed under the vector addition &
scalar multiplication from V .

Let (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in W_1$,

$$\Rightarrow 2x_1 + 3y_1 + \beta_1 = 0 \quad \& \quad 2x_2 + 3y_2 + \beta_2 = 0. \longrightarrow (*)$$

Then

$$(x_1, y_1, \beta_1) + (x_2, y_2, \beta_2) = (x_1 + x_2, y_1 + y_2, \beta_1 + \beta_2)$$

Then $2(x_1 + x_2) + 3(y_1 + y_2) + (\beta_1 + \beta_2)$

$$= 2x_1 + 2x_2 + 3y_1 + 3y_2 + \beta_1 + \beta_2$$

$$= (2x_1 + 3y_1 + \beta_1) + (2x_2 + 3y_2 + \beta_2)$$

$$= 0 + 0 \quad (\text{by } (*))$$

$$= 0$$

$$\therefore (x_1, y_1, \beta_1) + (x_2, y_2, \beta_2) \in W.$$



Let $(x, y, z) \in W$, and $c \in \mathbb{R}$

$$c(x, y, z) = (cx, cy, cz)$$

$$2cx + 3cy + cz = c(2x + 3y + z) = c0 = 0.$$

$\Rightarrow W_1$ is a subspace

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Is W_2 a subspace?

$$(x_1, y_1, z_1) \in W_2 \text{ and } (x_2, y_2, z_2) \in W_2$$

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$$2(x_1 + x_2) + 3(y_1 + y_2) + 3(z_1 + z_2) = \\ (2x_1 + 3y_1 + 3z_1) + (2x_2 + 3y_2 + 3z_2)$$

$$1 + 1 = 2$$

Hence W_2 is not closed under vector addition
& therefore not a subspace.

Problem 3: Let $S \subset \mathbb{R}$ and $W = \{ f \in \mathcal{F}(S, \mathbb{R}) : f(s_0) = 0 \}$

for a fixed $s_0 \in S$. Then prove that W is a
subspace of $\mathcal{F}(S, \mathbb{R})$.

Proof: Recall that $\mathcal{F}(S, \mathbb{R}) = \{ f: S \rightarrow \mathbb{R} \}$ with
vector addition $(f+g)(s) = f(s) + g(s)$ for $f, g \in \mathcal{F}(S, \mathbb{R})$

$$(cf)(s) = c(f(s)) \text{ where } f \in \mathfrak{F}(S, \mathbb{R}) \text{ & } c \in \mathbb{R}$$

Let $f, g \in W \Rightarrow f(s_0) = 0 \text{ & } g(s_0) = 0$.

Then for $f, g \in W$ and $c \in \mathbb{R}$

$$\begin{aligned} (f+g)(s_0) &= f(s_0) + g(s_0) = 0 + 0 = 0. \\ \Rightarrow f+g &\in W. \end{aligned}$$

$$\begin{aligned} (cf)(s_0) &= cf(s_0) = c0 = 0 \\ \Rightarrow cf &\in W. \end{aligned}$$

W is hence a subspace of $\mathfrak{F}(S, \mathbb{R})$. — ■.

Problem 4: Check whether the first vector is a linear combination of the other two vectors in the following:

$$(i) \left\{ (-2, 2, 2), (1, 2, -1), (-3, -3, 3) \right\} \text{ in } \mathbb{R}^3$$

$$(ii) \left\{ x^3 - 8x^2 + 4x, x^3 - 2x^2 + 3x - 1, x^3 - 2x + 3 \right\} \text{ in } \mathcal{P}_3(\mathbb{R}).$$

Solution: Want to check for the existence of $a, b \in \mathbb{R}$ s.t

$$(-2, 2, 2) = a(1, 2, -1) + b(-3, -3, 3)$$

$$= (a-3b, 2a-3b, -a+3b)$$

$$\begin{array}{l} \left. \begin{array}{l} a - 3b = -2 \\ 2a - 3b = 2 \\ -a + 3b = 2 \end{array} \right\} \Rightarrow \quad a - 2 = 2 \\ \Rightarrow a = 4 \Rightarrow 8 - 3b = 2 \\ \Rightarrow b = 2 \end{array}$$

$$\Rightarrow a = 4, b = 2$$

Hence $(-2, 2, 2) = 4(1, 2, -1) + 2(-3, -3, 3)$

(ii) $\left\{ x^3 - 8x^2 + 4x, x^3 - 2x^2 + 3x - 1, x^3 - 2x + 3 \right\}$ in $P_3(\mathbb{R})$

