Textbooks

1. Linear Algebra by Friedberg, Insel and Spence.
2. Linear Algebra done right by Sheldon Axler

By a scalar, we mean a Real number. We know that given two scalars a, b, we can add them to obtain another real numbers. We can also multiply them to get another real number. These operations satisfy

(i) Given a, b scalars $a+b = b+a$, $ab = ba$

(ii) Given a, b, c scalars $(a+b)+c = a+(b+c)$ & $(ab)c = a(bc)$

(iii) $\exists$ 0 s.t $a+0 = a$ for every scalar a.

(iv) $\exists$ 1 s.t $a \cdot 1 = a$ " " " ".

(v) Given any scalar a, $\exists$ an additive inverse b s.t $a+b=0$.

lly given any non-zero scalar a, $a \cdot 1/a = 1$.

(vi) Given scalars a, b, c , $a(b+c) = ab + ac$.

The set of all scalars is called the Field of Scalars. It is sometimes denoted by $\mathbb{F}$.

In this course our field of scalars is the set of real numbers (denoted by $\mathbb{R}$).

Vector Spaces.

Consider $\mathbb{R}^2 := \{(x, y) : x, y \in \mathbb{R}\}$. ordered tuples.

$(2, 3)$ is different from $(3,2)$.

$$(1,0) + (3, 4) = (4, 4).$$

$$2(3.5, 2) = (7, 4).$$

$$(2,5) + (0,0) = (2,5)$$

$$\mathbb{R}^3 := \{ (x, y, z) : x, y, z \in \mathbb{R} \}$$ Ordered tuples.

$$(2, 3, 1) + (5, 6, 4) = (7, 9, 5).$$

$$4 (1, 0, 1) = (4, 0, 4).$$

Definition of a Vector space.

A vector space V is a set with two operations, called vector addition & scalar multiplication s.t

* Given two elts $v_1, v_2 \in V$, the vector addition $v_1 + v_2$ is an element of V. (i.e. V is closed under vector addition).
* Given a scalar a and an elt. $v \in V$, the scalar multiplication av gives an elt. in V. (i.e. V is closed under scalar multiplication.).

and such that the following properties are satisfied:

Property I : For $v_1, v_2 \in V$, $v_1 + v_2 = v_2 + v_1$ (Commutativity)

Property II : Given $v_1, v_2, v_3 \in V$, $(v_1+v_2)+v_3 = v_1 + (v_2+v_3)$.
(Associativity).

Property III : $\exists$ an element $0 \in V$ s.t $v + 0 = v \ \forall v \in V$.
0 is called the zero vector. (Additive identity)

Property IV : Given $v \in V$, $\exists$ $w \in V$ s.t $v + w = 0$
(Additive inverse).

Property V : For every $v \in V$, $1v = v$ where 1 is the scalar multiplicative identity (Multiplicative identity).

Property VI : Given scalars a, b and $v \in V$

$$b(av) = (ab)v$$ (Multiplication is associative).

Property VII : Given a scalar a and $v_1, v_2 \in V$

$$a(v_1 + v_2) = av_1 + av_2$$

(Distributivity).

Property VIII Given scalar a, b and $v \in V$, then

$$(a+b)v = av + bv \quad \text{(Multiplication is linear)}$$

<u>Examples:</u> $\mathbb{R}^n := \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$

$$(x_1, x_2, \ldots, x_n) + (y_1, \ldots, y_n) := (x_1+y_1, x_2+y_2, \ldots, x_n+y_n) \in \mathbb{R}^n.$$

For c, $\quad c(x_1, \ldots, x_n) := (cx_1, cx_2, \ldots, cx_n) \in \mathbb{R}^n$

Is Multiplication linear?

