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The finite volume method for Richards equation

Robert Eymard [a], Michaël Gutnic [b,c,*] and Danielle Hilhorst [c]

[a] *Ecole Nationale des Ponts et Chaussées, 6 et 8 Avenue Blaise Pascal, Cité Descartes, Champs-sur-Marne, F-77455 Marne-la-Vallée Cedex 2, France*
E-mail: eymard@enpc.fr

[b] *Institut de Recherche Mathématique Avancée, Université Louis Pasteur, 7 rue René Descartes, F-67084 Strasbourg, France*
E-mail: michael.gutnic@math.u-strasbg.fr

[c] *Laboratoire de Mathématiques, Analyse Numérique et EDP, CNRS et Université de Paris-Sud (bât. 425), F-91405 Orsay Cedex, France*
E-mail: danielle.hilhorst@math.u-psud.fr

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In this paper we prove the convergence of a finite volume scheme for the discretization of an elliptic–parabolic problem, namely Richards equation $\beta(P)_t - \operatorname{div}(K(\beta(P)) \times \nabla(P+z)) = 0$, together with Dirichlet boundary conditions and an initial condition. This is done by means of a priori estimates in L^2 and the use of Kolmogorov's theorem on relative compactness of subsets of L^2.

Keywords: flow in porous media, Richards equation, finite volume methods, convergence of approximate solutions, discrete a priori estimates, Kolmogorov's theorem

AMS subject classification: 35k55, 65M12, 65N12, 65N22, 76M25, 76S05

1. Introduction

In this paper we consider the problem

$$(\mathcal{P})\begin{cases} c(u)_t = \Delta u + \operatorname{div}\big(K\big(c(u)\big)\nabla z\big) & \text{in } Q_T = \Omega \times (0,T), & (1.1)\\ u = u^D & \text{on } \partial\Omega \times (0,T), & (1.2)\\ c\big(u(x,0)\big) = c\big(u_0(x)\big) & \text{for all } x \in \Omega, & (1.3)\end{cases}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and T is a positive constant. We denote the space coordinates by $x = (x_1, x_2, \ldots, x_{N-1}, z)$ and assume that the functions c and K satisfy the hypotheses (cf. figures 3 and 2)

(H_c) c is a Lipschitz continuous nondecreasing function with Lipschitz constant L_c;

(H_K) K is a nondecreasing Lipschitz continuous function with Lipschitz constant L_K;

* Corresponding author.

and that the initial condition u_0 and the boundary data u^D satisfy the hypotheses

(H_0) $u_0 \in L^\infty(\Omega)$, $\|u_0\|_{L^\infty(\Omega)} = U_0$;

(H_D) u^D is Lipschitz continuous on $\overline{\Omega}$ with Lipschitz constant L_D.

Equation (1.1) changes type in $\Omega \times \mathbb{R}^+$: it is elliptic in the regions where $c(u)$ is constant and parabolic elsewhere. Since we do not expect the solution to be smooth, we define a weak solution of problem $(\mathcal{P})$ as follows.

Definition 1.1. A function u is a weak solution of problem $(\mathcal{P})$ if

(i) $u - u_D \in L^2(0, T; H_0^1(\Omega))$;

(ii) $c(u) \in L^\infty(Q_T)$;

(iii) u satisfies the integral identity

$$\int_0^T \int_\Omega \bigl\{c\bigl(u(x,t)\bigr) - c\bigl(u_0(x)\bigr)\bigr\}\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \bigl\{\nabla u(x,t)\nabla\psi(x,t) + K\bigl(c\bigl(u(x,t)\bigr)\bigr)\nabla z \nabla\psi(x,t)\bigr\}\,\mathrm{d}x\,\mathrm{d}t, \quad (1.4)$$

for all $\psi \in L^2(0,T; H_0^1(\Omega))$ such that $\psi_t \in L^\infty(Q_T)$, $\psi(\cdot, T) = 0$.

It follows from Otto [23] that $(\mathcal{P})$ has at most one weak solution.

Elliptic–parabolic equations have been studied a lot from the theoretical point of view. We refer, in particular, to the articles by van Duijn and Peletier [7], Hulshof [19], Hulshof and Wolanski [20], Alt and Luckhaus [1] and Otto [23]. They prove the existence and the uniqueness of the solution of boundary value problems for the equation (1.1), as well as regularity properties of the interface between saturated and unsaturated regions.

For numerical studies we refer to Hornung [18] for the discretization of Richards equation by the finite difference method and to Knabner [22] for its discretization by means of the finite element method. Kelanemer [21] and Chounet et al. [5] implement a mixed finite element method and Knabner et al. [12] apply a slighty different finite volume scheme than the one presented here.

Equation (1.1) is a basic equation in environmental sciences. We show in section 2 how it arises from Richards equation which models flow in groundwater. The purpose of this paper is to prove the convergence of a time implicit finite volume approximation for problem $(\mathcal{P})$.

Finite volume schemes have first been developed by engineers in order to study complex coupled physical phenomena where the conservation of extensive quantities (such as masses, energy, impulsion, etc.) must be carefully respected by the approximate solution. Another advantage of such schemes is that a large variety of meshes can be used. The basic idea is the following: one integrates the partial differential equations in each control volume and then approximates the fluxes across the volume boundaries. The finite volume method is one of the most popular methods among engineers performing computations in hydrology. Therefore it is of crucial importance to be able to present convergence proofs for precisely this method.

In section 3, we introduce the finite volume scheme and define the approximate problems $(\mathcal{P}_{h,k})$. We use upwinding in the discretization of the convection; this is not always done by hydrogeologists in practical applications. However, Forsyth and Kropinski [13] and Fuhrmann and Langmach [14] show that in some concrete geological examples, upwinding permits one to avoid numerical oscillations that are present otherwise. We prove the existence and uniqueness of the solution $u_{h,k}$ of problem $(\mathcal{P}_{h,k})$. The uniqueness proof is based on the fact that the discrete semigroup corresponding to $c(u(t))$ satisfies a contraction property whereas the existence proof uses arguments based on the topological degree.

In sections 4 and 5, we derive a priori estimates. We obtain an L^∞-bound for $c(u_{h,k})$ and prove an estimate on $u_{h,k}$ for a discrete norm corresponding to a norm in $L^2(0,T;H^1(\Omega))$. We then deduce estimates on differences of space translates of $u_{h,k}$ and on differences of space and time translates of $c(u_{h,k})$, which imply that the sequence $\{c(u_{h,k})\}$ is relatively compact in $L^2(Q_T)$. A basic ingredient that we use to obtain these estimates is a discrete form of Poincaré's inequality which we recall in section 3.

From these estimates, we deduce in section 6 the existence of a subsequence of $\{u_{h,k}\}$ which converges to a function $u \in L^2(0,T;H^1(\Omega))$ weakly in $L^2(Q_T)$ and such that $\{c(u_{h,k})\}$ converges to a function χ strongly in $L^2(Q_T)$. Finally, we prove that $\chi = c(u)$ and that u is the unique weak solution of problem $(\mathcal{P})$.

Finally, in section 7, we present numerical results for two numerical tests.

For other articles about the convergence of the finite volume method for elliptic or parabolic equations, we refer to Baughman and Walkington [2], Herbin [17] and Eymard et al. [10].

2. The physical context

A basic equation in environmental sciences is Richards equation

$$\beta(P)_t = \operatorname{div}\big(K\big(\beta(P)\big)\nabla(P+z)\big), \tag{2.1}$$

considered in the domain Q_T, where $P = P(x,t)$ denotes the liquid pressure and $\beta = \beta(P)$ the saturation in a porous medium (a typical profile of saturation is given in figure 1). It allows one to compute the pressure P as well as the velocity q of the groundwater flow which we define below.

Richards equation is a consequence of two physical laws:

(i) the equation for the mass balance

$$\frac{\partial}{\partial t}\beta(P) + \operatorname{div} q = 0, \tag{2.2}$$

where q is the liquid velocity, and

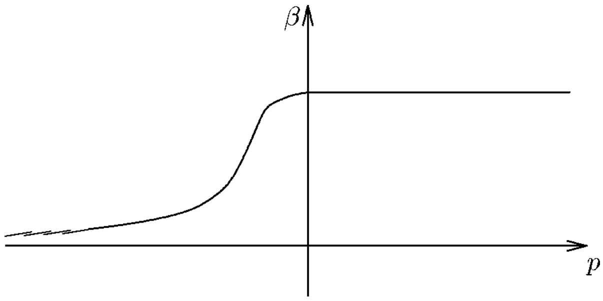

Figure 1. Typical saturation β.

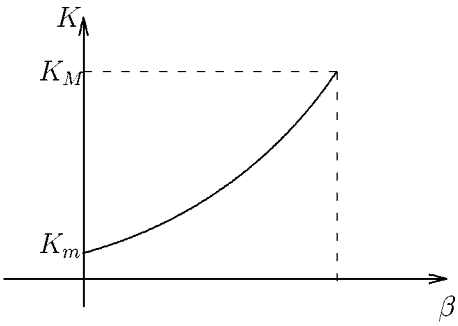

Figure 2. Typical permeability K.

(ii) Darcy's law

$$q = -K\big(\beta(P)\big)\nabla(P + z), \tag{2.3}$$

where K denotes the permeability of the medium (cf. figure 2) and where the term $K(\beta(P))\nabla z$ models the gravity effects.

If we substitute Darcy's law (2.3) into equation (2.2), we obtain Richards equation (2.1). Next we perform Kirchhoff's transformation. We set

$$F(s) := \int_0^s K\big(\beta(\tau)\big)\,\mathrm{d}\tau$$

and suppose that the function F is invertible. Then we set

$$u = F(P) \quad \text{in } Q_T,$$

and then

$$c(u) = c\big(F(P)\big) = \beta(P),$$

where the function c is either qualitatively similar to the function β or has a support which is bounded from the left as in figure 3. We finally obtain equation (1.1).

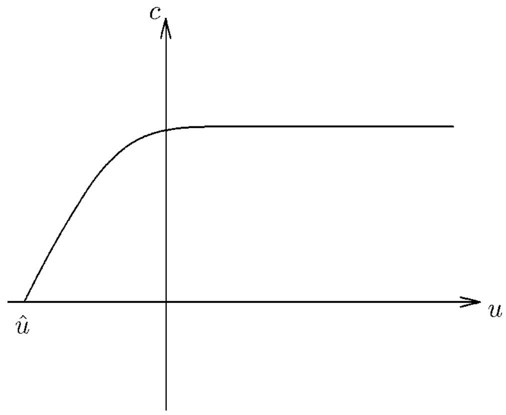

Figure 3. A profile of c.

3. The finite volume scheme

In this section, we construct approximate solutions of problem $(\mathcal{P})$. To that purpose, we introduce a time implicit discretization and a finite volume scheme for the discretization in space. Let $\mathcal{T}$ be a mesh of Ω. The elements of $\mathcal{T}$ will be called control volumes in what follows. For any $(p,q) \in \mathcal{T}^2$ with $p \neq q$, we denote by $e_{pq} = \overline{p} \cap \overline{q}$ their common interface; it is included in a hyperplane of $\mathbb{R}^N$ which neither intersects p nor q. Then $m(e_{pq})$ denotes the measure of e_{pq} for the Lebesgue measure of the hyperplane, $\boldsymbol{n}_{pq}$ denotes the unit vector normal to e_{pq}, oriented from p to q and $g_{pq} = \nabla z \cdot \boldsymbol{n}_{pq}$ denotes the cosine between $\boldsymbol{n}_{pq}$ and ∇z.

The set of boundary volumes is denoted by $\partial\mathcal{T} = \{p \in \mathcal{T},\ \mathrm{meas}(\partial p \cap \partial\Omega) \neq 0\}$ and for all $p \in \partial\mathcal{T}$, we denote by e_p the intersection of the boundary of p and the boundary of Ω, i.e., $e_p = \partial p \cap \partial\Omega$.

We denote by $\mathcal{E}$ the set of pairs of adjacent control volumes together with the set of pairs (p, e_p) for all $p \in \partial\mathcal{T}$, that is,

$$\mathcal{E} = \{(p,q) \in \mathcal{T}^2,\ p \neq q,\ m(e_{pq}) \neq 0\} \cup \{(p,e_p),\ p \in \partial\mathcal{T}\}.$$

For all $p \in \mathcal{T} \setminus \partial\mathcal{T}$, $N(p) = \{q \in \mathcal{T},\ (p,q) \in \mathcal{E}\}$ denotes the set of neighbors of p and for all $p \in \partial\mathcal{T}$, $N(p) = \{q \in \mathcal{T},\ (p,q) \in \mathcal{E}\} \cup \{e_p\}$ denotes the set of neighbors of p including the common boundary of p and Ω.

Furthermore, for all $p \in \mathcal{T}$, we denote by $m(p)$ the measure of p in $\mathbb{R}^N$.

We use the notation

$$h := \max_{p \in \mathcal{T}} \delta(p), \tag{3.1}$$

where $\delta(p)$ denotes the diameter of p, and suppose that there exists a family of points $x_p \in \Omega$ such that

$$(\mathrm{H}_{\mathcal{T}})\begin{cases} x_p \in p & \text{for all } p \in \mathcal{T}, \\ \dfrac{x_q - x_p}{|x_q - x_p|} = \boldsymbol{n}_{pq} & \text{for all } (p,q) \in \mathcal{T}^2. \end{cases}$$

Remark 3.1. In the case of a triangular mesh, x_p is the intersection of the perpendicular bisectors of the interfaces. Therefore, hypothesis $(\mathrm{H}_{\mathcal{T}})$ requires that the maximum value of the angles of each triangle is bounded from above by $\pi/2$.