Want to check if $\exists a, b \in \mathbb{R}$ s.t

$$x^3 - 8x^2 + 4x = a(x^3 - 2x^2 + 3x - 1) + b(x^3 - 2x^2 + 3x + 3) \rightarrow (*)$$

$$= (a+b)x^3 + (-2a)x^2 + (3a-2b)x + (3b-a)$$

\Rightarrow

$$a + b = 1$$

$$-2a = -8$$

$$3a - 2b = 4$$

$$3b - a = 0$$

$$\rightarrow a = 4 \Rightarrow b = 1 - 4 = -3.$$

$$12 + 6 = 18 \neq 4$$

Hence $\nexists a, b \in \mathbb{R}$ s.t $(*)$ is satisfied.

Problem 5: Let S_1, S_2 be subsets of a vector space V .

Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$.

Give an example when (i) $\text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$

(ii) $\text{span}(S_1 \cap S_2) \subsetneq \text{span}(S_1) \cap \text{span}(S_2)$.

Solution: Let $v \in \text{span}(S_1 \cap S_2)$

i.e. $\exists v_1, \dots, v_k \in S_1 \cap S_2$ and $a_1, \dots, a_k \in \mathbb{R}$

$$\text{s.t } v = a_1v_1 + a_2v_2 + \dots + a_kv_k.$$

$$v_1, \dots, v_k \in S_1 \cap S_2 \Rightarrow v_1, \dots, v_k \in S_1$$

$$\Rightarrow a_1 v_1 + \dots + a_k v_k \in \text{span}(S_1)$$

By a similar argument $a_1 v_1 + \dots + a_k v_k \in \text{span}(S_2)$

$$\Rightarrow a_1 v_1 + \dots + a_k v_k \in \text{span}(S_1) \cap \text{span}(S_2)$$

$$\Rightarrow v \in \text{span}(S_1) \cap \text{span}(S_2)$$

Hence $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$.

Example: (i) In \mathbb{R}^3 consider

$$S_1 = \{(1, 0, 0), (0, 1, 0), (1, 1, 0)\}$$

$$S_2 = \{(0, 1, 0), (1, 1, 0), (1, 2, 0)\}.$$

$$S_1 \cap S_2 = \{(0, 1, 0), (1, 1, 0)\}$$

$$\text{span}(S_1 \cap S_2) = \{(x, y, z) \in \mathbb{R}^3 : z=0\} = W$$

$$\text{span}(S_1) = W = \text{span}(S_2).$$

(ii) In \mathbb{R}^2 , let

$$S_1 = \{(1, 0), (0, 1)\}$$

$$S_2 = \{(1, 1), (1, -1)\}.$$

$$\text{Span}(S_1) = \text{Span}(S_2) = \mathbb{R}^2$$

$$S_1 \cap S_2 = \emptyset \quad \& \quad \text{Span}(S_1 \cap S_2) = \{0\}.$$

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Problem 6: Check whether the following set

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

is linearly dependent in $M_{3 \times 2}(\mathbb{R})$

Solution: To check whether S is linearly dependent,

we want to check for the existence of $a_1, a_2, \dots, a_5 \in \mathbb{R}$

$$\text{s.t } a_1 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + a_4 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + a_5 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a_1 + a_4 & a_1 + a_5 \\ a_2 + a_4 & a_2 + a_5 \\ a_3 + a_4 & a_3 + a_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

i.e.

$$\begin{aligned} a_1 + a_4 &= 0 \\ a_2 + a_4 &= 0 \\ a_3 + a_4 &= 0 \end{aligned}$$

$$a_1 + a_5 = 0$$

$$a_2 + a_5 = 0$$

$$a_3 + a_5 = 0$$

$$\Rightarrow a_1 = a_2 = a_3$$

$$a_4 = -a_1$$

$$\Rightarrow a_1 = a_2 = a_3$$

$$\& a_5 = -a_1$$

If $a_1 = 1$, then

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} + (-1) \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} + (-1) \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

= $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Hence S is linearly dependent.

