MODULE 2

RANDOM VARIABLE AND ITS DISTRIBUTION

PROBLEMS

1. Let $\mathcal{B}_1$ denote the Borel sigma-field of subsets of $\mathbb{R}$ and let $-\infty < x < y < \infty$. Define $\mathcal{B}_{[x,y]} = \{[x,y] \cap B : B \in \mathcal{B}_1\}$. Show that:
(i) $\{a\} \in \mathcal{B}_1, \forall a \in \mathbb{R}$;
(ii) If C is a countable subset of $\mathbb{R}$, then $C \in \mathcal{B}_1$;
(iii) $\mathcal{B}_{[x,y]}$ is a sigma-field of subsets of $[x,y]$.

2. Let Ω be a given set and let $X: \Omega \to \mathbb{R}$ be a given function. Define $X^{-1}: \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\Omega)$ by $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$, where, for a set $S, \mathcal{P}(S)$ denotes the power set of S. Let $A, B \in \mathcal{P}(\mathbb{R})$ and let $A_\alpha \in \mathcal{P}(\mathbb{R}), \alpha \in \Lambda$, where $\Lambda \subseteq \mathbb{R}$ is an arbitrary index set. Show that:
(i) $X^{-1}(A - B) = X^{-1}(A) - X^{-1}(B)$;
(ii) $X^{-1}(B^c) = \left(X^{-1}(B)\right)^c$;
(iii) $X^{-1}(\bigcup_{\alpha \in \Lambda} A_\alpha) = \bigcup_{\alpha \in \Lambda} X^{-1}(A_\alpha)$;
(iv) $X^{-1}(\bigcap_{\alpha \in \Lambda} A_\alpha) = \bigcap_{\alpha \in \Lambda} X^{-1}(A_\alpha)$;
(v) $A \cap B = \phi \Rightarrow X^{-1}(A) \cap X^{-1}(B) = \phi$.

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X: \Omega \to \mathbb{R}$ be a function. In each of the following cases, verify whether or not X is a r.v..
(i) $\Omega = \{-2, -1, 0, 1, 2, 3\}$,
$\mathcal{F} = \{\phi, \Omega, \{0\}, \{-1, 1\}, \{-2, -1, 1, 2, 3\}, \{-2, 0, 2, 3\}, \{-1, 0, 1\}, \{-2, 2, 3\}\}$
and $X(\omega) = \omega^2, \omega \in \Omega$;
(ii) $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$,
$\mathcal{F} = \left\{\phi, \Omega, \{\omega_1\}, \{\omega_2\}, \{\omega_2, \omega_3, \omega_4\}, \{\omega_1, \omega_3, \omega_4\}, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\right\}$,
$X(\omega_1) = 0, X(\omega_2) = X(\omega_3) = 1$ and $X(\omega_4) = 2$;
(iii) $\Omega = [0,1], \mathcal{F} = \mathcal{B}_{[0,1]}$, where $\mathcal{B}_{[0,1]}$ is as defined in Problem 1, and

$$X(\omega) = \begin{cases} \omega, & \text{if } \omega \in \left[0, \frac{1}{2}\right] \\ \omega - \frac{1}{2}, & \text{if } \omega \in \left(\frac{1}{2}, 1\right] \end{cases}.$$

4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X: \Omega \to \mathbb{R}$ be a r.v.. In each of the following cases determine whether or not $Y: \Omega \to \mathbb{R}$ is a r.v.:

(i) $Y(\omega) = |X(\omega)|, \omega \in \Omega$;
(ii) $Y(\omega) = \left(X(\omega)\right)^2, \omega \in \Omega$;
(iii) $Y(\omega) = \sqrt{X(\omega)}, \omega \in \Omega$, where $X^{-1}\left((-\infty, 0)\right) = \phi$;
(iv) $Y(\omega) = \max(X(\omega), 0)\,, \omega \in \Omega$;
(v) $Y(\omega) = \max(-X(\omega), 0)\,, \omega \in \Omega$.

