2.2 Derivative at a Point

Wednesday, September 6, 2023

Objectives:

1. Instantaneous Rate of change (IRC) continued
2. Computing the derivative at a point.
3. Understanding the limit definition of derivative.

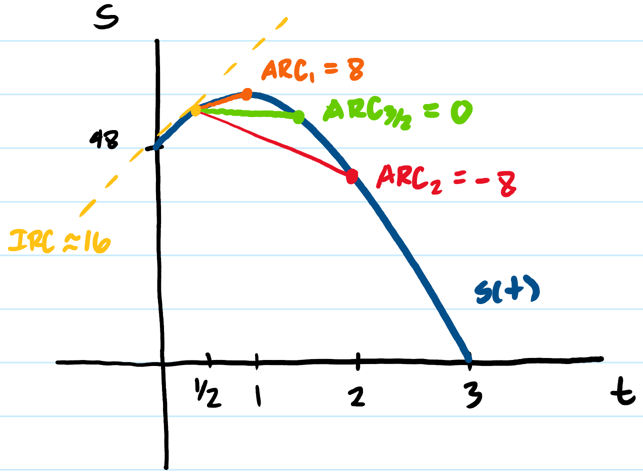
Recall the Average Rate of Change (ARC)

$$ARC = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \begin{array}{l}\to \text{change in } y \\ \to \text{change in } x\end{array}$$

or

$$ARC = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \begin{array}{l}\to \text{change in output} \\ \to \text{change in input}\end{array}$$

Example: $s(t) = 64 - 16(t-1)^2$



$$ARC_n = \frac{\overset{\substack{\text{end point}\\ \downarrow}}{s(n)} - \overset{\substack{\text{start point}\\ \downarrow}}{s(1/2)}}{n - 1/2} \to \text{interval}$$

$$IRC_{1/2} = \lim_{n \to 1/2} \frac{s(n) - s(1/2)}{n - 1/2}$$

$$= \lim_{n \to 1/2} \frac{64 - 16(n-1)^2 - [64 - 16(1/2 - 1)^2]}{n - 1/2}$$

$$= \lim_{n \to 1/2} \frac{64 - 16(n-1)^2 - [64 - 16(1/2 - 1)^2]}{n - 1/2}$$

$$= \lim_{n \to 1/2} \frac{-16(n-1)^2 + 4}{n - 1/2} \rightarrow \text{expanded } (n-1)^2$$

$$= \lim_{n \to 1/2} \frac{-16(n^2 - 2n + 1) + 4}{n - 1/2}$$

$$= \lim_{n \to 1/2} \frac{-16n^2 + 32n - 16 + 4}{n - 1/2}$$

$$= \lim_{n \to 1/2} \frac{-16n^2 + 32n - 12}{n - 1/2}$$

$$= \lim_{n \to 1/2} \frac{-4(2n-1)(2n-3)}{n - 1/2} \cdot \left(\frac{2}{2}\right)$$

$$= \lim_{n \to 1/2} \frac{-8\cancel{(2n-1)}(2n-3)}{\cancel{2n-1}}$$

$$= -8(2(1/2) - 3)$$

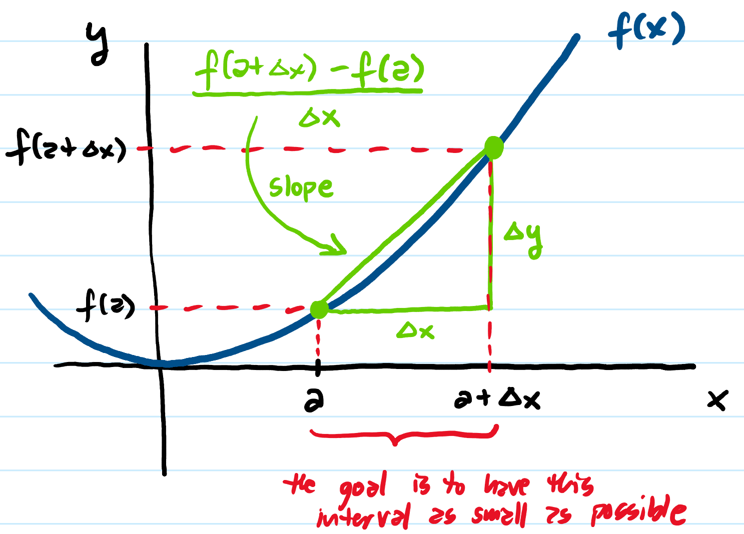
$$IRC_{1/2} = 16$$

Instantaneous Rate of Change (IRC)

$$IRC_a = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$ at a point a,

where f is a function in the interval $[a, b]$.