We denote $d_{pq} = |x_q - x_p|$ and define the transmissivity by $T_{pq} = m(e_{pq})/d_{pq}$. If $p \in \partial\mathcal{T}$ and $q = e_p$, we define

$$T_{pq} = T_{p,e_p} = \frac{m(e_p)}{d_{p,e_p}},$$

where $d_{p,e_p} = |x_{e_p} - x_p|$ and x_{e_p} is a point of e_p. We remark that hypothesis $(\mathrm{H}_{\mathcal{T}})$ means that e_{pq} and the segment $[x_p, x_q]$ are orthogonal.

Remark 3.2. Obviously, T_{pq} is symmetric in p and q, that is, $T_{pq} = T_{qp}$ for all $(p,q) \in \mathcal{E}$.

The time implicit finite volume scheme is defined by the following equations, in which $k > 0$ denotes the time step.

(i) The initial condition for the scheme is given by

$$u_p^0 = \frac{1}{m(p)} \int_p u_0(x)\,\mathrm{d}x, \tag{3.2}$$

for all $p \in \mathcal{T}$.

(ii) The discrete equation

$$m(p)\,\frac{c(u_p^{n+1}) - c(u_p^n)}{k} = \sum_{q \in N(p)} T_{pq}\big(u_q^{n+1} - u_p^{n+1}\big) + \sum_{q \in N(p)} m(e_{pq}) g_{pq} K\big(c\big(\big\{u^{n+1}\big\}_{pq}\big)\big), \tag{3.3}$$

for all $p \in \mathcal{T}$, $n \in \{0, \ldots, [T/k]\}$, where

$$\big\{u^{n+1}\big\}_{pq} = \begin{cases} u_p^{n+1} & \text{if } g_{pq} < 0, \\ u_q^{n+1} & \text{if } g_{pq} > 0. \end{cases} \tag{3.4}$$

The discrete Dirichlet condition is defined in the following way. For all $p \in \partial\mathcal{T}$ and for $q = e_p = \partial p \cap \partial\Omega$, we set

$$u_{e_p}^{n+1} = u_{e_p}^D = u^D(x_{e_p}), \tag{3.5}$$

where x_{e_p} is a point of e_p.

We remark that one cannot use an explicit finite volume scheme to solve Richards equation since c can be constant on an interval of $\mathbb{R}^+$, so it has no inverse function.

The numerical scheme (3.2)–(3.4) allows us to build an approximate solution, $u_{h,k} : \Omega \times \mathbb{R}^+ \mapsto \mathbb{R}$ given for all $p \in \mathcal{T}$ and all $n \in \{0, \dots, [T/k]\}$ by

$$u_{h,k}(x,t) = u_p^{n+1}, \quad \text{for all } x \in p, \ \text{for all } t \in \bigl(nk, (n+1)k\bigr]. \tag{3.6}$$

Since we have to deal with the inhomogeneous Dirichlet boundary condition $u = u^D$ on $\partial\Omega \times (0,T]$, we are led to consider the new unknown function

$$v_{h,k} = u_{h,k} - u_h^D, \tag{3.7}$$

where

$$u_h^D(x) = \begin{cases} u_p^D := u^D(x_p) & \text{if } x \in p, \\ u_{e_p}^D := u^D(x_{e_p}) & \text{if } x \in e_p. \end{cases} \tag{3.8}$$

Therefore,

$$v_{h,k}(x) = \begin{cases} v_p^n = u_p^n - u_p^D & \text{if } x \in p, \\ v_{e_p}^n = 0 & \text{if } x \in e_p. \end{cases} \tag{3.9}$$

With these notations (3.3) can be rewritten as

$$\begin{aligned} m(p)\,\frac{c(u_p^{n+1}) - c(u_p^n)}{k} &= \sum_{q \in N(p)} T_{pq}\bigl(v_q^{n+1} - v_p^{n+1}\bigr) + \sum_{q \in N(p)} T_{pq}\bigl(u_q^D - u_p^D\bigr) \\ &\quad + \sum_{q \in N(p)} m(e_{pq}) g_{pq} K\Bigl(c\bigl(\bigl\{u^{n+1}\bigr\}_{pq}\bigr)\Bigr), \end{aligned} \tag{3.10}$$

for all $p \in \mathcal{T}$ and $n \in \{0, \dots, [T/k]\}$. The discrete problem $(\mathcal{P}_{h,k})$ is given by initial condition (3.2), boundary condition (3.5) or (3.9) and either the discrete equation (3.3) or the discrete equation (3.10).

Next we state some estimates for u_h^D.

Lemma 3.3. The function u_h^D satisfies the L^2-estimate

$$\bigl\|u_h^D\bigr\|_{L^2(\Omega)}^2 = \sum_{p \in \mathcal{T}} m(p)\bigl(u_p^D\bigr)^2 \leqslant m(\Omega)\bigl\|u^D\bigr\|_{C(\bar{\Omega})}^2, \tag{3.11}$$

as well as the "discrete H^1-estimate"

$$\sum_{(p,q) \in \mathcal{E}} T_{pq}\bigl(u_q^D - u_p^D\bigr)^2 \leqslant C\, m(\Omega). \tag{3.12}$$

Proof. Since inequality (3.11) is obvious, we only prove the estimate (3.12). Since the function u^D is Lipschitz continuous, it follows that

$$\sum_{(p,q) \in \mathcal{E}} T_{pq}\bigl(u_q^D - u_p^D\bigr)^2 \leqslant \sum_{(p,q) \in \mathcal{E}} T_{pq} L_D^2 d_{pq}^2 \leqslant L_D^2 \sum_{(p,q) \in \mathcal{E}} m(e_{pq}) d_{pq}. \tag{3.13}$$

By a geometrical argument, we remark that

$$\sum_{(p,q)\in\mathcal{E}} m(e_{pq})d_{pq} \leqslant Cm(\Omega), \tag{3.14}$$

which we substitute in (3.13) to deduce (3.12). □

In the next lemma we give an equivalent form for the discrete convection term.

Lemma 3.4. For all $p \in \mathcal{T}$ and all $n \in \{0,\dots,[T/k]\}$, we have that

$$\begin{aligned}\sum_{q\in N(p)} m(e_{pq})g_{pq}K\bigl(c\bigl(\{u^{n+1}\}_{pq}\bigr)\bigr) \\ = \sum_{q\in N(p)} m(e_{pq})g^+_{pq}\bigl(K\bigl(c\bigl(u^{n+1}_q\bigr)\bigr) - K\bigl(c\bigl(u^{n+1}_p\bigr)\bigr)\bigr),\end{aligned} \tag{3.15}$$

where $g^+_{pq} = \max(g_{pq}, 0)$. Therefore, the discrete equation (3.3) can also be written in the form

$$\begin{aligned}m(p)\frac{c(u^{n+1}_p) - c(u^n_p)}{k} = {}& \sum_{q\in N(p)} T_{pq}\bigl(u^{n+1}_q - u^{n+1}_p\bigr) \\ &+ \sum_{q\in N(p)} m(e_{pq})g^+_{pq}\bigl(K\bigl(c\bigl(u^{n+1}_q\bigr)\bigr) - K\bigl(c\bigl(u^{n+1}_p\bigr)\bigr)\bigr).\end{aligned} \tag{3.16}$$

Proof. Let n be fixed in $\{0,\dots,[T/k]\}$. We have that

$$\begin{aligned}\sum_{q\in N(p)} m(e_{pq})g_{pq}K\bigl(c\bigl(\{u^{n+1}\}_{pq}\bigr)\bigr) = {}& \sum_{q\in N(p)} m(e_{pq})g^+_{pq}K\bigl(c\bigl(\{u^{n+1}\}_{pq}\bigr)\bigr) \\ &- \sum_{q\in N(p)} m(e_{pq})g^-_{pq}K\bigl(c\bigl(\{u^{n+1}\}_{pq}\bigr)\bigr).\end{aligned}$$

In view of the definition of $\{u^{n+1}\}_{pq}$, we obtain

$$\begin{aligned}\sum_{q\in N(p)} m(e_{pq})g_{pq}K\bigl(c\bigl(\{u^{n+1}\}_{pq}\bigr)\bigr) = {}& \sum_{q\in N(p)} m(e_{pq})g^+_{pq}K\bigl(c\bigl(u^{n+1}_q\bigr)\bigr) \\ &- \sum_{q\in N(p)} m(e_{pq})g^-_{pq}K\bigl(c\bigl(u^{n+1}_p\bigr)\bigr).\end{aligned} \tag{3.17}$$

We add and subtract $\sum_{q\in N(p)} m(e_{pq})g^+_{pq}K(c(u^{n+1}_p))$ on the right-hand side of (3.17). This yields

$$\begin{aligned}\sum_{q\in N(p)} m(e_{pq})g_{pq}K\bigl(c\bigl(\{u^{n+1}\}_{pq}\bigr)\bigr) = {}& \sum_{q\in N(p)} m(e_{pq})g^+_{pq}\bigl(K\bigl(c\bigl(u^{n+1}_q\bigr)\bigr) - K\bigl(c\bigl(u^{n+1}_p\bigr)\bigr)\bigr) \\ &+ K\bigl(c\bigl(u^{n+1}_p\bigr)\bigr) \sum_{q\in N(p)} m(e_{pq})g_{pq}.\end{aligned} \tag{3.18}$$

In view of the definition of g_{pq}, we remark that

$$\sum_{q\in N(p)} m(e_{pq})g_{pq} = \sum_{q\in N(p)} \int_{e_{pq}} \nabla z \cdot \boldsymbol{n}_{pq}\,\mathrm{d}s = \int_p \Delta z\,\mathrm{d}x = 0, \tag{3.19}$$

which we substitute in (3.18) to deduce (3.15). $\square$

Before proving the existence and uniqueness of the solution of the discrete problem $(\mathcal{P}_{h,k})$, we recall a discrete form of Poincaré's inequality and prove a useful identity which we often use in the sequel. For the proof of Poincaré's inequality we refer to Eymard et al. [9].

Lemma 3.5 (Discrete Poincaré inequality). Suppose that hypothesis $(\mathrm{H}_{\mathcal{T}})$ is satisfied and let v_h be a piecewise constant function defined by

$$v_h(x) = v_p \quad \text{if } x \in p,\ p \in \mathcal{T}, \tag{3.20}$$

and

$$v_h(x) = 0 \quad \text{if } x \in \partial\Omega. \tag{3.21}$$

Then

$$|v_h|^2_{L^2(\Omega)} = \sum_{p\in\mathcal{T}} m(p)v_p^2 \leqslant \delta^2(\Omega) \sum_{(p,q)\in\mathcal{E}} T_{pq}\,(v_q - v_p)^2. \tag{3.22}$$

Lemma 3.6 (Discrete integration by parts). Suppose that hypothesis $(\mathrm{H}_{\mathcal{T}})$ is satisfied. Let v_h be a piecewise constant function defined by (3.20) and (3.21) and E_{pq} any expression depending on p and q such that $E_{pq} = -E_{qp}$ for all $(p,q) \in \mathcal{T}^2$. Then the following equality holds:

$$-\sum_{p\in\mathcal{T}} \sum_{q\in N(p)} E_{pq}v_p = \sum_{(p,q)\in\mathcal{E}} E_{pq}(v_q - v_p). \tag{3.23}$$

Proof. In view of the definitions of $\mathcal{T}$, $N(p)$ and $\partial\mathcal{T}$, we have that

$$-\sum_{p\in\mathcal{T}} \sum_{q\in N(p)} E_{pq}v_p = -\sum_{(p,q)\in\mathcal{T}^2} E_{pq}v_p - \sum_{p\in\partial\mathcal{T}} E_{p,e_p}v_p. \tag{3.24}$$

We permute p and q on the right-hand side of (3.24). The antisymmetric property of E_{pq} implies that

$$-\sum_{p\in\mathcal{T}} \sum_{q\in N(p)} E_{pq}v_p = \sum_{(p,q)\in\mathcal{T}^2} E_{pq}v_q - \sum_{p\in\partial\mathcal{T}} E_{p,e_p}\,v_p. \tag{3.25}$$

Next we sum (3.24) and (3.25). This gives

$$-\sum_{p\in\mathcal{T}} \sum_{q\in N(p)} E_{pq}v_p = \frac{1}{2} \sum_{(p,q)\in\mathcal{T}^2} E_{pq}(v_q - v_p) - \sum_{p\in\partial\mathcal{T}} E_{p,e_p}\,v_p.$$

Since $v_{e_p} = 0$ for all $p \in \partial\mathcal{T}$, we can rewrite $-v_p = v_{e_p} - v_p$. In view of the definition of $\mathcal{E}$, we finally obtain (3.23). $\square$

Theorem 3.7. Suppose that the hypotheses (H$_c$), (H$_K$), (H$_0$), (H$_D$) and (H$_\mathcal{T}$) are satisfied. There exists a unique solution of the discrete problem ($\mathcal{P}_{h,k}$).

