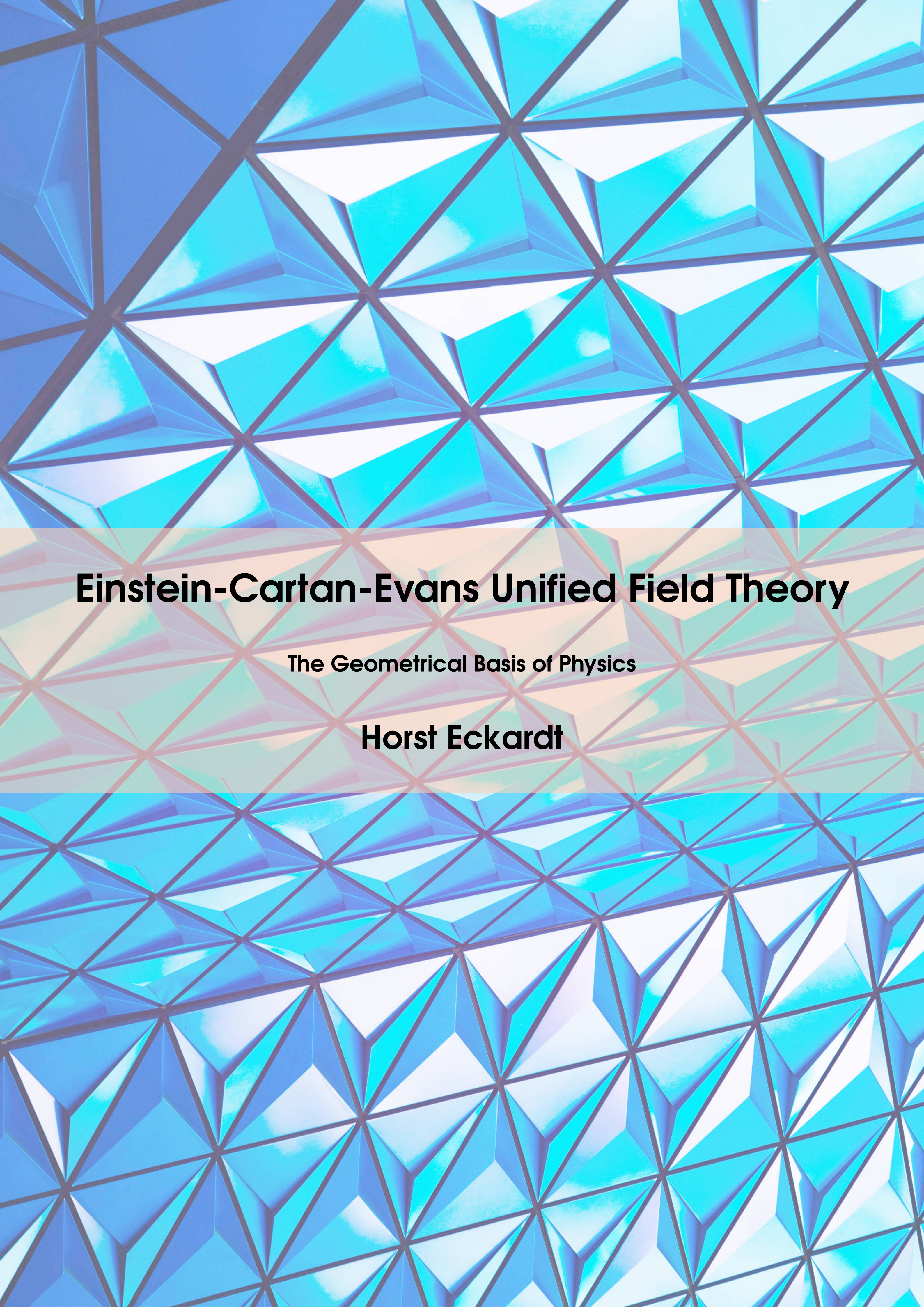

Einstein-Cartan-Evans Unified Field Theory

The Geometrical Basis of Physics

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In Memoriam
Myron W. Evans
(1950 - 2019)

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1. Introduction

Geometry is visible everywhere in daily life. It appears in objects that have been engineered in any form. We are familiar with geometry, since it has been used for centuries (Fig. 1.1). Also, in pure sciences like mathematics and physics, it plays an important role. The mathematical description of geometry consists of the logic elements of geometry itself, for example, the geometric constructions for triangles (Fig. 1.2). This type of logical treatment dates back to the beginning of recorded time, which is assumed be around 3500 B.C., when the first written documents appeared in Mesopotamia. For earlier times, we have to rely on documents of stone, like the pyramids in Egypt, which are probably much older than commonly assumed. The Cheops pyramid has been charted in detail, and correlations have been found to the circumference of the earth, hinting that geometry had had an important role even in the Stone Age. At that time, Europe had a flourishing Celtic culture, from which numerous stone relics exist, and the runes used by the druids were geometric signs.

Ancient philosophy, in particular natural philosophy, culminated in Greece. Pythagoras is said to have been the first founder of mathematics, and we all know the Pythagorean theorem. In Athens, where democracy was born, the “triumvirate” Socrates, Plato and Aristotle founded classical philosophy, starting at about 400 B.C. Their schools were valid for about a thousand years. Euclid, who wrote the pivotal treatise on geometric reasoning, Elements, was a member of the Platonic school.

During medieval times, knowledge from the Roman Empire was preserved by monasteries of the ecclesia and by Arabian philosophers and mathematicians. The Renaissance, which began in Italy in the 14th century and spread to the rest of Europe in the 15th and 16th centuries, was both a rebirth of ancient knowledge, and the beginning of modern empirical natural philosophy. This philosophy is connected with Galileo Galilei, who constituted the method of experimental proofs, and to Johann Kepler, who established our modern heliocentric model of the solar system, first presented as a hypothesis by Nicolaus Copernicus.

Since the 17th century, the mathematical description of physics has made great progress. Isaac Newton published the law of gravitation, which actually goes back to his mentor Robert Hooke. This represented huge progress in natural philosophy, because celestial events could now be predicted mathematically, although this has become completely possible only since the advent of computers.

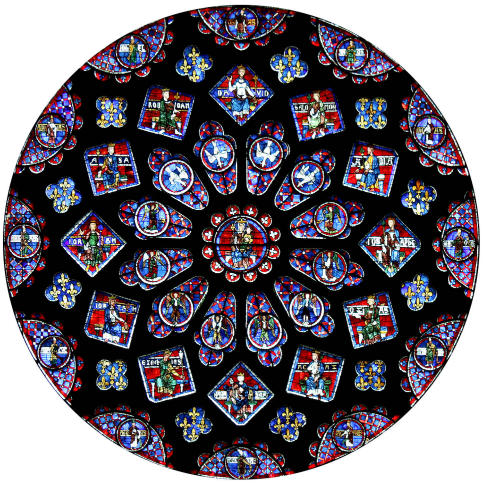


Figure 1.1: Example of geometry: rosette window in the cathedral of Chartres.

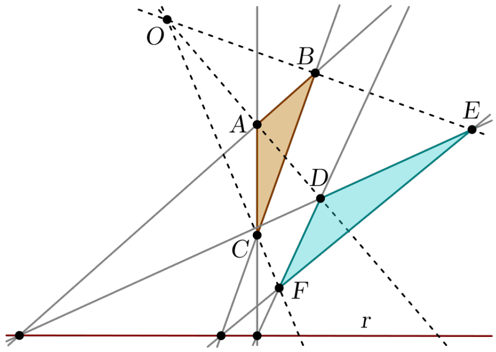


Figure 1.2: Example of geometry: triangles.

The 18th century was the golden age of mechanics. Newton's laws, and their generalizations by Lagrange and Hamilton, paved the way for mathematical physics for the next 300 years. Initially, geometry was used mainly for describing the motion of bodies and particles. The emergence of quantum mechanics in the 20th century extended geometry to the atomic and subatomic realm. For example, the atoms that comprise solids and molecules exhibit a geometrical structure (Fig. 1.3) which is essential for their macroscopic properties. Similar arguments hold for electrodynamics (see, e.g., Fig. 1.4). Faraday's lines of force describe a close-range effect, which was a basis for the geometrical description of electrodynamics, culminating in Maxwell's equations.

The use of geometry changed again at the beginning of the 20th century, when Einstein introduced his theory of general relativity, in which he based physics on non-Euclidean geometry. Gravitation was no longer described by a field imposed externally on space and time, but instead the "spacetime" itself was considered to be an object of description, and altered so that force-free bodies move on a virtually straight line (geodetic line) through space. Spacetime was considered to be curved, and the curving described the laws of gravitation. Along with this interpretation, geometry was considered to be an abstract concept described by numbers and mathematical functions. This approach is known as analytical geometry, and its simplest form uses coordinate systems and vectors.

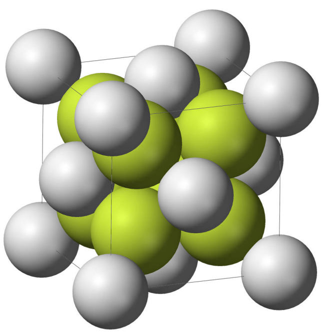


Figure 1.3: Example of geometry: unit cell of Fluorite crystal. White: Calcium, green: Fluorine.

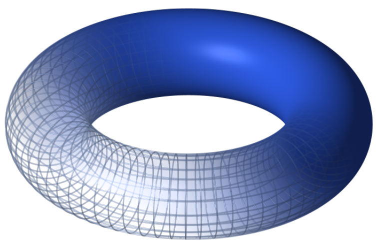


Figure 1.4: Example of geometry: torus.

Einstein's geometrical concept was the first paradigm shift in physics since Newton had introduced his laws of motion, 300 years earlier. Experimental validation of Einstein's general relativity has been rare, and has mainly concerned the solar system, like the deflection of light by the sun and the precession of the orbit of Mercury. In spite of this limitation, the theory was taken as a basis for cosmology, from which the existence of the big bang and dark matter was later extrapolated.

Unfortunately, this approach to cosmology has introduced self-contradictory inconsistencies. For example, the concept that the speed of light is an absolute upper limit is treated like a dogma in contemporary physics, and thus immune to rational argument. However, to explain the first expansive phase of the universe, one has to assume that this happened with an expansion velocity faster than the speed of light. This example is only one of the criticisms of Einstein that have yet to be answered properly and scientifically.

Later, after Einstein's death, the so-called velocity curve of galaxies was observed by astronomers. This means that stars in the outer arms of galaxies do not move according to Newton's law of gravitation, but have a constant velocity. However, Einstein's theory of general relativity is not able to explain this behavior. Both theories (Einstein and Newton) break down in cosmic dimensions. When a theory does not match experimental data, the scientific method requires that the theory be improved or replaced by a better concept. In the case of galactic velocity curves, however, it was "decided" that Einstein is right and that there has to be another reason why stars behave in this way. Dark matter that interacts through gravity and is distributed in a way that accounts for observed orbits was then postulated. Despite an intensive search for dark matter, even on the sub-atomic level, nothing has been found that could interact with ordinary matter through gravity, but not interact with observable electromagnetic radiation, such as light. Sticking with Einstein's theory seems to be a pipe dream, but nobody in the scientific community dares to abandon this non-working theory.

The members of the AIAS institute, Myron Evans at the head, took over the task of developing a new theory of physics that overcomes the problems in Einstein's general relativity. Shortly after the year 2000, Myron Evans developed the "Einstein Cartan Evans theory" (ECE theory [1, 2, 3]) as a replacement, and was even able to unify this with electrodynamics and quantum mechanics. This lead to significant progress in several fields of physics, and the most significant aspects are described in this text book.

ECE theory is based entirely on geometry, as was Einstein's general theory of relativity. Therefore, Einstein is included in the name of this new theoretical approach. Both theories take the geometry of spacetime (three space dimensions, plus one time dimension) as their basis. While Einstein thought that matter curves spacetime and assumed matter to be a "source" of fields, we will see that ECE theory is based entirely on the field concept and does not need to introduce external sources. This idea of sources created a number of difficulties in Einstein's theory.

Another reason for these difficulties is that Einstein made a significant mathematical error in his original theory (1905 to 1915), because all of the necessary information was not yet available. Riemann inferred the metric around 1850, and Christoffel inferred the idea of connection around the 1860s. The idea of curvature was inferred at the beginning of the twentieth century, by Levi Civita, Ricci, Bianchi and colleagues in Pisa. However, torsion was not inferred until the 1920s, by Cartan and his colleagues in Paris.

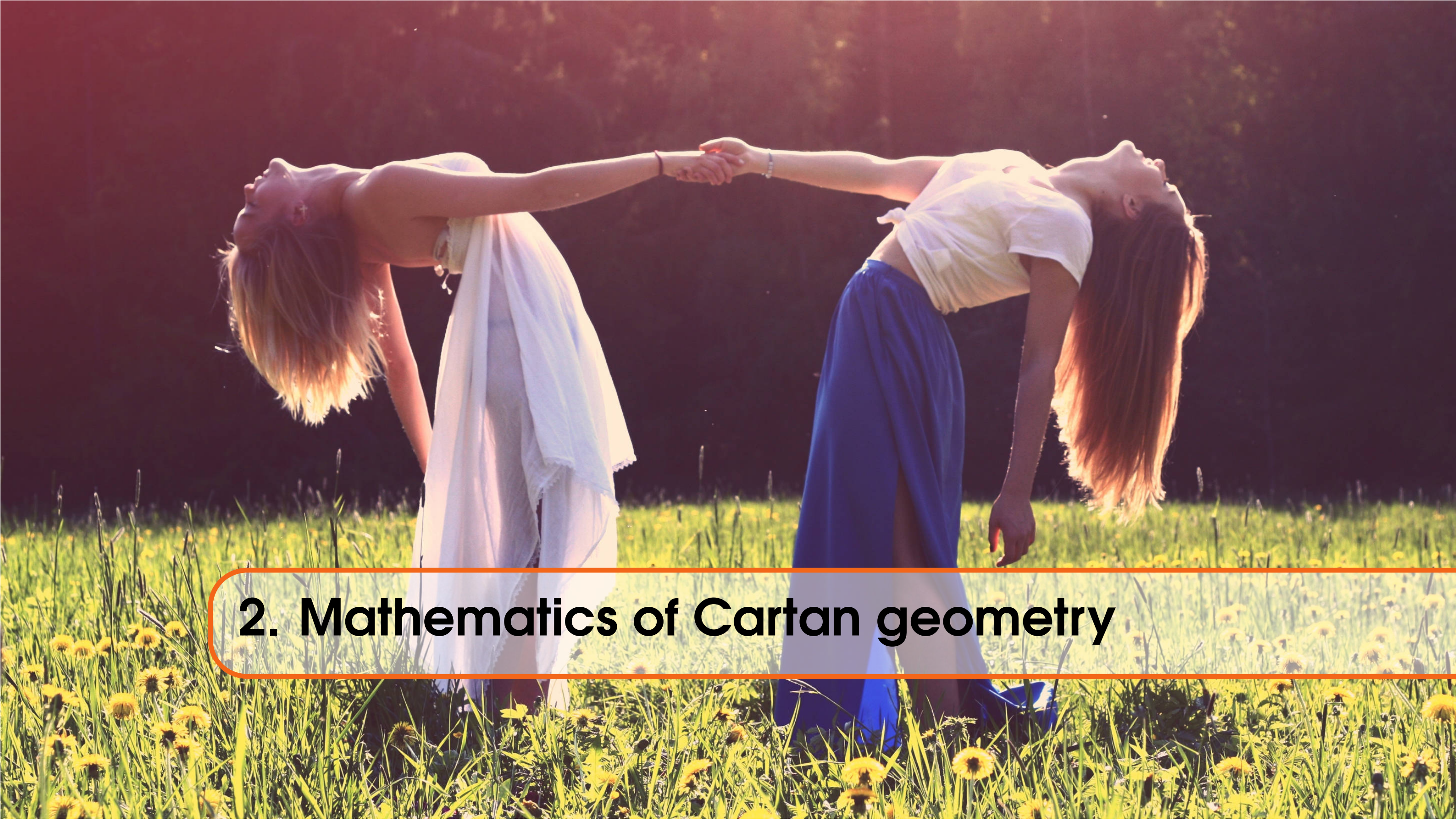
Therefore, in 1915, when Einstein published his field equations, Riemann geometry contained only curvature, and there was no way of determining that the Christoffel connection must be antisymmetric or at least asymmetric. The arbitrary decision to use a symmetric connection was made into an axiom, and the inferences of Einstein's theory ended up being based on incorrect geometry. Omission of torsion leads to many problems, as has been shown by the AIAS Institute, in great detail [4].

Torsion is a twisting of space, which turns out to be essential and inextricably linked to curvature, because if the torsion is zero then the curvature vanishes [4]. In fact, torsion is even more important than curvature, because the unified laws of gravitation and electrodynamics are basically physical interpretations of twisting, which is formally described by the torsion tensor.

ECE theory unifies physics by deriving all of it directly and deterministically from Cartan geometry, and doing so without using adjustable parameters. Spacetime is completely specified by curvature and torsion, and ECE theory uses these underlying fundamental qualities to derive all of physics from differential geometry, and to predict quantum effects without assuming them (as postulates) from the beginning. It is the first (and only) generally covariant, objective and causal unified field theory.

This book first introduces the mathematics on which ECE theory is based, so that the foundations of the theory can be explained systematically. Mathematical details are kept to a minimum, and explained only as far as is necessary to ensure understanding of the underlying Cartan geometry. This allows the fundamental ECE axioms and theorems to be introduced in a simple and direct manner. The same equations are shown to hold for electrodynamics, gravitation, mechanics and fluid dynamics, which places all of classical physics on common ground. Physics is then extended to the microscopic level by introducing canonical quantization and quantum geometry. The quantum statistics used is classically deterministic. There is no need for renormalization and quantum electrodynamics. All known effects, up to and including the structure of the vacuum, can be explained within the ECE axioms, which are based on Cartan geometry. This is the great advancement that this textbook will explain and clarify.

Part One: Geometry



2. Mathematics of Cartan geometry

2.1 Coordinate transformations

Before we can discuss the foundations of non-Cartesian and Cartan geometry on a mathematical level, we need to review the basics of analytical geometry.

2.1.1 Coordinate transformations in linear algebra

To start our discussion of geometry, we first recapitulate some basics of linear algebra. Cartan geometry is a generalization of these concepts, in a sense. Points in space are described by coordinates which are n-tuples for an n-dimensional vector space. The tuple components are numbers and describe how a point in space is reached by putting parts (for example yardsticks) in different directions together. The directions are called base vectors. For a three-dimensional Euclidian space we have the base vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.1}$$

A point with coordinates (X,Y,Z) is allocated to a vector

$$\mathbf{X} = X\mathbf{e}_1 + Y\mathbf{e}_2 + Z\mathbf{e}_3. \tag{2.2}$$

We have the freedom to choose any base in a vector space, rectangular or not, but when vector analysis is applied to the vector space, it is beneficial to have a rectangular basis. The basis vectors have to be normalized so that this is an orthonormal basis.

A question arises as to what happens when the basis vectors are changed. The position of points in the vector space should be independent of the basis, and we will encounter this fundamental requirement often in Cartan geometry. The coordinates will change when the basis changes. An important part of linear algebra deals with describing this mathematically. Taking the above basis vectors $\mathbf{e}_i$, a new basis $\mathbf{e'}_i$ in an n-dimensional vector space will be a linear combination of the original basis:

$$\mathbf{e}'_i = \sum_{j=1}^{n} q_{ij}\mathbf{e}_j \tag{2.3}$$

where the coefficients q_{ij} represent a matrix, the so-called transformation matrix. The above equation can therefore be written as a matrix equation

$$\begin{pmatrix} \mathbf{e}'_1 \\ \vdots \\ \mathbf{e}'_n \end{pmatrix} = \mathbf{Q} \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} \tag{2.4}$$

with

$$\mathbf{Q} = (q_{ij}) \tag{2.5}$$

and the unit vectors formally arranged in a column vector. $\mathbf{Q}$ must be of rank n and invertible. In Eq. (2.4) the unit vectors can be written with their components as row vectors. Denoting the j-th component of the unit vector $\mathbf{e}_i$ by $(e_i)_j = e_{ij}$, we then can set up a matrix from the unit vectors and write (2.4) in the form

$$\begin{pmatrix} e'_{11} & \dots & e'_{1n} \\ \vdots & & \vdots \\ e'_{n1} & \dots & e'_{nn} \end{pmatrix} = \mathbf{Q} \begin{pmatrix} e_{11} & \dots & e_{1n} \\ \vdots & & \vdots \\ e_{n1} & \dots & e_{nn} \end{pmatrix}. \tag{2.6}$$

Then the basis transformation is a matrix multiplication by $\mathbf{Q}$. The matrix for the inverse transformation is obtained by multiplying (2.4) or (2.6) by the inverse matrix $\mathbf{Q}^{-1}$:

$$\begin{pmatrix} e_{11} & \dots & e_{1n} \\ \vdots & & \vdots \\ e_{n1} & \dots & e_{nn} \end{pmatrix} = \mathbf{Q}^{-1} \begin{pmatrix} e'_{11} & \dots & e'_{1n} \\ \vdots & & \vdots \\ e'_{n1} & \dots & e'_{nn} \end{pmatrix}. \tag{2.7}$$

Multiplying $\mathbf{Q}$ with $\mathbf{Q}^{-1}$ gives the unit matrix which can be expressed by the Kronecker symbol:

$$\mathbf{Q}\,\mathbf{Q}^{-1} = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{pmatrix} = (\delta_{ij}). \tag{2.8}$$

■ **Example 2.1** The rotation of bases by an angle ϕ in a two-dimensional vector space can be described by the rotation matrix

$$\mathbf{Q} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{2.9}$$

The basis of unit vectors $(1,0)$, $(0,1)$ is then transformed to the new basis vectors

$$\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \tag{2.10}$$

this means

$$\mathbf{e}'_1 = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad \mathbf{e}'_2 = \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}. \tag{2.11}$$

Both basis sets are depicted in Fig. 2.1. ■

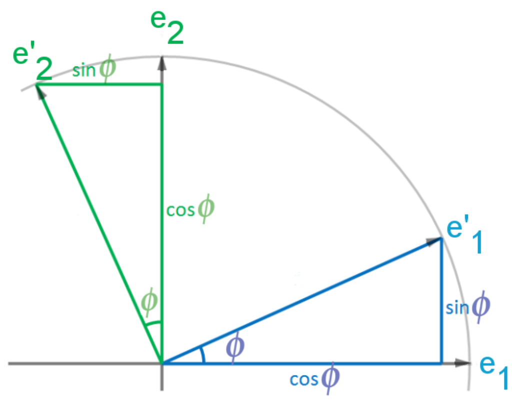


Figure 2.1: Basis vector rotation by an angle ϕ.

