2.6 Differentiability

Wednesday, September 20, 2023

Objectives:

1. What are the conditions for a function to be differentiable at a point?

Previously.

- Limit Definition of Derivative
  Suppose we have $f(x)$, then

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example:

- Given $g(t) = \frac{t}{t+1}$, find $\frac{dg}{dt}$ using LOD.

$$\frac{dg}{dt} = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(t+h)}{(t+h)+1} - \left(\frac{t}{t+1}\right)}{h}$$

$$= \lim_{h \to 0} \left(\frac{1}{h}\right)\left(\frac{t+h}{t+h+1} - \frac{t}{t+1}\right)$$

$$= \lim_{h \to 0} \left(\frac{1}{h}\right)\left(\frac{(t+h)(t+1) - t(t+h+1)}{(t+h+1)(t+1)}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{t^2 + ht + th + t - (t^2 + th + t)}{(t+h+1)(t+1)}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{\cancel{t^2} \cancel{+ht} + th \cancel{+t} \cancel{-t^2} \cancel{+th} \cancel{+t}}{(t+h+1)(t+1)}\right)$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left(\frac{\cancel{h}}{(t+h+1)(t+1)}\right)$$

$$= \lim_{h \to 0} \frac{1}{(t+h+1)(t+1)}$$

$$= \frac{1}{(t+1)(t+1)}$$

$$= \frac{1}{(t+1)(t+1)}$$

$$\frac{dg}{dt} = \frac{1}{(t+1)^2}$$

Note: the function $g(t) = \frac{t}{t+1}$ is not continuous at $t=-1$.

So, $$\frac{d}{dt}g(-1) = \lim_{h \to 0} \frac{g(-1+h) - g(-1)}{h}$$

$\vdots$ steps from before

$$= \lim_{h \to 0} \frac{1}{(-1+h+1)(-1+1)}$$

$$\frac{d}{dt}g(-1) = DNE.$$

Differentiability

For $f(x)$ be differentiable at a point $x=a$
the limit definition needs to exist at that point,
$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ need to exist at $x=a$.

Continuity and Differentiability

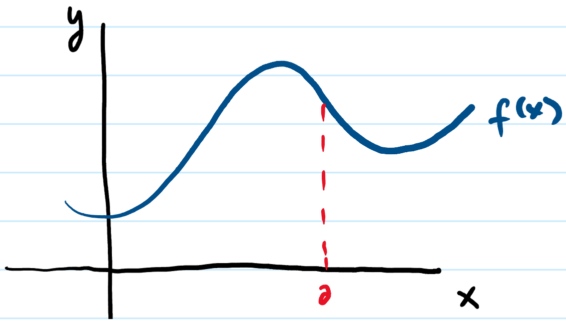
Theorem: If $f(x)$ is differentiable at $x=a$, then it is continuous at $x=a$.

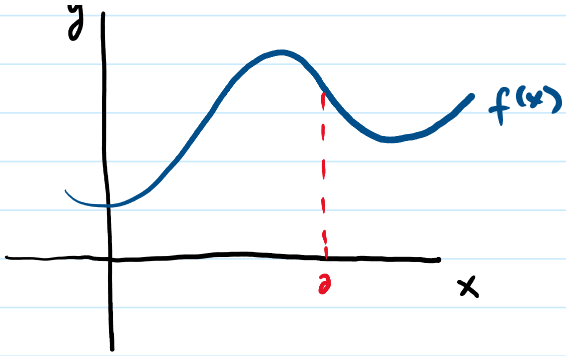
* the opposite is not true *

If $f(x)$ is continuous at $x=a$, then it may or may not be differentiable at $x=a$.

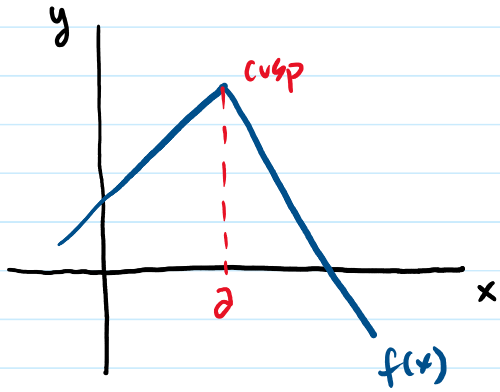
Examples:

- Continuous and differentiable at $x=a$.





- Continuous but not differentiable at $x=a$.



- not continuous and not differentiable at $x=a$.