Derivative at a point



Slope of secant line from point $(a, f(a))$ to $(a+\Delta x, f(a+\Delta x))$ is

$$\frac{f(a+\Delta x) - f(a)}{\Delta x} \quad \begin{matrix} \rightarrow \text{change in output} \\ \rightarrow \text{change in input} \end{matrix}$$

Have $\Delta x \to 0$ to find the derivative of f at point a.

$$\frac{df}{dx}(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} \longrightarrow \text{the limit definition of derivative at point } a$$

$\frac{df}{dx}(a)$ ↓ derivative of f at point a

Alternative notation: $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Examples:

- Use the limit definition of derivative at point a to find $f'(5)$ where $f(x) = \frac{1}{2}x + 3$.

$$f'(5) = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2}(5+h) \cancel{+3} - [\frac{1}{2}(5) \cancel{+3}]}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2}(5+h) - 5/2}{h}$$

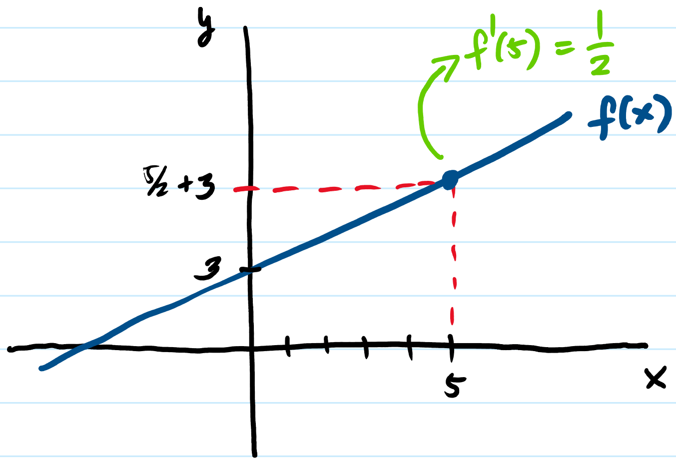
$$= \lim_{h \to 0} \frac{\cancel{5/2} + h/2 - \cancel{5/2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}/2}{\cancel{h}}$$

$$= \lim_{h \to 0} \frac{1}{2}$$

$$f'(5) = \frac{1}{2}$$

2



This makes sense since $f(x)$ is a line with slope $1/2$.

- Find $f'(2)$ where $f(x) = \frac{1}{x}$.

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$

use common denominator method

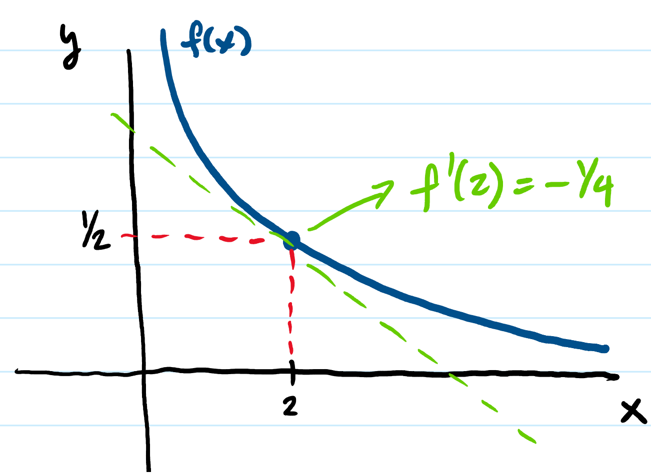
$$= \lim_{h \to 0} \left(\frac{1}{2+h} - \frac{1}{2}\right)\left(\frac{1}{h}\right)$$

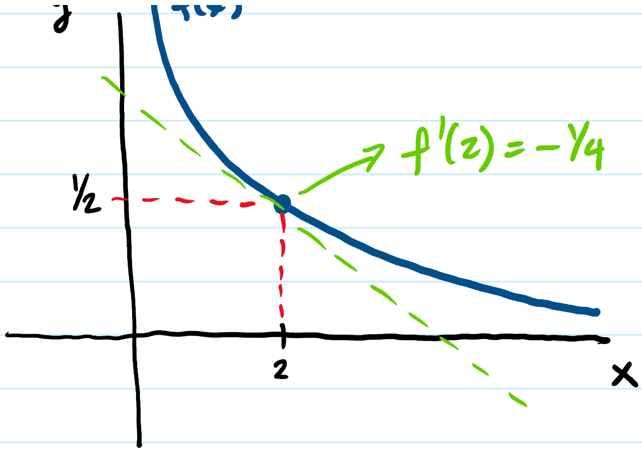
$$= \lim_{h \to 0} \frac{2-(2+h)}{(2+h)(2)}\left(\frac{1}{h}\right)$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{(2+h)(2)\cancel{h}}$$

$$= \lim_{h \to 0} \frac{-1}{(2+h)(2)}$$

$$f'(2) = -\frac{1}{4}$$





y
f(x)
$f'(2) = -1/4$
1/2
2
x