Variation of Parameters

Tuesday, November 14, 2023

Recall: 2nd - Order Linear Form

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

where P, Q, R, and G are continuous functions of t.
functions of t.

Recall: If P, Q, and R are constants,
that is

$$py'' + qy' + ry = G(t)$$

then we can use method of undetermined coefficients if $G(t) \neq 0$ and $G(t)$ is of the form given in the table.

The Method of Variation of Parameters

Example:

(1) Consider the following 2nd-order linear ODE.

$$y'' - 2y' + y = \frac{e^t}{1+t^2}. \quad G(t) = \frac{e^t}{1+t^2}, \quad P(t) = 1$$

$\longrightarrow$ We have constant coefficients.

Homogeneous Case: $\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\hookrightarrow \lambda_1 = \lambda_2 = 1$ (repeated eigenvalues)

$$y_h(t) = C_1 e^t + C_2 t e^t$$

$\hookrightarrow y_1(t) = e^t$

$\hookrightarrow y_2(t) = te^t$

$\longrightarrow$ Compute the Wronskian:

$$W(t) = \det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{bmatrix}\right)$$

$$= (t+1)e^{2t} - te^{2t}$$

$$W(t) = e^{2t}$$

$\longrightarrow$ Compute integral $y_2(t) \int \frac{G(t)\, y_1(t)}{P(t)\, W(t)}\, dt = te^t \int \left(\frac{e^t}{1+t^2}\right) \frac{e^t}{e^{2t}}\, dt$

$$= te^t \int \frac{1}{1+t^2}\, dt$$

$$A(t) = te^t \arctan(t)$$

$\longrightarrow$ Compute integral $-y_1(t) \int \frac{G(t)\, y_2(t)}{P(t)\, W(t)}\, dt = -e^t \int \left(\frac{e^t}{1+t^2}\right) \frac{te^t}{e^{2t}}\, dt$

$$= -e^t \int \frac{t}{1+t^2}\, dt$$

$$B(t) = -e^t\left(\frac{1}{2}\ln(1+t^2)\right)$$

$\longrightarrow$ Particular Solution:

$$y_p(t) = A(t) + B(t)$$

$$y_p(t) = te^t \arctan(t) - \frac{e^t}{2}\ln(1+t^2)$$

$\longrightarrow$ General Solution:

$$y(t) = y_h(t) + y_p(t)$$

$$y(t) = C_1 e^t + C_2 te^t + te^t \arctan(t) - \frac{e^t}{2}\ln(1+t^2)$$

(2) Consider the following 2nd-order linear ODE.

$t^2 y'' - 2ty' + 2y = t\ln(t) \qquad G(t) = t\ln(t) \qquad P(t) = t^2$

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$$t^2 y'' - 2ty' + 2y = t\ln(t) \quad , \quad G(t) = t\ln(t) \quad , \quad P(t) = t^2$$

Suppose that $y_h(t) = C_1 t + C_2 t^2$.

$\hookrightarrow y_1(t) = t$

$\hookrightarrow y_2(t) = t^2$

$\longrightarrow$ Compute Wronskian:

$$W(t) = \det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} t & t^2 \\ 1 & 2t \end{bmatrix}\right)$$

$$= 2t^2 - t^2$$

$$W(t) = t^2$$

$$\longrightarrow y_2(t) \int \frac{G(t)\, y_1(t)}{P(t)\, W(t)}\, dt = t^2 \int \frac{t\ln(t)}{t^2}\left(\frac{t}{t^2}\right) dt$$

$$= t^2 \int \frac{\ln(t)}{t^2}\, dt$$

$$= t^2 \left(-\frac{\ln(t) + 1}{t}\right)$$

$$A(t) = -t\,(\ln(t) + 1)$$

$$\longrightarrow -y_1(t) \int \frac{G(t)\, y_2(t)}{P(t)\, W(t)}\, dt = -t \int \frac{t\ln(t)}{t^2}\left(\frac{t^2}{t^2}\right) dt$$

$$= -t \int \frac{\ln(t)}{t}\, dt$$

$$= -t\left(\frac{\ln^2(t)}{2}\right)$$

$$B(t) = -\underline{t\ln^2(t)}$$

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$\longrightarrow$ $y_p(t) = A(t) + B(t)$

$$y_p(t) = -t(\ln(t)+1) - \frac{t\ln^2(t)}{2}$$

$\longrightarrow$ $y(t) = y_h(t) + y_p(t)$

$$y(t) = C_1 t + C_2 t^2 - t(\ln(t)+1) - \frac{t\ln^2(t)}{2}$$