<u>Claim:</u> $(a+b)(x_1, \dots, x_n) = a(x_1, \dots, x_n) + b(x_1, \dots, x_n)$

$$
\begin{aligned}
(a+b)(x_1, \dots, x_n) &= ((a+b)x_1, \dots, (a+b)x_n). \\
&= (ax_1 + bx_1, \dots, ax_n + bx_n) \\
&= (ax_1, \dots, ax_n) + (bx_1, \dots, bx_n) \\
&= a(x_1, \dots, x_n) + b(x_1, \dots, x_n)
\end{aligned}
$$

Example 2: The scalars $\mathbb{R}$ is a vector space.

with vector addition as the usual addition of real numbers and the scalar multiplication is the mult. of real numbers

Example 3: Let $V = \{0\}$

Vector addition: $0 + 0 = 0$

Scalar multiplication: $c0 = 0$.

Check that the properties I to $\overline{\underline{\text{VIII}}}$ are satisfied.

This vector space is called as the zero vector space.

Example 4: Let $\mathbb{C} = \{ a + ib : a, b \in \mathbb{R} \}$.

$$(a+ib) + (c+id) := (a+c) + i(b+d)$$

For $a \in \mathbb{R}$, $(c+id) \in \mathbb{C}$, then

$$a(c+id) := (ac) + i(ad).$$

Exercise: Properties I to $\overline{\underline{\text{VIII}}}$ are satisfied.

An element of a vector space V is referred to as a vector in V.

<u>Example 5</u> : Let $V = \mathcal{P}_n(\mathbb{R}) = \{$ polynomials of deg $\leq n\}$. where n is a positive integer.

$\mathcal{P}_2(\mathbb{R})$ has elts like x^2+1, $2x+3$, 4

ax^2+bx+c where $a, b, c \in \mathbb{R}$.

$$(x^2+4x+3) \quad + \quad (-4x+2) \quad = \quad x^2+5.$$

Vector addition:

$$(a_0+a_1x+\cdots+a_nx^n)+(b_0+b_1x+\cdots+b_nx^n)$$

$$:= (a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n.$$

where $a_i, b_i \in \mathbb{R}$

Scalar multiplication:

For a scalar c & $p(x) = a_0+a_1x+\cdots+a_nx^n$

define $c(a_0+\cdots+a_nx^n) := ca_0+\cdots+ca_nx^n.$

Exercise: Properties I to $\overline{\text{VIII}}$ are satisfied.

Example 6: Let $V = \mathcal{P}(\mathbb{R})$ be the set of all polynomials (in the indeterminate x) with co-efficients in $\mathbb{R}$.

With the usual addition & scalar multiplication $\mathcal{P}(\mathbb{R})$ is a vector space.

Example 7: Let $V = \mathcal{C}(\mathbb{R})$ where

$$\mathcal{C}(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \;:\; f \text{ is continuous} \}$$

Vector addition:

For $f, g \in \mathcal{C}(\mathbb{R})$

$$(f+g)(x) := f(x) + g(x)$$

Scalar multiplication:

$$(cf)(x) := c\, f(x)$$

$\mathcal{C}(\mathbb{R})$ is closed under vector addition & scalar mult.

Exercise: Properties I to VIII are satisfied.

Example 8: Let $V = \mathcal{F}(\mathbb{R})$

$\mathcal{F}(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} \}$ (set of all functions).

define vector addition & scalar mult. as in Example 7.

Exercise: $\mathcal{F}(\mathbb{R})$ is a vector space.

Example 9: $\mathbb{R}^{\infty}$ be the set of all infinite seqences.

An elt. in $\mathbb{R}^{\infty}$ will be $(1, 2, 5, 6, 1, 4, \ldots\ldots)$

$$\mathbb{R}^\infty := \{ (x_1, x_2, \dots) : x_i \in \mathbb{R} \}.$$

Exercise: $\mathbb{R}^\infty$ is a vector space with vector addition and scalar multiplication defined similar to the ones in $\mathbb{R}^n$ (coordinate-wise).

Example 10: Let $V = M_{m\times n}(\mathbb{R})$.

where $M_{m\times n}(\mathbb{R})$ is the set of all $m\times n$ matrices.

$$M_{m\times n}(\mathbb{R}) := \left\{ \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} : a_{ij} \in \mathbb{R} \right\}.$$

Vector addition & scalar mult. is defined coordinate wise.

