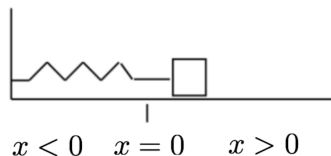## Spring-Mass Motion Investigation

In this problem we use Newton's Law of motion ($\sum F = ma$) to develop a system of rate of change equations in order to be able to describe, explain, and predict the motion of a mass attached to a spring.

$$x < 0 \quad x = 0 \quad x > 0$$

1. Depending on the values for parameters like the stiffness of the spring k, the weight of the object attached to the spring m, and the amount of friction, different behaviors may be possible. Imagine for a set spring and mass you vary the amount of friction on the surface. What do you imagine the various position versus velocity graphs would look like? Provide rough sketches.
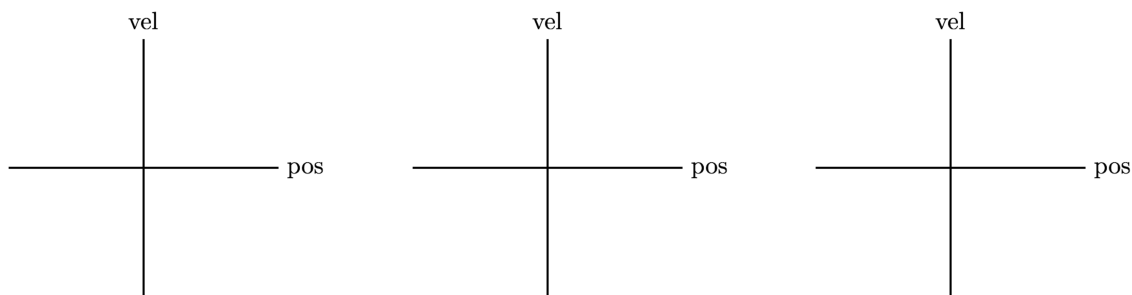
2. Use Newton's Law of motion to develop a rate of change equation to model the motion of an object on a spring. Assume that the only forces acting on the object are the spring force ($-kx$, where k is the spring constant) and the friction force (assumed to be proportional to the velocity, namely $-b\frac{dx}{dt}$, where b is the damping coefficient).

3. Application of Newton's Law of Motion to the spring-mass situation in the previous problem results in the following:

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0,$$

where x is the position of the object attached to the end of the spring, m is the mass of the object, b is the friction parameter (also called damping coefficient), and k is the spring constant. Because $\frac{dx}{dt} = y$, where y is the velocity, and $\frac{dy}{dt} = \frac{d^2x}{dt^2}$, we can converting this to a system of two differential equations as follows:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\frac{k}{m}x - \frac{b}{m}y$$

Use the GeoGebra applet https://ggbm.at/vT5tgWrg to investigate the motion of the object as depicted in the phase plane when $m = 1$, the spring constant $k = 2$, and the friction parameter, b, varies between 0 and 4. In particular, how does the vector field (and corresponding behavior of the mass) change when the friction parameter increases from 0 to say 2, 2.3, 3, or 3.8? Use the space below to record your observations.
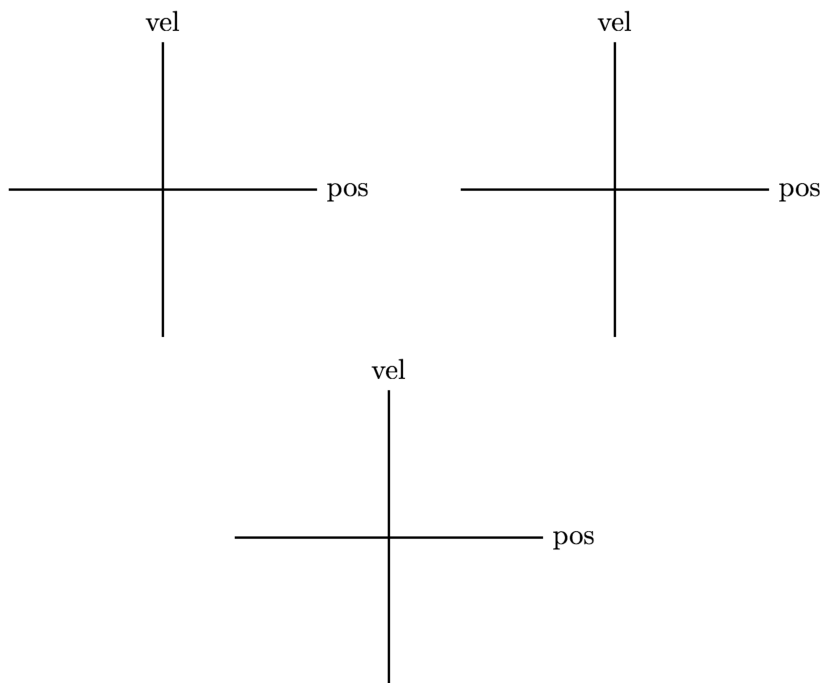
4. Joey and Kara set the friction parameter to 3, resulting in the following system:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -2x - 3y$$

