# Ordinary Differential Equations

*A Primer on Dynamics and Systems*

**Chris Hallstrom**
**University of Portland**

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# Preface

This book is designed to be used as a textbook for a standard one-semester introduction to ordinary differential equations. At the University of Portland, this course is MTH 321, *Introduction to Differential Equations*, taken primarily by engineering, physics, and mathematics students. An emphasis is placed on applications over the more theoretical aspects of the subject and the only prerequisite for the course is one year of calculus.

This particular text differs from other texts in several respects. First, it emphasizes a dynamical systems approach in which qualitative analysis is used to supplement the standard techniques for finding explicit solutions. Fundamental methods like separation of variables and techniques for solving first-order linear equations are covered, but so too are topics such as equilibria, stability and bifurcations. Secondly, the book includes a *systems* approach to second-order linear equations.

In writing this book, I have relied primarily on course notes that I have developed in teaching introductory differential equations courses over the past ten years. A few particular texts have influenced my approach, and this book certainly reflects those influences. I have listed the most prominent of these books in the Further Reading section at the end. I have also relied heavily on Jiří Lebl's book, *Notes on Diffy Qs* [JL], particularly in learning how to write a book in LaTeX.

The bulk of this text was written during a sabbatical leave from the University of Portland in Fall 2010; I certainly am grateful to UP for the opportunity to get this project underway. I would also like to acknowledge the support and suggestions offered by my colleagues, especially Hannah Callender, Craig Swinyard, Chris Lee, and Greg Hill.

Chris Hallstrom
Department of Mathematics
University of Portland

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# Chapter 1

# Introduction to Differential Equations

One of the main themes in calculus is the significance of the derivative. For instance, we know that the derivative of a function gives us the slope of the tangent line which allows us to do things like construct linear approximations. We also know that if the derivative of a function is positive (or negative) on some interval then the function is increasing (or decreasing) on that interval. This knowledge can help us sketch a graph or to locate local minima and maxima; in both cases, information about the derivative tells us more about the behavior or properties of the function.

In calculus, we typically start with a given function, find its derivative, and then use the derivative to learn more about the function. In this course, we turn this around. Suppose that all we have is some information about a function's derivative, can we figure out what the function is? In some cases, the answer will be yes. But even if we can't find an explicit formula for a function, we can still deduce many of its properties or otherwise describe its behavior. When dealing with applications, this is often good enough. This is the central problem in the study of differential equations: given some information about the derivative(s) of a function, what more can we say about the function's properties?

## 1.1 Terminology and Basic Ideas

What is a differential equation? Simply put, a **differential equation** is any equation involving a function (which is typically unknown) and one or more of its derivatives. Here is a simple example:

$$f'(t) = \cos(t). \tag{1.1}$$

We know from calculus that the function $f(t) = \sin(t)$ satisfies this equation. However, we also know that any function of the form $f(t) = \sin(t) + c$ will also work. In calculus, we call this process of finding $f(t)$ *antidifferentiating*. This example is almost *too* simple because we very quickly see what the unknown function is and it's easy to forget that in the original equation (1.1), the function $f(t)$ is unknown – it is only because we happen to remember our calculus facts that we know what it is.

Here's a more interesting example. Consider an object falling through the atmosphere. From physics, we know that the motion is described by Newton's second law which relates the force applied to an object to the acceleration it experiences. If $x(t)$ represents the position of the object at time $t$ then acceleration is the second derivative $x''(t)$ and Newton's law is expressed by the differential equation

$$F = mx''.$$

If we note that acceleration is simply the rate of change (i.e. the time derivative) of the velocity $v(t)$ then we can rewrite the equation of motion as a different (but equivalent) differential equation:

$$F = mv'.$$

Let's assume that the only forces acting on our object are a constant gravitational force $F_g$ and a drag force $F_d$ due to air resistance. If we set up our coordinate system so that downward is the positive direction, then we have $F_g = mg$. Suppose also that we know that the drag force on our object increases the faster the object is moving and then assume the simplest possible relationship that captures this: we take $F_d$ to be proportional to the object's velocity

$$F_d = -\gamma v.$$

The negative sign represents the fact that the drag force is opposing the motion, tending to slow the object down. The positive proportionality constant $\gamma$ is called the drag coefficient. Substituting these forces into the equation and rewriting to put $v'$ on the left hand side gives us the differential equation

$$v' = g - \frac{\gamma}{m}v. \tag{1.2}$$

which describes the motion of our object. The goal at this point is to either find an explicit formula for $v(t)$ or to otherwise learn more about the behavior of the function $v(t)$.

Note that unlike (1.1), the unknown function $v(t)$ appears on the right hand side of equation (1.2) which means that we cannot simply integrate (antidifferentiate) as we did before. The fact that the unknown function typically appears on the right-hand side is essentially what makes these kinds of differential equations much more challenging to solve than the standard calculus problems we're used to. In most cases, we will not be able to find an explicit solution!

A comment about the notation: in writing (1.2) we dropped the explicit reference to the independent variable by writing $v$ instead of $v(t)$. This is simply a matter of convenience; the reader is expected to remember that the velocity function is a function of time. Another clue that $v$ is a function is that its derivative $v'$ appears on the left hand side.

### Terminology and Notation

This course deals with **ordinary differential equations** , also known as **ODE**s. The term *ordinary* in this context means that we are working with functions of a single independent variable. If you've had a course in multivariable calculus, you know that for functions that depend on more than one independent variable we need to introduce the concept of *partial* derivatives. The analysis of **partial differential equations** (**PDE**s) is quite a bit different in many ways and the subject of an entirely different course.

We will refer to the unknown function as the **dependent** or **state variable**. In many of the applications that we will see, the state variable will be a function of time. Since a derivative can be thought of as an instantaneous rate of change, we can use differential equations to model the behavior of many physical systems that change over time. This is precisely why differential equations are often very effective in modeling phenomena in the physical world.[1]

Many applications involve additional quantities that don't change over time (assuming time is the independent variable) but might be adjusted from example to example – through natural causes for instance, or by running a series of experiments in which we change the constant somehow. These types of constants are called **parameters**.

In our falling body problem, for example, the parameters are the physical constants $g$, $\gamma$, and $m$. For the purposes of a particular falling object we could simply plug in numerical values for these constants. On the other hand, we might want to know how our solutions would change for different values of these constants. For this reason, we typically will not plug in numerical values until *after* we have found a solution. For instance, if we leave the gravitational constant as $g$, rather than plugging in a value of 9.8 $m/s^2$, our equation for a falling object is just as valid on the surface of Mars as it is on the surface of Earth. In fact, determining precisely how a solution depends upon the given parameters is an important aspect in the study of differential equations.

We've already mentioned the distinction between ODEs and PDEs. Here are some additional concepts and terms that we will use to describe different types of differential equations. As we will see, many of the techniques that we will develop will apply only to particular types of equations. We therefore need to be able to recognize what kind of equation we're dealing with so that we know what methods to apply.

The most basic way to describe an ODE is by the number of derivatives that appear. The **order** of a differential equation is simply the order of the highest derivative that appears in the equation. For example, if we write Newton's law of motion as $F = mv'$, this is a first-order equation. On the other hand, if we express velocity as the time derivative of position then Newton's second law becomes the second order equation $F = mx''$.

A standard way of writing a differential equation is by solving for the highest-order derivative and putting everything else on the right-hand side.[2] A generic first-order equation would then look

[1] See Eugene Wigner's essay "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" in Communications in Pure and Applied Mathematics, vol. 13, No. I (February 1960) for more thoughts on this subject.

[2] Of course, it's not always possible to solve for the highest-order derivative. Here's a more general way of

like this:

$$y' = f(t, y).$$

This is sometimes called **normal** or **standard form**. Here, $f$ is simply short-hand notation for "the right-hand side". It's written as a function of $t$ and $y$ because in general, those variables will appear on the right-hand side. If $t$ does not appear explicitly on the right-hand side, then we say that the equation is **autonomous** and the standard form becomes

$$y' = f(y). \tag{1.3}$$

If the right hand side does depend explicitly on $t$ then the equation is called **nonautonomous**.

### 1.1.1 Solutions to Differential Equations

A **solution** to a differential equation is any function that satisfies the given equation – meaning that the equation is true if the unknown function is replaced by the solution (and false if it's replaced by any other function). Typically, we do not know what these functions are beforehand so when we say that we're "solving" a differential equation, we mean that we're trying to find these unknown functions. One of the goals for this course is to learn some of the basic techniques for finding solutions to differential equations.

In some cases, such as (1.1), we can find explicit functions that solve the differential equations using standard calculus techniques. In other cases, we may be able to find a solution but not one expressed in terms of familiar functions. For example, consider this calculus problem:

$$y' = e^{-t^2}. \tag{1.4}$$

It turns out that there is no antiderivative that can be expressed in terms of so-called elementary functions. The best we can do is to simply write

$$y(t) = \int e^{-t^2}\, dt.$$

So even though we don't have a nice name for the function $y(t)$, we have essentially solved this antidifferentiation problem by finding an expression for a function that satisfies (1.4). In still other cases, it may not actually be possible to find any explicit solutions. For these problems our goal is to discover as much as we can about the *behavior* of the solutions using the information provided by the differential equation. This kind of *qualitative* analysis will be the subject of later sections.

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expressing a first-order differential equation:

$$F(t, y, y') = 0$$

where $F$ is simply shorthand for "everything on the left hand side".

### Exponential Solutions

As we have already mentioned, the study of differential equations starts to move beyond calculus when we investigate equations in which the unknown function appears on both the left and right hand sides. One of the most basic examples is

$$\frac{dy}{dt} = y. \tag{1.5}$$

It's important to understand that we cannot simply anti-differentiate on the right hand side. Because the derivative on the left is taken with respect to the variable $t$, we must integrate both sides of the equation with respect $t$, not $y$, to undo this derivative. But since we don't know what the function $y(t)$ is, it's impossible to evaluate its integral!

In the next chapter, we will learn techniques for systematically finding solutions to this and a few other equations. However, this particular example is simple enough, and we have enough calculus experience that it's not hard to guess the answer. It helps if we think about the question that is implied by this differential equation:

*What function is the same as its derivative?*

After thinking for a moment about the functions that you know, you probably realized that the exponential function

$$y(t) = e^t$$

satisfies this equation; in other words, it is a solution. You may have also realized that this function multiplied by any constant will also work so that:

$$y(t) = Ce^t$$

is a solution for any value of $C$. We say that this represents an infinite family of solutions.

Let's generalize this example a bit. Suppose that the right hand side of the differential equation includes a constant factor:

$$\frac{dy}{dt} = ky. \tag{1.6}$$

If we think about what happens when we differentiate exponential functions, you can see that $y(t) = e^{kt}$ is a solution, as is any constant times this exponential function. In fact, the family of functions

$$y(t) = Ce^{kt}. \tag{1.7}$$

represents all possible solutions to equation (1.6). We indicate this by saying (1.7) is the **general solution** of (1.6).

This particular differential equation will come up often enough that we don't want to have to stop and work out the solution every time it comes up. **From now on, whenever you see an equation that looks like (1.6), you should immediately recognize that the solution is the exponential function (1.7).**

### Checking Solutions

One of the nice things about differential equations is that if you think you have found a solution, it's usually fairly straightforward to check to see if you are correct: simply plug your solution into each side of the differential equation and verify that both sides are equal.

In Section 1.1, for example, we discussed an equation describing the motion of an object falling through the atmosphere. Just to make things a bit more concrete, let's suppose we have some specific values for the parameters, giving the differential equation

$$v' = 10 - v/5. \tag{1.8}$$

Later on in Chapter 2, we will learn methods for finding the solution, but for now let's imagine that someone else tells us that the functions

$$v(t) = Ce^{-t/5} + 50$$

are solutions (where $C$ is an arbitrary constant).

How can we tell if this is correct? To verify this claim, we simply compute the left-hand and right-hand sides of (1.8) separately and then compare to see if they're the same. Step one is to compute the left-hand side of the differential equation by differentiating the given $v(t)$:

$$v'(t) = -(C/5)e^{-t/5}. \tag{1.9}$$

We put this aside and move on to step two: computing the right-hand side of (1.8). Plugging in the given $v(t)$ and simplifying we get

$$10 - v/5 = 10 - \left(Ce^{-t/5} + 50\right)/5 = 10 - (C/5)e^{-t/5} - 10 = -(C/5)e^{-t/5}. \tag{1.10}$$

Finally, we compare the results. Since the right-hand side (1.10) is in fact the same as the left-hand side (1.9), we have verified that the given function $v(t)$ is in fact a solution of the differential equation (1.8).

**Example 1.1.1.** Show that $u(t) = te^{-t}$ is a solution to the differential equation

$$u' = -u + e^{-t}. \tag{1.11}$$

Starting with left hand side of equation (1.11), we have

$$u' = (te^{-t})' = e^{-t} - te^{-t} \qquad \text{(LHS)}$$

On the other hand, looking at the right hand side, we have

$$-u + e^{-t} = -te^{-t} + e^{-t} \qquad \text{(RHS)}$$

Now we can compare. Since we ended up with the same thing in both cases, we can say that $u'$ does in fact equal $-u + e^{-t}$. In other words, $u(t) = te^{-t}$ is a solution.

**Example 1.1.2.** Determine whether $y(t) = e^{-t} + C$ is the general solution to the equation $y' + y = 0$. If we plug the given function into the left hand side of the differential equation, we get

$$y' + y = (-e^{-t}) + (e^{-t} + C) = C.$$

Since we end up with $C$, not 0, we conclude that the given function is **not** the general solution.

### 1.1.2 The Initial Value Problem

In the previous section, we verified that the differential equation

$$v' = 10 - v/5$$

has solutions

$$v(t) = Ce^{-t/5} + 50. \tag{1.12}$$

Because the constant $C$ can take on any value, the expression for $v(t)$ given in (1.12) represents not just one solution but an infinite *family* of solutions. Figure 1.1 shows the graphs of $v(t)$ for several different values of $C$.

When this infinite family of solutions represents all possible solutions to the differential equation, we call this the **general solution**.[3] If we want to specify a single solution, however, we can do so by identifying a specific point in the $t - v$ plane through which the solution curve passes. For instance, only one of the curves shown in Figure 1.1 passes through the point $(0, 0)$. If we think about the physical meaning of the variables in this example, this solution corresponds to an object that has a velocity $v = 0$ when $t = 0$. Because the independent variable is time, we can think of this extra information as an **initial condition** or **initial value** that the solution must satisfy. We will use this terminology even in examples for which the independent variable is not time.

To find the particular solution that satisfies a given initial condition, we simply plug these values into the solution and solve for the constant $C$. For instance, plugging $v(0) = 0$ into (1.12) gives

$$v(0) = Ce^{-0/5} + 50 = 0$$

[3] A word of caution: just because a solution contains arbitrary constants does not mean it represents all possible solutions; in this case we cannot say that we have the general solution.

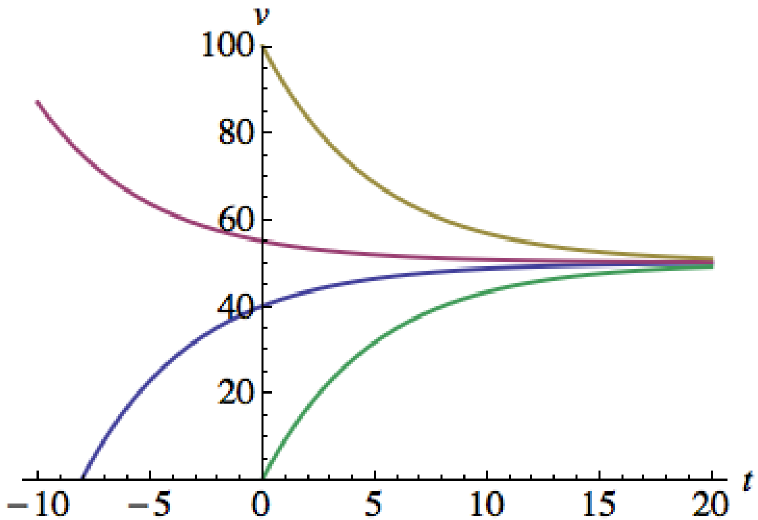



**Figure 1.1:** Time series plots of solutions to $v' = 10 - v/5$ on the interval $-10 < 0 < 20$ for several different values of $C$.

which implies $C = -50$. The particular solution that satisfies the given initial condition is therefore

$$v(t) = -50e^{-t/5} + 50.$$

If we don't have a specific numerical value in mind, we can still express express our solutions in terms of an arbitrary initial condition. This is helpful if we want to understand how the solutions depend upon the choice of initial value. In many cases, we will use use $t = 0$ as the initial time, but we can just as well start the clock at any other value, denoted $t_0$; we would then express our initial condition as $v(t_0) = v_0$. Once we have a solution that contains an arbitrary constant, we can simply plug in our initial conditions and solve for the constant.

In our example above we can express the solution in terms of the general initial value $v(0) = v_0$ by writing

$$v(0) = Ce^{-0/5} + 50 = v_0$$

and solving to get $C = v_0 - 50$. Then the solution can be written in terms of the initial value $v_0$:

$$v(t) = (v_0 - 50)e^{-t/5} + 50. \tag{1.13}$$

**Example 1.1.3.** Find the solution to $y' = 2y$ that satisfies $y(0) = 10$. First, we recall that the general solution to this differential equation is

$$y(t) = Ce^{2t}.$$

To satisfy the initial condition, we plug in $y(0) = 10$:

$$10 = Ce^{2(0)} = C.$$

So $C = 10$ and the particular solution is $y(t) = 10e^{2t}$.

Note that in these two examples, the differential equations are first order and the most general solution has one arbitrary constant. As we will see, the general solution to a second order equation will have two arbitrary constants. Roughly speaking, to solve a differential equation we must "undo" each of the derivatives with an integration. Since two derivatives means two integration constants, we will need two extra conditions to specify a specific solution to a second order equation. This principle applies to differential equations of higher orders as well.

In this text, the extra conditions that we need to determine a particular solution will typically come from specifying the value of the solution *and perhaps its derivatives* at some particular time. We will refer to the differential equation, together with the condition (or set of conditions) at an initial time $t_0$ as an **initial value problem** or **IVP** for short.

Specifying initial values is not the only way to impose the extra conditions needed to determine a particular solution however. Consider a second order differential equation. Instead of specifying the value of the function and its derivative at $t_0$, we could instead specify the value of the function at two different times, $t_0$ and $t_1$. We refer to these type of problems as **boundary value problems** or **BVP** for short.

## 1.2 Modeling with Differential Equations

As we've already seen, differential equations can be extremely useful in describing the natural world. Of course the real-world is complicated – filled with all sorts of competing and interrelated factors – and so we make certain assumptions in order to simplify things to the point where we can actually solve or otherwise analyze our system.

Going back to our example of an object falling through the atmosphere, for instance, we pretended that the gravitational acceleration is a constant. This is a fiction – but it's a very useful fiction and for many applications (finding the the trajectory of a soccer ball or the speed of a skydiver, e.g.) the error introduced by making this assumption is so small that it won't significantly impact any predictions we make based on our model. If we ever find that the error is too big, we can always go back and revise our model to account for the fact that the Earth is not flat.

There is something of an art to constructing mathematical models and we could easily spend an entire course on just this topic. In our context, modeling plays a critical role in determining where differential equations come from. At the same time, once we have a solution or otherwise understand the behavior of a solution, we can use this information to help us understand the validity of our models. We will not dwell on this issue, but throughout the course we will look at a few important examples which will help give us some context for interpreting the solutions to our equations.

The process of constructing a model can be summarized by the following steps:

1. Define the problem and identify the relevant quantities. State the assumptions and relationships between the quantities being studied. This is where the science comes in; often the assumptions and statements used here are given to us (the mathematicians) by someone better versed in the details of the particular system (biologists, economists, physicists, etc.).

2. Define variables and any parameters to be used in the model. When working with any model, it's important to be aware of which quantities are constant and which can change.

3. Translate the statements in Step 1 into equations involving the quantities stated in Step 2.

4. Analyze or solve the equations. Remember that in many cases it will not be possible to find an explicit solution; instead the goal is to understand the *behavior* of the solutions.

5. Compare the results of our analysis with observations and revise model if necessary.

### 1.2.1 Masses and Springs

One of the most important models that we will consider is the mass-spring equation. Consider an object with mass $m$ lying on a table attached to a fixed wall by a spring. At time $t = 0$, the object is displaced a distance $x_0$ from its equilibrium position and released. Ignoring friction with the table, we know that the object will oscillate around the equilibrium with a fixed frequency.

To construct the model, we use Newton's second law of motion. Letting $x(t)$ denote the displacement from equilibrium at time $t$, we have

$$F = mx''.$$

If we ignore friction or any other damping in the system, the only force acting on the object is the restoring force of the spring. If the spring obeys Hooke's law[4] this tells us that the restoring force is proportional to the displacement, i.e.

$$F = -kx.$$

There is a minus sign since the force acts in the direction opposite that of the displacement. In other words, a displacement to the right (positive $x$) results in a force acting to the left (negative $F$) and vice versa. The proportionality constant $k$ is referred to as the spring constant; it measures the stiffness of the spring.

The differential equation is then written

$$mx'' = -kx$$

[4]Note that Hooke's law is just one possible model for the behavior of a spring. In a nonlinear spring, for instance, the restoring force is proportional to some other power of the displacement.

Since this is a second-order equation, we need two additional conditions to determine a particular solution. These are given by the initial conditions

$$x(0) = x_0, \qquad x'(0) = v_0.$$

In many applications, of course, it is important to include the effects of damping, i.e. friction. For instance, we might assume that the damping force is proportional to the velocity of the object. Adding this to Hooke's law gives the differential equation

$$mx'' + \gamma x' + kx = 0. \tag{1.14}$$

It is customary to write this equation with all the terms on the left-hand side.

### 1.2.2 Electric Circuits

Another important second-order equation is one that models the flow of electric current in a simple circuit in which a resistor, a capacitor and an inductor are connected in series as shown in Figure 1.2. The current $I$, measured in amperes, is a function of time. The resistance $R$ (ohms), the capacitance C (farads) and the inductance (henrys) are all positive constants. A voltage $V(t)$ is also applied.

A physically relevant quantity is the total charge $Q$ on the capacitor at time $t$. The relationship between $Q$ and $I$ is

$$I = \frac{dQ}{dt}.$$

The flow of current is governed by Kirchhoff's second law and from the basic laws of electricity, we have the differential equation

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = V(t).$$

If we substitute for $I$, we have the second-order equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V(t).$$

Notice that the left hand side of this equation has the exact same mathematical form as the mass-spring equation discussed in the previous section. The only difference is the physical interpretation of the coefficients.

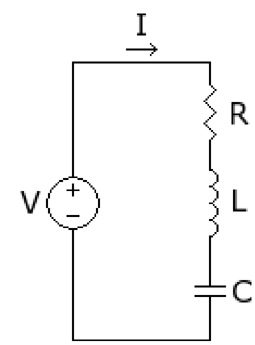



**Figure 1.2:** RLC series circuit driven by an applied voltage $V(t)$.

### 1.2.3 Population Models

Trying to describe how the population of some species changes over time can be quite challenging. Unlike many problems from physics, there are seldom any clear-cut or well understood mechanisms that govern how a given species interacts with its environment. Instead, models in population ecology are typically based on observations and assumptions that may only be reasonable in a limited context.

### Exponential Growth

Suppose that a population of brown marmorated stink bugs is observed over a period of time and is found to obey the following principle:

*The rate of change of the population is proportional to the current population level.*

In a nutshell: the more stinkbugs there are, the faster they reproduce. If $p(t)$ represents the population at time $t$, this principle can be translated into the differential equation

$$p' = kp. \tag{1.15}$$

As we know, the solution to this equation is $p(t) = ce^{kt}$. We can also express this solution in terms of an initial population by plugging in $p(0) = p_0$:

$$p(t) = p_0 e^{kt}. \tag{1.16}$$

Of course, we don't expect this to be a realistic model for long periods of time. In an actual ecosystem, we know that any population would be limited by the availability of resources (food), the presence of predators, etc. However, if a population is at low enough level so that resources are abundant and if there are otherwise no checks on the population growth, we may expect to see exponential growth for some interval of time.

### Logistic Model

The exponential model described above can be modified to account for limited resources. Suppose that we wanted a model that satisfied the following conditions:

(i) If $p(t)$ is small, the exponential growth model is not unreasonable, i.e. $p' \approx kp$.

(ii) Once $p(t)$ gets large enough, however, we expect the growth rate $p'$ to be negative.

We are now looking for a differential equation of the form

$$p' = f(p)$$

where the function $f(p)$ looks linear near the origin (condition (i)) but turns around and goes back to zero for larger $p$ (condition (ii)). A simple (and differentiable) function that satisfies these two conditions is a quadratic, which leads us to the model

$$p' = rp\left(1 - \frac{p}{K}\right). \tag{1.17}$$

We could further modify this model by including additional terms to account for other factors that can affect the population. Some of these examples will be considered in the examples and exercises in subsequent chapters.

### 1.2.4 Newton's Law of Heat Transfer

Suppose we have an object with some initial temperature $T_0$ that is placed in an environment at some other temperature $T_e$. We will assume that the temperature of the object is uniform and also that its temperature doesn't noticeably affect the surrounding temperature. For example, a cup of coffee sitting on your desk will cool off over time without noticeably affecting the room temperature. Newton's law of cooling (heating) says that the rate of change of the object's temperature, $T(t)$ is proportional to the difference between its temperature and the surrounding temperature. Translating this statement into a differential equation:

$$T' = k(T_e - T). \tag{1.18}$$

Notice that if the temperature of the object is greater than then surrounding temperature then the quantity $T_e - T$ is negative and since we expect the object to cool off in this case, the implication is that the proportionality constant $k$ is positive. On the other hand, if the temperature of the object is less than the surrounding temperature, then $T_e - T$ is positive, which means that $T' > 0$ indicating that the temperature will increase. This shows that this single model describes both heating and cooling scenarios.

Note that this model does not assume that the external temperature $T_e$ is a constant. For example, we can use Newton's law of cooling to describe the temperature $T(t)$ inside a room or house $T_e$ given by a fluctuating external temperature.

### 1.2.5 Mixing Problems

Suppose you have one or more containers or reservoirs into which some substance is simultaneously being added and removed. By keeping track of the net rate of change, we can construct a model for the total amount of the substance in the container. The following is a typical example

> **Example 1.2.1.** A tank initially contains 5 lbs. of salt dissolved in 100 gallons of water. A salt water solution with a concentration of 0.1 lbs/gal is pumped in at a rate of 2 gal/min. The tank is simultaneously drained at the same rate. Assume that the tank is well mixed so that there is a uniform concentration throughout. Find a differential equation that describes the amount of salt at time $t$.

The basic set-up for these mixing problems is:

$$\text{Net Rate of Change} = \text{Rate In} - \text{Rate Out}.$$

The first step is to identify the quantity that we're keeping track of. It is usually easier to work in terms of total amounts, rather than concentrations. So in the example above, we can let $S(t)$

represent the amount of salt in the tank at any given time. The left hand side of the differential equation will therefore be $S'(t)$.

We can also see that the two terms on the right hand side need to be rates of change of the amount of salt, in units of lbs per min. The units used for time are suggested by the rate given in the statement of the problem. The total rate in is given by the flow rate in (2 gal/min) times the concentration (0.1 lbs/gal). The total rate out is given by the flow rate out (also 2 gal/min) times the concentration *in the tank*, which is $S/100$. Note that if the flow rate in and the flow rate out aren't equal then the volume of the tank will not be constant!

We now have the initial value problem

$$\begin{aligned} S' &= 0.2 - \frac{S}{50} \\ S(0) &= 5. \end{aligned}$$

Note that the 5 lbs. of salt that are initially in the tank becomes the initial condition. A common error in setting up these types of problems is to try to force this quantity into the differential equations itself. This equation can be solved using the same substitution technique illustrated in the Newton's law of heating example.

### 1.2.6 Non-dimensional Models

Typically, when we construct a differential equation to model some real-world phenomenon, the variables and parameters have units. For example, in a population model, time might have units of days, months, or years depending on the context, and the population variable itself might carry units such as "1000s of individuals" or even "kilograms" of biomass. It is often useful, however, to rewrite a differential equation in terms of quantities that are **unitless** (or **non-dimensional**).

There are several reasons for doing this. First of all, if the magnitudes of the quantities are very large or very small, it can be awkward to perform calculations or to construct graphs. In these cases, a dimensionless form of the model can be found that makes it easier to work with and compare values. Secondly, the non-dimensional form of a differential equation typically reduces the number of parameters, making the equations simpler to work with. Finally, the non-dimensional form of a model can often help clarify the role that the parameters play in determining the behavior of the solutions.

To illustrate how to construct a non-dimensional model, we will take another look at the logistic population model described in Section 1.2.3. The initial value problem as originally formulated is

$$\begin{aligned} p' &= rp\left(1 - \frac{p}{K}\right) \qquad (1.19) \\ p(0) &= p_0. \end{aligned}$$

The independent variable $t$ has units of time, and the dependent variable $p$ is measured in number of individuals. In addition, there are three parameters: the growth rate $r$ with units of inverse time and the carrying capacity $K$ and initial population $p_0$, both with the same units as $p$.

The first step is to introduce new **non-dimensional** variables that do not have units. We accomplish this by dividing the original variables by quantities that have the same units. For instance, if we define $\tau = rt$ then because $r$ has units of inverse time, the new variable $\tau$ is a unitless measure of time. Similarly, if we define $y = p/K$, then because $p$ and $K$ both are measured in terms of number of animals, the new variable $y$ is unitless measure of population. Note that often there will be more than one way to define new unitless variables. For instance, we could have used $p/p_0$ instead.

