HW Set 4 - SOLUTIONS

1. (a) $\frac{dy}{dt} = 2y - t$ (not separable ⤳ use reverse prod rule).

$y' - 2y = -t$

Mult by u: $y'u - 2yu = -tu$ $\Rightarrow$ $y'u + yu' = -tu$ where $u' = -2u$

$\Rightarrow$ $u = ce^{-2t}$ Pick particular: $u = e^{-2t}$

$y'(e^{-2t}) - 2e^{-2t}y = -tu = -te^{-2t}$

$\int (ye^{-2t})' dt = \int -te^{-2t} dt$ By parts: $u = -t$ $du = -1\, dt$

$= \frac{1}{2}te^{-2t} - \int \frac{1}{2}e^{-2t} dt$ $dv = e^{-2t}dt$ $v = -\frac{1}{2}e^{-2t}$

$\int u\,dv = uv - \int v\,du$

$ye^{-2t} = \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} + c$

$\Rightarrow y = e^{2t}\left(\frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} + c\right)$, c: any constant

$\boxed{y = \frac{1}{2}t + \frac{1}{4} + ce^{2t}}$

(b) $\frac{dy}{dt} = \frac{-y}{t} + 2$ (again: not separable into $f(y)g(t)$)

$y' + \frac{1}{t}y = 2$

$y'u + \frac{y}{t}u = 2u$ $\Rightarrow$ $y'u + yu' = 2u$ where $u' = \frac{u}{t}$

$\frac{du}{dt} = \frac{u}{t}$ $\Rightarrow \int \frac{1}{u} du = \int \frac{1}{t} dt$

$\Rightarrow \ln|u| = \ln|t| + c$

$|u| = c|t|$, $c \geq 0$

$\Rightarrow u = ct$, c = any constant

Pick $u = t$

$y'(t) + y\underbrace{t'}_{} = 2t$

$y'u + yu'$

$\int (yt)' dt = \int 2t\, dt$

$yt = t^2 + c$

$\boxed{y = t + \frac{c}{t}}$

(c) $\frac{dP}{dx} = P\sin x$ (Separable.. hooray!)

$\int \frac{1}{P} dP = \int \sin x\, dx$

$\ln|P| = -\cos x + c$ $\Rightarrow$ $|P| = ce^{-\cos x}$, $c \geq 0$

$\boxed{P = Ce^{-\cos x}}$ c = any constant

(d) $\frac{dq}{dt} = \cos t$ (separable :))

$\int dq = \int \cos t \, dt$

$\boxed{q = \sin t + C}$

$\boxed{2.}$ Solve (i) using sep. of var., and (ii) using reverse prod-rule.

(i) $\frac{dy}{dt} = 2y+1$; $y(0)=2$

$\int \frac{1}{2y+1} dy = \int dt$

$\frac{1}{2}\ln|2y+1| = t + c$

$\ln|2y+1| = 2t + c_1$

$|2y+1| = e^{c_1}e^{2t} = c_2 e^{2t}$, $c_2 \geq 0$

$2y+1 = c_3 e^{2t}$, c_3 = any constant

$y = \frac{1}{2}(c_3 e^{2t} - 1)$ gen'l soln

IVP: $y(0)=2$

$2 = \frac{1}{2}(c_3 e^0 - 1)$

$4 = c_3 - 1$, $c_3 = 5$

$\boxed{\therefore\ y(t) = \frac{1}{2}(5e^{2t} - 1)}$

(ii) Reverse product rule

$y' - 2y = 1$

$y'u - 2yu = u \Rightarrow y'u + yu' = u$ where $u' = -2u$

$\Rightarrow$ let $u = e^{-2t}$

$y'(e^{-2t}) + (e^{-2t})'y = e^{-2t}$

$\int (ye^{-2t})' dt = \int e^{-2t} dt$

$ye^{-2t} = -\frac{1}{2}e^{-2t} + C$

$y = e^{2t}(-\frac{1}{2}e^{-2t} + C) = -\frac{1}{2} + Ce^{2t}$

gen'l soln (same!)