Now that we understand the basis transformation, we want to find the transformation law for vectors. The components of vectors in one base, the coordinates, are transformed to the components in another base. From the definition (2.2) a vector with coordinates x_i can be written as

$$\mathbf{X} = \sum_i x_i \mathbf{e}_i \tag{2.12}$$

and may be transformed to a representation in a second basis with coordinates x'_i:

$$\mathbf{X}' = \sum_i x'_i \mathbf{e}'_i. \tag{2.13}$$

Since the vector should remain the same in both bases, we can set $\mathbf{X} = \mathbf{X}'$. Inserting the basis transformations into this relation, one finds that the transformation law of coordinates is

$$\mathbf{X}' = \mathbf{Q}^{-1}\,\mathbf{X} \tag{2.14}$$

or in coordinates:

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix} = \mathbf{Q}^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \tag{2.15}$$

We notice the important result that the coordinates transform with the inverse matrix compared to the basis vectors and vice versa.

■ **Example 2.2** The transformation matrix of coordinates for the rotation in two dimensions is

$$\mathbf{Q}^{-1} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{2.16}$$

This can easily be seen because the reverse rotation is by an angle $-\phi$. Then the sine function reverses sign but the cosine function does not. The vectors on the basis axes are transformed to

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \cos\phi \\ -\sin\phi \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix}. \tag{2.17}$$

■

Comparing with (2.9), we see that the columns of the transformation matrix $\mathbf{Q}$ represent the coordinates of the transformed unit *vectors*, not the *basis*. In general:

$$\begin{pmatrix} x_{11} & \dots & x_{n1} \\ \vdots & & \vdots \\ x_{1n} & \dots & x_{nn} \end{pmatrix} = \mathbf{Q}^{-1} \begin{pmatrix} 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{pmatrix} \tag{2.18}$$

where $x_{ij} = (x_i)_j$ denotes the j$^{\text{th}}$ component of the transformed unit vector $\mathbf{e}_i$. Please notice that the index scheme for x_{ij} is transposed compared to the usual matrix definition.

2.1.2 General coordinate transformations and coordinate differentials

In the framework of general relativity, coordinate transformations are mappings from one vector space to another. These mappings are multidimensional functions. In the preceding section we restricted ourselves to linear transformations (or mappings), while in general relativity we operate with nonlinear transformations.

Space is described by a four-dimensional manifold, using advanced mathematics. However, in this book we do not develop these concepts in any great extent, but only explain the parts that are required for a basic understanding. The mathematical details can be found in textbooks on general relativity, for example, see [5]-[9].

In this book, we use one time-coordinate plus three space-coordinates for general relativity, with indices numbered from 0 to 3. Such vectors are also called 4-vectors. The functions and maps (later: the tensors) defined on this base space are functions of the coordinates: $f(x_i)$, i=0...3. In particular, coordinate transformations can be described in this form. Let's consider two coordinate systems A and B which describe the same space and are related by a nonlinear transformation. Let X_i be the components of a 4-vector $\mathbf{X}$ in space A and Y_i the components of a 4-vector $\mathbf{Y}$ in space B. The coordinate transformation function $f : \mathbf{X} \to \mathbf{Y}$ then can be expressed as a functional dependence of the components:

$$Y_i = f_i(X_j) = Y_i(X_j) \tag{2.19}$$

for all components i of f and all pairs i, j. In the following, we consider the transformations between a rectangular, orthonormal coordinate system, defined by basis vectors $(1,0,...),(0,1,...)$, etc., and coordinates

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \tag{2.20}$$

and a curvilinear coordinate system with coordinates

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}. \tag{2.21}$$

The transformation functions from the curvilinear to the cartesian coordinate system may be defined by

$$X_i = X_i(u_j) \tag{2.22}$$

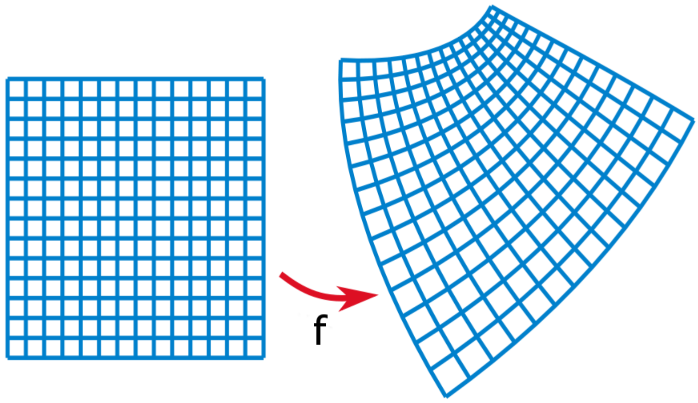


Figure 2.2: Transformation to curvilinear coordinates.

as discussed above. The inverse transformations define the so-called coordinate functions of u:

$$u_i = u_i(X_j). \tag{2.23}$$

The functions $u_i = constant$ define coordinate surfaces, see Fig. 2.2, for example.

The degree of change in each direction is given by the change of arc length and is expressed by the *scale factors*

$$h_i = \left|\frac{\partial \mathbf{X}}{\partial u_i}\right|. \tag{2.24}$$

The unit vectors in the curvilinear space are computed by

$$\mathbf{e}_i = \frac{1}{h_i}\frac{\partial \mathbf{X}}{\partial u_i}. \tag{2.25}$$

The *tangent vector* of the coordinate curves at each point of space is defined by

$$\nabla u_i = \sum_j \frac{\partial u_i}{\partial X_j}\mathbf{e}_j. \tag{2.26}$$

We require that curvilinear coordinate system be orthonormal at each point of space. This can be assured by the condition that the tangent vectors of the coordinate curves at each point fulfill the requirement

$$\nabla u_i \cdot \nabla u_j = \delta_{ij}. \tag{2.27}$$

The scale factors can alternatively be expressed by the modulus of the tangent vector:

$$h_i = \frac{1}{|\nabla u_i|}. \tag{2.28}$$

■ **Example 2.3** We consider the transformation from cartesian coordinates to spherical coordinates in Euclidean space. The curvilinear coordinates of a point in space are (r, θ, ϕ), where r is the radius, θ the polar angle and ϕ the azimuthal angle, see Fig. 2.3. The cartesian coordinates are (X, Y, Z). The transformation equations from the curvilinear to the rectangular coordinate system are

$$\begin{aligned} X &= r\sin\theta\cos\phi \\ Y &= r\sin\theta\sin\phi \\ X &= r\cos\theta \end{aligned} \tag{2.29}$$

and the inverse transformations are

$$\begin{aligned}
u_r &= r = \sqrt{X^2+Y^2+Z^2} \\
u_\theta &= \theta = \arccos \frac{Z}{\sqrt{X^2+Y^2+Z^2}} \\
u_\phi &= \phi = \arctan \frac{Y}{X}.
\end{aligned} \tag{2.30}$$

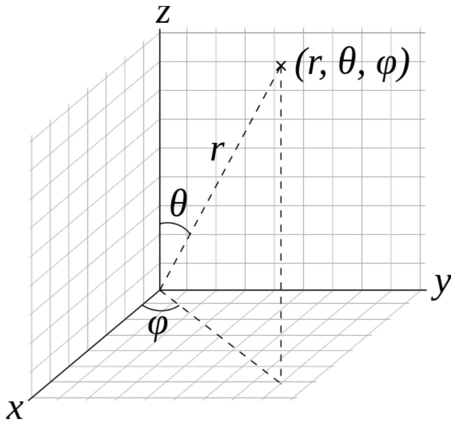


Figure 2.3: Spherical polar coordinates[1]

The vector of scale factors (2.24) is

$$\mathbf{h} = \begin{pmatrix} 1 \\ r \\ r\sin\theta \end{pmatrix} \tag{2.31}$$

and the matrix of column unit vectors (2.25) is

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \begin{pmatrix} \cos\phi\,\sin\theta & \sin\phi\,\sin\theta & \cos\theta \\ \cos\phi\,\cos\theta & \sin\phi\,\cos\theta & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}. \tag{2.32}$$

The components of $\mathbf{h}$ have to be positive. The sine function may have positive and negative values, but in spherical coordinates the range of θ is between 0 and π, therefore this function is always positive. ■

As explained, the coordinate systems are chosen in a way that ensures that the length of vectors is conserved. This must also hold for time-dependent processes. For example, a distance vector changing over time is

$$\Delta\mathbf{X} = \mathbf{v}\Delta t - \mathbf{X}_0 \tag{2.33}$$

where $\mathbf{v}$ is the velocity vector of a mass point and $\mathbf{X}_0$ is an offset. The squared distance is

$$s^2 = v^2\Delta t^2 - (\mathbf{X}_0)^2. \tag{2.34}$$

We notice that a minus sign appears in front of the space part of s^2. This is different from pure "static" Euclidean 3-space, where we have

$$s_E^2 = X^2+Y^2+Z^2. \tag{2.35}$$

[1]Source: Wikimedia Commons - https://commons.wikimedia.org/wiki/File:3D_Spherical.svg#/media/File:3D_Spherical.svg

Now we generalize Eq. (2.34). When the differences in time as well as in space between two points are infinitesimally different, we can write the distance between these points with coordinate differentials:

$$ds^2 = c\ dt^2 - dX^2 - dY^2 - dZ^2 \tag{2.36}$$

where ds is the differential line element. We have added a factor c to the time coordinate t so that all coordinates have the physical dimension of length. In the same way, we can express the line element in another coordinate system, say u coordinates:

$$ds^2 = (du_0)^2 - (du_1)^2 - (du_2)^2 - (du_3)^2. \tag{2.37}$$

So far we have dealt with a Euclidian 3-space, augmented by a time component. More generally, the above equations can be written in the form

$$ds^2 = \sum_{ij} \eta_{ij}\ dx_i\ dx_j \tag{2.38}$$

where η_{ij} represents a matrix of constant coefficients directly leading to the result (2.36) or (2.37):

$$(\eta_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.39}$$

Formally, we can write the coordinates as a 4-column vector

$$(x^\mu) = \begin{pmatrix} ct \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} \tag{2.40}$$

where μ runs from 0 to 3 and is written as an upper index. We can do the same for the coordinate differentials:

$$(dx^\mu) = \begin{pmatrix} c\,dt \\ dX^1 \\ dX^2 \\ dX^3 \end{pmatrix}. \tag{2.41}$$

At this point, we should notice that the determinant of the matrix (2.39) is -1. The η matrix is called the *metric* of the space, here the time-extended flat Euclidean space, also called *Minkowski space*. Obviously, the metric is negative definite. Sometimes η is defined with reverse signs but the result is the same. At this point, we enter the realm of special relativity, but we need not deal with Lorentz transformations in this book. Since the spacetime metric is an essential physical quantity in general relativity as well as in ECE theory, we introduce special relativity only under under the view point that the line element ds is independent of the coordinate system. Later we will see that this leads to the gamma factor of special relativity. This is the only formalism in common between Einstein's relativity and ECE theory. We will come back to this when physical situations are considered where very high velocities occur. This requires a relativistic treatment (in the sense of special relativity).

2.1.3 Transformations in curved spaces

So far, we have done linear algebra in Euclidean spaces, but now we are extending the concepts of the preceding section to curved spaces. This means that equidistant coordinate values do not describe line elements equal in length. But we should be warned: Using such coordinate systems does not mean that space is “curved” in any way. According to Example 2.1 in the preceding section, a curvilinear coordinate system can perfectly describe a Euclidean “flat” space.

Below, we consider two coordinate systems existing in the same space, denoted by primed and un-primed differentials dx and dx'. According to Eqs. (2.22, 2.23) we have a functional dependence between both coordinates:

$$x^\mu = x^\mu(x'^\nu) \tag{2.42}$$

and

$$x'^\nu = x'^\nu(x^\mu). \tag{2.43}$$

Differentiating these equations gives

$$dx'^\mu = \sum_\nu \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu, \tag{2.44}$$

$$dx^\mu = \sum_\nu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu. \tag{2.45}$$

To make these equations similar to the transformations in linear algebra (see section 2.2.2), we define transformation matrices

$$\alpha^\mu_\nu = \frac{\partial x^\mu}{\partial x'^\nu}, \tag{2.46}$$

$$\overline{\alpha}^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \tag{2.47}$$

so that any vector V with components V^μ in one coordinate system can be transformed to a vector V' in the other coordinate system by

$$V'^\mu = \alpha^\mu_\nu \, V^\nu, \tag{2.48}$$

$$V^\mu = \overline{\alpha}^\mu_\nu \, V'^\nu. \tag{2.49}$$

These matrices, however, are not elements of linear algebra but matrix functions, because we are not working with linear transformations. α is the inverse matrix function of $\overline{\alpha}$ and vice versa. This means:

$$\sum_\rho \alpha^\mu_\rho \, \overline{\alpha}^\rho_\nu = \delta^\mu_\nu \tag{2.50}$$

with the Kronecker delta

$$\delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}. \tag{2.51}$$

Here we have written α with an upper and lower index intentionally. This allows us to introduce the *Einstein summation convention*: if the same index appears as an upper and a lower index on one side of an equation, this index is summed over. Such an index is also called a *dummy index*. We will use this feature intensively, when tensors are introduced later. With this convention, which we will use without notice in the future, we can write:

$$\alpha^{\mu}_{\rho}\overline{\alpha}^{\rho}_{\nu} = \delta^{\mu}_{\nu}. \tag{2.52}$$

Since space is not necessarily flat, the metrical coefficients of (2.39) are not constant, and non-diagonal terms may appear. This general metric is conventionally called $g_{\mu\nu}$ and defined by the line element as before:

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu}. \tag{2.53}$$

For a flat space with cartesian coordinates we have

$$g_{\mu\nu} = \eta_{\mu\nu}. \tag{2.54}$$

■ **Example 2.4** We compute an example for a transformation matrix. Using example 2.3 (transformation between cartesian coordinates and spherical polar coordinates), we have by (2.29), (2.30):

$$\begin{aligned} x^1 &= r\sin\theta\cos\phi \\ x^2 &= r\sin\theta\sin\phi \\ x^3 &= r\cos\theta \end{aligned} \tag{2.55}$$

and the inverse transformations

$$\begin{aligned} x'^1 &= r = \sqrt{(x^1)^2+(x^2)^2+(x^3)^2} \\ x'^2 &= \theta = \arccos\frac{x^3}{\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}} \\ x'^3 &= \phi = \arctan\frac{x^2}{x^1}. \end{aligned} \tag{2.56}$$

The transformation matrix is according to (2.46):

$$\alpha^1_1 = \frac{\partial x^1}{\partial x'^1} = \frac{\partial}{\partial r}(r\sin\theta\cos\phi) = \sin\theta\cos\phi \tag{2.57}$$

$$\alpha^1_2 = \frac{\partial x^1}{\partial x'^2} = \frac{\partial}{\partial \theta}(r\sin\theta\cos\phi) = r\cos\theta\cos\phi$$

etc. ...

resulting in the 3x3 matrix

$$\alpha = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}. \tag{2.58}$$

Obviously, this matrix is not symmetric and even has a zero on the main diagonal. Nonetheless, it is of rank 3 and is invertible, as can be checked. We omit the details here, since the inverse matrix is a bit complicated. The determinant of α is $r^2\sin\theta$, the determinant of the inverse matrix $\overline{\alpha}$ is $1/(r^2\sin\theta)$. By insertion, one can check that

$$\alpha\cdot\overline{\alpha} = \mathbf{1}. \tag{2.59}$$

This example is available as code for the computer algebra system Maxima [23]. ■

■ **Example 2.5** As a further example, we will compute the metric of the coordinate transformation of the previous example (2.4), see code [24]. So far, we have no formal method given to do this. The simplest way for Euclidean spaces is the method going back to Gauss. If the metric **g** (a matrix) is known for one coordinate system x^μ, the invariant line element of a surface (which is hypothetical in our case) is given by

$$ds^2 = \left[dx^1 dx^2 dx^3\right] \mathbf{g} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{bmatrix}. \tag{2.60}$$

The metrical matrix belonging to another coordinate system x'^μ is then computable by

$$\mathbf{g}' = \mathbf{J}^T\, \mathbf{g}\, \mathbf{J} \tag{2.61}$$

where **J** is the Jacobian of the coordinate transformation:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^1}{\partial x'^3} \\ \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^2}{\partial x'^3} \\ \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^2} & \frac{\partial x^3}{\partial x'^3} \end{bmatrix}. \tag{2.62}$$

Comparing this with Eq. (2.57), we see that the transformation matrix α is identical with the Jacobian, so we can also write:

$$\mathbf{g}' = \alpha^T\, \mathbf{g}\, \alpha. \tag{2.63}$$

The metric of the cartesian coordinates is simply

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.64}$$

and can be inserted into (2.63), together with α from the preceding example. The result is

$$\mathbf{g}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{bmatrix} \tag{2.65}$$

for the metric of the spherical coordinates. Written as the line element, this is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2. \tag{2.66}$$

The metric is symmetric in general, and diagonal in most relevant cases. We will learn other methods of determining the metric in curved spaces during the course of this book. ■

2.2 Tensors

Now that we have explained coordinate transformations and their matrix representations, including the metric, to some extent, we will extend this formalism from vectors to tensors. First, we have to define what a tensor is, and then we can see how they are transformed.

In section 1.1.3 we introduced the formalism of writing matrices and vectors by indexed quantities, with upper or lower index, where this position was chosen more or less arbitrarily, for example to fulfill the Einstein summation convention. Now let's introduce k-dimensional objects (k ranging from 0 to any integer number) with upper and lower indices of the form

$$T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m}. \tag{2.67}$$

T has n upper indices μ_i and m lower indices ν_i with $n+m=k$. It is not required that all upper indices appear first, for example

$$T^{3}_{1}{}^{30} \tag{2.68}$$

is a valid object. The indices μ_i,ν_i represent the coordinate indices for each dimension, ranging from 0 to k-1 by definition. In the above example we have $k=4$, so

$$T^{3}_{5}{}^{40} \tag{2.69}$$

would not be a valid object. For $k=2$ such an object represents a matrix, for $k=1$ a vector and for $k=0$ (without index) a scalar value. A tensor is defined by objects of type (2.67) which adhere to a certain transformation behavior of the upper and lower indices. Given a coordinate transformation α^{μ}_{ρ} between two coordinate systems, this transformation has to be applied for each index of a tensor separately. For example, a 2-dimensional tensor T may be transformed to T' by

$$T'^{\mu\nu} = \alpha^{\mu}_{\rho}\ \alpha^{\nu}_{\lambda}\ T^{\rho\lambda}. \tag{2.70}$$

We further require that for lower indices we use the inverse transformation matrices:

$$T'_{\mu\nu} = \overline{\alpha}^{\rho}_{\mu}\ \overline{\alpha}^{\lambda}_{\nu}\ T_{\rho\lambda} \tag{2.71}$$

and, consequently, for mixed cases:

$$T'^{\mu}{}_{\nu} = \alpha^{\mu}_{\rho}\ \overline{\alpha}^{\lambda}_{\nu}\ T^{\rho}{}_{\lambda}\ . \tag{2.72}$$

Please notice that the α matrices are defined by the differentials of the transformation, see Eqs. (2.44, 2.45).

In section 2.1.1 we have seen that, if α^{μ}_{ρ} transforms the basis vectors, then the inverted matrix $\overline{\alpha}^{\nu}_{\lambda}$ transforms the coordinates of vectors. Therefore, the upper indices of tensors transform like coordinates, while the lower indices transform like the basis. Upper indices are also called *contravariant indices*, while lower indices are called *covariant indices*. A tensor containing both types of indices is called a *mixed index tensor*.

