Analyzing Equilibriums of 1st-Order ODEs

Objectives:
1. Stability of equilibriums
2. Linearization
3. Equilibrium analysis

Recall: Finding tangent line equation at a point given $f(x)$.

Given $y=f(x)$ and point (x_1, y_1).
The tangent line at point (x_1, y_1) of $f(x)$ is

$$y - y_1 = m(x - x_1)$$

where $m = f'(x_1)$.

So, at $x=a$, $y_1 = f(a)$ and $m = f'(a)$
with tangent line $L(x) = f(a) + f'(a)(x-a)$.

Recall: Equilibrium solutions

* Definition: Let $\frac{dy}{dt} = f(y)$ be an autonomous ODE of some continuous function $y(t)$ and $f(y)$ is a function that describes the change.
Suppose y^* is a critical point. Then,
1. $f(y^*) = 0$ and
2. $y(t) = y^*$ is an equilibrium solution.

Motivating Example

Given the ODE $\frac{dy}{dt} = y^4 - y^3 - y^2 + y$, $\longrightarrow$ 1st-order, homogeneous, autonomous, non-linear

determine the equilibriums and identify their stability.

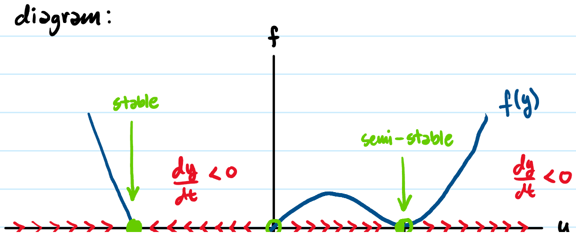
* Let $f(y) = y^4 - y^3 - y^2 + y$. Note that $\frac{dy}{dt} = f(y)$.

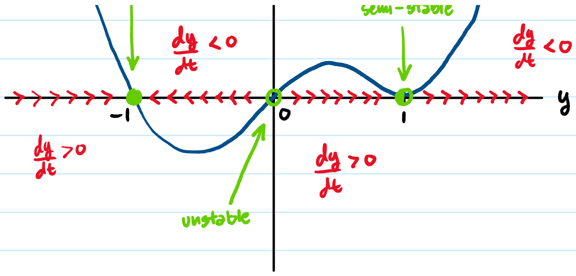
* Factorize $f(y)$, which is $f(y) = y(y+1)(y-1)^2$.

* Find critical points: $0 = y(y+1)(y-1)^2 \longrightarrow y_1^* = -1$, $y_2^* = 0$, $y_3^* = 1$

* Write down equilibrium solutions:
$y(t) = y_1^* \longrightarrow y(t) = -1$
$y(t) = y_2^* \longrightarrow y(t) = 0$
$y(t) = y_3^* \longrightarrow y(t) = 1$

* Phase diagram:





Linearization

* A way to identify equilibrium stability analytically.

* From the example: $\frac{dy}{dt} = y(y+1)(y-1)^2$

$$\downarrow$$

$$f(y) = y(y+1)(y-1)^2$$

Equilibrium solutions: $y(t) = -1$, $y(t) = 0$, $y(t) = 1$

Linearization on the equilibriums: $\frac{\partial f}{\partial y} = 4y^3 - 3y^2 - 2y + 1$

- $y^* = -1$; $\left.\frac{\partial f}{\partial y}\right|_{y=-1} = 4(-1)^3 - 3(-1)^2 - 2(-1) + 1 = -4 < 0$

  $\downarrow$

  $y(t) = -1$ is stable

- $y^* = 0$; $\left.\frac{\partial f}{\partial y}\right|_{y=0} = 4(0)^2 - 3(0) - 2(0) + 1 = 1 > 0$

  $\downarrow$

  $y(t) = 0$ is unstable

- $y^* = 1$; $\left.\frac{\partial f}{\partial y}\right|_{y=1} = 4(1) - 3(1) - 2(1) + 1 = 0$

  $\downarrow$

  $y(t) = 1$ might be semi-stable

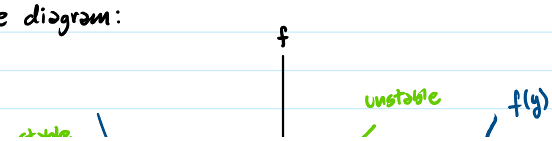
To make sure: $\frac{\partial^2 f}{\partial y^2} = 12y^2 - 6y - 2$

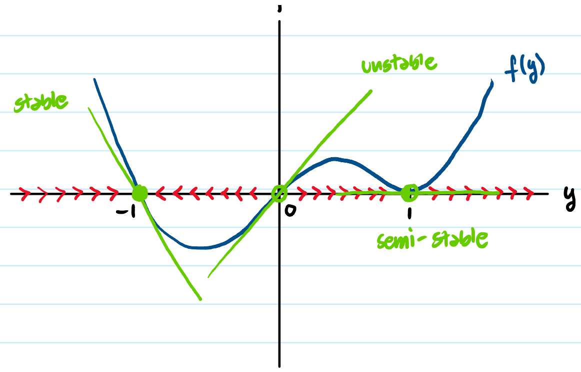
Then, $\left.\frac{\partial^2 f}{\partial y^2}\right|_{y=1} = 12(1)^2 - 6(1) - 2 = 4 > 0$

$\downarrow$

concave up
meaning $f(t) = 1$ is semi-stable
with surrounding solutions increasing.

* Phase diagram:





Stability of Equilibriums

Let $\frac{dy}{dt} = f(y)$ and $y(t) = y^*$ be an equilibrium solution.

- $y(t) = y^*$ is stable if $\underset{dy/dt > 0}{\rightarrow\rightarrow\rightarrow} \underset{y^*}{\bullet} \underset{dy/dt < 0}{\leftarrow\leftarrow\leftarrow}$ and $\left.\frac{\partial f}{\partial y}\right|_{y=y^*} < 0$
- $y(t) = y^*$ is unstable if $\underset{dy/dt < 0}{\leftarrow\leftarrow\leftarrow} \circ \underset{dy/dt > 0}{\rightarrow\rightarrow\rightarrow}$ and $\left.\frac{\partial f}{\partial y}\right|_{y=y^*} > 0$
- $y(t) = y^*$ is semi-stable if $\underset{dy/dt > 0}{\rightarrow\rightarrow\rightarrow} \circ \underset{dy/dt > 0}{\rightarrow\rightarrow\rightarrow}$ or $\underset{dy/dt < 0}{\leftarrow\leftarrow\leftarrow} \circ \underset{dy/dt < 0}{\leftarrow\leftarrow\leftarrow}$ and $\left.\frac{\partial f}{\partial y}\right|_{y=y^*} = 0$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{y=y^*} > 0 \qquad\qquad \left.\frac{\partial^2 f}{\partial y}\right|_{y=y^*} < 0$$