The next step is to substitute these new variables into the original differential equation. For example, wherever $p$ appears in the original, we can replace it with the quantity $Ky$. The time variable $t$ does not appear explicitly, but the derivative is taken with respect to $t$ so we need to rewrite it so that we're differentiating with respect to $\tau$ instead. This is done with the chain rule:

$$\frac{dp}{dt} = \frac{dp}{d\tau}\frac{d\tau}{dt} = \frac{d(yK)}{d\tau}\frac{d(rt)}{dt} = Kr\frac{dy}{d\tau}.$$

Then substituting this into the left hand side of (1.19), and using the substitution $p = Ky$ on the right, we get

$$Kry' = r(yK)\left(1 - \frac{yK}{K}\right) = rKy(1-y).$$

or dividing by $Kr$,

$$y' = y(1-y).$$

Comparing this differential equation to the original, we see that it's much simpler. In particular, there are no parameters! If we rewrite the initial condition, we see that $y(0) = p(0)/K = p_0/K$. If we define $\alpha = p_0/K$, we have the final non-dimensional form of the model:

$$y' = y(1-y) \tag{1.20}$$

$$y(0) = \alpha \tag{1.21}$$

In addition to being simpler to work with, this form of the differential equation shows us that the behavior of the solutions depends upon only a single parameter, $\alpha$. In terms of the original parameters, what matters is the ratio $p_0/K$.

## 1.3 Qualitative Analysis

There are many different ways to analyze a differential equation. One approach is to try and find an explicit solution and many introductory textbooks on ordinary differential equations are devoted to developing techniques to do this. This text will explore some standard techniques, however we also want to acknowledge that most differential equations do not have solutions that we can express in terms of elementary functions. In these cases, we still want to be able to say something about the *behavior* of the solutions. For instance, are they bounded or do they diverge to infinity as $t \to \infty$?

Do they oscillate or do they converge to a steady state? Even in cases where we do have an explicit solution, these questions aren't always straightforward to answer.

Fortunately, the differential equation itself tells us quite a lot about the behavior of the solution even when we cannot find an explicit form of the solution. This kind of investigation into the behavior of the solutions can be described as **qualitative analysis**, meaning that we are trying to say something about the behavior of the system without actually solving the ODE.

## Direction Fields

One tool that's extremely helpful in investigating the behavior of solutions to differential equations without knowing them explicitly is the **direction field**, also known as the **slope field**. In calculus, we learned that the derivative can be interpreted as the slope of the tangent line at a point on the graph of a function. This means that given a differential equation

$$u'(t) = f(t, u)$$

we may pick any point in the $t$-$u$ plane and plug the coordinates $t$ and $u$ into the right hand side to obtain the slope of the tangent line to the solution curve at that point. We then can sketch the tangent line (or a small segment of the tangent line), which then gives us information about the shape of the solution curve.

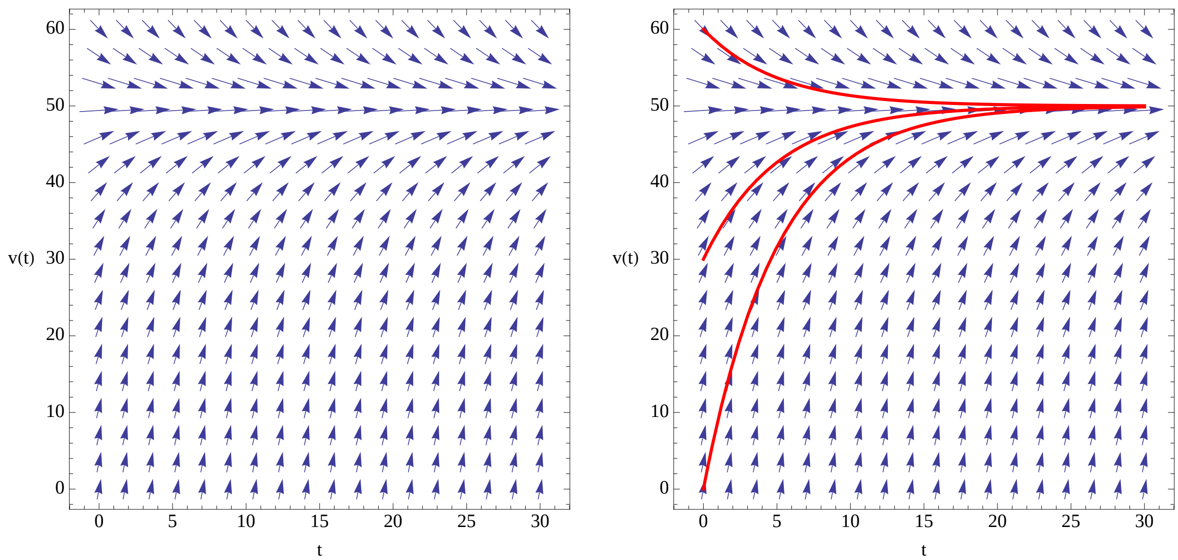
If we take a large collection of points in the $t$-$u$ plane and compute $f(t, u)$ at all of them, then we know the slopes of the tangent lines at all of those points and if we draw a small segment of the tangent line, we can visualize the solution curves that fit those tangent lines. This graph is what we call the **slope field**.

For example, consider the differential equation

$$\frac{dv}{dt} = 10 - \frac{v}{5}.$$

In this example, the right-hand side of the DE does not explicitly depend on $t$. This means that the slopes of the tangent lines (i.e. the derivative $v'$) only depend on the value of $v$. Figure 1.3 on the facing page shows the slope field, with a few solution curves superimposed. Here, the segments of the tangent lines are drawn as arrows to help us visualize the solutions: we simply follow the arrows.

Obviously the computer is very useful for generating slope fields. One particularly useful tool is a powerful and easy to use java app called DFIELD. You can download this tool from `http://math.rice.edu/~dfield/dfpp.html`.




**(a)** Slope field without solution curves.

**(b)** Slope field with solution curves.

**Figure 1.3:** Slope fields for $v' = 10 - v/5$.

### 1.3.1 Equilibria and Stability

Although slope fields can provide much useful information about the behavior of solutions, they cannot give a complete picture of a differential equation. For one thing, we are limited to looking at one window at a time determined by the intervals on the axes on which we have chosen to compute derivatives. Outside this window, we can only guess at the behavior of the solutions. In particular, the slope fields don't tell us about the behavior of the solutions as the independent variable goes to infinity. Another difficulty is that if our differential equation has any parameters we must choose numerical values for them in order to compute numerical values for the slopes of the tangent lines and we lose insight into how the behavior of solutions might depend on particular parameter values.

In this section, we develop some ideas and tools for further analyzing the behavior of our differential equations. Consider the previous example

$$\frac{dv}{dt} = 10 - \frac{v}{5} \tag{1.22}$$

where $v$ represents the velocity of an object falling through the atmosphere. The slope field shown in Figure 1.3b certainly suggests that over time all solutions approach a limiting value of $v = 50$. However, we would like to make a more rigorous argument that does not rely solely on graphical evidence.

Notice what the differential equation tells us if the object happens to be falling with a velocity precisely equal to 50. In this case, the right-hand side of equation (1.22) becomes zero, which means that the *left* side of the equation is also zero, i.e. that $v' = 0$; in other words the velocity is not changing. This tells us that the constant function $v(t) = 50$ is a solution. This special solution is called an **equilibrium solution**. We will sometimes use the notation $v^*$ to indicate an equilibrium solution.

Now suppose that at some moment this object is falling at a speed greater than 50. In this case, the right-hand side of (1.22) would be negative, implying that $v'$ is negative. So even though we don't know what the function $v(t)$ is, we do know that it's a *decreasing* function; in other words, the object is slowing down. Similarly, if at any time our object happens to be falling with a speed that's *less* than 50, then the right-hand side of the equation is positive and therefore $v'$ is positive. This makes $v(t)$ an *increasing* function; our object is speeding up. This suggests that all non-equilibrium solutions are approaching the equilibrium solution $v^* = 50$ as $t \to \infty$. Consequently, we call this equilibrium an **attractor** or **stable** equilibrium. More precisely, since the solutions converge to $v^*$ as $t \to \infty$ we will describe this as an **asymptotically stable** equilibrium.[5]

It's worth pointing out that in this particular example, we can find a physical interpretation of our analysis. If we set the right-hand side of (1.22) equal to zero, we're saying that the gravitational acceleration is exactly balanced by air resistance. If you happen to have direct experience with objects falling through the atmosphere, you might recognize this equilibrium solution as *terminal velocity*. If an object is traveling faster than terminal velocity, then we expect air resistance to slow it down. On the other hand, if the object is traveling slower, then we expect gravitational acceleration would speed it up. In this respect, the differential equation agrees with what we already know about how the world behaves.

The concepts of equilibria and stability arise in the context of **autonomous** equations like (1.22). Recall from Section 1.1 that a differential equation is autonomous if it has the form

$$\frac{dy}{dt} = f(y). \tag{1.23}$$

Equations of this form are of particular interest because then the slope-field does not depend upon time which means that we can use them to visualize the solutions (otherwise it would provide only a snapshot in time of the solution behavior).

Another consequence of autonomous equations is that if we can find values of $y$ that make the right-hand side zero (i.e. for which $f(y) = 0$) then the derivative $y'$ is zero *for all time* meaning that we can find the equilibrium solutions by finding the roots of the equation $f(y) = 0$. We will therefore limit our discussion of equilibrium solutions to autonomous equations. We will also make an assumption that equilibria are always **isolated**, meaning that if $y^*$ is an equilibrium then there is always some interval containing $y^*$ which contains no other equilibria.

After finding the equilibrium solutions of an equation, the next step is to investigate the behavior of other solutions. In our free-fall example, we saw that other solutions were either increasing or decreasing, depending on whether they started out below or above the equilibrium value. To

[5] We will clarify the distinction between stable and *asymptotically* stable equilibria in Chapter 3.

determine this behavior, we needed only to determine how the sign of $f(y)$ depended on the value of $y$. In practice, a simple way to visualize the sign of $f(y)$ is to look at a graph of $f(y)$.

For example, suppose that the graph of $f(y)$ shown in Figure 1.4. In addition to seeing that there are three equilibria (at the points $a$, $b$, and $c$) we also see where $f$ is positive and where its negative. If $y < a$ or $b < y < c$, for instance, then $f(y)$ is positive and so by (1.23), the solution $y(t)$ is increasing. On the other hand, if $a < y < b$ or $c < y$ then $f(y)$ is negative and therefore $y(t)$ is decreasing.

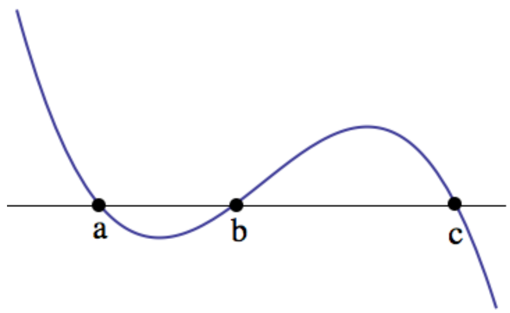



**Figure 1.4:** Given a differential equation of the form $y' = f(y)$, we can look at the graph of $f(y)$ versus $y$ to help us determine the behavior of the system. The roots of $f(y)$, marked $a$, $b$, and $c$ are equilibria solutions. Whether the solution $y(t)$ is increasing or decreasing is determined by the sign of $f$.

### Phase Lines

Once we have found all of the equilibria and determined the behavior of the non-equilibrium solutions, either by looking at the graph of $f(y)$ or otherwise, it's often helpful to summarize this information with the **phase line**. To draw the phase line, we simply draw a horizontal axis representing the values of the solution (i.e. $y$ values) and mark all of the equilibrium solutions as points on the axis. Then indicate whether non-equilibrium solutions are increasing or decreasing by drawing an arrow to the right or left respectively. For example, the phase line for the example above is shown in Figure 1.5.

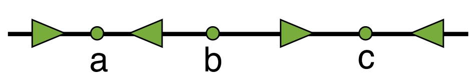



**Figure 1.5:** Phase line corresponding to Figure 1.4. For $y < a$ or $b < y < c$, $f(y) > 0$ and therefore $y$ is increasing. This is indicated in the phase line with an arrow to the right. Similarly, for $a < y < b$ or $c < y$, $f(y) < 0$ and therefore $y$ is decreasing, indicated with an arrow to the left.

**Example 1.3.1.** If $p(t)$ represents the population of some species as a function of time, one possible model for the change in population is the initial value problem

$$p'(t) = rp(1 - p/K) \tag{1.24}$$

$$p(0) = p_0 \tag{1.25}$$

where $r$ and $K$ are positive parameters.

We first look for the equilibrium solutions. Setting the right-hand side of (1.24) equal to zero, we see immediately that there are two equilibria solutions, $p^* = 0$ and $p^* = K$. The first equilibrium makes perfect sense for a population: if $p(t)$ ever equals zero, we would not expect it to change (the population is extinct). To investigate the second equilibrium, we look at the graph of $f(p)$. As we see in Figure 1.6, if $0 < p < K$ then $f(p)$ is positive and therefore $p' > 0$, which means $p$ is increasing. Similarly, if $K < p$ then $f(p)$ is negative and therefore $p'$ is negative meaning that $p$ is decreasing.

This information is summarized by the phase line shown in Figure 1.7. Any solutions that start near the equilibrium $p^* = 0$ will be increasing, i.e. moving *away* from the equilibrium. In this case, we say that the equilibrium is unstable. On the other hand, solutions that start near the equilibrium $p^* = K$ will move towards it, making $p^* = K$ a stable equilibrium. This equilibrium represents the **carrying capacity** of the environment: below this level, the population will continue to increase, but above it and the population is no longer sustainable.

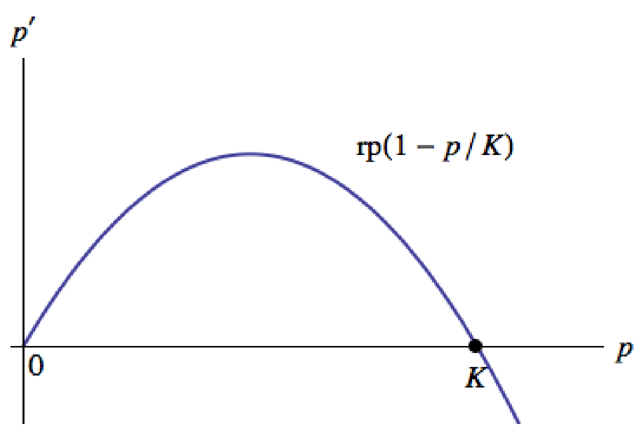



**Figure 1.6:** Graph of $f(p) = rp(1 - p/K)$. Since $f(p) = 0$ at $p^* = 0$ and $p^* = K$, these are the equilibrium solutions. If $0 < p < K$ then $f(p) > 0$ and therefore $p' > 0$. If $K < p$ then $f(p) < 0$ and therefore $p' < 0$.

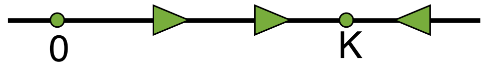



**Figure 1.7:** Phase line for the logistic equation $p' = rp(1 - p/K)$. The equilibrium $p^* = 0$ is unstable and the equilibrium $p^* = K$ is stable.

### Linear Stability Theorem

For some problems, it may be difficult to determine what the graph of $f(y)$ looks like. For example, if the right hand side includes parameters, then it may not be clear how the possible values of the parameters affect sign of $f(y)$. In cases like this, we would like a way to determine stability without having to graph $f(y)$. The key is to understand exactly what the graph has to do with stability. As in the example above, an equilibrium at $y = a$ is stable if the graph goes from positive to negative values. In other words $f(y)$ was *decreasing* at $y = a$ and therefore $f'(a) < 0$. Similarly, if $f(y)$ is *increasing* at the equilibrium, this would indicate instability. Thus if we can determine the value of the derivative $f'$ at an equilibrium point we can determine the stability without graphing.

**Theorem 1.3.1.** ***[Linear Stability]*** *Consider the initial value problem*

$$\frac{dy}{dt} = f(y). \tag{1.26}$$

*Suppose that there is an isolated*[6] *equilibrium at $y = y^*$. If $f'(y^*) < 0$ then $y^*$ is stable and if $f'(y^*) > 0$ then $y^*$ is unstable. If $f'(y^*) = 0$, then the equilibrium may be stable or unstable (the linear stability test is inconclusive).*

**Example 1.3.2.** In the previous example, we saw that for the logistic population model

$$\begin{aligned} p'(t) &= rp(1 - p/K) \\ p(0) &= p_0 \end{aligned}$$

where $r$ and $K$ are positive parameters that there are equilibria at $p^* = 0$ and $p^* = K$. To determine the stability using the linear stability theorem, we look at the value of the derivative $f'(p) = r - 2rp/K$ at the equilibria. For $p^* = 0$, we see $f'(0) = r$ which is positive, making this equilibrium unstable. On the other hand, for $p^* = K$, we have $f'(K) = r - 2rK/K = -r$ which is negative, indicating that the equilibrium is stable.

[6]Recall that the term *isolated* means that we can find an interval around $y^*$ which contains no other equilibrium.

**Example 1.3.3.** Consider the differential equation

$$x' = rx + ax^2 - x^3$$

where $a$ and $r$ are parameters. By inspection, we see that there is an equilibrium solution $x^* = 0$. To investigate the stability of this equilibrium, we could try to graph the function $f(x) = rx + ax^2 - x^3$. Although we know that $f(x)$ is a cubic, it's difficult to determine how the two parameters affect the graph. Instead, we use the linear stability theorem. Since $f'(x) = r + 2ax - 3x^2$, we see that $f'(0) = r$. Therefore the equilibrium $x^* = 0$ is stable if $r < 0$ and unstable if $r > 0$. If $r = 0$, the theorem does not apply, but in this case we have $f(x) = ax^2 - x^3$ which is easier to graph by cases (since there's now only one parameter). If either $a < 0$ or $a > 0$ then we have double root at $x = 0$ which makes the origin semi-stable. If $a = 0$ and $r = 0$, then $f(x) = -x^3$ and the origin is stable.

### 1.3.2 Bifurcations

One important aspect to our analysis is determining how equilibria and stability depend upon any parameters present in the differential equation. For particular values of a parameter, we can always look at the graph of $f(y)$ as described above, or sketch the slope field. However, in many applications, we may not want to fix the values of the parameters before hand. In terms of equilibria and stability, we need to determine how the parameter values effect the graph of $f(y)$.

If we take a look at the logistic population model in Example 1.3.1 on page 28, we can ask how changing the value of the parameter $K$ will affect the stability analysis. As we can see from Figure 1.6 on page 28, a change in $K$ (assuming it is still positive) will not change the basic shape of the parabola; we will always have a parabola opening downwards with roots at $p = 0$ and $p = K$. Furthermore, the first equilibria ($p = 0$) will be unstable while the second ($p = K$) remains stable. In short, changing the value of $K$ does not affect the qualitative analysis. Similarly, changing the value of $r$ (assuming it remains positive) does not change the qualitative analysis.

But now consider the differential equation

$$p'(t) = p(1 - p) - h. \tag{1.27}$$

This is a logistic population model that includes a constant term $h > 0$ that represents the effects of harvesting on the population. In order to illustrate the effect of the parameter $h$, we have set the values of $r$ and $K$ equal to one. We want to examine how changes in this parameter value might affect the qualitative analysis.

We will approach this question in two ways. The first method is to simply try to find the equilibria solutions analytically by solving the equation $f(p) = 0$. In this example, we have the equation

$$p(1-p) - h = 0$$

which we can rewrite as

$$p^2 - p + h = 0.$$

Then by the quadratic formula, the equilibrium solutions are given by

$$p = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4h}. \tag{1.28}$$

Because the equilibria are given by solving a quadratic, we know that the number of solutions depends on the discriminant $1 - 4h$; if this value is positive, we have two equilibria and if it's negative there are no equilibria. When it is precisely zero, we have one equilibrium. The change in behavior occurs when the discriminant is zero, so solving $1 - 4h = 0$ gives

$$h = \frac{1}{4}.$$

This tells us that as the parameter $h$ increases from 0 to values above $1/4$, the number of equilibria changes from two to zero.

This kind of significant change in the qualitative behavior is known as a **bifurcation**. Specifically, any time that a change in a parameter value results in a change in either the *number* of equilibria or in the *stability* of the equilibria, we say that a bifurcation has occurred. The value of the parameter at which the change occurs is known as a **bifurcation value**. In our example above, the bifurcation occurred at $h = 0.25$, so that is the bifurcation value. Depending on the example, there may be more than one bifurcation value.

For problems like this one in which it is possible to solve for the equilibrium solutions explicitly, finding the bifurcation values is usually straightforward. However, in many other examples it can be difficult or impossible to solve for the equilibrium values explicitly. We can still find the bifurcation values, however, if we analyze carefully how the parameter affects the graph of $f(y)$.

Let's examine how the parameter $h$ affects the graph of $f(p) = p(1-p) - h$. We already have seen in Figure 1.6 on page 28 the graph in the case $h = 0$. For non-zero values of $h$, however, the graph is simply shifted vertically. Figure 1.8 on the following page shows several graphs of $f(p)$ for different values of $h$. The top-most graph corresponds to $h = 0$ and we see there are two equilibria: one at $p = 0$ (unstable) and one at $p = 1$ (stable). Now as $h$ increases to 0.125, the graph shifts down slightly. There are still two equilibria, one stable and one unstable, so the qualitative picture has not changed.

However, when we look at $h = 0.25$, we see that now there is only a single root and because the graph lies under the $p$ axis for all other values of $p$, this equilibria is semi-stable. As we increase $h$

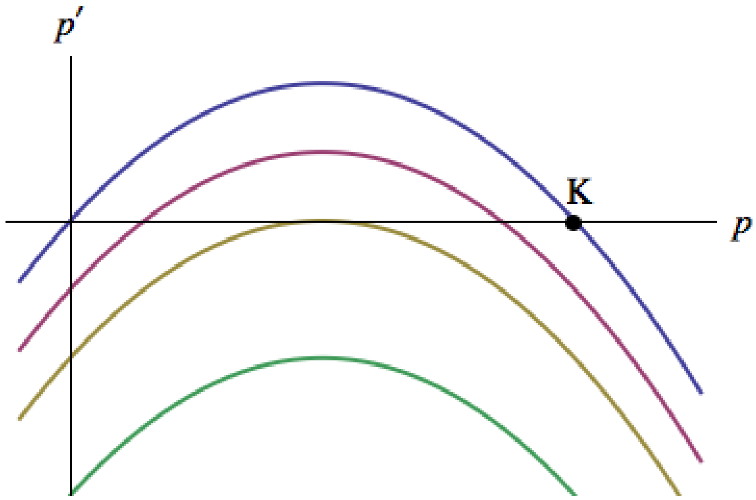



**Figure 1.8:** Graph of $f(p)$ for values of the parameter $h$ equal to $0, 0.125, 0.25$ and $0.5$ (from top to bottom).

still further, the graph shifts further down, and there are no longer any equilibrium solutions. We have found the bifurcation value.

In this example, it was easy to see how changes in the parameter value effected the graph. In other problems, however, it may not be so clear. To understand our graphical analysis a bit more, notice that the bifurcation occurred precisely when the graph of the parabola was tangent to the horizontal axes. For values of $h$ just below the bifurcation value, the graph shifts up slightly and we have two equilibria; for values just above the bifurcation value, the graph shifts down slightly and we have no equilibria. This suggests that we can find bifurcations by looking for values of the parameter that produce graphs in which a local minimum (or maximum) of $f(y)$ lies on the horizontal axis. To do this we:

1. Find the local minima and maxima in the usual way: by setting $f'(y) = 0$ and solving for $y$.
2. Enforce the condition that the minima or maxima found in step 1 lies on the horizontal axis by plugging the $y$ value found in step 1 into $f(y) = 0$. Solving for $h$ then gives the bifurcation values.

As an example, consider the one-parameter family

$$y' = f(y) = y(1-y)^2 + h. \tag{1.29}$$

In this equation, $f(y)$ is a cubic polynomial and therefore there is no simple way of solving for $y$ explicitly.[7] To find the bifurcations, we analyze the graph of $f(y)$ to see how it is affected by

[7]Although there actually is a formula that gives the roots of a cubic equation, it is not simple and except in rare cases, does not shed much light on the situation.

changes in the parameter. Figure 1.9 shows the graph of $f(y)$ for $h$ slightly greater than zero, for $h$ equal to zero, and $h$ slightly less than zero. We can see from the three graphs that a bifurcation occurs at $h = 0$ because at that value of the parameter, the graph of $f(y)$ has a local minimum that lies on the $y$-axis.

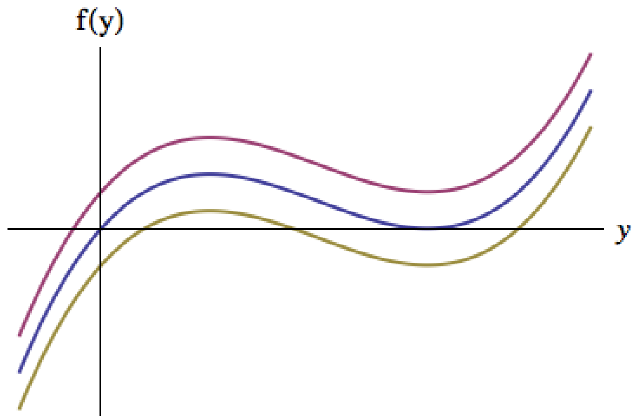



**Figure 1.9:** Graph of $f_h(y)$ for values of the parameter $h$ equal to 0 as well as values slightly above and slightly below this value.

Another bifurcation will occur as $h$ decreases further. To find the value of $h$ that produces this bifurcation, we follow the steps outlined above. First, we locate the local maximum by finding the critical points. Solving

$$f'(y) = 1 - 4y + 3y^2 = 0 \tag{1.30}$$

we get $y^* = 1$ and $y^* = 1/3$. From Figure 1.9 we see that the maximum is located at $y^* = 1/3$. Next, we find the value of the parameter such that maximum value is zero (i.e. so that the maximum sits on the $y$-axis). Plugging in,

$$y^*(1 - y^*)^2 + h = (1/3)(2/3)^2 + h = 0 \tag{1.31}$$

and solving gives the bifurcation value $h = -4/27$. Figure 1.10 shows the graph of $f_h(y)$ for $h$ slightly less than, equal to, and slightly greater than this value.

To summarize: we can use two method to find bifurcations. First of all, we can try to find the equilibrium explicitly by solving the equation $f(y) = 0$. Then try to determine how changing the values of the parameter might change either the number or stability of the equilibria. A second method is to analyze how the parameters affect the graph of $f(y)$.

If an equation has more than one parameter, then finding and analyzing bifurcations can be tricky. We will therefore focus primarily on equations that have a single parameter $h$, or in cases where there are more than one parameter (such as the logistic model in dimensional form), we will only vary one parameter at a time. We can indicate when differential equation depends upon a particular parameter with notation such as

$$y' = f_h(y). \tag{1.32}$$

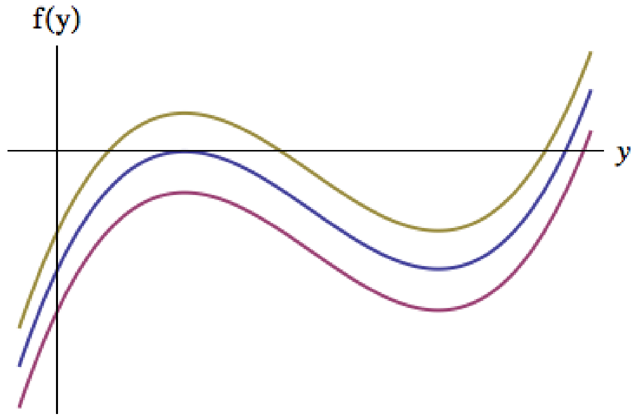



**Figure 1.10:** Graph of $f_h(y)$ for values of the parameter $h$ equal to $-4/27$ as well as values slightly above and slightly below this value.

Here the subscript on $f$ emphasizes that the right hand side depends upon a parameter $h$. Since we want to consider how the behavior of the solutions change as the parameter varies, we can consider this a **one-parameter family** of differential equations.

### Bifurcation Diagram

A helpful way of visualizing the effects of changing a parameter value is through the **bifurcation diagram**. Given the one-dimensional family of differential equations

$$y' = f_h(y)$$

the bifurcation diagram is a graph in the $h$-$y$ plane produced by plotting the equilibria values $y^*$ for each value of $h$. In other words, each vertical cross-section corresponds to a phase line produced by a fixed value of the parameter $h$. In the example of the logistic equation with harvesting, we can easily draw the bifurcation diagram by solving for the equilibria explicitly (see equation (1.28)) and graphing the two curves $p(h)$. In addition to showing the values of the equilibria as functions of the parameter, the bifurcation diagram also indicates the stability of each equilibria. In this example, the larger of the two equilibria is stable while the smaller is unstable. We indicate the unstable equilibria with a dotted curve.

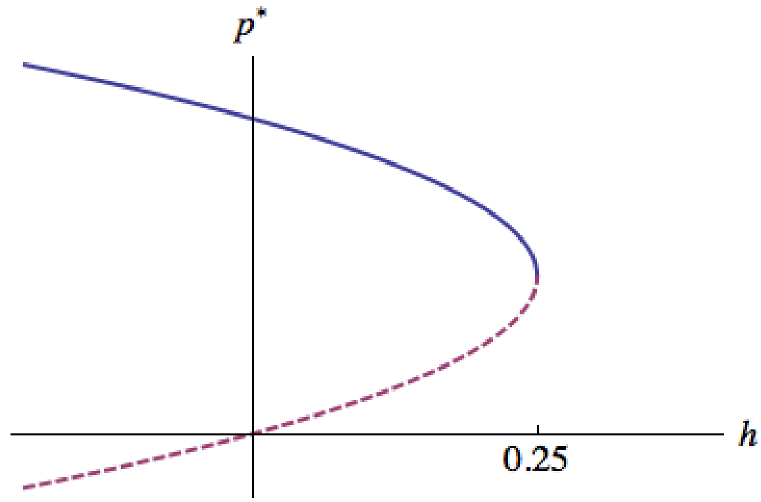



**Figure 1.11:** Bifurcation diagram for $p' = p(1-p) - h$. Reading the graph from left to right, we see that the number of equilibria for a given value of the parameter $h$ goes from 2 (if $h < 0.25$) to 0 (if $h > 0.25$). Thus there is a bifurcation at $h = 0.25$. The bifurcation diagram also indicates with a dotted curve which equilibrium is unstable.

## 1.4 Existence and Uniqueness

In this text, we focus on developing techniques for finding and analyzing solutions to differential equations. However, there are a few issues of a theoretical nature that need to be addressed. First, there is the question of *existence* of solutions: given a differential equation, how do we know that there are any solutions at all? It would be wise to address this question before we devote too much time to trying to find solutions. There is also a question of *uniqueness* of solutions. If we are able to find a solution, how do we know there aren't other solutions that start from the same initial condition but behave differently as time increases?