We conclude this section with the hint that the metric introduced in the previous section is also a tensor. Mathematically, more precisely, we would restrict the tensors then to live in metric spaces, but we won't bother too much with mathematical details in this textbook. The metric $g_{\mu\nu}$ in curved spaces is a symmetric matrix and a tensor of dimension 2. The inner product of two vectors v, w can be written with aid of the metric:

$$s = g_{\mu\nu}\ v^{\mu} w^{\nu}. \tag{2.73}$$

In Euclidean space with Cartesian coordinates, g is the unit matrix as demonstrated in Example (2.5). Indices of arbitrary tensors can be moved up and down via the relations

$$T^{\mu}{}_{\nu}\ = g_{\nu\rho}\ T^{\mu\rho} \tag{2.74}$$

and

$$T^{\mu}{}_{\nu}\ = g^{\mu\rho}\ T_{\rho\nu} \tag{2.75}$$

where $g^{\mu\rho}$ is the inverse metric:

$$g^{\mu\rho} g_{\nu\rho} = \delta^{\mu}{}_{\nu}\ . \tag{2.76}$$

■ **Example 2.6** We present several tensor operations. Tensors can be multiplied. Then the product has the union set of indices, for example

$$A^{\mu\nu}B_{\rho} = C^{\mu\nu}{}_{\rho}. \tag{2.77}$$

The order of multiplication of A and B plays a role. Therefore, such a product is only meaningful for tensors with a certain symmetry, for example the product tensor of two vectors:

$$v^{\mu}w^{\nu} = C^{\mu\nu}. \tag{2.78}$$

Here C is a symmetric tensor, i.e.

$$C^{\mu\nu} = C^{\nu\mu}. \tag{2.79}$$

Only tensors with the same rank can be added:

$$A^{\mu}{}_{\nu} + B^{\rho}{}_{\sigma} = C^{\alpha}{}_{\beta}. \tag{2.80}$$

The equation

$$A^{\mu}{}_{\nu} + B^{\rho\sigma}{}_{\tau} \;=?\quad C^{\alpha\beta}{}_{\tau} \tag{2.81}$$

is not compatible with the definition of tensors, and is therefore wrong. For further examples, see [5]. ■

2.3 Base manifold and tangent space

Now that we have seen an overview of the tensor formalism, we will consider the spaces on which these tensors are operating. A tensor can be considered as a function, for example

$$T^{\mu}{}_{\nu}\ : \mathbb{R}^4 \to \mathbb{R}^2 \tag{2.82}$$

which maps a 4-vector to a two-dimensional tensor field:

$$[ct, X, Y, Z] \to T^{\mu}{}_{\nu}\,(ct, X, Y, Z) \tag{2.83}$$

where the two indices of the tensor indicate that the image map is two-dimensional. We speak of "tensor field" in cases where a continuous argument range is mapped to a continuous image range which is different from the argument set. For example, T could be an electromagnetic field which is defined at each point of 4-space. If the set of arguments is not Euclidean, we require that, at each point of the argument set, a local neighborhood exists, which is homomorphous to an open subset of $\mathbb{R}^n$ where n is the dimension of the argument set. This is then called a *manifold*. Applying multiple tensor functions to a manifold means that several *maps* of the manifold exist. It is further required that the manifold is differentiable because we want to apply the differential calculus later. Assume that a point P is located within the valid local range of two different coordinate systems. Then the manifold is differentiable in P, if the Jacobian of the transformation between both coordinate systems is of rank n, the dimension of the manifold. For definition of scalar products, lengths, angles and volumes we need a metric structure for "measurements", and this requires the existence of a metric tensor. A differentiable manifold with a metric tensor is called *Riemannian manifold*.

■ **Example 2.7** In Fig. 2.4 an example for a 2-dimensional manifold is given: the surface of the earth. The geometry is non-Euclidean. For large triangles on the earth's surface, the sum of angles is different from 180°. A small region is mapped to a flat area where Euclidean geometry is re-established. This can be done for each point of a manifold within a neighborhood, but not globally for the whole manifold. ■

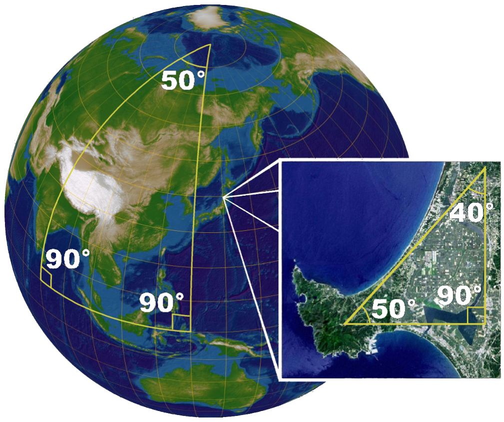


Figure 2.4: 2-dimensional manifold and mapping of a section to a plane segment.

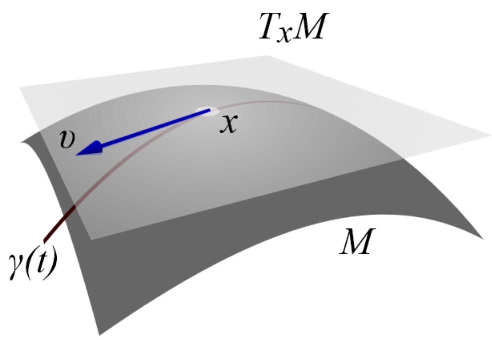


Figure 2.5: Tangential vector v to a 2-dimensional manifold M.

At each point of such a manifold a *tangential space* can be defined. This is a flat $\mathbb{R}^n$ space with the same dimension as the manifold. In Fig. 2.5 an example of a 2-dimensional manifold and tangent space is depicted. The manifold is denoted by M and the tangent space at the point x by T_xM. Such a tangent space (a plane) for example occurs for the motion of mass points along an orbital curve $\gamma(t)$.

The manifold can be covered by points with local neighborhoods and corresponding tangent spaces in each of these points. The set of all tangent spaces is called the *tangent bundle*. Changing the coordinate systems within the manifold means that the mapping from the manifold to the tangential space has to be redefined. A scalar product can be defined in the tangential space by use of the metric of the manifold.

Now we want to make the definition of tangent space independent of the choice of coordinates. The tangent space T_xM at a point x in the manifold can be identified with the space of directional derivative operators along curves through x. The partial derivatives $\frac{\partial}{\partial x^\mu} = \partial_\mu$ represent a suitable basis for the vector space of directional derivatives, which we can therefore safely identify with the tangent space.

Consider two manifolds M and N and a function $F: M \to N$ for a mapping of points of M to points in N. In M and N no differentiation is defined. However, we can define *coordinate charts* from the manifolds to their corresponding tangent spaces. These are the functions denoted by ϕ and ψ in Fig. 2.6. The coordinate charts allow us to construct a map between both tangent spaces:

$$\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n. \tag{2.84}$$

With aid of this construct we can define a partial derivative of f exploiting the indirection via the tangent spaces. For a point x^μ in $\mathbb{R}^m$ (the mapped point x of M) we define:

$$\frac{\partial f}{\partial x^\mu} := \frac{\partial}{\partial x^\mu}(\psi \circ f \circ \phi^{-1})(x^\mu). \tag{2.85}$$

In many application cases we have a curve in the manifold M described by a parameter λ. This could be the motion of a mass point in dependence of time. Similarly, as above, we can define the derivative of function f according to λ by using the chain rule:

$$\frac{df}{d\lambda} := \frac{dx^\mu}{d\lambda}\partial_\mu f. \tag{2.86}$$

As can be seen, there is a summation over the indices μ and the ∂_μ can be considered as a basis of the tangent space. This is sometimes applied in mathematical textbooks (for example [10]).

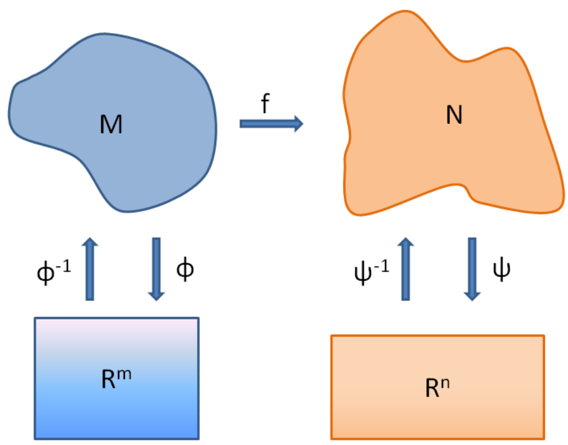


Figure 2.6: Mapping between two manifolds and tangent spaces.

n-forms

There is a special class of tensors, called n-forms. These comprise all completely anti-symmetric covariant tensors. In an n-dimensional space, there are 0-forms, 1-forms, ..., n-forms. All higher forms are zero by the antisymmetry requirement. A 2-form F can be constructed, for example, by two 1-forms (co-vectors) a and b:

$$F_{\mu\nu} = \frac{1}{2}(a_\mu b_\nu - a_\nu b_\mu). \tag{2.87}$$

By index raising this can be rewritten to the form

$$F'^{\mu\nu} = \frac{1}{2}(a'^\mu b'^\nu - a'^\nu b'^\mu) \tag{2.88}$$

with

$$a'^\mu = g^{\mu\nu} a_\nu, \text{ etc.} \tag{2.89}$$

Introducing a square bracket for an antisymmetric index permutation:

$$[\mu\nu] \rightarrow \mu\nu - \nu\mu \tag{2.90}$$

we can also write this in the form

$$F_{\mu\nu} = \frac{1}{2} a_{[\mu} b_{\nu]}. \tag{2.91}$$

In general, we can define

$$T_{[\mu_1\mu_2\dots\mu_n]} = \frac{1}{n!}(T_{\mu_1\mu_2\dots\mu_n} + \text{alternating sum over permutations of } \mu_1\dots\mu_n). \tag{2.92}$$

The antisymmtric tensor may contain further indices which are not permuted.

■ **Example 2.8** Consider a tensor $T_{\mu\nu\rho\sigma}{}^{\tau}$ being antisymmetric in the first three indices. Then we have

$$T_{[\mu\nu\rho]\sigma}{}^{\tau} = \frac{1}{6}(T_{\mu\nu\rho\sigma}{}^{\tau} - T_{\mu\rho\nu\sigma}{}^{\tau} + T_{\rho\mu\nu\sigma}{}^{\tau} - T_{\nu\mu\rho\sigma}{}^{\tau} + T_{\nu\rho\mu\sigma}{}^{\tau} - T_{\rho\nu\mu\sigma}{}^{\tau}). \tag{2.93}$$

By utilizing the antisymmetry of the first two indices, we can simplify this expression to

$$\begin{aligned} T_{[\mu\nu\rho]\sigma}{}^{\tau} =& \frac{1}{6}\left(T_{\mu\nu\rho\sigma}{}^{\tau} - (-T_{\mu\nu\rho\sigma}{}^{\tau}) + T_{\rho\mu\nu\sigma}{}^{\tau} - (-T_{\rho\mu\nu\sigma}{}^{\tau}) + T_{\nu\rho\mu\sigma}{}^{\tau} - (-T_{\nu\rho\mu\sigma}{}^{\tau})\right) \\ =& \frac{1}{3}\left(T_{\mu\nu\rho\sigma}{}^{\tau} + T_{\rho\mu\nu\sigma}{}^{\tau} + T_{\nu\rho\mu\sigma}{}^{\tau}\right). \end{aligned} \tag{2.94}$$

This is the sum of indices μ, ν, ρ cyclically permuted. ■

With the help of antisymmetrization, we can define the *exterior product* or *wedge product*. Given a p-form a and q-form b, we define the antisymmetric product by the $\wedge$ (*wedge*) operator:

$$(a \wedge b)_{\mu_1\dots\mu_{p+q}} := \frac{(p+q)!}{p!q!} a_{[\mu_1\dots\mu_p} b_{\mu_{p+1}\dots\mu_{p+q}]}. \tag{2.95}$$

For example, the wedge product of two 1-forms is

$$(a \wedge b)_{\mu\nu} = 2a_{[\mu}b_{\nu]} = a_\mu b_\nu - a_\nu b_\mu. \tag{2.96}$$

The wedge product is associative:

$$(a \wedge (b+c))_{\mu\nu} = (a \wedge b)_{\mu\nu} + (a \wedge c)_{\mu\nu}. \tag{2.97}$$

Mathematicians like to omit the indices if it is clear that an equation is written for forms. Thus the last equation can also be written as

$$a \wedge (b+c) = a \wedge b + a \wedge c \tag{2.98}$$

in a short-hand notation. Another property is that wedge products are not commutative. For a p-form a and a q-form b it is

$$a \wedge b = (-1)^{pq} b \wedge a \tag{2.99}$$

and for a 1-form:

$$a \wedge a = 0. \tag{2.100}$$

These features may justify the name "exterior product" as a generalization of a vector product in three dimensions.

An important operation on forms is applying the *Hodge dual*. First we have to define the Levi-Civita symbol in n dimensions:

$$\varepsilon_{\mu_1\dots\mu_n} = \begin{cases} 1 & \text{if } \mu_1\dots\mu_n \text{ is an even permutation of } 0,\dots(n-1), \\ -1 & \text{if } \mu_1\dots\mu_n \text{ is an odd permutation of } 0,\dots(n-1), \\ 0 & \text{otherwise.} \end{cases} \tag{2.101}$$

The determinant of a matrix can be expressed by this symbol. If $M^{\mu}_{\mu'}$ is a $n \times n$ matrix, the determinant $|M|$ obeys the relation

$$\varepsilon_{\mu'_1 \dots \mu'_n} |M| = \varepsilon_{\mu_1 \dots \mu_n} M^{\mu_1}_{\mu'_1} \cdots M^{\mu_n}_{\mu'_n} \tag{2.102}$$

or, restricting to one permutation at the left-hand side:

$$|M| = \varepsilon_{\mu_1 \dots \mu_n} M^{\mu_1}_{1} \cdots M^{\mu_n}_{n}. \tag{2.103}$$

The Levi-Civita symbol is defined in any coordinate system in the same way, not undergoing a coordinate transformation. Therefore, it is not a tensor. The symbol is totally antisymmetric, i.e. when any two indices are interchanged, the sign changes. All elements where one index appears twice are zero because the index set must be a permutation.

The Levi-Civita symbol can also be defined with upper indices in the same way. Then the determinant (2.102/2.103) takes the form

$$\varepsilon^{\mu'_1 \dots \mu'_n} |M| = \varepsilon^{\mu_1 \dots \mu_n} M^{\mu'_1}_{\mu_1} \cdots M^{\mu'_n}_{\mu_n} \tag{2.104}$$

or

$$|M| = \varepsilon^{\mu_1 \dots \mu_n} M^{1}_{\mu_1} \cdots M^{n}_{\mu_n}. \tag{2.105}$$

We can construct a tensor from the Levi-Civita symbol by multiplying it with the square root of the modulus of the metric (in Minkowski space the metric is negative definite, therefore we have to take the modulus). To show this, we start with the transformation equation of the metric tensor

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu} \tag{2.106}$$

and apply the determinant. With the product rule of determinants this can be written as

$$|g_{\mu'\nu'}| = \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right| \left| \frac{\partial x^{\nu}}{\partial x^{\nu'}} \right| |g_{\mu\nu}| = \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right|^2 |g_{\mu\nu}| \tag{2.107}$$

or

$$\left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right| = \sqrt{\frac{|g_{\mu'\nu'}|}{|g_{\mu\nu}|}} \tag{2.108}$$

where the left-hand side represents the determinant of the Jacobian. Using the special case

$$M^{\mu'}_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \tag{2.109}$$

and inserting this into (2.104) we obtain

$$\varepsilon^{\mu'_1 \dots \mu'_n} \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| = \varepsilon^{\mu_1 \dots \mu_n} \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_n}}{\partial x^{\mu_n}}. \tag{2.110}$$

The determinant of the inverse Jacobian is

$$\left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right| = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|^{-1}, \tag{2.111}$$

therefore we obtain from (2.110) with inserting (2.108):

$$\varepsilon^{\mu_1'\dots\mu_n'}\frac{1}{\sqrt{|g_{\mu'\nu'}|}} = \varepsilon^{\mu_1\dots\mu_n}\frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu_n'}}{\partial x^{\mu_n}}\frac{1}{\sqrt{|g_{\mu\nu}|}}. \tag{2.112}$$

So $\varepsilon^{\mu_1\dots\mu_n}/\sqrt{|g|}$ transforms like a tensor, and therefore is a tensor, by definition. The corresponding covariant tensor transforms as

$$\varepsilon_{\mu_1'\dots\mu_n'}\sqrt{|g_{\mu'\nu'}|} = \varepsilon_{\mu_1\dots\mu_n}\frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}}\cdots\frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}\sqrt{|g_{\mu\nu}|}. \tag{2.113}$$

Indices can be raised and lowered as usual by multiplying with metric elements.

With this behavior of the Levi-Civita symbol in mind, we define the Hodge-Dual of a tensorial form as follows. Assume a n-dimensional manifold, a p-dimensional sub-manifold $p < n$, and a tensor p-form A. We then define

$$\widetilde{A}_{\mu_1\dots\mu_{n-p}} := \frac{1}{p!}|g|^{-1/2}\,\varepsilon^{\nu_1\dots\nu_p}{}_{\mu_1\dots\mu_{n-p}}\,A_{\nu_1\dots\nu_p}. \tag{2.114}$$

The tilde superscript $\widetilde{\ }$ is called the *Hodge dual operator*. In the mathematical literature this is mostly denoted by an asterisk as prefix-operator ($*A$) but this is a very misleading notation, therefore we prefer the tilde superscript. The Hodge dual can be rewritten with a Levi-Civita symbol with only covariant components by

$$\widetilde{A}_{\mu_1\dots\mu_{n-p}} = \frac{1}{p!}|g|^{-1/2}\,g^{\nu_1\sigma_1}\cdots g^{\nu_p\sigma_p}\,\varepsilon_{\sigma_1\dots\sigma_p\mu_1\dots\mu_{n-p}}\,A_{\nu_1\dots\nu_p}. \tag{2.115}$$

In this book, we will mostly use a somewhat simpler form where a contravariant tensor is transformed into a covariant tensor and vice versa. The factors $g^{\nu_1\sigma_1}$, etc., can be used to raise the indices of $A_{\nu_1\dots\nu_p}$:

$$\widetilde{A}_{\mu_1\dots\mu_{n-p}} = \frac{1}{p!}|g|^{-1/2}\,\varepsilon_{\nu_1\dots\nu_n}\,A^{\nu_1\dots\nu_p}, \tag{2.116}$$

$$\widetilde{A}^{\mu_1\dots\mu_{n-p}} = \frac{1}{p!}|g|^{1/2}\,\varepsilon^{\nu_1\dots\nu_n}\,A_{\nu_1\dots\nu_p} \tag{2.117}$$

where the sign of the exponent of $|g|$ has been changed according to (2.113). As an example, in four-dimensional space we use $n = 4, p = 2$. Then Hodge duals of the A form are

$$\widetilde{A}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\,\varepsilon_{\mu\nu\sigma\rho}\,A^{\sigma\rho}, \tag{2.118}$$

$$\widetilde{A}^{\mu\nu} = \frac{1}{2}|g|^{1/2}\,\varepsilon^{\mu\nu\sigma\rho}\,A_{\sigma\rho}. \tag{2.119}$$

The Hodge dual $\widetilde{A}$ is linearly independent on the original form A. We will use the Hodge dual when deriving the theorems of Cartan geometry and the field equations of ECE theory.

2.4 Differentiation

We have already used some types of differentiation in the preceding sections, but only in the "standard" way within Euclidean spaces. Now we will extend this to curved spaces (manifolds) and to the calculus of p-forms.

2.4.1 Covariant differentiation

So far we have already used partial derivatives of tensors and parametrized derivatives. This, however, is not sufficient to define a general type of derivative in curved spaces of manifolds. Partial derivatives depend on the coordinate system. What we need is a "generally covariant" derivative that keeps its form under coordinate transformations and passes into the partial derivative for Euclidean spaces.

To retain linearity, the covariant derivative should have the form of a partial derivative plus a linear transformation. The latter corrects the partial derivative in such a way that covariance is ensured. The linear transformation depends on the coordinate indices. We define for the *covariant derivative* of an arbitrary vector field V^ν:

$$D_\mu V^\nu := \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{2.120}$$

where the $\Gamma^\nu_{\mu\lambda}$ are functions and called the *connection coefficients* or *Christoffel symbols*. In contrast to an ordinary partial derivative, the covariant derivative of a vector component V^ν depends on all other components via the sum with the connection coefficients (observe the summation convention!). The covariant derivative has tensor properties by definition, therefore Eq. (2.120) is a tensor equation, transducing a (1,0) tensor into a (1,1) tensor, and we can apply the transformation rules for tensors:

$$D_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} D_\mu V^\nu = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \left(\frac{\partial}{\partial x^\mu} V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \right). \tag{2.121}$$

On the other hand, we can apply the transformation to Eq. (2.120) directly:

$$D_{\mu'} V^{\nu'} = \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'}. \tag{2.122}$$

The single terms on the right-hand side transform as follows:

$$\begin{aligned} \partial_{\mu'} V^{\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu \right) \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial}{\partial x^\mu} V^\nu, \end{aligned} \tag{2.123}$$

$$\Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'} = \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda, \tag{2.124}$$

where the product rule has been applied in the first term. Eqs. (2.122) and (2.121) can be equated. The term with the partial derivative of V^ν cancels out and we obtain:

$$\Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} V^\lambda = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda. \tag{2.125}$$

Here we have replaced the dummy index ν by λ in the term with the mixed partial derivative. This is a common operation for tensor equations. Another common operation is multiplying a tensor equation by an indexed term and summing over one or more free indices (i.e. making the previously independent index into a dummy index). Multiplying the last equation by $\frac{\partial x^\lambda}{\partial x^{\lambda'}}$ then gives

$$\Gamma^{\nu'}_{\mu'\lambda'} V^\lambda = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} V^\lambda \tag{2.126}$$

so that we approach an equation of determining the transformation of the connection coefficients. The last equation holds for any vector V^λ, therefore the eqution must hold for the coefficients of V^λ directly. Thus, we obtain the transformation equation for the connection coefficients:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \tag{2.127}$$

Obviously, the Gammas do not transform as a tensor, the last term prevents this. The Gammas are not a tensor, therefore indices of Gamma cannot be raised and lowered by multiplying with metric elements and we need not put too much effort into maintaining the order of upper and lower indices.

