Trigonometric Integrals Worksheet

MTH 202A - Fall 2022 - University of Portland

Instructions: Provide complete solutions for each problem. Show steps clearly and write your solutions with standard mathematical notations.

Goals: By using trigonometric identities combined with integration by substitution/parts we'd like to evaluate integrals of the form $\int \sin^j(x) cos^k(x)dx$ (for integer values of j and k), and $\int \tan^k(x)\sec^j(x)dx$ (for integer values of k and j). Work together to discover the techniques that work for these anti-derivatives.

1. Warm-up problem: $\int \cos^4(x)\sin(x)\,dx$
2. $\int \sin^3(x)dx$ (Hint: Use the identity $\sin^2 x + \cos^2 x = 1$. Then make a substitution.)
3. $\sin^5(x)\cos^2(x)\,dx$ (Hint: Write $\sin^5(x)$ as $\sin^4(x)\sin(x)$.)
4. $\int \sin^7(x)\cos^5(x)\,dx$.
5. In general, how would you go about trying to evaluate $\int \sin^j(x)\cos^k(x)\,dx$, where j is odd? (Hint: consider the previous three problems.)
6. Note that the same kind of trick works when the power on $\cos x$ is odd. To check that you understand, what trig identity and what integration technique would you use to integrate $\int \cos^3(x)\sin^2(x)\,dx$?
7. Now what if the power on $\cos(x)$ and $\sin(x)$ are both even? Find $\int \sin^2(x)\,dx$, in each of the following two ways:
   a. Use the identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$.
   b. Integrate by parts, with $u = \sin(x)$ and $dv = \sin(x)dx$.
   c. Show that your answers to parts (a) and (b) above are the same. Hint: Use a double angle formula.
   d. How would you evaluate the integral $\int \sin^2(x)\cos^2(x)\,dx$?
8. Do the integral in problem (2), above, again, but this time by parts, using $u = \sin^2(x)$ and $dv = \sin(x)dx$.
9. Can you show your answers to problems (2) and (8) above are the same? It's another great trigonometric identity.
10. We also would like to be able to solve integrals of the form $\int \tan^k(x)\sec^j(x)\,dx$. These two functions play well with each other, since the derivative of $\tan(x)$ is $\sec^2(x)$ and the derivative of $\sec(x)$ is $\sec(x)\tan(x)$, and since there is a Pythagorean identity relating them. It sometimes works to use $u = \tan(x)$ and it sometimes works to use $u = \sec(x)$. Based on the values of k and j, which substitution should you use? Are there cases for which neither substitution works? Discuss these ideas and evaluate the following integrals.
    a. $\tan^5(x)\sec^2(x)\,dx$
    b. $\tan^4(x)\,dx$
    c. $\sec^5(x)\,dx$
    d. $\tan^2(x)\sec^2(x)\,dx$