They notice that graphs of solutions in the position-velocity plane seem to get pulled into the origin along a straight line. Help Joey and Kara figure out how to use algebra to find the slope of this straight line.
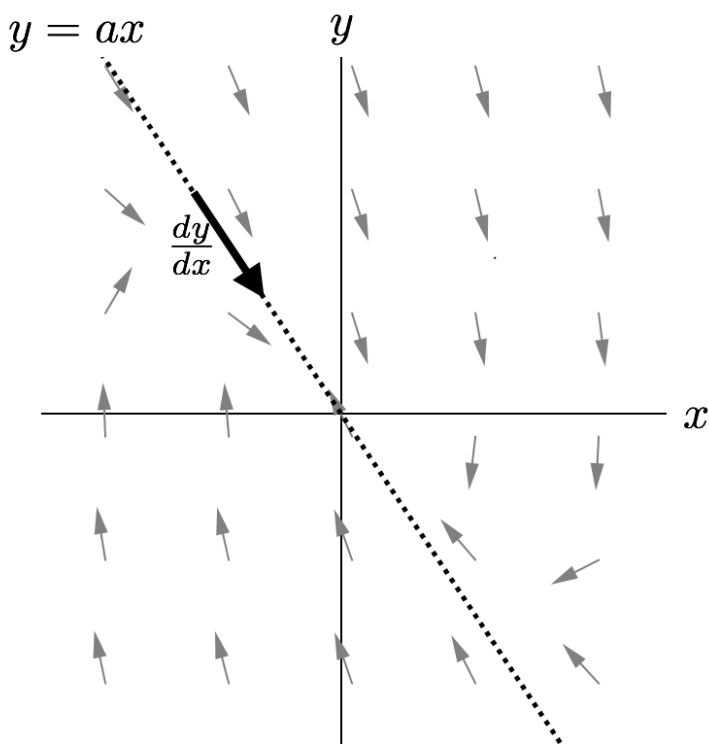
5. Continuing their investigation with the friction parameter set to 3, Kara and Joey are working to find the slope of the observed straight line. Joey sets up the equation $\dfrac{-2x - 3y}{y} = \dfrac{y}{x}$ and Kara sets up the equation $\dfrac{-2x - 3ax}{ax} = a$. Interpret Joey's and Kara's equations and then solve both.
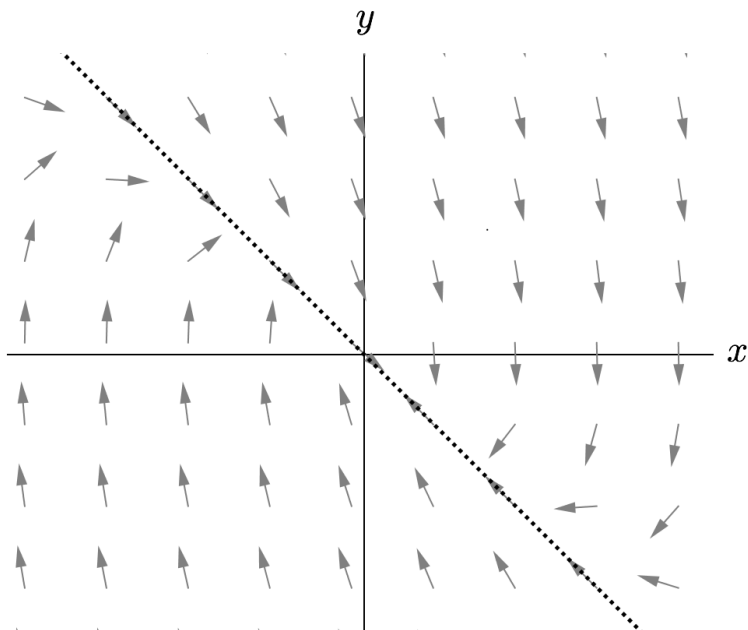

