Series Convergence Tests Guide

Divergence Test

For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim\limits_{n\to\infty} a_n$. This test cannot prove convergence of a series.

If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive.

If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.

Geometric Series

Geometric Series of the forms $\displaystyle\sum_{n=1}^{\infty} ar^{n-1}$ or $\displaystyle\sum_{n=0}^{\infty} ar^n$. This test applies to any geometric series that can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the ratio.

If $|r| < 1$, the series converges to $\dfrac{a}{1-r}$.

If $|r| \geq 1$, the series diverges.

p-Series

Series of the form $= \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ where p is a real number.

If $p > 1$, the series converges.

If $p \leq 1$, the series diverges.

Integral Test

If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\displaystyle\int_N^{\infty} f(x)dx$. This test is limited to those series for which the corresponding function f can be easily integrated.

If $\displaystyle\int_N^{\infty} f(x)dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.

If $\displaystyle\int_N^{\infty} f(x)dx$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Comparison Test

For $\sum_{n=1}^{\infty} a_n$ with non-negative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$. Typically used for a series similar to a geometric or p -series. It can sometimes be difficult to find an appropriate series.

If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Limit Comparison Test

For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

This test is typically used for a series similar to a geometric or p -series. Often easier to apply than the comparison test.

If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Alternating Series

For a series of the forms $\displaystyle\sum_{n=1}^{\infty}(-1)^n b_n$ or $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1} b_n$,

if $b_{n+1} \leq b_n$ for all $n \geq 1$ and

$\lim_{n\to\infty} bn \to 0$,

then the series converges. This test only applies to alternating series.

Ratio Test

For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

If $0 \leq \rho < 1$, the series converges absolutely.

If $\rho > 1$ or $\rho = \infty$, the series diverges.

If $\rho = 1$, the test is inconclusive.

This test is often used for series involving factorials or exponentials.

Root Test

For any series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

If $0 \leq \rho < 1$, the series converges absolutely.

If $\rho > 1$ or $\rho = \infty$, the series diverges.

If $\rho = 1$, the test is inconclusive.

This test is often used for series where $|a_n| = b_n^n$.

Absolute Convergence Test

- A series $\sum_{n=1}^{\infty} a_n$ exhibits absolute convergence if $\sum_{n=1}^{\infty} |a_n|$ converges.
- A series $\sum_{n=1}^{\infty} a_n$ exhibits conditional convergence if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.