Linear Transformation

Let V and W be vector spaces. A function $T: V \longrightarrow W$ is called a linear transformation if

(i) $T(v_1 + v_2) = T(v_1) + T(v_2) \quad \forall \; v_1, v_2 \in V$

(ii) $T(cv) = c\,T(v) \quad \forall \; c \in \mathbb{R}$ and $v \in V$

Examples: 1) $T: V \longrightarrow W$ be given by $Tv = 0_W$

This is called the linear transformation. $Null(T) = V$

Lemma: Let $T: V \to W$ be a function between vector spaces V and W. Then T is a linear transformation iff $T(v_1 + cv_2) = T(v_1) + cT(v_2)$ $\forall$ $v_1, v_2 \in V$ and $c \in \mathbb{R}$.

In example 1 $T(v_1 + cv_2) = 0_W = \underbrace{Tv_1}_{0_W} + c\underbrace{Tv_2}_{0_W}$.

2) Let $I: V \to V$ be the function given by $Iv = v$ $\forall v \in V$.

Check that I is a linear transformation. $\text{Null}(I) = \{0\}$

(Called the identity linear transformation).

3) $T : \mathbb{R} \longrightarrow \mathbb{R}$ be given by $T(x) = mx$ for a fixed real number $m \neq 0$.

Null$(T) = \{0\}$.

$$T(x_1 + cx_2) = m(x_1 + cx_2) = mx_1 + c\,mx_2$$
$$= Tx_1 + c\,Tx_2$$

4) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad T((x, y)) = (x + y,\ 2x + 3y)$

$$T((x_1, y_1) + c(x_2, y_2)) = T((x_1 + cx_2,\ y_1 + cy_2))$$
$$= (x_1 + c_2x_2 + y_1 + cy_2,\ 2(x_1 + cx_2) + 3(y_1 + cy_2)).$$

$$= (x_1 + y_1 , 2x_1 + 3y_1) + (cx_2 + cy_2 , 2cx_2 + 3cy_2).$$

$$= T((x_1, y_1)) + c(x_2 + y_2, 2x_2 + 3y_2)$$

$$= T((x_1, y_1)) + c\,T((x_2, y_2)).$$

5) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ where

$$T(x_1, x_2) = (x_1 + 2x_2, 3x_1 + 4x_2, 9x_1 + 10x_2)$$

Check that T is a linear transformation

If we write vectors in $\mathbb{R}^n$ as columns

$$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 9x_1 + 10x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 9 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

6) Let A be an $m \times n$ matrix.

Define $$T\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^m$$

matrix multiplication.

T is then a linear transformation.

7) Let $D : \mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R})$ where

$D(p(x)) = p'(x)$. $Null(D) = \{c \in \mathbb{R}\}$.

Then D is a linear transformation.

Observe $D : \mathcal{P}_4(\mathbb{R}) \longrightarrow \mathcal{P}_3(\mathbb{R})$ is also a linear transformation

where $D(p(x)) = p'(x)$.

8) $\mathbb{R}^\infty := \{ (x_1, x_2, \dots) : x_i \in \mathbb{R} \}$

Let $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be defined by

$$T((x_1, x_2, \dots)) = (0, x_1, x_2, \dots)$$

Check that T is a linear Transformation

T is called the right shift coordinate.

$T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be given by

$T(x_1, x_2, \dots)$

$= (x_2, x_3, \dots)$

Null $(T) =$

$\{ (x_1, x_2, \dots) : x_1 = 0 \}$

Lemma: Let $T : V \to W$. Then $T(0_V) = 0_W$

Proof: $T(0_V) = T(0_V + 0_V)$

$= T(0_V) + T(0_V)$

Adding the additive inverse of $T(0_V)$ to both sides

$0_W = T(0_V)$. ■

Null Space

Let $T: V \to W$ be a linear transformation. Then the null space of T, denoted by $\text{Null}(T)$, is the set

$$\mathrm{Null}(T) := \{ v \in V : Tv = 0 \}.$$

Lemma: $\mathrm{Null}(T)$ is a subspace of V.

Let $v, v_1, v_2 \in \mathrm{Null}(T)$ and $c \in \mathbb{R}$. Then

$$T(v_1 + v_2) = Tv_1 + Tv_2 = 0 \implies v_1 + v_2 \in \mathrm{Null}(T)$$

$$T(cv) = cTv = 0 \implies cv \in \mathrm{Null}(T).$$

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Proposition: Let $T : V \to W$ be a linear transformation. Then T is injective iff $\mathrm{Null}(T) = \{0\}$.

Proof: $(\Rightarrow)$ Assume T is injective.

Let $v \in \text{Null}(T) \Rightarrow Tv = 0 = T0$

$\Rightarrow$ $v = 0$ because T is injective.