6. Place your finger on the dotted line starting in the second quadrant and trace out the path that the mass takes, as represented in the phase plane. Describe what happens to your finger and relate this to the motion of the mass.

7. Joey found another straight line solution when the friction parameter b was set to 2. Use algebra to find the slope or explain why he is mistaken.

8. In your investigation of the spring-mass system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x - 3y$$

you should have found that when the friction parameter was equal to 3, solutions with initial conditions that are either on the line $y = -x$ or on the line $y = -2x$ head directly toward the origin along a straight path.
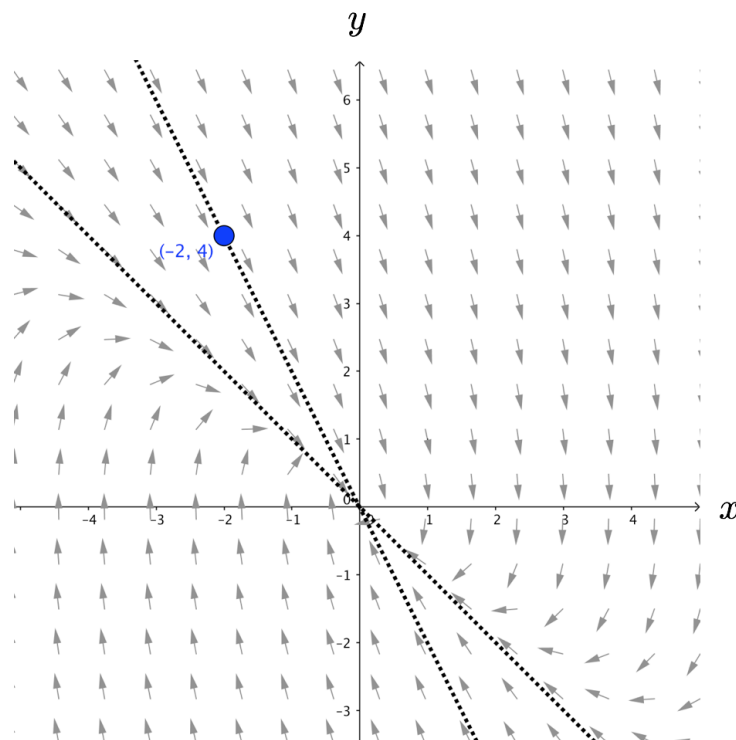
For the initial condition $(-2, 4)$, what are the equations for $x(t)$ and $y(t)$? Hint: substitute $y = -2x$ and $x = -y/2$ into dx/dt and dy/dt, respectively.

9. Susan notices that the $x(t)$ and $y(t)$ equations have the same exponent, and then makes the conjecture that along any straight line solution, $x(t)$ and $y(t)$ **must** have the same exponent. Do you agree with her conjecture? Why or why not?