IVP: $2 = -\frac{1}{2} + Ce^0$

$\frac{5}{2} = C$

$\boxed{y(t) = -\frac{1}{2} + \frac{5}{2}e^{2t}}$

same as above :)

$\boxed{3.}$ (a) $\frac{dy}{dt} = 2y - 3e^{-t}$ Cannot use separation of var. Can use reverse prod.

$y' = g(t)y + r(t)$ ✓

(b) $\frac{dy}{dt} = -0.2(75-y)$ Separable ⇝ can use SoV

Linear 1st order ⇝ can use reverse prod.

(c) $\frac{dy}{dt} = y^2+1$ Separable ⇒ use SoV

Not linear! (y^2) ⇒ cannot use rev. prod.

(d) $\frac{dy}{dt} = e^t y - \cos t$ Not separable ⇒ can't use SoV

Linear $y' = g(t)y + r(t)$ → can use rev. prod.

Answers will vary.

4. (a) Solvable using sep of var only

⇒ Want it not to be linear $\boxed{\frac{dy}{dt} = (y^2+1)(\sin t)}$ works.

(b) Reverse prod rule works but not SoV (must be linear, not separable)

$\boxed{\frac{dy}{dt} = 3t^2y + 4}$

(e) Either method works: $\boxed{\frac{dy}{dt} = (y+5)t}$

separated, but also linear.

5. Initially: 90 # salt in 20 gal water. Saltwater in @ 3 gal/min @ concen. 2 # salt/gal. Well-mixed brine flows out @ 3 gal/min.

Let $S(t)$ = amt of salt at time t (in #)

$\frac{dS}{dt}$ = rate in − rate out $= \left(\frac{2\,\#\text{ salt}}{1\text{ gal}}\right)\left(\frac{3\text{ gal}}{1\text{ min}}\right) - \left(\frac{S\,\#\text{ salt}}{20\text{ gal}}\right)\left(\frac{3\text{ gal}}{1\text{ min}}\right)$

$\therefore \frac{dS}{dt} = 6 - \frac{3S}{20}$

$= \frac{120-3S}{20}$ Separable!

$\int \frac{20}{120-3S}\,dS = \int dt \Rightarrow -\frac{1}{3} 20 \ln|120-3S| = t + C$

$$\ln|120-3S| = -\frac{3}{20}t + C$$

$$|120-3S| = Ce^{-3/20\, t}, \quad C \geq 0$$

$$120-3S = Ce^{-3/20\, t} \quad C = \text{any const}$$

$$S = \frac{120 - Ce^{-3/20\, t}}{3}$$

IVP: 90# salt at $t=0$.

$$90 = \frac{120 - Ce^{0}}{3} = 40 - C$$

$$\Rightarrow C = -50$$

$$\therefore S(t) = \frac{120 + 50e^{-3/20\, t}}{3}$$

$$\Rightarrow S(6) = \frac{120 + 50e^{-18/20}}{3} \approx \boxed{46.8 \text{ lbs}}$$

(seems plausible ∵ concentration coming in is weaker so mix gets diluted.)

$\boxed{9.}$ Tank A: starts w/ 15gal and 6# salt.

coming in: $\left(\frac{1 \text{ \# salt}}{\text{gal}}\right)\left(2 \frac{\text{gal}}{\text{min}}\right)$, out: $\left(\frac{S}{15+t} \frac{\text{\# salt}}{\text{gal}}\right)\left(\frac{1 \text{ gal}}{\text{min}}\right)$

(a) set up + solve IVPs for each of the following tanks.