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Problem 7: Let S be a finite set of non-zero polynomials
in $P(\mathbb{R})$ s.t. no two polynomials have the same

degree. Then prove that S is linearly independent.

Proof: Let P_1 be the polynomial in S with least degree. Let $\deg(P_1) = d_1$.

Let P_2 be the poly in S with the least degree $> d_1$,
 $\deg(P_2) = d_2$

After finitely many steps, we have

$$S = \{P_1, P_2, \dots, P_k\} \text{ s.t } \deg(P_i) = d_i$$

$$\text{and } d_1 < d_2 < d_3 < \dots < d_k.$$

Suppose $\underbrace{a_1 p_1 + a_2 p_2 + \dots + a_k p_k = 0} \quad \text{--- (*)}$

Co-eff. of x^{d_k} = $a_k b_k$

where b_k = co-eff. of x^{d_k} in $p_k(x)$.

$$\Rightarrow a_k b_k = 0 \Rightarrow a_k = 0 \quad (\text{since } b_k \neq 0 \text{ implies that } \deg(p_k) \neq d_k).$$

$$\Rightarrow a_1 p_1 + a_2 p_2 + \dots + a_{k-1} p_{k-1} = 0$$

By a similar argument $a_{k-1} = 0$

By repeating the above process, we
have $a_k = a_{k-1} = \dots = a_1 = 0$

$\Rightarrow S$ is linearly independent.

Problem 8: Determine whether the following subsets are
bases for the subspace $W = \{ax^2 + bx + c \in P_2(\mathbb{R}) : 2a + b + c = 0\}$
in $P_2(\mathbb{R})$.

$$(i) \quad S = \{x^2 - 2, x^2 + 2x - 4\}$$

$$(ii) \quad S = \{3x^2 - 2x - 4, x^2 - x - 1, -2x^2 + x + 3\}$$

$$(iii) \quad S = \{x^2 - 2, -2x^2 + x + 3\}.$$

Solution: (i) $S = \{x^2 - 2, x^2 + 2x - 4\}$

Let $a, b \in \mathbb{R}$ s.t $a(x^2 - 2) + b(x^2 + 2x - 4) = 0$

$$\Rightarrow (a+b)x^2 + 2bx + (-2a - 4b) = 0$$

$$\Rightarrow 2b = 0 \Rightarrow b = 0$$

$$a + b = 0 \Rightarrow a = 0$$

Hence S is linearly independent

$$\Rightarrow \dim(W) \geq 2$$

If S is not a spanning set, then

$\exists p(n) \in W$ s.t $p \notin \text{span}(S)$

$S \cup \{p\}$ is linearly ind.

$$\Rightarrow \dim(W) = 3 = \dim(\mathcal{P}_2(\mathbb{R})).$$

$\Rightarrow W = \mathcal{P}_2(\mathbb{R})$ but this is a contradiction

Hence S is a spanning set.

$\Rightarrow S$ is a basis of W .

$\Rightarrow \dim(W) = 2$.

(ii) S cannot be linearly independent,
the size of any linearly independent subset of
 W cannot be greater than $\dim(W)$.
 $\therefore S$ is not a basis.

(iii) By a corollary to the replacement theorem,
to check whether S is a basis, it is
enough to check for linear independence
since S has $\dim(W) = 2$ elts.

$$S = \{x^2 - 2, -2x^2 + x + 3\}.$$

$$\begin{aligned} \text{If } a(x^2 - 2) + b(-2x^2 + x + 3) &= 0 \\ \Rightarrow (a-2b)x^2 + bx + (3b-2a) &= 0 \\ a-2b=0, \quad b=0, \quad 3b-2a=0 \end{aligned}$$

$$\Rightarrow a=0, b=0.$$

Hence S is a basis of W .

Problem 9: For a fixed scalar $c \in \mathbb{R}$, let $W = \{ p \in P_n(\mathbb{R}) : p(c) = 0 \}$ be a subspace of $P_n(\mathbb{R})$.

Determine the dimension of W .

