Let $A$ be an $m \times n$ matrix. Then we define a map $L_A : \mathbb{R}^n \to \mathbb{R}^m$ be given by

$$L_A x := Ax$$

Check that $L_A$ is a linear transformation.

<u>Proposition:</u> Let $A$ be an $m \times n$ matrix. Then the matrix of $L_A$ w.r.t the standard basis is $A$.

<u>Proof</u>: Let $\alpha = (e_1, \dots, e_n)$ be the std. basis of $\mathbb{R}^n$
& $\beta$ be the std. basis of $\mathbb{R}^m$.

WTS $[L_A]_\alpha^\beta = A$.

Observe that for $x \in \mathbb{R}^n$ $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Rightarrow x = xe_1 + \dots + xe_n$

$\Rightarrow \quad x = [x]^\alpha$. $\quad$ lly $\quad y = [y]^\beta \quad \forall \; y \in \mathbb{R}^m \longrightarrow (*)$

$$[L_A x]^\beta = [L_A]_\alpha^\beta [x]^\alpha$$

$$(*) \Rightarrow \quad L_A x = [L_A]_\alpha^\beta x \quad \forall\, x \in \mathbb{R}^n.$$

$$L_A x = Ax \quad \forall\, x \in \mathbb{R}^n$$

$$\Rightarrow \quad Ax = [L_A]_\alpha^\beta x \quad \forall\, x \in \mathbb{R}^n. \longrightarrow (**)$$

Exercise: Prove that $A = [L_A]_\alpha^\beta$

Proposition: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. & $\alpha, \beta$ denote the std basis of $\mathbb{R}^n$ & $\mathbb{R}^m$ resp. Then

$$L_{[T]_\alpha^\beta} = T.$$

Proof: We know that $[Tx]^\beta = [T]_\alpha^\beta [x]^\alpha$

$$Tx = [T]_\alpha^\beta x. = L_{[T]_\alpha^\beta} x \quad \forall\ x \in \mathbb{R}^n$$

$$\therefore \quad T = L_{[T]_\alpha^\beta}. \qquad \blacksquare$$

Corollary: Let $A, B, C$ be $l \times k$, $m \times l$ and $n \times m$ matrices respectively. Then $(CB)A = C(BA)$.

$$\mathbb{R}^k \xrightarrow{L_A} \mathbb{R}^l \xrightarrow{L_B} \mathbb{R}^m \xrightarrow{L_C} \mathbb{R}^n$$

<u>Proof:</u> We know that

$$(L_C L_B) L_A = L_C (L_B L_A).$$

Let $\alpha$, $\beta$, $\gamma$ and $\delta$ be the ordered std. basis of $\mathbb{R}^k$, $\mathbb{R}^l$, $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

$$[(L_C L_B) L_A]_\alpha^\delta = [L_C (L_B L_A)]_\alpha^\delta .$$

$$L.H.S = \left([L_C L_B]_\beta^\delta\right)[L_A]_\alpha^\beta = \left([L_C]_\gamma^\delta [L_B]_\beta^\gamma\right)[L_A]_\alpha^\beta$$

$$= (CB)A.$$

$$R.H.S = [L_C]^{\delta}_{\gamma}\left([L_B L_A]^{\gamma}_{\alpha}\right) = (L_C)^{\delta}_{\gamma}\left([L_B]^{\gamma}_{\beta}[L_A]^{\beta}_{\alpha}\right).$$

$$= C(BA).$$

$$\Rightarrow (CB)A = C(BA).$$

<u>Exercise</u>: 1) $L_{AB} = L_A L_B$

2) $L_{A+B} = L_A + L_B.$

**Definition:** A linear transformation $T: V \to W$ is said to be invertible if $\exists$ a linear transformation $S: W \to V$ s.t $ST = I_V$ & $TS = I_W$ (where $I_V$ and $I_W$ are the identity maps of $V$ & $W$ resp.)

The map $S$ is called the inverse of $T$.

**Lemma:** Let $T: V \to W$ be an invertible linear transformation & suppose $S$ & $S'$ are two inverses.

Then $S = S'$.

<u>Proof</u>: $S = SI_W = S(TS') = (ST)S'$
$= I_V S' = S'$ ——— ∎

The unique inverse of an invertible linear transformation $T$ is denoted by $T^{-1}$.

We say that two vector spaces $V$ & $W$ are isomorphic if $\exists$ an invertible linear transformation $T: V \longrightarrow W$.