It's tempting to dismiss these as theoretical matters without any practical importance. However, questions of existence and uniqueness are often central to the question of how well our equations are doing in describing the world, in other words how effective is our model? If the domain of a solution is limited to a finite interval, for example, that can often tell us something about the validity of the assumptions we used to construct our model. And if a solution is not unique, how can we use our model to make predictions?

**Example 1.4.1. The leaky bucket. (Hubbard and West 1991, p. 159).** Consider a bucket with a hole in the bottom. At some initial time, suppose the bucket is full of water. To describe the rate at which the water level drops, we can use conservation of energy to derive the equation

$$\frac{dh}{dt} = -C\sqrt{h} \tag{1.33}$$

where $C$ is a constant that depends on physical parameters such as gravitational acceleration and the size and shape of the hole and the bucket. The solution to this equation is

$$h(t) = \begin{cases} \frac{C^2}{4}(t - t_e)^2 & \text{for } 0 \leq t \leq t_e \\ 0 & \text{for } t > t_e \end{cases} \tag{1.34}$$

where $t_e$ is the time it takes for the bucket to become empty (i.e. $h(t_e) = 0$).

Now, given any initial water level $h_0$, this solution tells us the value of $h$ at any later time. However, these solutions are not unique because for any initial value of $h$, the solution will eventually becomes zero, i.e. the bucket will be empty. Thinking about it another way, if at some time $t$, we observe that the bucket is empty, can we determine how long ago the bucket was full? No, of course not! The non-uniqueness of the differential equation means that we cannot use the model to say anything about the state of the bucket when we look too far backwards in time.

Fortunately, we can state a general theorem for first-order equations that tells us something about existence and uniqueness of solutions. We will state this result without proof and refer to it through out the text as needed.

**Theorem 1.4.1.** ***[Existence and Uniqueness]*** *Consider the initial value problem*

$$\frac{dy}{dt} = f(t, y)$$
$$y(t_0) = y_0.$$

*Suppose that the function $f(t, y)$ is continuous on the open rectangle $R = \{(t, y) : a < t < b, c < y < d\}$ in the ty-plane and that $(t_0, u_0)$ lies in the rectangle R. Then the initial value problem has a solution $y(t)$ defined on some interval $(\alpha, \beta)$ that contains $t_0$ (and where $(\alpha, \beta)$ is contained in $(a, b)$). Furthermore, if the partial derivative $\partial f/\partial y$ is also continuous on R then the solution to this initial value problem is unique.*[8]

This theorem is general in the sense it applies to any first-order equation. Although the proof is accessible to anyone with a solid background in calculus, it's a bit beyond the scope of this course. The interested reader is referred to the very nice outline in the text by Boyce & DiPrima. For the special case where the differential equation is *linear*, however, we can show existence by simply finding a formula for the solution. We will do this in the next chapter.

[8]Note that the condition that $\partial_y f$ be continuous is only needed for uniqueness. For existence, we only need the continuity of $f$. In some texts, this theorem is split into two separate theorems along these lines.

**Example 1.4.2.** For the leaky bucket model (1.33) we have $f(t,h) = -C\sqrt{h}$, which is continuous at all $h \geq 0$. The existence-uniqueness theorem then implies that a solution exists for any initial condition. However, $\partial f/\partial h = -C/(2\sqrt{h})$ which is not continuous at $h = 0$. This tells us that if we pick $h = 0$ as our initial condition, we cannot be guaranteed that solutions are unique.

### Interval of Existence

One feature of the existence and uniqueness theorem that is worth pointing out is that that the theorem doesn't say anything about how large or small the interval of existence $(\alpha, \beta)$ might be (other than it's contained in the rectangle $R$ described above).

For example, consider the initial value problem

$$y' = 1 + y^2$$
$$y(0) = 0.$$

Since $f(t,y) = 1 + y^2$ is continuous for all $(t,y)$, we are assured that there is a solution for any initial condition. Furthermore, since $\partial f/\partial y = 2y$ is continuous everywhere we know that any solution we find is unique.

In the next chapter we will learn how to find the solution, but for the time being, it is easy to verify that the function

$$y(t) = \tan(t)$$

is the solution that satisfies the given initial value. Notice that as $t$ increases, the value of this solution increase to infinity as $t$ approaches $\pi/2$. In the context of differential equations, we say that this solution exhibits a "blow-up" in finite time. Since the solution to an initial value problem extends forward (and backwards) from the initial value, we would say that the interval of existence in this example is $-\pi/2 < t < \pi/2$. Notice that the differential equation itself provides no clues that the interval of existence is finite.

Note that this is slightly different from what we would say if we were looking for the *domain* of the function $y(t) = \tan(t)$. In that case, we would say that since the tangent function is undefined if $t = \pi/2 + n\pi$ for any integer $n$, the domain of the function $\tan(t)$ is all real numbers except odd multiples of $\pi/2$. In contrast, the interval of existence is limited to the largest interval containing the initial time for which the solution exists: $(-\pi/2, \pi/2)$.

### Applications of Uniqueness

One useful application of the Existence and Uniqueness Theorem is in comparing the behaviors of two more more solutions to a given differential equation. Suppose that $y_1(t)$ and $y_2(t)$ are both solutions to

$$\frac{dy}{dt} = f(t, y).$$

and suppose that these solutions intersect at some point in time (say at $t = t_0$). In other words, suppose that $y_1(t_0) = y_2(t_0)$. Then both solutions satisfy the same initial value problem and therefore (assuming $f$ is nice) they must be the same function! This implies that solutions cannot intersect (provided that the conditions of the Existence and Uniqueness Theorem continue to hold). This is very useful, particularly when we are using slope fields or other graphical methods to visualize solutions.

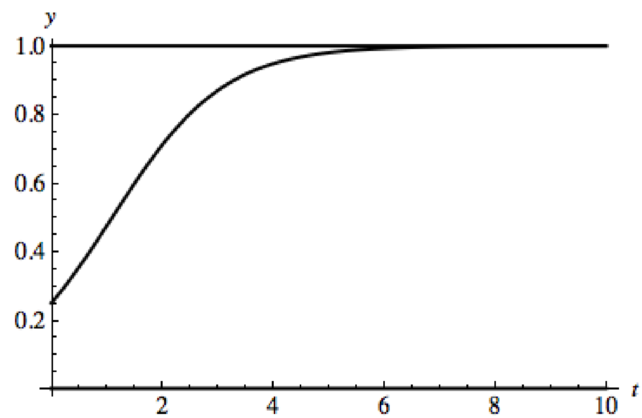



**Figure 1.12:** Equilibrium solutions $y^* = 0$ and $y^* = 1$ and solution satisfying $y(0) = 0.25$.

As an example, consider the logistic population model

$$y' = y(1 - y).$$

We know that this equation has two equilibrium solutions $y_1(t) = 0$ and $y_2(t) = 1$. We also know that for $0 < y_0 < 1$, the ODE implies that $y' > 0$ and therefore solution are increasing. If we use a slope field or otherwise approximate a solution with initial condition $y(0) = 0.25$ we see a solution increasing towards the equilibrium at $y^* = 1$. By uniqueness, we know that these solutions never intersect and therefore the increasing solution must in fact approach the equilibrium solution in the limit as $t$ goes to infinity.

## 1.5 Exercises

**1.1:** Consider the equation $\cos(x) = x$. Does this equation have any solutions? How many? How do you know?

**1.2:** Consider the differential equation $y'' + y' - 2y = 0$. Is the function $y(t) = e^t$ a solution? Is $y(t) = \sin(t)$ a solution?

**1.3:** Write down the general solutions to the following differential equations:

a. $w' = 4w$.

b. $2g' + g = 0$

c. $\frac{dp}{dt} = 0.1p$

**1.4:** By playing with exponential functions, find at least two solutions to each of the following equations. You may need to modify your initial guess to find a solution.

a. $y'' - y' - 2y = 0$

b. $y' - 2ty = 0$

c. $y'' - 4y' + 4y = 0$

**1.5:** Determine whether the function $y(t) = 1 + 2t$ is a solution to the differential equation

$$\frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t}.$$

**1.6:** For what value(s) of the constant $n$ is the function $u(t) = t^n$ a solution to the differential equation $2tu' = u$?

**1.7:** Consider the differential equation $u' = p(t)u + q(t)$. Is it true that the sum of two solutions is also a solution? Is a constant times a solution also a solution? Answer these questions again in the case where $q(t) = 0$.

**1.8:** Verify that the function $y(t) = -1/(t + c)$ is a solution to the ODE $y' = y^2$ for any value of $c$. Find a solution to this equation that satisfies the initial condition $y(0) = 1$. Can you find a solution that satisfies the initial condition $y(0) = 0$?

**1.9:** Verify that the function $y(t) = \tan(t)$ is a solution to the initial value problem

$$\begin{aligned} y' &= 1 + y^2 \\ y(0) &= 0. \end{aligned}$$

**1.10:** Verify that the initial value problem $u' = 1 - u^2$ has a solution $u(t) = \dfrac{e^{2t} - 1}{e^{2t} + 1}$ that satisfies the initial condition $u(0) = 0$.

**1.11:** Show that $u(t) = \ln(t + C)$ is a solution of the differential equation $u' = e^{-u}$ for any value of the constant $C$. Plot this function for a few representative values of $C$; include the particular solution that satisfies the initial condition $u(0) = 0$.

**1.12:** In the case where $T_e$ is constant, we can solve equation (1.18) by use a clever substitution. Define a new variable $y = T_e - T$ and substitute into the differential equation to get the differential equation

$$y' = -ky$$

Write down the solution to this and then rewrite in terms of the original variable $T$.

**1.13:** Verify that the function

$$y(t) = 1 - \frac{t^3}{3!} + \frac{t^6}{6!} - \frac{t^9}{9!} + \dots$$

is a solution to the differential equation $y''' + y = 0$. Here are a few questions that we could ask about this function, how many can you answer? How many roots does it have? What is the limit as $t \to \infty$? What does the graph look like? Is it periodic? Is it bounded? This problem is meant to illustrate that having an explicit formula for the solution of a differential equation is not always helpful.

**1.14:** Show that if $u_1$ and $u_2$ are both solutions to the differential equation $u' + p(t)u = 0$, then the ratio $u_1/u_2$ is a constant (assuming $u_2$ is nonzero).

**1.15:** Find the family of solutions for the differential equation

$$\frac{dy}{dx} = 2\sin(x)\cos(x).$$

**1.16:** Show that for any value of the constant $c$, the function $u(t) = (t + c)e^{-t}$ is a solution to the differential equation

$$u' = -u + e^{-t}.$$

**1.17:** Write down two differential equations for which the function $y(t) = 1/t$ is a solution. This first should be a calculus problem (i.e. the function $y$ should not appear on right hand side). For the second equation, rewrite the right hand side in terms of the function $y$.

**1.18:** By thinking about functions you know, find a solution to the differential equation

$$\frac{d^2y}{dt^2} = -y.$$

**1.19:** Determine the values of $r$ such that the following differential equations has a solution of the form $y = e^{rt}$.

(a) $y' + 2y = 0$

(b) $y'' - y = 0$

**1.20:** Show that if $u_1$ is a solution to the differential equation $u' = p(t)u$ and $u_2$ is a solution to $u' = p(t)u + q(t)$ then the sum $u = u_1 + u_2$ is a solution to the equation $u' = p(t)u + q(t)$.

**1.21:** Suppose that the velocity of a falling object satisfies the differential equation $v' = 10 - \frac{v^2}{250}$.

(a) If the initial velocity is $v = 0$, what is the rate of change $v'$ at that moment? Is the velocity increasing or decreasing?

(b) If the initial velocity is $v = 100$, what is the rate of change $v'$ at that moment? Is the velocity increasing or decreasing?

(c) At what speed must the object be falling so that $v' = 0$?

(d) Make a sketch of $v$ versus $t$ that shows how velocity changes over time for several different values of the initial velocity.

(e) Suppose that initially the object is at rest (i.e. $v = 0$). Estimate the velocity one second later (assume that $t$ has units of seconds). Estimate the velocity after five seconds have elapsed (i.e. $t = 5$).

**1.22:** Suppose that the velocity of a falling object satisfies the differential equation $v' = 10 - v/5$, with an initial velocity $v(0) = 0$. Use a direction field and use this to estimate the velocity of this object at $t = 5$. How long will it take until the object's velocity is 95% of the equilibrium value?

**1.23:** The following table shows the land area in Australia colonized by the American marine toad every five years from 1939-1974.

| Year | Area occupied (km$^2$) |
|---|---:|
| 1939 | 32,800 |
| 1944 | 55,800 |
| 1949 | 73,600 |
| 1954 | 138,000 |
| 1959 | 202,000 |
| 1964 | 257,000 |
| 1969 | 301,000 |
| 1974 | 584, 000 |

Model the migration of this toad using an exponential growth equation

$$\frac{dA}{dt} = kA$$

where $A(t)$ is the land area occupied at time $t$ to make predictions about the land area occupied in 2010, 2050, and 2100 by doing the following:

(a) solve the initial-value problem

(b) determine a reasonable value for the constant $k$ (there are many ways to do this)

(c) use your solution and value of $k$ to compute the predicted areas at the corresponding times

(d) compare your solutions to the actual data.

(This problem is taken from the article "Teaching Differential Equations With a Dynamical Systems Viewpoint", Paul Blanchard, *College Mathematics Journal*, Volume 25, no. 5, November, 1994, pp. 385-395.)

**1.24:** For the following equations, find the equilibria and sketch the phase line. Determine the stability of all equilibria.

(a) $u' = u^2(3 - u)$

(b) $u' = 2u(1 - u) - u/2$

(c) $u' = (4 - u)(2 - u)^3$.

**1.25:** At low population densities it may be difficult for an animal to reproduce because of limited opportunities to find mates. A population model for this scenario is the Allee model

$$p' = rp\left(\frac{p}{a} - 1\right)\left(1 - \frac{p}{K}\right) \qquad 0 < a < K$$

and $r$ is positive. Graph $p'$ vs. $p$, determine the equilibrium populations, and draw the phase line. Determine the stability of each equilibrium. Sketch a few generic time series plots for different initial conditions.

**1.26:** For the following equations that contain a parameter $h$, find the equilibria in terms of $h$ and determine their stability. Construct a bifurcation diagram showing how the equilibria and/or stability depend on $h$.

(a) $u' = hu - u^2$

(b) $u' = (1 - u)(u^2 - h)$

**1.27:** Find the bifurcation values $\alpha$ for the differential equation

$$y' = y^6 - 2y^3 + \alpha.$$

Sketch the bifurcation diagram.

**1.28:** Find the bifurcation values for $y' = r - y/2 - e^{-y}$. Sketch the bifurcation diagram.

**1.29:** Find the bifurcation values for $y' = 1 + ry - y^3$. Sketch the bifurcation diagram.

**1.30:** Consider the equation $u' = (\lambda - b)u - au^3$ where $a$ and $b$ are fixed positive constants and $\lambda$ is a parameter that is allowed to vary.

(a) Show that if $\lambda < b$ then there is a single equilibrium that is asymptotically stable.

(b) In the case where $\lambda > b$, find all equilibria and determine their stability.

(c) Sketch the bifurcation diagram.

**1.31:** Suppose that a deer population grows logistically while being harvested at a rate proportional to the size of the population. The equation modelling this growth is

$$p' = rp(1 - p/K) - \lambda p.$$

Explain the effects on the equilibrium deer population when $\lambda$ is slowly increased from a small value to a large value. Include a bifurcation diagram.

**1.32:** When a mass of 0.3 kg is placed on a spring hanging from the ceiling, it elongates the spring 15 cm. What is the stiffness $k$ of the spring?

**1.33:** A car of mass $m$ is moving at speed $V$ when it has to brake. The brakes apply a constant force $F$ until the car comes to rest. How long does it take the car to stop? How far does the car go before stopping? Now use $m = 1000$kg and $F = 6500$ N to compare the time and distance required to stop if $V = 30$ mph vs. 35 mph.

**1.34:** In describing the motion of a pendulum, let $\theta$ denote the angular displacement from the vertical (i.e. if the pendulum is hanging straight down then $\theta = 0$). If the amplitude of the oscillations of a pendulum is small, then $\sin\theta$ is nearly equal to $\theta$ (why?) and the pendulum is described by the differential equation

$$\theta'' + (g/l)\theta = 0$$

where $l$ is the length of the pendulum and $g$ is the usual gravitational acceleration constant.

(a) Show that this equation has a solution of the form $\theta = A\cos\omega t$ for some value of $\omega$ (in other words, find the value of $\theta$, in terms of $l$ and $g$, for which this is a solution). What is the period of the oscillations exhibited by this solution?

(b) Suppose a 650 lb wrecking ball is suspended on a 60 ft cable from the top of a crane. The ball, hanging vertically at rest against the building, is pulled back a small distance and then released. How long does it take to strike the building?

**1.35:** The Allee equation is a variation on the logistic population model that includes three positive parameters:

$$p' = rp\left(\frac{p}{a} - 1\right)\left(1 - \frac{p}{K}\right) \qquad 0 < a < K$$

What are the units of each quantity? Nondimensionalize this equation by introducing a unitless population $\widetilde{p} = p/K$ and time $\tau = rt$. How many unitless parameters are there in the nondimensional equation?

**1.36:** Suppose that the population of grasshoppers in your backyard is described by

$$p' = mp - \frac{ap}{1 + bp}s$$

where $a$, $b$, $m$, and $s$ are all positive parameters. Assume $s$ is unitless and denote the initial grasshopper population by $p_0$.

(a) Identify the units of each quantity.

(b) Reformulate the model using dimensionless variables and show that the behavior of the population is determined by a single parameter.

(c) Describe the long-time behavior of this grasshopper population.

**1.37:** Consider a population that is governed by the differential equation

$$p' = rp\left(1 - \frac{p}{K}\right) - \frac{ap^2}{p^2 + b^2}.$$

where the parameters $r$, $K$, $a$, and $b$ are all positive. What are the units of each quantity? Introduce non-dimensional variables $\widetilde{p} = p/b$ and $\tau = at/b$ to reformulate the equation into non-dimensional form.

**1.38:** Consider the non-dimensional differential equation

$$p' = p(a - bp) - \frac{p^2}{p^2 + 1}$$

where $a$ and $b$ are positive parameters and $p > 0$ is a non-dimensional population. We notice that $p^* = 0$ is an equilibrium, and so if we want to find any other non-zero equilibrium, we can factor out the $p$ and solve the equation

$$(a - bp) = \frac{p}{p^2 + 1}.$$

Unfortunately, this is a nasty equation to try to solve explicitly! Instead, we will try to simply count how many solutions (i.e., equilibria) there are.

(a) Graph the functions $a - bp$ and $p/(p^2 + 1)$ on the same axes for a few different values of the constants $a$ and $b$. Show that depending on the values of $a$ and $b$, there can be 1, 2, or 3 additional (nonzero) equilibria.

(b) If $a = 0.5$, estimate the value of $b$ at which a bifurcation occurs.

(c) Now using the value of $b$ that you found in the previous part, find a new approximate value of $a$ at which another bifurcation occurs.

**1.39:** Solve the initial value problem in section 1.2.5 using the substitution method described in Section 1.2.4.

**1.40:** A model for the water level $h(t)$ in a leaky bucket is given by the differential equation

$$h' = -\sqrt{h}.$$

(a) Find the equilibrium and determine its stability.

(b) Show that the function $h(t) = (\sqrt{h_0} - t/2)^2$ is a solution to this equation which satisfies the initial condition $h(0) = h_0$. On the same set of axes, sketch a few time-series plots for a few different values of $h_0$.

(c) Suppose that $h_0 = 9$. How full is the bucket at $t = 2$? At $t = 8$?

(d) Comment on the validity of this model. How well do you think it describes actual leaky buckets?

# Chapter 2

# Solving First-Order Equations

In this chapter, we will look at some specific techniques for solving certain first-order differential equations. Specifically, we will focus on two special cases:

- Separable Equations, which have the form $y' = f(y)g(t)$

- Linear Equations, which have the the form $y' = p(t)y + q(t)$

While there are many other types of first order equations that can also be solved with specialized techniques (and many textbooks do cover these methods), we will see that these two cases alone cover a wide range of applications. Furthermore, these two methods are sufficient for providing the conceptual framework needed to analyze higher order equations and systems.

## 2.1 Separable Equations

The first type of first-order equation that we will consider is the **separable equation** which is anything that can be written in the form $y' = f(y)g(t)$. The name comes from the observation that the dependent and independent variables are separated via multiplicative factors on the right-hand side. Note that in some cases, the right-hand side might need to be manipulated before it appears in this form. For example, the equation

$$y' = y + ty$$

is separable because we can rewrite it as $y' = y(1 + t)$ by factoring.

### 2.1.1 Calculus Revisited

A special case of separable equations occurs when $f(y) \equiv 1$ as in the example

$$\frac{dy}{dt} = \cos t.$$

As we have already discussed, we can solve this simply by anti-differentiating to obtain the solutions

$$y(t) = \sin(t) + C$$

where $C$ is any constant.

Since the unknown function $y$ does not appear on the right hand side, the differential equation is essentially a *calculus* problem. We note that in a calculus course, these types of problems are typically designed so that an antiderivative can be found; after all, one of the goals of such a course is to learn the techniques for finding indefinite integrals. In general, however, there is no guarantee that we will be able to find an explicit antiderivative. This does not mean that there isn't a solution – it simply means that we do not have a simple expression of the solution in terms of familiar functions.

For example, consider the equation

$$\frac{dy}{dt} = e^{-t^2}. \tag{2.1}$$

We can try to solve for the unknown function $y$ by anti-differentiating:

$$y(t) = \int e^{-t^2}\, dt + C. \tag{2.2}$$

At this point, we start trying out various integration techniques in order to simplify this expression. We might try looking it up in a table of integrals, or use software like Maple or Mathematica. As it turns out, however, none of these strategies will work for this problem. The fact is, there is no antiderivative in terms of elementary functions[1] and the best we can do is leave the function $y(t)$ expressed in terms of an integral. This example illustrates an important point, which is that many equations cannot be solved explicitly because they involve integrals that we either don't know how to do or cannot be simplified at all.

An alternative to Equation (2.2) is to use the Fundamental Theorem of Calculus to express $y(t)$ in terms of a definite integral. As we can verify by differentiation, the function

$$y(t) = \int_a^t e^{-s^2}\, ds + C \tag{2.3}$$

[1]The term *elementary function* refers to the types of functions that we're familiar with from calculus: polynomials, exponential and logarithmic functions, sines and cosines; as well as combinations of these obtained by adding, multiplying, etc.

satisfies the differential equation for any choice of the constant $a$ (choosing different values for $a$ is equivalent to changing the value of $C$). Using a definite integral to write the solution is particularly useful when working with an initial value. Suppose that we are given the initial condition $y(t_0) = y_0$. Plugging these values into (2.3) we get

$$y_0 = \int_a^{t_0} e^{-s^2}\, ds + C$$

and if we let $a = t_0$ and $C = y_0$, we get

$$y(t) = \int_{t_0}^{t} e^{-s^2}\, ds + y_0$$

as a solution to (2.1) with the given initial condition.

In some cases, integrals such as this show up often enough that they are given names. For example the **error function** (or **erf** for short) is defined as

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2}\, ds. \tag{2.4}$$

This is an important function for applications of probability and statistics.[2]

As we know from calculus, even when integrals *can* be simplified, it isn't always easy to figure out how to do so; sometimes an integral that looks impossible can in fact be done via some sneaky substitution. The bottom line is that this is not a course in integration techniques. We will attempt to simply our answers as much as possible, but at the same time, we will not worry too much about integration techniques. We will make use of resources such as calculators, integral tables, and computer tools like Maple, Mathematica, or Wolfram Alpha to help us simplify integrals but we should also recognize that some integrals cannot be simplified and in those cases, we must leave the solution in terms of an integral.

### 2.1.2 Autonomous Equations

Another special case occurs when the right-hand side does not depend explicitly on the independent variable – in other words when $g(t) \equiv 1$ in the general form of a separable equation. We start with the following simple example:

$$\frac{dy}{dt} = ky. \tag{2.5}$$

In section 1.1.1 we noted that this differential equation has solutions of the form $y = Ce^{kt}$, which we found by basically guessing: we thought about the functions that we're familiar with and

[2]The coefficient of $2/\sqrt{\pi}$ in front of the integral is there simply to make the limit of the error function as $t \to \infty$ equal to 1.

what happens when we differentiate them and we realized that exponential functions satisfy the differential equation. This may have felt slightly unsatisfying and you may have wished that there was some procedure that one could follow that would give us these solutions without having to make any guesses.

Because the unknown function $y(t)$ appears on the right hand side we cannot integrate (2.5) as it is currently written. However, if we rewrite the differential equation it is possible to integrate. This can be done by dividing both sides by the quantity $y$:

$$\frac{1}{y}\frac{dy}{dt} = k.$$

To do this, we need to make the assumption that the solution $y$ is not zero – we will return to this point later on. Now if we integrate both sides with respect to the independent variable $t$,

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int k\,dt,$$

we end up with an integral on the left-hand side that we can evaluate using the chain rule; the result is

$$\ln|y| = kt + C. \tag{2.6}$$

At this point, we make two observations:

1. Both integrals generated a constant of integration, but since they're arbitrary, we can simply collect them both into the single constant $C$ on the right hand side.

2. We used the chain rule to perform the integration on the left hand side. If your calculus is still rusty, and this isn't clear to you, try thinking about going in the other direction. In other words, if we were given the expression

   $$\ln|y(t)|$$

   and asked to differentiate *with respect to $t$*, then we would use the chain rule to get

   $$\frac{1}{y(t)}\frac{dy}{dt}.$$

   What we have done in this example is to go the other direction.

Now returning to (2.6), we still need to solve for the unknown function $y$. If we exponentiate both sides we get

$$|y| = e^{kt+C} = e^{kt}e^{C}$$

(using properties of exponents for that last step). Next, we can drop the absolute value sign on the left if we simply include the possibility that the right hand side could be positive or negative:

$$y = \pm e^{C}e^{kt}.$$

Note that we also rearranged the terms on the right. Finally, we observe that $C$ is an arbitrary constant that came from performing an integral. This means that the quantity $\pm e^C$ is itself an arbitrary constant (well, not completely arbitrary – it can never be zero). We can therefore can rename this quantity, say $\widetilde{C}$, and our solution is then

$$y(t) = \widetilde{C}e^{kt}. \tag{2.7}$$

where $\widetilde{C}$ is any nonzero real number.

We now must return to the subtle point that we made early on – that all along we were assuming that the solution $y$ was not zero. What would happen if in fact $y$ *was* zero? Well, if we look at the original differential equation $y' = ky$ we see that if $y$ ever did equal zero, then $y'$ would be zero as well, meaning that **the constant function** $y(t) = 0$ **is a solution to the differential equation**. If we allow the constant $\widetilde{C}$ to equal zero, then (2.7) represents all solutions.

At this point, we're done. The function given by (2.7) represents an infinite family of solutions to the differential equation. In practice, however, we will usually rewrite the solution slightly. The name $\widetilde{C}$ doesn't exactly roll off the tongue and so we will recycle the name "$C$" (which we used earlier in the example) to represent the constant in the solution, which is now written $y(t) = Ce^{kt}$.

Armed with this example, we can begin to generalize this technique. As we saw, the key step was to rewrite the differential equation in a way that allowed us to use the chain rule to find an antiderivative. We did that by dividing both sides by $y$, which allowed us to integrate both sides of the equation. Now suppose that we had a differential equation of the form

$$\frac{dy}{dt} = f(y). \tag{2.8}$$

As usual, $f$ here simply stands for "the right hand side" (recall that an equation of this form, in which the right hand side does not depend explicitly on $t$, is called an **autonomous** equation). In this case, we can rewrite the differential equation like this:

$$\frac{1}{f(y)}\frac{dy}{dt} = 1 \tag{2.9}$$

which allows us to integrate:

$$\int \frac{1}{f(y)}\frac{dy}{dt}\,dt = \int 1\,dt = t + c. \tag{2.10}$$

In principle, we can now integrate on the left hand side and solve for $y$ to get our solution. Of course, there is no guarantee that we will be able to do either of those things, but whether or not we can do the integral and solve for $y$ is really a question of calculus and/or algebra. As far as the differential equation is concerned, we have shown that any first-order autonomous equation has a solution.

**Example 2.1.1.** To find the general solution to the logistic equation

$$p' = rp\left(1 - \frac{p}{K}\right)$$

we rewrite the equation as

$$\frac{p'}{p(1 - p/K)} = r.$$

This allows us to then integrate both sides of the equation (with respect to the independent variable $t$). using a partial fraction decomposition (the details are left for the reader).

### 2.1.3 Separable Equations

We now consider the most general form of a separable equation

$$\frac{dy}{dt} = f(y)g(t). \tag{2.11}$$

Following the method described in the previous section, we divide both sides by the quantity $f(y)$ with the result that again we end up with something that we can, in principle, integrate. We call this technique **separation of variables** and we say that if a differential equation can be written in this form, it is **separable**.

**Example 2.1.2.** Solve the initial value problem

$$y' = \frac{t^2}{1 - y^2} \qquad y(0) = 0.$$

Separating variables, we have

$$(1 - y^2)y' = t^2$$

and integrating with respect to the independent variable $t$ then gives

$$y - \frac{1}{3}y^3 = \frac{1}{3}t^3 + C.$$

Plugging in the initial value gives $C = 0$, so the solution is determined by the equation

$$3y - y^3 = t^3.$$

Since we cannot solve for $y$ explicitly we simply stop here and say that this equation defines the solution *implicitly*. Actually, it is possible to solve a cubic equation explicitly, but the result is so messy that it's not very helpful to do so except in the simplest cases.

### Implicitly Defined Functions

As we saw in the previous example, it is sometimes difficult, if not impossible to solve for the unknown function explicitly. This, however, does not mean that there is no solution! It simply means that we cannot write it in a nice way using functions that we're familiar with. In these cases, we will be content to write down an equation which the solution satisfies and simply say that this equations defines the solution for us[3].

When it comes to visualizing functions that are defined implicitly, software such as Mathematica can help considerably. We can ask the software to graph a curve defined by an equation as in the example above, without having to define the function explicitly. For example, Mathematica's *ContourPlot* command was used to produce Figure 2.1 which shows the solution from Example 2.1.2.