So far, we have investigated covariant derivatives of a contravariant vector (Eq. (2.120)). The theory can be extended to covariant vectors of 1-forms ω_ν:

$$D_\mu \omega_\nu := \partial_\mu \omega_\nu + \bar{\Gamma}^\lambda_{\mu\nu} \omega_\lambda \tag{2.128}$$

where $\bar{\Gamma}$ is a connection coefficient being a priori different from Γ. It can be shown [5] that, for consistency reasons, $\bar{\Gamma}$ is the same as Γ except for the sign:

$$\bar{\Gamma}^\lambda_{\mu\nu} = -\Gamma^\lambda_{\mu\nu}. \tag{2.129}$$

Please note that the summation indices are different between (2.120) and (2.128). Now that we have a covariant derivative for contravariant and covariant components, the covariant derivative for arbitrary (k,m) tensors is defined as follows:

$$\begin{aligned} D_\sigma T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_m} := {} & \partial_\sigma T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_m} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\ldots\nu_m} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\mu_3\ldots\mu_k}{}_{\nu_1\ldots\nu_m} + \ldots \\ & - \Gamma^\lambda_{\sigma\nu_1} T^{\mu_1\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_m} - \Gamma^\lambda_{\sigma\nu_2} T^{\mu_1\ldots\mu_k}{}_{\nu_1\lambda\nu_3\ldots\nu_m} - \ldots \end{aligned} \tag{2.130}$$

By applying the covariant derivative, a (k,m) tensor is transformed into a (k,m+1) tensor. It is also possible to take the covariant derivative of a scalar function. Since no indices are defined for the connection in this case, we define for a scalar function ϕ:

$$D_\mu \phi := \partial_\mu \phi. \tag{2.131}$$

As we have seen, the connection coefficients are not a tensor. It is, however, easy to make a tensor of them by taking the antisymmetric sum of the lower indices:

$$T^\lambda{}_{\mu\nu} := \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \tag{2.132}$$

This is called the *torsion tensor*. When applying the transformation (2.127) for the difference of Gammas, the last term vanishes because the order in the mixed partial derivative is arbitrary. The torsion tensor is antisymmetric by definition. In four dimensions it can be written out for each index λ as

$$(T^\lambda{}_{\mu\nu}) = \begin{bmatrix} 0 & T^\lambda{}_{01} & T^\lambda{}_{02} & T^\lambda{}_{03} \\ -T^\lambda{}_{01} & 0 & T^\lambda{}_{12} & T^\lambda{}_{13} \\ -T^\lambda{}_{02} & -T^\lambda{}_{12} & 0 & T^\lambda{}_{23} \\ -T^\lambda{}_{03} & -T^\lambda{}_{13} & -T^\lambda{}_{23} & 0 \end{bmatrix}. \tag{2.133}$$

There are six independent components per λ. We will see later that this is one of the basis elements of Cartan geometry. A connection that is symmetric in its lower indices is torsion-free.

For completeness, we give the definition of the *Riemann curvature tensor*, which is also defined by the connection coefficients, but in a more complicated manner:

$$R^\lambda{}_{\rho\mu\nu} := \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}. \tag{2.134}$$

The tensor is antisymmetric in its last two indices. If it is written in pure covariant form $R_{\lambda\rho\mu\nu} = g_{\tau\lambda}\, R^\tau{}_{\rho\mu\nu}$ and the manifold is torsion-free, the Riemann tensor is also antisymmetric in its first two indices. This property will, however, not be used in Cartan geometry.

2.4.2 Metric compatibility and parallel transport

A fundamental property of vectors in physics is that they must be independent of their coordinate representation. From Euclidean space, we know that a rotation of a vector leaves its length and orientation against other vectors constant. In curved manifolds this is not necessarily the case anymore. Whether the length of a vector is preserved depends on the metric tensor. A parallel transport of a vector is depicted in Fig. 2.7. On a spherical surface, a vector is parallel transported from the north pole to a point on the equator in two ways: 1) moved directly along a meridian (red; right) and 2) moved first along another meridian and then along an equatorial latitude (left; blue). Obviously, the results are different, so this naive procedure is not compatible with a spherical manifold.

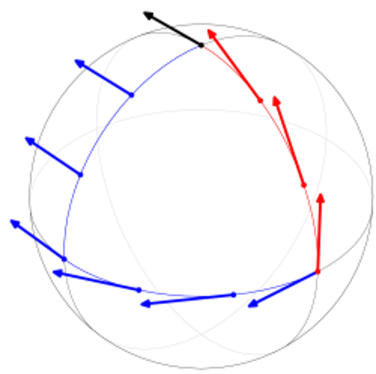


Figure 2.7: Parallel transport of a vector on a sphere.

Let us formalize the process to define parallel transport in a compatible way. A path is a displacement of a vector V^ν whose coordinates are parameterized, say by a parameter λ:

$$V^\nu = V^\nu(\lambda) \;\text{ at point } x^\nu(\lambda). \tag{2.135}$$

This can be considered as moving the vector (which is a tensor) along a predefined path. We define the covariant derivative along the path by

$$\frac{D}{d\lambda} := \frac{dx^\mu}{d\lambda} D_\mu, \tag{2.136}$$

where $\frac{dx^\mu}{d\lambda}$ is the tangent vector of the path. This gives us a method for specifying a *parallel transport* of V. This transport condition is fulfilled if the covariant derivative along the path vanishes:

$$\frac{D\,V^\nu}{d\lambda} = \frac{dx^\mu}{d\lambda} D_\mu V^\nu = \frac{dx^\mu}{d\lambda}(\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho) = 0. \tag{2.137}$$

Since the tangent vector cannot vanish (we would not have a path anymore), it follows that the covariant derivative of the tensor must vanish:

$$D_\mu V^\nu = 0. \tag{2.138}$$

This is the condition for parallel transport. It is fulfilled if and only if the covariant derivative along a path vanishes. This holds for any tensor. In particular we can choose the metric tensor and require it to be parallel transported:

$$D_\sigma g_{\mu\nu} = 0. \tag{2.139}$$

This is called *metric compatibility*. It is also said that the connection is metrically compatible because it is contained in the covariant derivative. It means that the metric tensor is covariantly constant everywhere and can be parallel transported. If this requirement were omitted, we would have difficulties defining meaningful physics in a manifold, for example, norms of vectors would not be constant but change during translations or rotations.

Example 2.9 We show that the inner product of two vectors is preserved if the vectors can be parallel transported. The inner product of vectors V^μ and W^ν is $g_{\mu\nu}V^\mu W^\nu$. Its covariant path derivative is

$$\frac{D}{d\lambda}\left(g_{\mu\nu}V^\mu W^\nu\right) = \left(\frac{D}{d\lambda}g_{\mu\nu}\right)V^\mu W^\nu + g_{\mu\nu}\left(\frac{D}{d\lambda}V^\mu\right)W^\nu + g_{\mu\nu}V^\mu\left(\frac{D}{d\lambda}W^\nu\right) = 0 \tag{2.140}$$

because all three tensors are parallel transported, by definition. In the same way, one can prove that, if $g_{\mu\nu}$ can be parallel transported, then so can its inverse $g^{\mu\nu}$:

$$\begin{aligned} 0 = \frac{D}{d\lambda}g_{\mu\nu} &= \frac{D}{d\lambda}\left(g_{\mu\sigma}\, g_{\rho\nu}\, g^{\rho\sigma}\right) \\ &= \frac{D}{d\lambda}\left(g_{\mu\sigma}\right) g_{\rho\nu}\, g^{\rho\sigma} + g_{\mu\sigma}\frac{D}{d\lambda}\left(g_{\rho\nu}\right) g^{\rho\sigma} + g_{\mu\sigma}\, g_{\rho\nu}\frac{D}{d\lambda}\left(g^{\rho\sigma}\right). \end{aligned} \tag{2.141}$$

The first two terms in the last line vanish by definition, and consequently the third term has to vanish. ■

The concept of parallel transport allows us to find the equation for geodesics. A *geodesic* is the generalization of a straight line in Euclidean space. Mass points without external forces move this way. In a curved manifold, the motion follows the curving of space and therefore is not a straight line. We can find the equation of geodesics by requiring that the path parallel transports its own tangent vector. This is in analogy to flat space where the tangent vector is parallel to its line vector. From (2.137) then we have

$$\frac{D}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \tag{2.142}$$

which can be written

$$\frac{dx^\mu}{d\lambda}D_\mu\frac{dx^\nu}{d\lambda} = \frac{dx^\mu}{d\lambda}\left(\frac{\partial}{\partial x^\mu}\frac{dx^\nu}{d\lambda} + \Gamma^\nu_{\mu\rho}\frac{dx^\rho}{d\lambda}\right) = 0 \tag{2.143}$$

and, by replacement of $\frac{\partial}{\partial x^\mu}$ by $\frac{\partial\lambda}{\partial x^\mu}\frac{d}{d\lambda}$, simplifies to

$$\frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\mu\rho}\frac{dx^\mu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \tag{2.144}$$

which is the *geodesic equation*. In flat space, the Gammas vanish and Newton's law $\ddot{x} = 0$ for an unconstrained motion is regained.

Given a path in the manifold, covariant derivatives can be used to describe the deviation of a tensor from being parallel transported. Consider a round-trip as depicted in Fig. 2.8. A tensor is moved counter-clockwise along its covariant tangent vector D_μ, then D_ν, and afterwards back to its starting point in the reverse order. In the case where the tensor is parallel transportable, all derivatives vanish. However, this will not be the case in general. The *commutator* of two covariant derivatives is defined as

$$[D_\mu, D_\nu] := D_\mu D_\nu - D_\nu D_\mu \tag{2.145}$$

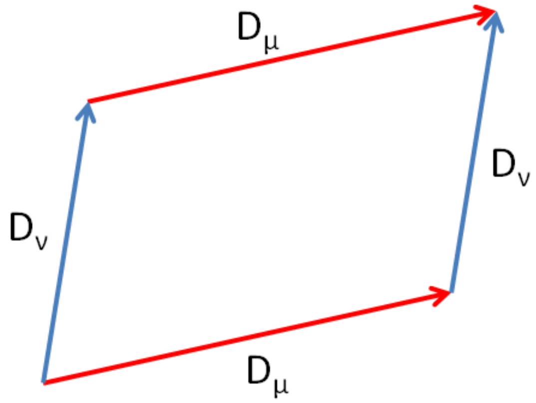


Figure 2.8: Closed loop for composition of two covariant derivatives.

and describes the difference of both paths with respect to the covariant derivative. We can apply this to a vector V^ρ and evaluate the terms:

$$\begin{aligned}
[D_\mu, D_\nu]V^\rho &= D_\mu D_\nu V^\rho - D_\nu D_\mu V^\rho \\
&= \partial_\mu (D_\nu V^\rho) - \Gamma^\lambda_{\mu\nu} D_\lambda V^\rho + \Gamma^\rho_{\mu\sigma} D_\nu V^\sigma \\
&\quad - \partial_\nu (D_\mu V^\rho) + \Gamma^\lambda_{\nu\mu} D_\lambda V^\rho - \Gamma^\rho_{\nu\sigma} D_\mu V^\sigma \\
&= \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\nu\sigma}) V^\sigma + \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma - \Gamma^\lambda_{\mu\nu} \partial_\lambda V^\rho - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} V^\sigma \\
&\quad + \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda \\
&\quad - \partial_\nu \partial_\mu V^\rho - (\partial_\nu \Gamma^\rho_{\mu\sigma}) V^\sigma - \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma + \Gamma^\lambda_{\nu\mu} \partial_\lambda V^\rho + \Gamma^\lambda_{\nu\mu} \Gamma^\rho_{\lambda\sigma} V^\sigma \\
&\quad - \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} V^\lambda \\
&= \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \right) V^\sigma - (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) D_\lambda V^\rho .
\end{aligned} \tag{2.146}$$

Comparing the last line with the definitions of curvature tensor (2.134) and torsion tensor (2.132), it can be written:

$$[D_\mu, D_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu}\, D_\lambda V^\rho . \tag{2.147}$$

Interestingly, the commutator of covariant derivatives of a vector depends linearly on the vector itself and its tangent vector, where the coefficients are the curvature and torsion tensors. In the case of no torsion, there would be no dependence on a derivative of V^ρ at all. The action of $[D_\mu, D_\nu]$ can be applied to a tensor of arbitrary rank. In general, it is

$$\begin{aligned}
[D_\rho, D_\sigma] X^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_m} = {} & R^{\mu_1}{}_{\lambda\rho\sigma} X^{\lambda \mu_2 \ldots \mu_k}{}_{\nu_1 \ldots \nu_m} + R^{\mu_2}{}_{\lambda\rho\sigma} X^{\mu_1 \lambda \ldots \mu_k}{}_{\nu_1 \ldots \nu_m} + \cdots \\
& - R^\lambda{}_{\nu_1 \rho\sigma} X^{\mu_1 \ldots \mu_k}{}_{\lambda \nu_2 \ldots \nu_m} - R^\lambda{}_{\nu_2 \rho\sigma} X^{\mu_1 \ldots \mu_k}{}_{\nu_1 \lambda \ldots \nu_m} - \cdots \\
& - T^\lambda{}_{\rho\sigma}\, D_\lambda X^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_m} .
\end{aligned} \tag{2.148}$$

We have seen that the curvature and torsion tensors depend on the connection coefficients directly. To describe the geometry of a manifold, one must know these coefficients. The geometry is typically defined by a coordinate transformation. However, there is no direct way to derive the connection coefficients from the coordinate transformation equations. In Example (2.8) we have seen that the metric tensor can be derived from the Jacobian, which contains the derivatives of the coordinate transformations. Therefore, what we need is a relation between the metric and the connection, from which the connection coefficients can be derived, when the metric is known. Such a relation is given by the metric compatibility condition:

$$D_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - \Gamma^\lambda_{\sigma\mu} g_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu} g_{\mu\lambda} = 0. \tag{2.149}$$

For a space of four dimensions, this tensor equation represents $4^3 = 64$ single equations. The first of them (for the diagonal metric elements) read:

$$\begin{aligned}
&\frac{\partial}{\partial x^0} g_{00} - 2\Gamma^0_{00}\, g_{00} = 0 \\
&-\Gamma^1_{00}\, g_{11} - \Gamma^0_{01}\, g_{00} = 0 \\
&-\Gamma^2_{00}\, g_{22} - \Gamma^0_{02}\, g_{00} = 0 \\
&-\Gamma^3_{00}\, g_{33} - \Gamma^0_{03}\, g_{00} = 0 \\
&\qquad\qquad \dots
\end{aligned} \tag{2.150}$$

You should keep in mind that the metric is symmetric, and therefore not all equations are linearly independent. It is difficult to see how many independent equations remain. Computer algebra (code available [8]) tells us that one half (24 equations) are dependent on the other 24 equations. Therefore, we can predefine 24 Gammas arbitrarily. A solution is, for example,

$$\begin{aligned}
\Gamma^0_{00} &= \frac{\frac{\partial}{\partial x^0}\, g_{00}}{2\, g_{00}} \\
\Gamma^0_{01} &= -\frac{g_{11}}{g_{00}} A_{25} \\
\Gamma^0_{02} &= -\frac{g_{22}}{g_{00}} A_{43} \\
\Gamma^0_{03} &= -\frac{g_{33}}{g_{00}} A_{40} \\
\Gamma^0_{10} &= \frac{\frac{\partial}{\partial x^1}\, g_{00}}{2\, g_{00}} \\
&\dots
\end{aligned} \tag{2.151}$$

with

$$\begin{aligned}
\Gamma^1_{00} &= A_{25} \\
\Gamma^2_{00} &= A_{43} \\
\Gamma^3_{00} &= A_{40} \\
&\dots
\end{aligned} \tag{2.152}$$

where the A_i are the predefined parameters, and they may even be functions of x^μ. From the first equation of (2.150), it can be seen that assuming $\Gamma^0_{00} = 0$ is not a good choice, because this would impose the restriction $\frac{\partial g_{00}}{\partial x^0} = 0$ on the metric a priori. Therefore, the diagonal elements of the lower pair of indices of Gamma do not vanish in general. By comparing the solutions for Γ^0_{01} andΓ^0_{10} in (2.151) it is obvious that the Gammas are not symmetric in the lower indices.

Having found the connection coefficients, we can construct the curvature and torsion tensors (2.134) and (2.132). While the coefficients go into the curvature tensor as is, the torsion tensor itself depends only on the antisymmetric part of the Gammas. Each 2-tensor or connection coefficient can be split into a symmetric and antisymmetric part:

$$\Gamma^\rho_{\mu\nu} = \Gamma^{\rho(S)}_{\mu\nu} + \Gamma^{\rho(A)}_{\mu\nu} \tag{2.153}$$

with

$$\Gamma^{\rho(S)}_{\mu\nu} = \Gamma^{\rho(S)}_{\nu\mu}, \tag{2.154}$$
$$\Gamma^{\rho(A)}_{\mu\nu} = -\Gamma^{\rho(A)}_{\nu\mu}.$$

For the torsion tensor we have

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda(A)}_{\mu\nu} - \Gamma^{\lambda(A)}_{\nu\mu} = 2\,\Gamma^{\lambda(A)}_{\mu\nu}, \tag{2.155}$$

the symmetric part does not enter the torsion. This motivates the imposition of additional antisymmetry requirements on the Gammas, instead of choosing 24 elements arbitrarily. So, in addition to the metric compatibility equation (2.149), we define 24 extra equations

$$\Gamma^{\rho}_{\mu\nu} = -\Gamma^{\rho}_{\nu\mu} \tag{2.156}$$

for all pairs $\mu \neq \nu$ with $\mu > \nu$. This reduces the number of free solution parameters from 24 to 4 (see code [26]). The situation gets quite complicated when non-diagonal elements in the metric are present [27]. Alternatively, we could even force a purely symmetric connection by requiring that

$$\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}. \tag{2.157}$$

Then there are no free parameters anymore, and all Gammas are uniquely defined, where 24 of them turn out to be zero. However, in this case, torsion is zero and we will run into irretrievable conflicts with geometrical laws, as we will see in subsequent sections. There is a reason for leaving a certain variability in the connection: the theorems of Cartan geometry have to be satisfied, which imposes additional conditions on curvature and torsion, and thereby on the connection.

For completeness, we describe how the symmetric connection coefficients are computed in Einsteinian general relativity. Starting with Eq. (2.149), this equation is written three times with permuted indices:

$$\partial_\sigma g_{\mu\nu} - \Gamma^{\lambda}_{\sigma\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\sigma\nu} g_{\mu\lambda} = 0, \tag{2.158}$$
$$\partial_\mu g_{\nu\sigma} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma} g_{\nu\lambda} = 0,$$
$$\partial_\nu g_{\sigma\mu} - \Gamma^{\lambda}_{\nu\sigma} g_{\lambda\mu} - \Gamma^{\lambda}_{\nu\mu} g_{\sigma\lambda} = 0.$$

Subtracting the second and third equation from the first and using the symmetry of the connection gives

$$\partial_\sigma g_{\mu\nu} - \partial_\mu g_{\nu\sigma} - \partial_\nu g_{\sigma\mu} + 2\,\Gamma^{\lambda}_{\mu\nu} g_{\lambda\sigma} = 0, \tag{2.159}$$

and multiplying the equation by $g^{\sigma\rho}$ gives for the Gamma term:

$$(\Gamma^{\lambda}_{\mu\nu} g_{\lambda\sigma})\, g^{\sigma\rho} = \Gamma^{\lambda}_{\mu\nu}(g_{\lambda\sigma}\, g^{\sigma\rho}) = \Gamma^{\lambda}_{\mu\nu}\delta^{\rho}_{\lambda} = \Gamma^{\rho}_{\mu\nu}. \tag{2.160}$$

From (2.159) then follows

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}). \tag{2.161}$$

The symmetric connection is determined completely by the metric, in accordance with our earlier result from the single equation of metric compatibility.

All derivations in this section were exemplified with a diagonal metric. They remain true if non-diagonal elements are added, but the solutions become much more complex. Imposing additional symmetry or antisymmetry conditions on the connection may lead to results differing from those for a diagonal metric.

It should be noted that the metric of a given geometry of a manifold is not unique, and depends on the choice of coordinate system. Recalling the examples above, Euclidean space can be described by cartesian or spherical coordinates which lead to different metric tensors. However, the spacetime structure is the same, only the numerical addressing of points changes, as do the coordinates of vectors. However, the vectors as physical objects (position and length) remain the same.

