

Undetermined Coefficients for Nonhomogeneous Linear ODEs

Objectives :

1. Introduce the method of Undetermined Coefficients

Recall: 2nd-order linear ODE with constant coefficients

$$y'' + py' + qy = f(t)$$

where p & q are constants, and $f(t)$ is a function.

If $f(t) = 0$, then it is homogeneous.

If $f(t) \neq 0$, then it is nonhomogeneous.

- To determine the general solution:

→ Find y_h , the homogeneous solution

→ Find y_p , the particular solution

Then, $y = y_h + y_p$ is the general solution.

- To determine the homogeneous solution $y_h(t)$:

$$r^2 + pr + q = 0 \quad \text{← characteristic equation}$$

- If r is distinct real roots $r_1 \neq r_2$, then

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

- If r is a repeated real root $r = r_1 = r_2$, then

$$y = C_1 e^{rt} + C_2 t e^{rt}$$

- If r is a complex conjugate root $r = \alpha + \beta i$

$$y = e^{\alpha t} (C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

The Method of Undetermined Coefficients

* This method is a way to solve for $y_p(t)$ by using an Ansatz (or "educated guess") based on the form of $f(t)$.

* These are Ansatz of common functions.

→ If $f(t) = C$, a constant, then $y_p(t) = A_0$, also a constant.

→ If $f(t) = A_n t^n$ where $P_n(t) = P_n t^n + P_{n-1} t^{n-1} + \dots + P_1 t + P_0$,

an nth-order polynomial, then $y_p(t) = A_n t^n + A_{n-1} t^{n-1} + \dots + A_1 t + A_0 = A_n t^n + P_n(t)$,
also an nth-order polynomial.

Note that $A_n t^n$ needs to be a complete nth-order polynomial with all different
coefficients $A_n, A_{n-1}, \dots, A_1, A_0$, regardless whether $f(t)$ is a complete polynomial.

→ If $f(t) = e^t$, an exponential function, then $y_p(t) = A_0 e^t$,
also an exponential.

Note that the exponent in $f(t)$ needs to be the same as the exponent in $y_p(t)$.

→ If $f(t) = \cos(t)$ or $f(t) = \sin(t)$ or $f(t) = \cos(t) + \sin(t)$, then
 $y_p(t) = A_0 \cos(t) + B_0 \sin(t)$.

Note that the inside function of sin or cos in $f(t)$ need to be the same
as the inside function in $y_p(t)$.

→ See table in the worksheet for more common Ansatz functions.

- If when the Ansatz have common terms in $y_h(t)$,
then the initial guess of $y_p(t)$ need to be multiplied by t .
Check repeatedly until terms of $y_p(t)$ does not have common terms in $y_h(t)$.

* Determining the Coefficients of the Ansatz.

→ If $y_p(t)$ is an Ansatz to $y'' + py' + qy = f(t)$ with
undetermined coefficients, the process of determining
the coefficients is to take y'_p and y''_p and plug-in into
the ODE, then solve for the coefficients using algebra.

Examples:

$$\textcircled{1} \quad y'' + y' = 1$$

$$\begin{aligned} \rightarrow \text{Homo Case: } r^2 + r &= 0 \\ r(r+1) &= 0 \rightarrow r_1 = 0 \\ &\quad r_2 = -1 \end{aligned}$$

$$y_h(t) = C_1 e^{0t} + C_2 e^{-t} = C_1 + C_2 e^{-t} \rightarrow \text{homogeneous solution}$$

$$\rightarrow \text{Non-homo Case: } f(t) = 1$$

"guess": $y_p = A_1 t$

\downarrow

we multiply by t
because the homogeneous has a constant

$$y'_p = A_1$$

$$y''_p = 0$$

$$y'' + y' = 1 \rightarrow 0 + A_1 = 1 \rightarrow A_1 = 1$$

$$y_p(t) = t \rightarrow \text{particular solution}$$

$$\rightarrow y(t) = C_1 + C_2 e^{-t} + t \rightarrow \text{general solution}$$

$$\textcircled{2} \quad y'' - y = \sin(t)$$

$$\rightarrow \text{Homo case: } r^2 - 1 = 0 \\ r^2 = 1 \rightarrow r_1 = +1 \\ r_2 = -1$$

