MODULE 6

RANDOM VECTOR AND ITS JOINT DISTRIBUTION

PROBLEMS

1. (i) Let $F: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$F(x,y) = \begin{cases} 1, & \text{if } x + 2y \geq 1 \\ 0, & \text{if } x + 2y < 1 \end{cases}.$$

Does $F(\cdot,\cdot)$ define a distribution function?

(ii) Let $F: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$F(x,y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } x + y < 1 \text{ or } y < 0 \\ 1, & \text{otherwise} \end{cases}.$$

Does $F(\cdot,\cdot)$ define a distribution function?

(iii) Let $F_{X,Y}(\cdot,\cdot)$ be the distribution function of some two-dimensional random vector (X, Y), and let $F_X(\cdot)$ and $F_Y(\cdot)$, respectively, be the marginal distribution functions of X and Y. Define $H(x,y) = \min\{F_X(x), F_Y(y)\}, (x,y) \in \mathbb{R}^2$ and $G(x,y) = \max\{F_X(x) + F_Y(y) - 1, 0\}, (x,y) \in \mathbb{R}^2$. Prove that:

(a) $G(\cdot,\cdot)$ and $H(\cdot,\cdot)$ are each distribution functions and that their marginal distribution functions are the same as those of $F_{X,Y}(\cdot,\cdot)$;

(b) $G(x,y) \leq F_{X,Y}(x,y) \leq H(x,y), \forall (x,y) \in \mathbb{R}^2$.

(***Note:*** Let the random variable X have distribution function $F_X(\cdot)$ and let $Y = g(X)$ have distribution function $F_Y(\cdot)$, where $g(\cdot)$ is some Borel function. If $g(\cdot)$ is strictly increasing (decreasing), then $F_{X,Y}(x,y) = U(x,y)$ $\left(F_{X,Y}(x,y) = L(x,y)\right)$.

2. Let the random vector $\underline{X} = (X_1, X_2)$ have the joint distribution function

$$F_{X_1,X_2}(x_1,x_2)=\begin{cases}0, & \text{if } x_1<0 \text{ or } x_2<0\\ \frac{x_1x_2}{8}, & \text{if } 0\le x_1<1, 0\le x_2<2 \text{ or } 1\le x_1<2, 0\le x_2<1\\ \frac{x_1}{4}, & \text{if } 0\le x_1<1, x_2\ge 2\\ \frac{1}{2}+\frac{x_1x_2}{8}, & \text{if } 1\le x_1<2, 1\le x_2<2\\ \frac{1}{2}+\frac{x_1}{4}, & \text{if } 1\le x_1<2, x_2\ge 2\\ \frac{x_2}{4}, & \text{if } x_1\ge 2, 0\le x_2<1\\ \frac{1}{2}+\frac{x_2}{4}, & \text{if } x_1\ge 2, 1\le x_2<2\\ 1 & \text{if } x_1\ge 2, x_2\ge 2\end{cases}.$$

Find $P(\{(X_1,X_2)=(0,0)\})$ and $P(\{(X_1,X_2)=(1,1)\})$. Is $\underline{X}=(X_1,X_2)$ of absolutely continuous type?

3. Let the random vector (X,Y) have the p.m.f.

$$f_{X,Y}(x,y)=\begin{cases}\frac{(x+y+k-1)!}{x!\,y!\,(k-1)!}\theta_1^x\theta_2^y(1-\theta_1-\theta_2)^k, & \text{if } (x,y)\in\mathbb{Z}_+\times\mathbb{Z}_+\\ 0, & \text{otherwise}\end{cases},$$

where $k\ge 1$ is an integer, $0<\theta_i<1, i=1,2,$ $\theta_1+\theta_2<1$ and $\mathbb{Z}_+=\{0,1,2,\ldots\}$. Find the marginal p.m.f.s of X and Y and the conditional distributions. (***Note:*** A distribution with above p.m.f. is called a bivariate negative binomial distribution).

4. Three balls are randomly placed in three empty boxes B_1, B_2 and B_3. Let N denote the total number boxes which are occupied and let X_i denote the number of balls in the box B_i, $i=1,2,3$.
   (i) Find the joint p.m.f. of (N,X_1);
   (ii) Find the joint p.m.f. of (X_1,X_2);
   (iii) Find the marginal p.m.f.s of N and X_2;
   (iv) Find the marginal p.m.f. of X_1 from the joint p.m.f. of (X_1,X_2).

