4.7 L'Hopital's Rule

Wednesday, November 15, 2023

Objectives:

1. Review limits
2. Introduce L'Hopital's rule for evaluating limits.

Recall: Evaluating limits

- $\lim_{x \to 4} \frac{x^2-16}{x-4} \to \frac{0}{0}$ (indeterminate)

↳ Algebraically (by factorization)

$$\lim_{x \to 4} \frac{\cancel{(x-4)}(x+4)}{\cancel{x-4}} = \lim_{x \to 4} x+4 = 8$$

- $\lim_{x \to \infty} \frac{4x^2-5x}{1-3x^2} \to \frac{\infty}{\infty}$ (indeterminate)

↳ Algebraically (by "multiplying by 1" technique)

$$\lim_{x \to \infty} \frac{4x^2-5x}{1-3x^2} \frac{(1/x^2)}{(1/x^2)} = \lim_{x \to \infty} \frac{4 - \overset{0}{\cancel{5/x}}}{\underset{0}{\cancel{1/x^2}} - 3} = -\frac{4}{3}$$

- $\lim_{x \to 0} \frac{\sin(x)}{x} \to \frac{0}{0}$ (indeterminate)

↳ showed by ST (can't do algebraically)

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

- $\lim_{x \to \infty} \frac{e^x}{x^2} \to \frac{\infty}{\infty}$ (indeterminate)

↳ exponential grows faster than polynomial

$$\lim_{x \to \infty} \frac{e^x}{x^2} = DNE$$

What if $\lim_{x \to 0} \frac{e^x}{x^2} = UND \to \frac{1}{0}$ (undefined)

What if $\lim_{x \to -\infty} \frac{e^x}{x^2} = 0 \to \frac{1}{\infty}$ (goes to 0)

L'Hopital's Rule

Goal: Evaluating <u>indeterminate</u> limits using derivative rules.

The Rule: If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$

where a can be any real number or infinite,
then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

where $f'(x)$ and $g'(x)$ are the derivatives of $f(x)$ and $g(x)$ respectively.

Examples:

- $$\lim_{x\to 4} \frac{x^2-16}{x-4} \to \frac{0}{0}$$

$$\xrightarrow{LH} \lim_{x\to 4} \frac{2x}{1} = \frac{2(4)}{1} = 8$$

- $$\lim_{x\to\infty} \frac{4x^2-5x}{1-3x^2} \to \frac{\infty}{\infty}$$

$$\xrightarrow{LH} \lim_{x\to\infty} \frac{8x-5}{-6x} \to \frac{\infty}{\infty}$$

$$\xrightarrow{LH} \lim_{x\to\infty} \frac{8}{-6} = \frac{8}{6} = \frac{4\cdot\cancel{2}}{3\cdot\cancel{2}} = -\frac{4}{3}$$

- $$\lim_{x\to 0} \frac{\sin(x)}{x} \to \frac{0}{0}$$

$$\xrightarrow{LH} \lim_{x\to 0} \frac{\cos(x)}{1} = 0$$

- $$\lim_{x\to\infty} \frac{e^x}{x^2} \to \frac{\infty}{\infty}$$

$$\xrightarrow{LH} \lim_{x\to\infty} \frac{e^x}{2x} \to \frac{\infty}{\infty}$$

$$\xrightarrow{LH} \lim_{x\to\infty} \frac{e^x}{2} = \text{DNE}$$

Other indeterminate Forms

a. $0(\pm\infty)$
b. 1^{∞}
c. 0^{0}
d. ∞^{0}
e. $\infty - \infty$

Examples:

- $\lim_{x \to -\infty} xe^{x} \rightarrow (-\infty)(0)$

$$\hookrightarrow \lim_{x \to -\infty} \frac{e^{x}}{1/x} \rightarrow \frac{0}{0}$$

$$\xrightarrow{LH} \lim_{x \to -\infty} \frac{e^{x}}{-1/x^{2}} \rightarrow \frac{0}{0}$$

$$\xrightarrow{LH} \lim_{x \to -\infty} \frac{e^{x}}{2/x^{3}} \rightarrow \frac{0}{0}$$

$\xrightarrow{LH}$ ... go on forever.

Alternative: $\lim_{x \to -\infty} \frac{x}{1/e^{x}} = \lim_{x \to -\infty} \frac{x}{e^{-x}} \rightarrow \frac{-\infty}{\infty}$

$$\xrightarrow{LH} \lim_{x \to -\infty} \frac{1}{-e^{-x}} = 0$$

- $\lim_{x \to \infty} x^{1/x} \rightarrow \infty^{0}$

Use log technique
let $y = x^{1/x}$.

$$\ln(y) = \ln(x^{1/x})$$

$$\ln(y) = \frac{1}{x}\ln(x) \longrightarrow \text{law of logs}$$

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{1}{x}\ln(x) \rightarrow \frac{\infty}{\infty}$$

LH

$$\lim_{x\to\infty} \frac{1/x}{1} = 0$$

$$\lim_{x\to\infty} \ln(y) = 0 \longrightarrow \text{recall that } y = x^{1/x} \text{ and } e^{\ln(y)} = y$$

$$e^{\lim_{x\to\infty} \ln(y)} = e^{0}$$

$$\lim_{x\to\infty} y = 1$$

$$\lim_{x\to\infty} x^{1/x} = 1$$