4.6 Related Rates

Thursday, November 9, 2023

Objectives:

1. Introducing how to solve related rates problems.

Main Idea of Related Rates

Combine word problems together with implicit differentiation.

Recall: Implicit Differentiation

Given an implicitly defined function

$f(x,y) = c$ for some constant c,

let let $y = y(x) \longrightarrow \frac{dy}{dx} = y'$.

Quick Example:

- Given $x^3y^5 + 3x = 8y^3 + 1$

$$3x^2y^5 + x^3 5y^4 y' + 3 = 24y^2y'$$

$\downarrow$ solve for y'

$$y' = \frac{3x^2y^5 + 3}{24y^2 - 5x^3y^4}$$

- Suppose $x = x(t)$ & $y = y(t) \longrightarrow \frac{dx}{dt} = x'$, $\frac{dy}{dt} = y'$

$$3x^2y^5x' + x^3 5y^4y' + 3x' = 24y^2y'$$

→ solve for x':

$$x' = \frac{24y^2y' - x^3 5y^4y'}{3x^2y^5 + 3}$$

→ solve for y':

$$y' = \frac{3x^2y^5x' + 3x'}{24y^2 - x^3 5y^4}$$

→ solve for $\frac{y'}{x'} = \frac{dy/dt}{dx/dt} = \frac{dy}{dx}$

$$\frac{y'}{x'} = \frac{dy}{dx} = \frac{3x^2y^5 + 3}{24y^2 - 5x^3y^4}$$

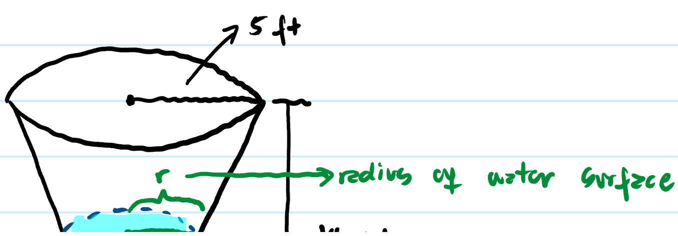
Now, we apply implicit differentiation to related rates problems.

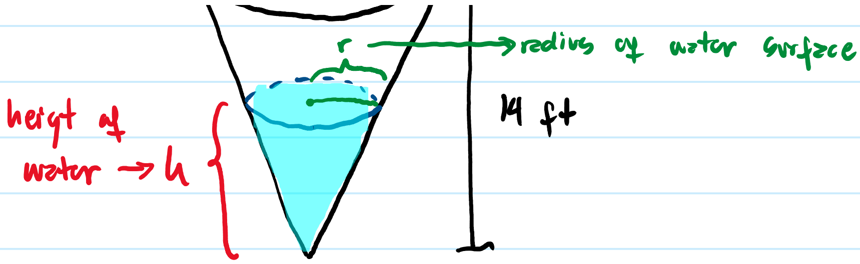
Example Problem 1:

A tank of water in the shape of a cone is leaking water at a constant rate of 2 ft³/min. The base radius of the tank is 5 ft and the height of the tank is 14 ft

a. At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?

b. At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?





a. Goal: Find $\frac{dh}{dt}$ given $h = 6$ ft and $\frac{dV}{dt} = -2\ \frac{ft^3}{min}$.

Water volume : $V = \frac{1}{3}\pi r^2 h$ ⟶ shape of a cone.

Let $V = V(t)$, $r = r(t)$, $h = h(t)$.

$$\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

$\frac{dV}{dt} = -2\ \frac{ft^2}{min}$ $h = 6ft$ Goal

* we don't know r and $\frac{dr}{dt}$.

Finding r and $\frac{dr}{dt}$: What is the relationship of r and h?

↓

we know that r changes as h changes.
we know the tanks dimensions.

Ratio : $\frac{\text{radius}}{\text{height}} \rightarrow \frac{5}{14} = \frac{r}{h} \longrightarrow r = \frac{5}{14}h$

tank water

$V = \underline{1}\,\pi\left(\underline{5}\,h\right)^2 = \underline{25}\,\pi h^3$

$$V = \frac{1}{3}\pi\left(\frac{5}{14}h\right)^2 = \frac{25}{588}\pi h^3$$

$$\frac{dV}{dt} = \frac{25}{588}\pi\left(3h^2\right)\frac{dh}{dt}$$

plug in $\frac{dV}{dt} = -2$ & $h = 6$

$$-2 = \frac{25}{588}\pi\left(3(6)^2\right)\frac{dh}{dt}$$

solve for $\frac{dh}{dt}$

$$-2 = \frac{25}{196}\pi(36)\frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{98}{225\pi}\ \frac{ft}{min}$$

b. We know that $\frac{h}{r} = \frac{14}{5} \longrightarrow h = \frac{14}{5}r$.

Let $h = h(t)$ and $r = r(t)$.

$$\frac{dh}{dt} = \frac{14}{5}\frac{dr}{dt}$$

$\frac{dh}{dt} = -\frac{98}{225\pi}\ \frac{ft}{min}$ at 6 ft and $\frac{dV}{dt} = -2\ \frac{ft^3}{min}$

$$-\frac{98}{225\pi} = \frac{14}{5}\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{5}{14}\left(-\frac{98}{225\pi}\right) = -\frac{7}{45\pi}\ \frac{ft}{min}.$$