Proposed Paths of Descent

A group of scientists at the Federal Aviation Association has come up with the following two different rate of change equations to predict the height of a helicopter as it nears the ground:

$$\frac{dh}{dt} = -h \qquad \text{and} \qquad \frac{dh}{dt} = -h^{\frac{1}{3}}$$

For both rate of change equations h is in feet and t is in minutes. The scientists, of course, want their models to predict that a helicopter actually lands - but do either or both of the proposed models predict this?

1. Getting familiar with the differential equations:

   (a) Just by examining the rate of change equations, what can you say about the height of the helicopter as predicted by $\frac{dh}{dt} = -h$ and by $\frac{dh}{dt} = -h^{\frac{1}{3}}$? More specifically, as h approaches zero, what can you say about $\frac{dh}{dt}$ and what does that imply about whether the model predicts that the helicopter lands?

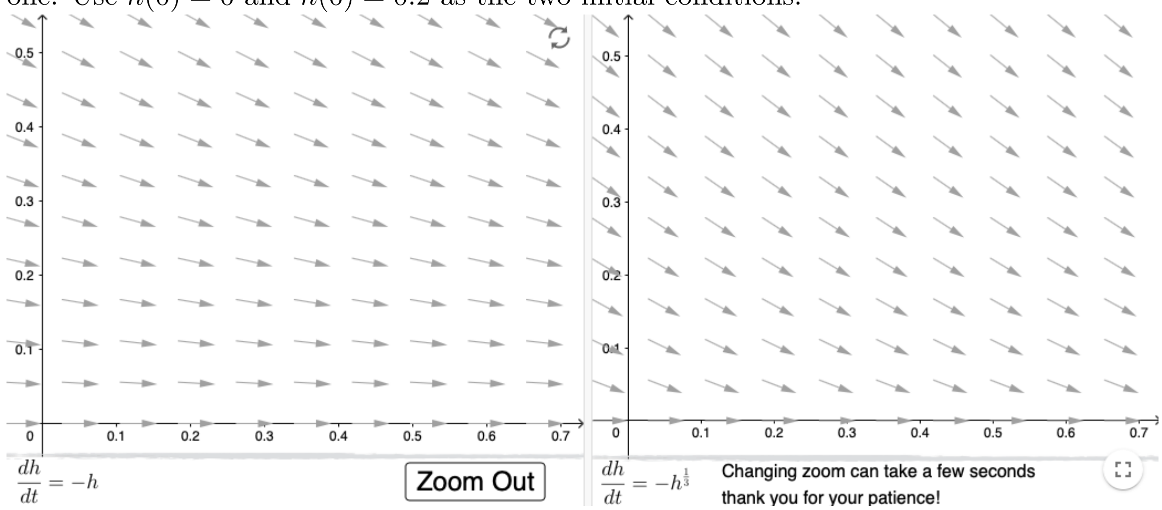
   (b) Sketch your best guess for a height versus time solution graph for each rate of change equation.

2. (a) What do each of the proposed rate of change equations say about the solution to the differential equation if the helicopter is already on the ground? Explain and sketch a corresponding graph of height versus time on the same set of axes from part 1b.

   (b) Interpret the initial condition $h(0) = 0$, and explain why $h(t) = 0$ should be a solution to each differential equation under this initial condition.

3. (a) Use the Geogebra applet, https://ggbm.at/dJsACfAN, to investigate the slope fields. What do the slope fields suggest about whether the model predicts if the helicopters will land? How do the slope fields compare with your sketches from part 1b?

   (b) On the zoomed in version of the two slope fields from the applet, sketch solution curves for each one. Use $h(0) = 0$ and $h(0) = 0.2$ as the two initial conditions.