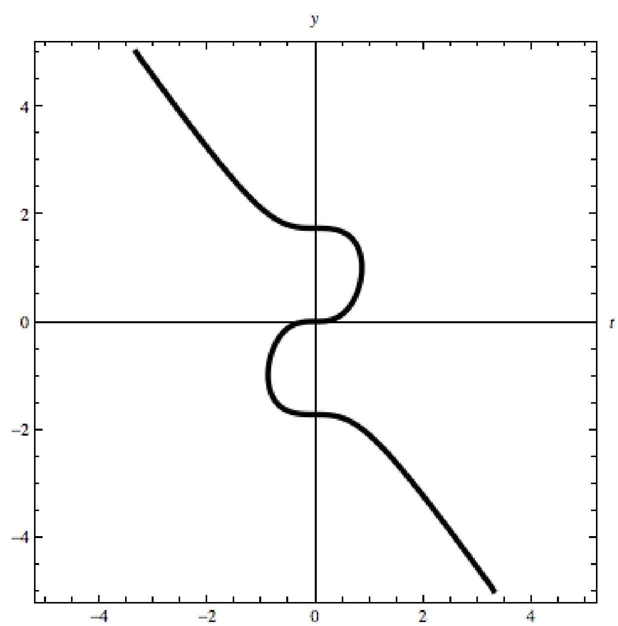



**Figure 2.1:** Graph of the curve defined by the equation $3y - y^3 = t^3$.

[3]We have encountered this idea before. Consider the following equation for the unknown number $x$:

$$\cos(x) = x.$$

We cannot solve this equation for $x$, yet we know that there is exactly one real number that satisfies this equation. We can say that this equation defines this number implicitly.

### 2.1.4 Interval of Existence

In addition to finding the solution to a differential equation, we may also wish to determine the interval of existence of the solution – in other words, the time interval for which the solution is valid. In the cases where the solution can be found explicitly, this is often a straightforward question to answer. However, things become slightly more difficult when the solution cannot be found explicitly. Even in these cases, however, we can use information from the differential equation to tell us something about the existence of solutions.

Consider our example from above. Looking at the original differential equation, we see that the derivative (and hence the solution) is undefined if $y^2 = 1$. We can then use our implicit equation for $y$ to determine the values of $t$ for which the solution equals 1 or $-1$. For instance, plugging in $y = 1$, we get

$$3 - 1 = t^3$$

which implies $t^3 = 2$. Similarly, plugging in $y = -1$ gives $t^3 = -2$. The two values $\pm\sqrt[3]{2}$ separate the real line into three intervals and the the solution interval is the one that contains the initial value (in this case $(t, y) = (0, 0)$). In other words, the interval of existence is

$$(-\sqrt[3]{2}, \sqrt[3]{2}).$$

Here is another example that illustrates a subtle distinction between the interval of existence for the solution to a differential equation and the domain of a function.

**Example 2.1.3.** Consider the differential equation

$$\frac{dy}{dt} = 1 + y^2.$$

with initial value $y(0) = 0$. Separating variables and integrating, we get

$$\int \frac{1}{1+y^2}\frac{dy}{dt}\,dt = \int 1\,dt.$$

which gives us the solution $y(t) = \tan(t)$. What is the interval of existence? Starting at the point specified by the initial condition, we may extend the solution forward and backwards in time as long as the solution remains defined and continuous, in other words as far as $\pi/2$ in either direction. The interval of existence is therefore $(-\pi/2, \pi/2)$. If we were discussing the function $y(t) = \tan(t)$ outside of the context of this differential equation, we would say that it has a domain of all real numbers except for the points $\pi/2 + n\pi$ for integers $n$.

## 2.2 Linear Equations

Unfortunately, most differential equations are not separable and therefore cannot be solved via the method of the previous section. For instance, the simple differential equation

$$y' = y + t$$

cannot be rewritten in a way that allows us to integrate both sides with respect to the independent variable; it is not separable and we need new techniques to handle equations such as this. We begin with some terminology. If a first-order differential equation can be written in the form

$$\frac{dy}{dt} = p(t)y + q(t). \tag{2.12}$$

then we say the differential equation is **linear**. If an equation is not linear, we say that it is **nonlinear**. Linear equations are quite important in the study of differential equations. For one thing, we will be able to find a general formula for the solution of any linear equation. More importantly, however, nonlinear equations can generally be approximated reasonably well by linear equations and therefore they may be used to model a wide array of real-world phenomena.

If the quantity $q(t) \equiv 0$ then we say the linear equation is **homogeneous**; otherwise the equation is **nonhomogeneous**. In the case of the homogeneous linear equation

$$\frac{dy}{dt} = p(t)y,$$

we can solve using separation of variables: dividing by the unknown function $y$ and integrating, we have

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int p(t)\,dt$$

integrating the left hand side

$$\ln|y(t)| = \int p(t)\,dt + C$$

and solving for $y$

$$y(t) = Ce^{\int p(t)\,dt.}$$

We have now solved the differential equation. For any particular example, the hardest part will be computing the integral of $p(t)$. Remember that in some cases, it may not be possible to simplify this integral – in which case we simply leave it as it is.

We now return to the general, nonhomogeneous linear equation (2.12). There are a variety of methods available for solving this equation and depending on the particular problem, some methods are easier to use than others. In this chapter, we will emphasize two techniques: *variation of parameters* and *undetermined coefficients*. Later, in Chapter 5, we will see yet another method.

### 2.2.1 Variation of Parameters

The first method that we will discuss for solving nonhomogenous linear equations is known as **variation of parameters**. This method has two significant features. First, it will always work. So if you're the kind of person that doesn't want to have to think about which method might be best suited for a particular problem then you can always rely on this method. Secondly, this method introduces a clever idea that will come up later on when we're solving second-order equations.

The basic idea is to solve a related homogeneous equation and then use that solution as a basis for finding the solution to the original nonhomogenous equation (the one we're actually trying to solve). Recall that the general nonhomogeneous first-order linear equation is

$$\frac{dy}{dt} = p(t)y + q(t). \tag{2.13}$$

We will call the related equation

$$\frac{dy}{dt} = p(t)y, \tag{2.14}$$

formed by simply dropping the $q(t)$ term, the **associated** homogeneous equation; because it is homogenous, we can solve it. Let $y_h$ be the solution to this new equation[4]:

$$y_h = Ce^{\int p(t)\,dt}$$

Next, we will guess that the solution to the nonhomogenous equation is similar to this. In particular, we will replace the arbitrary constant $C$ with a *function* $C(t)$ so that the solution we're looking for now has the form

$$y = C(t)e^{\int p(t)\,dt}.$$

We then simply plug this guess into the equation and (hopefully) solve for $C(t)$. Note that at this point (without having done this before) there's no guarantee that this will be successful; we are simply going to give this a try and see what happens. Plugging $y$ into 2.13 (and remembering that $C(t)$ is now a function and so we have to use the product rule on the left hand side) we get

$$C'(t)e^{\int p(t)\,dt} + C(t)e^{\int p(t)\,dt}p(t) = p(t)\left(C(t)e^{\int p(t)\,dt}\right) + q(t)$$

which simplifies to

$$C'(t)e^{\int p(t)\,dt} = q(t)$$

or more simply,

$$C'(t) = q(t)e^{-\int p(t)\,dt}.$$

At this point, we can integrate to solve for $C(t)$:

$$C(t) = \int q(t)e^{-\int p(t)\,dt}\,dt + k$$

[4]The subscript $h$ here just emphasizes that the function $y_h$ is a solution to the associated homogeneous equation, and not the original equation that we're interested in.

where $k$ is an arbitrary constant of integration. We can now replace $C(t)$ in the solution 2.2.1 with this expression and we're done. The solution is

$$\begin{aligned} y(t) &= \left( \int q(t) e^{-\int p(t)\,dt}\,dt + k \right) e^{\int p(t)\,dt} \\ &= e^{\int p(t)\,dt} \int q(t) e^{-\int p(t)\,dt}\,dt + k e^{\int p(t)\,dt}. \end{aligned} \tag{2.15}$$

Equation (2.15) is basically a formula for the solution, which tells us that a solution always exists (provided that the integrals exist, of course). Rather than trying to memorize this formula, however, I recommend that you learn the steps that produced this formula. Although it looks messy when written in terms of arbitrary coefficients $p(t)$ and $q(t)$, it's not so bad when the method is applied to a specific example.

**Example 2.2.1.** Consider the differential equation

$$y' = 2y + 1.$$

The associated homogeneous equation is $y' = 2y$ which has the solution $y_h = Ce^{2t}$. To apply the method of variation of parameters, we now replace the arbitrary constant $C$ with an unknown function $C(t)$ and look for a solution of the form

$$y = C(t)e^{2t}. \tag{2.16}$$

Plugging the solution into both sides of the differential equation:

$$C'e^{2t} + 2Ce^{2t} = 2(Ce^{2t}) + 1.$$

At this point, the quantity $2Ce^{2t}$ cancels. Note that in applying this method, a similar cancellation should always occur. If there is no cancellation, that's a good clue that something's not correct. After the cancellation, we can solve for $C'$:

$$C' = e^{-2t}$$

and integrate to obtain

$$C = -\frac{1}{2}e^{-2t} + k.$$

Here, $k$ is an arbitrary constant of integration. Now plugging this expression in for $C(t)$ back into 2.16 we get the solution

$$y(t) = \left( -\frac{1}{2}e^{-2t} + k \right) e^{2t} \tag{2.17}$$

$$= -\frac{1}{2} + ke^{2t}. \tag{2.18}$$

**Example 2.2.2.** Solve the linear equation

$$y' = -y + \cos(t)$$

with initial condition $y(0) = 0$.

The associated homogeneous equation is $y' = -y$ which has the solution $y_h = Ce^{-t}$. We then replace the constant $C$ with an unknown function $C(t)$ and look for a solution of the form

$$y = C(t)e^{-t}.$$

Plugging this into the the differential equation, we have

$$C'e^{-t} - Ce^{-t} = -Ce^{-t} + \cos(t)$$

which simplifies to the equation

$$C' = e^t \cos(t).$$

We can now integrate both sides to solve for $C(t)$:

$$C(t) = \frac{1}{2}e^t(\cos(t) + \sin(t)) + k.$$

Next, plugging this into the solution, we have

$$\begin{aligned} y(t) &= \left(\frac{1}{2}e^t(\cos(t) + \sin(t)) + k\right)e^{-t} \\ &= \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) + ke^{-t}. \end{aligned}$$

If we did not have an initial condition, we'd be finished at this point. However, since we must also make sure that $y(0) = 0$, we need to plug these values into the general solution we found to determine $k$:

$$0 = \frac{1}{2} + k$$

which means $k = -1/2$. The particular solution is therefore:

$$y(t) = \frac{1}{2}\left(\cos(t) + \sin(t) - e^{-t}\right).$$

### 2.2.2 Undetermined Coefficients

Although the method of variation of parameters has the advantage that it will always result in a solution, it has the disadvantage that implementing the method requires us to evaluate an integral. For some problems, a simpler technique is available which will allow us to find a solution without performing integrations. The downside to this new method is that it does not always work.

To motivate this new method, take a close look at the solution that we found in example 2.2.1. Notice that the solution can be written $y = y_h - \frac{1}{2}$ where $y_h$ is the solution to the associated homogeneous equation. Notice also that in this example, the coefficient $q(t)$ is a constant. In other words, the difference between the solution to the nonhomogenous equation and the associated homogeneous equation is a constant, *just like the coefficient* $q(t)$.

This suggests the following strategy for solving the nonhomogeneous equation:

1. Find the solution, $y_h$, to the associated homogeneous equation.
2. Guess a particular solution $y_p$ based on what type of function $q(t)$ is. The term *particular* here means that this solution satisfies some particular initial value.
3. Form the general solution by adding together: $y = y_h + y_p$.

The challenging part will be guessing the particular solution $y_p$. The general guideline is to guess a function that looks like $q$. As we will see, this will not always be possible. However, for those problems for which we can guess $y_p$, this method almost always is easier to carry out than variation of parameters. In particular, we will not have to do as many integrals.

Suppose for example that $q(t)$ is an exponential function. In this case, our guess $y_p$ should be a function such that when we plug it into the differential equation

$$y' = py + q$$

all of the terms will be similar exponentials (so that the right and left hand sides of the equation are the same). Now here's the question: what kind of function, when differentiated, results in an exponential? Answer: an exponential function! And if we add together several exponential functions? The result is still an exponential function!

In other words, if we guess an exponential function for $y_p$, then when we plug this guess into the differential equation, the result has a reasonable chance of giving us $q(t)$. All we need to do is include a coefficient to give us some freedom in matching the left and right hand sides. This "undetermined" coefficient is what gives this method its name.

**Example 2.2.3.** Find a particular solution for the nonhomogenous equation

$$y' = -y + 6e^{2t}.$$

In this example, $q(t)$ is the exponential function $6e^{2t}$ which suggests the particular solution $y_p = Ae^{2t}$. Then plugging this guess into the differential equation gives

$$\begin{aligned} 2Ae^{2t} &= -Ae^{2t} + 6e^{2t} \\ 3Ae^{2t} &= 6e^{2t} \\ A &= 2 \end{aligned}$$

So the particular solution is $y_p = 2e^{2t}$ and the general solution is

$$y = Ce^{-t} + 2e^{2t}.$$

**Example 2.2.4.** Find a particular solution for the nonhomogenous equation

$$y' = -y + t^2 - 1.$$

Now $q(t)$ is a polynomial function (specifically, a second order polynomial), so we guess a particular solution $y_p = At^2 + Bt + C$. Notice that even though $q(t)$ does not include a first-order term, our guess does. This is because when we plug our guess into the DE and differentiate we will generate first-order terms and so our guess must include a first-order term to take this into account.

Plugging our guess into the differential equation then gives

$$2At + B = -At^2 - Bt - C + t^2 - 1$$

and collecting like terms

$$At^2 + (2A + B)t + (B + C) = t^2 - 1$$

and then matching coefficients

$$A = 1 \qquad 2A + B = 0 \qquad B + C = -1.$$

We now have three algebraic equations for the three unknown coefficients. Solving them in order, we find

$$A = 1$$
$$B = -2$$
$$C = 1.$$

So the particular solution is

$$y_p = t^2 - 2t + 1.$$

**Example 2.2.5.** Find a particular solution for the nonhomogeneous equation

$$y' = 2y + \cos(2t).$$

Since $q(t)$ is a cosine function, our guess should also include a cosine function. However, because of the derivative term on the left-hand side, we should also include a sine term in our guess (if we did not, then we would not be able to match both sine and cosine terms on either side of the equation). Letting $y_p = A\cos(2t) + B\sin(2t)$ and plugging this guess into the left-hand side, we get

$$-2A\sin(2t) + 2B\cos(2t) = 2A\cos(2t) + 2B\sin(2t) + \cos(2t)$$

and collecting sine and cosine terms on the left,

$$(-2A - 2B)\cos(2t) + (2B - 2A)\sin(2t) = \cos(2t).$$

Then matching the cosine terms on the left and right hand sides, we have

$$-2A - 2B = 1$$

while matching sine terms gives us

$$2B - 2A = 0.$$

We then solve these two equations for the two unknowns and find that $A = -1/4$ and $B = -1/4$ so the particular solution is

$$y_p = -\frac{1}{4}\cos(2t) - \frac{1}{4}\sin(2t).$$

**Example 2.2.6.** Find a particular solution for the nonhomogeneous equation

$$y' = y + 2e^t.$$

As in example 2.2.3, the right hand side is an exponential, so we might guess

$$y_p = Ae^t$$

as our particular solution. However, upon substituting this into the equation, we find

$$\begin{aligned} Ae^t &= Ae^t + 2e^t \\ 0 &= 2e^t \\ 0 &= 2 \end{aligned}$$

Because we end up with a statement that is clearly not true, we conclude that the function $y_p = Ae^t$ cannot be a solution to the differential equation (for any value of the coefficient $A$). In other words, our guess did not work! Why not? If we look at the associated homogeneous equation

$$y' = y$$

we see that it has the solution $y_h = Ce^t$. So our guess above for the particular solution was in fact a solution to the homogeneous equation – it cannot possible be a solution to the nonhomogeneous equation as well!

At this point, let's finish the problem by finding the solution using variation of parameters. If we look for a particular solution of the form $y_p = v(t)e^t$ where $v(t)$ is an unknown function, then plugging this into the differential equation we find

$$\begin{aligned} (v'e^t + ve^t) &= ve^t + 2e^t \\ v' &= 2. \end{aligned}$$

and so $v(t) = 2t + c$. This means that the function $y_p$ is given by

$$\begin{aligned} y_p &= (2t + c)\, e^t \\ &= 2te^t + ce^t. \end{aligned}$$

Since this is a solution for any choice of the constant $c$ may take the simplest case $y_p = 2te^t$.

This last example illustrates an important point about making guesses for the particular solution. In cases where the function that we might otherwise guess (based on the form of the function $q(t)$) is *already* a solution to the associated homogeneous equation, the general principle is to include an extra factor of $t$ in our guess. For instance, instead of the guess $y_p = Ae^t$, the correct guess would be a particular solution of the form $y_p = Ate^t$. In some cases, it may be necessary to apply this principle more than once!

In general, we will be able to guess the particular solution as long as $p(t)$ is a constant and $q(t)$ is an exponential function, a polynomial, a sine or cosine function, or a sum or product of these kinds of functions. For any other type of $q$, it is usually too complicated to try to guess $y_p$. Here are some guidelines.

- If $q(t)$ is an exponential function, i.e. has the form $ae^{rt}$, then we should guess a particular solution
$$y_p = Ae^{rt}$$
for some unknown coefficient $A$.

- If $q(t)$ is a polynomial function, i.e. has the form $a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0$ then we should guess a particular solution
$$y_p = A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0.$$

- If $q(t)$ is a sine or cosine function, i.e has the form $a \sin(\omega t)$ or $a \cos(\omega t)$, then we should guess a particular solution with the same form:
$$y_p = A \sin(\omega t) + B \cos(\omega t).$$

- If $q(t)$ is a sum (or product) of any of the above, then $y_p$ should also be a sum (or product) of the guesses outlined above.

It is important that $p(t)$ is a constant – otherwise the product $py$ will not be the same *type* of function as the other terms in the equation. In the case where $p(t)$ is not constant, it is generally necessary to use variation of parameters.

## 2.3 Exercises

**2.1:** Use the method of variation of parameters to find the solution to the following equations:

(a) $y' = 2y + \cos(2t)$

(b) $u' = u + te^{-t}$

(c) $f' = tf - t^2$

**2.2:** Consider the first-order autonomous equation

$$\frac{dy}{dt} = ay + b$$

where $a$ and $b$ are constants. Use methods described in this chapter to find the solution.

**2.3:** A small turkey is taken out of the refrigerator set to 40°F and placed into an oven at 350°F. If $h = 0.15$ per hour is the heat loss coefficient for the turkey, how long will it take until the internal temperature is 165°F?

**2.4:** Solve the initial value problem

$$y' = e^{\sin t} \qquad y(0) = 0$$

and sketch the solution on the interval $[0, 10]$.

**2.5:** A 5 gallon bucket is full of pure water. Suppose we begin dumping salt into the bucket at a rate of 1/4 lbs. per minute. At the same time, we open a spigot at the bottom allowing the bucket to drain at a rate of 1/2 gallon per minute while adding additional pure water to keep the bucket full. Assume the salt water is well-mixed.

(a) How much salt is in the bucket after a very long time?

(b) How much salt is in the bucket after 1 minute? 10 minutes?

**2.6:** Fill in the details to derive the solution given in Example 2.1.1.

**2.7:** Find an explicit form of the general solution for the following differential equations:

(a) $u' = \dfrac{2u}{t+1}$

(b) $u' + u + \frac{1}{u} = 0$

(c) $y' = \dfrac{t+1}{y-1}$

(d) $y' = t(1 + y)^2$

(e) $y' = e^{t+y}$

**2.8:** Draw the phase line for the equation $y' = y(4 - y^2)$. Then solve the equation with the initial condition $y(0) = 1$.

**2.9:** Recall that a circle of radius $R$ centered at $(a, b)$ is described by the equation $(x-a)^2+(y-b)^2 = R^2$.

(a) Explain why this equation does not define a function.

(b) On the other hand, we say that this equation can be used to define several functions *implicitly*. What does this mean?

(c) Use the chain rule to find the slope of a tangent line at a fixed point $(x_0, y_0)$ on the graph of one of these implicit functions.

**2.10:** The graph of the equation $y^2 = x^3 - x$ is a called an elliptic curve.

(a) Give a formula for the slope of a tangent line to a point $(x_0, y_0)$ on this curve. Find the points at which the tangent lines are vertical.

(b) Suppose that a function $f(x)$ is defined implicitly by the curve $y^2 = x^3 - x$. What differential equation does $f(x)$ satisfy? (Again, use the chain rule.)

(c) Sketch the slope-field (either by hand or using DFIELD) for the differential equation in (b). Plot the elliptic curve $y^2 = x^3 - x$ on the same plane.

**2.11:** Find the solution in implicit form for the equation

$$y' = \frac{4 - 2t}{3y^2 - 5}$$

with initial condition $y(1) = 3$ and plot the solution. What is the interval of existence?

**2.12:** An arrow with mass $m$ is shot vertically upward with an initial velocity 160 ft/sec. It is subject to both a gravitational force $mg$ as well as air resistance with magnitude $\frac{mv^2}{800}$ where $v$ is the arrow's velocity. How high does the arrow go?

**2.13:** Find the general solutions to the following differential equations:

(a) $y' = 2y + 2e^{2t} + t^2$

(b) $y' = ay + e^{at}$ where $a$ is a constant

(c) $ty' + y = e^t$

(d) $y' = 3y + e^{-t}\cos(2t)$

(e) $y' = -y + 2\sin^2(t)$

**2.14:** Find the value of $y_0$ for which the solution to

$$y' = y + 1 + 3\sin(t) \qquad y(0) = y_0$$

remains finite as $t \to \infty$.

**2.15:** Solve the initial value problem

$$\begin{gathered} 2y' + ty = 2 \\ y(0) = 1 \end{gathered}$$

expressing the solution in terms of a definite integral. What is the limit of the solution as $t \to \infty$?

**2.16:** In very cold weather the thickness of ice on a pond increases at a rate inversely proportional to its thickness. If the ice is initially 0.05 inches thick and 4 hours later it is 0.075 inches thick, how thick will it be in 10 hours?

**2.17:** A decaying battery generating $200e^{-5t}$ volts is connected in series with a 20 ohm resistor and a 0.01 farad capacitor. The charge on the capacitor is then given by the differential equation

$$Rq' + \frac{1}{C}q = 200e^{-5t}$$

where $R = 20$ and $C = 0.01$. If the charge $q$ is initially equal to 0 at time $t = 0$, find the solution $q(t)$. Show that the charge reaches a maximum and find the time at which the maximum is reached.

**2.18:** When brought to the emergency room, an accident victim has 3 liters of blood and is still losing blood at a rate of 0.25 liters per hour. He is immediately given continuous blood transfusions at 0.5 liters per hour, and an antibiotic drug is administered intravenously at 0.5 grams per hour. Four hours later, the bleeding has stopped, and the transfusions continue for an additional two hours. Determine the amount of the antibiotic in the patient's blood at the time that the transfusions stop.

**2.19:** Consider the initial value problem

$$y' = ty(4-y)/(1+t) \qquad y(0) = y_0 > 0.$$

(a) Describe the long-time (i.e. $t \to \infty$) behavior of the solution.

(b) If $y_0 = 2$, find the time $T$ at which the solution first reaches the value 3.99.

(c) Find a range of initial values for which the solution lies in the interval $3.99 < y < 4.01$ by the time $t = 2$.

**2.20:** Find the solutions to the following initial value problems.

(a) $y' = y + 2te^{2t} \qquad y(0) = 1$

(b) $ty' + 2y = t^2 - t + 1 \qquad y(1) = 1/2 \quad (t > 0)$

(c) $u' = -u + 2\sin^2(t) \qquad u(0) = 0$

**2.21:** A tank that can hold 500 gal. originally contains 200 gal. of water with 100 lbs. of salt in solution. Water containing 1 lb. of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to drain at a rate of 2 gal/min. Find the amount of salt in the tank at the moment when the solution begins to overflow.

**2.22:** A sky diver weighing 180 lb. falls vertically downward from an altitude of 5000 ft. and opens the parachute after 10 seconds of free fall. Assume that the force of air resistance is $0.75|v|$ when the parachute is closed and $12|v|$ when the parachute is open. Velocity $v$ is measured in ft/sec.

(a) Find the speed of the sky diver when the parachute opens.

(b) Find the distance fallen before the parachute opens.

(c) From the moment that the parachute opens, how long does it take to reach the ground?

**2.23:** A house is initially at 12° C when the power goes out. The outside temperature varies according to

$$T_e = 9 + 10\cos 2\pi t$$

where $t$ is measured in days. The heat-loss coefficient is $k = 3$ (this is the proportionality constant in Newton's law of heat transfer). Find a formula for the temperature inside the house and graph it along with $T_e$ on the same axes. What is the time lag between the maximum inside and outside temperatures?

# Chapter 3

# Systems of Differential Equations

Up to this point we have focused on problems involving a single dependent variable. However, the world is complicated and most real applications involve multiple interacting quantities, each of which may be governed by a separate differential equation. In this chapter, we will look at differential equations that involve more than a single dependent variable.

## 3.1 Introduction to Systems

Suppose we have two quantities, $x(t)$ and $y(t)$, each of which changes over time according to some differential equation:

$$x' = f(x, y) \tag{3.1}$$

$$y' = g(x, y). \tag{3.2}$$

Because each differential equation involves both $x$ and $y$, the two equations are said to be **coupled** and because each equation depends on two unknown functions we cannot solve the equations separately. A set of two or more coupled equations is usually referred to as a **system** of equations and a **solution** to a system refers to a collection of solutions to the individual differential equations (i.e. the set $x(t)$ and $y(t)$ together is called a solution to the system).

Because we have two equations for the dependent functions $x(t)$ and $y(t)$, this set of equations is said to be **two-dimensional**. Although we will focus primarily on two-dimensional systems the general principles that we learn will apply just as well to higher dimensional systems. As before, the system is called **autonomous** if the functions $f$ and $g$ do not have an explicit dependence on the the independent variable $t$. Because the analysis is simpler, we will limit our discussion to autonomous systems.[1]

There are two basic approaches in analyzing a system of equations. The first is to try to understand how each of the state variables $x(t)$ and $y(t)$ depend on $t$. For instance, if we are able to solve equations (3.1-3.2), we can then graph $x(t)$ vs. $t$ as well as $y(t)$ vs. $t$. As in the case with a single differential equation, these graphs are referred to as **time series** plots. On the other hand, we can try to investigate the *relationship* between the variables $x(t)$ and $y(t)$. If we plot the parametric curve $(x(t), y(t))$ in the $x$-$y$ plane, for instance, we might gain insight into how the two state variables interact, something that is often difficult to discern from the time-series plots alone. This solution curve is called a **trajectory** or **orbit** and the $x$-$y$ plane is referred to as the **phase plane**. In many applications, it is precisely this relationship between the state variables that is of interest.

### Predator-Prey Population Model

As an example of how systems of equations arise, we will look at a model for two populations of interacting species. In previous chapters, we saw a very simple population model that exhibits exponential growth as well as the logistic model that takes into account the fact that realistically, a species' growth is limited by finite resources. Another way in which population models can be made more realistic is to take into account the fact that any species typically interacts with one or more other species within a complex network of predator and prey relationships.

A very simple model of this type of interaction is the **Lotka-Volterra** model which was first developed to model the population of a food fish (e.g. sardines) and various predator species (e.g. sharks) that consume them. To construct the model, we make the following assumptions:

- The presence of sharks keeps the sardine population far below its limit capacity. This suggests that without sharks, the sardines will exhibit exponential growth.

- The shark population relies on sardines for its food source. This suggests that in the absence of sardines, the shark population will decrease exponentially.

- The rate of predation, per predator, is proportional to the sardine population. In other words, predation will tend to decrease the sardine population at a rate that is proportional to the product of the two populations.

- The growth rate of the shark population, per sardine, is proportional to the shark population. In other words, the presence of the sardines tends to increase the shark population at a rate proportional to the product of the two populations.

[1] This is actually not a serious restriction. It is always possible to convert a nonautonomous system of equations into an autonomous system by treating $t$ as if it were a new dependent variable and then adding another differential equation, in essence increasing the dimension of the system.

Denoting the sardine population $x(t)$ and the shark population $y(t)$, the above assumptions give rise to the following differential equations:

$$x' = ax - bxy \tag{3.3}$$
$$y' = -cy + dxy. \tag{3.4}$$

where the parameters $a, b, c, d$ are all positive.

### Higher Order Equations

Another way in which systems of equations arise is through the analysis of higher order differential equations. Consider Newton's second law of motion. We have already seen how it can be expressed as a differential equation by writing the acceleration as the second derivative of position:

$$F(x, x', t) = mx''$$

One drawback to expressing Newton's law in this way is that we might not have a good intuition for acceleration. However, if we introduce a new variable, namely the velocity $v = x'(t)$, and use the fact that $x'' = v'$ then we can express Newton's law as a first order equation for $v$ instead of a second order equation for $x$. The trade-off is that we now have two variables (position and velocity) to keep track of, each with its own differential equation:

$$x' = v \tag{3.5}$$
$$v' = \frac{1}{m}F(x, v, t). \tag{3.6}$$

Although we now have two equations instead of one, the hope is that this system of equations will give us a better intuitive understanding of the dynamics than a single second order equation.

This idea can be applied to any differential equation of second or higher order. By introducing new variables as needed, we can always reformulate such an equation as a system of first-order equations. This means that once we have a framework for analyzing systems, we in principle handle equations of any order.