Proof. To begin with, we prove that problem ($\mathcal{P}_{h,k}$) has at most one solution. Let n be fixed and assume that for u_p^n given, there are two solutions u_{1p}^{n+1} and u_{2p}^{n+1} satisfying (3.3) and (3.16). Since g_{pq}^+ and T_{pq} are nonnegative and since c and K are nondecreasing functions, we deduce from (3.16) that

$$\frac{m(p)}{k}\,c\big(u_{ip}^{n+1}\big) \leqslant \frac{m(p)}{k}\,c\big(u_p^n\big) + \sum_{q\in N(p)} T_{pq}\big(\max\big(u_{1q}^{n+1}, u_{2q}^{n+1}\big) - u_{ip}^{n+1}\big) + \sum_{q\in N(p)} m(e_{pq})g_{pq}^+\big[K\big(c\big(\max\big(u_{1q}^{n+1}, u_{2q}^{n+1}\big)\big)\big) - K\big(c\big(u_{ip}^{n+1}\big)\big)\big], \tag{3.26}$$

for $i = 1, 2$. We remark that for all $p \in \mathcal{T}$ $\max(u_{1p}^{n+1}, u_{2p}^{n+1}) = u_{1p}^{n+1}$ or $\max(u_{1p}^{n+1}, u_{2p}^{n+1}) = u_{2p}^{n+1}$. Therefore, the inequalities (3.26) lead to

$$\begin{aligned}
&\frac{m(p)}{k}\,c\big(\max\big(u_{1p}^{n+1}, u_{2p}^{n+1}\big)\big) \\
&\quad\leqslant \frac{m(p)}{k}\,c\big(u_p^n\big) + \sum_{q\in N(p)} T_{pq}\big(\max\big(u_{1q}^{n+1}, u_{2q}^{n+1}\big) - \max\big(u_{1p}^{n+1}, u_{2p}^{n+1}\big)\big) \\
&\qquad+ \sum_{q\in N(p)} m(e_{pq})g_{pq}^+\big[K\big(c\big(\max\big(u_{1q}^{n+1}, u_{2q}^{n+1}\big)\big)\big) - K\big(c\big(\max\big(u_{1p}^{n+1}, u_{2p}^{n+1}\big)\big)\big)\big].
\end{aligned} \tag{3.27}$$

Using a similar argument we obtain

$$\begin{aligned}
&\frac{m(p)}{k}\,c\big(\min\big(u_{1p}^{n+1}, u_{2p}^{n+1}\big)\big) \\
&\quad\geqslant \frac{m(p)}{k}\,c\big(u_p^n\big) + \sum_{q\in N(p)} T_{pq}\big(\min\big(u_{1q}^{n+1}, u_{2q}^{n+1}\big) - \min\big(u_{1p}^{n+1}, u_{2p}^{n+1}\big)\big) \\
&\qquad+ \sum_{q\in N(p)} m(e_{pq})g_{pq}^+\big[K\big(c\big(\min\big(u_{1q}^{n+1}, u_{2q}^{n+1}\big)\big)\big) - K\big(c\big(\min\big(u_{1p}^{n+1}, u_{2p}^{n+1}\big)\big)\big)\big].
\end{aligned} \tag{3.28}$$

We subtract (3.28) from (3.27) and remark that for all nondecreasing functions f we have that $f(\max(a,b)) - f(\min(a,b)) = |f(a) - f(b)|$. We deduce that

$$\begin{aligned}
&\frac{m(p)}{k}\,\big|c\big(u_{1p}^{n+1}\big) - c\big(u_{2p}^{n+1}\big)\big| \\
&\quad\leqslant \sum_{q\in N(p)} T_{pq}\big(\big|u_{1q}^{n+1} - u_{2q}^{n+1}\big| - \big|u_{1p}^{n+1} - u_{2p}^{n+1}\big|\big)
\end{aligned}$$

$$+\sum_{q\in N(p)} m(e_{pq})g^+_{pq}\big[\big|K\big(c\big(u^{n+1}_{1q}\big)\big)-K\big(c\big(u^{n+1}_{2q}\big)\big)\big| \\ -\big|K\big(c\big(u^{n+1}_{1p}\big)\big)-K\big(c\big(u^{n+1}_{2p}\big)\big)\big|\big]. \tag{3.29}$$

We now apply a formula similar to (3.15) to the last term on the right-hand side of (3.29) and sum the result over $p \in \mathcal{T}$. This yields, also using the inequality $|a|-|b| \leqslant |a-b|$,

$$\sum_{p\in\mathcal{T}} \frac{m(p)}{k}\big|c\big(u^{n+1}_{1p}\big)-c\big(u^{n+1}_{2p}\big)\big| \\ \leqslant \sum_{p\in\mathcal{T}}\sum_{q\in N(p)} T_{pq}\big(\big|u^{n+1}_{1q}-u^{n+1}_{2q}\big|-\big|u^{n+1}_{1p}-u^{n+1}_{2p}\big|\big) \\ +\sum_{p\in\mathcal{T}}\sum_{q\in N(p)} m(e_{pq})g_{pq}\big|K\big(c\big(\big\{u^{n+1}_1\big\}_{pq}\big)\big)-K\big(c\big(\big\{u^{n+1}_2\big\}_{pq}\big)\big)\big|. \tag{3.30}$$

Since $T_{pq}=T_{qp}$ and $g_{pq}=-g_{qp}$ for all $(p,q)\in\mathcal{T}^2$, we deduce that the terms in both sums on the right-hand side of (3.30) are antisymmetric in (p,q) so that the terms for (p,q) and the terms for (q,p) cancel for all $(p,q)\in\mathcal{T}^2$. Moreover, u_1 and u_2 satisfy the same boundary conditions, so that the only terms which are left involve pairs of the form (p,e_p) for $p\in\partial\mathcal{T}$. We obtain, in view of (3.26),

$$\sum_{p\in\mathcal{T}} \frac{m(p)}{k}\big|c\big(u^{n+1}_{1p}\big)-c\big(u^{n+1}_{2p}\big)\big| \\ \leqslant -\sum_{p\in\partial\mathcal{T}}\big[T_{p,e_p}\big|u^{n+1}_{1p}-u^{n+1}_{2p}\big|+m(e_p)g^-_{p,e_p}\big|K\big(c\big(u^{n+1}_{1p}\big)\big)-K\big(c\big(u^{n+1}_{2p}\big)\big)\big|\big]\leqslant 0.$$

Hence, we deduce that

$$c\big(u^{n+1}_{1p}\big)=c\big(u^{n+1}_{2p}\big), \tag{3.31}$$

for all $p\in\mathcal{T}$. We subtract the equations (3.16) for u^{n+1}_{1p} and u^{n+1}_{2p}, substitute (3.31), multiply the result by $u^{n+1}_{1p}-u^{n+1}_{2p}$ and sum over $p\in\mathcal{T}$. We obtain

$$\sum_{p\in\mathcal{T}}\sum_{q\in N(p)} T_{pq}\big(\big(u^{n+1}_{1q}-u^{n+1}_{2q}\big)-\big(u^{n+1}_{1p}-u^{n+1}_{2p}\big)\big)\big(u^{n+1}_{1p}-u^{n+1}_{2p}\big)=0.$$

Since u^{n+1}_1 and u^{n+1}_2 satisfy the same boundary conditions, we may apply first the identity in lemma 3.6 and then the discrete Poincaré inequality (3.22). This yields

$$\sum_{p\in\mathcal{T}} m(p)\big(u^{n+1}_{1p}-u^{n+1}_{2p}\big)^2=0$$

and, finally,

$$u^{n+1}_{1p}=u^{n+1}_{2p},$$

for all $p\in\mathcal{T}$.

Next we prove the existence of the solution of problem $(\mathcal{P}_{h,k})$. For this purpose, we consider a sequence of smooth strictly increasing functions c_ε such that $|c'_\varepsilon| \leqslant L_c$ and c_ε converges to c uniformly on $\mathbb{R}$. To begin with we prove the existence of a unique solution of the problem

$$\frac{m(p)}{k} c_\varepsilon\big(u_p^\varepsilon\big) - \sum_{q\in N(p)} T_{pq}\big(u_q^\varepsilon - u_p^\varepsilon\big) - \sum_{q\in N(p)} m(e_{pq}) g_{pq} K\big(c_\varepsilon\big(\{u^\varepsilon\}_{pq}\big)\big) \\ = \frac{m(p)}{k} c_\varepsilon\big(u_p^n\big). \tag{3.32}$$

Let $U^\varepsilon = (u_p^\varepsilon)_{p\in\mathcal{T}}$ be a solution of (3.32) and $U^n = (u_p^n)_{p\in\mathcal{T}}$. Let $\mathcal{P}$ be the number of elements of $\mathcal{T}$. Then U^ε and U^n may be viewed as vectors of $R^{\mathcal{P}}$. We rewrite (3.32) as

$$\mathcal{F}\big(U^\varepsilon\big) - \mathcal{G}\big(U^\varepsilon\big) = \mathcal{C}\big(U^n\big), \tag{3.33}$$

where $\mathcal{F}$ corresponds to the first and the second terms on the left-hand side of (3.32) and $\mathcal{G}$ corresponds to the third term. We suppose that

$$-M \leqslant c_\varepsilon\big(u_p^n\big) \leqslant M, \tag{3.34}$$

for all $p \in \mathcal{T}$. It follows from the same argument as in the proof of lemma 4.1 below that

$$-M \leqslant c_\varepsilon\big(u_p^\varepsilon\big) \leqslant M. \tag{3.35}$$

Since c_ε is strictly increasing we deduce from (3.35) that

$$\big|u_p^\varepsilon\big| \leqslant U_M, \tag{3.36}$$

for all $p \in \mathcal{T}$, where $U_M = \max(|c_\varepsilon^{-1}(-M)|, |c_\varepsilon^{-1}(M)|)$. Next, let $\mathcal{B} = B(0, r)$ be a ball of $\mathbb{R}^{\mathcal{P}}$ with $r > \sqrt{\mathcal{P}}\, U_M$. For the purpose of contradiction, we assume that (3.33) has a solution U^ε on $\partial\mathcal{B}$. Then $|U^\varepsilon| = r$. By the choice of r, this yields a contradiction with (3.36). Thus equation (3.33) has no solution on the boundary of $\mathcal{B}$. Moreover, in view of hypothesis (H_K) and by the choice of c_ε, $\mathcal{F}$ and $\mathcal{G}$ are continuous maps from $\mathbb{R}^{\mathcal{P}}$ into $\mathbb{R}^{\mathcal{P}}$. Therefore we can define the topological degree of the application $\mathcal{F} - \mathcal{G}$ on the set $\mathcal{B}$ associated to $\mathcal{C}(U^n)$, that is, $d(\mathcal{F} - \mathcal{G}, \mathcal{B}, \mathcal{C}(U^n))$ (see [6]). Next, we consider the application $\mathcal{H}$ defined by

$$\mathcal{H} : [0,1] \times \mathcal{B} \to \mathbb{R}^{\mathcal{P}}, \quad \big(\lambda, U^\varepsilon\big) \mapsto \mathcal{F}\big(U^\varepsilon\big) - \lambda\mathcal{G}\big(U^\varepsilon\big).$$

$\mathcal{H}$ is continuous from $[0,1]\times\mathcal{B}$ into $\mathbb{R}^{\mathcal{P}}$. Moreover, we can prove as for equation (3.33) that for all $\lambda \in [0,1]$

$$\mathcal{F}\big(U^\varepsilon\big) - \lambda\mathcal{G}\big(U^\varepsilon\big) = \mathcal{C}\big(U^n\big)$$

has no solution on the boundary of $\mathcal{B}$. Therefore, in view of the property of invariance of the topological degree [6, theorem 3.1 (d3)], we deduce that

$$d\big(\mathcal{F} - \mathcal{G}, \mathcal{B}, \mathcal{C}\big(U^n\big)\big) = d\big(\mathcal{F}, \mathcal{B}, \mathcal{C}\big(U^n\big)\big). \tag{3.37}$$

It follows from Eymard et al. [11] that the equation

$$\mathcal{F}(U^\varepsilon) = \mathcal{C}(U^n)$$

has exactly one solution $\widetilde{U}^\varepsilon = (\widetilde{u}_p^\varepsilon)_{p\in\mathcal{T}}$. By the choice of c_ε, $\mathcal{F}$ is differentiable. Moreover, the Jacobian matrix of $\mathcal{F}$ in $\widetilde{U}^\varepsilon$ is strictly diagonal dominant so that $\widetilde{U}^\varepsilon$ is a regular value of $\mathcal{F}$, and it follows from Deimling [6, definition 2.1] that

$$d(\mathcal{F}, \mathcal{B}, \mathcal{C}(U^n)) = \operatorname{sgn}(\mathcal{J}_{\mathcal{F}}(\widetilde{U}^\varepsilon)) \neq 0.$$

Hence we deduce from (3.37) that

$$d(\mathcal{F} - \mathcal{G}, \mathcal{B}, \mathcal{C}(U^n)) \neq 0.$$

Since $\mathcal{F}$ and $\mathcal{G}$ are continuous on $\mathbb{R}^{\mathcal{P}}$, $\mathcal{F} - \mathcal{G}$ is continuous on $\overline{\mathcal{B}}$ and we deduce in view of the properties of the topological degree [6, theorem 3.1 (d4)] that

$$(\mathcal{F} - \mathcal{G})^{-1}(\mathcal{C}(U^n)) \neq \emptyset,$$

so that there exists at least one solution of (3.32). Using exactly the same argument as the one leading to the uniqueness of the solution of (3.3), we deduce that this solution, which we denote by u_p^ε, is unique.