5. Let X_1 and X_2 have the joint p.m.f.

$$f_{\underline{X}}(x_1,x_2)=\begin{cases}\left(\frac{2}{3}\right)^{x_1+x_2}\left(\frac{1}{3}\right)^{2-x_1-x_2}, & \text{if } (x_1,x_2)=(0,0),(0,1),(1,0),(1,1)\\ 0, & \text{otherwise}\end{cases}.$$

   (i) Find the joint p.m.f. of $Y_1=X_1-X_2$ and $Y_2=X_1+X_2$;
   (ii) Find the marginal p.m.f.s of Y_1 and Y_2;

(iii) Find $E(Y_1^2 Y_2)$.

6. Let $\underline{X} = (X_1, X_2)$ have the joint p.m.f.

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} \frac{x_1 x_2}{36}, & \text{if } x_1 = 1,2,3 \\ 0, & \text{otherwise} \end{cases},$$

and let $Y_1 = X_1 X_2$ and $Y_2 = X_2$.

(i) Find the joint p.m.f. of (Y_1, Y_2);

(ii) Find the marginal p.m.f. of Y_1.

(iii) Find $P(\{X_1 + X_2 = 4\})$.

7. Suppose that $X_1, \dots, X_n$ are i.i.d. random variables and that $P(X_1 = 0) = 1 - p = 1 - P(X_1 = 1)$, for some $p \in (0, 1)$. Let X denote the number of $X_1, \dots, X_n$ that are as large as X_1. Find the p.m.f. of X.

8. Suppose that the number, X, of eggs laid by a bird has the $P(\lambda)$ distribution (the Poisson distribution with mean λ), and the probability that an egg would finally develop is $p \in (0,1)$; here $\lambda > 0$. Further suppose that eggs develop independently of each other. Show that the number, Y, of eggs surviving has the $P(\lambda p)$ distribution. Also, find the conditional distribution of X given $Y = y$, where $y \in \{0, 1, 2, \dots\}$.

9. Let the random vector (X, Y) have the joint p.d.f.

$$f_{X,Y}(x, y) = \begin{cases} \frac{\Gamma(\theta_1 + \theta_2 + \theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} x^{\theta_1 - 1} y^{\theta_2 - 1} (1 - x - y)^{\theta_3 - 1}, & \text{if } x > 0, y > 0, x + y < 1 \\ 0, & \text{otherwise} \end{cases},$$

where $\theta_i > 0, i = 1, 2, 3$. Find the marginal p.d.f.s of X and Y and the conditional p.d.f.s. (***Note:*** A distribution with above p.d.f. is called a bivariate beta distribution).

10. Let the random variable $\underline{X} = (X_1, X_2)$ have the joint p.m.f.

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} \frac{x_1 + 2x_2}{18}, & \text{if } (x_1, x_2) \in \{1, 2\} \times \{1, 2\} \\ 0, & \text{otherwise} \end{cases}.$$

Determine the conditional mean and conditional variance of X_2 given $X_1 = x_1,\ x_1 \in \{1, 2\}$.

11. Let $\underline{X} = (X_1, X_2, X_3)$ be a random vector with joint p.m.f.

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \begin{cases} \frac{1}{4}, & \text{if } (x_1,x_2,x_3) \in A \\ 0, & \text{otherwise} \end{cases},$$

where $A = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$.

(i) Are X_1, X_2, X_3 independent?
(ii) Are X_1, X_2, X_3 pairwise independent?
(iii) Are $X_1 + X_2$ and X_3 independent?

12. Let X and Y be two random variables such that $(\{X \in \{0,1\}\}) = P(\{Y \in \{0,1\}\}) = 1$. If $P(\{X = 1, Y = 1\}) = P(\{X = 1\})P(\{Y = 1\})$, show that X and Y are independent random variables.

13. Five cards are drawn at random without replacement from a deck of 52 cards. Let the random variables X_1, X_2 and X_3, respectively, denote the number of spades, the number of hearts and the number of diamonds among the five drawn cards.
(i) Find the joint p.m.f. of (X_1, X_2, X_3);
(ii) Are random variables X_1, X_2 and X_3 independent?

