Let V be a finite dimensional vector space and let β be an ordered basis of V. Let β' be another ordered basis. Given $v \in V$, how are the column representation of v w.r.t β, denoted by $[v]^{\beta}$ is related to $[v]^{\beta'}$?

To study this, consider $I_V : \underset{\substack{\uparrow \\ \text{ordered} \\ \text{basis } \beta}}{V} \longrightarrow \underset{\substack{\uparrow \\ \text{ordered} \\ \text{basis } \beta'}}{V}$ the identity linear transformation.

Consider the matrix $[I_V]_{\beta}^{\beta'}$.

We know that $I_V v = v$

Then $[I_V v]^{\beta'} = [I_V]_{\beta}^{\beta'} [v]^{\beta}$.

$\Rightarrow$ $[v]^{\beta'} = [I_V]_{\beta}^{\beta'} [v]^{\beta}$.

The matrix $[I_V]_{\beta}^{\beta'}$ is called the change of basis matrix from β to β'.

Proposition: $\left[I_V\right]_{\beta}^{\beta'}$ is invertible.

Proof: The matrix corresponding to an invertible linear transformation is invertible. I_V is an inv. lin. trans.
Thus $\left[I_V\right]_{\beta}^{\beta'}$ is hence an invertible.

* If $\dim V = n$, then $\left[I_V\right]_{\beta}^{\beta'}$ is an $n \times n$ matrix.

* Exercise: Prove that the change of basis matrix $[I_V]_{\beta}^{\beta'}$ is the inverse of the change of basis matrix $[I_V]_{\beta'}^{\beta}$ from β' to β.

Example: Let us consider $V = \mathbb{R}^2$ with $\beta = (e_1, e_2)$, the standard basis and $\beta' = ((1,1), (1,-1))$.

$$e_1 = I_V e_1 = (1,0) = \frac{1}{2}(1,1) + \frac{1}{2}(1,-1).$$

$$e_2 = I_V e_2 = (0,1) = \frac{1}{2}(1,1) + \left(\frac{-1}{2}\right)(1,-1)$$

$$\Rightarrow \quad [I_V]_{\beta}^{\beta'} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

Let $v = (x, y) \in \mathbb{R}^2$ then $[v]^{\beta} = \begin{pmatrix} x \\ y \end{pmatrix}$

note that $v = (x, y) = \frac{(x+y)}{2}(1,1) + \frac{(x-y)}{2}(1,-1).$

$$[v]^{\beta'} = \begin{pmatrix} (x+y)/2 \\ (x-y)/2 \end{pmatrix}$$

$$[I_V]_{\beta}^{\beta'} [v]^{\beta} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x+y)/2 \\ (x-y)/2 \end{pmatrix} = [v]^{\beta'}$$

Let $T : V \to V$ be a linear transformation. A linear transformation V to itself is called a <u>linear operator</u>.

Let β and β' be ordered basis of V.

Let us study how $[T]_{\beta}^{\beta}$ and $[T]_{\beta'}^{\beta'}$ are related.

$$\begin{tikzcd} \underset{\beta'}{V} \arrow[r, "I_V"] \arrow[rrr, bend right, "T"'] & \underset{\beta}{V} \arrow[r, "T"] & \underset{\beta}{V} \arrow[r, "I_V"] & \underset{\beta'}{V} \end{tikzcd}$$

$$I_V T I_V = T$$

$$[T]_{\beta'}^{\beta'} = [I_V T I_V]_{\beta'}^{\beta'} = [I_V]_{\beta}^{\beta'} [T]_{\beta}^{\beta} [I]_{\beta'}^{\beta}$$