Next we prove the existence of a solution u_p^{n+1} of (3.3). We replace u_p^ε by $v_p^\varepsilon + u_p^D$ in (3.32), multiply by v_p^ε and sum the result over $p \in \mathcal{T}$. This yields

$$\begin{aligned}
&-\sum_{p\in\mathcal{T}} \sum_{q\in N(p)} T_{pq}(v_q^\varepsilon - v_p^\varepsilon)v_p^\varepsilon \\
&\quad = \sum_{p\in\mathcal{T}} \sum_{q\in N(p)} T_{pq}(u_q^D - u_p^D)v_p^\varepsilon + \sum_{p\in\mathcal{T}} \sum_{q\in N(p)} m(e_{pq})g_{pq}K(c_\varepsilon(\{u^\varepsilon\}_{pq}))v_p^\varepsilon \\
&\qquad - \frac{1}{k}\sum_{p\in\mathcal{T}} m(p)(c_\varepsilon(u_p^\varepsilon) - c_\varepsilon(u_p^n))v_p^\varepsilon.
\end{aligned} \tag{3.38}$$

We perform the discrete integration by parts of lemma 3.6 on the left-hand side of (3.38) and for the first two terms on the right-hand side of (3.38). We then use the Cauchy–Schwarz inequality and the inequalities

$$ab \leqslant a^2 + \frac{1}{4}b^2 \quad \text{and} \quad ab \leqslant \delta^2(\Omega)a^2 + \frac{1}{4\delta^2(\Omega)}b^2$$

to obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{(p,q)\in\mathcal{E}} T_{pq}(v_q^\varepsilon - v_p^\varepsilon)^2 \\
&\quad \leqslant \sum_{(p,q)\in\mathcal{E}} T_{pq}(u_q^D - u_p^D)^2 + \sum_{(p,q)\in\mathcal{E}} m(e_{pq})d_{pq}(K(c_\varepsilon(\{u^\varepsilon\}_{pq})))^2 \\
&\qquad + \frac{\delta^2(\Omega)}{k^2}\sum_{p\in\mathcal{T}} m(p)(c_\varepsilon(u_p^\varepsilon) - c_\varepsilon(u_p^n))^2 + \frac{1}{4\delta^2(\Omega)}\sum_{p\in\mathcal{T}} m(p)(v_p^\varepsilon)^2.
\end{aligned} \tag{3.39}$$

In view of estimate (3.12) and of the maximum principle on $c_\varepsilon(u_p^\varepsilon)$ given by (3.35) together with property (3.14), the first three terms on the right-hand side of (3.39) are bounded independently of ε. Applying the Poincaré inequality (3.22) to the left-hand side of (3.39), we finally deduce that

$$\frac{1}{4\delta^2(\Omega)}\sum_{p\in\mathcal{T}} m(p)\big(v_p^\varepsilon\big)^2 \leqslant C,$$

where the constant C does not depend on ε, so that in view of estimate (3.11)

$$\big|u_p^\varepsilon\big| \leqslant \widetilde{C} = C(h),$$

for all $p \in \mathcal{T}$. Thus there exists u_p^{n+1} and a subsequence $u_p^{\varepsilon'}$ such that

$$u_p^{\varepsilon'} \to u_p^{n+1}, \tag{3.40}$$

for all $p \in \mathcal{T}$, as $\varepsilon' \to 0$. In view of the convergence of $u_p^{\varepsilon'}$ (3.40) and of the uniform convergence of $c_{\varepsilon'}$ to c, we have, letting ε' tend to 0 in (3.32), that u_p^{n+1} satisfies (3.3). □

The mathematical problem is to study the convergence of $\{u_{h,k}\}$ to the weak solution of problem $(\mathcal{P})$ as h and k tend to zero.

4. A priori estimates

In this section we first prove that $c(u_{h,k})$ satisfies a discrete maximum principle and then derive an estimate for $u_{h,k}$ in a discrete space analogous to $L^2(0,T;H^1(\Omega))$.

Lemma 4.1. Let $u_{h,k}$ be the solution of problem $(\mathcal{P}_{h,k})$ and suppose that the hypotheses (H$_c$), (H$_K$), (H$_0$) and (H$_D$) are satisfied. Let

$$M = \max\big(\big\|c(u_0)\big\|_{L^\infty(\Omega)}, \big\|c\big(u^D\big)\big\|_{L^\infty(\partial\Omega)}\big).$$

Then for all $p \in \mathcal{T}$ and $0 \leqslant n \leqslant [T/k]$, we have that

$$-M \leqslant c\big(u_{h,k}(x,t)\big) \leqslant M \quad \text{for all} \;\; x \in p,\; t \in \big(nk,(n+1)k\big]. \tag{4.1}$$

Proof. We make use of an induction argument. Set $n \in \{0,\ldots,[T/k]\}$ and suppose that $-M \leqslant c(u_p^n) \leqslant M$ for all $p \in \mathcal{T}$. We consider equation (3.16). Since there is a finite number of control volumes, there exist p_0 and p_1 such that

$$u_{p_0}^{n+1} \leqslant u_p^{n+1} \leqslant u_{p_1}^{n+1},$$

for all $p \in \mathcal{T}$, which implies that

$$c\big(u_{p_0}^{n+1}\big) \leqslant c\big(u_p^{n+1}\big) \leqslant c\big(u_{p_1}^{n+1}\big), \tag{4.2}$$

for all $p \in \mathcal{T}$. If p_0 and p_1 are strictly interior then, for all $q \in \mathcal{T}$, we have that

$$u_{p_0}^{n+1} \leqslant u_q^{n+1} \leqslant u_{p_1}^{n+1}.$$

Therefore, in view of the monotonicity of c and K we deduce that, for all $(p,q) \in \mathcal{E}$,

$$K\big(c\big(u_{p_0}^{n+1}\big)\big) \leqslant K\big(c\big(u_q^{n+1}\big)\big) \leqslant K\big(c\big(u_{p_1}^{n+1}\big)\big).$$

In particular, we get

$$\sum_{q\in N(p_0)} T_{p_0,q}\big(u_q^{n+1} - u_{p_0}^{n+1}\big) + \sum_{q\in N(p_0)} m(e_{pq}) g_{pq}^{+}\big(K\big(c\big(u_q^{n+1}\big)\big) - K\big(c\big(u_{p_0}^{n+1}\big)\big)\big) \geqslant 0 \tag{4.3}$$

and

$$\sum_{q\in N(p_1)} T_{p_1,q}\big(u_q^{n+1} - u_{p_1}^{n+1}\big) + \sum_{q\in N(p_1)} m(e_{pq}) g_{pq}^{+}\big(K\big(c\big(u_q^{n+1}\big)\big) - K\big(c\big(u_{p_1}^{n+1}\big)\big)\big) \leqslant 0. \tag{4.4}$$

Next we substitute (4.3) and (4.4) in (3.16). Together with (4.2) this yields

$$c\big(u_{p_0}^{n}\big) \leqslant c\big(u_{p_0}^{n+1}\big) \leqslant c\big(u_p^{n+1}\big) \leqslant c\big(u_{p_1}^{n+1}\big) \leqslant c\big(u_{p_1}^{n}\big), \tag{4.5}$$

for all $p \in \mathcal{T}$. Now if p_0 is not strictly interior then (4.3) is not necessarily true. Then either $u^D(x_{e_{p_0}}) \geqslant u_{p_0}^{n+1}$ and we follow exactly the same argument as above to obtain

$$c\big(u_{p_0}^{n}\big) \leqslant c\big(u_p^{n+1}\big), \tag{4.6}$$

for all $p \in \mathcal{T}$, or $u^D(x_{e_{p_0}}) \leqslant u_{p_0}^{n+1}$ and in view of the monotonicity of c and by the choice of p_0, we have that

$$c\big(u^D(x_{e_{p_0}})\big) \leqslant c\big(u_{p_0}^{n+1}\big) \leqslant c\big(u_p^{n+1}\big), \tag{4.7}$$

for all $p \in \mathcal{T}$. By the same argument, if p_1 is not strictly interior, we get either

$$c\big(u_p^{n+1}\big) \leqslant c\big(u_{p_1}^{n}\big), \tag{4.8}$$

or

$$c\big(u_p^{n+1}\big) \leqslant c\big(u_{p_1}^{n+1}\big) \leqslant c\big(u^D(x_{e_{p_1}})\big), \tag{4.9}$$

for all $p \in \mathcal{T}$. In the case of (4.5), (4.6) or (4.8), we make use of the induction hypothesis, while in the case of (4.7) and (4.9), by definition of M, we have that $c(u^D(x_{e_{p_1}})) \leqslant M$ and $c(u^D(x_{e_{p_0}})) \geqslant -M$. Thus in all cases, we deduce that

$$-M \leqslant c\big(u_p^{n+1}\big) \leqslant M,$$

for all $p \in \mathcal{T}$, which implies the result of lemma 4.1. □

Lemma 4.2. Suppose that the hypotheses (H_c), (H_K),(H_0), (H_D) and ($\mathrm{H}_{\mathcal{T}}$) are satisfied. There exists a positive constant C such that

$$\sum_{n=0}^{[T/k]} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\big(v_q^{n+1} - v_p^{n+1}\big)^2 \leqslant C. \tag{4.10}$$

Proof. We multiply equation (3.10) by kv_p^{n+1}, and sum the result over $n = 0, \dots, [T/k]$ and $p \in \mathcal{T}$ to obtain

$$\begin{aligned} &\sum_{n=0}^{[T/k]} \sum_{p\in\mathcal{T}} m(p)\big(c\big(u_p^{n+1}\big) - c\big(u_p^n\big)\big)u_p^{n+1} - \sum_{n=0}^{[T/k]} k \sum_{p\in\mathcal{T}} \sum_{q\in N(p)} T_{pq}\big(v_q^{n+1} - v_p^{n+1}\big)v_p^{n+1} \\ &\quad= \sum_{n=0}^{[T/k]} \sum_{p\in\mathcal{T}} m(p)\big(c\big(u_p^{n+1}\big) - c\big(u_p^n\big)\big)u_p^D + \sum_{n=0}^{[T/k]} k \sum_{p\in\mathcal{T}} \sum_{q\in N(p)} T_{pq}\big(u_q^D - u_p^D\big)v_p^{n+1} \\ &\qquad+ \sum_{n=0}^{[T/k]} k \sum_{p\in\mathcal{T}} \sum_{q\in N(p)} m(e_{pq})g_{pq}K\big(c\big(\big\{u^{n+1}\big\}_{pq}\big)\big)v_p^{n+1}. \end{aligned} \tag{4.11}$$

To begin with we consider the first term on the left-hand side of (4.11). Let $B \in C(\mathbb{R})$ be defined by

$$B(s) = c(s)s - \int_0^s c(\tau)\,\mathrm{d}\tau = \int_0^s \big(c(s) - c(\tau)\big)\mathrm{d}\tau.$$

For all $p \in \mathcal{T}$ and $n \in \mathbb{N}$ such that $0 \leqslant n \leqslant [T/k]$, we have that

$$B\big(u_p^{n+1}\big) - B\big(u_p^n\big) = \big(c\big(u_p^{n+1}\big) - c\big(u_p^n\big)\big)u_p^{n+1} - \int_{u_p^n}^{u_p^{n+1}} \big(c(\tau) - c\big(u_p^n\big)\big)\mathrm{d}\tau. \tag{4.12}$$

Since c is nondecreasing, the last term on the right-hand side of (4.12) is nonnegative. Therefore we multiply inequality (4.12) by $m(p)$ and sum over $n = 0, \dots, [T/k]$ and $p \in \mathcal{T}$ to obtain

$$\sum_{n=0}^{[T/k]} \sum_{p\in\mathcal{T}} m(p)\big(B\big(u_p^{n+1}\big) - B\big(u_p^n\big)\big) \leqslant \sum_{n=0}^{[T/k]} \sum_{p\in\mathcal{T}} m(p)\big(c\big(u_p^{n+1}\big) - c\big(u_p^n\big)\big)u_p^{n+1}. \tag{4.13}$$

All the terms in the sum on n on the left-hand side of (4.13) simplify except for the first and the last ones. Besides, in view of the definition of B and of the monotonicity of c, $B(s) \geqslant 0$ for all $s \in \mathbb{R}$, so that

$$\sum_{p\in\mathcal{T}} m(p)\,B\big(u_p^{[T/k]+1}\big) \geqslant 0, \tag{4.14}$$

and we finally deduce that

$$-\sum_{p\in\mathcal{T}} m(p)B\big(u_p^0\big) \leqslant \sum_{n=0}^{[T/k]} \sum_{p\in\mathcal{T}} m(p)\big(c\big(u_p^{n+1}\big) - c\big(u_p^n\big)\big)u_p^{n+1}. \tag{4.15}$$

From the monotonicity of c and since $|u_p^0| \leqslant U_0$ and $|c(u_p^0)| \leqslant M$, we deduce that

$$B\big(u_p^0\big) \leqslant \big|c\big(u_p^0\big) - c(0)\big|\big|u_p^0\big| \leqslant \big(M + \big|c(0)\big|\big)U_0, \tag{4.16}$$

for all $p \in \mathcal{T}$. We multiply (4.16) by $m(p)$ and sum the result over $p \in \mathcal{T}$ to obtain

$$\sum_{p\in\mathcal{T}} m(p)B\big(u_p^0\big) \leqslant U_0 m(\Omega)\big(M + \big|c(0)\big|\big) = C(u_0, \Omega, c). \tag{4.17}$$

Next we consider the first term on the right-hand side of (4.11). Since u^D does not depend on time, all the terms in the sum on n simplify except for the first and the last ones. Then we apply the Cauchy–Schwarz inequality to deduce that

$$\begin{aligned}
&\sum_{n=0}^{[T/k]} \sum_{p\in\mathcal{T}} m(p)\big(c\big(u_p^{n+1}\big) - c\big(u_p^n\big)\big)u_p^D \\
&\quad\leqslant \left(\sum_{p\in\mathcal{T}} m(p)\big(c\big(u_p^{[T/k]+1}\big) - c\big(u_p^0\big)\big)^2\right)^{1/2} \left(\sum_{p\in\mathcal{T}} m(p)\big(u_p^D\big)^2\right)^{1/2}.
\end{aligned}$$

In view of lemma 4.1, which states the maximum principle for $c(u_{h,k})$, and in view of estimate (3.11), there finally holds

$$\sum_{n=0}^{[T/k]} \sum_{p\in\mathcal{T}} m(p)\big(c\big(u_p^{n+1}\big) - c\big(u_p^n\big)\big)u_p^D \leqslant 2Mm(\Omega)\big\|u^D\big\|_{C(\overline{\Omega})} = C\big(u^D, c, \Omega\big). \tag{4.18}$$

Finally we perform the discrete integration by parts of lemma 3.6 for the last term on the left-hand side of (4.11). For the last two terms on the right-hand side of (4.11) we perform the same computation and also use the inequalities

$$|ab| \leqslant a^2 + \frac{b^2}{4} \quad \text{and} \quad |ab| \leqslant d_{pq}a^2 + \frac{b^2}{4d_{pq}}.$$

Substituting inequalities (4.15) and (4.17) as well as (4.18), we finally obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{n=0}^{[T/k]} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\big(v_q^{n+1} - v_p^{n+1}\big)^2 \\
&\quad\leqslant C(u_0, \Omega, c) + C\big(u^D, c, \Omega\big) + \sum_{n=0}^{[T/k]} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\big(u_q^D - u_p^D\big)^2 \\
&\qquad + \sum_{n=0}^{[T/k]} k \sum_{(p,q)\in\mathcal{E}} m(e_{pq})d_{pq}K\big(c\big(\big\{u^{n+1}\big\}_{pq}\big)\big)^2.
\end{aligned} \tag{4.19}$$

In view of estimate (3.12), the second term on the right-hand side of (4.19) is bounded, whereas in view of the maximum principle for $c(\{u^{n+1}\}_{pq})$ of lemma 4.1, of hypothesis (H_K) and of property (3.14), the last term on the right-hand side of (4.19) is bounded as well. This implies (4.10). □

One can then apply lemma 4.2 and the discrete Poincaré inequality to deduce the following result.