14. Consider a sample of size 3 drawn with replacement from an urn containing 3 white, 2 black and 3 red balls. Let the random variables X_1 and X_2, respectively, denote the number of white balls and the number of black balls in the sample. Determine whether or not X_1 and X_2 are independent.

15. The joint p.d.f. of (X, Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & \text{if } 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}.$$

(i) Verify whether X and Y are independent;
(ii) Find the marginal p.d.f.s. of X and Y;
(iii) Find $P\left(\left\{0 < X < \frac{1}{2}, \frac{1}{4} < Y < 1\right\}\right)$ and $P(\{X + Y < 1\})$.

16. The joint p.d.f. of (X, Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} ce^{-(2x+3y)}, & \text{if } 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases},$$

where c is a real constant.

(i) Find the value of the constant c;
(ii) Verify whether X and Y are independent;
(iii) Find the marginal p.d.f.s of X and Y;

(iv) Find $P\left(\left\{X < \frac{Y}{2}\right\}\right)$.

17. Let f and g be two p.d.f.s with respective distribution functions F and G. Define $h: \mathbb{R}^2 \to [0, \infty)$ as

$$h(x, y) = [1 + \alpha\{2F(x) - 1\}\{2G(y) - 1\}]f(x)g(y),$$

where $\alpha \in [-1,1]$.

(i) Show that h is a p.d.f. of some random vector (X, Y);

(ii) Show that the marginal p.d.f.s of X and Y are f and g, respectively;

(iii) Does there exist a value of $\alpha \in [-1,1]$ such that X and Y are independent?

18. Let $\underline{X} = (X_1, X_2, X_3)$ be a random vector with joint p.d.f.

$$f_{\underline{X}}(x_1, x_2, x_3) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{\left(x_1^2+x_2^2+x_3^2\right)}{2}} \left(1 + x_1x_2x_3 e^{-\frac{\left(x_1^2+x_2^2+x_3^2\right)}{2}}\right), \quad x_i \in \mathbb{R}, i = 1,2,3.$$

(i) Are X_1, X_2, X_3 independent?

(ii) Are X_1, X_2, X_3 pairwise independent?

(iii) Find the marginal p.d.f.s of $(X_1, X_2), (X_1, X_3)$, and (X_2, X_3).

19. A point X_1 is chosen at random from the interval $(0,1)$ and then a point X_2 is chosen at random from the interval $(0, X_1)$. Compute $P(\{X_1 + X_2 \geq 1\})$ and find the conditional mean $E(X_1|X_2 = x_2), x_2 \in (0,1)$.

20. With the help of a counter example, show that if the random variables X_1 and X_2 are uncorrelated, then this does not, in general, imply that X_1 and X_2 are independent.

21. Let $\underline{X} = (X_1, X_2)$ be a random vector having the p.d.f.

$$f(x_1, x_2) = \begin{cases} \dfrac{1}{2x_1^2 x_2}, & \text{if } 1 < x_1 < \infty, \dfrac{1}{x_1} < x_2 < x_1 \\ 0, & \text{otherwise} \end{cases}.$$

(i) Find the marginal p.d.f.s of X_1 and X_2;

(ii) Find the conditional means and variances of X_1 given $X_2 = x_2$ ($x_2 \in (0, \infty)$) and X_2 given $X_1 = x_1$ ($x_1 \in (1, \infty)$);

(iii) Are X_1 and X_2 independent random variables?

(iv) Find $\mathrm{Corr}(X_1, X_2)$;

(v) Find $P(\{X_2 < \frac{1}{2}\}|\{X_1 = 2\})$;

(vi) Find $P(\{X_2 < \frac{1}{2}\}|\{X_1 > 3\})$.