5. Consider a random experiment of two independent tosses of a coin so that the sample space is $\Omega = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$ with obvious interpretations of outcomes $\text{HH}, \text{HT}, \text{TH}$ and TT. Let $\mathcal{F} = \mathcal{P}(\Omega)$ (the power set of Ω) and let $P(\cdot)$ be a probability measure defined on $\mathcal{F}$ such that $P(\{\text{HH}\}) = p^2, P(\{\text{HT}\}) = P(\{\text{TH}\}) = p(1-p)$ and $P(\{\text{TT}\}) = (1-p)^2$, where $p \in (0,1)$. Define the function $X: \Omega \to \mathbb{R}$ by $X(\{\text{HH}\}) = 2, X(\{\text{HT}\}) = X(\{TH\}) = 1$ and $X(\{\text{TT}\}) = 0$, i.e., $X(\omega)$ denotes the number of Hs (heads) in ω. Show that X a r.v. and find the probability space $(\mathbb{R}, \mathcal{B}_1, P_X)$ induced by X.

6. A card is drawn at random from a deck of 52 playing cards so that the sample space consists of names of 52 cards (e.g., jack of heart, ace of spade, etc,). Let $\mathcal{F} = \mathcal{P}(\Omega)$ (the power set of Ω). Define $X: \Omega \to \mathbb{R}$ by

$$X(\omega) = \begin{cases} 5, & \text{if } \omega \text{ is an ace} \\ 4, & \text{if } \omega \text{ is a king} \\ 3, & \text{if } \omega \text{ is a queen.} \\ 2, & \text{if } \omega \text{ is a jack} \\ 1, & \text{otherwise} \end{cases}$$

Show that X is a r.v. and find the probability space $(\mathbb{R}, \mathcal{B}_1, P_X)$ induced by X.

7. Let X_1, X_2 and X_3 be three random variables with respective distribution functions F_1, F_2 and F_3, where

$$F_1(x) = \begin{cases} 0, & \text{if } x < -1 \\ \dfrac{x+2}{4}, & \text{if } -1 \le x < 1 \\ 1, & \text{if } x \ge 1 \end{cases}; \quad F_2(x) = \begin{cases} 0, & \text{if } x < -1 \\ \dfrac{x+1}{4}, & \text{if } -1 \le x < 0 \\ \dfrac{x+3}{4}, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \ge 1 \end{cases};$$

and

$$F_3(x) = \begin{cases} 0, & \text{if } x < -2 \\ \frac{1}{3}, & \text{if } -2 \le x < 0 \\ \frac{1}{2}, & \text{if } 0 \le x < 5 \\ \frac{1}{2} + \frac{(x-5)^2}{2}, & \text{if } 5 \le x < 6 \\ 1, & \text{if } x \ge 6 \end{cases}.$$

(i) Sketch the graph of $F_1(x)$ and compute $P\left(\left\{-\frac{1}{2} < X_1 \le \frac{1}{2}\right\}\right), P(\{X_1 = 0\}), P(\{X_1 = 1\}), P(\{-1 \le X_1 < 1\})$ and $P(\{-1 < X_1 < 1\})$;

(ii) Compute $P\left(\left\{\frac{1}{4} \le X_2 \le \frac{3}{4}\right\}\right), P\left(\left\{X_2 \ge \frac{1}{2}\right\}\right), P(\{X_2 \ge 0\})$ and $P\left(\left\{0 < X_2 \le \frac{1}{2}\right\}\right)$;

(iii) Compute $P(\{-2 \le X_3 < 5\}), P\left(\left\{0 < X_3 < \frac{11}{2}\right\}\right)$ and the conditional probability $P\left(\left\{\frac{3}{2} < X_3 \le \frac{11}{2}\right\} | \{X_3 > 2\}\right)$.

8. Do the following functions define distribution functions?

(i) $F_1(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \le x \le \frac{1}{2} \\ 1, & \text{if } x > \frac{1}{2} \end{cases}$; (ii) $F_2(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-x}, & \text{if } x \ge 0 \end{cases}$;

and

(iii) $F_3(x) = \frac{1}{2} + \frac{\tan^{-1}(x)}{\pi}, -\infty < x < \infty.$

9. Let $F: \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - \frac{2}{3}e^{-\frac{x}{2}} - \frac{1}{3}e^{-\left[\frac{x}{2}\right]}, & \text{if } x \ge 0 \end{cases},$$

where, for $y \in \mathbb{R}, [y]$ denotes the largest integer $\le y$. Show that F is a d.f. of some r.v. X. Compute $P(\{X > 4\}), P(\{X = 4\}), P(\{X \ge 4\}), P(\{X = 3\})$ and $P(\{3 \le X < 6\})$.