Lemma 4.3. Suppose that the hypotheses (H_c), (H_K), (H_0), (H_D) and ($\mathrm{H}_\mathcal{T}$) are satisfied. There exists a positive constant C such that

$$\left\|v_{h,k}\right\|_{L^2(Q_T)} \leqslant C \tag{4.20}$$

and

$$\left\|u_{h,k}\right\|_{L^2(Q_T)} \leqslant C. \tag{4.21}$$

5. Estimates on differences of space and time translates

Throughout this section, we suppose that the hypotheses (H_c), (H_K), (H_0), (H_D) and ($\mathrm{H}_\mathcal{T}$) are satisfied. First we deduce from lemma 4.2 an estimate on differences of space translates of the approximate solution $u_{h,k}$. We then derive estimates on differences of space and time translates of the function $c(u_{h,k})$ which imply that the sequence $\{c(u_{h,k})\}$ is relatively compact in $L^2(Q_T)$.

Lemma 5.1. There exists a positive constant C such that

$$\int_{\Omega_\xi\times(0,T)} \big(u_{h,k}(x+\xi,t) - u_{h,k}(x,t)\big)^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant |\xi|\big(|\xi| + 2h\big)C \tag{5.1}$$

and

$$\int_{\Omega_\xi\times(0,T)} \big(v_{h,k}(x+\xi,t) - v_{h,k}(x,t)\big)^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant |\xi|\big(|\xi| + 2h\big)C, \tag{5.2}$$

for all $\xi \in \mathbb{R}^N$, where $\boldsymbol{\Omega}_\xi = \{x \in \boldsymbol{\Omega},\ [x, x+\xi] \subset \boldsymbol{\Omega}\}$.

Proof. Inequality (5.2) follows from estimate (4.10). We refer to [10, lemma 3.3] for a complete proof. Inequality (5.1) then immediately follows. □

Corollary 5.2. We have that

$$\int_{\Omega_\xi\times(0,T)} \big(c(u_{h,k})(x+\xi,t) - c(u_{h,k})(x,t)\big)^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant |\xi|\big(|\xi| + 2h\big)L_c^2 C, \tag{5.3}$$

for all $\xi \in \mathbb{R}^N$, where $\boldsymbol{\Omega}_\xi = \{x \in \boldsymbol{\Omega},\ [x+\xi, x] \subset \boldsymbol{\Omega}\}$, and C is the same constant as in lemma 5.1.

Proof. This result follows from the fact that c is a Lipschitz continuous function with Lipschitz constant L_c. $\square$

First we give two technical lemmas which are necessary for the proof of the estimate on time translates.

Lemma 5.3. Let $T > 0$, $\tau \in (0, T)$, $k > 0$ be given and $(a^n)_{n\in\mathbb{N}}$ be a family of nonnegative real values. Then

$$\int_0^{T-\tau} \sum_{\substack{n\in\mathbb{N},\\ t<nk\leqslant t+\tau}} a^n \,\mathrm{d}t \leqslant \tau \sum_{n=0}^{[T/k]-1} a^n. \tag{5.4}$$

Proof. We introduce the characteristic function χ defined by $\chi(n, t_1, t_2) = 1$ if $t_1 < nk \leqslant t_2$ else $\chi(n, t_1, t_2) = 0$. Then we have that

$$\int_0^{T-\tau} \sum_{\substack{n\in\mathbb{N},\\ t<nk\leqslant t+\tau}} a^n \,\mathrm{d}t \leqslant \sum_{n=0}^{[T/k]-1} a^n \int_0^{T-\tau} \chi(n, t, t+\tau)\,\mathrm{d}t \leqslant \sum_{n=0}^{[T/k]-1} a^n \tau. \qquad \square$$

Lemma 5.4. Let $T > 0$, $\tau \in (0, T)$, $\zeta \in [0, \tau]$, $k > 0$ be given and $(a^n)_{n\in\mathbb{N}}$ be a family of nonnegative real values. Then

$$\int_0^{T-\tau} \sum_{\substack{n\in\mathbb{N},\\ t<nk\leqslant t+\tau}} a^{[(t+\zeta)/k]} \,\mathrm{d}t \leqslant \tau \sum_{n=0}^{[T/k]} a^n. \tag{5.5}$$

Proof. We have that

$$\int_0^{T-\tau} \sum_{\substack{n\in\mathbb{N},\\ t<nk\leqslant t+\tau}} a^{[(t+\zeta)/k]} \,\mathrm{d}t \leqslant \sum_{m=0}^{[T/k]} \int_{mk}^{(m+1)k} a^m \sum_{n=0}^{[T/k]} \chi(n, t-\zeta, t-\zeta+\tau)\,\mathrm{d}t,$$

and for all $m \in \mathbb{N}$

$$\begin{aligned}
&\int_{mk}^{(m+1)k} \sum_{n=0}^{[T/k]} \chi(n, t-\zeta, t-\zeta+\tau)\,\mathrm{d}t \\
&\quad= \int_0^{k} \sum_{n=0}^{[T/k]} \chi(n-m, t-\zeta, t-\zeta+\tau)\,\mathrm{d}t = \sum_{n=0}^{[T/k]} \int_{-nk}^{k-nk} \chi(-m, t-\zeta, t-\zeta+\tau)\,\mathrm{d}t \\
&\quad\leqslant \int_{\mathbb{R}} \chi(-m, t-\zeta, t-\zeta+\tau)\,\mathrm{d}t \leqslant \int_{\zeta+\tau-mk}^{\zeta-mk} \mathrm{d}t \leqslant \tau,
\end{aligned}$$

which concludes the proof. $\square$

We now consider differences of time translates of the function $c(u_{h,k})$.

Lemma 5.5. There exists a positive constant C such that

$$\int_{\Omega\times(0,T-\tau)} \big(c(u_{h,k})(x,t+\tau) - c(u_{h,k})(x,t)\big)^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \tau C, \tag{5.6}$$

for all $\tau \in (0,T)$.

Proof. Let $\tau \in (0,T)$ and $t \in (0, T-\tau)$. We set

$$\mathcal{A}(t) = \int_\Omega \big(c(u_{h,k})(x,t+\tau) - c(u_{h,k})(x,t)\big)^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Applying definition (3.6) yields

$$\mathcal{A}(t) = \sum_{p\in\mathcal{T}} m(p)\Big(c\big(u_p^{[(t+\tau)/k]+1}\big) - c\big(u_p^{[t/k]+1}\big)\Big)^2,$$

which also reads as

$$\mathcal{A}(t) = \sum_{p\in\mathcal{T}} \Bigg[\Big(c\big(u_p^{[(t+\tau)/k]+1}\big) - c\big(u_p^{[t/k]+1}\big)\Big) \times \sum_{\substack{n\in\mathbb{N},\\ t<nk\leqslant t+\tau}} m(p)\Big(c\big(u_p^{n+1}\big) - c\big(u_p^{n}\big)\Big)\Bigg]. \tag{5.7}$$

We now substitute the equation for the scheme (3.10) into (5.7). This gives

$$\begin{aligned}\mathcal{A}(t) = {} & \sum_{\substack{n\in\mathbb{N},\\ t<nk\leqslant t+\tau}} k \sum_{p\in\mathcal{T}} \sum_{q\in N(p)} T_{pq}\big(v_q^{n+1} - v_p^{n+1}\big)\Big(c\big(u_p^{[(t+\tau)/k]+1}\big) - c\big(u_p^{[t/k]+1}\big)\Big) \\ & + \sum_{\substack{n\in\mathbb{N},\\ t<nk\leqslant t+\tau}} k \sum_{p\in\mathcal{T}} \sum_{q\in N(p)} T_{pq}\big(u_q^D - u_p^D\big)\Big(c\big(u_p^{[(t+\tau)/k]+1}\big) - c\big(u_p^{[t/k]+1}\big)\Big) \\ & + \sum_{\substack{n\in\mathbb{N},\\ t<nk\leqslant t+\tau}} k \sum_{p\in\mathcal{T}} \sum_{q\in N(p)} m(e_{pq}) g_{pq} K\Big(c\big(\{u^{n+1}\}_{pq}\big)\Big) \\ & \qquad \times \Big(c\big(u_p^{[(t+\tau)/k]+1}\big) - c\big(u_p^{[t/k]+1}\big)\Big).\end{aligned}$$

We perform the discrete integration by parts of lemma 3.6 and then use inequalities of the form

$$\pm 2ab \leqslant ca^2 + \frac{b^2}{c}$$

to deduce that

$$\mathcal{A}(t) \leqslant \mathcal{A}_D(t) + \mathcal{A}_K(t) + \mathcal{A}_v(t) + \frac{3}{2}\,\mathcal{A}_c(t,\tau) + \frac{3}{2}\,\mathcal{A}_c(t,0), \tag{5.8}$$

where

$$\begin{aligned}
\mathcal{A}_D(t) &= \sum_{\substack{n\in\mathbb{N},\\ t<nk\leqslant t+\tau}} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\big(u_q^D - u_p^D\big)^2,\\
\mathcal{A}_K(t) &= \sum_{\substack{n\in\mathbb{N},\\ t<nk\leqslant t+\tau}} k \sum_{(p,q)\in\mathcal{E}} m(e_{pq}) d_{pq} K\big(c\big(\big\{u^{n+1}\big\}_{pq}\big)\big)^2, \qquad (5.9)\\
\mathcal{A}_v(t) &= \sum_{\substack{n\in\mathbb{N},\\ t<nk\leqslant t+\tau}} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\big(v_q^{n+1} - v_p^{n+1}\big)^2,
\end{aligned}$$

and

$$\mathcal{A}_c(t,\theta) = \sum_{\substack{n\in\mathbb{N},\\ t<nk\leqslant t+\tau}} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\big(c\big(u_q^{[(t+\theta)/k]+1}\big) - c\big(u_p^{[(t+\theta)/k]+1}\big)\big)^2. \qquad (5.10)$$

First we integrate $\mathcal{A}_D(t)$ from 0 to $T-\tau$. Using lemma 5.3 we obtain

$$\int_0^{T-\tau} A_D(t)\,\mathrm{d}t \leqslant \tau \sum_{n=0}^{[T/k]-1} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\big(u_q^D - u_p^D\big)^2. \qquad (5.11)$$

In view of lemma 3.3, we finally deduce that

$$\int_0^{T-\tau} \mathcal{A}_D(t)\,\mathrm{d}t \leqslant \tau C m(\Omega) \sum_{n=0}^{[T/k]-1} k \leqslant \tau T C m(\Omega). \qquad (5.12)$$

Next we consider $\mathcal{A}_K(t)$. In view of the maximum principle in lemma 4.1, we have that $K\big(c\big(\{u^{n+1}\}_{pq}\big)\big) \leqslant K_M = K(M)$ for all $(p,q)\in\mathcal{E}$. We integrate $\mathcal{A}_K(t)$ from 0 to $T-\tau$. Using lemma 5.3, we obtain

$$\int_0^{T-\tau} \mathcal{A}_K(t)\,\mathrm{d}t \leqslant K_M^2 \tau \sum_{n=0}^{[T/k]-1} k \sum_{(p,q)\in\mathcal{E}} m(e_{pq}) d_{pq} \leqslant \tau T C m(\Omega) K_M^2. \qquad (5.13)$$

We now turn to the study of $\mathcal{A}_c(t,\theta)$, which we also integrate from 0 to $T-\tau$. In view of lemma 5.4, this yields

$$\int_0^{T-\tau} \mathcal{A}_c(t,\theta)\,\mathrm{d}t \leqslant \tau \sum_{n=0}^{[T/k]} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\big(c\big(u_q^{n+1}\big) - c\big(u_p^{n+1}\big)\big)^2$$

for $\theta = 0, \tau$. We then use the Lipschitz continuity of c, the definition (3.9) and apply lemmas 3.3 and 4.2 to deduce that

$$\int_0^{T-\tau} \mathcal{A}_c(t,\theta)\,\mathrm{d}t \leqslant \tau C \qquad (5.14)$$

for $\theta = 0,\ \tau$.

Last, we integrate $\mathcal{A}_v(t)$ from 0 to $T-\tau$. In view of lemma 5.3, we obtain

$$\int_0^{T-\tau} \mathcal{A}_v(t)\,\mathrm{d}t \leqslant \tau \sum_{n=0}^{[T/k]} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\big(v_q^{n+1} - v_p^{n+1}\big)^2.$$

It follows from lemma 4.2 that

$$\int_0^{T-\tau} \mathcal{A}_v(t)\,\mathrm{d}t \leqslant \tau C. \tag{5.15}$$

Substituting the inequalities (5.12)–(5.15) into (5.8), we deduce estimate (5.6). □

Corollary 5.6. There exists a subsequence $\{c(u_{h_m,k_m})\}$ of $\{c(u_{h,k})\}$ and a function $\chi \in L^2(Q_T)$ such that

$$c(u_{h_m,k_m}) \to \chi \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \tag{5.16}$$

as h_m and k_m tend to zero.