Another important example of a second order equation is the one that describes the motion of a mass on a spring. If we let $y(t)$ denote the displacement from equilibrium of an object with mass $m$ then the differential equation describing the motion is

$$my'' + \gamma y' + ky = 0 \tag{3.7}$$

where $\gamma$ is a damping coefficient, and $k$ is the spring constant. If we introduce $v = y'$ then the original differential equation can be rewritten as the system of equations

$$y' = v \tag{3.8}$$
$$v' = -\frac{k}{m}y - \frac{\gamma}{m}v. \tag{3.9}$$

One important reason for rewriting a higher order equation as a system is that in many cases, the systems approach can help illuminate relationships between the states variables which can often tell us something about the dynamics of the system. Consider the case of a mass on a spring with values $m = 1$, $\gamma = 0$, and $k = 1$ chosen for simplicity. In this case we obtain the second order differential equation

$$y'' + y = 0.$$

Although we have not yet learned any particular techniques for solving second order equations, this one is simple enough that we can guess that the solutions are the sine and cosine functions. For instance, we can easily verify that $y(t) = \cos(t)$ is one solution. If we express this second order equation as a system, we have

$$y' = v \tag{3.10}$$
$$v' = -y \tag{3.11}$$

and using our previous solution $y(t) = \cos(t)$, we now have $v(t) = -\sin(t)$. Figure 3.1 shows the graphs of the solutions $y(t)$ and $v(t)$.

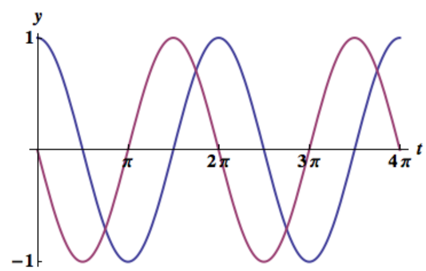



**Figure 3.1:** Time series graphs of the solutions $y(t)$ and $v(t)$.

We can see the relationship between $y$ and $v$ directly by plotting the the parametric curves $y(t) = \cos(t)$ and $v(t) = -\sin(t)$ in the $y$-$v$ plane. In this particular example, we note that

$$y^2 + v^2 = \cos^2(t) + \sin^2(t) = 1.$$

In other words, the points $(y(t), v(t))$ in the $y$-$v$ plane always lie on a circle of radius 1 centered at the origin. This view of the solution curve gives a very clear picture of the relationship between position $y$ and velocity $v$. For instance, Figure 3.2 shows us that the velocity is at a maximum precisely when the position is zero. Similarly, at the moment that position is maximized, the velocity is zero. Of course this information is contained in Figure 3.1 as well, but it is much easier to see in Figure 3.2.

By itself, the solution curve in the phase plane does not give us complete information. For instance, it's impossible to tell from Figure 3.2 alone where on the trajectory the solution lies at any given value of $t$. However, in many applications, it is precisely this relationship between the state variables that gives us the best understanding of the dynamics of the system. It is helpful, therefore, to consider both of these types of graphs when analyzing a system of equations.

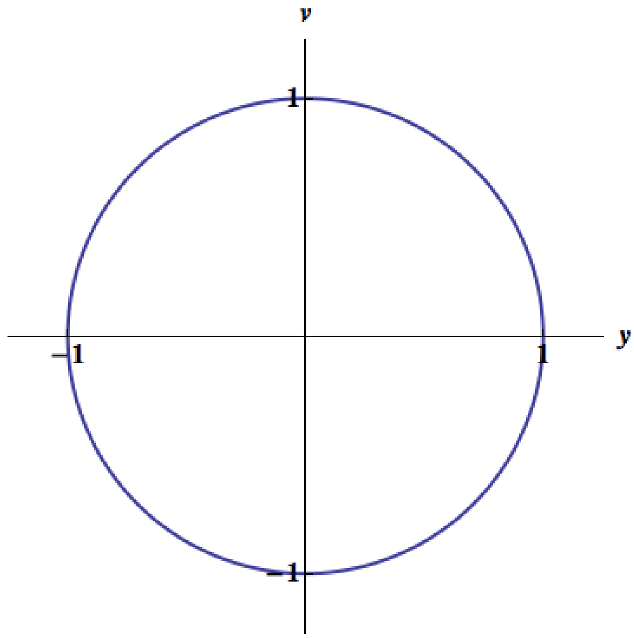



**Figure 3.2:** Graph of solution curve in the $y$-$v$ plane.

## 3.2 Qualitative Methods

As was the case with first-order equations, most examples cannot be solved explicitly. To handle these cases, we need qualitative methods that can shed light on the behavior of systems without the need to find explicit solutions. Later in the chapter, we will investigate in more detail an important special case for which we can find solutions.

### 3.2.1 Equilibria and Nullclines

One of the most basic ways we can analyze a system is to look for possible equilibria. Generalizing from the one-dimensional case, equilibria are found by setting the right-hand sides of all the equations to zero. For example, in the predator-prey model discussed in Section 3.1, we look for values of $x$ and $y$ such that $x' = y' = 0$. Setting the right-hand side of (3.3) equal to zero and solving, we see that $x' = 0$ if $x = 0$ or if $y = a/b$. Similarly, setting the right-hand side of (3.4) equal to zero, we see that $y' = 0$ if $y = 0$ or if $x = c/d$. We therefore have two equilibria solutions: one when both populations are zero, the other when $x = c/d$ and $y = a/b$.

To analyze the system further, we can find the **nullclines**. These are the curves in the $x$-$y$ plane that are determined by the conditions $x' = 0$ or $y' = 0$ considered separately. Note that where the nullclines intersect, both conditions are met, corresponding to equilibrium solutions. In our example, the nullclines for $x' = 0$ are the $y$-axis and the horizontal line $y = a/b$ and the nullclines for $y' = 0$ are the $x$-axis and the vertical line $x = c/d$.

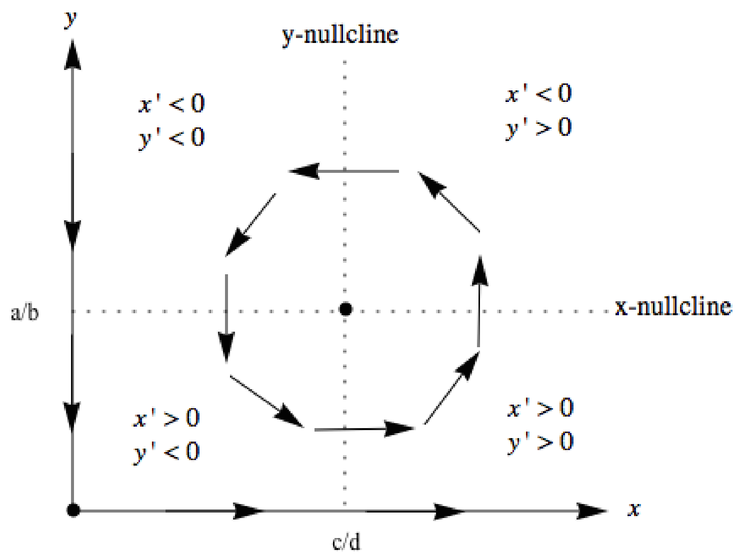



**Figure 3.3:** Nullclines (dashed) showing the behavior of $x$ and $y$ in different regions of the first quadrant.

Since we're only really interested in the case where both $x$ and $y$ are positive, we will focus on those nullclines that separate the first quadrant into four regions:

- If $x < c/d$ and $y > a/b$ then $x' < 0$ and $y' < 0$.
- If $x < c/d$ and $y < a/b$ then $x' > 0$ and $y' < 0$.
- If $x > c/d$ and $y < a/b$ then $x' > 0$ and $y' > 0$.
- If $x > c/d$ and $y > a/b$ then $x' < 0$ and $y' > 0$.

Figure 3.3 summarizes our qualitative understanding of the dynamics up to this point. Note that the general behavior of $x$ and $y$ indicated in Figure 3.3 is not precise. Although the arrows indicate some kind of cyclical behavior, we cannot conclude from this picture alone whether the actual solutions are closed loops or instead spirals either approaching or diverging away from the non-zero equilibrium. To get a more precise picture of the solutions requires still further analysis.

### 3.2.2 Phase Portraits

In addition to illuminating relationships between the state variables in a system, the phase plane point of view also provides an important method for visualizing solutions, particularly in those cases where the equations cannot be solved explicitly.

Suppose that $x(t)$ and $y(t)$ are solutions to the system of differential equations (3.1-3.2) and we let $\mathbf{u} = (x(t), y(t))$ denote the parametric curve in the $x$-$y$ plane determined by these solutions. If we think of this curve as the path of a moving particle, then $dx/dt$ and $dy/dt$ are the horizontal and vertical velocities of this particle and the **vector field F** defined by

$$\mathbf{F}(x, y) = (x', y') = (f(x, y), g(x, y)) \tag{3.12}$$

shows us the direction that the particle will travel at any given moment. More precisely, if we fix a point $(x, y)$ then the vector $\mathbf{F} = (f(x, y), g(x, y))$ points in the direction in which the particle will move. This means that by sketching the vector field, we can sketch a solution curve by simply following the arrows (in the same way that we did with direction fields in Chapter 1).

Actually, if all we want to do is sketch the trajectory of a solution, we only need the *direction* of the vector field arrows, not their magnitudes. Since vector fields typically have vectors of varying lengths, it can sometimes get confusing if the vector field arrows overlap or if they are very small. To remedy this, we will usually normalize the vector $\mathbf{F}$ by dividing by the magnitude $|\mathbf{F}|$ to get vectors of unit length. The result is called the **direction field**.

As an example, let's look at the Lotka-Volterra system described in Section 3.1. To make things concrete, we pick values for the coefficients and consider the system

$$x' = 2x - xy \tag{3.13}$$

$$y' = -2y + xy. \tag{3.14}$$

If we pick a point in the $xy$ plane, say $(1, 1)$, and plug the $x$ and $y$ values into the vector field, we find

$$\frac{\mathbf{F}(1,1)}{|\mathbf{F}|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

which tells us the solution curve that passes through this point is moving down and to the right. By plotting a variety of these direction field arrows in the $xy$ plane, we gain a clear picture of the solution curves by simply following the arrows. Figure 3.4 on the next page shows the direction field with several specific solution curves traced out.

As a final note, we point out the importance of working with autonomous systems when working with direction fields and phase planes. Because the system (3.13-3.14) is autonomous, the direction field shown in Figure 3.4 is time independent. This is what allows us to visualize the solution by following the arrows. If the vector field changed over time, then Figure 3.4 would only represent a snapshot in time and we would not be able to sketch the trajectories.

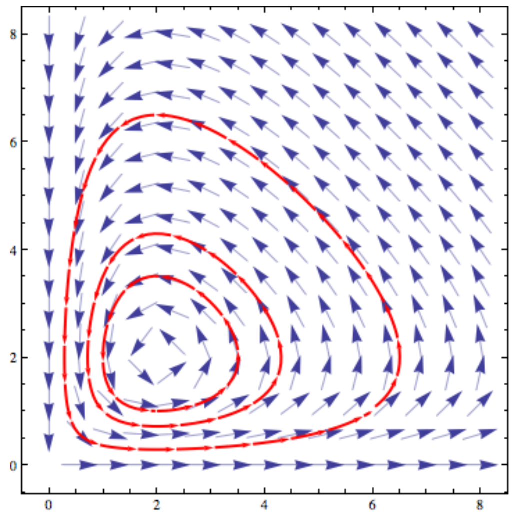


**Figure 3.4:** Direction field for the Lotka-Volterra system (3.13-3.14). The solution curves correspond to different initial conditions.

## 3.3 Linear Systems

We now turn to the special case of a system of linear differential equations with constant coefficients. In addition to having the nice feature that we will be able to find solutions for these systems, we will also see that they also have a special role to play in the analysis of more complicated systems. As you may recall from calculus, we can use the linear approximation of a function to understand the local behavior of a function. The situation with systems of equations is somewhat similar: the local behavior of a nonlinear system can often be approximated by a linear approximation of the system.

### 3.3.1 Definitions and Notation

Consider the system of differential equations

$$x' = ax + by \tag{3.15}$$
$$y' = cx + dy \tag{3.16}$$

where the coefficients $a, b, c$ and $d$ are constants and the dependent variables $x(t)$ and $y(t)$ are functions of $t$. The system is said to be **linear** because the right hand sides depend only on the first powers of $x$ and $y$.[2] and as we have already mentioned, because we have two dependent variables the system is called **two-dimensional**.

The first step is to rewrite our linear system of equations (3.15-3.16) in a more compact form using vector and matrix notation.[3] First, we collect the dependent variables into a single vector **u**

[2]We will give another, more precise, definition of linear later on.
[3]Some basic definitions and properties are given in Appendix B.

that we will call the **solution vector** and we let $\mathbf{u}'$ refer to the vector formed by the derivatives of the two state variables:

$$\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{u}' = \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

We also define $\mathbf{A}$ to be the $2 \times 2$ **coefficient matrix**:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By definition of matrix multiplication (see Appendix B), the product of the matrix $\mathbf{A}$ and the vector $\mathbf{u}$ is given by

$$\mathbf{Au} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

which means that the original system of differential equations (3.15-3.16) can be written as

$$\mathbf{u}' = \mathbf{Au}. \tag{3.17}$$

Just as we did for the one-dimensional problem, we can specify a particular solution by including the additional constraint of an initial condition. Since the solution is comprised of two components, $x(t)$ and $y(t)$, the initial condition is really *two* conditions, one for each component:

$$x(t_0) = x_0 \tag{3.18}$$

$$y(t_0) = y_0. \tag{3.19}$$

We can denote these conditions in vector form:

$$\mathbf{u}(t_0) = \mathbf{u_0} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{3.20}$$

Expressing our system of equations using vector notation has several benefits. First of all, the notation gives us a very economical way of writing a system of linear equations. For a two-dimensional system, this might not seem like significant gain, but many practical applications can include dozens of dependent variables, in which case a systems approach is absolutely necessary. Perhaps more importantly, however, using vectors and matrices to represent our system of equations allows us to use tools and techniques from linear algebra. Again, for our purposes, we will not assume a background in linear algebra and will instead introduce and define anything that we need as we go along.

### Equilibrium Solutions

Suppose that $\mathbf{u}^* = (x^*, y^*)$ is an equilibrium solution for our system (3.17), meaning that

$$\mathbf{Au}^* = \mathbf{0} \tag{3.21}$$

or in component form

$$ax^* + by^* = 0 \tag{3.22}$$
$$cx^* + dy^* = 0. \tag{3.23}$$

To solve for $x^*$ and $y^*$, we can solve the first equation for $y^*$ (assuming that $b \neq 0$):

$$y^* = -\frac{a}{b}x^*$$

and plug into the second equation to get

$$cx^* - d\left(\frac{a}{b}\right)x^* = 0$$

which can be rewritten as

$$(ad - bc)x^* = 0.$$

From this we conclude that $x^* = 0$ or $ad - bc = 0$. More to the point, if $ad - bc \neq 0$ then it must be the case that $x^* = 0$ (which then implies $y^* = 0$) and so the only equilibrium solution is obvious one (i.e. the origin). On the other hand, if $ad - bc = 0$ then we can have *nontrivial* equilibrium solutions. The quantity $ad - bc$ is called the **determinant** of the coefficient matrix $\mathbf{A}$. We will denote the determinant as $\det \mathbf{A}$.

Note that in solving (3.22) for $y$, we made the assumption that $b \neq 0$. This assumption isn't really crucial because in the case where $b$ does equal 0 we can simply solve (3.23) for $y^*$ instead and proceed under the assumption that $d \neq 0$. In the case where both $b$ and $d$ vanish then $ad - bc = 0$ and the conclusion reached above still holds.

To illustrate how ideas from linear algebra can come into play here, let's go back to (3.21). Readers that have seen some linear algebra will know that if the determinant of the matrix $\mathbf{A}$ is nonzero then the inverse $\mathbf{A}^{-1}$ exists and therefore

$$\begin{aligned}
\mathbf{A}^{-1}\left(\mathbf{A}\mathbf{u}^*\right) &= \mathbf{A}^{-1}\mathbf{0} \\
\left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{u}^* &= \mathbf{0} \\
\mathbf{I}\mathbf{u}^* &= \mathbf{0} \\
\mathbf{u}^* &= \mathbf{0}.
\end{aligned}$$

This shows that the only solution is the trivial zero vector. On the other hand, if $\det \mathbf{A} = 0$ then the inverse of $\mathbf{A}$ does not exist and there must be nonzero equilibrium solutions.

### 3.3.2 Partially Decoupled Equations

To help motivate some of the techniques that we will develop in the next section, let's look at an example in which only one of the equations depends on both dependent variables. Consider the following system of two differential equations

$$\begin{aligned} x' &= x + 2y \\ y' &= -y \end{aligned}$$

Although the first equation involves both $x$ and $y$, the second equation is a simple first-order equation involving only the variable $y$ and so can easily see that the general solution is $y(t) = C_1 e^{-t}$. Now that we know what $y(t)$ is, we can simply plug it back into the first equation, giving the nonhomogeneous linear equation

$$x' = x + 2C_1 e^{-t}$$

which we can solve to get $x(t) = C_2 e^t - C_1 e^{-t}$.

To express what we have just learned in terms of vector notation, we have found the solution vector

$$\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -C_1 e^{-t} + C_2 e^t \\ C_1 e^{-t} \end{pmatrix}. \tag{3.24}$$

It will be informative to rewrite this slightly. First, we will separate the vector into a sum of two vectors, one that depends on the constant $C_1$ and the other depending on $C_2$. Then, we factor out the exponential. This gives us the solution

$$\mathbf{u} = \begin{pmatrix} -C_1 e^{-t} + C_2 e^t \\ C_1 e^{-t} \end{pmatrix} = \begin{pmatrix} -C_1 e^{-t} \\ C_1 e^{-t} \end{pmatrix} + \begin{pmatrix} C_2 e^t \\ 0 \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.25}$$

Writing it this way will help us understand how the solution depends upon the constants and will also help us interpret the geometry of the solution. Actually, (3.25) represents an infinite *family* of solutions since each choice of the constants $C_1$ and $C_2$ gives us a different solution. Let's first consider the special case where $C_2 = 0$. Then we have the solutions

$$\mathbf{u} = C_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Geometrically, these solutions are a family of vectors that are all multiples of the vector $(-1, 1)$.[4] In other words, for all $C_1$ and all values of $t$, the solution $\mathbf{u}(t)$ lies on the straight line between the origin and the point $(-1, 1)$. We can express all of these solutions in terms of a single solution by defining

$$\mathbf{u_1} = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{3.26}$$

[4]To be consistent with the notation, we should really write the vector like this: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. However, this can make the typesetting look awkward. To get around this, we will often write what is known as the **transpose** of a vector. Most of the time it will be clear from context what we mean, but if we need to make it clear that we're using the transpose we can write it as $(-1, 1)^T$.

Similarly, if we consider the particular value $C_1 = 0$ and leave $C_2$ arbitrary then we have the solutions

$$\mathbf{u} = C_2 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which are always parallel to the vector $(1, 0)$. Again, we have a family of straight-line solutions, all of which are multiples of the particular solution

$$\mathbf{u_2} = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.27}$$

With the notation that we have introduced, the general solution (3.25) can be written in terms of the particular solutions $\mathbf{u_1}$ and $\mathbf{u_2}$:

$$\mathbf{u} = C_1 \mathbf{u_1} + C_2 \mathbf{u_2} \tag{3.28}$$

As was the case with first order equations, we may select a particular solution by supplying some additional information which we can use to determine the constants $C_1$ and $C_2$. Geometrically, this can be done by specifying a point in the $(x, y)$ plane that the solution passes through. For example, suppose we want to find the solution to the differential equation that passes through the point $(0, 1)$ at time $t = 0$. We then plug the values

$$x(0) = 0 \tag{3.29}$$

$$y(0) = 1 \tag{3.30}$$

into the solution and obtain

$$\mathbf{u} = \begin{pmatrix} -C_1 + C_2 \\ C_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We then solve these two equations for the constants and find $C_1 = 1$ and $C_2 = 1$ giving the particular solution

$$\mathbf{u} = \begin{pmatrix} -e^{-t} + e^t \\ e^{-t} \end{pmatrix}. \tag{3.31}$$

### Straight-Line Solutions

There is important interpretation of these exponential solutions that comes from looking at them geometrically. If we go to the phase plane and draw the solutions $\mathbf{u_1}$ and $\mathbf{u_2}$, we see that their trajectories lie on two (straight) lines through the origin. This tells us that if we start with any initial condition on either of these lines, we obtain a solution that remains on that line for all time. The direction field for this system, along with these straight line solutions is shown in Figure 3.5.

It turns out that even if we didn't already have the explicit solutions for this system, we can find these special straight-line solutions. The key geometric observation is that for straight-line solutions, the solution vectors are "lined-up" with the vector field in the sense that the vector field at a point $(x, y)$ on such a line must point in the same (or opposite) direction as the vector $\mathbf{v}$ that points

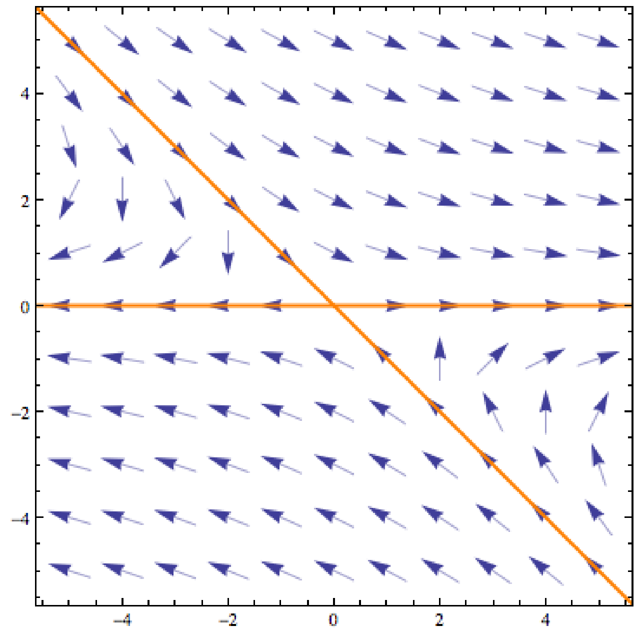



**Figure 3.5:** Direction field for the system $\mathbf{u}' = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \mathbf{u}$ with straight line solutions superimposed.

from the origin to $(x, y)$, or in other words, the vector field and $\mathbf{v}$ are parallel. Since the vector field at the point $(x, y)$ is given by $\mathbf{Av}$, this translates to the equation

$$\mathbf{Av} = \lambda \mathbf{v}.$$

for some number $\lambda$. If $\lambda > 0$ then the vector field points in the same direction as $\mathbf{v}$ (away from the origin) and therefore our straight-line solutions will diverge to infinity. On the other hand, if $\lambda < 0$ then the vector field points in the opposite direction (towards the origin) and the straight-line solutions converge to the origin.

### 3.3.3 Initial Value Problem

Recall that for the example in the previous section, we were able to express our solution as

$$\mathbf{u} = C_1 \mathbf{u_1} + C_2 \mathbf{u_2}. \tag{3.32}$$

The form of this general solution suggests a strategy for solving the general initial value problem. The plan is as follows:

1. First, we (somehow) find two specific solutions $\mathbf{u_1}$ and $\mathbf{u_2}$. These solutions must be *independent* in a sense that we will make precise later on. For the 2 dimensional case, we can just

take this to mean that they should not be multiples of each other. We will return later on to the question of exactly how to find $\mathbf{u_1}$ and $\mathbf{u_2}$, but for the moment we will just assume that we have found them. We will also use the term *fundamental solutions* to refer to $\mathbf{u_1}$ and $\mathbf{u_2}$.

2. Chances are that neither of the particular solutions found in step 1 satisfy the given initial conditions. Therefore we will try to construct the desired solution out of these two. We will show that if $\mathbf{u_1}$ and $\mathbf{u_2}$ are any two solutions to the differential equation $\mathbf{u}' = \mathbf{A}\mathbf{u}$ then so is the linear combination

$$\mathbf{u} = c_1\mathbf{u_1} + c_2\mathbf{u_2}. \tag{3.33}$$

3. Finally, we will show that it is possible to choose values for $c_1$ and $c_2$ such that the linear combination (3.33) satisfies any initial condition. This means that (3.33) represents the *general solution* to the differential equation. In verifying this step, we will make a precise definition of the term *independent*.

### Principle of Linearity

Here we focus on step (2) in our strategy. Let us suppose that we have somehow found two solutions $\mathbf{u_1}$ and $\mathbf{u_2}$ to the linear system (3.17). We want to show that the linear combination defined by (3.33) is also a solution. Plugging $\mathbf{u} = c_1\mathbf{u_1} + c_2\mathbf{u_2}$ into the left hand side, we have

$$\begin{aligned}
\mathbf{u}' = (c_1\mathbf{u_1} + c_2\mathbf{u_2})' &= c_1\mathbf{u_1}' + c_2\mathbf{u_2}' \\
&= c_1\mathbf{A}\mathbf{u_1} + c_2\mathbf{A}\mathbf{u_2} \\
&= \mathbf{A}(c_1\mathbf{u_1} + c_2\mathbf{u_2}) \\
&= \mathbf{A}\mathbf{u}
\end{aligned}$$

which verifies that $\mathbf{u}$ is a solution. This property of our differential equation is summarized as follows:

> Suppose $\mathbf{u}' = \mathbf{A}\mathbf{u}$ is a linear system of differential equations.
>
> 1. If $\mathbf{u}(t)$ is a solution and $k$ is any constant, then $k\mathbf{u}(t)$ is also a solution.
> 2. If $\mathbf{u_1}(t)$ and $\mathbf{u_2}(t)$ are two solutions then $\mathbf{u_1}(t) + \mathbf{u_2}(t)$ is also a solution.

Essentially, this property is what defines the term *linear*, in the sense that we call a system of equations *linear* precisely when these properties are satisfied.

### Satisfying the Initial Conditions

As we have just seen, if we have two solutions $\mathbf{u_1}$ and $\mathbf{u_2}$, the linearity principle gives us a way of constructing new solutions – in fact an infinite number of new solutions since the constants $c_1$ and $c_2$ are arbitrary. We now need to verify that we can choose the constants in such a way that the linear combination satisfies any given initial condition.

Plugging the general initial condition $\mathbf{u}(t_0) = \mathbf{u_0} = (x_0, y_0)$ into our solution, we have

$$c_1\mathbf{u_1}(t_0) + c_2\mathbf{u_2}(t_0) = \mathbf{u_0} \tag{3.34}$$

or in component form,

$$\begin{aligned} c_1x_1(t_0) + c_2x_2(t_0) &= x_0 \\ c_1y_1(t_0) + c_2y_2(t_0) &= y_0. \end{aligned}$$

In order to simplify the notation, we will drop the reference to $t_0$ at this point and remember that we are evaluating the functions at this particular value of the independent variable. We now have two equations for the two unknown constants and we can solve these simultaneous equations in the usual way. For instance, if we solve for $c_2$ in the first equation:

$$c_2 = \frac{x_0 - c_1x_1}{x_2} \tag{3.35}$$

then we can plug it into the second equation:

$$c_1y_1 + \left(\frac{x_0 - c_1x_1}{x_2}\right)y_2 = y_0$$

and then solve this for $c_1$:

$$c_1 = \frac{y_0x_2 - x_0y_2}{y_1x_2 - x_1y_2}. \tag{3.36}$$

Once that we have $c_1$, we can plug it back into (3.35) and solve:

$$\begin{aligned} c_2 &= \frac{x_0}{x_2} - \frac{c_1x_1}{x_2} \\ &= \frac{x_0}{x_2} - \frac{x_1}{x_2}\left(\frac{y_0x_2 - x_0y_2}{y_1x_2 - x_1y_2}\right) \\ &= \frac{x_0(y_1x_2 - x_1y_2) - x_1(y_0x_2 - x_0y_2)}{x_2(y_1x_2 - x_1y_2)} \\ &= \frac{x_0y_1x_2 - y_0x_1x_2}{x_2(y_1x_2 - x_1y_2)} \\ &= \frac{x_0y_1 - y_0x_1}{y_1x_2 - x_1y_2}. \end{aligned} \tag{3.37}$$

At this point, we have successfully found values of the constants $c_1$ and $c_2$ that give us a solution to the differential equation that also satisfies the initial conditions. Notice also that the denominator in both of the expressions (3.36) and (3.37) is the same. This tells us that we can solve for $c_1$ and $c_2$ provided that this expression $(x_1 y_2 - y_1 x_2)$ is not zero. This essentially serves as a condition on the solutions $\mathbf{u_1}$ and $\mathbf{u_2}$ that we can use as a definition of the term "independent" introduced earlier.

To make this a bit more precise, we make the following definitions. Given two solutions $\mathbf{u_1}$ and $\mathbf{u_2}$ to the linear system $\mathbf{u}' = \mathbf{A}\mathbf{u}$, the **Wronskian** is defined by

$$W(t) = x_1(t)y_2(t) - y_1(t)x_2(t). \tag{3.38}$$

Note that this quantity can also be expressed as the determinant of a matrix whose columns are given by the two solution vectors:

$$W(t) = \det\begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix}.$$

This observation may be generalized to define the Wronskian for higher-dimensional systems.

We say that two solutions $\mathbf{u_1}$ and $\mathbf{u_2}$ are **independent** (and therefore the initial value can be satisfied) if their Wronskian is nonzero at $t_0$, i.e. $W(t_0) \neq 0$. We can also refer to a pair of independent solutions as **fundamental solutions**.

Before concluding this section, we need to address one subtle point. In solving for $c_1$ and $c_2$ above, we only addressed the need for $W$ to be nonzero. However, we also divided by $x_2$, so it appears that we may need to impose a condition that $x_2$ is not zero. However this is not actually the case. If $x_2(t_0)$ were zero, then we would have solved for $c_1$ in the first equation and then used that to find $c_2$ using the second equation.

## 3.4 Eigenvalues and Eigenvectors

At this point, we have a general strategy for solving systems of linear equations and the initial value problem, provided that we are able to find two particular solutions. We now return to the question of how to find these first two solutions. To help motivate our method, we go back to our example of the partially decoupled system

$$\mathbf{u}' = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\mathbf{u} \tag{3.39}$$

and recall that we found the general solution

$$\mathbf{u} = c_1\mathbf{u_1} + c_2\mathbf{u_2} \tag{3.40}$$

where $\mathbf{u_1}$ and $\mathbf{u_2}$ are the particular solutions

$$\mathbf{u_1} = e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\mathbf{u_2} = e^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We observe that both $\mathbf{u_1}$ and $\mathbf{u_2}$ can be expressed as an exponential function times a constant vector. This suggests that in general, we can look for solutions that have the form

$$\mathbf{u}(t) = e^{\lambda t}\mathbf{v} \tag{3.41}$$

where $\lambda$ is a real number and $\mathbf{v}$ is a constant vector. Plugging this into our linear differential equation gives

$$\lambda e^{\lambda t}\mathbf{v} = \mathbf{A}e^{\lambda t}\mathbf{v}$$

and dividing by the exponential and rearranging gives

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}.$$

If we can find a number $\lambda$ and $\mathbf{v}$ that satisfy this equation, then we will have a solution of the form (3.41) and if we can find more than one such solution, we can use the linearity principle to form the general solution.

