# 3.1 Powers and Polynomials

Monday, September 25, 2023

Objectives:

1. Finding equation of tangent line.
2. Higher order derivatives of polynomials

Recall: Power Rule of Derivatives

Suppose we have a function of the form

$$f(x) = x^n \text{ where } n \text{ is a real number,}$$

then $\frac{df}{dx} = nx^{n-1}$ or $f' = nx^{n-1}$. (first derivative)

Recall: Qualitative Analysis of $f(x)$ (polynomials)

Given $f(x)$.

$\longrightarrow$ look for roots, meaning solve $f(x) = 0$.

$\longrightarrow$ Find $f'(x)$, then find critical points.

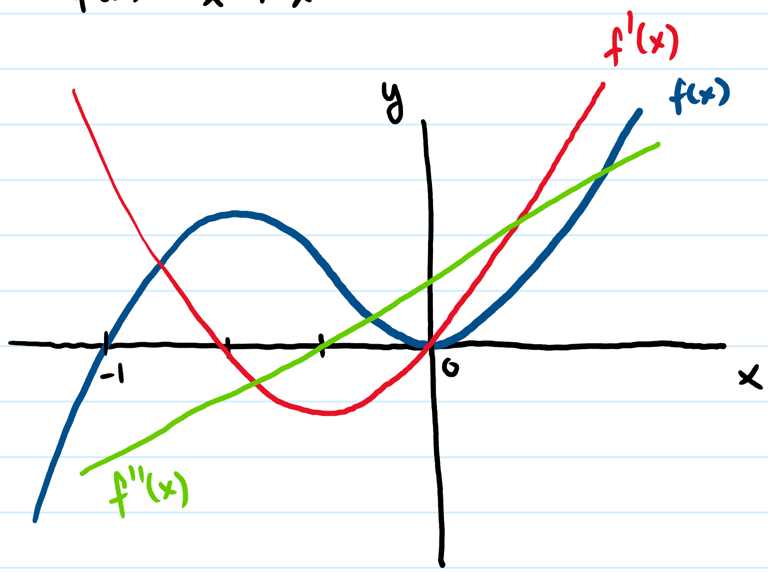
- critical points is when $y' = 0$
- Find $x$ when $y' > 0 \to$ increasing $f(x)$
- Find $x$ when $y' < 0 \to$ decreasing $f(x)$

$\longrightarrow$ Find $f''(x)$, then find inflection points.

- inflection points is when $y'' = 0$
- Use critical points from $y' = 0$, say $x_0$ is a critical point
  - $\longrightarrow$ If $f''(x_0) < 0$, then $x_0$ is concave down
  - $\longrightarrow$ If $f''(x_0) > 0$, then $x_0$ is concave up
- If $f''(x_0) < 0$, $f''(y_0) > 0$, and $y''(x^*) = 0$, then $x^*$ is an inflection point.
- If $x^*$ is an inflection point and $y'(x^*) = 0$, then $x^*$ is a saddle point.

Example:

- $f(x) = x^3 + x^2$




$f(x) = x^3 + x^2 \longrightarrow f(x) = 0$ when $x = -1$ and $x = 0$
$f'(x) = 3x^2 + 2x \longrightarrow f'(x) = 0$ when $x = -2/3$ and $x = 0$
$f''(x) = 6x + 2 \longrightarrow f''(x) = 0$ when $x = -1/3$

Increasing $f(x)$ when $f'(x) > 0$, if $x < -1$ & $x > 0$.
decreasing $f(x)$ when $f'(x) < 0$, if $-1 < x < 0$.
critical points when $f'(x) = 0$, if $x = -1$ and $x = 0$

Concave up $f(x)$ when $f''(x) > 0$, if $x = 0$.
Concave down $f(x)$ when $f''(x) < 0$, if $x = -1$.
Inflection point $f(x)$ when $f''(x) = 0$, if $x = -1/3$.

## Higher Derivatives of Polynomials

Suppose we are given $f(x) = x^n$ where $n$ is a real number, then

$$\frac{d^1}{dx^1} f(x) = n x^{n-1} \quad \text{(1st derivative by the power rule)}$$

$\hookrightarrow (n-1)$

also

$$\frac{d^2}{dx^2} f(x) = n(n-1) x^{n-2} \quad \text{(2nd derivative)}$$

$$\frac{d^3}{dx^3} f(x) = n(n-1)(n-2) x^{n-3} \quad \text{(3rd derivative)}$$

$dx^2$

$$\frac{d^3}{dx^3} f(x) = n(n-1)(n-2)\,x^{n-3} \quad \text{(3rd derivative)}$$

$$\frac{d^4}{dx^4} f(x) = n(n-1)(n-2)(n-3)\,x^{n-4}$$

$\vdots$ Let $k$ be a positive integer.

$$\frac{d^k}{dx^k} f(x) = \underbrace{n(n-1)(n-2)(n-3)\cdots(n-(k-1))}_{\text{Binomial Coefficients}}\, x^{n-k}$$

$$\frac{d^k}{dx^k} f(x) = \frac{n!}{(n-k)!}\, x^{n-k}$$

Examples:

- Find $\frac{d^5}{dx^5} f(x)$ of $f(x) = x^7 \longrightarrow n=7,\ k=5,\ n>k$

$$\frac{d^5}{dx^5} x^7 = \frac{7!}{(7-5)!} x^{7-5} = \frac{7!}{2!} x^2 = \frac{7!}{2!} x^2$$

- Find $\frac{d^7}{dx^7} f(x)$ of $f(x) = x^5 \longrightarrow n=5,\ k=7,\ n<k$

be careful! $\frac{d^7}{dx^7} x^5 = \frac{5!}{(5-7)!} x^{5-7}$

$(-2)!$ is undefined

$\downarrow$

$f' = 5x^4,\ f'' = 5\cdot4x^3,\ f''' = 5\cdot4\cdot3x^2,$

$f^{(4)} = 5\cdot4\cdot3\cdot2x,\ f^{(5)} = 5\cdot4\cdot3\cdot2\cdot1,\ f^{(6)} = 0$

## Kth derivative of $f(x) = x^n$

Kth derivative

$\Big\{\ n!\quad x^{n-k}\quad \text{if}\quad 0 \le k \le n$

kth derivative

$$f^{(k)}(x) = \begin{cases} \dfrac{n!}{(n-k)!} x^{n-k} & \text{if } 0 \le k < n \\ 0 & \text{if } k > n \\ k! & \text{if } k = n \end{cases}$$