22. Let (X, Y) be a random vector such that the p.d.f. of X is

$$f_X(x) = \begin{cases} 4x(1-x^2), & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

and, for fixed $x \in (0,1)$, the conditional p.d.f. of Y given $X = x$ is

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{2y}{1-x^2}, & \text{if } x < y < 1 \\ 0, & \text{otherwise} \end{cases}.$$

(i) For $y \in (0,1)$, find conditional p.d.f. of X given $Y = y$;

(ii) Find $E(X|Y = \frac{1}{2})$ and $\mathrm{Var}\left(X|Y = \frac{1}{2}\right)$;

(iii) Find $P\left(\left\{0 < Y < \frac{1}{3}\right\}\right)$ and $P\left(\left\{\frac{1}{3} < Y < \frac{2}{3}\right\} \middle| \left\{X = \frac{1}{2}\right\}\right)$.

23. Let (X, Y) be a random vector with joint p.m.f. given by:

	$f_{X,Y}(x,y)$			
$y \downarrow x \rightarrow$	1	2	3	4
4	.08	.11	.09	.03
5	.04	.12	.21	.05
6	.09	.06	.08	.04

(i) Find the conditional p.m.f. of X, given $Y = 5$;

(ii) Find the probabilities $P(\{X + Y \leq 8\})$, $P(\{X + Y > 7\})$, $P(\{XY \leq 14\})$, $P(\{XY > 18\})$, $P(\{X = 3\}|\{Y = 5\})$ and $P(\{Y = 5\}|\{X = 3\})$;

(iii) Find $\mathrm{Corr}(X, Y)$.

24. Let $X_1, \ldots, X_n$ be n random variables with $E(X_i) = \mu_i, \mathrm{Var}(X_i) = \sigma_i^2$ and $\rho_{ij} = \mathrm{Corr}\left(X_i, X_j\right), i, j = 1, \ldots, n, i \neq j$. For real numbers $a_i, b_i, i = 1, \ldots, n$, define $Y = \sum_{i=1}^{n} a_i X_i$ and $Z = \sum_{i=1}^{n} b_i X_i$. Find $\mathrm{Cov}(Y, Z)$.

25. Let X_1, X_2 and X_3 be three independent random variables each with a variance σ^2. Define the random variables

$$W_1 = X_1, \quad W_2 = \frac{\sqrt{3}-1}{2}X_1 + \frac{3-\sqrt{3}}{2}X_2, \qquad \text{and } W_3 = (\sqrt{2}-1)X_2 + \left(2-\sqrt{2}\right)X_3.$$

Find $\mathrm{Corr}(W_1, W_2)$, $\mathrm{Corr}(W_1, W_3)$ and $\mathrm{Corr}(W_2, W_3)$.

26. Let X and Y be jointly distributed random variables with $E(X) = E(Y) = 0, E(X^2) = E(Y^2) = 2$ and $\mathrm{Corr}(X, Y) = 1/3$. Find $\mathrm{Corr}\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right)$.

27. Let (X, Y) have the joint p.m.f. given by:

(x, y)	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)
$f_{X,Y}(x, y)$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{4}{15}$

and $f_{X,Y}(x, y) = 0$, elsewhere. Find $\rho = \text{Corr}(X, Y)$.

28. Let X_1, X_2 and X_3 be three random variables with means, variances and correlation coefficients denoted by μ_1, μ_2, μ_3; $\sigma_1^2, \sigma_2^2, \sigma_3^2$ and ρ_{12}, ρ_{13}, ρ_{23} , respectively. If $E\big((X_1 - \mu_1)|X_2 = x_2, X_3 = x_3\big) = b_2(x_2 - \mu_2) + b_3(x_3 - \mu_3)$, for some constants b_2 and b_3, determine b_2 and b_3 in terms of the variances and correlation coefficients.

29. Let $X_1, \ldots, X_n$ denote a random sample, where $X_1, \ldots, X_n$ are positive with probability one. Show that

$$E\left(\frac{X_1 + X_2 + \cdots + X_k}{X_1 + X_2 + \cdots + X_n}\right) = \frac{k}{n},\ k = 1,2, \ldots, n.$$

30. Let $X_1, \ldots, X_n$ be a random sample of absolutely continuous type random variables. If the expectation of X_1 is finite and the distribution of X_1 is symmetric about $\mu \in (-\infty, \infty)$ then show that

(i) $X_{r:n} - \mu \overset{d}{=} \mu - X_{n-r+1:n}$, $r = 1, \ldots, n$;
(ii) $E(X_{r:n} + X_{n-r+1:n}) = 2\mu, r = 1, \ldots, n$;
(iii) $E\left(X_{\frac{n+1}{2}:n}\right) = \mu$, if n is odd;
(iv) $P\left(X_{\frac{n+1}{2}:n} > \mu\right) = \frac{1}{2}$, if n is odd.