■ **Example 2.10** We compute the connection for the spherical coordinate system (r, θ, ϕ) for three cases: general connection, antisymmetrized connection, and symmetrized connection. This example is available as Maxima code [25]. The metric tensor is from Example 2.5:

$$(g_{\mu\nu}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{bmatrix}. \tag{2.162}$$

Since the metric is not time-dependent, indices run from 1 to 3. This gives $3^3 = 27$ equations from metric compatibility (2.149), and the first equations are

$$\begin{aligned} -2\,\Gamma^1_{11} &= 0 \\ -\Gamma^2_{11}\,r^2 - \Gamma^1_{12} &= 0 \\ -\Gamma^3_{11}\,r^2\sin^2(\theta) - \Gamma^1_{13} &= 0 \\ -\Gamma^2_{11}\,r^2 - \Gamma^1_{12} &= 0 \\ 2r - 2\,\Gamma^2_{12}\,r^2 &= 0 \\ &\dots \end{aligned} \tag{2.163}$$

The solution (obtained by computer algebra) contains 9 free parameters $A_1, \dots, A_9$. There are 27 solutions in total. Some of them are:

$$\begin{aligned} &\Gamma^1_{11} = 0 \\ &\Gamma^1_{13} = -A_9\,r^2\sin^2(\theta) \\ &\Gamma^1_{31} = 0 \\ &\Gamma^1_{33} = -A_3\,r^2\sin^2(\theta) \\ &\Gamma^2_{12} = \frac{1}{r} \\ &\Gamma^2_{21} = -\frac{A_4}{r^2} \\ &\Gamma^3_{23} = \frac{\cos(\theta)}{\sin(\theta)} \\ &\Gamma^3_{32} = A_2 \\ &\dots \end{aligned} \tag{2.164}$$

With 9 additional antisymmetry conditions, the solutions are

$$
\begin{aligned}
&\Gamma^1_{11} = 0 \\
&\Gamma^1_{13} = \Gamma^1_{31} = 0 \\
&\Gamma^1_{23} = -\Gamma^1_{32} = -A_{10} \\
&\Gamma^2_{12} = -\Gamma^2_{21} = \frac{1}{r} \\
&\Gamma^3_{23} = -\Gamma^3_{32} = \frac{\cos(\theta)}{\sin(\theta)} \\
&\ldots
\end{aligned}
\tag{2.165}
$$

There is only one free parameter A_{10} left. A certain similarity to the general solution is retained, but with antisymmetry. If symmetric connection coefficients are enforced, most Gammas are zero. The only non-zero coefficients are:

$$
\begin{aligned}
&\Gamma^1_{22} = -r \\
&\Gamma^1_{33} = -r\sin^2(\theta) \\
&\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \\
&\Gamma^2_{33} = -\cos(\theta)\sin(\theta) \\
&\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \\
&\Gamma^3_{23} = \Gamma^3_{32} = \frac{\cos(\theta)}{\sin(\theta)}
\end{aligned}
\tag{2.166}
$$

This example is often found in textbooks of general relativity. If all coordinates have the physical dimension of length, then the connection coefficients have the same physical dimension. In this example we have angles and lengths, therefore the physical dimensions differ. ■

2.4.3 Exterior derivative

So far, we have dealt with covariant derivatives of tensors. Now we want to extend the concept of derivatives to n-forms. We already know that a partial derivative of a tensor does not conserve the tensor properties. Therefore, we will define an appropriate derivative for n-forms. We have already introduced antisymmetric forms in section 2.3. It is useful to define a derivative on these objects that conserves antisymmetry and tensor properties. A partial derivative for one coordinate generates an additional index in a tensor, therefore a p-form is extended to a (p+1)-form by the definition

$$(d \wedge A)_{\mu_1 \ldots \mu_{p+1}} := (p+1)\partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]}. \tag{2.167}$$

This (p+1)-form is a tensor, irrespective of what A is. The simplest exterior derivative is that of a scalar function $\phi(x_\mu)$ which is

$$(d \wedge \phi)_\mu = \partial_\mu \phi, \tag{2.168}$$

in other words, this is the gradient of ϕ. Another example is the definition of the electromagnetic field in tensor form $F_{\mu\nu}$ as a 2-form (see example 2.11 below). It is derived as an exterior derivative of a 1-form, the vector potential A_μ:

$$F_{\mu\nu} := (d \wedge A)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.169}$$

The tensor character of exterior derivatives can be seen by applying the transformation law (2.123) to a (0,1) tensor V for example:

$$\frac{\partial}{\partial x^{\mu'}} V_{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\mu}} V_{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\nu}}{\partial x^{\nu'}} V_{\nu} \right) \tag{2.170}$$
$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu}}{\partial x^{\mu} \partial x^{\nu'}} V_{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial}{\partial x^{\mu}} V_{\nu}.$$

The first term in the second line should not appear if this were a tensor transformation. It can be rewritten to

$$\frac{\partial^2 x^{\nu}}{\partial x^{\mu'} \partial x^{\nu'}} V_{\nu} \tag{2.171}$$

and now is symmetric in μ' and ν'. Since the exterior derivative only contains antisymmetric sums of both indices, all these terms vanish because partial derivatives are commutable. Therefore, $d \wedge V_\nu$ transforms like a tensor, and so do all n-forms.

An important property of an exterior derivative is that its two-fold application is zero:

$$d \wedge (d \wedge A) = 0. \tag{2.172}$$

The reason is the same as above, the partial derivatives are commutable, summing up to zero in all antisymmetric sums.

■ **Example 2.11** We describe Maxwell's homogeneous field equations in form notation and transform this to the well-known vector form (code [28]). The homogeneous laws are the Gauss law and the Faraday law. In tensor notation they are condensed into one equation:

$$d \wedge F = 0 \tag{2.173}$$

or with indices

$$(d \wedge F)_{\mu\nu\rho} = 0. \tag{2.174}$$

Because F is a 2-form, the exterior derivative of F is a 3-form. The electromagnetic field tensor is antisymmetric and defined by the contravariant tensor

$$F^{\mu\nu} = \begin{bmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{bmatrix} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix} \tag{2.175}$$

where E^i are the components of the electric field and B^i those of the magnetic field. It is $E^1 = E_X, E^2 = E_Y$, etc. To be able to apply the exterior derivative, we first have to transform this tensor to covariant form. Since classical electrodynamics takes place in a Euclidean space, we use the Minkowski metric to lower the indices:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2.176}$$

Then the covariant field tensor is

$$F_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}F^{\rho\sigma} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -cB^3 & cB^2 \\ -E^2 & cB^3 & 0 & -cB^1 \\ -E^3 & -cB^2 & cB^1 & 0 \end{bmatrix}. \tag{2.177}$$

Compared to the contravariant form, only the signs of the electric field components have changed. Working out the exterior derivative for $\mu=0, \nu=1, \rho=2$, we obtain

$$(d\wedge F)_{012} = \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} - \partial_0 F_{21} - \partial_1 F_{02} - \partial_2 F_{10}. \tag{2.178}$$

Because F is antisymmetric, the negative summands are equal to the positive summands with reversed sign so that we have

$$(d\wedge F)_{012} = 2(\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01}), \tag{2.179}$$

this is twice the cyclic sum of indices. Since (μ, ν, ρ) must be a subset of (0, 1, 2, 3) only the combinations
(0, 1, 2)
(0, 1, 3)
(0, 2, 3)
(1, 2, 3)
are possible, leading to four equations for $d\wedge F$. Setting $F_{01} = E_X$ etc. leads to the four equations:

$$\begin{aligned} 2(c\partial_0 B^3 + \partial_1 E^2 - \partial_2 E^1) &= 0 \\ 2(-c\partial_0 B^2 + \partial_1 E^3 - \partial_3 E^1) &= 0 \\ 2(c\partial_0 B^1 + \partial_2 E^3 - \partial_3 E^2) &= 0 \\ 2(c\partial_1 B^1 + c\partial_2 B^2 + c\partial_3 B_3) &= 0 \end{aligned} \tag{2.180}$$

or, written with cartesian components and simplified:

$$\begin{aligned} \partial_t B_Z + \partial_X E_Y - \partial_Y E_X &= 0 \\ \partial_t B_Y - \partial_X E_Z + \partial_Z E_X &= 0 \\ \partial_t B_X + \partial_Y E_Z - \partial_Z E_Y &= 0 \\ \partial_X B_X + \partial_Y B_Y + \partial_Z B_Z &= 0 \end{aligned} \tag{2.181}$$

where we have used $\partial_0 = 1/c \cdot \partial_t$. Comparing these equations with the curl operator:

$$\nabla \times \mathbf{V} = \begin{bmatrix} \partial_Y V_Z - \partial_Z V_Y \\ -\partial_X V_Z + \partial_Z V_X \\ \partial_X V_Y - \partial_Y V_X \end{bmatrix} \tag{2.182}$$

the first three equations of (2.181) contain the third, second and first line of this operator and can be written in vector form:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{2.183}$$

which is the Faraday law. The fourth equation of (2.181) is the Gauss law

$$\nabla \cdot \mathbf{B} = 0. \tag{2.184}$$

We conclude this example with the hint that the inhomogeneous Maxwell equations (Coulomb law and Ampère-Maxwell law) cannot be written as an exterior tensor derivative due to the current terms. In those cases, a formulation similar to that in the next example has to be used. ■

■ **Example 2.12** As an example involving the Hodge dual (code [29]), we derive the homogeneous Maxwell equations from a tensor notation containing the Hodge dual of the electromagnetic field tensor introduced in the preceding example, 2.11. In tensor notation, the equation is:

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0 \tag{2.185}$$

and involves the Hodge dual of the 4 x 4 field tensor, defined as follows:

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = \begin{bmatrix} 0 & -cB^1 & -cB^2 & -cB^3 \\ cB^1 & 0 & -E^3 & E^2 \\ cB^2 & E^3 & 0 & -E^1 \\ cB^3 & -E^2 & E^1 & 0 \end{bmatrix}. \tag{2.186}$$

Indices are raised using the Minkowski metric (2.176):

$$\widetilde{F}^{\mu\nu} = \eta^{\mu\kappa}\eta^{\nu\rho}\widetilde{F}_{\kappa\rho}. \tag{2.187}$$

Therefore, the covariant Hodge dual is:

$$\widetilde{F}^{\mu\nu} = \begin{bmatrix} 0 & cB^1 & cB^2 & cB^3 \\ -cB^1 & 0 & -E^3 & E^2 \\ -cB^2 & E^3 & 0 & -E^1 \\ -cB^3 & -E^2 & E^1 & 0 \end{bmatrix}, \tag{2.188}$$

for example:

$$\widetilde{F}_{01} = \frac{1}{2}(\varepsilon_{0123}F^{23} + \varepsilon_{0132}F^{32}) = F^{23} \tag{2.189}$$

and

$$\widetilde{F}^{01} = \eta^{00}\eta^{11}\widetilde{F}_{01} = -\widetilde{F}_{01}. \tag{2.190}$$

The homogeneous laws of classical electrodynamics are obtained as follows, by choice of indices. The Gauss law is obtained by choosing:

$$\nu = 0 \tag{2.191}$$

and so

$$\partial_1\widetilde{F}^{10} + \partial_2\widetilde{F}^{20} + \partial_3\widetilde{F}^{30} = 0. \tag{2.192}$$

In vector notation this is

$$\nabla \cdot \mathbf{B} = 0. \tag{2.193}$$

The Faraday law of induction is obtained by choosing:

$$\nu = 1, 2, 3 \tag{2.194}$$

and consists of three component equations:

$$\begin{aligned} &\partial_0\widetilde{F}^{01} + \partial_2\widetilde{F}^{21} + \partial_3\widetilde{F}^{31} = 0 \\ &\partial_0\widetilde{F}^{02} + \partial_1\widetilde{F}^{12} + \partial_3\widetilde{F}^{32} = 0 \\ &\partial_0\widetilde{F}^{03} + \partial_1\widetilde{F}^{13} + \partial_2\widetilde{F}^{23} = 0. \end{aligned} \tag{2.195}$$

These can be condensed into one vector equation, which is

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0. \tag{2.196}$$

The differential form, tensor and vector notations are summarized as follows:

$$d \wedge F = 0 \rightarrow \partial_\mu \widetilde{F}^{\mu\nu} = 0 \rightarrow \nabla \cdot \mathbf{B} = 0 \tag{2.197}$$
$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0.$$

The homogeneous laws of classical electrodynamics are most elegantly represented by the differential form notation, but most usefully represented by the vector notation. ■

Exterior covariant derivative

So far, we have seen that exterior derivatives are antisymmetric sums of partial derivatives applied to n-forms. The question now is what happens if we want to combine the concept of the exterior derivative with a covariant derivative. This is a generalization of the concept, which should be more appropriate to curved manifolds where covariant derivatives play an important role for their description, for example, to define commutators as in section 2.4.2. We can define an exterior covariant derivative by creating an (n+1)-form from an n-form A:

$$D \wedge A := (D \wedge A)_{\mu_1 \dots \mu_{n+1}} = D_{[\mu} \wedge A_{\nu_1 \cdots \nu_n]}. \tag{2.198}$$

For a 1-form A_ν this then is

$$\begin{aligned} D \wedge A = (D \wedge A)_{\mu\nu} = D_{[\mu} \wedge A_{\nu]} &= \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda - \partial_\nu A_\mu + \Gamma^\lambda_{\nu\mu} A_\lambda \\ &= \partial_{[\mu} A_{\nu]} - (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) A_\lambda \end{aligned} \tag{2.199}$$

and with the definition (2.132) of the torsion tensor this can be written:

$$D \wedge A = \partial_{[\mu} A_{\nu]} - T^\lambda{}_{\mu\nu} A_\lambda. \tag{2.200}$$

Since the right-hand side is a tensor, $D \wedge A$ is also a tensor. The equation can be written in form notation:

$$D \wedge A = d \wedge A - T\ A. \tag{2.201}$$

We will extend this concept further in the next chapter.

2.5 Cartan geometry

Having developed the basics of Riemannian geometry, including torsion, we now approach the central point of this book: Cartan geometry. This will be the mathematical foundation of all fields of physics, as we will see.

2.5.1 Tangent space, tetrads and metric

By using Riemannian geometry as a basis, we have available nearly all tools that we need to develop the geometry that is called Cartan geometry and is the basis of ECE theory. We now need to set our focus on tangent spaces. In section 2.1 we dealt with coordinate transformations in the base manifold. The tangent space at a point x in the base manifold was introduced as a Minkowski space of the same dimension for the local neighborhood of x. A vector V^μ defined in the base manifold can be transformed to a vector in tangent space denoted by V^a. We introduce Latin indices to denote vectors and tensors in tangent space. A vector in the base manifold can be transformed to the corresponding one in the tangent space by a transformation matrix q. This is similar to introduction of the transformation matrix α in Eqs. (2.46 ff.), but with the difference that the transformation takes place between two different spaces. The basic transformation is

$$V^a = q^a{}_\mu V^\mu \tag{2.202}$$

with transformation matrix elements $q^a{}_\mu$. This is the basis of Cartan geometry, and q is called the *tetrad*. q transforms between the base manifold and tangent space. The inverse transformation is $q^{-1} = (q^\mu{}_a)$, producing a vector in the base manifold:

$$V^\mu = q^\mu{}_a V^a. \tag{2.203}$$

If the metric of the tangent space η_{ab} is transformed to the base manifold (this is a (0,2) tensor), the result must be the metric of the base manifold $g_{\mu\nu}$ by definition:

$$g_{\mu\nu} = n\, q^a{}_\mu q^b{}_\nu\, \eta_{ab}, \tag{2.204}$$

and inversely:

$$\eta_{ab} = \frac{1}{n}\, q^\mu{}_a q^\nu{}_b\, g_{\mu\nu}, \tag{2.205}$$

where n is the dimension of the base manifold. Since q is a coordinate transformation, the product of q and its inverse has to be the unit matrix:

$$\mathbf{q}\mathbf{q}^{-1} = \mathbf{1} \tag{2.206}$$

which, written in component form, is

$$q^a{}_\mu q^\nu{}_a = \delta^\nu_\mu, \tag{2.207}$$
$$q^a{}_\mu q^\mu{}_b = \delta^a_b. \tag{2.208}$$

The sum of the diagonal elements of (2.206), called the *trace*, is the dimension of the spaces between which the transformation takes place:

$$q^a{}_\mu q^\mu{}_a = n. \tag{2.209}$$

However, this kind of summed product will often occur in our calculations and it is beneficial to let the result be unity:

$$q^a{}_\mu q^\mu{}_a := 1. \tag{2.210}$$

Therefore we introduce a scaling factor of $1/\sqrt{n}$ to the tetrad elements and $\sqrt{n}$ to the inverse tetrad elements:

$$q^a{}_\mu \to \frac{1}{\sqrt{n}}\, q^a{}_\mu, \tag{2.211}$$
$$q^\mu{}_a \to \sqrt{n}\, q^\mu{}_a. \tag{2.212}$$

Thus, the conditions (2.204) and (2.205) remain satisfied.

■ **Example 2.13** We consider the transformation to spherical polar coordinates, Eq. (2.58) from Example (2.4):

$$\alpha = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}. \tag{2.213}$$

The inverse transformation is

$$\alpha^{-1} = \begin{bmatrix} \cos(\phi)\sin(\theta) & \sin(\phi)\sin(\theta) & \cos(\theta) \\ \frac{\cos(\phi)\cos(\theta)}{r} & \frac{\sin(\phi)\cos(\theta)}{r} & -\frac{\sin(\theta)}{r} \\ -\frac{\sin(\phi)}{r\sin(\theta)} & \frac{\cos(\phi)}{r\sin(\theta)} & 0 \end{bmatrix} \tag{2.214}$$

as can be seen from code example [30]. To make this transformation a tetrad from a cartesian base manifold to a Euclidian tangent space with spherical polar coordinates, we have to set

$$\mathbf{q} = \frac{1}{\sqrt{3}}\,\alpha, \tag{2.215}$$

$$\mathbf{q}^{-1} = \sqrt{3}\,\alpha^{-1}. \tag{2.216}$$

Then we have

$$\mathbf{q}\mathbf{q}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.217}$$

which is the unit matrix as required. ■

2.5.2 Derivatives in tangent space

We will now investigate the differential calculus in tangent space and how it is connected to that of the base manifold. The tangent space at a point x is a Euclidian space, and we could argue that this allows us to use ordinary differentiation. To define a derivative, we have to construct infinitesimal transitions from the neighborhood of x. For a point $y \neq x$, however, another tangent space is defined because of the definition of tangent spaces. Therefore, the curved structure of the base manifold has to be respected in the definition of the derivatives in tangent space. In the base manifold, we defined the covariant derivative for this purpose, see Eq. (2.120):

$$D_\mu V^\nu := \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{2.218}$$

where the partial derivatives ∂_μ and the connection coefficients $\Gamma^\nu_{\mu\lambda}$ are operating on a vector V^λ in the base manifold. We can do the same definition for a vector V^a in tangent space, but the connection coefficients are different here:

$$D_\mu V^a := \partial_\mu V^a + \omega^a{}_{\mu b} V^b. \tag{2.219}$$

The role of the connection coefficients is taken over by other coefficients called *spin connections* $\omega^a{}_{\mu b}$. These have the same number of indices as the Γs but transform in the tangent space. Therefore they have two Latin indices. The name "spin connection" comes from the fact that this can be used to define covariant derivatives of spinors, which is actually impossible using the Γ connection coefficients. The derivative D_μ itself is defined with respect to the base manifold and therefore has a Greek index. This also has to be present in the spin connection to maintain the indices as required for a tensor expression.

Covariant derivatives of a mixed index tensor are defined in a way so that the indices of tangent space are accompanied by a spin connection and the indices of the base manifold by a Christoffel connection, for example:

$$D_\mu V^a_{\ \nu} = \partial_\mu V^a_{\ \nu} + \omega^a_{\ \mu b} V^b_{\ \nu} - \Gamma^\lambda_{\ \mu\nu} V^a_{\ \lambda} \,. \tag{2.220}$$

or

$$D_\mu X^{ab}_{\ \ c\nu} = \partial_\mu X^{ab}_{\ \ c\nu} + \omega^a_{\ \mu d} X^{db}_{\ \ c\nu} + \omega^b_{\ \mu d} X^{ad}_{\ \ c\nu} - \omega^d_{\ \mu c} X^{ab}_{\ \ d\nu} - \Gamma^\lambda_{\ \mu\nu} X^{ab}_{\ \ c\lambda} \,. \tag{2.221}$$

In the second example d and λ are dummy indices. The summations over lower (contravariant) indices have a minus sign for both the spin connection and Christoffel connection terms. The spin connections are not tensors, as this holds for the Γ connections. However, the expressions with covariant derivatives are tensors.

2.5.3 Exterior derivatives in tangent space

In section 2.4.3 we introduced exterior derivatives. These are n-forms based on covariant derivatives. Considering a mixed-index tensor V^a_μ, we can interpret this as a vector-valued 1-form where a is the index of the vector component. So V^a would be a short notation of this 1-form. The concept of antisymmetric n-forms has been introduced in section 2.3. An exterior derivative of n-forms has been introduced in section 2.4.3, where a p-form is extended to a (p+1)-form by introducing the antisymmetric derivative operator $d\wedge$, see Eq. (2.167):

$$(d\wedge A)_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\ldots\mu_{p+1}]}. \tag{2.222}$$

We can extend this concept to the tangent space. First, the definition of the covariant derivative can be extended to mixed-index tensors by giving A one or more indices of tangent space:

$$(d\wedge A^b)_{\mu_1\ldots\mu_{p+1}} := (p+1)\partial_{[\mu_1} A^b_{\ \mu_2\ldots\mu_{p+1}]}. \tag{2.223}$$

This definition stands on its own, but in curved manifolds it becomes important to define a covariant exterior derivative of p-forms by basing this definition on the covariant derivative operator D_μ. In form notation, this kind of covariant derivative is written as

$$(D\wedge A^b)_{\mu_1\ldots\mu_{p+1}} := (p+1)D_{[\mu_1} A^b_{\ \mu_2\ldots\mu_{p+1}]}. \tag{2.224}$$

where the D's at the right-hand side are the “usual" covariant derivaties of coordinate index μ_1, etc., as defined in (2.220) for example. A may be a tensor of an arbitrary number of Greek and Latin indices, as before. The lower Greek indices define the p-form. In short indexless notation we can also write:

$$D\wedge A := (p+1)D_{[\mu_1} A_{\mu_2\ldots\mu_{p+1}]}. \tag{2.225}$$

We will come back to this short-hand notation later. For example Eq. (2.220) with exterior covariant derivative and coordinate indices $\mu \in \{0,1,2\}$ reads:

$$\begin{aligned} D\wedge V^a &= (D\wedge V^a)_{\mu\nu} \\ &= 2\,(D_0 V^a_{\ 1} + D_1 V^a_{\ 2} + D_2 V^a_{\ 0} - D_1 V^a_{\ 0} - D_2 V^a_{\ 1} - D_0 V^a_{\ 2}) \\ &= 2\,(D_0(V^a_{\ 1} - V^a_{\ 2}) + D_1(V^a_{\ 2} - V^a_{\ 0}) + D_2(V^a_{\ 0} - V^a_{\ 1})) \end{aligned} \tag{2.226}$$

where the “normal” covariant derivatives are defined as before, for example:

$$D_0 V^a_{\ 1} = \partial_0 V^a_{\ 1} + \omega^a_{\ 0b} V^b_{\ 1} - \Gamma^\lambda_{\ 01} V^a_{\ \lambda} \,. \tag{2.227}$$

The antisymmetry of the 2-form (2.226) requires

$$(D\wedge V^a)_{\mu\nu} = -(D\wedge V^a)_{\nu\mu} \tag{2.228}$$

from which it follows that interchanging the indices μ and ν gives the negative result of (2.226). That this is the case can be seen directly from the second line of the equation.