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Solution: We know that $p(c) = 0$

iff

$$p(x) = (x - c) q(x)$$

Claim:

$$S = \{x - c, x(x - c), x^2(x - c), \dots, x^{n-1}(x - c)\}$$

is a basis of W .

in W

That S is linearly independent, follows from problem 7.

$$\Rightarrow \dim(W) \geq n.$$

$$\text{If } \dim(W) > n$$

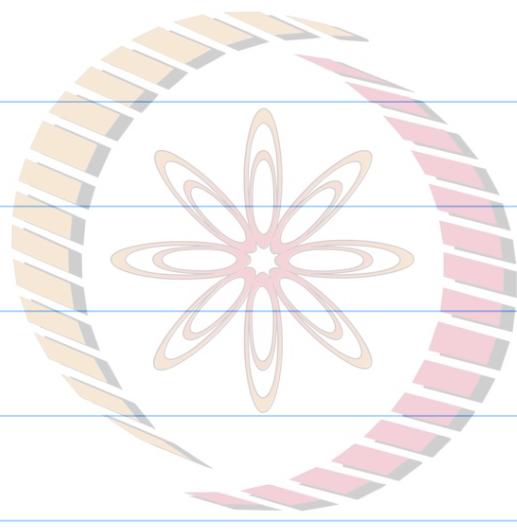
$$\Rightarrow \dim(W) = n+1 = \dim(P_n(\mathbb{R})).$$