31. Let $X_1, \ldots, X_n$ be a random sample and let $E(X_1)$ be finite.
    (i) Find the conditional expectation $E(X_1|X_1 + \cdots + X_n = t)$, where $t \in \mathbb{R}$ is such that the conditional expectation is defined.
    (ii) If X_1 is of absolutely continuous type and $(\pi_1, \ldots, \pi_n)$ is a permutation of $(1, \ldots, n)$, find $P\big(X_{\pi_1} < \cdots < X_{\pi_n}\big)$.

32. Let X_1 and X_2 be i.i.d. $N(0,1)$ random variables and let $Y = X_1 + X_2, Z = X_1^2 + X_2^2$.
    (i) Show that the m.g.f. of (Y, Z) is $M_{Y,Z}(t_1, t_2) = \frac{e^{\frac{t_1^2}{1-2t_2}}}{1-2t_2}$, $t_2 < \frac{1}{2}$;
    (ii) Using (a), find $\text{Corr}(Y, Z)$.

33. Suppose that the lifetimes of electric bulbs manufactured by a manufacturer follows exponential distribution with mean of 50 hours. Eight such bulbs are chosen at random.
    (i) Find the probability that, among eight chosen bulbs, 2 will last less than 40 hours, 3 will last anywhere between 40 and 60 hours, 2 will last anywhere between 60 and 80 hours and 1 will last for more than 80 hours;
    (ii) Find the expected number of bulbs in the lot of 8 chosen bulbs with lifetime between 60 and 80 hours;
    (iii) Find the expected number of bulbs in the lot of 8 chosen bulbs with lifetime between 60 and 80 hours, given that the number of bulbs in the lot with lifetime anywhere between 40 and 60 hours is 2.

34. Suppose that $\underline{X} \sim \text{Mult}(30, \theta_1, \theta_2, \theta_3, \theta_4)$. Find the conditional probability mass function of (X_1, X_2, X_3, X_4) given that $X_1 + X_2 + X_3 + X_4 = 28$.

35. Let $\underline{X} = (X_1, X_2)$ have the joint p.d.f.

$$f_{\underline{X}}(x_1, x_2) = \phi(x_1)\phi(x_2)[1 + \alpha(2\Phi(x_1) - 1)(2\Phi(x_2) - 1)], x_i \in \mathbb{R}, i = 1, 2,$$

where $|\alpha| \leq 1$.
    (i) Verify that $f_{\underline{X}}(x_1, x_2)$ is a p.d.f.;
    (ii) Find the marginal p.d.f.s of X_1 and X_2;
    (iii) Is (X_1, X_2) jointly normal?

36. Let $\underline{X} = (X_1, X_2) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ and, for real constants a_1, a_2, a_3, and a_4 ($a_i \neq 0$, $i = 1, 2, 3, 4, a_1 a_4 \neq a_2 a_3$), let $Y = a_1 X_1 + a_2 X_2$ and $Z = a_3 X_1 + a_4 X_2$.
    (i) Find the joint p.d.f. of (Y, Z) ;
    (ii) Find the marginal p.d.f.s. of Y and Z.

37. Let X and Y be i.i.d. $N(0, \sigma^2)$ random variables.
    (i) Find the joint p.d.f. of (U, V), where $U = aX + bY$ and $V = bX - aY (a \neq 0, b \neq 0)$;
    (ii) Show that U and V are independent;
    (iii) Show that $\frac{X+Y}{\sqrt{2}}$ and $\frac{X-Y}{\sqrt{2}}$ are i.i.d. $N(0, \sigma^2)$ random variables.

38. Let $\underline{X} = (X_1, X_2) \sim N_2(0, 0, 1, 1, \rho)$.
    (i) Find the m.g.f. of $Y = X_1 X_2$;
    (ii) Using (i), find $E(X_1^2 X_2^2)$;
    (iii) Using conditional distribution of X_1 given X_2, find $E(X_1^2 X_2^2)$.