Let $Q = [I_V]_{\beta}^{\beta'}$

Then $$[T]_{\beta'}^{\beta'} = Q[T]_{\beta}^{\beta} Q^{-1}$$

Two matrices A and B are said to be similar if there exists an invertible matrix Q s.t

$$A = QBQ^{-1}.$$

Let $V_1, V_2, \ldots, V_n$ be vector spaces over $\mathbb{R}$. Consider the product of $V_1, \ldots, V_n$

$$V_1 \times V_2 \times \cdots \times V_n := \{ (v_1, v_2, \ldots, v_n) : v_i \in V_i \}$$

Vector addition:

Let $(v_1, v_2, \ldots, v_n)$ and $(v_1', v_2', \ldots, v_n') \in V_1 \times V_2 \times \cdots \times V_n$.

$$(v_1, v_2, \ldots, v_n) + (v_1', v_2', \ldots, v_n') := (v_1 + v_1', v_2 + v_2', \ldots, v_n + v_n').$$

Scalar multiplication :

Let $c \in \mathbb{R}$ and $(v_1, v_2, \ldots, v_n) \in V$.

Define $c(v_1, v_2, \ldots, v_n) := (cv_1, cv_2, \ldots, cv_n)$

With these operations, $V_1 \times V_2 \times \cdots \times V_n$ is a vector space. (Exercise).

The identity vector in $V_1 \times V_2 \times \cdots \times V_n$ is

$(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})$

identity of V_1 of V_2 V_n.

Given $(v_1, \dots, v_n) \in V_1 \times \cdots \times V_n$, the element $(-v_1, -v_2, \dots, -v_n)$ is the additive inverse of $(v_1, \dots, v_n)$.

Examples: Consider $\mathbb{R}^2 \times \mathbb{R}^3$.

$$\mathbb{R}^2 \times \mathbb{R}^3 = \{((x_1, x_2), (x_3, x_4, x_5)) : x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}\}.$$

Define $T : \mathbb{R}^2 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^5$

$$T((x_1, x_2), (x_3, x_4, x_5)) := (x_1, x_2, x_3, x_4, x_5).$$

Check that T is an isomorphism.

Let $\beta = \{((1,0),(0,0,0)), ((0,1),(0,0,0)), ((0,0),(1,0,0)), ((0,0),(0,1,0)), ((0,0),(0,0,1))\}$

Then β is a basis of $\underbrace{\mathbb{R}^2}\times\underbrace{\mathbb{R}^3}$.

* Consider the vector space $P_2(\mathbb{R}) \times \mathbb{R}^2$

An element will be $(x^2+2, (2,3))$

Exercise : Check that $\{(1,(0,0)), (x,(0,0)), (x^2,(0,0)), (0,(1,0)), (0,(0,1))\}$ is a basis of $P_2(\mathbb{R})\times\mathbb{R}^2$.

Proposition: Let $V_1, V_2, \ldots, V_n$ be finite dimensional vector spaces over $\mathbb{R}$. Then

$$\dim(V_1 \times V_2 \times \cdots \times V_n) = \dim(V_1) + \dim(V_2) + \cdots + \dim(V_n).$$

Proof: Let β_i be an ordered basis of V_i.

Let $\dim(V_i) = m_i$.

Then $\# \beta_i = m_i$ ($\#$ Cardinality)

Let $\beta = \{(v_1, 0, \ldots, 0) : v_1 \in \beta_1\} \cup \{(0, v_2, 0, \ldots 0) : v_2 \in \beta_2\}$
$\cup \ldots \cup \{(0,0,\ldots, v_n) : v_n \in \beta_n\}$.

$\# \beta = m_1 + m_2 + \cdots + m_n$

Claim: β is a linearly independent set.

Then $\displaystyle\sum_{i=1}^{n} \sum_{j=1}^{m_i} a_{ij_i} (0,0,\ldots, v_{ij_i}, 0 \ldots, 0) = 0$

v_{ij_i} ↖ j_i^{th} basis elt. in β_i

$$= \left(\sum_{j_1=1}^{m_1} a_{1j_1} v_{1j_1}, \sum_{j_2=1}^{m_2} a_{2j_2} v_{2j_2}, \ldots, \sum_{j_n=1}^{m_n} a_{nj_n} v_{nj_n} \right) = 0$$

$$\Rightarrow \forall i \quad \sum_{j_i=1}^{m_i} a_{ij_i} v_{ij_i} = 0 \text{ in } V_i$$

$$\Rightarrow \forall i, \forall j_i \quad a_{ij_i} = 0$$

$\Rightarrow$ β is linearly independent.