$$y_h(t) = C_1 e^t + C_2 e^{-t} \rightarrow \text{homogeneous solution}$$

$$\rightarrow \text{Non homo case: } f(t) = \sin(t)$$

$$\text{"guess": } y_p = A \sin(t) + B \cos(t)$$

$$y'_p = A \cos(t) - B \sin(t)$$

$$y''_p = -A \sin(t) - B \cos(t)$$

$$y'' - y = \sin(t) \rightarrow -A \sin(t) - B \cos(t) - (A \sin(t) + B \cos(t)) = \sin(t) \\ -2A \sin(t) - 2B \cos(t) = \sin(t)$$

$$\begin{aligned} \sin(t); \quad -2A &= 1 \rightarrow A = -\frac{1}{2} \\ \cos(t); \quad -2B &= 0 \rightarrow B = 0 \end{aligned}$$

$$\text{so, } y_p(t) = \left(-\frac{1}{2}\right) \sin(t) \rightarrow \text{particular solution}$$

$$\rightarrow y(t) = C_1 e^t + C_2 e^{-t} + \left(-\frac{1}{2}\right) \sin(t) \rightarrow \text{general solution}$$

$$\textcircled{3} \quad y'' - y' - 2y = 3t^2 - 1$$

$$③ y'' - y' - 2y = 3t^2 - 1$$

$$\rightarrow \text{Homogeneous case: } r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0 \rightarrow r_1 = 2, r_2 = -1$$

$$y_h(t) = C_1 e^{2t} + C_2 e^{-t}$$

$$\rightarrow \text{Non homogeneous case: } f(t) = 3t^2 - 1 \rightarrow \text{2nd-order polynomial}$$

"guess": $y_p = At^2 + Bt + C$

$$y'_p = 2At + B$$

$$y''_p = 2A$$

$$y'' - y' - 2y = 3t^2 - 1 \rightarrow 2A - (2At+B) - 2(At^2+Bt+C) = 3t^2 - 1$$

$$2A - 2At - B - 2At^2 - 2Bt - 2C = 3t^2 - 1$$

$$-2At^2 - (2A+2B)t - 2A - B - 2C = 3t^2 - 1$$



$$t^2: -2A = 3$$

$$t: -(2A+2B) = 0$$

$$\text{constants: } -2A - B - 2C = 1$$

} 3 equations
3 unknowns

↓ convert to
matrix vector form

$$\begin{bmatrix} A & B & C \\ -2 & 0 & 0 \\ -2 & -2 & 0 \\ -2 & -1 & -2 \end{bmatrix}_{3 \times 3} \begin{bmatrix} A \\ B \\ C \end{bmatrix}_{3 \times 1} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}_{3 \times 1}$$



↓ use linear algebra or technology to solve for $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$.

$$A = -\frac{3}{2}, B = \frac{3}{2}, C = \frac{1}{4}$$

$$y_p(t) = -\frac{3}{2}t^2 + \frac{3}{2}t - \frac{1}{4}$$

$$\rightarrow y(t) = C_1 e^{2t} + C_2 e^{-t} - \frac{3}{2} t^2 + \frac{3}{2} t - \frac{7}{4}$$

$$④ y'' - y' - 2y = 2e^{-3t}$$

$$\rightarrow \text{Homo case: } r^2 - r - 2 = 0 \rightarrow r_1 = 2, r_2 = -1 \rightarrow y_h(t) = C_1 e^{2t} + C_2 e^{-t}$$

$$\rightarrow \text{Non homo case: } f(t) = 2e^{-3t}$$

"guess": $y_p = A e^{-3t}$

$$y'_p = -3A e^{-3t}$$

$$y''_p = 9A e^{-3t}$$

$$y'' - y' - 2y = 2e^{-3t} \rightarrow 9Ae^{-3t} + 3Ae^{-3t} - 2Ae^{-3t} = 2e^{-3t}$$