Exercise: $M_{m\times n}(\mathbb{R})$ is a vector space.

Non-Examples:

* Let $V = \{\text{Polynomials of degree} = n\}$.

$$(x^n + 2) + (-x^n + 5) = 7$$

* $\mathbb{R}^+ = \{\text{all positive real numbers}\}$. with vector addition & scalar mult. as in $\mathbb{R}$.

Not closed under scalar mult.

<u>Proposition:</u> Let V be a vector space and suppose $u, v, w \in V$. If $u + w = v + w$, then

$u = v.$

Proof: Let $(-w)$ denote the additive inverse of w.

then $u + w = v + w$

$\Rightarrow (u+w) + (-w) = (v+w) + (-w).$

$\Rightarrow u + (w + (-w)) = v + (w + (-w))$. (Since vector addition is associative)

$\Rightarrow u + 0 = v + 0$ (Since $-w$ is the additive inverse of w)

$\Rightarrow u = v$ (since 0 is the additive identity).

Hence we have proved the result. $\blacksquare$

Exercise: Prove the following. In any given vector space V

(i) $\underset{\text{Real Number}}{0}\, v = 0$ ← zero vector in V. $\forall\, v \in V$.

(ii) $(-1)\, v = -v \quad \forall\, v \in V$ (where $-v$ is the additive inverse of v)

Proposition: The additive identity & additive inverse of any vector is unique.

<u>Proof</u>: Suppose 0 and $0'$ are two additive identities.

$$0 = 0 + 0' = 0'$$

(since $0'$ is an additive identity) (since 0 is the additive)

Exercise : Check that inverse of any vector $v \in V$ is unique.

Let V be a vector space. We say that W is a (vector) subspace of V if $W \subseteq V$ and if the following two conditions are satisfied:

(1) (W is closed under vector addition): If $w_1, w_2 \in W \; (\subseteq V)$ then $w_1 + w_2 \in W$. (Note that the vector addition is borrowed from V).

(2) (W is closed under scalar mult.): If c is a scalar and $w \in W$, then $cw \in W$ (Scalar mult. from V).

Lemma: If W is a subspace of a vector space V, then W is also a vector space (with vector addition & scalar mult. the same as in V).

Proof: By definition, W is closed under addition & scalar multiplication.

Note that every property except III and IV are automatically satisfied in W since the same is satisfied by the addition & scalar mult. in V.

Property III.

By an exercise, we know that $0w = 0$. zero vector in V.

Then by (2) in the defn of a subspace, we have $0w \in W \Rightarrow 0 \in W$. (Hence the zero vector of V is in W and is the zero vector of W as well).

Property IV.

Note that $(-1)w = -w$ (Additive inverse of w in V). Hence by (2) in the defn of a subspace, we have $-w \in W$.

Example 1 : Consider $W = \{ (x, y, 0) : x, y \in \mathbb{R} \}$.

Then $W \subset \mathbb{R}^3$

For $(x_1, y_1, 0)$ & $(x_2, y_2, 0) \in W$,

$$(x_1, y_1, 0) + (x_2, y_2, 0) = (x_1 + x_2, y_1 + y_2, 0) \in W.$$

$|||^{ly}$ W is closed under scalar multiplication.

Hence W is a subspace of $\mathbb{R}^3$.

Example 2: $\mathcal{P}_2(\mathbb{R}) \subseteq \mathcal{P}_3(\mathbb{R})$

With the usual vector space operations, $\mathcal{P}_2(\mathbb{R})$ is a subspace of $\mathcal{P}_3(\mathbb{R})$

In fact $\mathcal{P}_k(\mathbb{R})$ is a subspace of $\mathcal{P}_\ell(\mathbb{R})$ where $\ell \geq k$.

All these are subspaces of $\mathcal{P}(\mathbb{R})$.

$\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$. (Not a subspace).