Exercise: β is indeed a spanning set.

Hence $\dim(V_1 \times \cdots \times V_n) = \dim(V_1) + \cdots + \dim(V_n)$.

Let V be a vector space and $\mathcal{U}$ be a subspace of V.

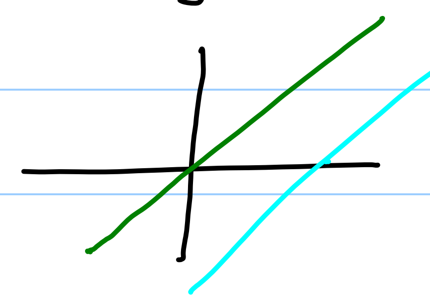
For $v \in V$, define the set

$$v + \mathcal{U} := \{ v + u \ : \ u \in \mathcal{U} \}.$$

<u>Example:</u> In $\mathbb{R}^2$, consider $\mathcal{U} = \{ (x, x) : x \in \mathbb{R} \}$

Let $v = (2, 0)$. Then

$$v + \mathcal{U} = \{ (x+2, x) : x \in \mathbb{R} \}$$



We define the quotient space $V/\mathcal{U}$ (V modulo $\mathcal{U}$)

$$V/\mathcal{U} := \{ v + \mathcal{U} : v \in V \}.$$

An element of $V/\mathcal{U}$ is called an affine subset of V.

In the above example, $\mathbb{R}^2/\mathcal{U} = \{$ lines in $\mathbb{R}^2$ of slope $1 \}$.

Proposition: Let V be a vector space and $\mathcal{U}$ a subspace of V. Then for $v_1, v_2 \in V$, $v_1 + \mathcal{U} = v_2 + \mathcal{U}$ if and only if $v_1 - v_2 \in \mathcal{U}$.

Proof: $(\Leftarrow)$ If $v_1 - v_2 \in \mathcal{U} \Rightarrow v_1 - v_2 = u \in \mathcal{U}$

$\Rightarrow \quad v_1 = (v_2 + u)$

Consider an element in $v_1 + \mathcal{U}$, say $v_1 + u_1$.

Then $v_1 + u_1 = (v_2 + u) + u_1 = v_2 + (u + u_1)$

$\in v_2 + \mathcal{U}$.

$$\Rightarrow \quad v_1 + U \subset v_2 + U.$$

$|||^{ly}$ $\quad v_2 + U \subset v_1 + U.$

$$\Rightarrow \quad v_1 + U = v_2 + U.$$

($\Rightarrow$) Suppose $v_1 + U = v_2 + U$. Let $v_1 + u_1 \in v_2 + U \Rightarrow \exists\, u_2 \in U$ s.t. $v_1 + u_1 = v_2 + u_2$

$$\Rightarrow \quad v_1 - v_2 = u_2 - u_1 \in U \quad \text{———} \quad \blacksquare.$$

Exercise: If $(v_1+\mathcal{U}) \cap (v_2+\mathcal{U}) \neq \phi$, then $v_1+\mathcal{U} = v_2+\mathcal{U}$.

Vector addition on $V/\mathcal{U}$.

Let $v_1+\mathcal{U}$ and $v_2+\mathcal{U}$ be elts in $V/\mathcal{U}$.

