Recall that a matrix $A = (a_{ij})$ is a diagonal matrix if every off-diagonal entry is zero.

i.e. $a_{ij} = 0$ for $i \neq j$.

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

Let $\mathrm{diag}(a_1, \ldots, a_n)$ denote the matrix

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & a_n \end{pmatrix}$$

eg: $\operatorname{diag}(1,2,3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Consider two diagonal matrices

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & a_n \end{pmatrix} + \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & b_n \end{pmatrix} = \begin{pmatrix} a_1+b_1 & 0 & \cdots & 0 \\ 0 & a_2+b_2 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & a_n+b_n \end{pmatrix}$$

i.e. $$\mathrm{diag}(a_1, \ldots, a_n) + \mathrm{diag}(b_1, \ldots, b_n) = \mathrm{diag}(a_1+b_2, \ldots, a_n+b_n).$$

For a scalar c, $$c\,\mathrm{diag}(a_1, \ldots, a_n) = \mathrm{diag}(ca_1, \ldots, ca_n).$$

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & a_n \end{pmatrix} \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & b_n \end{pmatrix} = \begin{pmatrix} a_1b_1 & 0 & \cdots & 0 \\ 0 & a_2b_2 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & a_nb_n \end{pmatrix}$$

$$\mathrm{diag}(a_1, \ldots, a_n)\,\mathrm{diag}(b_1, \ldots, b_n) = \mathrm{diag}(a_1 b_1, \ldots, a_n b_n).$$

$$\left(\mathrm{diag}(1,2,3)\right)^n = \mathrm{diag}(1^n, 2^n, 3^n).$$

$$\left(\mathrm{diag}(a_1, \ldots, a_n)\right)^m = \mathrm{diag}(a_1^m, \ldots, a_n^m).$$

Let $p(x) = x^3 + 2x + 3$

$$p(A) := A^3 + 2A + 3I_n$$

If $A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$

$$p(A) = \operatorname{diag}(a_1^3, a_2^3, \ldots, a_n^3) + \operatorname{diag}(2a_1, \ldots, 2a_n) + \operatorname{diag}(3, \ldots, 3).$$

$$= \operatorname{diag}(a_1^3 + 2a_1 + 3, \ldots, a_n^3 + 2a_n + 3)$$

$$= \operatorname{diag}(p(a_1), \ldots, p(a_n)).$$

Lemma: Let A be a diagonal matrix. Then the rank of A is equal to the number of its non-zero entries.

Let $A = \text{diag}(a_1, \dots, a_n)$ be a diagonal matrix. Consider the linear transformation L_A. Then $L_A v$

$$L_A v = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 x_1 \\ a_2 x_2 \\ \vdots \\ a_n x_n \end{pmatrix}$$

Let $\beta = \{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Then

$L_A e_1 = a_1 e_1$, $L_A e_2 = a_2 e_2$, $\ldots$, $L_A e_n = a_n e_n$.

Let $T : V \to V$ be a linear operator on V (i.e. T is a linear transformation from V to itself.)

eg: (*) Consider $T = I_V$. Then $I_V v = v \;\; \forall \, v \in V$.

(*) Let $T = \lambda I_V$. Then $Tv = \lambda v. \;\; \forall \, v \in V$.

Definition of an eigenvector:

Let $T: V \to V$ be a linear trans. We say that a non-zero vector $v \in V$ is an eigenvector of T if $Tv = \lambda v$ for some scalar λ. The scalar λ is called the eigenvalue corresponding to the eigenvector v.

Example 1: Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$T(x, y) = (2x, 3y).$$

Let $v_1 = (1,0)$. Then $Tv_1 = (2,0) = 2v_1$

||ly if $v_2 = (0,1)$, then $Tv_2 = 3(0,1) = 3v_2$.

The standard basis are examples of eigenvectors.

In fact any vector of the type $(a,0)$ is an eigenvector of T with eigenvalue 2. $[T(a,0) = (2a,0) = 2(a,0)$

$(1,1)$ is not eigenvector of T.