4. Solve the following initial value problems:

   (a) $\dfrac{dh}{dt} = -h$

   (i) $h(0) = 2$ (ii) $h(0) = 0$ (*Hint*: Use problem 2b)

   (b) $\dfrac{dh}{dt} = -h^{\frac{1}{3}}$

   (i) $h(0) = 2$ (ii) $h(0) = 0$ (*Hint*: Use problem 2b)

5. (a) For each differential equation, interpret the results from problem 4 in terms of whether the model predicts the helicopter will ever touch the ground. If so, at what time?

(b) For each differential equation, suppose we know $h\left(\left(\frac{2}{3}\right)^{\frac{3}{2}}\right) = 0$. In each case, what is $h(0)$?

(c) For each differential equation, interpret the results from problem 4 in terms of whether graphs of (i) and (ii) will ever touch or cross.

The Uniqueness Theorem

As we saw with the helicopters when $h\left(\frac{2}{3}^{3/2}\right) = 0$, it became difficult to predict a value for $h(0)$, because the initial value problem $\frac{dh}{dt} = -h^{\frac{1}{3}}; \quad h\left(\frac{2}{3}^{3/2}\right) = 0$ had multiple solutions. When solutions touch or cross, we lose the predictive power of differential equations. How could we have known that we are safe to use the predictive power of a differential equation without needing the exact solutions? To answer that question, we need the **Uniqueness Theorem** which gives conditions under which an initial value problem has exactly one solution.

In the formal language of differential equations, the term "unique" or "uniqueness" refers to whether or not two solution functions ever touch or cross each other. Using this terminology, the two solutions you found to $\frac{dh}{dt} = -h$ are unique while the two solutions you found to $\frac{dh}{dt} = -h^{\frac{1}{3}}$ are not unique. Fortunately, one does not have to always analytically solve a differential equation to determine if solutions will or will not be unique. The **Uniqueness Theorem** sets out conditions for when solutions are unique.

Theorem. Let $f(x, y)$ be a real valued function which is continuous on the rectangle

$$R = \{(x, y) : |x - x_0| \leq a, |y - y_0| \leq b\}.$$

Assume f has a partial derivative with respect to y and that this partial derivative $\partial f/\partial y$ is also continuous on the rectangle R. Then there exists an interval

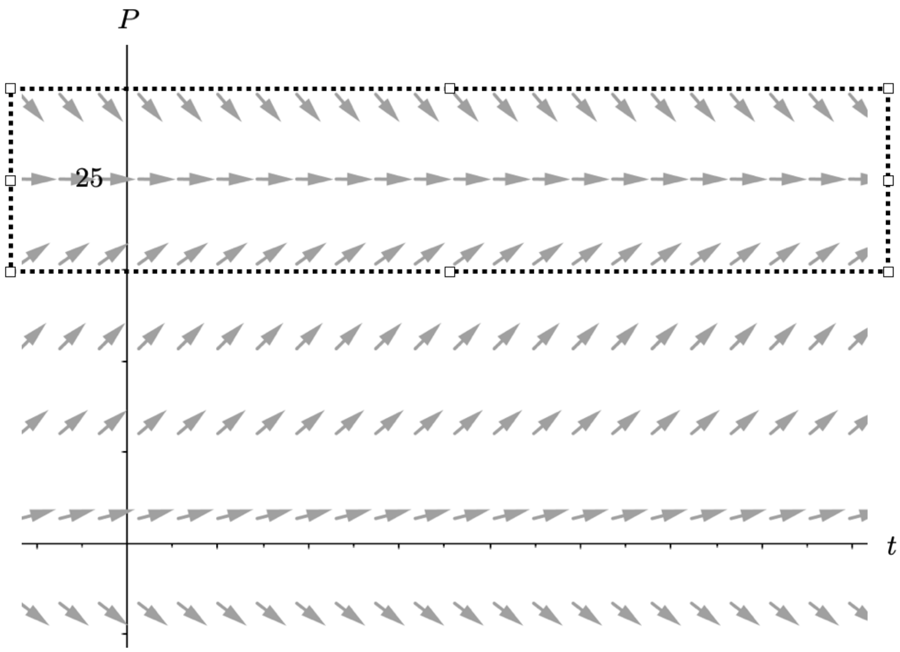
$$I = [x_0 - \Delta x, x_0 + \Delta x] \text{ (with } \Delta x \leq a)$$

such that the initial value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

has a unique solution $y(x)$ defined on the interval I.