Example 3 : $\quad W = \{ (x, y, z) \in \mathbb{R}^3 : \quad x + y + z = 0 \}$.

Let (x_1, y_1, z_1) & $(x_2, y_2, z_2) \in W$.

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$$(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2)$$

$$= 0 + 0 = 0$$

Check that W is closed under scalar multiplication.

Example 4: Let $W = \mathcal{P}_{even}(\mathbb{R})$ where

$$\mathcal{P}_{even}(\mathbb{R}) = \{ p(x) \in \mathcal{P}(\mathbb{R}) : p(-x) = p(x) \}.$$

Claim: $\mathcal{P}_{even}(\mathbb{R})$ is a subspace of $\mathcal{P}(\mathbb{R})$.

Let $h(x) = p(x) + q(x)$

But $h(-x) = p(-x) + q(-x) = p(x) + q(x) = h(x)$

$\Rightarrow \quad h \in P_{even}(\mathbb{R})$

Let $h(x) = c\, p(x)$ where c - scalar & $p(x) \in P_{even}(\mathbb{R})$.

$$h(-x) = c\, p(-x) = c\, p(x) = h(x)$$

$$\Rightarrow h \in P_{even}(\mathbb{R}).$$

Hence the claim is established

Example 5: Consider W = diagonal matrices of size n.

i.e
$$\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & 0 \cdots & 0 \\ \vdots & & & 0 \\ 0 & \cdots & 0 & a_n \end{pmatrix} \text{ where } a_i \in R.$$

$$W \subset M_{n\times n}(\mathbb{R})$$

Exercise: W is a subspace of $M_{n\times n}(\mathbb{R})$.

<u>Example 0</u> : Every vector space is a subspace of itself.

Then set $W = \{0\}$ is a subspace of V.

where 0 is the additive identity of V.

Any subspace of a vector space which is not V itself or the zero vector space is called a proper subspace.

Lemma: The intersection of two subspaces of a vector space is a subspace.

Proof: Let W_1 & W_2 be subspaces of V.

Clearly $W_1 \cap W_2 \subset V$.

Let $w_1, w_2 \in W_1 \cap W_2 \Rightarrow w_1, w_2 \in W_1$

& hence $w_1 + w_2 \in W_1$

$|||^{ly}$ $w_1 + w_2 \in W_2$

$\Rightarrow$ $w_1 + w_2 \in W_1 \cap W_2$.

$|||^{ly}$ $W_1 \cap W_2$ is closed under scalar multiplication.

Exercise: Let $W = \{(x, y, z) : ax + by + cz = 0\}$ where a, b & c are scalars. Prove that W is a subspace of $\mathbb{R}^3$.

Let $v = (1,1,0)$ be a vector in $\mathbb{R}^3$.

Clearly $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ which contains v.

Suppose $W = \{v\}$ $\quad v+v = (2,2,0) \notin W$

$\pi(1,1,0) = (\pi, \pi, 0)$

Let $U = \{ av : a \in \mathbb{R} \}$.

Then every subspace of $\mathbb{R}^3$ should contain $\mathcal{U}$.

Exercise: $\mathcal{U}$ is a subspace of $\mathbb{R}^3$.

Let $v = (1,1,0)$ & $w = (1,0,3)$. $\in \mathbb{R}^3$

Notice that every scalar multiple of v should be in any subspace that contains v. Similarly any scalar mult. of w should be in any subspace that contains w.

av , bw where $a \in \mathbb{R}$, $b \in \mathbb{R}$ will be in W if W contains v & w.

Then $av + bw \in W$

Let $U = \{ av + bw : a \in \mathbb{R}, b \in \mathbb{R} \}$

Then $U \subseteq W$.

Exercise : U is a subspace of $\mathbb{R}^3$.

The vectors of the type $av + bw$ where a & b are scalars are called linear combinations of v & w & the subspace U is called the span of v & w.

