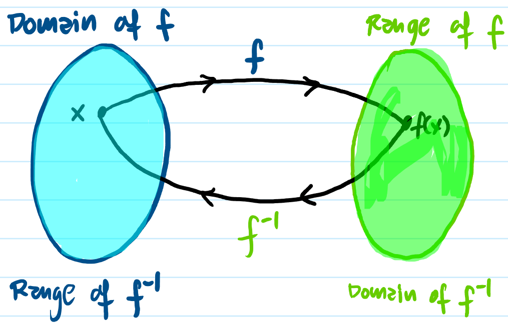
3.6 Inverse Functions

Wednesday, October 11, 2023

Objectives

1. Finding inverse functions
2. Finding derivatives of inverse functions

Recall: Intuition on Inverse Functions



This means that

1. $f(f^{-1}(x)) = x$
2. $f^{-1}(f(x)) = x$

Notation: Inverse of $f(x)$ is $f^{-1}(x)$.

* Note that $f^{-1}(x) \neq \frac{1}{f(x)}$
* $(f(x))^{-1} = \frac{1}{f(x)}$

Finding the Inverse of a Function

Example:

1. Given $f(x) = 3x - 2$. Find $f^{-1}(x)$.

$y = 3x - 2$ → rename $f(x)$ to y.

$x = 3y - 2$ → switch x with y and y with x.

$x + 2 = 3y$

$\frac{1}{3}(x+2) = y$ → solve for y.

$y = \frac{1}{3}(x+2)$

$f^{-1}(x) = \frac{1}{3}(x+2)$ → rename y into $f^{-1}(x)$

Checking: $f(f^{-1}(x)) = 3(f^{-1}(x)) - 2$

$= \cancel{3}\left(\frac{1}{\cancel{3}}(x+2)\right) - 2$

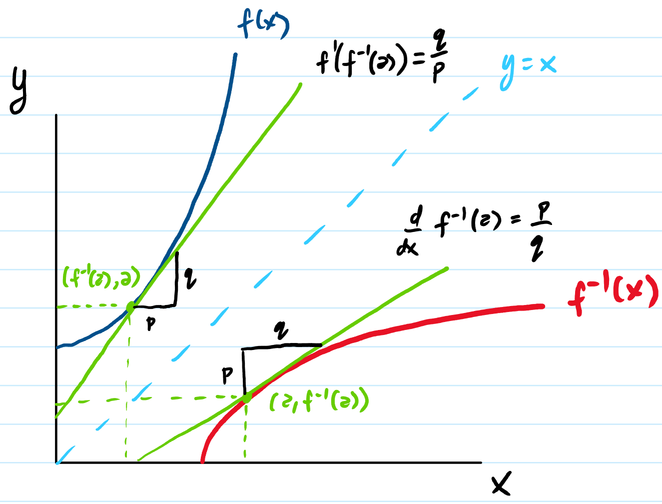
$= x + 2 - 2$

$f(f^{-1}(x)) = x$ ✓

Derivatives of Inverse Functions

Given $f^{-1}(x)$ which is an inverse of function $f(x)$.

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$



Short Proof: Let $y = f^{-1}(x)$.

$$y = f^{-1}(x)$$
$$f(y) = f(f^{-1}(x))$$
$$f(y) = x$$
$$\frac{d}{dx} f(y) = \frac{d}{dx} x$$ → take derivative of both side with respect to x (implicit differentiation)

chain rule ↓

$$\left(\frac{df}{dy}\right)\left(\frac{dy}{dx}\right) = 1$$
$$f'(y)\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{f'(y)}$$
$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Example:

2. Given $f(x) = \dfrac{e^{-3x}}{x^2+1}$. Find $\dfrac{d}{dx} f^{-1}(x)$ at $(-1, 0)$
   x y

$$f'(x) = \frac{(x^2+1)(-3e^{-3x}) - (e^{-3x})(2x)}{(x^2+1)}$$ → Quotient rule

$$f'(x) = -\frac{e^{-3x}(3x^2+2x+3)}{(x^2+1)^2}$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

$$= \frac{1}{-\dfrac{e^{-3y}(3y^2+2y+3)}{(y^2+1)^2}}$$ → let $y = f(x)$. So, we have point $(0, -1)$ using $f^{-1}(x)$.
y x

$$= \frac{(y^2+1)^2}{-e^{-3y}(3y^2+2y+3)}$$

↓

$$\frac{d}{dx} f^{-1}(-1) = \frac{(\cancel{0}^{\,0\,2}+1)^2}{-\cancel{e^{-3(0)}}_{1}(\cancel{3(0)^2}_{0}+\cancel{2(0)}_{0}+3)}$$

$$\frac{d}{dx} f^{-1}(-1) = -\frac{1}{3}$$