3.2 The Exponential Function

Thursday, September 28, 2023

Objectives:

1. Derivatives of the exponential and exponential function

Recall: Exponentials and logaritims.

exponential $\rightarrow$ $a^b = N$

logarithm $\rightarrow$ $\log_a(N) = b$

they are inverses $\rightarrow$ $a^{\log_a(x)} = x$

$\rightarrow$ $\log_a(a^x) = x$

natural log $\rightarrow$ $\ln(x)$ $\rightarrow$ base e.

$\rightarrow$ $\ln(e) = 1$ or $e^{\ln(1)} = 1$

$e^{\ln(e)} = e$

Recall: Limit Definition of the derivative

Given $f(x)$, then

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Derivative of the Exponential Function

Suppose $f(x) = a^x$,

then
$$\begin{aligned}
\frac{df}{dx} &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} \\
&= \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} \\
&= a^x \lim_{h \to 0} \frac{(a^h - 1)}{h} \\
&= a^x \lim_{h \to 0} \underbrace{\frac{a^{(0+h)} - a^0}{h}}
\end{aligned}$$

$$= \underbrace{\lim_{h\to 0} \frac{a^h - a^0}{h}}_{\frac{d}{dx}f(0)}$$

$$f'(x) = a^x f'(0)$$

what is this?

Definitions of e

1. $\displaystyle \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

2. $\displaystyle \lim_{h\to 0} \frac{e^h - 1}{h} = 1$

Facts: ① If $f(x) = e^x$, then $\displaystyle f'(0) = \lim_{h\to 0}\frac{e^h - 1}{h} = \ln(e) = 1$

So, if $a = e$, then

$$f(x) = e^x, \quad f'(x) = a^x f'(0) = e^x \ln(e) = e^x.$$

In general, $f(x) = a^x$, $f'(x) = a^x \ln(a)$.

Derivatives of Logarithmic functions

Facts: ② If $f(x)$ and $g(x)$ are inverses of each other, then

$$g'(x) = \frac{1}{f'(g(x))}$$

So, if $f(x) = e^x$ and $g(x) = \ln(x)$, then

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{e^{g(x)}} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}.$$

So, if $g(x) = \ln(x)$, then $g'(x) = \dfrac{1}{x}$ when $x > 0$,

or if $g(x) = \ln(|x|)$, then $g'(x) = \dfrac{1}{x}$ when $x \neq 0$.

or if $g(x) = \ln(|x|)$, then $g'(x) = \frac{1}{x}$ when $x \neq 0$.

Note that this is for $a = e$ only. How do we generalized it to a.

Facts: ③ change of base formula

$$\log_a b = \frac{\log_c b}{\log_c a} \quad \text{(from base } a \text{ to base } c\text{)}$$

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(a)}\right)$$

$$= \frac{1}{\ln(a)} \frac{d}{dx}(\ln x)$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln(a)}$$

Summary:

- $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx} a^x = a^x \ln(a)$
- $\frac{d}{dx} \ln(x) = \frac{1}{x}$
- $\frac{d}{dx} \log_a(x) = \frac{1}{x\ln(a)}$

Examples:

- Find $f'(x)$ given $f(x) = 3^x$.

  $f(x) = 3^x \rightarrow a = 3 \rightarrow f'(x) = 3^x \ln(3)$

- Find $f'(x)$ given $f(x) = \log_2(x)$.

$$f(x) = \log_2(x) = \frac{1}{x\ln(2)}$$