10. Let $F(\cdot)$ and $G(\cdot)$ be two distribution functions. Verify whether or not the following functions are distribution functions:

(i) $H(x) = F(x) + G(x)$; (ii) $H(x) = \max\big(F(x), G(x)\big)$; (iii) $H(x) = \min\big(F(x), G(x)\big)$.

11. (i) Let $F_1(\cdot), \dots, F_n(\cdot)$ be distribution functions and let $a_1, \dots, a_n$ be positive real numbers satisfying $\sum_{i=1}^{n} a_i = 1$. Show that $G(x) = \sum_{i=1}^{n} a_i F_i(x)$ is also a d.f.;
(ii) If $F(\cdot)$ is a d.f. and α is a positive real constant, then show that $G(x) = \big(F(x)\big)^{\alpha}$ and $H(x) = 1 - \big(1 - F(x)\big)^{\alpha}$ are also distribution functions.

12. Do there exist real numbers α, β, γ and δ such that the following functions become distribution functions?

(i) $F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x^2}{2}, & \text{if } 0 \le x < 1 \\ \frac{1}{2} + \alpha(x-1)^2, & \text{if } 1 \le x \le 2 \\ \beta + \frac{(x-2)^4}{7}, & \text{if } 2 < x \le 3 \\ 1, & \text{if } x > 3 \end{cases}$; (ii) $G(x) = \begin{cases} 0, & \text{if } x \le 0 \\ \gamma + \delta e^{-\frac{x^2}{2}}, & \text{if } x > 0 \end{cases}$.

13. Do the following functions define probability mass functions of some random variables of discrete type?

(i) $f_1(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in \{-1, 0, 1, 2\} \\ 0, & \text{otherwise} \end{cases}$; (ii) $f_2(x) = \begin{cases} \frac{e^{-1}}{x!}, & \text{if } x \in \{0, 1, 2, \cdots\} \\ 0, & \text{otherwise} \end{cases}$;

(iii) $f_3(x) = \begin{cases} \binom{50}{x} \left(\frac{1}{4}\right)^{x} \left(\frac{3}{4}\right)^{50-x}, & \text{if } x \in \{1, 2, \cdots, 50\} \\ 0, & \text{otherwise} \end{cases}$.

14. For each of the following, find the value of constant c so that $f(\cdot)$ is a p.m.f. of some discrete type r.v. (say X). Also, for each of the following, find $P(\{X > 2\}), P(\{X < 4\})$, and $P(\{1 < X < 2\})$:

(i) $f(x) = \begin{cases} c(1-p)^{x}, & \text{if } x \in \{1, 2, 3, \cdots\} \\ 0, & \text{otherwise} \end{cases}$; (ii) $f(x) = \begin{cases} \frac{c\lambda^{x}}{x!}, & \text{if } x \in \{1, 2, \cdots\} \\ 0, & \text{otherwise} \end{cases}$;

here $p \in (0,1)$ and $\lambda > 0$ are fixed constants.

15. Do the following functions define probability density functions of some random variables of absolutely continuous type?

(i) $f_1(x) = \begin{cases} \frac{9+x}{180}, & \text{if} - 10 < x < 10 \\ 0, & \text{otherwise} \end{cases}$; (ii) $f_2(x) = \begin{cases} \frac{(x^2+1)e^{-x}}{2}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$;

(iii) $f_3(x) = \begin{cases} \frac{2+\cos x}{2\pi}, & \text{if } 0 < x < \pi \\ 0, & \text{otherwise} \end{cases}$.