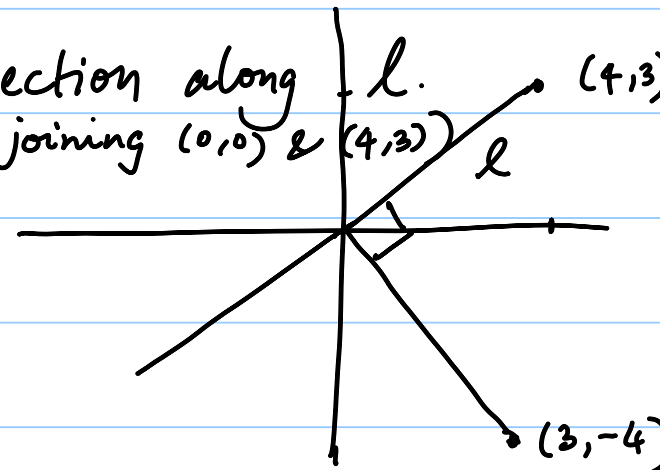
Example 2: If $T = I_V$, then every non-zero vector is an eigenvector with eigenvalue 1.

Example 3: If $T : V \to V$ is not injective. Let $v \in \mathrm{Null}(T)$ s.t. $v \neq 0$. Then

$$Tv = 0 = 0v.$$

Hence v is an eigenvector with eigenvalue 0.

Example 4: Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be reflection along ℓ.
(where ℓ is the line joining $(0,0)$ & $(4,3)$)



If $v_1 = (4,3)$

Then $Tv_1 = v_1$.

Let $v_2 = (3,-4)$. Then

$Tv_2 = (-3,4) = -v_2$.

Then T has eigenvectors v_1 with eigenvalue 1 and v_2 with eigenvalue -1.

Let A be an $n \times n$ matrix. We say that a vector v is an eigenvector of A with eigenvalue λ if v is an eigenvector of L_A with eigenvalue λ.

Example: Let $A = \operatorname{diag}(a_1, \ldots, a_n)$.

claim: The e_i is an eigenvector of A.

$L_A e_i = A e_i = a_i e_i$. Hence e_i is an eigenvector of L_A with eigenvalue a_i.

Definition of an eigenspace.

Let $T: V \longrightarrow V$ be a linear operator on V. Then the eigenspace of a scalar λ is the set of all vectors s.t $Tv = \lambda v$.

For $\lambda \in \mathbb{R}$,

$$Tv = \lambda v \iff Tv = \lambda I_V v$$

$$\iff (T - \lambda I_V) v = 0$$

$\Leftarrow$) $\quad v \in \mathrm{Null}(T - \lambda I_V)$.

Hence the eigenspace of λ is the null space of $T - \lambda I_V$. which is a subspace of V.

Observe that λ is an eigenvalue iff $\exists$ a non-zero vector $v \in \mathrm{Null}(T - \lambda I_V)$. iff $T - \lambda I_V$ is not injective.

Example 1: $\quad T(x, y) = (2x, 3y) \quad$ where $T : \mathbb{R}^2 \to \mathbb{R}^2$.

Consider $\left(T - 2I_V\right)(x, y) = (0, 3y)$

Clearly the subspace $\{y=0\}$ is contained in $N(T-2I_V)$

Illy $\{x=0\} \subseteq N(T-3I_V)$.

Exercise : $T-\lambda I_V$ is invertible for all $\lambda \neq 2, 3$.

<u>Proposition</u>: Let $T: V \to V$ be a linear operator on V which has finite dimension (say n). If $\beta = (v_1, \ldots, v_n)$ is an ordered basis of V consisting of

eigenvectors of T, then $[T]_\beta^\beta$ is a diagonal matrix.

Conversely if $[T]_\beta^\beta$ is a diagonal matrix corresponding to an ordered basis $\beta = (v_1, \ldots, v_n)$, then v_i are eigenvectors of T.

Proof: We have a basis $\beta = (v_1, \ldots, v_n)$ consisting of eigenvectors of T.

$$Tv_j = \lambda_j v_j \quad \text{where } \lambda_j \text{ is the eigenvalue of } v_j.$$

$$\Rightarrow \quad [Tv_j]^{\beta} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \lambda_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$ where λ_j is in the j^{th} row.

$\Rightarrow \quad [T]_{\beta}^{\beta} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ where λ_j is the eigenvalue of the eigenvector v_j.

Let $\beta = (v_1, \dots, v_n)$ be a basis s.t

$[T]_{\beta}^{\beta} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

i.e. $Tv_j = \lambda_j v_j$ for $j = 1, 2, \dots, n$.

$\Rightarrow (v_1, \dots, v_n)$ are eigenvectors corresponding to λ_j. —— $\blacksquare$

Definition: We say that a linear transformation $T: V \to V$ is diagonalizable if there exists a basis β such that $[T]_{\beta}^{\beta}$ is a diagonal matrix.