2.5.4 Tetrad postulate

Since the tangent space is uniquely related to the base manifold via the tetrad matrix $q^a{}_\mu$, the Γ-connections of the base manifold and spin connections of the tangent space are related to each other. To see how this is the case, we use the so-called *metric compatibility*, the statement that a vector must be the same when described in different coordinate systems. This is necessary for physical uniqueness, otherwise we would be dealing with a kind of mathematics that is not related to physical objects and processes. We introduced this concept in section 2.4.2 for vectors in the base manifold, and here we extend it to the tangent space in Cartan geometry.

Having this in mind, we can represent a covariant derivative of a tangent vector in two different ways. Denoting the orthonormal unit vectors in the base manifold by $\hat{e}_\nu$ and those of the tangent space by $\hat{e}_a$, we can write

$$DV = D_\mu V^\nu = (\partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda)\hat{e}_\nu \tag{2.229}$$

and

$$DV = D_\mu V^a = (\partial_\mu V^a + \omega^a{}_{\mu b} V^b)\hat{e}_a \tag{2.230}$$

for the same vector DV. In the latter case, one also speaks of a *mixed basis* because the derivative relates to the manifold as before. The latter equation can be transformed into the base manifold coordinates by transforming the coordinates V^a and the unit vectors $\hat{e}_a$ according to the rules introduced in section 2.5.1 and with renaming of dummy indices:

$$\begin{aligned} D_\mu V^a &= \left(\partial_\mu V^a + \omega^a{}_{\mu b} V^b\right)\hat{e}_a \\ &= \left(\partial_\mu (q^a{}_\nu V^\nu) + \omega^a{}_{\mu b} q^b{}_\lambda V^\lambda\right) q^\sigma{}_a \hat{e}_\sigma \\ &= q^\sigma{}_a \left(q^a{}_\nu \partial_\mu V^\nu + V^\nu \partial_\mu q^a{}_\nu + \omega^a{}_{\mu b} q^b{}_\lambda V^\lambda\right)\hat{e}_\sigma \\ &= \left(\partial_\mu V^\nu + q^\nu{}_a V^\lambda \partial_\mu q^a{}_\lambda + \omega^a{}_{\mu b} q^\nu{}_a q^b{}_\lambda V^\lambda\right)\hat{e}_\nu . \end{aligned} \tag{2.231}$$

Comparing with Eq. (2.229) then directly gives

$$\boxed{\Gamma^\nu{}_{\mu\lambda} = q^\nu{}_a \partial_\mu q^a{}_\lambda + q^\nu{}_a q^b{}_\lambda \omega^a{}_{\mu b} .} \tag{2.232}$$

Multiplying this equation with $q^\lambda{}_c$ and applying the same rules as above gives

$$q^\lambda{}_c \Gamma^\nu{}_{\mu\lambda} = q^\nu{}_a \omega^a{}_{\mu c} + q^\lambda{}_c q^\nu{}_a \partial_\mu q^a{}_\lambda \tag{2.233}$$

and multiplying with $q^b{}_\nu$ gives

$$q^b{}_\nu q^\lambda{}_c \Gamma^\nu{}_{\mu\lambda} = \omega^b{}_{\mu c} + q^\lambda{}_c \partial_\mu q^b{}_\lambda \tag{2.234}$$

which after renaming of indices is

$$\boxed{\omega^a{}_{\mu b} = q^a{}_\nu q^\lambda{}_b \Gamma^\nu{}_{\mu\lambda} - q^\lambda{}_b \partial_\mu q^a{}_\lambda .} \tag{2.235}$$

Thus, we have obtained the relations between both types of connections that we needed. Knowing one of them and the tetrad matrix allows us to compute the other connection.

We can further multiply Eq. (2.232) by $q^c{}_\nu$, obtaining (after applying the rules)

$$q^c{}_\nu \Gamma^\nu{}_{\mu\lambda} = \partial_\mu q^a{}_\lambda + q^b{}_\lambda \omega^a{}_{\mu b} . \tag{2.236}$$

As can be seen by comparison with (2.220), these are exactly the terms of the covariant derivative of the tensor $q^a{}_\nu$ in a mixed basis. It follows that

$$\boxed{D_\mu q^a{}_\nu = 0.} \tag{2.237}$$

This is called the *tetrad postulate*. It states that the covariant derivative of all tetrad elements vanishes.

This is a consequence of metric compatibility, which we postulated at the beginning of this section. As was shown earlier in Eq. (2.139), metric compatibility in the base manifold is defined by an analogue equation for the metric:

$$D_\sigma g_{\mu\nu} = 0. \tag{2.238}$$

If the space is Euclidean, we have

$$D_\sigma \eta_{\mu\nu} = 0 \tag{2.239}$$

for the Minkowski metric (2.176). Since this is also the metric for the tangent space, we can apply the corresponding definition of the covariant derivative:

$$D_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega^c{}_{\mu a} \eta_{cb} - \omega^c{}_{\mu b} \eta_{ac} = 0. \tag{2.240}$$

The Minkowski metric lowers the Latin indices of the spin connections so that we have

$$-\omega_{a\mu b} - \omega_{b\mu a} = 0 \tag{2.241}$$

or

$$\omega_{a\mu b} = -\omega_{b\mu a}. \tag{2.242}$$

Metric compatibility provides the property of antisymmetry for the spin connections. Notice that antisymmetry is only defined if the respective indices are all at the lower or upper positon. Despite this antisymmetry, the spin connection is not a tensor, as is also the case for the Γ connection. The symmetry properties of the Γ connection were discussed in section 2.4.2.

■ **Example 2.14** We compute some spin connection examples from Eq. (2.235). We need a given geometry defined by a tetrad and the Christoffel connection coefficients. We will use example 2.13, where we considered a transformation to spherical polar coordinates. We interpret this in such a way that the polar coordinates of the base manifold are transformed into cartesian coordinates of the tangent space. According to Eqs. (2.213) and (2.215) the tetrad matrix then is

$$\mathbf{q} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}. \tag{2.243}$$

The spin connections for spherical polar coordinates have been investigated in three variants in example 2.10:

1. a general connection,
2. a connection antisymmetrized in the non-diagonal lower indices,
3. a symmetric connection (used in Einsteinian relativity).

These functions for the Γs have to be inserted into Eq. (2.235), together with the tetrad elements of (2.243). Please notice that both the tetrad and inverse tetrad elements occur in (2.235). The $q^a{}_\nu$ are the elements of (2.243) and the $q^\nu{}_a$ are those of the inverted tetrad matrix, essentially Eq. (2.214).

The calculation is lengthy and has been automated in code example [31]. The results for case 1 (the general connection) are, for example:

$$\omega^{(1)}{}_{1(1)} = 0, \tag{2.244}$$
$$\omega^{(1)}{}_{1(2)} = -\sin(\theta)\,(A_9\, r \sin(\theta) + A_8 \cos(\theta)),$$
$$\omega^{(1)}{}_{1(3)} = A_8 \sin(\phi)\sin(\theta)^2 - A_9 \sin(\phi)\, r\cos(\theta)\sin(\theta) - \frac{A_7\cos(\phi)}{r}.$$

The As are constants contained in the Γs. Obviously they have different physical units, otherwise there would be problems in summation. In order to make it easier to distinguish between Latin and Greek indices, the numbers for Latin indices have been set in parentheses. For case 2 (above), the results are simpler:

$$\omega^{(1)}{}_{1(1)} = 0, \tag{2.245}$$
$$\omega^{(1)}{}_{1(2)} = \frac{A_{10}\cos(\theta)}{r^2\sin(\theta)},$$
$$\omega^{(1)}{}_{1(3)} = -\frac{A_{10}\sin(\phi)}{r^2},$$

and in case 3 (symmetric Christoffel connections), all spin connections vanish:

$$\omega^a{}_{\mu b} = 0, \tag{2.246}$$

indicating that there is no spin connection for a geometry without torsion. The antisymmetry holds even for the case where a and b are indices at different positions (upper and lower), because the metric in tangent space is the unit matrix. The antisymmetry has been checked using the code, and it is always

$$\omega^a{}_{\mu b} = -\omega^b{}_{\mu a} \tag{2.247}$$

as required. ■

2.5.5 Evans lemma

We now come to some more specifically relevant properties of Cartan geometry. The tetrad postulate can be modified to give a differential equation of second order for the tetrad elements. This equation is a wave equation and is fundamental for many fields of physics. The tetrad postulate (2.237) can be augmented by an additional derivative:

$$D^\mu(D_\mu q^a{}_\nu) = 0. \tag{2.248}$$

We introduced a covariant derivative with upper index in order to make μ a summation (dummy) index. Because the expression in the parentheses is a scalar function due to the tetrad postulate, we need not bother with how this derivative is defined, it reduces to a partial derivative by definition. So, we can write:

$$\partial^\mu(D_\mu q^a{}_\nu) = 0. \tag{2.249}$$

or

$$\partial^\mu(\partial_\mu q^a{}_\nu + \omega^a{}_{\mu b} q^b{}_\nu - \Gamma^\lambda{}_{\mu\nu} q^a{}_\lambda) = 0. \tag{2.250}$$

In a manifold with 4-vectors $[ct, X, Y, Z]$, the contravariant form of the partial derivative is defined in the usual way:

$$[\partial_0,\, \partial_1,\, \partial_2,\, \partial_3] = \left[\frac{1}{c}\frac{\partial}{\partial t},\, \frac{\partial}{\partial X},\, \frac{\partial}{\partial Y},\, \frac{\partial}{\partial Z}\right], \tag{2.251}$$

while the covariant form of the partial derivative is defined with sign changed for the spatial derivatives:

$$\left[\partial^0,\, \partial^1,\, \partial^2,\, \partial^3\right] = \left[\frac{1}{c}\frac{\partial}{\partial t},\, -\frac{\partial}{\partial X},\, -\frac{\partial}{\partial Y},\, -\frac{\partial}{\partial Z}\right]. \tag{2.252}$$

Therefore $\partial^\mu \partial_\mu$ is the d'Alambert operator

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} - \frac{\partial^2}{\partial Z^2}. \tag{2.253}$$

Then from Eq. (2.250) follows

$$\Box\, q^a{}_\nu + G^a{}_\nu = 0 \tag{2.254}$$

a wave equation with the tensor function

$$G^a{}_\nu = \partial^\mu(\omega^a{}_{\mu b} q^b{}_\nu) - \partial^\mu(\Gamma^\lambda{}_{\mu\nu} q^a{}_\lambda). \tag{2.255}$$

This equation can be made an eigenvalue equation by requiring $G^a{}_\nu$ to be split into a tetrad part and a scalar function R:

$$G^a{}_\mu = R q^a{}_\nu \tag{2.256}$$

with

$$R = q^\nu{}_a \left(\partial^\mu(\omega^a{}_{\mu b} q^b{}_\nu) - \partial^\mu(\Gamma^\lambda{}_{\mu\nu} q^a{}_\lambda)\right). \tag{2.257}$$

R contains only dummy indices and is a scalar function. Then (2.254) can be written as

$$\boxed{\Box\, q^a{}_\nu + R q^a{}_\nu = 0} \tag{2.258}$$

and is called the *Evans lemma*. It is a generally covariant eigenvalue equation. R plays the role of a curvature, as we will see in later chapters. The entire field of generally covariant quantum mechanics is based on this equation. The equation is highly non-linear, because R depends on the eigenfunction $q^a{}_\nu$ and the Christoffel and spin connections. In later chapters, in a first approximation, we will often assume that R is a constant.

2.5.6 Maurer-Cartan structure equations

The torsion and curvature tensors of Riemannian geometry can be transformed to 2-forms of Cartan geometry simply by defining

$$T^a{}_{\mu\nu} := q^a{}_\kappa T^\kappa{}_{\mu\nu}\,, \tag{2.259}$$

$$R^a{}_{b\mu\nu} := q^a{}_\rho q^\sigma{}_b R^\rho{}_{\sigma\mu\nu}. \tag{2.260}$$

Multiplication with tetrad elements replaces some Greek indices with Latin indices of the tangent space, so the torsion and curvature tensors defined in Eqs. (2.132) and (2.134) are made 2-forms of torsion and curvature. To these forms two foundational relations apply, which will be derived in this section, using the proof described in [11].

We first define forms of the Christoffel and spin connections similarly to (2.259) and (2.260):

$$\Gamma^a{}_{\mu\nu} := q^a{}_\lambda \, \Gamma^\lambda{}_{\mu\nu}, \tag{2.261}$$

$$\omega^a{}_{\mu\nu} := q^b{}_\nu \, \omega^a{}_{\mu b}. \tag{2.262}$$

These are both 2-forms as well. With these definitions, the tetrad postulate (2.237) can be formulated by inserting these definition into (2.236):

$$\Gamma^a{}_{\mu\nu} = \partial_\mu q^a{}_\nu + \omega^a{}_{\mu\nu}. \tag{2.263}$$

Inserting the definition of torsion

$$T^\kappa{}_{\mu\nu} := \Gamma^\kappa{}_{\mu\nu} - \Gamma^\kappa{}_{\nu\mu} \tag{2.264}$$

into (2.259) gives

$$T^a{}_{\mu\nu} = q^a{}_\kappa (\Gamma^\kappa{}_{\mu\nu} - \Gamma^\kappa{}_{\nu\mu}) = \Gamma^a{}_{\mu\nu} - \Gamma^a{}_{\nu\mu}, \tag{2.265}$$

and inserting relation (2.263) gives

$$T^a{}_{\mu\nu} = \partial_\mu q^a{}_\nu - \partial_\mu q^a{}_\nu + \omega^a{}_{\mu\nu} - \omega^a{}_{\mu\nu}. \tag{2.266}$$

This can be written with the $\wedge$ operator for antisymmetric forms, introduced in example 2.8 and section 2.4.3 as

$$(T^a)_{\mu\nu} = (d \wedge q^a)_{\mu\nu} + (\omega^a{}_b \wedge q^b)_{\mu\nu} \tag{2.267}$$

or, in short form notation:

$$\boxed{T^a = d \wedge q^a + \omega^a{}_b \wedge q^b} \tag{2.268}$$

which is called the *first Maurer-Cartan structure equation*.

The Riemann curvature tensor is defined

$$R^\lambda{}_{\rho\mu\nu} := \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}. \tag{2.269}$$

We define additional 1-forms of the Christoffel connection:

$$\Gamma^a{}_{\mu b} := q^a{}_\lambda q^\nu{}_b \Gamma^\lambda{}_{\mu\nu} \tag{2.270}$$

and from (2.263) we have

$$\Gamma^a{}_{\mu b} = q^\nu{}_b (\partial_\mu q^a{}_\nu + \omega^a{}_{\mu\nu}). \tag{2.271}$$

Then the curvature form (2.260) can be written:

$$R^a{}_{b\mu\nu} = \partial_\mu \Gamma^a{}_{\nu b} - \partial_\nu \Gamma^a{}_{\mu b} + \Gamma^a{}_{\mu c}\Gamma^c{}_{\nu b} - \Gamma^a{}_{\nu c}\Gamma^c{}_{\mu b}. \tag{2.272}$$

This is an antisymmetric 2-form that in form notation reads:

$$R^a{}_b = d \wedge \Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b. \tag{2.273}$$

The first term on the right-hand side is

$$d \wedge \Gamma^a{}_b = (d \wedge d \wedge q^a) q_b + d \wedge \omega^a{}_b = d \wedge \omega^a{}_b \tag{2.274}$$

because of the rule $d \wedge d \wedge a = 0$ for any form a. The second term of (2.273) is

$$\Gamma^a{}_c \wedge \Gamma^c{}_b = (q_c \, d \wedge q^a + \omega^a{}_c) \wedge (q_b \, d \wedge q^c + \omega^c{}_b). \tag{2.275}$$

The terms with the exterior derivative can be written with full indices as $q^\nu{}_c \partial_\mu q^a{}_\nu$, for example. The product is summed over by the dummy index ν.

From the Leibniz rule we find:

$$q^\lambda{}_c \partial_\mu q^a{}_\lambda + q^a{}_\lambda \partial_\mu q^\lambda{}_c = \partial_\mu (q^\lambda{}_c q^a{}_\lambda) = \partial_\mu \delta^a_c = 0, \tag{2.276}$$

therefore:

$$q^\lambda{}_c \partial_\mu q^a{}_\lambda = -q^a{}_\lambda \partial_\mu q^\lambda{}_c. \tag{2.277}$$

The summation on the left-hand and right-hand side can be contracted to functions

$$q^a{}_c = -q^a{}_c. \tag{2.278}$$

It follows

$$q^a{}_c = 0, \tag{2.279}$$

$$q^\nu{}_c \partial_\mu q^a{}_\nu = 0. \tag{2.280}$$

Therefore from (2.275):

$$\Gamma^a{}_c \wedge \Gamma^c{}_b = \omega^a{}_c \wedge \omega^c{}_b, \tag{2.281}$$

and with (2.274), we obtain from (2.273):

$$\boxed{R^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b} \tag{2.282}$$

which is called the *second Maurer-Cartan structure equation*. Using the definition of the exterior covariant derivative (2.198), the Maurer-Cartan structure equations can be written in the form

$$T^a = D \wedge q^a = d \wedge q^a + \omega^a{}_b \wedge q^b, \tag{2.283}$$

$$R^a{}_b = D \wedge \omega^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{2.284}$$

■ **Example 2.15** The validity of structure equations is demonstrated by an example of the transformation to spherical polar coordinates again. The tetrad was defined in example 2.13, and the spin connections in example 2.14. For the Gamma connections two versions were used: a general, asymmetric connection, and an antisymmetrized connection, as described in example 2.14. If we know the Gamma connection, we can compute the torsion form:

$$T^a{}_{\nu\mu} = q^a{}_\lambda \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \right) \tag{2.285}$$

and the Riemann form:

$$R^a{}_{b\mu\nu} = q^a{}_\sigma q^\mu{}_b \left(\partial_\mu \Gamma^\sigma{}_{\nu\rho} - \partial_\nu \Gamma^\sigma{}_{\mu\rho} + \Gamma^\sigma{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\rho} - \Gamma^\sigma{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\rho} \right). \tag{2.286}$$

This has been done in code example [32]. The antisymmetry of the form elements in the two last indexes is checked:

$$T^a{}_{\mu\nu} = -T^a{}_{\nu\mu}\,, \tag{2.287}$$
$$R^a{}_{b\mu\nu} = -R^a{}_{b\nu\mu}. \tag{2.288}$$

For example, we find with the antisymmetrized connection:

$$T^{(2)}{}_{11} = 0 \tag{2.289}$$
$$T^{(2)}{}_{13} = \frac{2\cos(\phi)\sin(\theta)}{\sqrt{3}} + \frac{2A_{10}\sin(\phi)\cos(\theta)}{\sqrt{3}r} \tag{2.290}$$
$$T^{(2)}{}_{31} = -\frac{2\cos(\phi)\sin(\theta)}{\sqrt{3}} - \frac{2A_{10}\sin(\phi)\cos(\theta)}{\sqrt{3}r} \tag{2.291}$$
$$R^{(1)}{}_{(3)11} = 0 \tag{2.292}$$
$$R^{(1)}{}_{(3)13} = -\frac{{A_{10}}^2\sin(\phi)\cos(\theta)}{r^3\sin(\theta)} \tag{2.293}$$
$$R^{(1)}{}_{(3)31} = \frac{{A_{10}}^2\sin(\phi)\cos(\theta)}{r^3\sin(\theta)} \tag{2.294}$$

Now all elements of the torsion and curvature form are computed, and we are ready to evaluate the right-hand sides of the structure equations (2.283) and (2.284), which in indexed form can be written:

$$D_\mu q^a{}_\nu - D_\nu q^a{}_\mu = \partial_\mu q^a{}_\nu - \partial_\nu q^a{}_\mu + \omega^a{}_{\mu b} q^b{}_\nu - \omega^a{}_{\nu b} q^b{}_\mu \tag{2.295}$$

and

$$D_\mu \omega^a{}_{\nu b} - D_\nu \omega^a{}_{\mu b} = \partial_\mu \omega^a{}_{\nu b} - \partial_\nu \omega^a{}_{\mu b} + \omega^a{}_{\mu c}\omega^c{}_{\nu b} - \omega^a{}_{\nu c}\omega^c{}_{\mu b}. \tag{2.296}$$

The covariant derivatives have been resolved according to their definitions for each permutation of (μ,ν). When the indices run over *all* values 1,2, this does not matter because the antisymmetry property sets all quantities with equal indices, for example $(\mu,\nu)=(1,1)$, to zero. In the code example [32], it is shown that the right-hand sides of the structure equations are equal to the definitions of the torsion and curvature form defined by (2.285) and (2.286). In addition, it is shown that re-computing the torsion and curvature tensors from their 2-forms gives the original tensors (2.264) and (2.269):

$$T^\rho{}_{\mu\nu} = q^\rho{}_a T^a{}_{\mu\nu}\,, \tag{2.297}$$
$$R^\sigma{}_{\rho\mu\nu} = q^\sigma{}_a\, q^b{}_\rho\, R^a{}_{b\mu\nu}. \tag{2.298}$$

■

3. The fundamental theorems of Cartan geometry

We have now arrived at a knowledge level in Cartan geometry that allows us to formulate the fundamental theorems of this geometry. Some of them are known for a longer time and have been mentioned in textbooks [12], but others have been found during the development of ECE theory. The theorems can easily be formulated in form notation but for the proofs we have to descend to the tensor notation and then climb to the form notation again.