Define $(v_1+\mathcal{U}) + (v_2+\mathcal{U}) := (v_1+v_2)+\mathcal{U}$

Scalar Multiplication on $V/\mathcal{U}$

Define for $c \in \mathbb{R}$ & $v + \mathcal{U} \in V/\mathcal{U}$

$$c(v + \mathcal{U}) := cv + \mathcal{U}.$$

Suppose $v_1 + \mathcal{U} = v_1' + \mathcal{U}$ & $v_2 + \mathcal{U} = v_2' + \mathcal{U}$

Then $(v_1 + \mathcal{U}) + (v_2 + \mathcal{U}) = (v_1 + v_2) + \mathcal{U}$

$(v_1' + \mathcal{U}) + (v_2' + \mathcal{U}) = (v_1' + v_2') + \mathcal{U}.$?

Consider $(v_1 + v_2) - (v_1' + v_2')$

$$= \underbrace{(v_1 - v_1')}_{\in \mathcal{U}} + \underbrace{(v_2 - v_2')}_{\in \mathcal{U}} \in \mathcal{U}$$

$$\therefore \quad (v_1 + v_2) + \mathcal{U} = (v_1' + v_2') + \mathcal{U}$$

A similar argument will tell us that scalar is well-defined.

Exercise: Check that $V/\mathcal{U}$ is a vector space w.r.t the vector addition & scalar mult. just defined.

The identity of V/U is given by $0 + U$ where 0 is the identity of V.
The inverse corresponding to $v + U$ is given by $(-v) + U$.

Let us define the quotient map $\pi : V \longrightarrow V/U$ by $\pi(v) = v + U$.

π - surjective.

$$\pi(v_1 + v_2) = (v_1 + v_2) + U = (v_1 + U) + (v_2 + U)$$
$$= \pi(v_1) + \pi(v_2).$$

$|||^{ly}$ check that $\pi(cv) = c\pi(v)$

Hence π is a linear map.

<u>Proposition</u>: Let V be a finite dimensional vector space and U be a subspace. Then

$$\dim(V/U) = \dim(V) - \dim(U).$$

<u>Proof</u>: Consider $\pi : V \to V/U$ is a linear map.

$$R(\pi) = V/U$$

By dimension theorem,

$$\dim(V/U) = \dim(V) - \dim(\text{null}(\pi)).$$

Claim: $\text{null}(\pi) = U$

Let $v \in \text{null}(\pi) \Rightarrow \pi(v) = v + U = 0 + U$

$\Rightarrow v \in U \Rightarrow \text{null}(\pi) \subset U$.

Let $u \in U$ then $u - 0 \in U \Rightarrow u + U = 0 + U$

$\Rightarrow \pi(u) = u + U = 0 + U \Rightarrow u \in \text{null}(\pi)$

Let $T : V \to W$ be a linear transformation.

Consider $\mathcal{U} = \text{null}(T)$. Now define

$\tilde{T} : V/\mathcal{U} \to W$. to be $\tilde{T}(v + \mathcal{U}) := Tv$

Suppose $v_1 + \mathcal{U} = v_2 + \mathcal{U} \Rightarrow v_1 - v_2 \in \mathcal{U} \Rightarrow T(v_1 - v_2) = 0$

$$\tilde{T}(v_1 + \mathcal{U}) = Tv_1 = Tv_2 = \tilde{T}(v_2 + \mathcal{U}).$$

$\Rightarrow \tilde{T}$ is well-defined.

Exercise: Check that $\tilde{T}$ is a linear transformation and $R(T) = R(\tilde{T})$.

Proposition: $\tilde{T}$ is an isomorphism from $V/\mathcal{U}$ onto $R(T)$.

Proof: Clearly $\tilde{T}$ is a linear map from $V/\mathcal{U}$ onto $R(\tilde{T})$
$= R(T)$

What is the $null(\tilde{T}) = ?$

Suppose $\tilde{T}(v+\mathcal{U}) = 0 \Rightarrow Tv = 0 \Rightarrow v \in null(T) = \mathcal{U}$

$\Rightarrow v + \mathcal{U} = 0 + \mathcal{U} \Rightarrow null(\tilde{T}) = \{0 + \mathcal{U}\}.$

Hence $\tilde{T}$ is injective.