**Proposition:** Let $T : V \to W$ be an invertible linear transformation. Then $T$ is bijective.

**Proof:** $T$- injective.

Suppose $v_1$ & $v_2$ are s.t. $Tv_1 = Tv_2$

Then $T^{-1}(Tv_1) = T^{-1}(Tv_2)$

$\Rightarrow (T^{-1}T)v_1 = (T^{-1}T)v_2 \Rightarrow I_V v_1 = I_V v_2$

$\Rightarrow v_1 = v_2$

Surjectivity: Let $w \in W$.

Suppose $v = T^{-1}w$. Then check that $Tv = w$

Proposition: Let $T: V \to W$ be a bijective linear transformation. The $T$ is invertible.

Proof: Let us define $S: W \to V$ as follows:

For $w \in W$. By surjectivity, $\exists$ a vector $v \in V$ s.t $Tv = w$. Define $Sw = v$. ( $S$ is well-defined by the injectivity of $T$ ).

By definition, $TS = I_W$

WTS $ST = I_V$. Observe for $v \in V$

$$T(STv) = (TS)Tv = (I_W)Tv = Tv$$

But $T$- injective $\Rightarrow$ $STv = v \quad \forall\, v \in V$.

$$\Rightarrow ST = I_V.$$

Finally, we want to check that $S$ is a linear transformation.

$$T(S(w_1 + w_2)) = w_1 + w_2 = (TS)w_1 + (TS)w_2$$

$$= T(Sw_1 + Sw_2)$$

By injectivity of $T$, we have

$$S(w_1 + w_2) = Sw_1 + Sw_2. \qquad \blacksquare$$

<u>Proposition</u>: Let $V$ & $W$ be finite dimensional vector spaces. Then $V$ & $W$ are isomorphic iff $\dim(V) = \dim(W)$.

<u>Proof</u>: $(\Rightarrow)$ Let $T : V \to W$ an invertible linear transformation.

$\Rightarrow$ $T$ - surjective i.e $R(T) = W$.

$T$ - injective $\Leftrightarrow$ $N(T) = \{0\}$

By dimension theorem,

$$\dim(V) = \dim(N(T)) + \dim(R(T)).$$
$$= 0 + \dim(W).$$

($\Leftarrow$) Let $\{v_1, \ldots, v_n\}$ & $\{w_1, \ldots, w_n\}$ be bases of $V$ and $W$ resp. $\exists$ a unique linear transformation $T : V \longrightarrow W$ s.t $Tv_j = w_j$ for $1 \leq j \leq n$.

Since $\{w_1, \ldots, w_n\}$ is a basis of $W$, we get that $R(T) = W \Rightarrow T$ - surjective.

By the dimension theorem,

$$\dim(V) = \dim(N(T)) + \dim(W)$$

$$\Rightarrow \dim(N(T)) = 0 \Rightarrow N(T) = \{0\}$$

$\Rightarrow T$ is injective.

$\therefore$ $T$ is an invertible linear transformation.

Hence, $V$ and $W$ are isomorphic vector spaces.

eg: $T: \mathbb{R}^3 \longrightarrow P_2(\mathbb{R})$, define

$$T(a,b,c) = a + bx + cx^2.$$

Exercise: Prove that every finite dimensional vector space is isomorphic to $\mathbb{R}^n$ for some $n$.

Exercise: $\mathbb{R}^n$ and $\mathbb{R}^m$ are isomorphic iff $n = m$.

Definition: Let $A$ be an $m \times n$ matrix. We say that an $n \times m$ matrix $B$ is the inverse of $A$ if $AB = I_m$ and $BA = I_n$. We then say that $A$ is an invertible matrix.

Exercise: The inverse of a matrix $A$ is unique.

Example: Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Then

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

Theorem: Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces $V$ & $W$. Suppose $\alpha$ and $\beta$ are ordered basis of $V$ & $W$ respectively. Then

$T$ is an invertible linear transformation if and only if $[T]_\alpha^\beta$ is an invertible matrix. Moreover, $\left([T]_\alpha^\beta\right)^{-1} = [T^{-1}]_\beta^\alpha$.

<u>Proof</u>: Let $\alpha = (v_1, \dots, v_n)$ and $\beta = (w_1, \dots, w_m)$ be ordered bases of $V$ and $W$ respectively.

$(\Rightarrow)$ Let $T$ be an invertible linear transformation. and $T^{-1} : W \to V$ be its inverse.