### 3.4.1 The Eigenvalue Problem

The vector equation that we have just seen arises in many contexts in mathematics and finding solutions of it is known as the **eigenvalue problem**.

**Definition.** Given a matrix $\mathbf{A}$, the number $\lambda$ is an **eigenvalue** of $\mathbf{A}$ if there exists a nonzero vector $\mathbf{v}$, called an **eigenvector**, such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}. \tag{3.42}$$

Our goal is to find values of $\lambda$ for which (3.42) has a nontrivial (i.e. nonzero) solution. We proceed by rewriting (3.42) as

$$(\mathbf{A} - \lambda\mathbf{I})v = 0 \tag{3.43}$$

where $\mathbf{I}$ is the identity matrix of the same size as the coefficient matrix $\mathbf{A}$. From our discussion above on equilibrium solutions, we know that the only way that this vector equation can have a nontrivial solution is if the matrix $\mathbf{A} - \lambda\mathbf{I}$ is singular, i.e. if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \tag{3.44}$$

This equation is known as the **characteristic equation**. From the definition of the determinant for a $2 \times 2$ matrix, the characteristic equation becomes

$$(a - \lambda)(d - \lambda) - bc = 0$$

which reduces to the quadratic equation

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0 \tag{3.45}$$

which is called the **characteristic polynomial**. The roots of this polynomial are the eigenvalues and once we find the eigenvalues, we can plug them back into (3.42) to find the eigenvectors.

#### Trace and Determinant

Because it will be useful later on, here is an alternative way of writing the characteristic polynomial. Given a square matrix the sum of its diagonal elements is defined as the **trace** of a matrix. We already have been introduced to the determinant a $2 \times 2$ matrix. Using these quantites, the characteristic polynomial above can be expressed as

$$\lambda^2 - \tau\lambda + \Delta = 0. \tag{3.46}$$

where $\tau = a + d$ is the trace and $\Delta = ad - bc$ is the determinant of the matrix $\mathbf{A}$. The solutions to this quadratic can then be written as

$$\lambda = \frac{\tau}{2} \pm \frac{1}{2}\sqrt{\tau^2 - 4\Delta}. \tag{3.47}$$

Depending on the value of $\tau^2 - 4\Delta$, there are three possibilities:

1. Two real and distinct eigenvalues if $\tau^2 - 4\Delta > 0$.
2. One (repeated) real eigenvalue if $\tau^2 - 4\Delta = 0$.
3. No real eigenvalues if $\tau^2 - 4\Delta < 0$.

We will now consider each of these three cases.

### 3.4.2 Two Real and Distinct Eigenvalues

We first consider the case where the characteristic polynomial has two real (and distinct) roots, which we denote $\lambda_1$ and $\lambda_2$. These are the eigenvalues of the system and for each eigenvalue, we can find a corresponding eigenvector which we denote $\mathbf{v_1}$ and $\mathbf{v_2}$.

If we define the functions $\mathbf{u_1} = e^{\lambda_1 t}\mathbf{v_1}$ and $\mathbf{u_2} = e^{\lambda_2 t}\mathbf{v_2}$ then as we showed above, these are two solutions to the system of differential equations. The general solution is then constructed by forming the linear combination of these two solutions:

$$\mathbf{u} = c_1 e^{\lambda_1 t}\mathbf{v_1} + c_2 e^{\lambda_2 t}\mathbf{v_2}. \tag{3.48}$$

Notice that the long-time behavior of the solutions is completely determined by the eigenvalues in the following sense. If both eigenvalues are negative, then the two terms in the solution both contain decaying exponentials and all solutions approach the origin as $t \to \infty$ regardless of the values of $c_1$ and $c_2$. Another way to say this is that the origin is an asymptotically stable equilibrium for any initial condition.

On the other hand, if one eigenvalue (say $\lambda_1$) is negative while the other ($\lambda_2$) is positive then the term with the decaying exponential will become negligible and therefore as time gets large the solution basically looks like the remaining term

$$\mathbf{u} \approx c_2 e^{\lambda_2 t}\mathbf{v_2}. \tag{3.49}$$

Not only do we conclude that the solution diverges to infinity, but that it does so *in the direction of* $\mathbf{v_2}$.

If both eigenvalues are positive, then the key is to compare the magnitude of the eigenvalues. If $\lambda_1 < \lambda_2$ then for large values of $t$, the exponential $e^{\lambda_2 t}$ will be much larger than $e^{\lambda_1 t}$. This means that the solution is dominated by the second term in the general solution

$$\mathbf{u} \approx c_2 e^{\lambda_2 t}\mathbf{v_2} \tag{3.50}$$

and again, we can say that the solution diverges in the direction of $\mathbf{v_2}$. The exception to these last two cases comes if we consider an initial condition that lies on the straight-line solution corresponding to $\mathbf{v_1}$, in which case $c_2 = 0$. Then the solution follows $v_1$, either converging to the origin if $\lambda_1 < 0$, or diverging to infinity in the direction of $\mathbf{v_1}$ if $\lambda_1$ is positive.

**Example 3.4.1.** Consider the system of differential equations

$$\begin{aligned} x' &= x + 2y \\ y' &= -y. \end{aligned}$$

In vector form, we have the equation

$$\mathbf{u}' = \mathbf{A}\mathbf{u}$$

where $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ is the solution vector and $\mathbf{A}$ is the coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

The characteristic equation is

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) = \det\left(\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) &= \det\begin{pmatrix} 1-\lambda & 2 \\ 0 & -1-\lambda \end{pmatrix} \\ &= (1-\lambda)(-1-\lambda) - 0 \cdot 2 \\ &= \lambda^2 - 1 = 0 \end{aligned}$$

and solving for $\lambda$, we have the two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$.

For each of these eigenvalues, we can then plug a value of $\lambda$ into equation (3.42) (or equivalently to (3.43)) to find the corresponding eigenvector. First for $\lambda_1 = 1$ we have

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \mathbf{v} = \mathbf{v}$$

which written in component form says

$$\begin{aligned} v_1 + 2v_2 &= v_1 \\ -v_2 &= v_2. \end{aligned}$$

(Here, $v_1$ and $v_2$ are the components of $\mathbf{v}$.) The second equation tells us that $v_2 = 0$ and if we plug this into the first equation we get $v_1 = v_1$ which is true no matter the value of $v_1$. This means that the first eigenvector is any vector of the form

$$\mathbf{v} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

To obtain a particular eigenvector, we are free to choose any value of $v_1$ (except zero, since $\mathbf{v}$ must be nonzero). To keep things simple, we will choose $v_1 = 1$, giving an eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Now that we have a solution to the eigenvalue problem, (3.41) tells us that

$$\mathbf{u}_1(t) = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is a solution to the differential equation.

Similarly, the second eigenvector is found by plugging the second eigenvalue into (3.42) and solving for $\mathbf{v}$. Since $\lambda_2 = -1$, we get

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \mathbf{v} = -\mathbf{v}$$

or

$$\begin{aligned} v_1 + 2v_2 &= -v_1 \\ -v_2 &= -v_2. \end{aligned}$$

The second equation gives us no information about $v_2$ while the first equation can be rewritten as $2v_1 = -2v_2$. Thus the second eigenvector has the form

$$\mathbf{v} = \begin{pmatrix} -v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

For the sake of simplicity, we choose $v_2 = 1$ and obtain the eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. By (3.41) we now have a second solution to our differential equation:

$$\mathbf{u}_2(t) = e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The general solution to the system can then be represented by the linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$:

$$\mathbf{u}(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

### 3.4.3 Repeated Real Eigenvalues

If the roots of the characteristic polynomial are repeated this only gives us one eigenvalue/eigenvector pair $(\lambda, \mathbf{v})$ and therefore only one solution

$$\mathbf{u}_1 = e^{\lambda t}\mathbf{v}.$$

In order to construct the general solution, however, we need a second (independent) solution. To help motivate the general procedure for the case of repeated eigenvalues, we will construct an example. As in the case above, we will use a partially decoupled system of differential equations that feature a repeated root to the characteristic polynomial. Referring back to the solution of the general characteristic polynomial, we will have repeated roots if the discriminant is zero:

$$(a - d)^2 + 4bc = 0.$$

Furthermore, to get a partially decoupled system, we can set $c = 0$, which then implies $a = d$. This suggests the following example.

**Example 3.4.2.** Solve the system of equations

$$\begin{aligned} x' &= x + y \\ y' &= y. \end{aligned}$$

From the second equation, we have $y(t) = c_1 e^t$. Then the first equation becomes

$$x' = x + c_1 e^t$$

which has the solution $x(t) = c_2 e^t + c_1 t e^t$. If we express the solution to the system in vector form, we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \left[ t e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] + c_2 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.51}$$

In this example, the characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$ and so we have a repeated eigenvalue $\lambda = 1$. As we can easily verify, the corresponding eigenvector is $\mathbf{v}^T = (1, 0)$ which gives us the solution

$$\mathbf{u}_1 = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We can also see from (3.51) that a second solution is given by

$$\mathbf{u}_2 = t e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which we can express as

$$\mathbf{u}_2 = t e^t \mathbf{v} + e^t \mathbf{w}$$

where $\mathbf{v}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$ and $\mathbf{w}$ is a second constant vector which we will refer to as a **generalized eigenvector**.

In general, we will solve the case of repeated eigenvalues by first finding a solution given by the eigen-pair $(\lambda, \mathbf{v})$ as before and then looking for a second solution in the form

$$\mathbf{u} = t e^{\lambda t} \mathbf{v} + e^{\lambda t} \mathbf{w}. \tag{3.52}$$

Plugging this into the original system $\mathbf{u}' = \mathbf{A}\mathbf{u}$ we get

$$e^{\lambda t} \mathbf{v} + t\lambda e^{\lambda t} \mathbf{v} + \lambda e^{\lambda t} \mathbf{w} = \mathbf{A} t e^{\lambda t} \mathbf{v} + \mathbf{A} e^{\lambda t} \mathbf{w}.$$

Canceling the exponential factors and simplifying, we arrive at the equation

$$\mathbf{A}\mathbf{w} = \lambda \mathbf{w} + \mathbf{v}. \tag{3.53}$$

Since we know the eigenvector $\mathbf{v}$, this equation may be solved to find $\mathbf{w}$.

**Example 3.4.3.** Solve the system

$$\mathbf{u}' = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{u}.$$

First, we find the eigenvalues by solving $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$, or equivalently

$$\det \begin{pmatrix} 3 - \lambda & 1 \\ -1 & 1 - \lambda \end{pmatrix} = (3 - \lambda)(1 - \lambda) + 1 = \lambda^2 - 4\lambda + 4 = 0.$$

The roots are $\lambda_1 = \lambda_2 = 2$, so we have a repeated eigenvalue $\lambda = 2$. The corresponding eigenvector is found by solving

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which implies the condition $v_1 + v_2 = 0$. One possible choice for eigenvector is

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and so one solution to the system is

$$\mathbf{u}_1 = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Next, we use the eigenvector $\mathbf{v}$ to find a generalized eigenvector by solving (3.53):

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

which implies the condition $w_1 + w_2 = 1$. One possible solution is $\mathbf{w}^T = (1, 0)$ giving a second solution to the system:

$$\mathbf{u}_2 = te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Finally, we can express the general solution to the system by constructing the linear combination of the two solutions that we have found:

$$\mathbf{u} = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left(te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right).$$

### 3.4.4 Complex Eigenvalues

The third possibility is that there are no real roots of the characteristic equation. It turns out that we can make progress in this case (and eventually end up with real valued solutions) by allowing our eigenvalues to have complex values.

Suppose that we have an eigenvalue $\lambda = \alpha + i\beta$ where $\alpha$ and $\beta$ are both real and we let $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ be the corresponding eigenvector. Then proceeding as before, we can form the solution

$$\mathbf{u}_1 = e^{\lambda t}\mathbf{v} = e^{(\alpha+i\beta)t}(\mathbf{a} + i\mathbf{b}) = e^{\alpha t}e^{i\beta t}(\mathbf{a} + i\mathbf{b}). \tag{3.54}$$

In addition, we can easily verify that as long as the coefficient matrix **A** has only real valued entries, then the complex conjugate $\overline{\lambda} = \alpha - i\beta$ is a second eigenvalue with corresponding eigenvector $\overline{\mathbf{v}} = \mathbf{a} - i\mathbf{b}$ and therefore

$$\mathbf{u}_2 = e^{(\alpha - i\beta)t}(\mathbf{a} - i\mathbf{b}) = e^{\alpha t}e^{-i\beta t}(\mathbf{a} - i\mathbf{b}) \tag{3.55}$$

is a second solution.

In principle, we could stop here since we have found two independent solutions. However, if our goal is to obtain real-valued solutions, we need to rewrite our solutions in such a way to get rid of the complex values. The key is to rewrite the exponential $e^{i\beta t}$ using Euler's formula:

$$e^{i\beta t} = \cos(\beta t) + i\sin(\beta t).$$

If you have not seen this relation between complex exponentials on the one hand and sines and cosines on the other, it's a very useful way of dealing with complex exponents. A derivation of this identity can be found in Appendix A.

Using Euler's formula to rewrite our complex-valued solution (3.54), we have

$$\mathbf{u}_1 = e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)](\mathbf{a} + i\mathbf{b}).$$

Expanding and collecting real and imaginary parts we have

$$\mathbf{u}_1 = e^{\alpha t}[\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b}] + ie^{\alpha t}[\cos(\beta t)\mathbf{b} + \sin(\beta t)\mathbf{a}]. \tag{3.56}$$

Similarly, the second solution (3.55) may be rewritten as

$$\mathbf{u}_2 = e^{\alpha t}[\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b}] - ie^{\alpha t}[\cos(\beta t)\mathbf{b} + \sin(\beta t)\mathbf{a}].$$

Notice that this is the complex conjugate of (3.56).

We are now in a position to find real-valued solutions. Taking advantage of the linearity of the system, we form the linear combinations

$$\widetilde{\mathbf{u}}_1 = \frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2) = e^{\alpha t}[\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b}] \tag{3.57}$$

$$\widetilde{\mathbf{u}}_2 = \frac{1}{2i}(\mathbf{u}_1 - \mathbf{u}_2) = e^{\alpha t}[\cos(\beta t)\mathbf{b} + \sin(\beta t)\mathbf{a}] \tag{3.58}$$

to construct new *real-valued* solutions. From this point on, we will drop the tildes and simply refer to these as the two fundamental solutions with which the general solution may be constructed. It is also worth pointing out that as long as we remember that these two solutions simply come from taking the real and imaginary parts of the complex-valued eigenvector, it is not be necessary to memorize them.

**Example 3.4.4.** Solve the system
$$\mathbf{u}' = \begin{pmatrix} 1 & -2 \\ 4 & -3 \end{pmatrix}\mathbf{u}.$$
We begin by finding the eigenvalues by solving
$$\det\begin{pmatrix} 1-\lambda & -2 \\ 4 & -3-\lambda \end{pmatrix} = 0.$$
The resulting characteristic equation is the quadratic $\lambda^2 + 2\lambda + 5 = 0$ which has solutions
$$\lambda = -1 \pm 2i.$$
Considering the first eigenvalue (given by the + sign), we find the corresponding eigenvector by plugging back into the eigenvalue equation and solving
$$\begin{pmatrix} 1-(-1+2i) & -2 \\ 4 & -3-(-1+2i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
to obtain the equation $(2-2i)v_1 - 2v_2 = 0$. Simplifying, this tells us that $v_2 = (1-i)v_1$ and so choosing $v_1 = 1$, we get the eigenvector
$$\mathbf{v} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\mathbf{a}} + i\underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{\mathbf{b}}.$$
Note that we have separated the eigenvector into real and imaginary parts. At this point, we simply refer back to the general expressions given by (3.57 - 3.58) and plug in values for $\alpha, \beta$, $\mathbf{a}$, and $\mathbf{b}$. The first solution is
$$\mathbf{u}_1 = e^{-t}\left[\cos(2t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin(2t)\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] = e^{-t}\begin{pmatrix} \cos(2t) \\ \cos(2t)+\sin(2t) \end{pmatrix}$$
and the second is
$$\mathbf{u}_2 = e^{-t}\left[\cos(2t)\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin(2t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = e^{-t}\begin{pmatrix} \sin(2t) \\ \sin(2t)-\cos(2t) \end{pmatrix}.$$
Again, the general solution is given by forming the linear combination of these two solutions:
$$\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2.$$

## 3.5 Phase Planes

In the previous section, we saw how to find the solutions to a linear systems by solving the eigenvalue problem. We now want to investigate the behavior of these solutions by sketching their trajectories in the phase plane.

### 3.5.1 Linear Systems with Real Eigenvalues

As we have seen, if a system has a real-valued eigenvalue then there is at least one straight-line solution. These solutions are easy to sketch, all we need is the eigenvector to tell us the direction of the line and the eigenvalue to tell us if solutions converge to the origin or diverge to infinity. More subtle is the behavior of the other solutions formed by the linear combination of straight-line solutions. As we shall see, the crucial thing in determining the behavior of the solutions is the sign of the eigenvalues.

#### Saddle Nodes

We first consider the case where we have one positive eigenvalue ($\lambda_1 > 0$) and one negative eigenvalue ($\lambda_2 < 0$) with corresponding eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$. The general solution is then written

$$\mathbf{u} = k_1 e^{\lambda_1 t}\mathbf{v_1} + k_2 e^{\lambda_2 t}\mathbf{v_2}. \tag{3.59}$$

Now consider the behavior of this solution as $t$ gets very large. Since $\lambda_2 < 0$, the second exponential gets very small and unless $k_1 = 0$, we conclude that the solution is approximately equal to the first term alone

$$\mathbf{u} \approx k_1 e^{\lambda_1 t}\mathbf{v_1} \tag{3.60}$$

indicating that solutions will approach the straight-line solution which diverges to infinity. We can do the same kind of analysis in reverse, taking the limit as $t$ approaches negative infinity. In this case, it is the first exponential $e^{\lambda_1 t}$ that goes to zero and now the solution approaches infinity along the other straight-line solution.

Consider the example

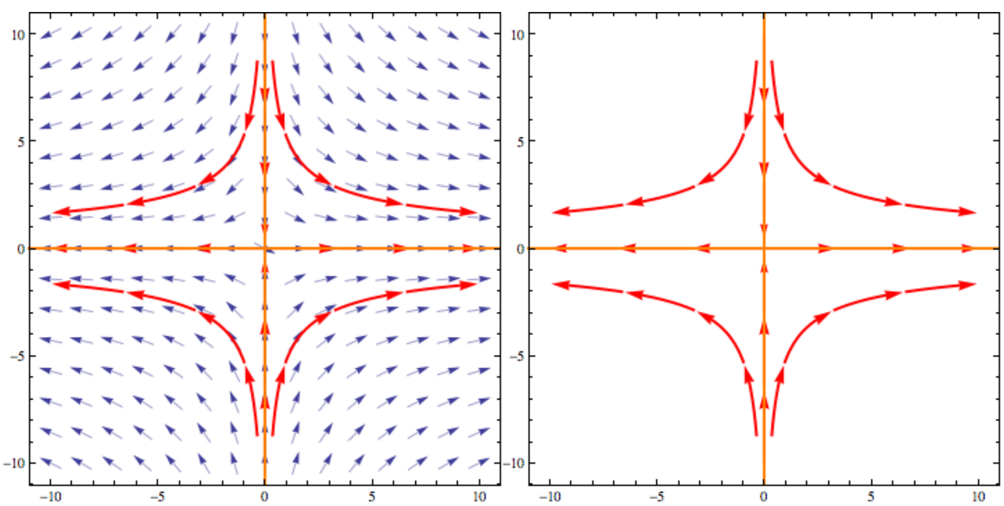
$$\mathbf{u}' = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{u}. \tag{3.61}$$

This is a particularly easy system to analyze because it is completely decoupled. The $x$-equation is $x' = 2x$ which has solution $x(t) = k_1 e^{2t}$. Similarly, the solution to the $y$-equation is $y(t) = k_2 e^{-t}$. We may therefore express the solution in the following vector form

$$\mathbf{u} = k_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.62}$$

If we consider an initial on the $y$-axis, so that $k_1 = 0$, we have solutions that converge to the origin along the $y$-axis. Similarly, if we take a point on the $x$-axis, making $k_2 = 0$, we see solutions that diverge along the $x$-axis. Any other solutions corresponding to starting at an initial point off the axes will have a behavior that combines these two behaviors: it will collapse onto the $x$-axis, while diverging in the $x$ direction.

Figure 3.6 shows two sketches of the phase plane for a saddle equilibrium at the origin: first with the direction field superimposed and then without the direction field. Notice that the general shape of the solution curves is determined by the straight-line solutions alone, whereas the direction field gives us a much more accurate picture of the solutions. In practice, we will typically use phase-plane sketches to help us understand the general behavior of the solutions. Therefore we will usually not show the direction field in the phase plane. Note also that in this example the straight-line solutions are aligned with the coordinate axes, but in other examples this may not be the case.




**(a)** With direction field superimposed.

**(b)** Without direction field.

**Figure 3.6:** Saddle node and solution curves.

### Stable Nodes

We now suppose that both eigenvalues are negative (but still distinct) $\lambda_1 < \lambda_2 < 0$. In this case, the general solution

$$\mathbf{u} = k_1 e^{\lambda_1 t}\mathbf{v_1} + k_2 e^{\lambda_2 t}\mathbf{v_2} \tag{3.63}$$

will converge to the origin for any initial condition. As with saddles, we sketch the phase plane by first finding the straight-line solutions corresponding to the eigenvectors. For initial conditions that do not lie on these straight-line solutions, we observe that since $\lambda_1 < \lambda_2 < 0$, the first term in (3.63) will go to zero faster than the second term meaning that as $t$ gets large, the solution is approximately given by the second term alone:

$$\mathbf{u} \approx k_2 e^{\lambda_2 t}\mathbf{v_2}. \tag{3.64}$$

Geometrically, this means that we expect that solution curves will quickly approach the straight-line determined by $\mathbf{v_2}$ and then curve towards the origin. Because all solutions tend towards the origin as $t$ approaches infinity, we say that the origin is **stable** More precisely, the origin in this case is **asymptotically** stable meaning that as $t$ goes to infinity, solutions actually converge to the origin. The term **sink** is also used to describe the origin in this case.

Consider the example

$$\mathbf{u}' = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{u}. \tag{3.65}$$

As before, we can solve this system easily because it is decoupled. In vector form, the solution is

$$\mathbf{u} = k_1 e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.66}$$

Since the exponential $e^{-2t}$ goes to zero faster than $e^{-t}$, the solutions that don't start on one of the axes will quickly be approximated by the second term which means that they will approach the origin in the direction of $(0, 1)^T$, i.e. along the $y$-axis. Figure 3.7 shows a sketch of the phase plane for a stable node. As mentioned above, we are typically only interested in the general behavior of the solution curves and therefore the solution curves should be viewed in terms of their general shape and behavior as $t \to \infty$ only.

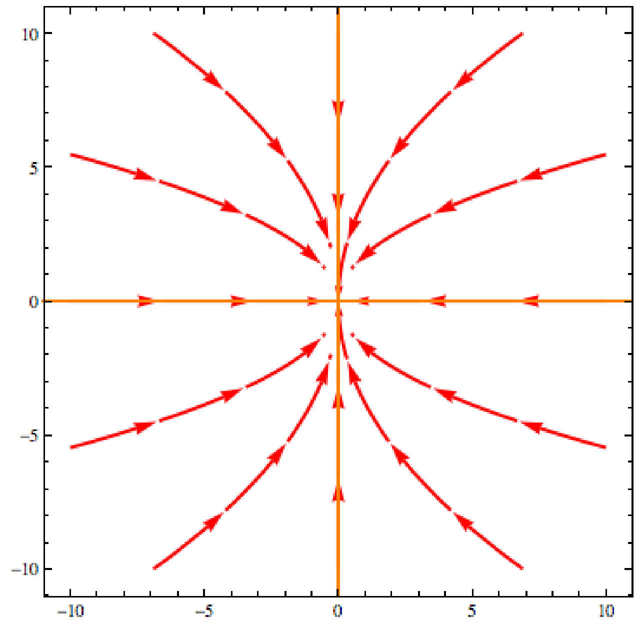



**Figure 3.7:** Stable node with several generic solution curves.

### Unstable Nodes

In the case where $0 < \lambda_1 < \lambda_2$, the phase plane is similar to the previous case, but with the solution trajectories heading in the opposite directions. In other words, all solutions will move away from the origin. For this reason, the origin is now described as **unstable**, or a source. If we consider the general solution

$$\mathbf{u} = k_1 e^{\lambda_1 t}\mathbf{v_1} + k_2 e^{\lambda_2 t}\mathbf{v_2} \tag{3.67}$$

we see that since $\lambda_1 < \lambda_2$, the solution will be dominated by the second term as $t$ gets large:

$$\mathbf{u} \approx k_2 e^{\lambda_2 t}\mathbf{v_2} \tag{3.68}$$

If we look at the example

$$\mathbf{u}' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{u}. \tag{3.69}$$

we have the solution

$$\mathbf{u} = k_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.70}$$

Since the exponential $e^{2t}$ goes to infinity faster than $e^t$, the solutions that don't start on one of the axes will quickly be dominated by the first term, meaning that as they diverge, they will tend to point in the direction of $(1, 0)^T$, i.e. along the $x$-axis. Figure 3.8 shows the phase plane for a typical unstable node.

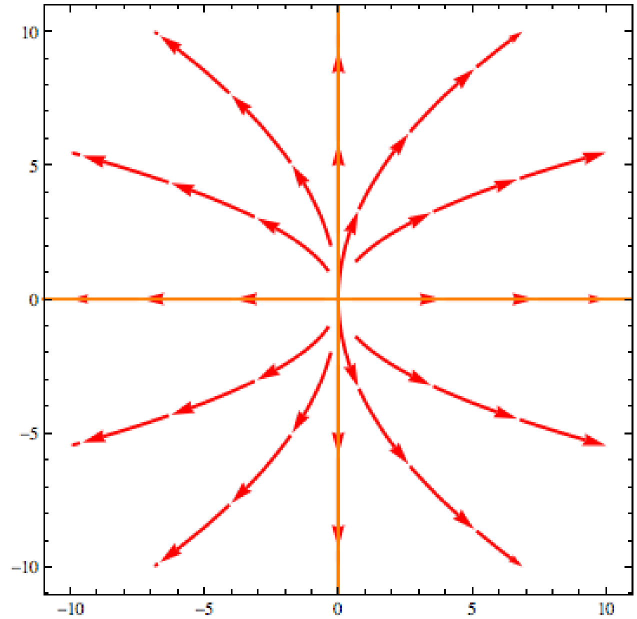



**Figure 3.8:** Unstable node with generic solution curves.

### 3.5.2 Complex Eigenvalues

In the case of real eigenvalues we saw that whether solutions converge to the equilibrium at the origin or diverge to infinity was determined by the sign of the eigenvalues. Although the solution curves are more difficult to draw precisely in the case of complex eigenvalues, their general behavior is fairly easy to describe. Suppose we have a linear system with a complex eigenvalues

$$\lambda = \alpha \pm i\beta$$

where $\beta \neq 0$. As we saw in Section 3.4.4, the real-valued solutions will be combinations of exponential and trigonometric functions:

$$\mathbf{u_1} = e^{\alpha t}[\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b}] \tag{3.71}$$

$$\mathbf{u_2} = e^{\alpha t}[\cos(\beta t)\mathbf{b} + \sin(\beta t)\mathbf{a}]. \tag{3.72}$$

If $\alpha > 0$ then the exponential in the solutions grows as $t \to \infty$, producing trajectories that spiral out to infinity. Conversely, if $\alpha < 0$ then the solutions go to zero as $t \to \infty$, producing trajectories that spiral into the origin. In the case $\alpha = 0$, the solutions are purely oscillatory and we expect periodic solutions. In this case, the period of oscillation is determined by combinations of $\sin\beta t$ and $\cos\beta t$. Since

$$\sin\beta(t + 2\pi/\beta) = \sin\beta t$$
$$\cos\beta(t + 2\pi/\beta) = \cos\beta t$$

these solutions have a period of $2\pi/\beta$. Whether the solutions spiral in a clockwise or counter-clockwise direction is best answered by looking at the direction field. In fact, it only requires a single nonzero direction vector to determine the direction of the spiral.

Consider the the linear system

$$\mathbf{u}' = \begin{pmatrix} 1 & -2 \\ 4 & -3 \end{pmatrix} \mathbf{u} \tag{3.73}$$

discussed in Section 3.4.4. As we saw, the eigenvalues were $\lambda = -1 \pm 2i$. Since the real part is negative, the trajectories will be spirals that approach the origin as $t \to \infty$. In this case, we say the origin is a **stable spiral**. To determine the direction of the spirals, we can look at the direction field at a particular point. For instance, if we plug the point $(1, 0)$ into the vector field (i.e. the right-hand side of the differential equation), we have

$$\begin{pmatrix} 1 & -2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

This vector points up and to the right, indicating that the solution curve is spiraling in a counter-clockwise direction. As with other sketches of the phase plane, we are primarily concerned with a qualitative picture and therefore we will not generally rely on the direction field to produce a precise sketch.

