Power Series Techniques

General Form of the Power Series

A series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ is a power series centered at $x = 0$.

A series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$ is a power series centered at $x = a$.

Convergence of Power Series

Consider the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$.

- If the power series converges at $x = a$, the radius of convergence is defined to be $R = 0$.
- If the power series converges for all x, then the radius of convergence is $R = \infty$.
- If the power series converges for values of x which $|x - a| < R$ or $a - R < x < a + R$, the radius of convergence is R.
- The interval of convergence is the interval $(a - R, a + R)$ including the end points where the power series converges.

Use the ratio test to determine the radius and interval of convergence.

- *Step 1:* Let $a_n = c_n(x-a)^n$ and $a_{n+1} = c_{n+1}(x-a)^{n+1}$.
- *Step 2:* Simplify ratio $\frac{|a_{n+1}|}{|a_n|} = \frac{|c_{n+1}(x-a)^{n+1}|}{|c_n(x-a)^n|} = \frac{c_{n+1}}{c_n}(x-a)$
- *Step 3:* Compute $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$.
- *Step 4:* Interpret the results.
  - If $\rho = 0$, the $R = \infty$. The power series converges for all x.
  - If $\rho = N \cdot |x - a|$, where N is a finite positive number, then $R = \frac{1}{N}$. The interval of convergence includes $(a - \frac{1}{N}, a + \frac{1}{N})$, and possibly the end points.
  - If $\rho \to \infty$, the $R = 0$. The power series converges at $x = a$ and nowhere else.
- *Step 5:* If interval of convergence is $(a - \frac{1}{N}, a + \frac{1}{N})$, the end points $x = a - \frac{1}{N}$ and $x = a + \frac{1}{N}$ may or may not converge. To determine whether the end points converge, enter them into the power series one at a time and use a series convergence test.

Taylor and Maclaurin Series

If $f(x)$ has derivatives of all orders at $x = a$, then the Taylor series for function $f(x)$ at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots.$$

The Taylor series for $f(x)$ at 0 is known as the Maclaurin series for $f(x)$.

Taylor Polynomials

If $f(x)$ has n derivatives at $x = a$, then the nth Taylor polynomial for $f(x)$ at a is

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The nth Taylor polynomial for $f(x)$ at 0 is known as the nth Maclaurin polynomial for $f(x)$.