39. Let $\underline{X} = (X_1, X_2)$ have the joint p.d.f.

$$f(x,y) = \begin{cases} \frac{1}{\pi} e^{-\frac{1}{2}(x^2+y^2)}, & \text{if } xy > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Show that $X_i \sim N(0,1), i = 1,2,$ but $\underline{X} = (X_1, X_2)$ does not have a bivariate normal distribution.

40. For a fixed $\rho \in (-1,1)$ and $\alpha \in (0,1)$, let the random variable (X,Y) have the joint p.d.f.

$$g_\rho(x,y) = \alpha f_\rho(x,y) + (1-\alpha) f_{-\rho}(x,y),$$

where $f_r(\cdot,\cdot), -1 < r < 1$, denotes the pdf of $N_2(0,0,1,1,r)$. Show that X and Y are normally distributed but the distribution of (X,Y) is not bivariate normal.

41. Consider the random vector (X,Y) as defined in Problem 39.
   (i) Find $\mathrm{Corr}(X,Y)$;
   (ii) Are X and Y independent?

42. Suppose that $\underline{X} \sim N_2(0,0,1,1,0)$. Find c_1 such that $P(-c_1 \le X_1 \le c_1, -c_1 \le X_2 \le c_1) = 0.95$.

43. (i) Let $(X,Y) \sim N_2(5,8,16,9,0.6)$. Find $P(\{5 < Y < 11\})|\{X = 2\}), P(\{4 < X < 6\})$ and $P(\{7 < Y < 9\})$;
   (ii) Let $(X,Y) \sim N_2(5,10,1,25,\rho)$, where $\rho > 0$. If $P(\{4 < Y < 16\})|\{X = 5\}) = 0.954$, determine ρ.

44. (i) Let $X \sim \mathrm{Bin}(n_1, p)$ and $Y \sim \mathrm{Bin}(n_2, p)$ be independent random variables. For $t \in \{0, 1, \ldots, \min(n_1, n_2)\}$, find the conditional distribution and conditional mean of X given $X + Y = t$.

   (ii) Let $X \sim \mathrm{P}(\lambda_1)$ and $Y \sim \mathrm{P}(\lambda_2)$be independent random variables. For $t \in \{0,1,\ldots\}$, find the conditional distribution and conditional mean of X given $X + Y = t$

45. Let X and Y be independent random variables with respective p.d.f.s.

$$f_X(x) = \begin{cases} \frac{1}{3}, & \text{if } 1 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} e^{-(y-2)}, & \text{if } y \geq 2 \\ 0, & \text{otherwise} \end{cases}.$$

Find the distribution function of $T = \frac{X}{Y}$ and hence find the p.d.f. of T.

46. Let X and Y be i.i.d. $U(0,1)$ random variables. Find the marginal p.d.f.s. of
    (i) $X + Y,\ X - Y,\ \frac{X+Y}{X-Y},\ |X - Y|$;
    (ii) $\min(X, Y),\ \max(X, Y), \frac{\min(X,Y)}{\max(X,Y)}$;
    (iii) $X^2 + Y^2$.

47. Let $W \sim Be(\alpha_1, \alpha_2)$ and $T \sim G(\alpha_1 + \alpha_2, \theta)$ be independent random variables. Using Example 10.2.11, show that $WT \sim G(\alpha_1, \theta)$.

48. Let X and Y be i.i.d. random variables with common p.d.f. $f(x) = \frac{c}{1+x^4}, -\infty < x < \infty$, where c is the normalizing constant. Find the p.d.f. of $Z = \frac{X}{Y}$.

49. Let X and Y be i.i.d. $N(0,1)$ random variables. Define the random variables R and Θ by $X = R\cos\Theta, Y = R\sin\Theta$.
    (i) Show that R and Θ are independent with $\frac{R^2}{2} \sim \text{Exp}(1)$ and $\Theta \sim U(0, 2\pi)$;
    (ii) Show that $X^2 + Y^2$ and $\frac{X}{Y}$ are independently distributed;
    (iii) Show that $\sin\Theta$ and $\sin 2\Theta$ are identically distributed and hence find the p.d.f. of $T = \frac{XY}{\sqrt{X^2+Y^2}}$;
    (iv) Find the distribution of $U = \frac{3X^2Y - Y^3}{X^2+Y^2}$.