10. (a) What are the $x(t)$ and $y(t)$ equations for the solution with initial condition $(-1, 2)$? What does the 3D graph of this solution look like? Sketch your 3D graphs. Then use this app to discuss how your graph compares to this one. https://www.geogebra.org/3d/s3gka36p. This app is preset for this problem.

   (b) If you multiplied $x(t)$ and $y(t)$ equations from problem 10a by some number, say -3 for example, is the result also a solution to the system of differential equations? Algebraically show that your conclusion is correct.

   (c) What are the $x(t)$ and $y(t)$ equations for *any* solution with initial condition along the line $y = -2x$?

11. For the initial condition $(-2, 2)$, what are the equations for $x(t)$ and $y(t)$? What are the $x(t)$ and $y(t)$ equations for *any* solution with initial condition along the line $y = -x$?

12. (a) Suppose you were to start with an initial condition somewhere in the second quadrant between the two straight line solutions, say at $(-4, 6)$. Sketch what you think the solution as viewed in the phase plane looks like and explain your reasoning.
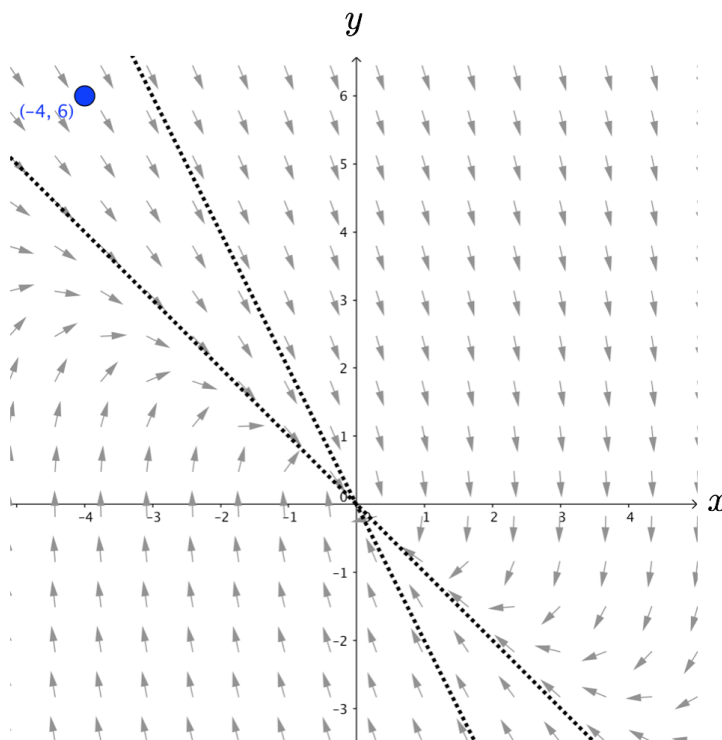
(b) Notice that $(-4, 6)$ is a linear combination of the initial conditions $(-2, 4)$ and $(-2, 2)$, that is, $(-4, 6) = (-2, 4) + (-2, 2)$. Show that the solution with the initial condition $(-4, 6)$ is also a linear combination of the solutions with initial conditions $(-2, 4)$ and $(-2, 2)$.

(c) According to your result in 12b, what does the solution in the phase plane look? Explain your reasoning.

13. (a) What are the $x(t)$ and $y(t)$ equations for the solution with initial condition $(2,5)$?

(b) **According to your $x(t)$ and $y(t)$ equations**, what does the solution in the phase plane look like? Explain your reasoning and provide a sketch. Use the GeoGebra applet, https://ggbm.at/cMSUC7qR to corroborate your conclusion.

(c) Develop an argument that almost all graphs of solutions in the phase plane head into the origin with a slope of -1.

14. As a review of this unit, answer the following questions for the following system

$$\frac{dx}{dt} = -3x + 2y$$
$$\frac{dy}{dt} = 6x + y$$

(a) Find the slopes of the straight line solutions.

(b) For each straight line, find a solution.

(c) Form the general solution.

(d) In the phase plane sketch the straight line solutions and several non-straight line solutions.