Proof. This is a consequence of Kolmogorov's theorem (see [3, theorem IV.25, p. 72; 10, lemma 3.5]). □

6. Convergence

In this section we prove the convergence of the approximate solution to the weak solution of problem ($\mathcal{P}$). To begin with we state a convergence result which will be useful in what follows.

Lemma 6.1. Let $\{u_m\}$ be such that u_m converges to u weakly in $L^2(Q_T)$ and $\{c(u_m)\}$ converges to a limit χ strongly in $L^2(Q_T)$ and a.e. in Q_T as $m \to \infty$. Then

$$\chi = c(u) \quad \text{a.e. in } Q_T. \tag{6.1}$$

Proof. We adapt a proof due to Alt and Luckhaus [1]. □

We are now in a position to present our main result.

Theorem 6.2. Let T be a fixed positive constant and suppose that the hypotheses (H_c), (H_0), (H_D) and ($\mathrm{H}_{\mathcal{T}}$) are satisfied. Then:

(i) $u_{h,k}$ converges to u weakly in $L^2(Q_T)$,

(ii) $c(u_{h,k})$ converges to $c(u)$ strongly in $L^2(Q_T)$,

as h and k tend to zero, where u is the unique weak solution of problem ($\mathcal{P}$).

Proof. From lemmas 4.1, 4.3, 5.5 and 6.1, and corollaries 5.2 and 5.6, we deduce the existence of a subsequence $\{u_{h_m,k_m}\}$ of $\{u_{h,k}\}$ and of a function $u \in L^2(Q_T)$ such that

$$\left.\begin{array}{lll}\text{(i)} & u_{h_m,k_m} & \text{converges to } u \text{ weakly in } L^2(Q_T),\\ \text{(ii)} & c(u_{h_m,k_m}) & \text{converges to } c(u) \text{ strongly in } L^2(Q_T),\end{array}\right\} \tag{6.2}$$

as h_m and k_m tend to zero. Next we show that u is the weak solution of problem ($\mathcal{P}$).

Let $m \in \mathbb{N}$. For the sake of simplicity, we set $\mathcal{T} = \mathcal{T}_m$, $h = h_m$ and $k = k_m$. Let T be a fixed positive constant and $\psi \in \mathbf{\Psi}$, where $\mathbf{\Psi}$ is defined by

$$\begin{aligned}\mathbf{\Psi} = \bigg\{ &\psi \in C^{2,1}\big(\overline{\Omega} \times [0,T]\big),\ \psi = \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial\Omega \times [0,T],\\ &\psi = 0 \text{ on } \Omega \times \{T\} \bigg\}.\end{aligned} \tag{6.3}$$

We multiply equation (3.10) by $k\,\psi(x_p, nk)$, and sum the result over $n = 0, \dots, [T/k] - 1$ and $p \in \mathcal{T}$. We deduce that

$$T_{1m} + T_{2m} + T_{3m} = 0, \tag{6.4}$$

with

$$\begin{aligned}T_{1m} = {} & \sum_{n=0}^{[T/k]-1} \sum_{p \in \mathcal{T}} m(p)\big(c\big(u_p^{n+1}\big) - c\big(u_p^n\big)\big)\psi(x_p, nk),\\ T_{2m} = {} & -\sum_{n=0}^{[T/k]-1} k \sum_{p \in \mathcal{T}} \sum_{q \in N(p)} T_{pq}\big(v_q^{n+1} - v_p^{n+1}\big)\psi(x_p, nk)\\ & - \sum_{n=0}^{[T/k]-1} k \sum_{p \in \mathcal{T}} \sum_{q \in N(p)} T_{pq}\big(u_q^D - u_p^D\big)\psi(x_p, nk),\end{aligned}$$

and

$$T_{3m} = -\sum_{n=0}^{[T/k]-1} k \sum_{p \in \mathcal{T}} \sum_{q \in N(p)} m(e_{pq}) g_{pq} K\big(c\big(\big\{u^{n+1}\big\}_{pq}\big)\big)\psi(x_p, nk).$$

We first consider T_{1m}. We add and subtract $c(u_p^{n+1})\,\psi(x_p, (n+1)k)$ in the following expression of T_{1m}. This gives

$$\begin{aligned}T_{1m} = {} & \sum_{p \in \mathcal{T}} m(p) c\big(u_p^{[T/k]}\big)\psi\big(x_p, [T/k]k\big) - \sum_{p \in \mathcal{T}} m(p) c\big(u_p^0\big)\psi(x_p, 0)\\ & - \sum_{n=0}^{[T/k]-1} \sum_{p \in \mathcal{T}} m(p) c\big(u_p^{n+1}\big)\big(\psi\big(x_p, (n+1)k\big) - \psi(x_p, nk)\big).\end{aligned} \tag{6.5}$$

Next we consider the first term on the right-hand side of (6.5). For m large enough, we necessarily have that $k = k_m < T$. Therefore, $0 \leqslant T - [T/k]k < k$. In view of lemma 4.1, we have that $c(u_p^{[T/k]+1}) \leqslant M$. Since $\psi \in C^1(\overline{\Omega} \times [0,T])$, there exists a positive constant $C_{1\psi}$, which only depends on ψ, T and Ω, such that

$$\left|\psi\big(x_p, [T/k]k\big)\right| = \left|\psi(x_p, T) - \psi\big(x_p, [T/k]k\big)\right| \leqslant C_{1\psi}\left|T - [T/k]k\right| \leqslant C_{1\psi}k.$$

Thus, for m large enough we have that

$$\sum_{p\in\mathcal{T}} m(p)c\big(u_p^{[T/k]}\big)\psi\big(x_p, [T/k]k\big) \leqslant C_{1\psi}Mkm(\Omega). \tag{6.6}$$

We deduce from (6.6) that the first term on the right-hand side of (6.5) tends to zero as k tends to zero. Next, we consider the second term on the right-hand side of (6.5). We set

$$\mathcal{A}_0 = \sum_{p\in\mathcal{T}} m(p)c\big(u_p^0\big)\psi(x_p, 0) - \int_\Omega c\big(u_0(x)\big)\psi(x, 0)\,\mathrm{d}x,$$

which we rewrite as

$$\mathcal{A}_0 = \sum_{p\in\mathcal{T}} \int_p \big(c\big(u_p^0\big)\psi(x_p, 0) - c\big(u_0(x)\big)\psi(x, 0)\big)\,\mathrm{d}x.$$

We now add and subtract terms of the form $c(u_p^0)\psi(x, 0)$ to obtain

$$\mathcal{A}_0 = \sum_{p\in\mathcal{T}} \int_p c\big(u_p^0\big)\big(\psi(x_p, 0) - \psi(x, 0)\big)\mathrm{d}x + \sum_{p\in\mathcal{T}} \int_p \big(c\big(u_p^0\big) - c\big(u_0(x)\big)\big)\psi(x, 0)\,\mathrm{d}x.$$

In view of lemma 4.1, we have that

$$|\mathcal{A}_0| \leqslant M \sum_{p\in\mathcal{T}} \int_p \left|\psi(x_p, 0) - \psi(x, 0)\right|\mathrm{d}x + L_c \sum_{p\in\mathcal{T}} \int_p \left|u_p^0 - u_0(x)\right|\left|\psi(x, 0)\right|\mathrm{d}x. \tag{6.7}$$

Since

$$\sum_{p\in\mathcal{T}} \int_p \left|\psi(x_p, 0) - \psi(x, 0)\right|\mathrm{d}x \leqslant m(\Omega)C_{1\psi}h,$$

we deduce that the first term on the right-hand side of (6.7) converges to zero as h tends to zero and finally that $\mathcal{A}_0$ tends to zero as h tends to zero. Last, we show that the third term on the right-hand side of (6.5) converges to

$$\int_0^T \int_\Omega c\big(u(x,t)\big)\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } h \text{ and } k \text{ tend to zero.}$$

We set

$$T_{4m} = \sum_{n=0}^{[T/k]-1} \sum_{p\in\mathcal{T}} m(p)c\big(u_p^{n+1}\big)\big(\psi\big(x_p,(n+1)k\big) - \psi(x_p,nk)\big) - \int_0^T \int_\Omega c\big(u(x,t)\big)\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Splitting the integrals into sums over $n = 0,\dots,[T/k]k$ and $p \in \mathcal{T}$, we obtain

$$\begin{aligned} T_{4m} = {} & \sum_{n=0}^{[T/k]-1} \sum_{p\in\mathcal{T}} \int_{nk}^{(n+1)k} \int_p c\big(u_p^{n+1}\big)\,\frac{\psi(x_p,(n+1)k) - \psi(x_p,nk)}{k}\,\mathrm{d}x\,\mathrm{d}t \\ & - \sum_{n=0}^{[T/k]-1} \sum_{p\in\mathcal{T}} \int_{nk}^{(n+1)k} \int_p c\big(u(x,t)\big)\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ & - \int_{[T/k]k}^T \int_\Omega c\big(u(x,t)\big)\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{6.8}$$

Since ψ_t is continuous on $\overline{\Omega}\times[0,T]$, there exists a constant C_{ψ_t} such that $|\psi_t(x,t)| \leqslant C_{\psi_t}$ for all $(x,t) \in \overline{\Omega}\times[0,T]$. Then in view of lemma 4.1 we deduce that

$$\left|\int_{[T/k]k}^T \int_\Omega c\big(u(x,t)\big)\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t\right| \leqslant m(\Omega)MC_{\psi_t}\big|T - [T/k]k\big|.$$

Now we add and subtract $c(u_{h,k}(x,t))\psi_t(x,t)$ to the second and the first term on the right-hand side of (6.8), respectively. Using the definition of $u_{h,k}$ (3.6), we get

$$\begin{aligned} |T_{4m}| \leqslant {} & \sum_{n=0}^{[T/k]-1} \sum_{p\in\mathcal{T}} \int_{nk}^{(n+1)k}\int_p \big|c\big(u_p^{n+1}\big) - c\big(u(x,t)\big)\big|\,\big|\psi_t(x,t)\big|\,\mathrm{d}x\,\mathrm{d}t \\ & + \sum_{n=0}^{[T/k]-1} \sum_{p\in\mathcal{T}} \int_{nk}^{(n+1)k}\int_p \big|c\big(u_p^{n+1}\big)\big|\left|\frac{\psi(x_p,(n+1)k) - \psi(x_p,nk)}{k} - \psi_t(x,t)\right|\mathrm{d}x\,\mathrm{d}t \\ & + m(\Omega)MC_{\psi_t}\big|T - [T/k]k\big|. \end{aligned} \tag{6.9}$$

The first term on the right-hand side of (6.9) is bounded from above by

$$\int_0^T \int_\Omega \big|c(u_{h,k})(x,t) - c(u(x,t))\big|\,|\psi_t(x,t)|\,\mathrm{d}x\,\mathrm{d}t$$

which converges to zero as h and k tend to zero by (6.2(ii)). In view of lemma 4.1, it is easy to see that the second term on the right-hand side of (6.9) tends to zero as well as h and k tend to zero. The last term on the right-hand side of (6.9) is bounded from above by Ck which converges to zero as k tends to 0. Thus,

$$\lim_{h,k\to 0} T_{4m} = 0,$$

which together with (6.5), (6.6) and the convergence of $\mathcal{A}_0$ to zero implies that

$$\lim_{h,k\to 0} T_{1m} = -\int_\Omega c\big(u_0(x)\big)\psi(x,0)\,\mathrm{d}x - \int_0^T \int_\Omega c\big(u(x,t)\big)\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

for all admissible test functions $\psi \in \boldsymbol{\Psi}$.