Let V be a vector space. A linear transformation $T : V \to \mathbb{R}$ is called a linear functional on V. Let V^* be the collection of all linear functionals on V.

Example: Let $V = \mathcal{C}([0,2\pi])$. Fix $h \in \mathcal{C}([0,2\pi])$.

Define $T : V \to \mathbb{R}$ given by

$$T(x) = \frac{1}{2\pi} \int_0^{2\pi} x(t)\, h(t)\, dt.$$

Then T is a linear functional on $\mathcal{C}([0,2\pi])$.

Vector addition on V^*

Consider $f, g \in V^*$. Recall

$(f+g)(v) := f(v) + g(v)$ is a linear functional.

Scalar multiplication
|||ly $(cf)(v) := c\, f(v)$ is $\checkmark$.

$\therefore$ V^* is closed under both these operations.

Then V^* is a vector space with these operations.

Let β be an ordered basis of V and δ be a basis of $\mathbb{R}$.

Let $f, g \in V^*$

Let us define the map

$$\Phi : V^* \longrightarrow M_{1\times n}(\mathbb{R})$$

$\Phi(f) = [f]_\beta^\delta$. We have already checked that Φ is a linear map and that it is invertible.

Hence V^* is isomorphic to $M_{1\times n}(\mathbb{R})$ which has dimension $n = \dim V$.

Hence $\dim(V^*) = \dim(V)$.

Hence V^* is isomorphic to V.

Let V be a finite dimensional vector space.
Let $\beta = (v_1, \dots, v_n)$ be an ordered basis of V.

Then for $v \in V$, consider $[v]^{\beta} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$.

Define $f_i : V \longrightarrow \mathbb{R}$ given by $f_i(v) = a_i$.

Check that $f_i \in V^*$.

The linear functional f_i is called the i^{th} coordinate function of β.

Theorem: Let V be a finite dim. vector space. Let $\beta = (v_1, \ldots, v_n)$ be an ordered basis of V. Let $\beta^* = \{f_1, \ldots, f_n\}$. Then β^* is a basis of V^*. Further, for $f \in V^*$

$$f = \sum_{i=1}^{n} f(v_i) f_i .$$

Proof: Enough to show that β^* is a spanning set.

Let $f \in V^*$.

Suppose $g = \sum f(v_i) f$. We want to show that $g = f$.

i.e $\quad g(v) = f(v) \quad \forall\, v \in V.$

Recall that f, g are linear.

$\therefore$ it is enough to show that $g(v_i) = f(v_i)$ for ever basis vector in β.

By defn $\quad g(v_i) = \left(\sum_{j=1}^{n} f(v_j) f_j \right)(v_i)$

$$= \sum_{j=1}^{n} f(v_j) f_j (v_i).$$

Check that $f_i(v_j) = 1$ if $i = j$

and $f_i(v_j) = 0$ if $i \neq j$

Hence $g(v_i) = f(v_i)$

Since the above is true $\forall i$, we have $f \equiv g$.

$\blacksquare$

The basis β^* is called the dual basis corresponding to β.

<u>Example:</u> Let $V = \mathbb{R}^2$ and $\beta = \{(1,1), (1,-1)\}$

Let $\{f_1, f_2\}$ be the dual basis.