Definition of Linear Combination & Span

Let $S \subset V$ where V is a vector space. A linear combination in S is a vector of the type

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

where a_i $i = 1, 2, \ldots, n$ are scalars & $v_i \in S$ where $i = 1, \ldots, n$.

The set of all linear combinations is called the span of S and is denoted as $\text{span}(S)$.

$$\text{span}(S) := \{ a_1 v_1 + \cdots + a_n v_n \; : \; a_i \in \mathbb{R} \;\&\; v_i \in \mathbb{R} \}.$$

$1 . v_1 = v_1 \in S$. $\qquad v_1 + v_2 \in S$.

If S is finite, i.e $S = \{v_1, \ldots, v_m\}$, then

$$\text{span}(S) := \{ a_1 v_1 + a_2 v_2 + \cdots + a_m v_m : a_i \in \mathbb{R} \}.$$

Exercise: Check that the definitions match.

Convention: $\quad \text{Span}(\phi) = \{0\}$.

Theorem: Let $S \subset V$ where V is a vector space. Then span(S) is a subspace of V which contains S. Span(S) is contained in a subspace of V which contains S.

Proof:

Span$(S) \subset V$ since every elt. of span(S) is given $a_1v_1 + \cdots + a_nv_n$ where $a_i \in \mathbb{R}$ & $v_i \in S$.

Since V is closed under scalar mult. $a_iv_i \in V$

& closed under vector addition gives that

$a_1v_1 + \cdots + a_nv_n \in V$.

(Let $a_1v_1 + \cdots + a_nv_n$, $b_1v_1 + \cdots + b_nv_n \in span(S)$

Then $(a_1v_1 + \cdots + a_nv_n) + (b_1v_1 + \cdots + b_nv_n)$

$= (a_1+b_1)v_1 + \cdots + (a_n+b_n)v_n \in span(S)$.

Let $(a_1v_1 + \cdots + a_nv_n)$ & $(b_1w_1 + \cdots + b_mw_m)$ be vectors in span(S). $) \longrightarrow (*)$

Then express each of these vectors as a linear combination of $\{v_1, \ldots, v_n\} \cup \{w_1, \ldots, w_m\}$ with 0 as the co-eff. of terms which were not present in the expression to begin with.

Then by using the observation $(*)$, span(S) is closed under vector addition.

Let c be a scalar & let $a_1v_1 + \cdots + a_nv_n \in \text{Span}(S)$

$$c(a_1v_1 + \cdots + a_nv_n) = (ca_1)v_1 + \cdots + (ca_n)v_n \in \text{span}(S).$$

$\Rightarrow$ span(S) is a subspace

Let $v \in S$. Then $1.v$ is a linear combination in S.

$1.v \in \text{span}(S)$. But $1v = v$

$\Rightarrow \quad v \in \text{span}(S) \Rightarrow S \subset \text{span}(S).$

Let W be a subspace of V which contains S.

Let $a_1v_1 + \cdots + a_kv_k \in \text{span}(S)$.

$v_1, v_2, \ldots, v_k \in W$

W - subspace $\Rightarrow$ $a_1v_1 \in W$, $a_2v_2 \in W$, $\ldots$, $a_kv_k \in W$.

& $(a_1v_1 + a_2v_2 + \cdots + a_kv_k) \in W$.

Hence $\text{span}(S) \subset W$. ——— $\blacksquare$.

We say that a subset $S \subset V$ spans the vector space V if $\text{span}(S) = V$. Then S is called a spanning set of V.

Example: $V = \mathbb{R}^3$ & $S = \{(1,0,0), (0,1,0), (0,0,1)\}$.

Then $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$.

$\Rightarrow \quad V \subset \text{span}(S)$.

Hence S is a spanning set of V.

$V = \mathbb{R}^2$ $\quad S = \{(1,2), (2,3), (3,4)\}.$

$(x, y) = a(1,2) + b(2,3)$

$x = a + 2b$ $\quad -(y - 2x) = b$

$y = 2a + 3b$ $\quad$ Calculate a.

$\Rightarrow S$ is a ~~s~~spanning set.