3.1 Cartan-Bianchi identity

The first theorem is called *Cartan-Bianchi identity* [12] and is known as the *first Bianchi identity* or simply the *Bianchi identity* in Riemannian geometry without torsion. We have added the name of Cartan to stress that this theorem connects torsion and curvature in Cartan geometry. In form notation, it reads:

$$D \wedge T^a = R^a{}_b \wedge q^b. \tag{3.1}$$

This is an equation of 3-forms. To prove this equation, we recast the left-hand side into the right-hand side. Inserting the definition of the exterior covariant derivative gives, for the left-hand side:

$$(D \wedge T^a)_{\mu\nu\rho} = (d \wedge T^a)_{\mu\nu\rho} + (\omega^a{}_b \wedge T^b)_{\mu\nu\rho}. \tag{3.2}$$

Since this is an antisymmetric 3-form, we can write in commutator notation (see section 2.3):

$$D_{[\mu} T^a{}_{\nu\rho]} = \partial_{[\mu} T^a{}_{\nu\rho]} + \omega^a{}_{[\mu b} T^b{}_{\nu\rho]}\,. \tag{3.3}$$

In example 2.8, we had seen that the six index permutations of a 3-form can be reduced to three cyclic permutations of the indices, by use of antisymmetry properties. Therefore, we obtain

$$\begin{aligned} D_{[\mu} T^a{}_{\nu\rho]} =& \partial_\mu T^a{}_{\nu\rho} + \partial_\nu T^a{}_{\rho\mu} + \partial_\rho T^a{}_{\mu\nu} \\ & + \omega^a{}_{\mu b} T^b{}_{\nu\rho} + \omega^a{}_{\nu b} T^b{}_{\rho\mu} + \omega^a{}_{\rho b} T^b{}_{\mu\nu}\,. \end{aligned} \tag{3.4}$$

Please notice that the lower b index of the spin connection is not included in the permutations, because it is a Latin index of tangent space.

Inserting the definition of torsion

$$T^a{}_{\nu\mu} = \Gamma^a{}_{\mu\nu} - \Gamma^a{}_{\nu\mu} = q^a{}_\lambda \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right) \tag{3.5}$$

then leads to

$$\begin{aligned} D_{[\mu} T^a{}_{\nu\rho]} =& \partial_\mu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right)\right] + \partial_\nu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\rho\mu} - \Gamma^\lambda{}_{\mu\rho}\right)\right] \\ &+ \partial_\rho \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right)\right] \\ &+ \omega^a{}_{\mu b} q^b{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) + \omega^a{}_{\nu b} q^b{}_\lambda \left(\Gamma^\lambda{}_{\rho\mu} - \Gamma^\lambda{}_{\mu\rho}\right) \\ &+ \omega^a{}_{\rho b} q^b{}_\lambda \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right). \end{aligned} \tag{3.6}$$

The first term in brackets can be written with help of the Leibniz theorem:

$$\partial_\mu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right)\right] = \left(\partial_\mu q^a{}_\lambda\right) \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) + q^a{}_\lambda \left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right). \tag{3.7}$$

Applying the tetrad postulate (2.236) in the form

$$\partial_\mu q^a{}_\lambda = q^a{}_\nu \Gamma^\nu{}_{\mu\lambda} - q^b{}_\lambda \omega^a{}_{\mu b}. \tag{3.8}$$

then gives

$$\begin{aligned} \partial_\mu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right)\right] =& \left(q^a{}_\nu \Gamma^\nu{}_{\mu\lambda} - q^b{}_\lambda \omega^a{}_{\mu b}\right) \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) \\ &+ q^a{}_\lambda \left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right). \end{aligned} \tag{3.9}$$

Adding the first and fourth term of (3.6) causes the terms with $\omega^a{}_{\mu b}$ to cancel out:

$$\begin{aligned} &\partial_\mu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right)\right] + \omega^a{}_{\mu b} q^b{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) \\ &= q^a{}_\sigma \Gamma^\sigma{}_{\mu\lambda} \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) + q^a{}_\lambda \left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right). \end{aligned} \tag{3.10}$$

Putting all terms of (3.6) together, we obtain

$$\begin{aligned} &D_{[\mu} T^a{}_{\nu\rho]} = \\ &\quad q^a{}_\sigma \Gamma^\sigma{}_{\mu\lambda} \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) + q^a{}_\lambda \left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right) \\ &+ q^a{}_\sigma \Gamma^\sigma{}_{\nu\lambda} \left(\Gamma^\lambda{}_{\rho\mu} - \Gamma^\lambda{}_{\mu\rho}\right) + q^a{}_\lambda \left(\partial_\nu \Gamma^\lambda{}_{\rho\mu} - \partial_\nu \Gamma^\lambda{}_{\mu\rho}\right) \\ &+ q^a{}_\sigma \Gamma^\sigma{}_{\rho\lambda} \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right) + q^a{}_\lambda \left(\partial_\rho \Gamma^\lambda{}_{\mu\nu} - \partial_\rho \Gamma^\lambda{}_{\nu\mu}\right). \end{aligned} \tag{3.11}$$

Rearranging the sum:

$$\begin{aligned} &D_{[\mu} T^a{}_{\nu\rho]} = \\ &\quad q^a{}_\lambda \left[\left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\nu \Gamma^\lambda{}_{\mu\rho}\right) + q^a{}_\lambda \left(\partial_\nu \Gamma^\lambda{}_{\rho\mu} - \partial_\rho \Gamma^\lambda{}_{\nu\mu}\right) + q^a{}_\lambda \left(\partial_\rho \Gamma^\lambda{}_{\mu\nu} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right)\right] \\ &+ q^a{}_\sigma \left[\Gamma^\sigma{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\rho} - \Gamma^\sigma{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\rho} + \Gamma^\sigma{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu} - \Gamma^\sigma{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} + \Gamma^\sigma{}_{\rho\lambda} \Gamma^\lambda{}_{\mu\nu} - \Gamma^\sigma{}_{\mu\lambda} \Gamma^\lambda{}_{\rho\nu}\right]. \end{aligned} \tag{3.12}$$

Now, in the first line the dummy index λ is replaced by σ:

$$\begin{aligned} &D_{[\mu}T^a{}_{\nu\rho]} = \\ &\quad q^a{}_\sigma \Big[\left(\partial_\mu \Gamma^\sigma{}_{\nu\rho} - \partial_\nu \Gamma^\sigma{}_{\mu\rho}\right) + q^a{}_\sigma \left(\partial_\nu \Gamma^\sigma{}_{\rho\mu} - \partial_\rho \Gamma^\sigma{}_{\nu\mu}\right) + q^a{}_\sigma \left(\partial_\rho \Gamma^\sigma{}_{\mu\nu} - \partial_\mu \Gamma^\sigma{}_{\rho\nu}\right) \\ &+ \Gamma^\sigma{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\rho} - \Gamma^\sigma{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\rho} + \Gamma^\sigma{}_{\nu\lambda}\Gamma^\lambda{}_{\rho\mu} - \Gamma^\sigma{}_{\rho\lambda}\Gamma^\lambda{}_{\nu\mu} + \Gamma^\sigma{}_{\rho\lambda}\Gamma^\lambda{}_{\mu\nu} - \Gamma^\sigma{}_{\mu\lambda}\Gamma^\lambda{}_{\rho\nu}\Big]. \end{aligned} \tag{3.13}$$

This expression can be compared to the definition of the Riemann tensor (2.269) with some renumbering:

$$R^\sigma{}_{\rho\mu\nu} := \partial_\mu \Gamma^\sigma{}_{\nu\rho} - \partial_\nu \Gamma^\sigma{}_{\mu\rho} + \Gamma^\sigma{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\rho} - \Gamma^\sigma{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\rho}. \tag{3.14}$$

Obviously (3.13) is the cyclic sum of the Riemann tensor:

$$D_{[\mu}T^a{}_{\nu\rho]} = q^a{}_\sigma\, R^\sigma{}_{[\rho\mu\nu]} = q^a{}_\sigma\, R^\sigma{}_{[\mu\nu\rho]}. \tag{3.15}$$

According to the procedure in Eqs. (2.270-2.273), the Riemann tensor can be written as a 2-Form:

$$R^a{}_{b\nu\rho} = q^a{}_\sigma q^\mu{}_b\, R^\sigma{}_{\mu\nu\rho}. \tag{3.16}$$

To bring the right-hand side of (3.15) into this form, we extend the Riemann tensor by a unity term according to rule (2.207):

$$q^\tau{}_b q^b{}_\mu = \delta^\tau_\mu \tag{3.17}$$

and re-associate the products:

$$q^a{}_\sigma\, R^\sigma{}_{\mu\nu\rho} = R^a{}_{\mu\nu\rho} = R^a{}_{\tau\nu\rho}(q^\tau{}_b\, q^b{}_\mu)\delta^\tau_\mu = (R^a{}_{\tau\nu\rho}q^\tau{}_b)\; q^b{}_\tau \delta^\tau_\mu = R^a{}_{b\nu\rho}\; q^b{}_\mu. \tag{3.18}$$

Re-introducing the cyclic sum we have

$$D_{[\mu}T^a{}_{\nu\rho]} = q^b{}_{[\mu}\, R^a{}_{b\nu\rho]} = R^a{}_{b[\mu\nu}\, q^b{}_{\rho]}. \tag{3.19}$$

which in form notation gives the Cartan-Bianchi identity:

$$\boxed{D \wedge T^a = R^a{}_b \wedge q^b.} \tag{3.20}$$

■ **Example 3.1** We check the Cartan-Bianchi identity by computing all required elements according to example 2.15 (the transformation from cartesian to spherical polar coordinates). The Cartan-Bianchi identity (3.20) can be written in indexed form according to (3.19):

$$D_\mu T^a{}_{\nu\rho} + D_\nu T^a{}_{\rho\mu} + D_\rho T^a{}_{\mu\nu} = R^a{}_{b\mu\nu}\, q^b{}_\rho + R^a{}_{b\nu\rho}\, q^b{}_\mu + R^a{}_{b\rho\mu}\, q^b{}_\nu. \tag{3.21}$$

Resolving the covariant derivatives according to (3.4) this finally gives:

$$\begin{aligned} &\partial_\mu T^a{}_{\nu\rho} + \partial_\nu T^a{}_{\rho\mu} + \partial_\rho T^a{}_{\mu\nu} + \omega^a{}_{\mu b} T^b{}_{\nu\rho} + \omega^a{}_{\nu b} T^b{}_{\rho\mu} + \omega^a{}_{\rho b} T^b{}_{\mu\nu} \\ &= R^a{}_{b\mu\nu}\, q^b{}_\rho + R^a{}_{b\nu\rho}\, q^b{}_\mu + R^a{}_{b\rho\mu}\, q^b{}_\nu \end{aligned} \tag{3.22}$$

for each index triple (μ,ν,ρ). The left-hand and right-hand sides of this equation are computed in code example [33], and comparison shows that both sides are equal. We note that this result is obtained for both forms of Gamma connections (unconstrained and symmetrized). The torsion tensor is the same for both forms, but the Gamma and spin connections are different. The Cartan-Bianchi identity holds, irrespective of this difference. ■

3.2 Cartan-Evans identity

In the preceding section, it has been shown that the Cartan-Bianchi identity is a rigorous identity of the Riemannian manifold in which ECE theory is defined. The *Cartan-Evans identity* [13, 14, 15] is a new identity of differential geometry, and is the counterpart of the Cartan-Bianchi identity in dual-tensor representation. Both identities will be identical with the ECE field equations as will be worked out in later chapters. The Cartan-Bianchi identity is valid in the Riemannian manifold, and Cartan geometry in the Riemannian manifold is well known to be equivalent to Riemann geometry, thought to be the geometry of natural philosophy (physics). The same holds for the Cartan-Evans identity, which reads

$$D\wedge \widetilde{T}^a = \widetilde{R}^a{}_b \wedge q^b. \tag{3.23}$$

The concept of the Hodge dual was introduced at the end of section 2.3, and use of the Hodge dual for Maxwell's equations was already discussed in example 2.12. In this section, we introduce a Hodge dual connection for use in the covariant Hodge dual derivative. Thereafter, the proof of the Cartan-Evans identity is worked out in full analogy to the proof of the Cartan-Bianchi identity.

As has been seen in previous sections, only the antisymmetric part of the Christoffel connection is essential for Cartan geometry. Restricting the connection to the antisymmetric part, we can define the Hodge dual of the Christoffel connection according to Eq. (2.114) by

$$\Lambda^\lambda{}_{\mu\nu} := \widetilde{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}\Gamma^\lambda{}_{\alpha\beta}, \tag{3.24}$$

where $|g|^{-1/2}$ is the inverse square root of the modulus of the determinant of the metric, a weighting factor, by which the Levi-Civita symbol $\varepsilon_{\alpha\beta\mu\nu}$ is made the totally antisymmetric unit tensor, see section 2.3. In (3.24) the Levi-Civita symbol appears with mixed upper and lower indices. Therefore, we have to raise the first two indices in accordance with (2.115):

$$\Lambda^\lambda{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}g^{\rho\alpha}g^{\sigma\beta}\varepsilon_{\rho\sigma\mu\nu}\Gamma^\lambda{}_{\alpha\beta}. \tag{3.25}$$

Since the totally antisymmetric tensor (based on the Levi-Civity symbol) does not change its form for any coordinate transformation, we can use the metric of Minkowski space $\eta_{\mu\nu}$ with $|g|=1$:

$$\Lambda^\lambda{}_{\mu\nu} = \frac{1}{2}\eta^{\rho\alpha}\eta^{\sigma\beta}\varepsilon_{\rho\sigma\mu\nu}\Gamma^\lambda{}_{\alpha\beta}. \tag{3.26}$$

In this way, a new connection $\Lambda^\lambda{}_{\mu\nu}$ is defined. It is well known that the connection does not transform as a tensor under the general coordinate transformation, but the antisymmetry in its lower two indices means that its Hodge dual may be defined for each upper index of the connection as in the equation above. The antisymmetry of the connection is the basis for the Cartan-Evans identity, a new and fundamental identity of differential geometry. In ECE theory, it will become the inhomogeneous field equation as was already indicated for the homogeneous Maxwell equation in example 2.12. Note carefully that the torsion is a tensor, but the connection is not a tensor. The same is true of the Hodge duals of the torsion and connection.

In Eq. (2.147) the fundamental commutator equation of Riemannian geometry was derived:

$$[D_\mu, D_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T^\lambda{}_{\mu\nu}\,D_\lambda V^\rho, \tag{3.27}$$

which holds for any vector V^ρ of the base manifold. Now take the Hodge duals of either side of Eq.

(3.27) using:

$$[D_\mu, D_\nu]_{\mathrm{HD}} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}[D_\alpha, D_\beta], \tag{3.28}$$

$$\widetilde{R}^\rho{}_{\sigma\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}R^\rho{}_{\sigma\alpha\beta}, \tag{3.29}$$

$$\widetilde{T}^\lambda{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}T^\lambda{}_{\alpha\beta}. \tag{3.30}$$

Thus:

$$[D_\alpha, D_\beta]_{\mathrm{HD}}V^\rho = \widetilde{R}^\rho{}_{\sigma\alpha\beta}V^\sigma - \widetilde{T}^\lambda{}_{\alpha\beta}\, D_\lambda V^\rho. \tag{3.31}$$

Re-label indices in Eq. (3.31) to give:

$$[D_\mu, D_\nu]_{\mathrm{HD}}V^\rho = \widetilde{R}^\rho{}_{\sigma\mu\nu}V^\sigma - \widetilde{T}^\lambda{}_{\mu\nu}\, D_\lambda V^\rho. \tag{3.32}$$

The left-hand side of this equation is defined by:

$$[D_\mu, D_\nu]_{\mathrm{HD}}V^\rho := D_\mu(D_\nu V^\rho) - D_\nu(D_\mu V^\rho) \tag{3.33}$$

where the covariant derivatives must be defined by the Hodge dual connection (which was defined in Eq. (3.24)):

$$D_\mu V^\rho = \partial_\mu V^\rho + \Lambda^\rho{}_{\mu\lambda}V^\lambda, \tag{3.34}$$

$$D_\nu V^\rho = \partial_\nu V^\rho + \Lambda^\rho{}_{\nu\lambda}V^\lambda. \tag{3.35}$$

Working out the algebra of torsion and curvature according to Eqs. (2.132, 2.134):

$$\widetilde{T}^\lambda{}_{\mu\nu} = \Lambda^\lambda{}_{\mu\nu} - \Lambda^\lambda{}_{\nu\mu}, \tag{3.36}$$

$$\widetilde{R}^\lambda{}_{\mu\nu\rho} = \partial_\mu\Lambda^\lambda{}_{\nu\rho} - \partial_\nu\Lambda^\lambda{}_{\mu\rho} + \Lambda^\lambda{}_{\mu\sigma}\Lambda^\sigma{}_{\nu\rho} - \Lambda^\lambda{}_{\nu\sigma}\Lambda^\sigma{}_{\mu\rho}. \tag{3.37}$$

These are the Hodge dual torsion and curvature tensors of the Riemannian manifold.

Now we prove the Cartan Evans identity as follows. The identity is:

$$D\wedge\widetilde{T}^a = \widetilde{R}^a{}_b\wedge q^b \tag{3.38}$$

or

$$d\wedge\widetilde{T}^a + \omega^a{}_b\wedge\widetilde{T}^b = \widetilde{R}^a{}_b\wedge q^b. \tag{3.39}$$

In tensorial notation, in the Riemannian manifold Eqs. (3.38, 3.39) become:

$$D_\mu\widetilde{T}^a{}_{\nu\rho} + D_\rho\widetilde{T}^a{}_{\mu\nu} + D_\nu\widetilde{T}^a{}_{\rho\mu} = \widetilde{R}^a{}_{\mu\nu\rho} + \widetilde{R}^a{}_{\rho\mu\nu} + \widetilde{R}^a{}_{\nu\rho\mu}, \tag{3.40}$$

which can be written with permutation brackets as

$$D_{[\mu}\widetilde{T}^a{}_{\nu\rho]} = \widetilde{R}^a{}_{[\mu\nu\rho]} = q^a{}_\sigma\,\widetilde{R}^\sigma{}_{[\mu\nu\rho]}. \tag{3.41}$$

This equation is formally identical to (3.15) with the following correspondences:

$$\begin{aligned} &T\to\widetilde{T}, \\ &R\to\widetilde{R}, \\ &\Gamma\to\Lambda. \end{aligned} \tag{3.42}$$

Therefore, the proof of the Cartan-Evans identity can proceed in full analogy to that of the Cartan-Bianchi identity in the previous section. Starting with the equivalent of the left-hand side of Eq. (3.15),

$$D_{[\mu}\widetilde{T}^{a}{}_{\nu\rho]} = \partial_{[\mu}\widetilde{T}^{a}{}_{\nu\rho]} + \omega^{a}{}_{[\mu b}\widetilde{T}^{b}{}_{\nu\rho]}\,, \tag{3.43}$$

it follows that this expression is equal to its right-hand side equivalent of (3.15):

$$q^{a}{}_{\sigma}\,\widetilde{R}^{\sigma}{}_{[\mu\nu\rho]}. \tag{3.44}$$

It follows the validity of Eqs. (3.40 / 3.41), which are the counterpart of (3.19):

$$D_{[\mu}\widetilde{T}^{a}{}_{\nu\rho]} = q^{a}{}_{\sigma}\widetilde{R}^{\sigma}{}_{[\mu\nu\rho]}. \tag{3.45}$$

In form notation, this is the Cartan-Evans identity:

$$\boxed{D\wedge\widetilde{T}^{a} = \widetilde{R}^{a}{}_{b}\wedge q^{b}.} \tag{3.46}$$

In the proof of the Cartan-Bianchi identity, the tetrad postulate (2.237) was used. For the Cartan-Evans identity, this has to be used in the form with the Λ connection:

$$\partial_{\mu}q^{a}{}_{\lambda} + q^{b}{}_{\lambda}\,\omega^{a}{}_{\mu b} - q^{a}{}_{\nu}\Lambda^{\nu}{}_{\mu\lambda} = 0. \tag{3.47}$$

Obviously, here the spin connection ω depends on the Λ connection, not the Γ connection. (It would have been best to use a different symbol for ω, but we stay with ω for convenience.)

In summary, all geometric elements for the Cartan-Evans identity are obtained from the following equation set:

$$\Lambda^{\lambda}{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\eta^{\rho\alpha}\eta^{\sigma\beta}\varepsilon_{\rho\sigma\mu\nu}\Gamma^{\lambda}{}_{\alpha\beta}, \tag{3.48}$$

$$\omega^{a}{}_{\mu b} = q^{a}{}_{\nu}q^{\lambda}{}_{b}\Lambda^{\nu}{}_{\mu\lambda} - q^{\lambda}{}_{b}\partial_{\mu}q^{a}{}_{\lambda}, \tag{3.49}$$

$$\widetilde{T}^{\lambda}{}_{\mu\nu} = \Lambda^{\lambda}_{\mu\nu} - \Lambda^{\lambda}_{\nu\mu}, \tag{3.50}$$

$$\widetilde{R}^{\lambda}{}_{\mu\nu\rho} = \partial_{\mu}\Lambda^{\lambda}_{\nu\rho} - \partial_{\nu}\Lambda^{\lambda}_{\mu\rho} + \Lambda^{\lambda}_{\mu\sigma}\Lambda^{\sigma}_{\nu\rho} - \Lambda^{\lambda}_{\nu\sigma}\Lambda^{\sigma}_{\mu\rho}. \tag{3.51}$$

Alternatively, the Hodge duals of curvature and torsion can be computed from the original quantities (based on the Γ connection):

$$\widetilde{R}^{\rho}{}_{\sigma\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}R^{\rho}{}_{\sigma\alpha\beta}, \tag{3.52}$$

$$\widetilde{T}^{\lambda}{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}T^{\lambda}{}_{\alpha\beta}. \tag{3.53}$$

The 2-forms of $\widetilde{T}^{a}$ and $\widetilde{R}^{a}{}_{b}$ are obtainable in the usual way by multiplying with tetrad elements:

$$\widetilde{R}^{a}{}_{b\mu\nu} = q^{a}{}_{\rho}q^{\sigma}{}_{b}\widetilde{R}^{\rho}{}_{\sigma\mu\nu}, \tag{3.54}$$

$$\widetilde{T}^{a}{}_{\mu\nu} = q^{a}{}_{\lambda}\widetilde{T}^{\lambda}{}_{\mu\nu}\,. \tag{3.55}$$

One of the novel inferences of the Cartan-Evans identity is that there is a Hodge dual connection in the Riemannian manifold in four dimensions. This is a basic discovery, and may be developed in pure mathematics using any type of manifold. However, that development is not of interest to physics by Ockham's Razor, and the need to test a theory against experimental data.