Informal Theorem. Suppose we want to know if the graphs of solution functions to a particular rate of change equation $\frac{dy}{dt} = f(t,y)$ are unique. That is, we have decided on a region of the y vs t plane that we care about. For example, suppose we want to know if graphs of solutions to $\frac{dP}{dt} = 2P\left(1 - \frac{P}{25}\right)$ ever touch or cross the equilibrium solution. Then the region we care about is shown in the dotted rectangle below.



Once we have this region established, IF the rate of change of the rate of change, $\frac{\partial}{\partial y}\left(\frac{dy}{dt}\right)$, is continuous in the region we care about. THEN the graphs of the solution functions in this region are guaranteed to be unique. That is, they do not touch or cross each other.

6. Discuss how this informal version captures the formal statement of the uniqueness theorem. How would you use this informal version to discuss the predictive power of $\frac{dh}{dt} = -h^{\frac{1}{3}}$?

7. If you are given a differential equation and determine that the conditions of the uniqueness theorem are NOT met in a specific range of y-values, what can you conclude about the graphs of solution functions within that range of y-values? Explain.

Homework Set 5

1. Suppose two planes start descending at the same time, one is directly above the other and both follow the same differential equation, $\frac{dh}{dt} = -h^{1/3}$. Is there any possibility of a midair collision? Will the initially higher one ever get below the initially lower one? Develop two different arguments to support your conclusion, one based on the uniqueness theorem and one based on the fact this differential equation only depends on h and hence graphs of solutions are related to each in a particular way.

2. In light of the **Uniqueness Theorem**, consider the population model
$$\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{12.5}\right).$$
If $P(0) < 12.5$, will the population ever reach 12.5? Explain.

3. For each differential equation, determine (with reasons) whether or not graphs of solution functions will ever touch any and all equilibrium solution functions (consider both positive and negative values of t).

(a) $\frac{dL}{dt} = .5(1-L)$ (b) $\frac{dy}{dt} = 0.3y\left(1 - \frac{y}{10}\right)$ (c) $\frac{dy}{dt} = -t+1$ (d) $\frac{dy}{dt} = y^{\frac{1}{2}}$

4. Suppose two students are memorizing a list according to the same model $\frac{dL}{dt} = 0.5(1-L)$ where L represents the fraction of the list that is memorized at any time t. According to the uniqueness theorem, will the student who starts out knowing none of the list ever catch up to the student who knows one-third of the list? Explain.

5. What values of p result in predictions that the helicopter will land in a finite amount of time for the model $\frac{dh}{dt} = -h^p$? Explain and show all work.

6. We could use the equation $\frac{dh}{dt} = h^{\frac{1}{3}}$ to model a helicopter taking off. Suppose we know that $h(0) = 0$ and $h(12) = 8$. What does $h(12) = 8$ mean? When did the helicopter take off? Why did you need to know that the $h(12) = 8$ to determine when the helicopter took off?

7. Do graphs of solution functions to the rate of change equation $\frac{dy}{dt} = |y|$ ever touch or cross the equilibrium solutions $y(t) = 0$?
   (a) What does the uniqueness theorem say about this question?
   (b) Answer this question by solving the rate of change equation. *Hint: you will need use the piecewise definition of $|y|$ to create two differential equations, one for each "piece" and then solve each one separately.*

8. (a) Go to the glossary and identify all terms that are relevant to this unit and list those terms here.
   (b) Are there other vocabulary terms that you think are relevant for this unit that were not included? If yes, list them.