Example: $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T(x,y) = (2x, 3y)$ is diagonizable.

Example: Let $A = \text{diag}(a_1, \ldots, a_n)$ be a diagonal matrix. Then L_A is diagonalizable.

Let us revisit example 4 above.

$T: \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection along ℓ which is the line joining 0 and $(4,3)$

Recall that $(4,3)$ & $(3,-4)$ are eigenvectors with eigenvalues 1 and -1 respectively.

Check that $(4,3)$ & $(3,-4)$ are linearly independent.

Let $\beta = ((4,3), (3,-4))$.

i.e $$[T]_\beta^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[T^2]_\beta^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = [I_V]_\beta^\beta$$

Hence $T^2 = I_V$

Let A be an $n \times n$ matrix. We say that A is diagonalizable if the linear transformation L_A is diagonalizable.

Example: All diagonal matrices are diagonalizable.

Proposition: Let A be an $n \times n$ matrix. Then A is diagonalizable iff $\exists$ a diagonal matrix D and an invertible matrix Q such that

$$A = QDQ^{-1}.$$

(Rephrasing : An $n \times n$ matrix is diagonalizable iff A is similar to a diagonal matrix).

Proof: Let us assume that A is diagonalizable.

Let $\beta' = (v_1, \ldots, v_n)$ be a basis of $\mathbb{R}^n$ s.t

$$[L_A]_{\beta'}^{\beta'} = \mathrm{diag}(a_1, \ldots, a_n) = D.$$

$$L_A = I_{\mathbb{R}^n} L_A I_{\mathbb{R}^n}$$ where $I_{\mathbb{R}^n}$ is the identity linear trans. in $\mathbb{R}^n$.

Let β be the standard basis of $\mathbb{R}^n$.

$$A = [L_A]_\beta^\beta = [I L_A I]_\beta^\beta = [I]_{\beta'}^{\beta} [L_A]_{\beta'}^{\beta'} [I]_\beta^{\beta'}$$

Let $Q = [I]_{\beta'}^{\beta}$, then $Q^{-1} = [I]_\beta^{\beta'}$

Hence $A = QDQ^{-1}$

To prove the converse, let

$A = QDQ^{-1}$ where D is a diagonal matrix and Q an invertible matrix.

Let $\beta = (e_1, \ldots, e_n)$ be std basis. Then

$De_j = \lambda_j e_j$ (where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$)

Consider $\beta' = (Qe_1, \ldots, Qe_n)$.

$(DQ^{-1})(Qe_j) = De_j = \lambda e_j$.

Then $QDQ^{-1}(Qe_j) = Q(\lambda e_j) = \lambda Qe_j$

i.e Qe_j is an eigenvector of QDQ^{-1}.

Claim: β' is a basis of $\mathbb{R}^n$.

Exercise

Hence QDQ^{-1} is a diagonal matrix w.r.t β'

i.e. A is diagonalizable.

Proposition: Let A be an $n \times n$ matrix. Suppose $\beta' = (v_1, \dots, v_n)$ are vectors in $\mathbb{R}^n$ s.t. $A v_j = \lambda_j v_j$ and s.t β is linearly independent, then

$$A = Q D Q^{-1} \text{ where}$$

$$Q = (v_1, \dots, v_n)$$

and $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ where λ_j is the eigenvalue of v_j.

Proof: From the proof of the previous proposition,

$$A = [L_A]_{\beta}^{\beta} = [I]_{\beta'}^{\beta} [L_A]_{\beta'}^{\beta'} [I]_{\beta}^{\beta'}$$

$[L_A]_{\beta'}^{\beta'} = \text{diag}(\lambda_1, \ldots, \lambda_n)$ where λ_j are

s.t $\quad Av_j = \lambda_j v_j$.

The j^{th} column of the change of basis matrix $[I]_{\beta'}^{\beta}$ is the column vector of v_j.

Therefore

$$Q := [I]_{\beta'}^{\beta} = (v_1, \ldots, v_n).$$

Proposition: A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if $\det(A - \lambda I_n) = 0$.

Proof: Let λ be an eigenvalue of A. i.e.

$\exists$ an non-zero vector v s.t $Av = \lambda v$

$\Rightarrow$ $(A - \lambda I_n) v = 0$

Hence $A - \lambda I_n$ is not injective.

i.e. $A - \lambda I_n$ is not invertible ($\mathbb{R}^n$ has dim n....)

