MODULE 5

SPECIAL ABSOLUTELY CONTINUOUS DISTRIBUTIONS AND THEIR PROPERTIES

PROBLEMS

1. Let $X \sim U(0, \theta)$, where θ is a positive integer and let $Y = X - [X]$, where $[x]$ is the largest integer $\leq x$. Show that $Y \sim U(0, 1)$.

2. Let $F(\cdot)$ be the d.f. of a r.v. X, where $P(X = 1) = p = 1 - P(X = 0)$. Find the distribution of $Y = F(X)$. Does $Y \sim U(0,1)$? Interpret your findings on light of Theorem 1.3 (i).

3. Let the r.v. X have the p.d.f
$$f(x) = \begin{cases} 6x(1-x), & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$
Show that $Y = X^2(3 - 2X) \sim U(0, 1)$.

4. Let $X \sim U(0, 1)$ and let $U = \min(X, 1 - X)$. Find the p.d.f. of $Y = (1 - U)/U$. Does Y has finite expectation?

5. (i) If $X \sim U(0, 1)$, find the distribution of $Y = -\lambda \ln X$, where $\lambda > 0$;
   (ii) Let the r.v. X have the Cauchy p.d.f. $f(x) = \pi^{-1}(1 + x^2)^{-1}, -\infty < x < \infty$. Find the p.d.f. of $Y = X^{-1}$.

6. If $X \sim U(0, \theta)$, where $\theta > 0$, find the distribution of $Y = \min(X, \theta/2)$. Calculate $P\left(\{\frac{\theta}{4} < Y < \frac{\theta}{2}\}\right)$.

7. Let $X \sim N(0, 1)$ and let
$$Y = \begin{cases} X, & \text{if } |X| \leq 1 \\ -X, & \text{if } |X| > 1 \end{cases}.$$
Find the distribution of Y.

8. Let $X \sim N(\mu, \sigma^2)$. Find the distribution function and probability density function of $Y = X^2$.

9. (i) If $X \sim N(12, 16)$, find $P(X \geq 20)$, (use $\Phi(2) = 0.9772$);
   (ii) If $X \sim N(\mu, \sigma^2)$, $P(9.6 \leq X \leq 13.8) = 0.7008$ and $P(X \geq 9.6) = 0.8159$, find μ, σ^2 and $P(\{X \geq 13.8 | X \geq 9.6\})$(use $\Phi(0.9) = 0.8159$ and $\Phi(1.2) = 0.8849$).

10. For $x > 0$, show that

$$(x^{-1} - x^{-3})\phi(x) < 1 - \Phi(x) < x^{-1}\phi(x).$$

(Hint: Use integration by parts in $(2\pi)^{\frac{1}{2}}\left(1 - \Phi(x)\right) = \int_x^\infty t^{-1}(te^{-\frac{t^2}{2}})dt$).

11. Let $Z \sim N(0, 1)$. Find $\mathrm{E}\left(Z\Phi(Z)\right)$ and $E(\left(Z^2\Phi(Z)\right)$. (Hint: Use the fact that $\phi^{'}(z) = -z\phi(z)$ and integrate by parts.)