$\Rightarrow$ $\text{Null}(T) = 0$.

$(\Leftarrow)$ Assume $\text{Null}(T) = \{0\}$. Then suppose v_1 and $v_2 \in V$

s.t $Tv_1 = Tv_2 \Rightarrow Tv_1 - Tv_2 = 0 \longrightarrow (*)$

$Tv_1 - Tv_2 = T(v_1 - v_2) = 0$

$\text{Null}(T) = \{0\} \Rightarrow v_1 - v_2 = 0 \Rightarrow v_1 = v_2.$

$\blacksquare$

Example : $T : \mathbb{R}^3 \longrightarrow \mathbb{R}$ where

$$T(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

Check that T is a linear transformation

Then $\text{Null}(T) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$.

Definition: The dimension of null(T) is called the nullity of T

Range of T.

Let $T: V \to W$ be a linear transformation. The set $\{ Tv : v \in V \}$ is called the range of T and denoted $R(T)$.

<u>Lemma</u>: $R(T)$ is a subspace of W.

Exercise.

Definition: The dimension of $R(T)$, where $T: V \to W$ is a linear transformation, is called the Rank of T.

Dimension Theorem:

Let V be a finite dimensional vectors space and $T: V \to W$ be a linear transformation. Then

$$\dim(V) = \text{rank}(T) + \text{nullity}(T).$$

Proof: Let $\dim(V) = n$ and $\text{nullity}(T) = k \le n$.

Let $\{v_1, \ldots, v_k\}$ be a basis of $\text{null}(T)$.

Extending this to a basis of V, we get

$\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$.

Goal: $\dim(V) = \text{rank}(T) + \text{nullity}(T)$.

or $\text{rank}(T) = \dim(V) - \text{nullity}(T)$

$= n - k$.

Claim: $\{Tv_{k+1}, \ldots, Tv_n\}$ is a basis of $R(T)$.

Let $w \in R(T)$. i.e. $\exists\ v \in V$ s.t. $Tv = w$

$\{v_1, \ldots, v_n\}$ is a basis $\Rightarrow v = a_1 v_1 + \cdots + a_n v_n$.

$$\Rightarrow Tv = T(a_1 v_1 + \cdots + a_n v_n)$$

$$= \underbrace{a_1 \cancelto{0}{Tv_1} + \cdots + a_k \cancelto{0}{Tv_k}} + a_{k+1} Tv_{k+1} + \cdots + a_n Tv_n$$

$$= a_{k+1} Tv_{k+1} + \cdots + a_n Tv_n.$$

Hence $\{Tv_{k+1}, \ldots, Tv_n\}$ is a spanning set of $R(T)$.

Linear Independence

Let $b_{k+1} T v_{k+1} + \cdots + b_n T v_n = 0$

$\Rightarrow T(b_{k+1} v_{k+1} + \cdots \; b_n v_n) = 0$

$\Rightarrow b_{k+1} v_{k+1} + \cdots + b_n v_n \in \mathrm{Null}(T)$

$\Rightarrow b_{k+1} v_{k+1} + \cdots + b_n v_n = b_1 v_1 + \cdots + b_k v_k$ for some $b_1, \ldots, b_k$.

$\Rightarrow (-b_1) v_1 + \cdots + (-b_k) v_k + b_{k+1} v_{k+1} + \cdots + b_n v_n = 0$.

$\Rightarrow b_j = 0 \Rightarrow b_{k+1} = \cdots = b_n = 0$

$\Rightarrow \{T v_{k+1}, \ldots, T v_n\}$ is linearly independent

$\Rightarrow \mathrm{Rank}(T) = n-k$. ——— $\blacksquare$.

Example: $D : \mathcal{P}_3(\mathbb{R}) \longrightarrow \mathcal{P}_2(\mathbb{R})$ where $D(p(x)) = p'(x)$.

$\dim(\mathcal{P}_3(\mathbb{R})) = 4$ $\quad$ $\text{Null}(D) = \{ c \in \mathbb{R} \}$

$\text{nullity}(D) = 1$

$\Rightarrow \text{Rank}(D) = 3 = \dim(\mathcal{P}_2(\mathbb{R}))$.

$\Rightarrow \text{Rang}(D) = \mathcal{P}_2(\mathbb{R})$.

Corollary: Let V & W be finite dimensional Vector spaces st $\dim(V) = \dim(W)$. Let $T : V \to W$ be a linear transformation

Then T is injective iff T is surjective.

Proof: T- injective $\Longleftrightarrow$ $Null(T) = \{0\}$ $\Longleftrightarrow$ $Nullity(T) = 0$

$\Longleftrightarrow$ $\dim(V) = Rank(T) = \dim(W)$

$\Longleftrightarrow$ $Range(T) = W$ $\Longleftrightarrow$ T - surjective — ■