Let V & W be vector spaces and suppose $S: V \to W$ and $T: V \to W$ are linear transformations. We define $S+T : V \to W$ to be

$$(S+T)v := Sv + Tv \quad \forall\, v \in V.$$

Proposition: With S & T as above, $S+T$ is a linear transformation.

Proof: For $v_1, v_2 \in V$,

$$(S+T)(v_1+v_2) = S(v_1+v_2) + T(v_1+v_2)$$
$$= Sv_1 + Sv_2 + Tv_1 + Tv_2$$
$$= Sv_1 + Tv_1 + Sv_2 + Tv_2$$
$$= (S+T)v_1 + (S+T)v_2.$$

Check that $(S+T)(av) = a(S+T)v$.

Let $T: V \to W$ be a linear transformation between vector spaces. Let $a \in \mathbb{R}$. The we define $(aT): V \to W$ to be

$$(aT)(v) := a(Tv)$$

Exercise: Check that (aT) is a linear transformation.

<u>Example</u>: $S: V \to V$ be $Sv = a_1 v$

and $T : V \to V$ be $Tv = a_2 v$

then $(S+T)v = (a_1 + a_2)v$

Let the space of all linear transformations between V & W be denoted by $\mathcal{L}(V, W)$. Then $\mathcal{L}(V, W)$ is a subset of $\mathcal{F}(V, W)$.

Proposition: The set $\mathcal{L}(V, W)$ is a subspace of $\mathcal{F}(V, W)$.

Proof: We just checked that $\mathcal{L}(V, W)$ is closed under addition & scalar multiplication.

Let $S : V \to W$ and $T : U \to V$ be linear transformations between vector spaces. Then we define the product (or composition) of linear transformations $ST : U \to W$ to be

$$(ST)u \quad := \quad S(Tu) \quad \forall \ u \in U.$$

Exercise: Prove that ST is a linear transformation.

Proposition:

Let V & W be finite dimensional vector spaces & suppose $\beta = (v_1, \ldots, v_m)$ and $\gamma = (w_1, \ldots, w_n)$. Let $S : V \to W$ and $T : V \to W$ be linear transformations.

$$[S+T]_\beta^\gamma = [S]_\beta^\gamma + [T]_\beta^\gamma$$

Proof: Let $[S]_\beta^\gamma = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & & a_{nm} \end{pmatrix}$. Then $Sv_j = a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{nj}w_n$

& $[T]_\beta^\gamma = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix}$

We know that $Tv_j = b_{1j}w_1 + \cdots + b_{nj}w_n$.

$$(S+T)v_j = (a_{1j}w_1 + \cdots + a_{nj}w_n) + (b_{1j}w_1 + \cdots + b_{nj}w_n)$$
$$= (a_{1j}+b_{1j})w_1 + \cdots + (a_{nj}+b_{nj})w_n$$

$$\begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & & \vdots \\ \vdots & & \\ a_{n1}+b_{n1} & & a_{nm}+b_{nm} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix} = [S]_\beta^\gamma + [T]_\beta^\gamma .$$

<u>Exercise</u>: Let $T: V \to W$ be a linear transformation as in the proposition above & suppose $a \in \mathbb{R}$.

Then $$[aT]_\beta^\gamma = a[T]_\beta^\gamma .$$

Proposition: Let U, V & W be finite dimensional vector spaces and suppose $\alpha = (u_1, \ldots, u_\ell)$, $\beta = (v_1, \ldots, v_m)$ & $\gamma = (w_1, \ldots, w_n)$ be ordered bases of U, V & W respectively. Then, if $S : V \to W$ and $T : U \to V$ are linear transformations, we have

$$[ST]_\alpha^\gamma = [S]_\beta^\gamma [T]_\alpha^\beta$$

Example: Let $L : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ be define

$$L(x_1, x_2, \ldots) := (x_2, x_3, \ldots)$$

& $$R(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots)$$

Then $$LR(x_1, x_2, \ldots) = (x_1, x_2, \ldots)$$

& $$RL(x_1, x_2, \ldots) = (0, x_2, x_3 \ldots)$$

Proof of proposition: $\alpha = (u_1, \ldots, u_l)$, $\beta = (v_1, \ldots, v_m)$ and $\gamma = (w_1, \ldots, w_n)$ be the ordered bases of U, V & W resp.

Let $[T]_\alpha^\beta = \begin{pmatrix} a_{11} & \cdots & a_{1\ell} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{m\ell} \end{pmatrix}$

Then $Tu_i = \sum_{j=1}^{m} a_{ji} v_j$

$[S]_\beta^\gamma = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix}$

Then $Sv_j = \sum_{k=1}^{n} b_{kj} w_k$

$$(ST)u_i = S(Tu_i) = S\left(\sum_{j=1}^{m} a_{ji} v_j\right)$$

$$= \sum_{j=1}^{m} a_{ji}(S v_j) = \sum_{j=1}^{m} a_{ji}\left(\sum_{k=1}^{n} b_{kj} w_k\right)$$

$$= \sum_{k=1}^{n}\left(\sum_{j=1}^{m} b_{kj} a_{ji}\right) w_k$$

Let $\quad c_{ki} = \sum_{j=1}^{n} b_{kj} a_{ji}$

$$(ST)u_i = \sum_{k=1}^{n} c_{ki} w_k$$

Hence

$$[ST]_{\alpha}^{\gamma} = \begin{pmatrix} c_{11} & \cdots & c_{1\ell} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{n\ell} \end{pmatrix}$$

Check that

$$\begin{pmatrix} b_{11} & \cdots & b_{m1} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1\ell} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{m\ell} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1\ell} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{n\ell} \end{pmatrix}$$

$\blacksquare$