Next we consider the term T_{2m}, which we rewrite in view of lemma 3.6 as

$$T_{2m} = \sum_{n=0}^{[T/k]-1} k\Bigg(\sum_{(p,q)\in\mathcal{E}} m(e_{pq})\big(v_q^{n+1} - v_p^{n+1}\big)\frac{\psi(x_q,nk) - \psi(x_p,nk)}{d_{pq}} + \sum_{(p,q)\in\mathcal{E}} m(e_{pq})\big(u_q^D - u_p^D\big)\frac{\psi(x_q,nk) - \psi(x_p,nk)}{d_{pq}}\Bigg).$$

Our purpose is to prove that T_{2m} converges to

$$-\int_0^T\int_\Omega u(x,t)\Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } h \text{ and } k \text{ tend to zero.}$$

In order to do so, we introduce T'_{2m} defined by

$$T'_{2m} = \sum_{n=0}^{[T/k]-1} \int_{nk}^{(n+1)k}\int_\Omega u_{h,k}(x,t)\Delta\big(\psi(x,nk)\big)\,\mathrm{d}x\,\mathrm{d}t,$$

where we split the integral on Ω into a sum over $p \in \mathcal{T}$. Since $\Delta\psi(x,nk)$ does not depend on t, we deduce that

$$T'_{2m} = \sum_{n=0}^{[T/k]-1} k\sum_{p\in\mathcal{T}} u_p^{n+1}\int_p \Delta\psi(x,nk)\,\mathrm{d}x, \tag{6.10}$$

which we rewrite as

$$T'_{2m} = \sum_{n=0}^{[T/k]-1} k\sum_{p\in\mathcal{T}}\sum_{q\in N(p)} u_p^{n+1}\int_{e_{pq}} \nabla\psi(s,nk)\cdot\boldsymbol{n}_{pq}\,\mathrm{d}s. \tag{6.11}$$

We replace u_p^{n+1} by $v_p^{n+1} + u_p^D$ in (6.11). Since

$$\int_{e_{pq}} \nabla\psi(s,nk)\cdot\boldsymbol{n}_{pq}\,\mathrm{d}s = -\int_{e_{pq}} \nabla\psi(s,nk)\cdot\boldsymbol{n}_{qp}\,\mathrm{d}s,$$

for all $(p,q) \in \mathcal{T}^2$ and

$$\int_{e_p} \nabla\psi(s,nk) \cdot \boldsymbol{n}_{pq}\,\mathrm{d}s = 0,$$

for all $p \in \partial\mathcal{T}$, we can use lemma 3.6 to deduce that

$$T'_{2m} = -\sum_{n=0}^{[T/k]-1} k\biggl(\sum_{(p,q)\in\mathcal{E}} \bigl(v_q^{n+1} - v_p^{n+1}\bigr) \int_{e_{pq}} \nabla\psi(s,nk) \cdot \boldsymbol{n}_{pq}\,\mathrm{d}s + \sum_{(p,q)\in\mathcal{E}} \bigl(u_q^D - u_p^D\bigr) \int_{e_{pq}} \nabla\psi(s,nk) \cdot \boldsymbol{n}_{pq}\,\mathrm{d}s\biggr).$$

In view of (6.10) and the weak convergence of $\{u_{h_m,k_m}\}$ to u as h_m and k_m tend to zero, we deduce that the term T'_{2m} converges to

$$\int_0^T\int_\Omega u(x,t)\Delta\psi(x,t))\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } h_m \text{ and } k_m \text{ tend to zero.}$$

Finally, we consider the term $T_{2m} + T'_{2m}$. It can be written as

$$T_{2m} + T'_{2m} = \sum_{n=0}^{[T/k]-1} k \sum_{(p,q)\in\mathcal{E}} m(e_{pq})\bigl(v_q^{n+1} - v_p^{n+1} + u_q^D - u_p^D\bigr) R_{pq}^n, \tag{6.12}$$

where, since $x_{e_p} \in \partial\Omega$ and therefore $\psi(x_{e_p}, nk) = 0$ for all $p \in \partial\mathcal{T}$, we have that

$$R_{pq}^n = \frac{\psi(x_q,nk) - \psi(x_p,nk)}{d_{pq}} - \frac{1}{m(e_{pq})} \int_{e_{pq}} \nabla\psi(s,nk) \cdot \boldsymbol{n}_{pq}\,\mathrm{d}s$$

for all $(p,q) \in \mathcal{E}$. In view of the definition of h (3.1) and of the regularity properties of ψ, there exists a positive constant $C_{2\psi}$ such that

$$\bigl|R_{pq}^n\bigr| \leqslant C_{2\psi} h.$$

We return to (6.12) and apply the Cauchy–Schwarz inequality to the right-hand side. This yields

$$\bigl|T_{2m} + T'_{2m}\bigr| \leqslant C_{2\psi} h \Biggl(\sum_{n=0}^{[T/k]-1} k \sum_{(p,q)\in\mathcal{E}} m(e_{pq}) d_{pq}\Biggr)^{1/2} \times \Biggl(\sum_{n=0}^{[T/k]-1} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\bigl(v_q^{n+1} - v_p^{n+1} + u_q^D - u_p^D\bigr)^2\Biggr)^{1/2}.$$

We then conclude from lemma 3.3 and the estimate (4.10) that $T_{2m} + T'_{2m}$ tends to zero as h and k tend to zero. Therefore,

$$\lim_{h,k\to 0} T_{2m} = -\int_0^T\int_\Omega u(x,t)\Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Finally, we consider T_{3m}. In order to prove the convergence of T_{3m} to the term

$$T_3' = \int_0^T \int_\Omega K\big(c\big(u(x,t)\big)\big) \nabla z \nabla \psi(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

we introduce the term

$$T_{3m}' = \sum_{n=0}^{[T/k]-1} k \sum_{p\in\mathcal{T}} \sum_{q\in N(p)} g_{pq} K\big(c\big(u_p^{n+1}\big)\big) \int_{e_{pq}} \psi(s,nk)\,\mathrm{d}s.$$

In view of the definition of g_{pq}, we can rewrite T_{3m}' as

$$T_{3m}' = \sum_{n=0}^{[T/k]-1} k \sum_{p\in\mathcal{T}} K\big(c\big(u_p^{n+1}\big)\big) \sum_{q\in N(p)} \int_{e_{pq}} \psi(s,nk) \nabla z \cdot \boldsymbol{n}_{pq}\,\mathrm{d}s.$$

Therefore,

$$T_{3m}' = \sum_{n=0}^{[T/k]-1} k \sum_{p\in\mathcal{T}} K\big(c\big(u_p^{n+1}\big)\big) \int_p \operatorname{div}\big(\psi(x,nk)\nabla z\big)\,\mathrm{d}x.$$

We remark that

$$\operatorname{div}\big(\psi(x,nk)\nabla z\big) = \nabla\psi(x,nk)\nabla z,$$

so that T_{3m}' can be written as

$$T_{3m}' = \sum_{n=0}^{[T/k]-1} k \sum_{p\in\mathcal{T}} K\big(c\big(u_p^{n+1}\big)\big) \int_p \nabla\psi(x,nk)\nabla z\,\mathrm{d}x.$$

Then we define

$$\phi(x,t) = \nabla\psi(x,nk), \quad \text{for all } t \in \big(nk,(n+1)k\big],\ n \in \big\{0,\ldots,[T/k]-1\big\}.$$

In view of the definitions of $u_{h,k}$ and ϕ, we can rewrite T_{3m}' as

$$T_{3m}' = \int_0^{[T/k]k} \int_\Omega K\big(c\big(u_{h,k}(x,t)\big)\big) \phi(x,t) \nabla z\,\mathrm{d}x\,\mathrm{d}t.$$

Next we compare T_{3m}' and T_3'. We have that

$$\begin{aligned} T_{3m}' - T_3' = {} & \int_0^T \int_\Omega K\big(c\big(u_{h,k}(x,t)\big)\big) \nabla z \phi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ & - \int_0^T \int_\Omega K\big(c\big(u(x,t)\big)\big) \nabla z \nabla\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ & - \int_{[T/k]k}^T \int_\Omega K\big(c\big(u_{h,k}(x,t)\big)\big) \nabla z \phi(x,t)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{6.13}$$

We add and subtract the term $K(c(u_{h,k}(x,t)))\nabla z\nabla\psi(x,t)$ to the right-hand side of (6.13). This gives

$$\begin{aligned}T'_{3m}-T'_3 = &\int_0^T\int_\Omega K\big(c\big(u_{h,k}(x,t)\big)\big)\nabla z\big(\phi(x,t)-\nabla\psi(x,t)\big)\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_\Omega \big(K\big(c\big(u_{h,k}(x,t)\big)\big)-K\big(c\big(u(x,t)\big)\big)\big)\nabla z\nabla\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_{[T/k]k}^T\int_\Omega K\big(c\big(u_{h,k}(x,t)\big)\big)\nabla z\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Using lemma 4.1, the regularity properties of ψ and the convergence property (6.2(ii)), we deduce that

$$\lim_{h,k\to 0} T'_{3m} = T'_3 = \int_0^T\int_\Omega K\big(c\big(u(x,t)\big)\big)\nabla z\nabla\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{6.14}$$

Next we show that $\lim_{h,k\to 0}|T_{3m}-T'_{3m}|=0$. For all $p\in\mathcal{T}$ and all $q\in N(p)$, we set

$$\psi_{pq}=\frac{1}{m(e_{pq})}\int_{e_{pq}}\psi(s,nk)\,\mathrm{d}s.$$

We add and subtract the terms $K(c(u_p^{n+1}))\psi(x_p,nk)$ and $K(c(\{u^{n+1}\}_{pq}))\psi_{pq}$ in the expression of $T'_{3m}-T_{3m}$. We obtain

$$\begin{aligned}T'_{3m}-T_{3m} = &\sum_{n=0}^{[T/k]-1} k\sum_{p\in\mathcal{T}}\sum_{q\in N(p)} m(e_{pq})g_{pq}K\big(c\big(\{u^{n+1}\}_{pq}\big)\big)\psi_{pq}\\ &+\sum_{n=0}^{[T/k]-1} k\sum_{p\in\mathcal{T}} K\big(c\big(u_p^{n+1}\big)\big)\psi(x_p,nk)\sum_{q\in N(p)} m(e_{pq})g_{pq}\\ &+\sum_{n=0}^{[T/k]-1} k\sum_{p\in\mathcal{T}}\sum_{q\in N(p)} m(e_{pq})g_{pq}\big(\psi(x_p,nk)-\psi_{pq}\big)\\ &\times\big(K\big(c\big(\{u^{n+1}\}_{pq}\big)\big)-K\big(c\big(u_p^{n+1}\big)\big)\big).\end{aligned} \tag{6.15}$$

We consider the terms on the right-hand side of (6.15). Since $\psi=0$ on $\partial\Omega\times[0,T]$ and since $g_{pq}=-g_{qp}$ for all $(p,q)\in\mathcal{T}^2$, the first term is equal to zero. Besides, we deduce from (3.19) that the second term is also equal to zero. Finally, we consider the last term, which we denote by T_{5m}. Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}T_{5m}^2\leqslant &\sum_{n=0}^{[T/k]-1} k\sum_{p\in\mathcal{T}}\sum_{q\in N(p)} m(e_{pq})d_{pq}\big(\psi(x_p,nk)-\psi_{pq}\big)^2\\ &\times\sum_{n=0}^{[T/k]-1} k\sum_{p\in\mathcal{T}}\sum_{q\in N(p)} T_{pq}\big(K\big(c\big(\{u^{n+1}\}_{pq}\big)\big)-K\big(c\big(u_p^{n+1}\big)\big)\big)^2.\end{aligned} \tag{6.16}$$

In view of inequality (3.14), of the regularity properties of ψ and of the Lipschitz continuity of functions K and c, we deduce from the estimates (3.12) and (4.10) that

$$T_{5m}^2 \leqslant Ch^2 \sum_{n=0}^{[T/k]-1} k \sum_{(p,q)\in\mathcal{T}^2} T_{pq}\big(u_q^{n+1} - u_p^{n+1}\big)^2 \leqslant Ch^2.$$

Returning to (6.15), it follows that

$$\big|T'_{3m} - T_{3m}\big| \leqslant Ch, \tag{6.17}$$

and as a consequence of (6.14) and (6.17) we deduce that

$$\lim_{h,k\to 0} T_{3m} = \int_0^T\!\!\int_\Omega K\big(c\big(u(x,t)\big)\big)\nabla z\nabla\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Passing to the limit into (6.4), we have thus shown that $\{u_{h_m,k_m}\}$ converges to a function u which satisfies the integral equality

$$\begin{aligned} &-\int_\Omega c\big(u_0(x)\big)\psi(x,0)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_\Omega c\big(u(x,t)\big)\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^T\!\!\int_\Omega u(x,t)\Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega K\big(c\big(u(x,t)\big)\big)\nabla z\nabla\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0, \end{aligned} \tag{6.18}$$

for all $\psi \in \mathbf{\Psi}$, where $\mathbf{\Psi}$ is defined by (6.3). By the definitions of $v_{h,k}$ and u_h^D and in view of lemmas 3.3 and 4.3, we deduce that the sequence $\{v_{h_m,k_m}\}$ converges to $v = u - u^D$ weakly in $L^2(Q_T)$. Next we show that $v \in L^2(0,T;H_0^1(\Omega))$ and thus that $u \in L^2(0,T;H^1(\Omega))$. We define $\widetilde{v}_{h,k}$ by

$$\begin{aligned} \widetilde{v}_{h,k} &= v_{h,k} \quad \text{a.e. in } \Omega \times (0,T), \\ \widetilde{v}_{h,k} &= 0 \quad\quad \text{a.e. in } \big(\mathbb{R}^N \setminus \Omega\big) \times (0,T). \end{aligned}$$

Therefore $\{\widetilde{v}_{h_m,k_m}\}$ converges weakly to $\widetilde{v}$ with

$$\begin{aligned} \widetilde{v} &= v \quad \text{a.e. in } \Omega \times (0,T), \\ \widetilde{v} &= 0 \quad \text{a.e. in } \big(\mathbb{R}^N \setminus \Omega\big) \times (0,T). \end{aligned}$$

Then for all $\xi \in \mathbb{R}^N$, $\xi \neq 0$ we have in view of lemma 5.1 that

$$\int_0^T \int_{\mathbb{R}^N} \frac{|\widetilde{v}_{h,k}(x+\xi,t) - \widetilde{v}_{h,k}(x,t)|^2}{|\xi|^2}\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{|\xi| + 2h}{|\xi|}\,C.$$

The functional

$$u \to \int_0^T\!\!\int_{\mathbb{R}^N} u^2(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

is convex and continuous from $L^2(\mathbb{R}^N \times (0,T))$ into $\mathbb{R}$. Therefore, it follows, for instance, from [8, corollary 2.2, p. 10] that it is also weakly lower semi-continuous. Thus,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} \frac{|\widetilde{v}(x+\xi,t) - \widetilde{v}(x,t)|^2}{|\xi|^2} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \liminf_{h_m,\ k_m \to 0} \int_0^T \int_{\mathbb{R}^N} \frac{|\widetilde{v}_{h_m,k_m}(x+\xi,t) - \widetilde{v}_{h_m,k_m}(x,t)|^2}{|\xi|^2} \,\mathrm{d}x\,\mathrm{d}t \leqslant C.
\end{aligned}$$