(e) Sketch these solutions in 3D. Use this app for help if needed: https://www.geogebra.org/3d/fmsshkd6. This version of the app needs to be programmed with your answer. The "curve" function has 6 inputs. They are $(x(t), y(t), t, t, a, b)$, where a and b are the bounds of time (e.g., 0 to 10).

(f) How would you classify the equilibrium point?

Homework Set 10

1. Consider the system from questions 3-7 from Unit 10. What is the smallest value of b for which we get solutions that, when viewed in the position-velocity plane, lie along a straight line? Algebraically support your conclusion.

2. Straight line Solutions for Systems of the Form

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

Systems of equations of the form above are a special type of **linear system**. Linear systems model important applications, such as the spring mass system. Moreover, it is possible to find the general solution for any such linear system. For each of the system of differential equations below, address the following questions:

- How many equilibrium solutions are there are and what are they?

- Are there solutions that, when viewed in the phase plane (i.e., the $x - y$ plane), lie along a straight line? If so, algebraically figure out the exact slope of the straight line(s).

- For those systems that do have solutions that, when viewed in the phase plane, lie along a straight line, figure out the exact $x(t)$ and $y(t)$ equations for any solution with initial condition on the straight line(s).

- For those systems that have straight line solutions, write down the general solution.

- How would you classify the equilibrium solution? Create terms if needed to classify any new types of equilibrium solutions and explain the meaning of your terms.

- For those systems of differential equations that do have solutions that, when viewed in the phase plane, lie along straight lines, what do these straight lines look like in 3D? Provide your best 3D sketch.

(a) $\begin{aligned}\frac{dx}{dt} &= -3x + 2y \\ \frac{dy}{dt} &= 6x + y\end{aligned}$
(b) $\begin{aligned}\frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= -x + y\end{aligned}$
(c) $\begin{aligned}\frac{dx}{dt} &= -2x - 2y \\ \frac{dy}{dt} &= -x - 3y\end{aligned}$
(d) $\begin{aligned}\frac{dx}{dt} &= 2x + 2y \\ \frac{dy}{dt} &= x + 3y\end{aligned}$

3. You figured out from our analysis on the previous problems, sometimes there are solutions in the phase plane that lie along a straight line headed directly towards or away from the equilibrium solution at the origin and sometimes there are not.

(a) Explain in words how you figure out whether there are any straight line solutions in the phase plane and if so, what the slopes of this line or lines are. Demonstrate how your approach works in general for linear systems of the form

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

(b) Explain in words how you figure out the $x(t)$ and $y(t)$ equations for any and all straight line solutions in the phase plane. Demonstrate how your approach works in general for linear systems of the form

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

(c) Explain in words why having two different straight line solutions is useful for finding the $x(t)$ and $y(t)$ equations for any initial condition.

4. Below is a vector field for the system of differential equations:

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = -4y$$

Straight line solutions lie along the line $y = 0$ (with positive exponent in the $x(t)$ and $y(t)$ equations) and along the line $y = -2x$ (with negative exponent in the $x(t)$ and $y(t)$ equations).
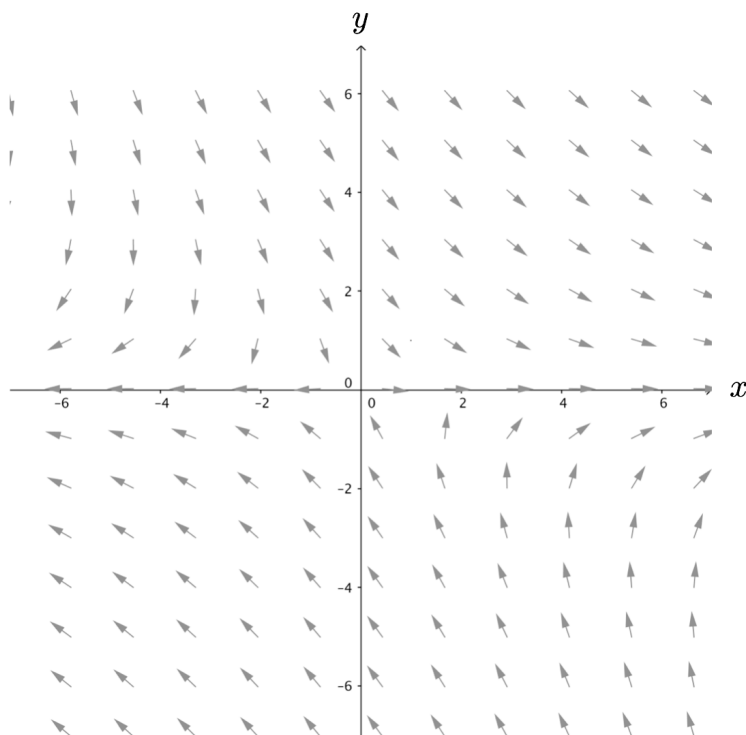
(a) Consider two different initial conditions, one at the point (1, 0) and one at the point (3, 0). Determine, with reasons, what happens to the graphs of the two solutions with these initial conditions as time progresses.

