4.3 Optimization Part 2

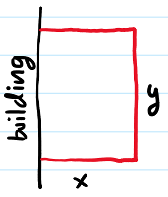
Monday, November 6, 2023

Wednesday, November 8, 2023

Objectives:

1. Continue introducing optimization
2. Basic optimization method with constraints.

Example Problem 1:



Goal: Find dimensions of the rectangle so that we can use 500 ft of fencing material.

Constraints: $2x + y = 500$ ft perimeter

Objective Function: $A(x) = xy$ area enclosed by fence

→ Solve for the constraint y.

$$2x + y = 500$$
$$y = 500 - 2x$$

→ Substitute it to $A(x)$.

$$A(x) = x(500 - 2x)$$
$$A(x) = 500x - 2x^2$$

→ domain : $-\infty < x < \infty$

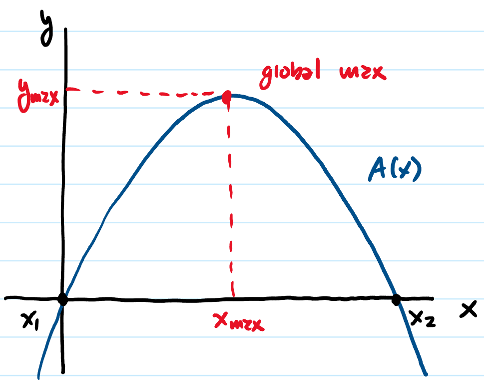
roots: $0 = 500x - 2x^2$

$0 = x(500 - 2x)$

↳ $x_1 = 0$

↳ $x_2 = 250$

} width can be in between $[0, 250]$



→ Find $\frac{dA}{dx}$ and set $\frac{dA}{dx} = 0$.

$$\frac{dA}{dx} = 500 - 4x$$

$$\frac{\quad}{dx}$$

$$0 = 500 - 4x$$
$$4x = 500$$
$$x = 125 \longrightarrow \text{critical point}$$

→ Is it max or min?

$$\frac{d^2A}{dx^2} = -4 \longrightarrow \text{always negative (concave down)}$$
So, critical point is global max.

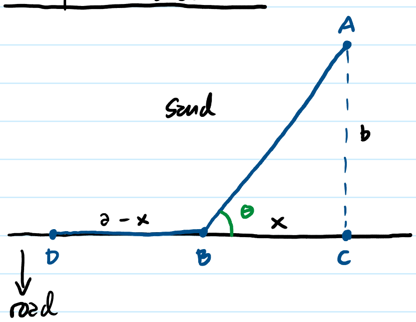
→ Critical point:
$x = 125$ ft width with max area of

$$A(125) = 500(125) - 2(125)^2 \approx 31250 \text{ ft}^2.$$

Dimensions: width → $x = 125$ ft
length → $2x + y = 500$
$$y = 500 - 2x$$
$$y = 500 - 2(125)$$
$$y = 250$$

So, we need the rectangle be 125×250 subject to 500 ft of fencing.

Example Problem 2:



Goal: Determine point B along the road where you should transition to minimize total travel time from point D to A.

→ v is speed on the road.
→ w is speed on the sand; $w < v$

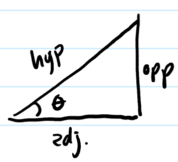
Constraints:
1. $\overline{DB}$ speed v.
2. $\overline{BA}$ speed $w < v$.
3. $\overline{DB}$ distance is $a - x$
4. $\overline{BA}$ distance $\sqrt{x^2 + b^2}$

Objective function: total travel time.

$$f(x) = \frac{a - x}{v} + \frac{\sqrt{x^2 + b^2}}{w}$$

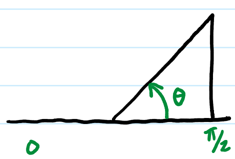
$$\text{velocity} = \frac{\text{distance}}{\text{time}} \longrightarrow \text{time} = \frac{\text{distance}}{\text{velocity}}$$

Angular Version:
Constraints:
- $\overline{BA}$ distance



$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{x}{\overline{BA}} \longrightarrow \overline{BA} = x\cos(\theta)$$

- Angle $0 \le \theta \le \pi/2$



Objective function:

- $$f(x, \theta) = \frac{a - x}{v} + \frac{x\cos(\theta)}{w}$$

Critical Points:

$$f'(x) = -\frac{1}{v} + \frac{x}{w\sqrt{x^2+b^2}}$$

$$\downarrow$$

$$0 = -\frac{1}{v} + \frac{x}{w\sqrt{x^2+b^2}}$$

$$\frac{1}{v} = \frac{x}{w\sqrt{x^2+b^2}}$$

$$\left(w\sqrt{x^2+b^2}\right)^2 = (xv)^2$$
$$w^2(x^2+b^2) = x^2v^2$$
$$w^2x^2+w^2b^2 = x^2v^2$$
$$w^2b^2 = x^2v^2 - w^2x^2$$
$$w^2b^2 = (v^2-w^2)x^2$$
$$\frac{w^2b^2}{v^2-w^2} = x^2$$

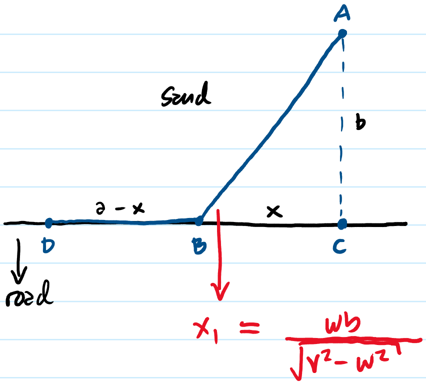
$$\downarrow$$

$$x_1 = \frac{wb}{\sqrt{v^2-w^2}} \longrightarrow \text{critical point}$$

where did a go?

$$f''(x) = \frac{b^2}{w(x^2+b^2)^{3/2}} \longrightarrow f''(x_1) > 0 \text{ (concave up) (minimum)}$$

* the initial point D: $f(0) = \frac{a}{v} + \frac{b}{w}$, $f'(0) = -1/v$
* the end point C: $f(a) = \frac{\sqrt{a^2+b^2}}{w}$, $f'(a) = -\frac{1}{v} + \frac{a}{w\sqrt{a^2+b^2}} \neq 0$
* $\overline{DC} = a$



Depending on where you start:

- If start close to C, then cut directly.
- If start away from C, then cut at a distance x_1.