$$10Ae^{-3t} = 2e^{-3t}$$

$$10A = 2$$

$$A = \frac{1}{5}$$

$$y_p(t) = \frac{1}{5} e^{-3t}$$

$$\rightarrow y(t) = C_1 e^{2t} + C_2 e^{-t} + \frac{1}{5} e^{-3t}$$

$$⑤ y'' - y' - 2y = 2\cos(3t)$$

$$\rightarrow \text{Homo case: } y_h(t) = C_1 e^{2t} + C_2 e^{-t}$$

$$\rightarrow \text{Non homo case: } f(t) = 2\cos(3t)$$

"guess": $y_p = A\cos(3t) + B\sin(3t)$

$$y'_p = -3A\sin(3t) + 3B\cos(3t)$$

$$y''_p = -9A\cos(3t) - 9B\sin(3t)$$

$$y'' - y' - 2y = 2\cos(3t) \rightarrow \text{simplifications...}$$

↓

$$(-11A - 3B)\cos(3t) + (3A - 11B)\sin(3t) = 2\cos(3t)$$

↓

$$(-11A - 3B) \cos(3t) + (3A - 11B) \sin(3t) = 2 \cos(3t)$$



$$\begin{aligned} \cos(3t): & \quad -11A - 3B = 2 \\ \sin(3t): & \quad 3A - 11B = 0 \end{aligned}$$

$\left. \begin{array}{l} 2 \text{ equations} \\ 2 \text{ unknowns} \end{array} \right\}$



$$\begin{bmatrix} A & B \\ -11 & -3 \\ 3 & -11 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

use linear algebra or technology to solve for $\begin{bmatrix} A \\ B \end{bmatrix}$



$$A = -\frac{11}{65}, \quad B = -\frac{3}{65}$$

$$y_p(t) = -\frac{11}{65} \cos(3t) - \frac{3}{65} \sin(3t)$$

$$\rightarrow y(t) = C_1 e^{2t} + C_2 e^{-t} - \frac{11}{65} \cos(3t) - \frac{3}{65} \sin(3t)$$

$$⑥ y'' - y' - 2y = t^2 e^t$$

$$\rightarrow \text{Homo Case: } y_h(t) = C_1 e^{2t} + C_2 e^{-t}$$

$$\rightarrow \text{Non homo case: } f(t) = t^2 e^t \rightarrow \text{polynomial} \times \text{exponential}$$

$$\text{"guess": } y_p = (At^2 + Bt + C)e^t$$

$$y'_p = e^t [At^2 + (2A+B)t + (B+C)]$$

$$y''_p = e^t [At^2 + (4A+B)t + (2A+2B+C)]$$

$$y'' - y' - 2y = t^2 e^t \rightarrow \text{simplifications...}$$

$$e^t [t^2(-2A) + t(2A - 2B) + (2A + B - 2C)] = t^2 e^t$$

$$t^2: -2A = 1$$

$$t: 2A - 2B = 0$$

$$\text{constant: } 2A + B - 2C = 0$$



$$\begin{bmatrix} -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} A & B & C \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 2 & -2 & 0 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A = -\frac{1}{2}, B = -\frac{1}{2}, C = -\frac{3}{4}$$

$$y_p(t) = e^t \left[-\frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4} \right]$$

$$\rightarrow y(t) = C_1 e^{2t} + C_2 e^{-t} + e^t \left[-\frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4} \right]$$

$$\textcircled{3} \quad y'' - 2y' + y = 3e^t$$

$$\rightarrow \text{Homogeneous case: } r^2 - 2r + 1 = 0 \\ (r-1)(r-1) = 0 \rightarrow r_1 = r_2 = 1$$

$$y_h(t) = C_1 e^t + C_2 t e^t$$

$$\rightarrow \text{Non homogeneous case: } f(t) = 3e^t$$

$$\text{"guess": } y_p = At^2 e^t$$

$$y'_p = 2At e^t + At^2 e^t$$

$$y''_p = 2Ae^t + 4At e^t + At^2 e^t$$

$$y'' - 2y' + y = 3e^t \rightarrow \text{simplifications...}$$

$$2Ae^t = 3e^t$$

$$2A = 3$$

$$A = \frac{3}{2}$$

$$y_p(t) = \frac{3}{2}t^2 e^t$$

$$\rightarrow y(t) = C_1 e^t + C_2 t e^t + \frac{3}{2}t^2 e^t$$