Tank B: $\frac{dS_B}{dt} = \left(\frac{0 \text{ \# salt}}{\text{gal}}\right)\left(2 \frac{\text{gal}}{\text{min}}\right) - \left(\frac{S_B}{15+t} \frac{\text{\# salt}}{\text{gal}}\right)\left(\frac{1 \text{ gal}}{\text{min}}\right)$

$$\therefore \boxed{\frac{dS_B}{dt} = \frac{-S_B}{15+t}}$$ separable

Solve: $\int \frac{dS_B}{S_B} = \int \frac{-dt}{15+t} \Rightarrow \ln|S_B| = -\ln|15+t| + C$

$$|S_B| = e^{\ln|15+t|^{-1} + C} = C|15+t|^{-1}$$

$$S_B = \frac{C}{15+t}$$

IVP: $S_B(0) = 6 = \frac{C}{15} \Rightarrow C = 90 \Rightarrow \boxed{S_B(t) = \frac{90}{15+t}}$

Tank C: $\dfrac{dS_c}{dt} = \left(\dfrac{1 \text{ lb salt}}{1 \text{ gal}}\right)\left(\dfrac{1 \text{ gal}}{1 \text{ min}}\right) - \left(\dfrac{S_c \ \# \text{ salt}}{\underbrace{15-t}_{\text{gal drop by 1 each min}} \text{ gal}}\right)\left(\dfrac{2 \text{ gal}}{1 \text{ min}}\right)$

$$\frac{dS_c}{dt} = 1 - \frac{2S_c}{15-t}$$

Reverse product: $S_c' + \dfrac{2}{15-t} S_c = 1$

$$S_c' u + \frac{2}{15-t} S_c u = u \qquad y'u + yu' = u \quad \text{where } u' = \frac{2u}{15-t}$$

Separable: $\displaystyle\int \frac{1}{2u}\,du = \int \frac{1}{15-t}\,dt$

$$\tfrac{1}{2}\ln|2u| = -\ln|15-t| + c$$

$$\ln|2u| = -2\ln|15-t| + c$$

$$|2u| = c|15-t|^{-2} \qquad c \geq 0$$

$$2u = \frac{c}{(15-t)^2}, \quad c = \text{any}$$

$$u = \frac{\cancel{c}\ 1}{2(15-t)^2}. \text{ Let } c=1.$$

$$S_c'u + S_c u' = u$$

$$\int (S_c u)'\,dt = \int u\,dt$$

$$\int \left(S_c \frac{1}{2(15-t)^2}\right)' dt = \int \frac{1}{2(15-t)^2}\,dt$$

$$S_c \cdot \frac{1}{\cancel{2}(15-t)^2} = \cancel{\tfrac{1}{2}}(15-t)^{-1} + c$$

$$S_c = (15-t) + c(15-t)^2$$

IVP: $S_c(0) = 6 = (15-0) + c(15-0)^2$

$6 = 15 + 225c, \quad c = -.04$

$$\Rightarrow \boxed{S_c(t) = 15 - t - .04(15-t)^2}$$

Tank D: starts w/ 0# salt, 6 gal water

$$\frac{dS_D}{dt} = \left(\frac{1\#}{\text{gal}}\right)\left(2\frac{\text{gal}}{\text{min}}\right) - \left(\frac{S_D}{15+t}\right)\left(\frac{1 \text{ gal}}{\text{min}}\right)$$

$$\frac{dS_D}{dt} = 2 - \frac{S_D}{15+t}$$

$\xrightarrow{\text{in-class work for Tank A}}$ $S_D(t) = \dfrac{30t + t^2 + C}{15+t}$

IVP: $S_D(0) = 0 = \dfrac{c}{15} \Rightarrow c = 0 \quad \therefore \boxed{S_D(t) = \dfrac{30t + t^2}{15+t}}$

(b) see graphs below.

- A, B, C all have same initial condition.
- C empties after 15 min b/c (rate out) > (rate in).
- A starts w/ $\frac{6 \# \text{ salt}}{15 \text{ gal}}$ and the concentration of salt goes up from here b/c incoming solution has more salt per gal.
- D also increases in overall salt for this ↑ reason, but starts off with less salt to begin with.
- B gets diluted over time b/c pure water enters, so salt decreases over time.