$$\Rightarrow W = P_n(\mathbb{R}). \quad \text{This is a contradiction}$$

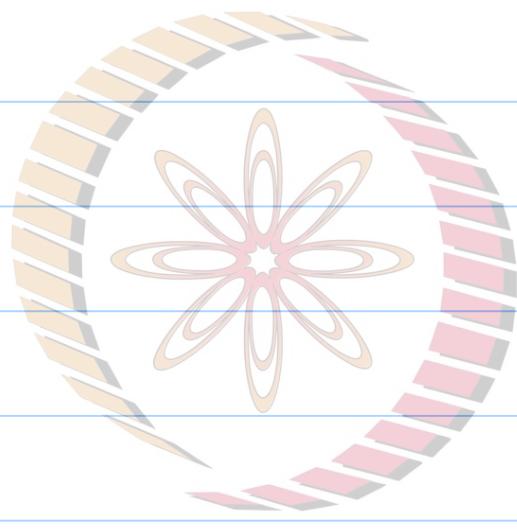
since $1(c) = 1 \neq 0$

$$\Rightarrow \dim(W) = n. \quad \blacksquare.$$

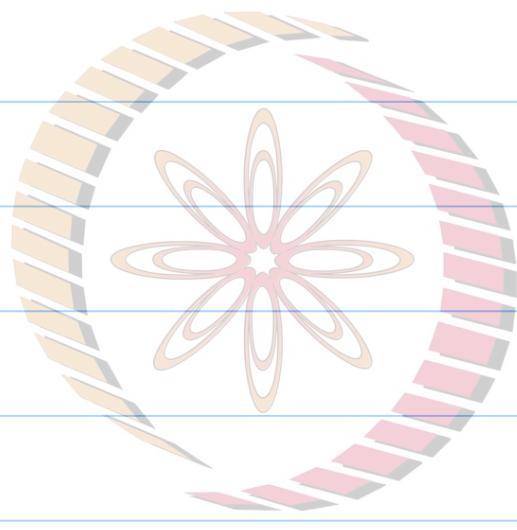
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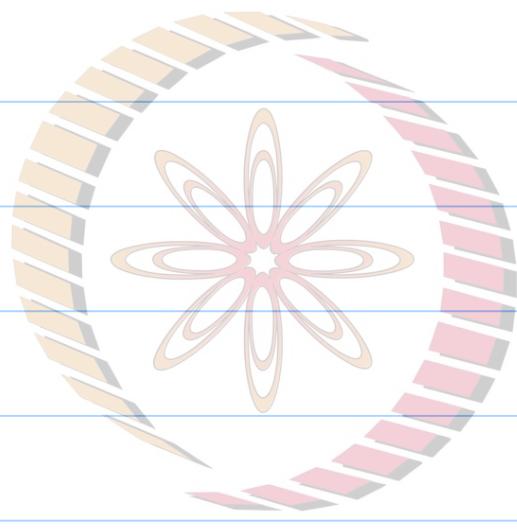
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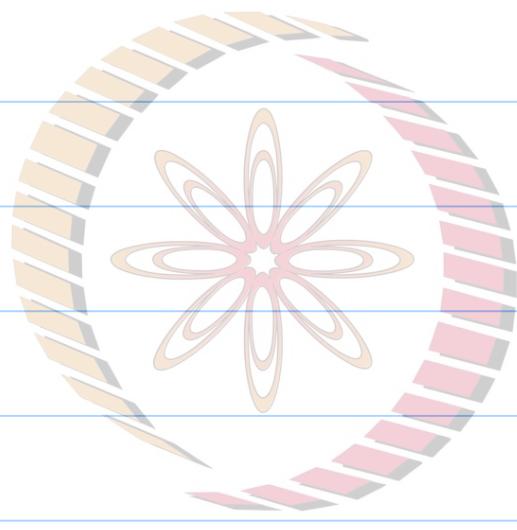
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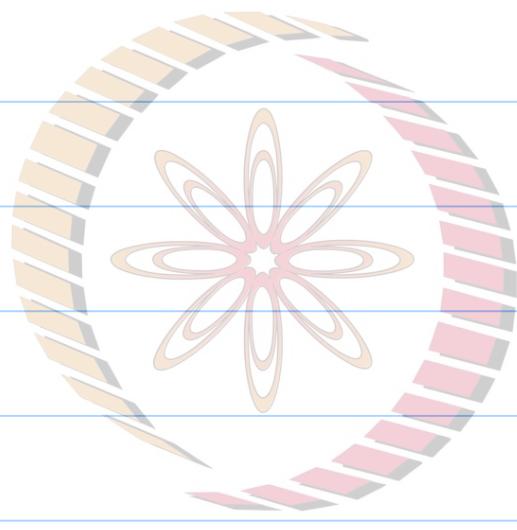
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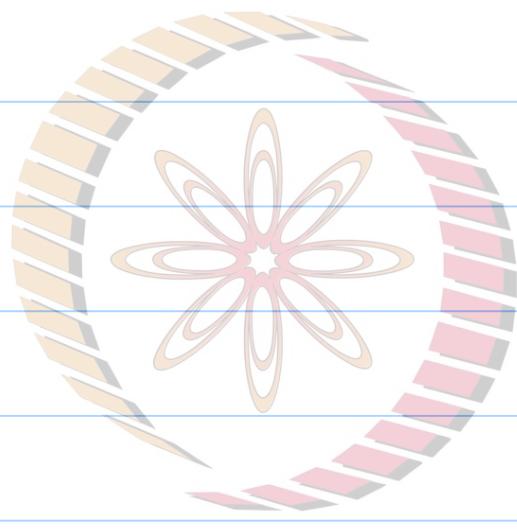
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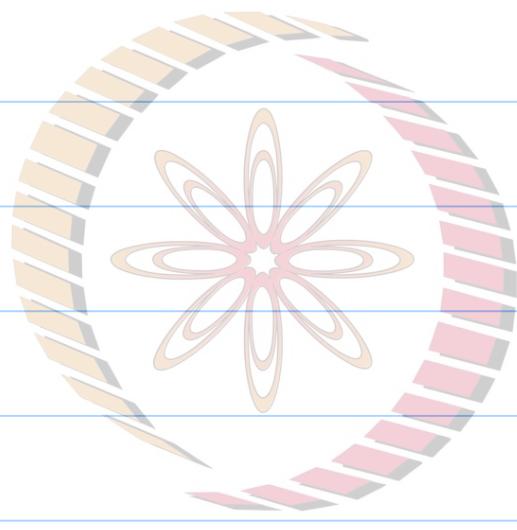
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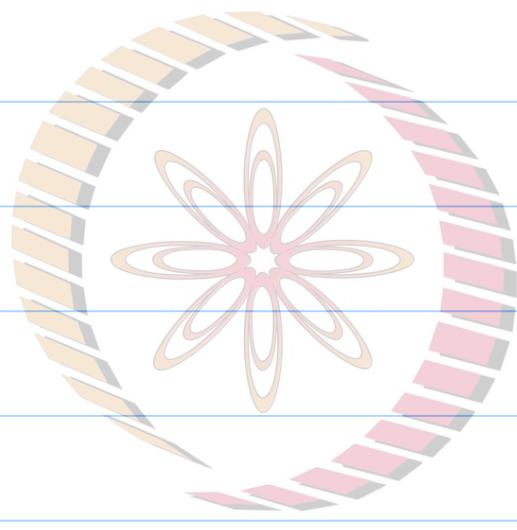
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