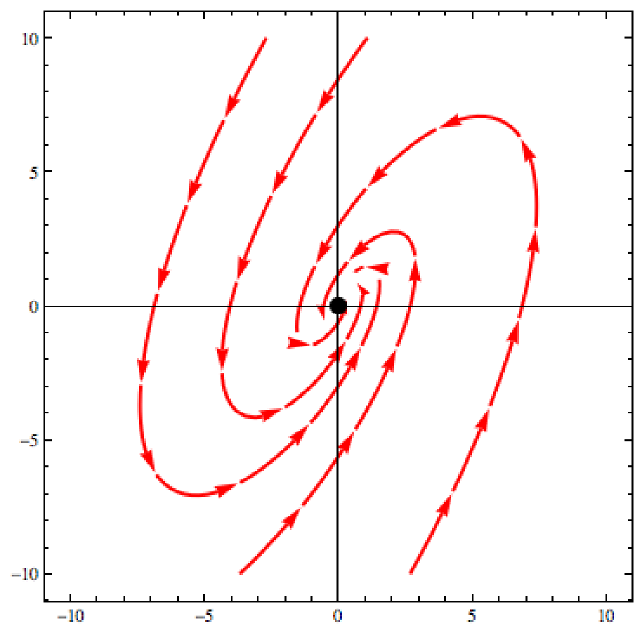



**Figure 3.9:** Stable spiral node with generic solution curves.

In the case where the real part of the eigenvalues is zero, there is neither exponential growth nor decay and the solutions will be periodic. The trajectories in the phase plane will be closed curves and we say that the origin is a **center**. Again, whether the trajectories travel clockwise or counterclockwise can be determined by looking at the direction field. Note that in the case of complex eigenvalues, we do not need the eigenvectors in order to sketch the phase plane.

Consider the linear system

$$\mathbf{u}' = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \mathbf{u}. \tag{3.74}$$

The eigenvalues for this system are $\lambda = \pm i$ and therefore we have periodic solutions. If we plug the point $(1, 0)$ into the vector field, we have

$$\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{3.75}$$

indicating that at the point $(1, 0)$, the solution curve is headed up and to the left, i.e. in a counterclockwise direction. Figure 3.10 shows a sketch of the phase plane, both with and without the direction field.

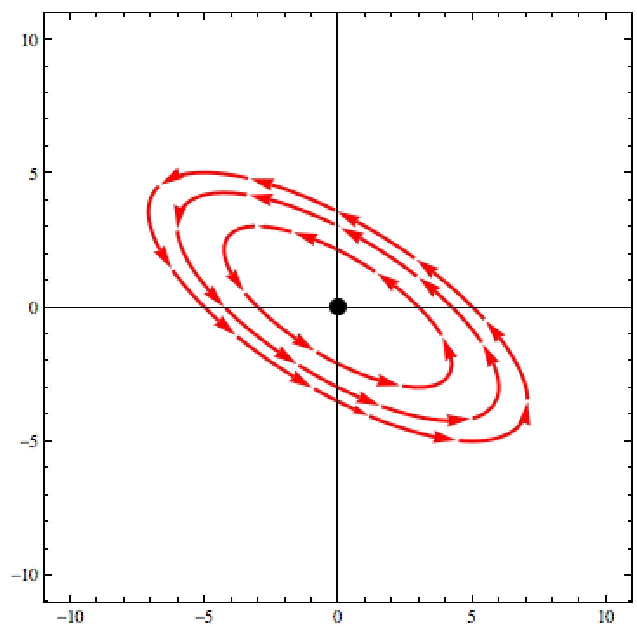



**Figure 3.10:** Center equilibrium with generic solution curves.

### Elliptical Trajectories

In the last section we stated that in the case of complex eigenvalues with zero real part, the solution curves in the phase plane are closed curves. If we graph the solutions parametrically, they look elliptical, but how can we determine that these curves are actually ellipses?

Here is one way to make this argument. First, we note that in order for the eigenvalues to be complex with zero real part, the coefficients must satisfy the conditions $d = -a$ and $-a^2 - bc > 0$ (see Section 3.4.1). The system can then be expressed

$$x' = ax + by$$
$$y' = cx - ay.$$

From these equations, we also have

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{cx - ay}{ax + by}$$

which we rewrite as

$$(cx - ay) - (ax + by)\frac{dy}{dx} = 0. \tag{3.76}$$

Next, we recall from analytic geometry that the general equation for an ellipse centered at the origin is

$$Ax^2 + Bxy + Cy^2 = D$$

where $4AC - B^2 > 0$. If we differentiate this quadratic with respect to $x$, we get

$$2Ax + By + (Bx + 2Cy)\,\frac{dy}{dx} = 0$$

which agrees with equation (3.76) if we set $A = c/2$, $B = -a$, and $C = -b/2$. Finally, we compute

$$4AC - B^2 = 4(c/2)(-b/2) - a^2 = -bc - a^2$$

which we know is a positive quantity from our condition that the eigenvalues are imaginary. This verifies that the trajectories are elliptical.

### 3.5.3 Repeated and Zero Eigenvalues

Systems that have either a repeated eigenvalue or zero eigenvalues are sometimes referred to as **degenerate** cases. It is helpful to think of these cases as boundary cases between the more common types of systems that we've already seen. For example, the case of repeated eigenvalues occurs when the discriminant in the solution to the characteristic polynomial is zero. This separates the case of complex valued eigenvalues from the case of two real eigenvalues.

#### Repeated Eigenvalue

We will illustrate this case with an example. Consider the system

$$\mathbf{u}' = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\mathbf{u}. \tag{3.77}$$

As we saw in section 3.4.3, there is a repeated eigenvalue $\lambda = 2$ with corresponding eigenvector $\mathbf{v} = (1, -1)$. Geometrically, this corresponds to a single straight-line solution that diverges from the origin. What about the other solutions? If we look at the direction field for this system, shown in Figure 3.11, we see trajectories that look like they're trying to spiral around the origin, however it appears as if the straight-line solution gets in the way. In the case of repeated eigenvalues, we say that the origin is an **improper node**.

This phase plane sketch is typical of the repeated eigenvalue case. In general, after finding the straight-line solution, we will simply sketch in solutions that spiral either into (if $\lambda < 0$) or away from (if $\lambda > 0$) the origin.

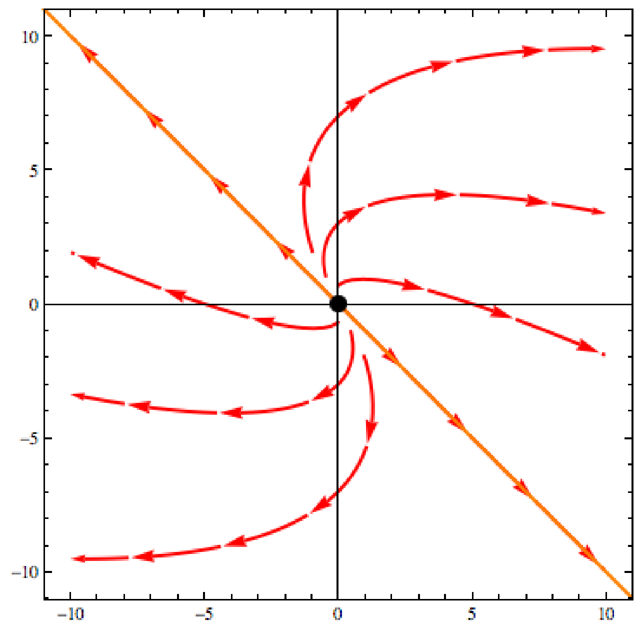



**Figure 3.11:** Degenerate node (repeated eigenvalues) with generic solution curves.

## Zero Eigenvalues

The final case to consider is when one or both of the eigenvalues is zero. We can think of this as a boundary case that lies between the cases of positive eigenvalue (and solutions that tend to infinity) and negative eigenvalues (and solutions that tend to the origin). Suppose we have a system with eigenvalues $\lambda_1 = 0$ and $\lambda_2 \neq 0$ and corresponding eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$. Then the general solution would be written

$$\mathbf{u} = k_1 e^{\lambda_1 t}\mathbf{v_1} + k_2 e^{\lambda_2 t}\mathbf{v_2} \tag{3.78}$$

but since $\lambda_1 = 0$, it simplifies to

$$\mathbf{u} = k_1\mathbf{v_1} + k_2 e^{\lambda_2 t}\mathbf{v_2}. \tag{3.79}$$

Now suppose that we consider an initial condition that makes $k_2 = 0$. In this case, the solution becomes $\mathbf{u} = k_1\mathbf{v_1}$ which is a constant. This means that every point that lies on the line determined by the eigenvector $\mathbf{v_2}$ is an equilibrium. If $\lambda_2 < 0$ then the second term in (3.78) goes to zero and the solutions tend to the equilibrium point $k_1\mathbf{v_1}$ along a line parallel to $\mathbf{v_2}$. Similarly, if $\lambda_2 > 0$ then the second term in (3.78) diverges and solutions move away from the equilibrium $k_1\mathbf{v_1}$ along a line parallel to $\mathbf{v_2}$.

**Example 3.5.1.** Consider the system

$$\mathbf{u}' = \begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix}\mathbf{u}. \tag{3.80}$$

Solving the characteristic equation, we find that this system has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$ with corresponding eigenvectors $(1, -3)$ and $(1, -1)$. Therefore the general solution is

$$\mathbf{u} = k_1\begin{pmatrix} 1 \\ -3 \end{pmatrix} + k_2 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{3.81}$$

Every point on the line $y = -x/3$, determined by the first eigenvector, is an equilibrium. The non-equilibrium solutions are straight lines that are parallel to the second eigenvector. Because the non-zero eigenvalue is positive, the solutions move away from the line of equilibria as shown in Figure 3.12.

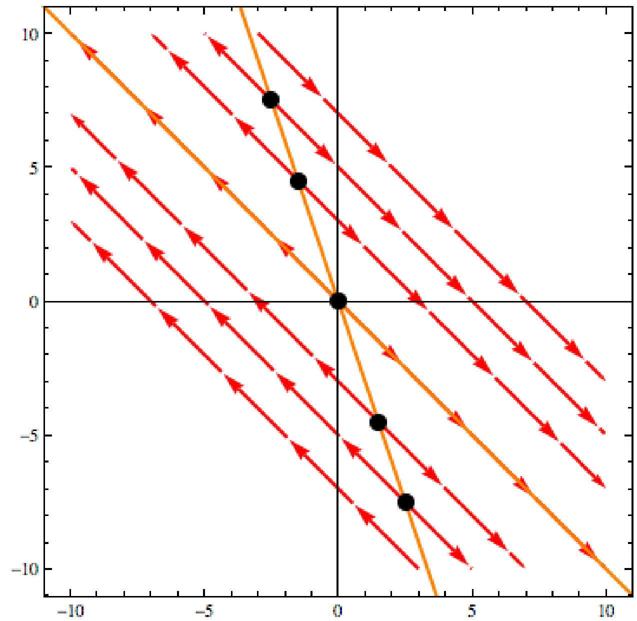



**Figure 3.12:** Solution trajectories for Example 3.5.1.

## 3.6 Stability and the Trace-Determinant Plane

In the previous sections, we saw a variety of solutions to different linear systems and as we saw, the behavior of the solutions (or equivalently the phase plane sketch) depends on the eigenvalues and whether they are positive, negative, complex valued, etc. This correspondence is summarized in the following table.

**Table 3.1:** Summary of behavior of linear homogeneous two dimensional systems.

| Eigenvalues | Behavior |
|---|---|
| distinct real and positive | source / unstable node |
| distinct real and negative | sink / stable node |
| real and opposite signs | saddle |
| purely imaginary | center point / ellipses |
| complex with positive real part | spiral source |
| complex with negative real part | spiral sink |
| repeated | improper node |
| zeros | degenerate |

Recall that in Section 3.4.1 we saw that the characteristic polynomial, and therefore the eigenvalues can be expressed in terms of the trace and determinant of the matrix **A**. In particular, the eigenvalues can be written

$$\lambda = \frac{\tau}{2} \pm \frac{1}{2}\sqrt{\tau^2 - 4\Delta} \tag{3.82}$$

where $\tau = a + d$ is the trace of the matrix **A** and $\Delta = ad - bc$ is the determinant of **A**. Expressing the eigenvalues in terms of the trace and determinant of **A** suggests that we can also describe the behavior of the solutions in terms of these matrix quantities.

Examining (3.82) we see that the key quantity is the discriminant $\tau^2 - 4\Delta$; if it is positive, we have nodes and if it is negative, we have spirals. Graphing the curve defined by the equation $\tau^2 - 4\Delta = 0$, we see ( Figure 3.13) a parabola that separates the $\tau$-$\Delta$ plane into two regions: if $\tau^2 - 4\Delta > 0$ we have two real eigenvalues while if $\tau^2 - 4\Delta < 0$ we have complex eigenvalues.

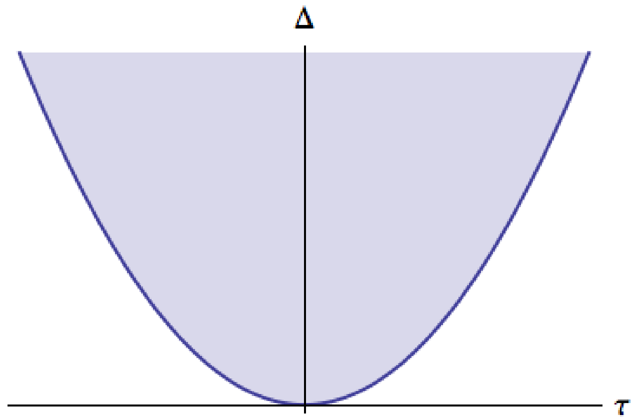



**Figure 3.13:** The trace-determinant plane is separated into two regions by the parabola $\Delta = \tau^2/4$.

In the case where $\tau^2 - 4\Delta < 0$, the complex valued eigenvalues have real part $\tau/2$. This implies that if $\tau < 0$, the spiral is stable, while if $\tau > 0$ the spiral is unstable. In the case $\tau = 0$, we have a center.

In the case where $\tau^2 - 4\Delta > 0$ we have two real eigenvalues and therefore either a saddle, stable or unstable node. If $\tau > 0$, then the eigenvalue

$$\lambda = \frac{\tau}{2} + \frac{1}{2}\sqrt{\tau^2 - 4\Delta}$$

is the sum of two positive terms and therefore is positive. Now consider the other eigenvalue

$$\lambda = \frac{\tau}{2} - \frac{1}{2}\sqrt{\tau^2 - 4\Delta}.$$

In the case where $\Delta = 0$, this eigenvalue is 0, so our system has one positive and one zero eigenvalue. If $\Delta > 0$, then

$$\tau^2 - 4\Delta < \tau^2.$$

Since we are considering the case where $\tau > 0$ we have

$$\sqrt{\tau^2 - 4\Delta} < \tau$$

and therefore

$$\frac{\tau}{2} - \frac{1}{2}\sqrt{\tau^2 - 4\Delta} > 0.$$

This means that both eigenvalues are positive and the origin is an unstable node.

On the other hand, suppose that $\tau > 0$, but $\Delta < 0$. Then

$$\tau^2 - 4\Delta > \tau^2$$

so that

$$\sqrt{\tau^2 - 4\Delta} > \tau$$

and therefore

$$\frac{\tau}{2} - \frac{1}{2}\sqrt{\tau^2 - 4\Delta} < 0.$$

In this case, we have one positive and one negative eigenvalue and the origin is a saddle. A similar analysis shows that if $\tau < 0$ and $\tau^2 - 4\Delta > 0$ then we have a stable node if $\Delta > 0$, a saddle node if $\Delta < 0$, and a line of equilibria (from a zero eigenvalue) if $\Delta = 0$. The different cases are summarized in Figure 3.14.

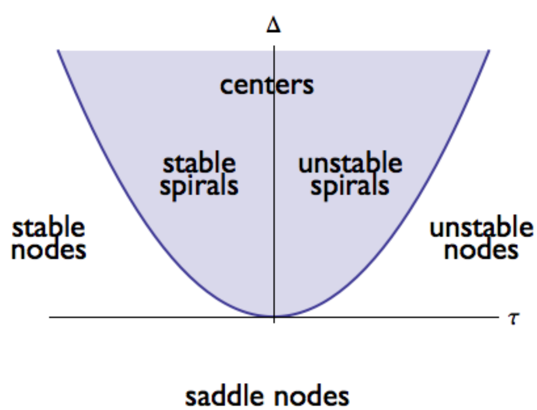



**Figure 3.14:** A summary of the different cases in the trace-determinant plane.

## 3.7 Nonhomogeneous Systems

Just as we did in the case of first order equations in Chapter Two, we can consider nonhomogeneous systems

$$\mathbf{u}' = \mathbf{A}\mathbf{u} + \mathbf{f}(t) \tag{3.83}$$

where

$$\mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \tag{3.84}$$

is a given vector function. As before, the coefficient matrix $\mathbf{A}$ is constant. In many examples, the nonhomogeneous term can be interpreted as some kind of applied force.

### 3.7.1 Undetermined Coefficients

For cases where the nonhomogeneous term $\mathbf{f}$ is relatively simple, we can often use the method of undetermined coefficients. As in Chapter 2, the idea is to make an educated guess as to the form of a particular solution to (3.83) based on the form of $\mathbf{f}$. In particular, if the vector components of $\mathbf{f}$ are comprised of sums or products of exponential functions, polynomials, or sines and cosines, then we can look for a particular solution $\mathbf{u_p}$ with a similar form. The rules for guessing $\mathbf{u_p}$ work precisely the way they did for single equations; however, if a term appears in one component of $\mathbf{f}(t)$, then the guess generally will have that term in each component. This means that the calculations can quickly become rather lengthy.

**Example 3.7.1.** Find a particular solution for the system

$$\mathbf{u}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{u} + \begin{pmatrix} 2t \\ e^t \end{pmatrix}. \tag{3.85}$$

For this forcing term, we guess a particular solution of the form

$$\mathbf{u_p} = \begin{pmatrix} a_1 t + b_1 + c_1 e^t \\ a_2 t + b_2 + c_2 e^t \end{pmatrix}.$$

Plugging this into the differential equation gives

$$\begin{pmatrix} a_1 + c_1 e^t \\ a_2 + c_2 e^t \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} a_1 t + b_1 + c_1 e^t \\ a_2 t + b_2 + c_2 e^t \end{pmatrix} + \begin{pmatrix} 2t \\ e^t \end{pmatrix}$$

and simplifying leads to the equations

$$\begin{aligned} a_1 + c_1 e^t &= -2a_1 t - 2b_1 - 2c_1 e^t + a_2 t + b_2 + c_2 e^t + 2t \\ a_2 + c_2 e^t &= a_1 t + b_1 + c_1 e^t - 2a_2 t - 2b_2 - 2c_2 e^t + e^t. \end{aligned}$$

Comparing the coefficients gives the simultaneous equations

$$
\begin{array}{rl}
t \text{ terms in first eqn:} & 0 = -2a_1 + a_2 + 2 \\
\text{constant terms in first eqn:} & a_1 = -2b_1 + b_2 \\
\text{exponential terms in first eqn:} & c_1 = -2c_1 + c_2 \\
t \text{ terms in second eqn:} & 0 = a_1 - 2a_2 \\
\text{constant terms in second eqn:} & a_2 = b_1 - 2b_2 \\
\text{exponential terms in second eqn:} & c_2 = c_1 - 2c_2 + 1.
\end{array}
$$

This system of equations can be solved to find

$$a_1 = 4/3 \quad a_2 = 2/3 \quad b_1 = -10/9 \quad b_2 = -8/9 \quad c_1 = 1/8 \quad c_2 = 3/8.$$

which gives the particular solution

$$\mathbf{u_p} = \frac{1}{9}\begin{pmatrix} 12t - 10 \\ 6t - 8 \end{pmatrix} + \frac{1}{8}e^t\begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

### 3.7.2 Variation of Parameters

In practice, we will seldom see any systems for which the method of undetermined coefficients will not work, but for the sake of completeness, we outline another method that does not require any guessing. This is the method of variation of parameters and it is based on the same principle that we saw when solving first-order nonhomogeneous equations.

We first define a **fundamental matrix** $\mathbf{\Phi}(t)$ to be a $2 \times 2$ matrix whose columns are two independent solutions of the associated homogeneous system $\mathbf{u} = \mathbf{A}\mathbf{u}$. As the reader can verify, the matrix $\mathbf{\Phi}$ satisfies the equation $\mathbf{\Phi}'(t) = \mathbf{A}\mathbf{\Phi}$ and consequently the general solution to the homogeneous equation can be expressed in the form

$$\mathbf{u_h}(t) = \mathbf{\Phi}\,\mathbf{c}$$

where $\mathbf{c}$ is a vector whose components are two arbitrary constants. Following the method introduced in Chapter 2, we will look for a solution to (3.83) of the form

$$\mathbf{u}(t) = \mathbf{\Phi}(t)\mathbf{c}(t)$$

where now $\mathbf{c}(t)$ is a vector whose components are two unknown *functions*. Differentiating this solution, we have

$$\begin{aligned}\mathbf{u}'(t) = \mathbf{\Phi}'(t)\mathbf{c}(t) + \mathbf{\Phi}(t)\mathbf{c}'(t) &= \mathbf{A}\mathbf{\Phi}\mathbf{c} + \mathbf{\Phi}\mathbf{c}' \\ &= \mathbf{A}\mathbf{u} + \mathbf{\Phi}\mathbf{c}'\end{aligned}$$

which combined with the fact that $\mathbf{u}$ is presumed to be a solution to (3.83) means that

$$\mathbf{f}(t) = \mathbf{\Phi}\mathbf{c}'.$$

At this point, the goal is to solve this system of equations for the vector $\mathbf{c}'$. For a general matrix equation, we would write

$$\mathbf{c}' = \mathbf{\Phi}^{-1}\mathbf{f}.$$

We know that the inverse of the fundamental matrix exists because by definition its columns are independent and therefore its determinant is nonzero. Finally, integrating gives

$$\mathbf{c}(t) = \int \mathbf{\Phi}^{-1}(t)\mathbf{f}(t)\,dt + \mathbf{k}.$$

where $\mathbf{k}$ is an arbitrary constant vector. We can then express the general solution to the nonhomogeneous system (3.83) by

$$\mathbf{u}(t) = \mathbf{\Phi}(t)\mathbf{k} + \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{f}(t)\,dt. \tag{3.86}$$

As was the case in Chapter 2, we now have a general formula for the solution and need only to plug in a fundamental matrix $\mathbf{\Phi}$ and the forcing $\mathbf{f}$ and then compute the integrals.

In the case where our system is two dimensional and the fundamental matrix is given by

$$\mathbf{\Phi} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix},$$

the inverse of this matrix is given by

$$\mathbf{\Phi}^{-1} = \frac{1}{W(t)}\begin{pmatrix} y_2 & -x_2 \\ -y_1 & x_1 \end{pmatrix}$$

where $W(t) = x_1y_2 - x_2y_1$ is the Wronskian.

**Example 3.7.2.** Find the general solution to the system

$$\mathbf{u}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 2t \\ e^t \end{pmatrix}.$$

As the reader can verify, the general solution to the homogenous equation is

$$\mathbf{u_h} = c_1 e^{-3t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore we can construct the fundamental matrix

$$\mathbf{\Phi} = \begin{pmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{pmatrix}.$$

The Wronskian is given by $W(t) = -e^{-3t}e^{-t} - e^{-3t}e^{-t} = -2e^{-4t}$ and therefore the inverse of the fundamental matrix is

$$\mathbf{\Phi}^{-1} = \frac{1}{W(t)}\begin{pmatrix} y_2 & -x_2 \\ -y_1 & x_1 \end{pmatrix} = -\frac{e^{4t}}{2}\begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-3t} & -e^{-3t} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} e^{3t} & -e^{3t} \\ -e^{t} & -e^{t} \end{pmatrix}.$$

Now plugging into the formula (3.86), we get the solution

$$\begin{aligned}
\mathbf{u} &= \mathbf{\Phi k} + \begin{pmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{pmatrix}\int -\frac{1}{2}\begin{pmatrix} e^{3t} & -e^{3t} \\ -e^{t} & -e^{t} \end{pmatrix}\begin{pmatrix} 2t \\ e^{t} \end{pmatrix} dt \\
&= \mathbf{\Phi k} - \frac{1}{2}\begin{pmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{pmatrix}\int \begin{pmatrix} 2te^{3t} - e^{4t} \\ -2te^{t} - e^{2t} \end{pmatrix} dt \\
&= \mathbf{\Phi k} - \frac{1}{2}\begin{pmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{pmatrix}\begin{pmatrix} (-8e^{3t} - 9e^{4t} + 24t)/36 \\ 2e^{t} - \frac{1}{2}e^{2t} - 2te^{t} \end{pmatrix} \\
&= \mathbf{\Phi k} + \frac{1}{9}\begin{pmatrix} 12t - 10 \\ 6t - 8 \end{pmatrix} + \frac{1}{8}e^{t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}
\end{aligned}$$

## 3.8 Exercises

**3.1:** Find all the equilibrium solutions for the system

$$\begin{aligned} x' &= 3x + y \\ y' &= -3x - y. \end{aligned}$$

What is the determinant of the coefficient matrix for this system?

**3.2:** Recall from Section 1.2.2 that an RLC circuit with no applied voltage is described by the equation

$$Lq'' + Rq' + \frac{1}{C}q = 0$$

where $q(t)$ is the charge across the capacitor.

(a) By introducing a new variable $I = q'(t)$, rewrite this equation as a system of two first order equations.

(b) Suppose $L = 1$, $R = 0$ and $C = 1/4$. If $q(0) = 8$ and $q'(0) = 0$, guess a solution $q(t)$.

(c) For the solution you found in part (b), show a time series plot ($q$ vs $t$) as well as a graph of $q$ vs $I$ in the phase plane.

**3.3:** The following system of equations is a model for the populations of two species that compete for limited resources.

$$\begin{aligned} x'(t) &= 2x\left(1 - \frac{x}{2}\right) - xy \\ y'(t) &= 3y\left(1 - \frac{y}{3}\right) - 2xy. \end{aligned}$$

Find and sketch the nullclines and find any equilibrium values. Use the nullclines to sketch the phase plane, indicating the trajectories of several possible solution. Based on your sketch, what can you say about the stability of the equilibria?

**3.4:** Verify that the functions $x(t) = 2e^t$ and $y(t) = -3e^t$ are solutions to the system

$$\begin{aligned} x' &= 4x + 2y \\ y' &= -3x - y \end{aligned}$$

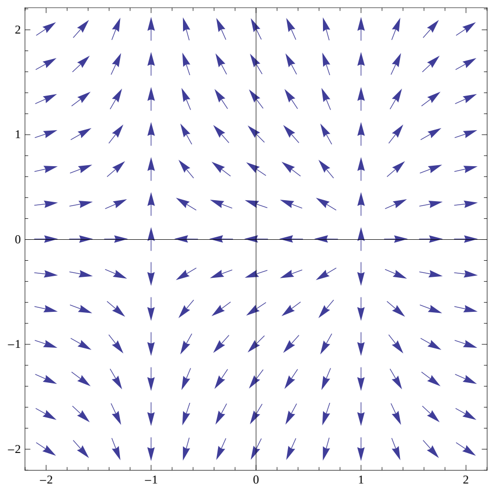
Plot the solution curve in the $x$-$y$ plane.

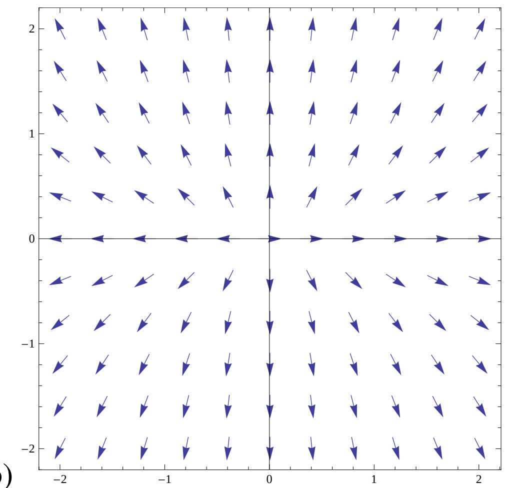
**3.5:** Find the general solution to the system

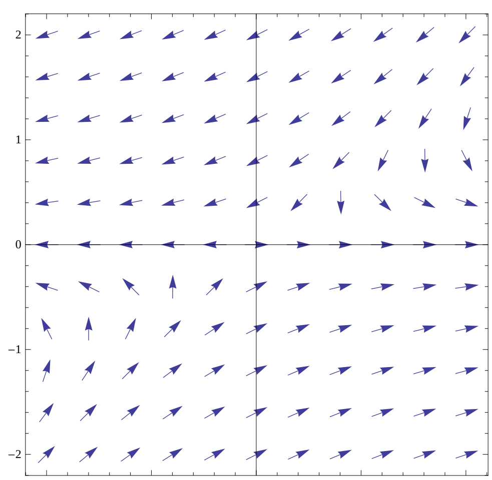
$$\begin{aligned} x' &= -x + 10y \\ y' &= -y \end{aligned}$$

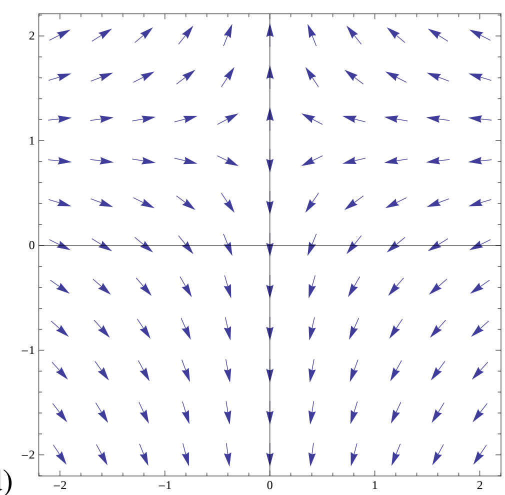
by decoupling the equations (i.e. solve the $y$ equation first). Then find the solution that satisfies the initial conditions $x(0) = 0$ and $y(0) = 5$. Sketch the time-series plots for the functions $x(t)$ and $y(t)$ and then sketch the solution curve in the $x$-$y$ plane.

**3.6:** For each of the four direction fields given below, find the corresponding system and describe briefly how you made your determination. You should do this without using a computer to generate the direction fields.