■ **Example 3.2** In analogy to example 3.1, we check the Cartan-Evans identity by computing all required elements according to example 2.15 (the transformation from cartesian to spherical polar coordinates). The Cartan-Evans identity (3.46) can be written in indexed form according to (3.45):

$$D_\mu \widetilde{T}^a{}_{\nu\rho} + D_\nu \widetilde{T}^a{}_{\rho\mu} + D_\rho \widetilde{T}^a{}_{\mu\nu} = \widetilde{R}^a{}_{b\mu\nu}\, q^b{}_\rho + \widetilde{R}^a{}_{b\nu\rho}\, q^b{}_\mu + \widetilde{R}^a{}_{b\rho\mu}\, q^b{}_\nu. \tag{3.56}$$

Resolving the covariant derivatives according to (3.4) finally gives:

$$\begin{aligned} &\partial_\mu \widetilde{T}^a{}_{\nu\rho} + \partial_\nu \widetilde{T}^a{}_{\rho\mu} + \partial_\rho \widetilde{T}^a{}_{\mu\nu} + \omega^a{}_{\mu b} \widetilde{T}^b{}_{\nu\rho} + \omega^a{}_{\nu b} \widetilde{T}^b{}_{\rho\mu} + \omega^a{}_{\rho b} \widetilde{T}^b{}_{\mu\nu} \\ &= \widetilde{R}^a{}_{b\mu\nu}\, q^b{}_\rho + \widetilde{R}^a{}_{b\nu\rho}\, q^b{}_\mu + \widetilde{R}^a{}_{b\rho\mu}\, q^b{}_\nu \end{aligned} \tag{3.57}$$

for each index triple (μ, ν, ρ). The left-hand and right-hand sides of this equation are computed in code examples [34, 35]. There is however a difference. While the Cartan-Bianchi identity holds for any dimension n of Riemannian space, introducing the Hodge dual for the Cartan-Evans identity constrains the dimension of the dual 2-forms to $n-2$. So, to obtain comparable equations for both identities, we have to use $n = 4$ in the example, leading to 2-forms of the Hodge duals, as well. We have to extend the transformation matrix α (Eq. (2.213)) by the 0-component (time coordinate), resulting in

$$\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & \cos\theta & -r\sin\theta & 0 \end{bmatrix}. \tag{3.58}$$

The time coordinate remains unaltered by the transformation. The n-dimensional metric tensor $\mathbf{g}$ can be computed from the tetrad by (2.204):

$$g_{\mu\nu} = n\, q^a{}_\mu q^b{}_\nu \eta_{ab}. \tag{3.59}$$

We obtain the metric tensor:

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix} \tag{3.60}$$

which has the modulus of the determinant

$$|g| = r^4 \sin^2\theta. \tag{3.61}$$

The Levi-Civita symbol $\varepsilon_{\alpha\beta\mu\nu}$ in four dimensions can be computed by the formula

$$\varepsilon_{a_0,a_1,a_2,a_3} = \mathrm{sig}(a_3 - a_0)\ \mathrm{sig}(a_3 - a_1)\ \mathrm{sig}(a_3 - a_2)\ \mathrm{sig}(a_2 - a_0)\ \mathrm{sig}(a_2 - a_1)\ \mathrm{sig}(a_1 - a_0). \tag{3.62}$$

Now we have all of the elements that we need to evaluate Eqs. (3.48-3.51). With these, both sides of Eq. (3.56) can be evaluated as was done in example 3.1. We do this in two code examples. In the first example we repeat the calculations of example 3.1 (Cartan-Bianchi identity) in four dimensions [34]. An interesting result is that the Gamma connection, obtained with additional antisymmetry conditions, has only 4 free parameters. This is similar to Einstein's theory, where the symmetric metric is only determined up to 4 parameters that can be chosen freely and represent "free choice of coordinates". In Cartan geometry, the metric is uniquely defined from the tetrad.

The "free choice" appears in the connections. Therefore, this choice is also present in torsion and curvature, and finally in the fundamental theorems. Some results from code [34] are:

$$
\begin{aligned}
\Gamma^0{}_{12} &= A_4 r^2 \\
\Gamma^3{}_{01} &= -\frac{A_2}{r^2 \sin^2\theta} \\
\omega^{(2)}{}_{1(3)} &= \frac{A_3 \cos\phi}{r^2} \\
T^2{}_{13} &= \frac{2A_3}{r^2} \\
R^0{}_{213} &= \frac{A_1 r \sin\theta - A_2 \cos\theta}{\sin\theta}
\end{aligned}
\tag{3.63}
$$

As in the preceding example, comparison of both sides of the Cartan-Bianchi identity shows that both sides are equal, in this case for four dimensions.

In the second code example [35], the Hodge dual connections Λ and ω and the tensors $\widetilde{T}$, $\widetilde{R}$ and their corresponding 2-forms are computed. We obtain, for example, for the Hodge dual connections and tensors:

$$
\begin{aligned}
\Lambda^0{}_{03} &= \frac{A_4}{\sin^2\theta} \\
\Lambda^3{}_{01} &= \frac{\cos\theta}{r^2 \sin^3\theta} \\
\omega^{(2)}{}_{1(3)} &= \frac{\sin\phi\,\left(r^2\cos\theta\sin^2\theta + A_2\sin\theta - A_1 r\cos\theta\right)}{r^3\sin\theta} \\
\widetilde{T}^2{}_{13} &= 0 \\
\widetilde{T}^2{}_{02} &= -\frac{2A_3}{r^4\sin^2\theta} \\
\widetilde{R}^0{}_{213} &= 0 \\
\widetilde{R}^0{}_{202} &= -\frac{2A_2 r^2\cos\theta\sin\theta - A_2{}^2}{r^4\sin^4\theta}
\end{aligned}
\tag{3.64}
$$

Inserting these quantities into both sides of the Cartan-Evans identity, we find that both sides are equal, thus the identity holds in the chosen example. ∎

3.3 Alternative forms of Cartan-Bianchi and Cartan-Evans identity

3.3.1 Cartan-Evans identity

We showed that the Cartan-Evans identity is based on the fundamental definition of the Hodge dual torsion and curvature, and adds three of them in cyclic permutation.

By using the definition

$$\widetilde{T}^a{}_{\mu\nu} = q^a{}_\lambda \widetilde{T}^\lambda{}_{\mu\nu} \tag{3.65}$$

it follows that:

$$D_\mu \widetilde{T}^a{}_{\nu\rho} = (D_\mu q^a{}_\kappa)\widetilde{T}^\kappa{}_{\nu\rho} + q^a{}_\kappa D_\mu \widetilde{T}^\kappa{}_{\nu\rho} \tag{3.66}$$

using the Leibniz rule. We use the tetrad postulate:

$$D_\mu q^a{}_\kappa = 0 \tag{3.67}$$

to find that:

$$D_\mu \widetilde{T}^a{}_{\nu\rho} = q^a{}_\kappa D_\mu \widetilde{T}^\kappa{}_{\nu\rho} \,. \tag{3.68}$$

It follows that:

$$D_\mu \widetilde{T}^\kappa{}_{\nu\rho} + D_\nu \widetilde{T}^\kappa{}_{\rho\mu} + D_\rho \widetilde{T}^\kappa{}_{\mu\nu} = \widetilde{R}^\kappa{}_{\mu\nu\rho} + \widetilde{R}^\kappa{}_{\nu\rho\mu} + \widetilde{R}^\kappa{}_{\rho\mu\nu} \tag{3.69}$$

which is the Cartan-Evans identity written in the base manifold only. This equation may be rewritten as:

$$D_\mu T^{\kappa\mu\nu} = R^\kappa{}_\mu{}^{\mu\nu}. \tag{3.70}$$

The easiest way to see this is to take a particular example:

$$D_1 \widetilde{T}^\kappa{}_{23} + D_3 \widetilde{T}^\kappa{}_{12} + D_2 \widetilde{T}^\kappa{}_{31} = \widetilde{R}^\kappa{}_{123} + \widetilde{R}^\kappa{}_{312} + \widetilde{R}^\kappa{}_{231} \tag{3.71}$$

and then to take Hodge dual terms with upper indices according to Eq. (2.117). The constant factors cancel out. For the Levi-Civita symbol, the relation holds:

$$\varepsilon^{\mu\nu\alpha\beta} = -\varepsilon_{\mu\nu\alpha\beta}, \tag{3.72}$$

so that the sign change also cancels out. Furthermore, for a two-fold Hodge dual of a tensor T, the relation

$$\widetilde{\widetilde{T}} = \pm T \tag{3.73}$$

is valid, so that any sign change of this kind also cancels out. We take the Hodge dual of (3.71) term by term. The Levi-Civita symbol effects that in the expressions

$$\varepsilon^{\mu\nu\alpha\beta} T^\kappa{}_{\alpha\beta} \tag{3.74}$$

the index pairs $(\mu\nu)$ and $(\alpha\beta)$ are mutually exclusive:

$$\begin{aligned} &\mu \neq \nu, \quad \alpha \neq \beta, \\ &\mu \notin \{\alpha, \beta\}, \\ &\nu \notin \{\alpha, \beta\}. \end{aligned} \tag{3.75}$$

In total, we obtain for the Hodge dual example of (3.71):

$$D_1 T^{\kappa 01} + D_2 T^{\kappa 02} + D_3 T^{\kappa 03} = R^\kappa{}_1{}^{01} + R^\kappa{}_2{}^{02} + R^\kappa{}_3{}^{03} \tag{3.76}$$

which is an example of Eq. (3.70), the alternative form of the Cartan-Evans identity:

$$\boxed{D_\mu T^{\kappa\mu\nu} = R^\kappa{}_\mu{}^{\mu\nu}} \,. \tag{3.77}$$

3.3.2 Cartan-Bianchi identity

Eq. (3.77) is the most useful format of the Cartan-Evans identity. The Cartan-Bianchi identity can also be rewritten into this format. From Eq. (3.19) follows:

$$D_\mu T^\kappa{}_{\nu\rho} + D_\nu T^\kappa{}_{\rho\mu} + D_\rho T^\kappa{}_{\mu\nu} = R^\kappa{}_{\mu\nu\rho} + R^\kappa{}_{\nu\rho\mu} + R^\kappa{}_{\rho\mu\nu} \tag{3.78}$$

which is identical to (3.69), except that these are the original tensors instead of the Hodge duals. Therefore, the same derivation as above leads to the alternative form of the Cartan-Bianchi-identity:

$$\boxed{D_\mu \widetilde{T}^{\kappa\mu\nu} = \widetilde{R}^\kappa{}_\mu{}^{\mu\nu}} \,. \tag{3.79}$$

It should be noted that in the above contravariant forms of both identities, the Hodge dual and original tensors are interchanged, compared to the covariant forms (3.19) and (3.45).

3.3.3 Consequences of the identities

At the end of this section we will investigate the implications of antisymmetry of the Gamma connection in Cartan geometry. The Gamma connection has to have at least antisymmetric parts with

$$\Gamma^{\lambda}{}_{\mu\nu} = -\Gamma^{\lambda}{}_{\nu\mu}. \tag{3.80}$$

If $\mu = \nu$, the commutator vanishes, as do the torsion and curvature tensors. If there are only symmetric parts in the connection:

$$\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu} \neq ?\, 0 \tag{3.81}$$

then torsion vanishes, leading to the special case of (3.77):

$$R^{\kappa}{}_{\mu}{}^{\mu\nu} = 0. \tag{3.82}$$

It has been shown by computer algebra [16, 17] that all of the metrics of the Einstein field equation in the presence of matter give the erroneous result:

$$R^{\kappa}{}_{\mu}{}^{\mu\nu} \neq ?\, 0, \tag{3.83}$$

$$D_{\mu} T^{\kappa\mu\nu} = ?\, 0. \tag{3.84}$$

This contradicts basic properties of Cartan geometry, the superset of Riemannian geometry, and therefore Eq. (3.77) is a constraint for theories like Einsteinian relativity, which are based on Riemannian geometry. This error has been perpetuated uncritically for nearly a hundred years, and allowed to create a defective cosmology that should be discarded by scholars. The cosmology of the Standard Model is baseless and incorrect, and should be replaced by ECE cosmology, which is based on torsion.

3.4 Further identities

There are some other identities which are not as significant to the field equations of ECE theory, but which represent new insights into Cartan geometry. They were developed as part of ECE theory, and we present them here, partially without proofs (which can be found in the Unified Field Theory (UFT) Section of www.aias.us).

3.4.1 Evans torsion identity (first Evans identity)

From the Cartan-Bianchi identity another identity can be derived, containing torsion terms only. This is the *Evans torsion identity* [14]. In explicit form it reads

$$T^{\kappa}{}_{\lambda\nu}\, T^{\lambda}{}_{\sigma\mu} + T^{\kappa}{}_{\lambda\mu}\, T^{\lambda}{}_{\nu\sigma} + T^{\kappa}{}_{\lambda\sigma}\, T^{\lambda}{}_{\mu\nu} = 0 \tag{3.85}$$

and can be written in short form with permutation brackets:

$$\boxed{T^{\kappa}{}_{\lambda[\nu}\, T^{\lambda}{}_{\sigma\mu]} = 0}\,. \tag{3.86}$$

The identity can be rewritten in form notation as

$$T^{\kappa}{}_{\lambda} \wedge T^{\lambda} = 0, \tag{3.87}$$

or by multiplying with $q^{a}{}_{\kappa}$:

$$\boxed{T^{a}{}_{\lambda} \wedge T^{\lambda} = 0}\,. \tag{3.88}$$

Here $T^{a}{}_{\lambda}$ is a 1-form and T^{λ} is a 2-form, making up a 3-form on the left-hand side. The proof consists mainly of inserting the definitions of torsion into the Cartan-Bianchi identity, and can be found in the literature [14].

3.4.2 Jacobi identity

The Jacobi identity [18] is an exact identity used in field theory and general relativity. It is an operator identity that applies to covariant derivatives and group generators [19] alike. It is very rarely proven in all detail, so we are providing the following complete proof. The *Jacobi identity* is a permuted sum of three covariant derivatives:

$$\boxed{[D_\rho,[D_\mu,D_\nu]]+[D_\nu,[D_\rho,D_\mu]]+[D_\mu,[D_\nu,D_\rho]]=0}\,. \tag{3.89}$$

For the proof, we expand the commutators on the-left hand side:

$$\begin{aligned} L.H.S =&[D_\rho,D_\mu D_\nu-D_\nu D_\mu]+[D_\nu,D_\rho D_\mu-D_\mu,D_\rho]+[D_\mu,D_\nu D_\rho-D_\rho D_\nu] \\ =&D_\rho(D_\mu D_\nu-D_\nu D_\mu)-(D_\mu D_\nu-D_\nu D_\mu)D_\rho \\ &+D_\nu(D_\rho D_\mu-D_\mu,D_\rho)-(D_\rho D_\mu-D_\mu,D_\rho)D_\nu \\ &+D_\mu(D_\nu D_\rho-D_\rho D_\nu)-(D_\nu D_\rho-D_\rho D_\nu)D_\mu, \end{aligned} \tag{3.90}$$

and this expansion is regarded as an expansion by algebra which sums up to zero:

$$\begin{aligned} L.H.S =&D_\rho D_\mu D_\nu-D_\rho D_\nu D_\mu-D_\mu D_\nu D_\rho+D_\nu D_\mu D_\rho \\ &+D_\nu D_\rho D_\mu-D_\nu D_\mu D_\rho-D_\rho D_\mu D_\nu+D_\mu D_\rho D_\nu \\ &+D_\mu D_\nu D_\rho-D_\mu D_\rho D_\nu-D_\nu D_\rho D_\mu+D_\rho D_\nu D_\mu \\ =&0, \end{aligned} \tag{3.91}$$

Q.E.D. The Jacobi identity can also be written in an alternative form:

$$\boxed{[[D_\mu,D_\nu],D_\rho]+[[D_\rho,D_\mu],D_\nu]+[[D_\nu,D_\rho],D_\mu]=0}\,. \tag{3.92}$$

3.4.3 Bianchi-Cartan-Evans identity

Einsteinian general relativity uses the second Bianchi identity, which is obtained from the covariant derivative of the first Bianchi identity. General relativity ignores torsion, but the same procedure can be applied to the Cartan-Bianchi identity of Cartan geometry, which contains both torsion and curvature. The result is the *Bianchi-Cartan-Evans identity* [20, 21, 22]:

$$\boxed{D_\mu D_\lambda T^\kappa{}_{\nu\rho}\;+D_\rho D_\lambda T^\kappa{}_{\mu\nu}\;+D_\nu D_\lambda T^\kappa{}_{\rho\mu}\;=D_\mu R^\kappa{}_{\lambda\nu\rho}+D_\rho R^\kappa{}_{\lambda\mu\nu}+D_\nu R^\kappa{}_{\lambda\rho\mu}}\,. \tag{3.93}$$

The proof was first carried out in UFT paper 88 [20] which is the most read paper of ECE theory. Two variants of this identity can be produced by cyclic permutation of (μ,ν,ρ). Eq. (3.93) is the correct "second Bianchi identity" augmented by torsion. In Einsteinian theory the incorrect version

$$D_\mu R^\kappa{}_{\lambda\nu\rho}+D_\rho R^\kappa{}_{\lambda\mu\nu}+D_\nu R^\kappa{}_{\lambda\rho\mu}\;=?\;0 \tag{3.94}$$

is used. This follows from (3.93) by arbitrarily omitting torsion. The Einstein-Hilbert field equation is derived from this erroneously truncated "second Bianchi identity" [20]. Therefore, all solutions of the Einstein-Hilbert field equation are inconsistent.

It should be noted that the Bianchi-Cartan-Evans identity gives no information beyond what is provided by the Cartan-Bianchi identity, because it is derived from the latter by differentiation and therefore not independent.

3.4.4 Jacobi-Cartan-Evans identity

The Jacobi identity can be used to derive another identity. When the terms of the Cartan-Bianchi identity are inserted into the Jacobi identity (3.89), the relation

$$\begin{aligned}
&\left([D_\rho,[D_\mu,D_\nu]]+[D_\nu,[D_\rho,D_\mu]]+[D_\mu,[D_\nu,D_\rho]]\right)V^\kappa \\
&=\left(D_\rho R^\kappa{}_{\lambda\mu\nu}+D_\nu R^\kappa{}_{\lambda\rho\mu}+D_\mu R^\kappa{}_{\lambda\nu\rho}\right)V^\lambda \\
&\quad-\left(T^\lambda{}_{\mu\nu}\,[D_\rho,D_\lambda]+T^\lambda{}_{\rho\mu}\,[D_\nu,D_\lambda]+T^\lambda{}_{\nu\rho}\,[D_\mu,D_\lambda]\right)V^\kappa \\
&=0
\end{aligned} \tag{3.95}$$

follows, where the Jacobi identity has been applied to an arbitrary vector V^κ of the base manifold. Because the Jacobi identity sums to zero, we obtain the equation

$$\begin{aligned}
&\left(D_\rho R^\kappa{}_{\lambda\mu\nu}+D_\nu R^\kappa{}_{\lambda\rho\mu}+D_\mu R^\kappa{}_{\lambda\nu\rho}\right)V^\lambda \\
&=\left(T^\lambda{}_{\mu\nu}\,[D_\rho,D_\lambda]+T^\lambda{}_{\rho\mu}\,[D_\nu,D_\lambda]+T^\lambda{}_{\nu\rho}\,[D_\mu,D_\lambda]\right)V^\kappa.
\end{aligned} \tag{3.96}$$

Further transformations, described in [22], give

$$\boxed{D_\rho R^\kappa{}_{\lambda\mu\nu}+D_\nu R^\kappa{}_{\lambda\rho\mu}+D_\mu R^\kappa{}_{\lambda\nu\rho}=T^\alpha{}_{\mu\nu}\,R^\kappa{}_{\lambda\rho\alpha}+T^\alpha{}_{\rho\mu}\,R^\kappa{}_{\lambda\nu\alpha}+T^\alpha{}_{\nu\rho}\,R^\kappa{}_{\lambda\mu\alpha}}, \tag{3.97}$$

which is called the *Jacobi-Cartan-Evans identity*.

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Examples of computer algebra (Maxima)

[23] Ex2.4.wxm

[24] Ex2.5.wxm

[25] Ex2.10.wxm

[26] CH02-diag-metric.wxm

[27] CH02-nondiag-metric.wxm

[28] Ex2.11.wxm

[29] Ex2.12.wxm

[30] Ex2.13.wxm

[31] Ex2.14.wxm

[32] Ex2.15.wxm

[33] Ex3.1.wxm

[34] Ex3.2a.wxm

[35] Ex3.2b.wxm