16. In each of the following, find the value of constant c so that $f(\cdot)$ is a p.d.f. of some r.v. (say X) of absolutely continuous type. Also, for each of the following, find $P(\{X > 3\}), P(\{X \le 3\})$, and $P(\{3 < X < 4\})$:

(i) $f(x) = \begin{cases} cxe^{-x^2}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$; (ii) $f(x) = \begin{cases} cxe^{-(x-2)}, & \text{if } x > 2 \\ 0, & \text{otherwise} \end{cases}$.

17. (i) Let X be a discrete type r.v. with support $S_X = \{0, 1, 2, 3, 4\}, P(\{X = 0\}) = P(\{X = 1\}) = \frac{1}{10}, P(\{X = 2\}) = P(\{X = 3\}) = P(\{X = 4\}) = \frac{4}{15}$. Find the d.f. of X and sketch its graph.

(ii) Let the r.v. X have the p.m.f.

$$f_X(x) = \begin{cases} \dfrac{x}{5050}, & \text{if } x \in \{1, 2, \cdots, 100\} \\ 0, & \text{otherwise} \end{cases}.$$

Show that the d.f. of X is

$$f_X(x) = \begin{cases} 0, & \text{if } x < 1 \\ \dfrac{[x]([x]+1)}{10100}, & \text{if } 1 \le x < 100 \\ 1, & \text{if } x \ge 100 \end{cases}.$$

Also compute $P(\{3 < X < 50\})$.

18. For each of the following p.d.f.s of some r.v. (say X) of absolutely continuous type, find the d.f. and sketch its graph. Also compute $P(\{|X| < 1\})$ and $P(\{X^2 < 9\})$.

(i) $f(x) = \begin{cases} \frac{x^2}{18}, & \text{if} - 3 < x < 3 \\ 0, & \text{otherwise} \end{cases}$; (ii) $f(x) = \begin{cases} \frac{x+2}{18}, & \text{if} - 2 < x < 4 \\ 0, & \text{otherwise} \end{cases}$.

(iii) $f(x) = \begin{cases} \frac{1}{2x^2}, & \text{if } |x| > 1 \\ 0, & \text{otherwise} \end{cases}.$

19. (i) Let X be a r.v. of absolutely continuous type with p.d.f.

$$f(x) = \begin{cases} cx^2, & \text{if} -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Compute the values of $c, P(\{X = 0\}), P(\{X > 0\}), P(\{X > 1/2\}), P(\{|X| > 1/2\}), P(\{1/2 < X < 3/4\}), P(\{1/2 < X < 2\})$ and the conditional probability $P(\{X < 3/4\}|\{X > 1/2\})$;

(ii) Let X be a r.v. of absolutely continuous type with p.d.f.

$$f(x) = \begin{cases} c(x+1)e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases},$$

where $\lambda > 0$ is a given constant. Compute the values of $c, P(\{X = 2\}), P(\{X > 2\}), P(\{X > 1\}), P(\{1 < X < 3\}), P(\{|X - 2| > 1\})$ and the conditional probability $P(\{X < 3\}|\{X > 1\})$.

20. Let X be a r.v. with d.f. $F_X(\cdot)$. In each of the following cases determine whether X is of discrete type or of absolutely continuous type. Also find the p.d.f./p.m.f. of X:

(i) $F_X(x) = \begin{cases} 0, & \text{if } x < -2 \\ \frac{1}{3}, & \text{if } -2 \le x < 0 \\ \frac{1}{2}, & \text{if } 0 \le x < 5 \\ \frac{3}{4}, & \text{if } 5 \le x < 6 \\ 1, & \text{if } x \ge 6 \end{cases}$; (ii) $F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-x}, & \text{if } x \ge 0 \end{cases}.$

21. Let the r.v. X have the d.f.

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{3}, & \text{if } 0 \le x < 1 \\ \frac{2}{3}, & \text{if } 1 \le x < 2 \\ 1, & \text{if } x \ge 2 \end{cases}.$$

Show that X is neither of discrete type nor of absolutely continuous type.

22. For the three d.f.s considered in Problems 20 and 21, find the decomposition $F_X(x) = \alpha F_d(x) + (1-\alpha)F_c(x), x \in \mathbb{R}$, where $\alpha \in [0,1], F_d$ is a d.f. of some r.v. of discrete type and F_c is a continuous d.f..