By a theorem from the previous week, we have
$\det(A - \lambda I_n) = 0$.

Suppose $\det(A - \lambda I_n) = 0 \Rightarrow A - \lambda I_n$ is not invertible
$\Rightarrow \exists\ v \neq 0$ in V s.t $(A - \lambda I_n)v = 0$

$\Rightarrow \quad Av = \lambda v.$

$\therefore \quad \lambda$ is an eigenvalue of A.

<u>Definition:</u> We call the expression $\det(A - \lambda I_n)$ as the characteristic polynomial of the matrix A and we shall denote this by $f(\lambda)$.

Hence the eigenvalues of A are precisely the roots of $f(\lambda)$.

Example: Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Characteristic polynomial of A

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}\right)$$

$$= -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1.$$

Hence eigenvalues are solns of $\lambda^2 - \lambda - 1 = 0$

$\Rightarrow$ $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$, $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$ are the eigenvalues of A.

$$A - \lambda_2 I = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} - \frac{1-\sqrt{5}}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -(1-\sqrt{5})/2 & 1 \\ 1 & (1+\sqrt{5})/2 \end{pmatrix}$$

$$(A - \lambda_1 I)\begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{aligned} -\frac{1-\sqrt{5}}{2}x + y &= 0 \\ x + \frac{1+\sqrt{5}}{2}y &= 0 \end{aligned}$$

$v_2 = \begin{pmatrix} -\frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$ is an eigenvector of λ_2

$v_1 = \begin{pmatrix} -\frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$ is an eigenvector of λ_1

Hence for the basis $\beta' = (v_1 , v_2)$, the

$$D = [L_A]_{\beta'}^{\beta'} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = Q D Q^{-1}$$

where $Q = (v_1 , v_2)$

Example: Let there be x pairs of juvenile rabbits and y pairs of adult rabbits which is captured in $\begin{pmatrix} x \\ y \end{pmatrix}$.

Assume that the population growth is given by

$$\begin{pmatrix} y \\ x+y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

After n years, what is the population of rabbits?

We know that

$A^n v_0$ has our answer.

We know that $A = QDQ^{-1}$

$$A^2 = QDQ^{-1}QDQ^{-1} = QD^2Q^{-1}$$

|||ly $A^3 = QD^2Q^{-1}QDQ^{-1} = QD^3Q^{-1}$

By induction $A^n = QD^nQ^{-1}$.

But $D = \mathrm{diag}(\lambda_1, \lambda_2)$ where $\lambda_1 = \frac{1+\sqrt{5}}{2}$ & $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

$D^n = \mathrm{diag}(\lambda_1^n, \lambda_2^n)$.

$$Q = \begin{pmatrix} -\frac{1-\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$$

$$Q^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ -1 & -\frac{1-\sqrt{5}}{2} \end{pmatrix}$$

$$Q^{-1} v_0 = \begin{pmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}$$

$$D^n Q^{-1} v_0 = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix} = \begin{pmatrix} \lambda_1^n/\sqrt{5} \\ -\lambda_2^n/\sqrt{5} \end{pmatrix}$$

$$Q = \begin{pmatrix} -\frac{1-\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$$

$$\underbrace{\left(\frac{1-\sqrt{5}}{2}\right)}_{\lambda_2} \cdot \underbrace{\left(\frac{1+\sqrt{5}}{2}\right)}_{\lambda_1} = -1$$

$$= \begin{pmatrix} \lambda_1^{-1} & \lambda_2^{-1} \\ 1 & 1 \end{pmatrix}$$

$$A^n v_0 = Q D Q^{-1} v_0 = \begin{pmatrix} \lambda_1^{-1} & \lambda_2^{-1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n/\sqrt{5} \\ -\lambda_2^n/\sqrt{5} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_1^{n-1} - \lambda_2^{n-1} \\ \lambda_1^n - \lambda_2^n \end{pmatrix}$$

Population of pairs of juvenile rabbits after n years

$$= (\lambda_1^{n-1} - \lambda_2^{n-1})/\sqrt{5}$$

of adult rabbits $= (\lambda_1^{n} - \lambda_2^{n})/\sqrt{5}$

$v_0 = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix}$ $v_1 = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$

$F_0 = 0, \quad F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad F_4 = 3, \quad \ldots$

are called fibonacci numbers.

Consider the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Characteristic poly is λ^2+1.