50. Let U_1 and U_2 be i.i.d. $U(0,1)$ random variables. Show that $X_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2)$ and $X_2 = \sqrt{-2\ln U_1}\sin(2\pi U_2)$ are i.i.d. $N(0,1)$ random variables. (This is known as *the Box-Muller transformation*).

51. Let $(X, Y) \sim N_2(0, 0, 1, 1, \rho)$.
    (i) Show that $P(\{X > 0, Y > 0\}) = \frac{1}{4} + \frac{\text{arc}\ \sin\rho}{2\pi}$. Also find $P(\{X < 0, Y < 0\}), P(\{X > 0, Y < 0\})$ and $P(\{X < 0, Y > 0\})$;
    (ii) Show that $P(\{XY > 0\}) = \frac{1}{2} + \frac{\text{arc}\ \sin\rho}{\pi}$ and $P(\{XY < 0\}) = \frac{1}{2} - \frac{\text{arc}\ \sin\rho}{\pi}$.

52. Let $X_1, \dots, X_n$ be a random sample from the Exp(1) distribution.

(i) Find the marginal distributions of $Y_1, \dots, Y_n$, where

$$Y_i = \frac{\sum_{j=1}^{i} X_j}{\sum_{j=1}^{i+1} X_j}, \; i = 1, \dots, n-1, \qquad Y_n = X_1 + \cdots + X_n;$$

(ii) Are $Y_1, \dots, Y_n$ independent?

53. Let X_1, X_2, X_3 be i.i.d. $G(m, 1)$ random variables. Let $Z_1 = X_1 + X_2 + X_3$, $Z_2 = \frac{X_2}{X_1+X_2+X_3}$ and $Z_3 = \frac{X_3}{X_1+X_2+X_3}$.
    (i) Show that Z_1 and (Z_2, Z_3) are independent and find marginal p.d.f.s. of Z_1, Z_2 and Z_3;
    (ii) Find $E(Z_1^2 Z_2 Z_3)$.

54. Let X_1 and X_2 be independent random variables with $X_i \sim \text{Bin}\left(n_i, \frac{1}{2}\right)$, $i = 1, 2$. Using the m.g.f. technique, find the distribution of $Y = X_1 - X_2 + n_2$.

55. Let $X_{1:n} \leq X_{2:n} \leq \cdots, \leq X_{n:n}$ be the set of order statistics associated with a random sample of size $n \, (\geq 2)$ from the $\text{Exp}(1)$ distribution.
    (i) Let $Z_1 = nX_{1:n}$, $Z_i = (n-i+1)(X_{i:n} - X_{i-1:n})$, $i = 2, \dots, n$. Show that $Z_1, \dots, Z_n$ are i.i.d. $\text{Exp}(1)$ random variables;
    (ii) Using (i), or otherwise, find $E(X_{r:n}), \text{Var}(X_{r:n})$ and $\text{Cov}(X_{r:n}, X_{s:n}), 1 \leq r < s \leq n$;
    (iii) Show that $X_{r:n}$ and $X_{s:n} - X_{r:n}$ are independent for any $s > r$;
    (iv) Find the p.d.f. of $X_{r+1:n} - X_{r:n}, r = 1, 2, \dots, n$.

56. Let $X_1, \dots, X_n$ be i.i.d. non-negative random variables ($P(\{X_1 \geq 0\}) = 1$) of the absolutely continuous type. If $E(|X_1|) < \infty$ and $M_n = \max(X_1, \dots, X_n)$, show that

$$E(M_n) = E(M_{n-1}) + \int_0^\infty \left(F(x)\right)^{n-1} \left(1 - F(x)\right) dx.$$

57. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics associated with a random sample of size $n \, (\geq 2)$ from the $U(0,1)$ distribution. Let $Y_i = \frac{X_{i:n}}{X_{i+1:n}}, i = 1, \dots, n-1$, and $Y_n = X_{n:n}$. Show that $Y_1, \dots, Y_n$ are independent and find the p.d.f of $Y_i, i = 1, \dots, n$.