Then

$$TT^{-1} = I_W \quad \& \quad T^{-1}T = I_V.$$

$$[TT^{-1}]_\beta^\beta = [I_W]_\beta^\beta = I_m$$

Recall that $\quad [TT^{-1}]_\beta^\beta = [T]_\alpha^\beta [T^{-1}]_\beta^\alpha$

$$\therefore \quad [T]_\alpha^\beta [T^{-1}]_\beta^\alpha = I_m.$$

A similar argument gives us that

$$[T^{-1}]_\beta^\alpha \, [T]_\alpha^\beta = I_n \longrightarrow (*)$$

And therefore $[T]_\alpha^\beta$ is invertible with inverse

$$\left([T]_\alpha^\beta\right)^{-1} = [T^{-1}]_\beta^\alpha \quad (\text{from } (*)).$$

($\Leftarrow$) Let $[T]_\alpha^\beta$ be an invertible matrix with inverse given by $B$

Let $\alpha = (v_1, \dots, v_n)$ and $\beta = (w_1, \dots, w_m)$

Let $B = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix}$

Define $S : W \longrightarrow V$ in the following manner.

Recall that to define a linear transformation from $W \to V$ it is enough to describe the fn $S$ on a basis of $W$.

Consider $\quad S w_j = b_{1j} v_1 + b_{2j} v_2 + \dots + b_{nj} v_j = u_j$

Then $\exists!$ linear transformation $S : W \to V$ s.t

$$S w_j = u_j .$$

Consider

$$[TS]_\beta^\beta = [T]_\alpha^\beta [S]_\beta^\alpha = [T]_\alpha^\beta B = I_m$$

$$= [I_W]_\beta^\beta$$

$\Rightarrow \quad TS = I_W$

$|||^{ly}$ $ST = I_V$

Therefore $S$ is the inverse of $T$

$\Rightarrow$ $T$ is an invertible linear transformation —— $\blacksquare$.

<u>Corollary</u>: Let $A$ be an $m \times n$ matrix. Then $A$ is invertible if and only if $L_A$ is invertible. Also $L_A^{-1} = L_{A^{-1}}$.

<u>Proof:</u> Let $\alpha$ and $\beta$ be the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ resp.

$$[L_A]_\alpha^\beta = A$$

$$\left([L_A]_\alpha^\beta\right)^{-1} = A^{-1} = [L_A^{-1}]_\beta^\alpha$$

But $$[L_{A^{-1}}]_\beta^\alpha = A^{-1}$$

$$\therefore \quad [L_{A^{-1}}]_\beta^\alpha = [L_A^{-1}]_\beta^\alpha$$

$$\Rightarrow \quad L_{A^{-1}} = L_A^{-1}. \qquad \blacksquare$$

Corollary: Let $A$ be an invertible $m \times n$ matrix. Then $m = n$.

Proof: $A$ is an invertible matrix $\Leftrightarrow$ $L_A : \mathbb{R}^n \to \mathbb{R}^m$ is an invertible linear transformation

$$\Rightarrow \dim(\mathbb{R}^n) = \dim(\mathbb{R}^m)$$

$$\Leftrightarrow \quad n = m.$$

Recall that $\mathcal{L}(V, W)$ denoted the vector space of all linear transformations from $V$ to $W$.

<u>Theorem</u>: Let $V$ and $W$ be finite dimensional vector spaces. Let $\dim(V) = n$ and $\dim(W) = m$. Then $\mathcal{L}(V, W)$ is isomorphic to $M_{m\times n}(\mathbb{R})$.

<u>Proof</u>: Let $\alpha = (v_1, \ldots, v_n)$ and $\beta = (w_1, \ldots, w_m)$

be bases of $V$ and $W$ respectively.

Define

$$\Phi : \mathcal{L}(V, W) \longrightarrow M_{m \times n}(\mathbb{R})$$

$$\Phi(T) = [T]_{\alpha}^{\beta}.$$

$$\Phi(S+T) = [S+T]_{\alpha}^{\beta} = [S]_{\alpha}^{\beta} + [T]_{\alpha}^{\beta}$$

$$= \Phi(S) + \Phi(T) \quad \forall \; S, T \in \mathcal{L}(V, W)$$

$$\Phi(cT) = c\,\Phi(T) \quad \forall\ T \in \mathcal{L}(V,W)\ \&\ c \in \mathbb{R}.$$

Exercise: $\text{Null}(\Phi) = 0$ ← zero linear transformation

$\Rightarrow \Phi$ is injective.

Surjectivity: Let $A$ be an $m \times n$ matrix.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Define $T : V \to W$ where

$$Tv_j = (a_{1j} w_1 + a_{2j} w_2 + \cdots + a_{mj} w_m).$$

$\exists!$ linear transformation $T : V \to W$ s.t

$$Tv_j = (a_{1j} w_1 + \cdots + a_{mj} w_m).$$

Check that $[T]_\alpha^\beta = A$.

i.e. $\Phi(T) = A$.

Thus $\Phi$ is an isomorphism. $\blacksquare$

Corollary: $\dim(\mathcal{L}(V, W)) = mn.$

NPTEL