We know that $f_1(1,1) = 1$ & $f_1(1,-1) = 0$

i.e $1 \cdot f_1(1,0) + 1\, f_1(0,1) = 1$

$1\, f_1(1,0) - 1\, f_1(0,1) = 0$

$\Rightarrow f_1(1,0) = 1/2, \; f_1(0,1) = 1/2.$

$\therefore \; f_1(x,y) = (x+y)/2.$

$|||^{ly} \quad f_2(x,y) = (x-y)/2.$

Theorem: Let $T: V \to W$ be a linear transformation. Then, the map $T^t : W^* \to V^*$ given by $T^*h := hT$ is a linear transformation. Suppose V & W are finite dim. & let α, β be ordered bases of V & W resp. Suppose α^*, β^* are the dual bases of α & β resp. Then

$$[T^t]_{\beta^*}^{\alpha^*} = \left([T]_{\alpha}^{\beta}\right)^t$$

Proof: Let $h \in W^*$. Then $T^t h = hT$

Since h & T are linear, hT is also a linear transformation

& $hT : V \to \mathbb{R}$ $\Rightarrow$ $T^t h \in V^*$.

Let $h, g \in W^*$ $T^t(g+h) = (g+h)T = gT + hT$

$= T^t g + T^t h$.

$\|^{ly}$ $T^t(ch) = c\, T^t h$.

Hence T^t is a linear transformation

Suppose V & W are finite dim. Let $\alpha = (v_1, \dots, v_n)$

and $\beta = (w_1, \dots, w_m)$ be ordered basis.

Let $\alpha^* = (f_1, \dots, f_n)$ & $\beta^* = (g_1, \dots, g_m)$

Suppose $\quad [T]_{\alpha}^{\beta} = A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$

Let $B = [T^t]_{\beta^*}^{\alpha^*} \qquad \alpha^* = (f_1, \dots, f_n) \qquad \beta^* = (g_1, \dots, g_m)$

The i^{th} column of B is the column repn of $[T^t g_i]^{\alpha^*}$

$$T^t g_i = \sum_{j=1}^{n} T^t g_i(v_j) f_j$$

Hence $[T^t g_i]^{\alpha^*} = \begin{pmatrix} T^t g_i(v_1) \\ \vdots \\ T^t g_i(v_n) \end{pmatrix} = \begin{pmatrix} g_i T v_1 \\ \vdots \\ g_i T v_n \end{pmatrix} = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix}$

$\Rightarrow \quad [T^t g_i]^{\alpha^*}$ is the i^{th} row of A.

Hence $\quad B = A^t$. $\qquad$ ——— $\blacksquare$.

Consider $\quad V^{**} = (V^*)^*$.

We know that $\quad \dim V^{**} = \dim V^* = \dim V$.

Let $\quad v \in V$. define

$v^{**} : V^* \to \mathbb{R}$ given by $v^{**}(f) := f(v)$.

$$v^{**}(f+g) = (f+g)(v) = f(v) + g(v) = v^{**}(f) + v^{**}(g).$$

<u>Theorem</u>: Define $\Psi : V \longrightarrow V^{**}$ defined by

$\Psi(v) = v^{**}$. Then Ψ is an isomorphism.

Let $v_1, v_2 \in V$. For $h \in V^*$

$$\Psi(v_1 + v_2)(h) = h(v_1 + v_2) = h(v_1) + h(v_2)$$

$$= \Psi(v_1)(h) + \Psi(v_2)(h)$$

$$\Rightarrow \quad \psi(v_1 + v_2) = \Psi(v_1) + \Psi(v_2).$$

Claim: Ψ is injective.

Suppose $v \in V$ s.t $v \neq 0$. Let $\beta = (v_1, \ldots, v_n)$ be an ordered basis with $v_1 = v$ & $\beta^* = (f_1, \ldots, f_n)$ be the dual basis.

Then $f_1(v_1) = 1$.

$\Rightarrow \quad v^{**}(f_1) = 1$.

$\Rightarrow \quad v^{**} \neq 0$.

If $\quad v \in \text{null}(\Psi)$

Then $\quad \Psi(v) = 0 \quad \Rightarrow \quad v(f) = 0 \;\; \forall f \in V^*.$

$$\Rightarrow \quad v = 0.$$

$\Rightarrow \quad \Psi$ is injective.

Since $\quad \dim(V) = \dim(V^{**})$

Hence Ψ is an isomorphism.