In particular,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} \frac{\widetilde{v}(x+\xi,t) - \widetilde{v}(x,t)}{|\xi|} \varphi(x,t) \,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \left(\int_0^T \int_{\mathbb{R}^N} \frac{|\widetilde{v}(x+\xi,t) - \widetilde{v}(x,t)|^2}{|\xi|^2} \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \left(\int_0^T \int_{\mathbb{R}^N} \varphi^2(x,t) \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \\
&\leqslant C \|\varphi\|_{L^2(\mathbb{R}^N \times (0,T))},
\end{aligned} \tag{6.19}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N \times (0,T))$. Changing variables on the left-hand side of (6.19) we obtain

$$\left| \int_0^T \int_{\mathbb{R}^N} \frac{\varphi(x-\xi,t) - \varphi(x,t)}{|\xi|} \widetilde{v}(x,t) \,\mathrm{d}x\,\mathrm{d}t \right| \leqslant C \|\varphi\|_{L^2(\mathbb{R}^N \times (0,T))}.$$

Setting $\xi = \varepsilon e_i$ with ε positive and letting $\varepsilon \downarrow 0$, we obtain for $i = 1, \ldots, N$,

$$\left| - \int_0^T \int_{\mathbb{R}^N} \frac{\partial \varphi(x,t)}{\partial x_i} \widetilde{v}(x,t) \,\mathrm{d}x\,\mathrm{d}t \right| \leqslant C \|\varphi\|_{L^2(\mathbb{R}^N \times (0,T))}.$$

Hence we deduce that

$$\frac{\partial \widetilde{v}}{\partial x_i} \in L^2\big(\mathbb{R}^N \times (0,T)\big).$$

Therefore $\widetilde{v} \in L^2(0,T; H^1(\mathbb{R}^N))$. Since also $\widetilde{v} = 0$ a.e. in $\mathbb{R}^N \backslash \Omega$, $v \in L^2(0,T; H_0^1(\Omega))$ and thus $u \in L^2(0,T; H^1(\Omega))$ satisfies

$$u = u^D \quad \text{on } \partial\Omega \times (0,T).$$

Integrating by parts the third term in the integral equation (6.18), we deduce that u satisfies

$$\begin{aligned}
&\int_0^T \int_\Omega \big(c\big(u(x,t)\big) - c\big(u_0(x)\big)\big) \psi_t(x,t) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \int_\Omega \big\{\nabla u(x,t) \nabla \psi(x,t) + K\big(c\big(u(x,t)\big)\big) \nabla z \nabla \psi(x,t)\big\} \mathrm{d}x\,\mathrm{d}t = 0,
\end{aligned}$$

for all $\psi \in \Psi$. Also using the density of Ψ in the set $\{\psi \in L^2(0,T;H_0^1(\Omega))$, $\psi_t \in L^\infty(Q_T)$, $\psi(\cdot,T) = 0\}$, we finally deduce that u coincides with the unique weak solution of problem $(\mathcal{P})$; in particular, the whole sequence $\{u_{h,k}\}$ converges to u. $\square$

7. Numerical tests

In this section we describe the effective computations in the case of two numerical tests. Since the numerical algorithm is completely implicit we have to linearize the discrete equation (3.3) at each time step. We do so by means of a Newton-like method, given as follows. We set

$$u_p^{n+1,0} = u_p^n, \quad \text{for all } p \in \mathcal{T},$$

and define $u_p^{n+1,\nu+1}$, $p \in \mathcal{T}$, by

$$\begin{aligned}
&\frac{m(p)}{k}\, c'\bigl(u_p^{n+1,\nu}\bigr)u_p^{n+1,\nu+1} - \sum_{q\in N(p)} T_{pq}\bigl(u_q^{n+1,\nu+1} - u_p^{n+1,\nu+1}\bigr)\\
&\quad = \frac{m(p)}{k}\, c'\bigl(u_p^{n+1,\nu}\bigr)u_p^{n+1,\nu} - m(p)\,\frac{c(u_p^{n+1,\nu}) - c(u_p^n)}{k}\\
&\qquad + \sum_{q\in N(p)} m(e_{pq})g_{pq}K\bigl(c\bigl(\bigl\{u^{n+1,\nu}\bigr\}_{pq}\bigr)\bigr).
\end{aligned} \tag{7.1}$$

This means solving a linear system with a sparse matrix. To that purpose we use a Bi-CGStab algorithm (see, for instance, [25]).

In the test case which we consider in the following, the Richards equation has the form

$$\beta(P)_t = \operatorname{div}\bigl(K_\beta(P)\nabla(P+z)\bigr).$$

The transformed equation is then given by

$$c(u)_t = \Delta u + \operatorname{div}\bigl(K_c(u)\nabla z\bigr).$$

7.1. *First numerical test: the Hornung–Messing problem*

We consider a horizontal flow in a homogeneous ground. We then have to solve a Richards equation without convection term. We take $\Omega = (0,1)\times(0,1)$ for the space domain. The saturation and the permeability functions are given by

$$\beta(P) := \begin{cases} \dfrac{\pi^2}{2} - 2\arctan^2(P), & P < 0,\\ \dfrac{\pi^2}{2}, & P \geqslant 0, \end{cases} \qquad K_\beta(P) := \begin{cases} \dfrac{2}{(1+P)^2}, & P < 0,\\ 2 & P \geqslant 0. \end{cases}$$

An exact solution of the Richards equation (2.1) is then given by

$$P(x,y,t) = \begin{cases} -\dfrac{1}{2}\,s, & s < 0, \\ -\tan\left(\dfrac{\mathrm{e}^s - 1}{\mathrm{e}^s + 1}\right), & s \geqslant 0, \end{cases} \qquad s = x - y - t.$$

The problem after Kirchhoff's transformation is given by problem $(\mathcal{P})$ with

$$c(u) = \beta(P) = \begin{cases} \dfrac{\pi^2}{2} - 2\arctan^2\left(\dfrac{u}{2-u}\right), & u < 0, \\ \dfrac{\pi^2}{2}, & u \geqslant 0, \end{cases}$$

and the following initial and boundary conditions:

$$\begin{aligned} &u^D(x,y,t) = u(x,y,t), && \text{for all } (x,y,t) \in \partial\Omega \times [0,T], \\ &c\big(u_0(x,y)\big) = c\big(u(x,y,0)\big), && \text{for all } (x,y) \in \Omega, \end{aligned}$$

where the solution u is given by

$$u(x,y,t) = \begin{cases} \dfrac{2\,h(x,y,t)}{1 + h(x,y,t)}, & h(x,y,t) < 0, \\ 2\,h(x,y,t), & h(x,y,t) \geqslant 0, \end{cases} \qquad \text{for all } (x,y,t) \in \Omega \times [0,T).$$

Note that the medium is unsaturated for $u \leqslant 0$ and saturated for $u \geqslant 0$ and that it becomes completely saturated for $t \geqslant 1$ s.

The saturation is shown in figure 4 at time $t = 0.2$ s and $t = 0.8$ s. We have used a uniform square mesh with $h = 0.04$ and the time step is equal to $k = 0.01$. Figure 5 represents the error in L^2-norm between the discrete and the continuous solutions; it

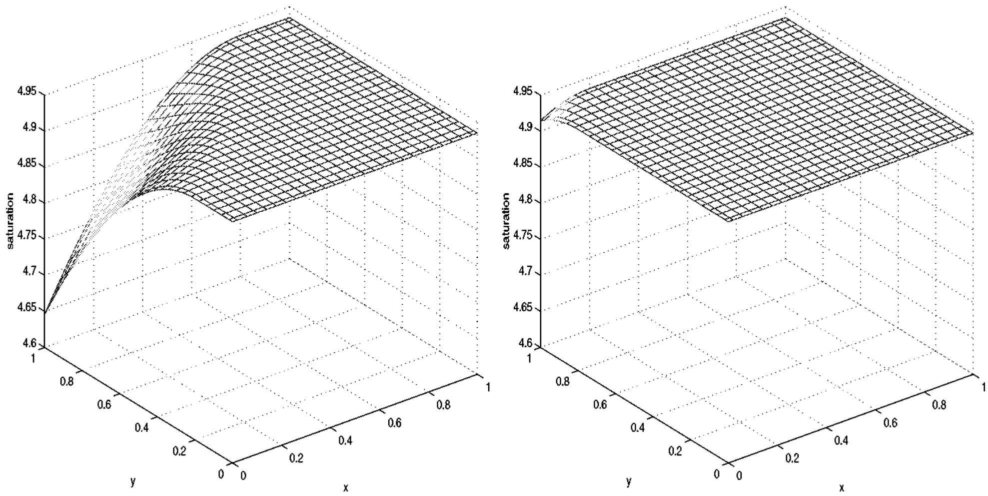

Figure 4. Saturation for $t = 0.2$ s and for $t = 0.8$ s.

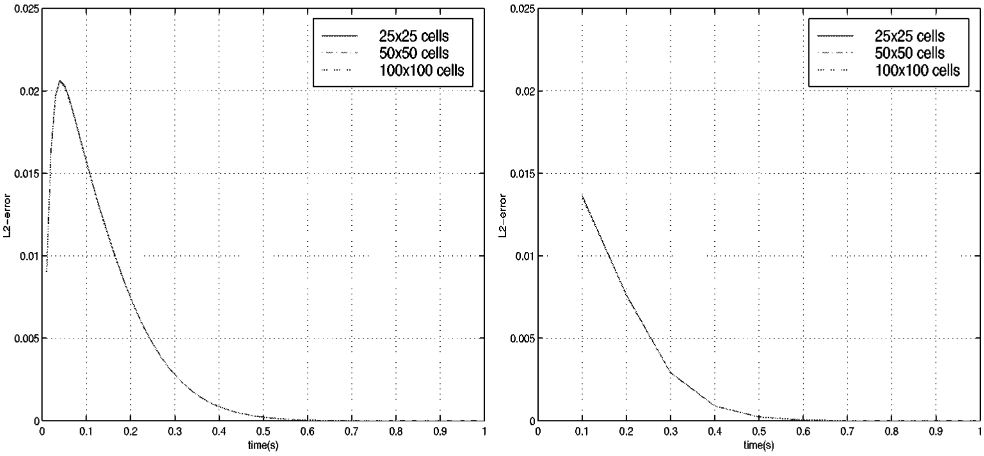

Figure 5. L^2-error between analytical and numerical solutions.

hardly varies when changing the space step h (say, 0.01, 0.02 or 0.04) and the time step k (say, 0.1 instead of 0.01).

7.2. *Second numerical test: the Haverkampf–Celia problem [4,16]*

We consider here the case of a sand ground represented by the space domain $\Omega = (0,1) \times (0,0.7)$. The saturation and permeability functions are given by

$$\beta(P) := \begin{cases} \dfrac{\alpha(\beta_s - \beta_r)}{\alpha + |100\,P|^{\delta}} + \beta_r, & P < 0, \\ \beta_s, & P \geqslant 0, \end{cases}$$

$$K_\beta(P) := \begin{cases} K_s \dfrac{A}{A + |100\,P|^{\xi}}, & P < 0, \\ K_s, & P \geqslant 0, \end{cases}$$

where $\alpha = 1.611\text{E}6$, $\beta_s = 0.287$, $\beta_r = 0.075$ and $\delta = 3.96$, and where $K_s = 9.444444\text{E}{-5}$ m/s, $A = 1.175\text{E}6$ and $\xi = 4.74$. From β and K_β, we have tabulated suitable values for the functions c and K_c.

We have taken here $\gamma = 1$, the initial condition $P(x,z,0) = -0.614$, a homogeneous Neumann boundary condition for $x = 0$ and $x = 1$, the flux condition $K_\beta(P)(\partial P/\partial n) = 3.694823\text{E}{-7}$ for $z = 0$ and the Dirichlet boundary condition $P = -0.614$ for $z = 0.7$.

In this test case, which is in fact one-dimensional, no analytical solution is known. Therefore, we have compared our numerical solution with those found by Ramarosy [24] with a finite volume method and by Chounet et al. [5] with a mixed finite element method. Our results are quite similar to theirs and we also obtain a good stability when varying the time step.

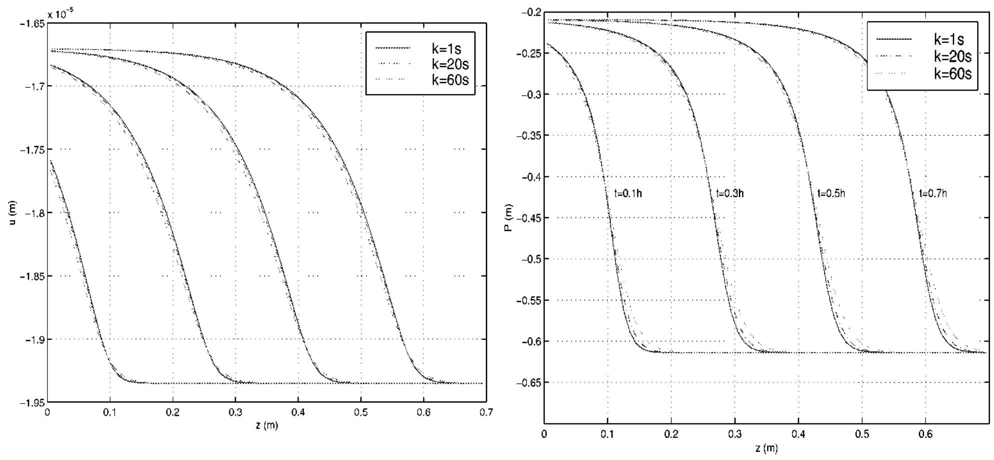

Figure 6. The front u and pressure P.

In figure 6 the function u and the pressure P, interpolated from the values of u, are respectively represented at the times $t = 0.1$ h, $t = 0.3$ h, $t = 0.5$ h and $t = 0.7$ h. We have taken here $\Delta z = 0.01$ and $k = 1$ s, 20 s and 60 s.

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