(b) Repeat problem 4a for the initial conditions (-1, 2) and (-3, 6).

5. Find a value or a range of values for the parameter n between -4 and 4 (including non-integer values) in the system of differential equations

$$\frac{dx}{dt} = -3x + ny$$
$$\frac{dy}{dt} = 6x + y$$

so that when you view solutions in the x-y plane there are

(a) exactly two different straight line solutions

(b) no straight line solutions

(c) exactly one straight line solution

(d) an infinite number of equilibrium solutions and an infinite number of straight line solutions

6. Consider the following system of differential equations:

$$\frac{dx}{dt} = 2x$$
$$\frac{dy}{dt} = 2y$$

(a) Without using technology, sketch many different solutions in the phase plane. Explain your reasoning.

(b) Unlike other systems of differential equations that we have been studying, this system can be solved using techniques from our study of 1-dimensional systems. What makes this system different?

(c) Find the general solution in two ways, one using separation of variables and the other using straight line techniques.

(d) Explain how the general solution can help you make sense of the solution graphs in the phase plane.

7. Without using technology, sketch many different solutions in the phase plane for the following system of differential equations. Explain your reasoning. [*Hint*: how many equilibrium solutions are there?]

$$\frac{dx}{dt} = -3x - \frac{1}{2}y$$
$$\frac{dy}{dt} = 6x + y$$

8. **A Swaying Skyscraper**: When the John Hancock Building in Boston, MA was first built it tended to sway back and forth (like an oscillating spring) so much so that people in the top floors experienced motion sickness. Similar to the spring mass system of rate of change equations, we can model the back and forth motion of the building by adding a gravity term, which turns out to be x^3, to the spring mass model. Notice how this system is similar to a linear system but has the addition of a nonlinear term x^3 term in $\frac{dy}{dt}$. As we will see in a later unit, the inclusion of this nonlinear term means that we are not able to use the straight line solution technique. However, we can still use our previous graphical approaches to explore solutions to help make predictions about the swaying motion of the building.

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x - y + x^3$$

In this simplified system of rate of change equations, x stands for the amount of displacement of the building from the vertical position at any time t and y stands for the horizontal velocity of the building at any time t. Use the GeoGebra Vector Field applet, https://ggbm.at/kkNXUVds, as a tool to explore solutions as viewed in the xy-plane (i.e., the phase plane).

(a) Determine all equilibrium solutions and explain the meaning of each one in terms of the swaying skyscraper.

(b) Provide a sketch of several representative curves in the phase plane and give an interpretation for the motion of the building for the different types of curves (e.g., does the building remain standing? If so, for what initial conditions? For what range of initial conditions is a disaster predicted?)

9. (a) Go to the glossary and identify all terms that are relevant to this unit and list those terms here.

(b) Are there other vocabulary terms that you think are relevant for this unit that were not included? If yes, list them.