(a)



(b)



(c)



(d)


(i) $x' = -x$, $y' = y - 1$

(ii) $x' = x^2 - 1$, $y' = y$

(iii) $x' = x + 2y$, $y' = -y$

(iv) $x' = 2x$, $y' = y$

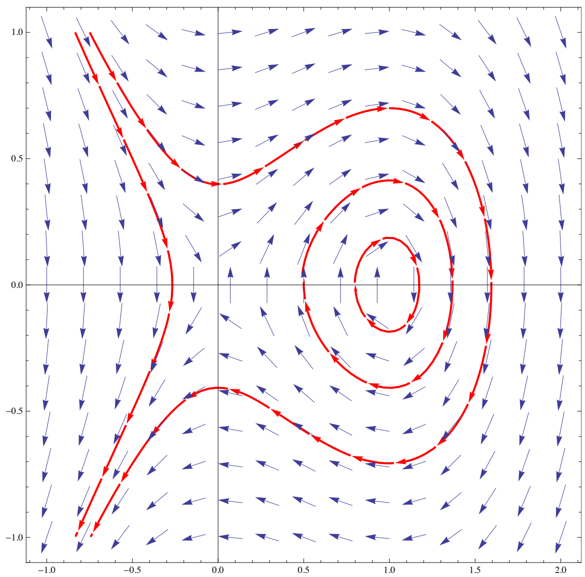
(v) $x' = x$, $y' = 2y$

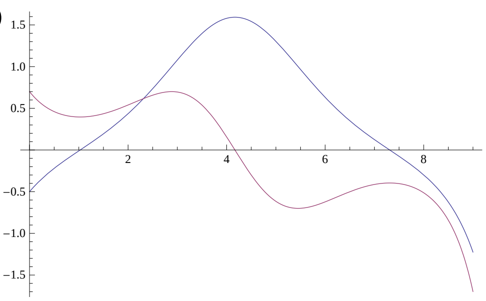
(vi) $x' = x - 1$, $y' = -y$

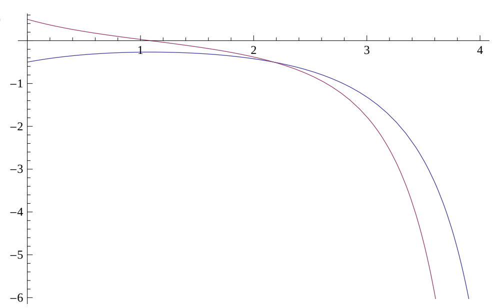
(vii) $x' = x^2 - 1$, $y' = -y$

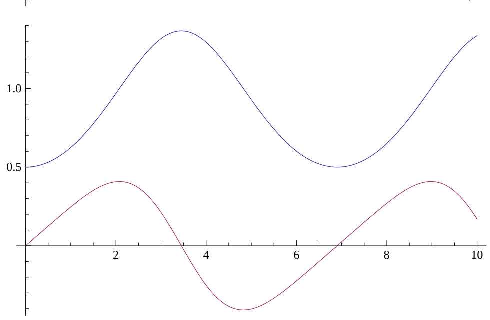
(viii) $x' = x - 2y$, $y' = -y$

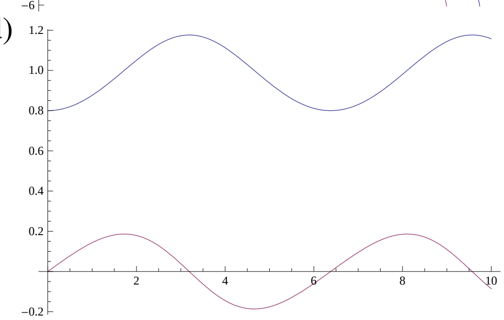
**3.7:** The following is a direction field showing solutions to a system of differential equations for four different initial conditions. Each of the four time series graphs below correspond to one of the four solutions shown in the phase plane. Match the solution curves with the time-series plots.




(a)



(b)



(c)



(d)


**3.8:** Consider the linear second order equation with constant coefficients:

$$y'' + py' + qy = 0.$$

(a) Write the first order system that corresponds to this equation.

(b) Show that if $q \neq 0$ then the origin is the only equilibrium.

**3.9:** Given the following:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

find the quantities

(a) $\mathbf{A} + \mathbf{B}$

(b) $\mathbf{B} - 4\mathbf{A}$

(c) $\det \mathbf{A}$

(d) $\mathbf{Bu}$

**3.10:** Consider the system

$$\mathbf{u}' = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \mathbf{u}$$

(a) Show that $\mathbf{u}_1 = e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{u}_2 = e^{-4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are fundamental solutions.

(b) Find a solution that satisfies the initial condition $\mathbf{u}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

**3.11:** Consider the linear system

$$\mathbf{u}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{u}.$$

(a) Show that the vector function

$$\mathbf{u} = \begin{pmatrix} te^{2t} \\ -(t+1)e^{2t} \end{pmatrix}$$

is a solution to the differential equation.

(b) Solve the initial value problem given $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

**3.12:** Consider the system

$$\begin{aligned} x' &= 1 \\ y' &= x \end{aligned}$$

(a) Show that $\mathbf{u} = \begin{pmatrix} t \\ t^2/2 \end{pmatrix}$ is a solution to this system.

(b) Show that $2\mathbf{u}$ is *not* a solution.

**3.13:** For each of the systems below, use a computer to help sketch the phase plane. Include any straight-line solutions you can find.

(a) $\mathbf{u}' = \begin{pmatrix} 3 & 2 \\ 0 & -2 \end{pmatrix}\mathbf{u}$

(b) $\mathbf{u}' = \begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix}\mathbf{u}$

(c) $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\mathbf{u}$

(d) $\mathbf{u}' = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}\mathbf{u}$

(e) $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{u}$

**3.14:** Find the eigenvalues and eigenvectors for each of the matrices in the previous problem.

**3.15:** Find the eigenvalues and eigenvectors for the following matrices:

(a) $\mathbf{A} = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$

(b) $\mathbf{B} = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}$

(c) $\mathbf{C} = \begin{pmatrix} 2 & -8 \\ 1 & -2 \end{pmatrix}$

**3.16:** For each of the systems below, find the general solution and then find the particular solution that satisfies the initial condition $\mathbf{u}(0) = (1, 0)$. Sketch the phase plane and include in your sketch the solution that passes through the point $(1, 0)$. You may use technology to verify your phase plane, but you should be able to sketch it without the use of technology.

(a) $\mathbf{u}' = \begin{pmatrix} 3 & 2 \\ 0 & -2 \end{pmatrix} \mathbf{u}$

(b) $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{u}$

(c) $\mathbf{u}' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{u}$

(d) $\mathbf{u}' = \begin{pmatrix} -2 & -2 \\ -1 & -3 \end{pmatrix} \mathbf{u}$

(e) $\mathbf{u}' = \begin{pmatrix} 2 & -8 \\ 1 & -2 \end{pmatrix} \mathbf{u}$

**3.17:** For the following system, find the eigenvalues in terms of the parameter $\alpha$. Then find the critical value (or values) of $\alpha$ for which the qualitative nature of the solutions changes.

$$\mathbf{u}' = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix} \mathbf{u}$$

**3.18:** Consider the second-order equation

$$y'' + py' + qy = 0$$

where the constants $p$ and $q$ are both positive.

(a) Write this equation as a first-order system of equations and express the eigenvalues in terms of $p$ and $q$.

(b) Show that all solutions will converge to zero. Hint: consider the three cases (two real eigenvalues, one repeated eigenvalue, and two complex eigenvalues) separately.

**3.19:** Find the solution to the linear system

$$\begin{aligned} x'(t) &= -x + y \\ y'(t) &= 4x - 4y \end{aligned}$$

with initial conditions $x(0) = 10$ and $y(0) = 0$. Sketch the solution in the phase plane.

**3.20:** Consider the linear system

$$\begin{aligned} x' &= -3x + 10y \\ y' &= -x + 3y \end{aligned}$$

Show that all solution curves in the phase portrait for this system are ellipses.

**3.21:** Suppose that the general solution of a linear system is

$$\begin{aligned} x(t) &= c_1 e^{-2t} + c_2 e^{4t} \\ y(t) &= -3c_1 e^{-2t} + c_2 e^{4t} \end{aligned}$$

(a) What is the system? In other words, write down the differential equations for which this is the solution.

(b) Sketch the phase plane for this system.

**3.22:** Find the general solution for the system

$$\mathbf{u}'(t) = \begin{pmatrix} -5 & 3 \\ 2 & -10 \end{pmatrix} \mathbf{u} + \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}.$$

**3.23:** Recall that given two solutions $\mathbf{u_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{u_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ to the linear system $\mathbf{u}' = \mathbf{A}\mathbf{u}$ the **Wronskian** is defined by

$$W(t) = x_1 y_2 - x_2 y_1.$$

(a) Show that $W(t)$ satisfies the differential equation

$$W'(t) = (a + d)W(t)$$

where $a$ and $d$ are the diagonal elements of the coefficient matrix $\mathbf{A}$.

(b) Solve the differential equation for $W$ in part (a) to show that if $W(t)$ is nonzero at any time $t$, then it must also be nonzero at $t = t_0$ (in other words, it is either always zero or its never zero).

**3.24:** Consider two interconnected tanks. Tank 1 initially contains 30 gallons of water and 25 oz. of salt, and Tank 2 initially contains 20 gallons of water and 15 oz. of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of 1.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into Tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from Tank 2 at a rate of 4 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

(a) Let $S_1(t)$ and $S_2(t)$, respectively, denote the amount of salt in each tank at time $t$. Write down the differential equations and initial conditions that model this flow process.

(b) Find the values of $S_1$ and $S_2$ at which the system is in equilibrium; represent these values by $S_1^*$ and $S_2^*$ respectively. Which tank do you think will approach its equilibrium value most rapidly?

(c) Introduce the new variables $x_1 = S_1 - S_1^*$ and $x_2 = S_2 - S_2^*$ to represent the the deviation from equilibrium values. Rewrite the system in terms of these new variables.

(d) Solve the system for $x_1$ and $x_2$ and graph the time-series plots for $x_1$ and $x_2$ (use the same set of axes).

(e) Estimate the time $T$ at which $|x_1(t)| \leq 0.5$ and $|x_2(t)| \leq 0.5$ for all $t \geq T$.

**3.25:** Two tanks each contain 20 L of salt water. A solution containing 4 g/L of salt flows into tank 1 at a rate of 3 L/min and the solution in tank 2 is being drained at the same rate. In addition, solution flows into tank 1 from tank 2 at a rate of 1 L/min and into tank 2 from tank 1 at a rate of 4 L/min. Initially, tank 1 contains 40 g of salt and tank 2 contains 20 g of salt. Find the amount of salt in each tank at time $t$.

# Chapter 4

# Second-Order Differential Equations

In Chapter 3 we saw how second (or higher) order equations can be expressed as a system of first order equations. Although there are many advantages to treating a second order equation as a system, many equations from physics are naturally expressed as second-order equations and it is often convenient to treat these examples directly. In this chapter, we outline some basic techniques for working directly with some specific second-order linear equations.

## 4.1 Special Cases

In some special cases, a second order equation can be transformed into a first order equation. In this section we will look at a few examples of how this happens. Since these cases occur with some frequency in the context of physics, we will describe these equations in terms of Newton's second law of motion. Suppose the state variable $x(t)$ represents the position of a particle at time $t$ then we have

$$mx'' = F(t, x, x')$$

where $F(t, x, x')$ is an applied force. Note that since $x(t)$ represents position, the first derivative $x'$ is velocity and $x''$ is acceleration. To simplify the notation, we will set the mass $m$ equal to one from here on.

#### Force is Independent of Position

In the case where the force does not depend on the position, the differential equation can be expressed as

$$x'' = F(t, x').$$

If we introduce a new variable to represent velocity, $v = x'$, then the differential equation can be written as

$$v' = F(t, v).$$

This is now a first order equation for the variable $v(t)$! If we can solve this equation for $v(t)$, then the position can be recovered by simply antidifferentiating

$$x(t) = \int v(t)\, dt + C.$$

**Example 4.1.1.** Consider the equation

$$x'' = 2tx'.$$

Introducing $v = x'$, the differential equation becomes

$$v' = 2tv.$$

This is a separable equation which we solve to find $v = Ce^{t^2}$. Then in terms of the original variable,

$$x(t) = \int v(t)\, dt = C \int e^{t^2}\, dt + D.$$

Since this integral cannot be simplified further, we leave the solution in this form.

**Force is Independent of Time**

If the force does not depend explicitly on the independent variable $t$, the differential equation has the form

$$x'' = F(x, x').$$

Again we introduce the velocity $v = x'$. Using the chain rule, we can rewrite the second derivative $x''(t)$ in terms of the variable $x$:

$$x'' = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

In essence, we are rewriting the differential equation as if $x$ was the independent variable. Now the differential equation becomes

$$v\frac{dv}{dx} = F(x, v).$$

Again, we have transformed the original second order differential equation into a first order equation for the variable $v(x)$. If we can solve this, then the we can recover the original variable $x(t)$ by solving $x'(t) = v(x)$ using separation of variables.

**Example 4.1.2.** Consider the equation

$$x'' = \frac{1}{2\sqrt{x}}x'.$$

Since the right-hand side does not depend explicitly on $t$, we introduce $v = x'$ and use the chain rule as described above to rewrite the differential equation as

$$v\frac{dv}{dx} = \frac{1}{2\sqrt{x}}v.$$

One solution is given by $v = 0$, which implies the solution $x(t) = C$. The other possibility is that

$$\frac{dv}{dx} = \frac{1}{2\sqrt{x}}.$$

Antidifferentiating gives

$$v(x) = \sqrt{x} + C_1$$

or

$$\frac{dx}{dt} = \sqrt{x} + C_1.$$

Separating variables and integrating then gives

$$\int \frac{1}{\sqrt{x} + C_1}\,dx = \int 1\,dt = t + C_2.$$

This integral can be evaluated using the substitution $u = \sqrt{x} + C_1$ to obtain

$$2(\sqrt{x} + C_1) - 2C_1 \ln|\sqrt{x} + C_1| = t + C_2.$$

This equation cannot be solved for $x$ explicitly, instead we consider this equation as an *implicit* definition of $x(t)$.

### Conservative Forces

Another important situation arises when the force depends only upon the position $x$. In this case, we say that the force $F(x)$ is **conservative** (for reasons that will become apparent). Using the same substitutions as in the previous case, the original differential equation can be rewritten as

$$v\frac{dv}{dx} = F(x).$$

This may be directly antidifferentiated to obtain

$$\frac{1}{2}v^2 = \int F(x)\,dx + E.$$

We have chosen to express the integration constant by the name $E$ because in many physical applications, this constant represents a physical energy. If we introduce the quantity

$$V(x) = -\int F(x)\,dx$$

(which can be interpreted as the **potential energy**), then we have the equation

$$\frac{1}{2}v^2 + \int F(x)\,dx = E. \tag{4.1}$$

Since the first term represents the kinetic energy of the system (recall that we have set the mass of the particle to 1), this equation states that the sum of the kinetic and potential energies is a constant. In other words, we have derived an *energy conservation law*. Note that the value of the constant $E$ is typically determined by initial conditions:

$$E = \frac{1}{2}v_0^2 + V(x_0).$$

Again, the original variable $x(t)$ may be recovered by solving the conservation equation (4.1) for $v$ and then antidifferentiating. More often, however, we will analyze the behavior of solutions in the phase plane by plotting the relationship between $x$ and $v$ from (4.1).

**Example 4.1.3.** Consider the mass-spring equation without damping:

$$mx'' + kx = 0.$$

The potential energy $V(x)$ is given by

$$V(x) = -\int(-kx)\,dx = \frac{1}{2}kx^2. \tag{4.2}$$

Note that because $E$ is already a constant of integration, we do not need to include a second constant when simplifying the integral in (4.2). We now have the conservation of energy equation

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E.$$

This equation describes a family of ellipses in the $x$-$v$ plane. Since the trajectories are closed curves, we know that the solutions, i.e. the position and velocity, are periodic, which agrees with our experience with undamped masses on springs.

## 4.2 Linear Equations with Constant Coefficients

Consider a second-order homogeneous linear equation with constant coefficients:

$$y'' + py' + qy = 0 \tag{4.3a}$$

together with initial values

$$y(t_0) = y_0 \tag{4.3b}$$
$$y'(t_0) = y_0'. \tag{4.3c}$$

Note that there is no loss of generality in having the leading coefficient be one since if there were a coefficient in front of the $y''$ term, we could just divide through by that constant and the result would be an equation in the form of (4.3a). Our goal now is to construct the solution to this initial value problem. Here is an outline of our method:

1. Find two *independent* solutions $y_1$ and $y_2$. In this context, this means that the two solutions are not multiples of one another.

2. Construct the linear combination
$$y = c_1y_1 + c_2y_2 \tag{4.4}$$
where $c_1$ and $c_2$ are arbitrary constants. Because of the linearity of the equation, this will also be a solution.

3. Finally, plug in the initial conditions and solve for the constants $c_1$ and $c_2$.

The first step is to find two independent solutions. The basic idea is to guess a solution of the form $y = e^{\lambda t}$. As we have seen, exponential solutions tend to come up quite often, so this isn't completely out of the blue. We begin by plugging our guess into the differential equation to see under what circumstances it might be a solution:

$$\begin{aligned} y'' + py' + qy &= 0 \\ (\lambda^2 e^{\lambda t}) + p(\lambda e^{\lambda t}) + qe^{\lambda t} &= 0 \end{aligned}$$

and dividing by the exponential

$$\lambda^2 + p\lambda + q = 0.$$

This shows that our guess is indeed a solution, provided that $\lambda$ is a solution to the quadratic equation

$$\lambda^2 + p\lambda + q = 0. \tag{4.5}$$

This is known as the **characteristic equation** and the solutions are given by the quadratic formula:

$$\lambda = -\frac{p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}. \tag{4.6}$$

As we know, there are three possibilities, depending on whether the quantity $p^2 - 4q$ is positive, negative, or zero. We will consider each of these possibilities in turn.

### Case 1: Two Real Roots ($p^2 - 4q > 0$)

In the case where $p^2 - 4q > 0$, there will be two real and distinct values of $\lambda$ that solve the characteristic equation. We will refer to them as $\lambda_1$ and $\lambda_2$. Recalling that the solutions to the characteristic equation are precisely the values of $\lambda$ for which our guess of an exponential function is a solution to the differential equation, this means that we have two solutions

$$y_1 = e^{\lambda_1 t} \quad \text{and} \quad y_2 = e^{\lambda_2 t} \tag{4.7}$$

and therefore the general solution is

$$\boxed{y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.} \tag{4.8}$$

**Example 4.2.1.** Find the solution to the differential equation

$$y'' - y' - 2y = 0 \tag{4.9}$$

that satisfies the initial conditions

$$y(0) = 1 \tag{4.10}$$

$$y'(0) = 0. \tag{4.11}$$

First we find the general solution by plugging in our guess $e^{\lambda t}$. This gives the characteristic equation

$$\lambda^2 - \lambda - 2 = 0.$$

Solving this, we find $\lambda_1 = 2$ and $\lambda_2 = -1$. Therefore the general solution is

$$y = c_1 e^{2t} + c_2 e^{-t}.$$

Then, plugging in the initial conditions, we get the two equations

$$1 = c_1 + c_2 \tag{4.12}$$

$$0 = 2c_1 - c_2. \tag{4.13}$$

Solving these, we find $c_1 = 1/3$ and $c_2 = 2/3$, so the final solution is

$$y = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

**Case 2: One Real Root** ($p^2 - 4q = 0$)

In the case where $p^2 - 4q = 0$, there is only one root to the characteristic equation, which we will refer to as $\lambda$. This gives us one solution to the differential equation: $y_1 = e^{\lambda t}$. To solve the initial value problem, we need to find a second independent solution.

There are a variety of ways to find a second solution. We will outline one method here; others will be explored in exercises. We will use a technique that we introduced in solving the first-order nonhomogeneous equation. Motivated by the fact that $y = ce^{\lambda t}$ is a solution for any value of the constant $c$, we will replace $c$ with an unknown function and see if we can find solutions.

Let $y = v(t)e^{\lambda t}$. Plugging this into our differential equation we have

$$y'' + py' + qy = 0$$

$$\left(v''e^{\lambda t} + 2v'\lambda e^{\lambda t} + v\lambda^2 e^{\lambda t}\right) + p\left(v'e^{\lambda t} + v\lambda e^{\lambda t}\right) + qve^{\lambda t} = 0$$

dividing by the exponentials

$$v'' + 2v'\lambda + v\lambda^2 + p(v' + v\lambda) + qv = 0$$

and rearranging

$$v'' + (2\lambda + p)v' + (\lambda^2 + p\lambda + q) = 0.$$

At this point, we recall two facts about $\lambda$. First, it is a solution to the characteristic equation and therefore

$$\lambda^2 + p\lambda + q = 0.$$

Secondly, since we are considering case 2, we know that $p^2 - 4q = 0$ and therefore $\lambda = -p/2$. Using these two facts, our equation for $v$ simplifies considerably:

$$v'' = 0.$$

Solving, we get $v = at + b$ where $a$ and $b$ are arbitrary constants. Plugging this into our solution, we get

$$y_2 = (at + b)e^{\lambda t}.$$

Although we could use this for our second independent solution, it is convenient to simplify. Recall that for a linear equation, if we add two solutions together, the result is another solution. More generally, any linear combination of two solutions is also a solution. Therefore if we take $y_2$ found above and add to it the solution $-be^{\lambda t}$, we get

$$y_3 = ate^{\lambda t}.$$

Similarly, we can divide this by the constant $a$ and the result is still a solution. Therefore the second independent solution, in simplest form is $y = te^{\lambda t}$ and the general solution in case 2 is

$$\boxed{y = c_1 e^{\lambda t} + c_2 t e^{\lambda t}} \tag{4.14}$$

**Case 3: Complex Roots** ($p^2 - 4q < 0$)

In the case where $p^2 - 4q < 0$, we know that the roots are complex valued:

$$\begin{aligned} \lambda &= -\frac{p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2} \\ &= -\frac{p}{2} \pm \frac{\sqrt{(-1)(4q - p^2)}}{2} \\ &= -\frac{p}{2} \pm i\frac{\sqrt{4q - p^2}}{2}. \end{aligned}$$

We now have the roots written in terms of a real and imaginary part, i.e. $\lambda = \alpha \pm i\beta$ and if we plug these values of $\lambda$ into our exponential solutions, we get

$$y = e^{\lambda t} = e^{(\alpha \pm i\beta)t} = e^{\alpha t} e^{\pm i\beta t}.$$

Using Euler's formula,

$$e^{i\beta t} = \cos(\beta t) + i\sin(\beta t)$$

we can rewrite our solutions in terms of real-valued functions. If we denote our two solutions as

$$\begin{aligned} y_1 &= e^{\alpha t} e^{i\beta t} \\ y_2 &= e^{\alpha t} e^{-i\beta t} \end{aligned}$$

then applying Euler's formula, we have

$$\begin{aligned} y_1 &= e^{\alpha t} \left(\cos(\beta t) + i \sin(\beta t)\right) \\ y_2 &= e^{\alpha t} \left(\cos(\beta t) - i \sin(\beta t)\right). \end{aligned}$$

Note that in simplifying $y_2$, we have made use of the fact that cosine is an even function and that sine is an odd function. Finally, we observe that given two solutions, any linear combination of them is also a solution. So for instance, we may form the linear combination

$$\frac{1}{2}\left(y_1 + y_2\right) = e^{\alpha t} \cos(\beta t)$$

to form a new real-valued solution. Similarly, the linear combination

$$\frac{1}{2i}\left(y_1 - y_2\right) = e^{\alpha t} \sin(\beta t)$$

is also a solution. As we can verify, these two solutions are independent and therefore we may use them instead of $y_1$ and $y_2$ above to form the general solution:

$$\boxed{y = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)} \tag{4.15}$$

where $\alpha = -p/2$ and $\beta = \sqrt{4q - p^2}/2$.

**Example 4.2.2.** Find the solution to the initial value problem

$$y'' + y' + y = 0 \tag{4.16}$$

$$y(0) = 1 \tag{4.17}$$

$$y'(0) = 0. \tag{4.18}$$

First we find the general solution by plugging in our guess $e^{\lambda t}$. This gives the characteristic equation

$$\lambda^2 + \lambda + 1 = 0.$$

Solving this, we find $\lambda_1 = -1/2 + i\sqrt{3}/2$ and $\lambda_2 = -1/2 - i\sqrt{3}/2$. Therefore the general solution is

$$y = c_1 e^{-t/2} \cos(\sqrt{3}/2t) + c_2 e^{-t/2} \sin(\sqrt{3}/2t).$$

Then, plugging in the initial conditions, we get the two equations

$$1 = c_1 \tag{4.19}$$

$$0 = -c_1/2 + \sqrt{3}c_2/2 \tag{4.20}$$

Solving for $c_2$, we find $c_2 = 1/\sqrt{3}$, so the final solution is

$$y = e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right).$$

## 4.3 The Nonhomogeneous Equation

Now that we know how to solve homogeneous second order linear equations, we turn to the nonhomogeneous case. To be precise, we consider the equation

$$y'' + py' + qy = g(t) \tag{4.21}$$

with initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$. Note that we are still considering the case where the coefficients $p$ and $q$ are constants. The function on the right hand side, $g(t)$, is often called the **forcing** or **source** term.

The strategy for finding the general solution to (4.21) comes out of the following observation. Let $y$ denote the general solution that we're looking for and suppose for a moment that somehow we find a *particular* solution to (4.21). Recall that the term "particular" here means that this solution satisfies some particular initial condition, not necessarily the one that we want. We denote this solution $y_p$. If we form the difference of these two solutions $y - y_p$ and plug it into the left hand side of the differential equation, we get

$$(y - y_p)'' + p(y - y_p)' + q(y - y_p) = (y'' + py' + qy) - \left(y_p'' + py_p' + qy_p\right) \tag{4.22}$$

but since $y$ and $y_p$ are both solutions to (4.21), this simplifies to

$$= g(t) - g(t) \tag{4.23}$$

$$= 0. \tag{4.24}$$

In other words, the quantity $y - y_p$ is a solution to the *homogeneous* equation, and as we saw in the previous section, we know how to find the general solution the homogeneous equation! To summarize, the general solution to (4.21) is given by

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t) \tag{4.25}$$

where $y_1$ and $y_2$ are independent solutions to the homogeneous equation (4.3a) and $y_p$ is *any* solution to the nonhomogeneous equation (4.21).

### 4.3.1 Undetermined Coefficients

In order to find the general solution to the nonhomogeneous equation, we need to find a single particular solution $y_p$. As we did for first-order equations, we will discuss two methods for doing this. The first method, known as the method of **undetermined coefficients**, involves making an educated guess. It has the benefit of being fairly straightforward, at least in the cases where $g(t)$ is a simple function. The downside to this method is that depending on the function $g(t)$, we may not be able to make a guess. With practice we will have a good idea of when this method will (or won't) work.

As we saw with first order equations, what we should guess for $y_p$ is based on the observation that when we plug our guess into the left hand side of the differential equation, the result should be a function that looks like $g(t)$. In other words, if we know what kind of function appears on the right-hand side, we should think about what kind of function – when plugged in to the left-hand side – will give us this result.

For example, suppose that $g(t)$ is an exponential function. In this case, our guess $y_p$ should be something such that when we plug it into the left-hand side of the differential equation

$$y'' + py' + qy$$

the result will be an exponential. Now the question is this: what kind of function, when differentiated, results in an exponential? Answer: an exponential function! And if we differentiate again? Still an exponential. And if we add together several exponential functions? The result is still an exponential!

Therefore, if we guess that $y_p$ is an exponential function, then when we plug this guess into the left hand side, the result has a reasonable chance of giving us $g(t)$. All we need to do is include a coefficient to give us some freedom in matching the left and right hand sides. This "undetermined" coefficient is what gives this method its name.

**Example 4.3.1.** Find a particular solution for the nonhomogenous equation

$$y'' + y' - 2y = 6e^{2t}.$$

In this example, the right-hand side is an exponential function. So we guess that our particular solution should also an exponential function, so that when we plug this guess into the left-hand side, we will be able to match the two sides of the differential equation. Let $y_p = Ae^{2t}$. Then plugging this guess into the left-hand side gives

$$\begin{aligned} 4Ae^{2t} + 2Ae^{2t} - 2Ae^{2t} &= 6e^{2t} \\ 4Ae^{2t} &= 6e^{2t} \\ 4A &= 6 \\ A &= \frac{3}{2}. \end{aligned}$$

So the particular solution is $y_p = \frac{3}{2}e^{2t}$.

**Example 4.3.2.** Find a particular solution for the nonhomogenous equation

$$y'' + y' - 2y = t^2 - 1.$$

Now the right-hand side is a polynomial function (specifically, a second order polynomial), so we guess that our particular solution should also be an polynomial. Let $y_p = At^2 + Bt + C$. Notice that even though $g(t)$ does not include a first-order term, our guess does. This is because when we plug our guess into the left-hand side, and differentiate, we will generate first-order terms, and so our guess must include a first-order term to take this into account. (Another way to think about it is to realize that there is a first-order term in $g$ – but the coefficient is zero! )

Plugging our guess into the differential equation then gives

$$(4A) + (2At + B) - 2(At^2 + Bt + C) = t^2 - 1$$

and rearranging terms

$$(-2A)t^2 + (2A - 2B)t + (4A + B - 2C) = t^2 - 1$$

and then matching coefficients gives

$$-2A = 1 \qquad 2A - 2B = 0 \qquad 4A + B - 2C = -1.$$

We now have three algebraic equations for the three unknown coefficients. Solving them in order, we find

$$\begin{aligned} A &= -\frac{1}{2} \\ B &= -\frac{1}{2} \\ C &= -\frac{7}{4}. \end{aligned}$$

So the particular solution is

$$y_p = -\frac{1}{2}t^2 - \frac{1}{2}t - \frac{7}{4}.$$

**Example 4.3.3.** Find a particular solution for the nonhomogeneous equation

$$y'' + y' - 2y = \cos(2t).$$

Since the right hand side is a cosine function, our guess should also include a cosine function. However, because of the first derivative term on the left-hand side, we also need to include a sine term in our guess (if we did not, then we would not be able to match both sine and cosine terms on either side of the equation). So letting $y_p = A\cos(2t) + B\sin(2t)$ and plugging this guess into the left-hand side, we get

$$-4A\cos(2t) - 4B\sin(2t) - 2A\sin(2t) + 2B\cos(2t) - 2A\cos(2t) - 2B\sin(2t) = \cos(2t)$$

and collecting sine and cosine terms on the left,

$$(-6A + 2B)\cos(2t) + (-6B - 2A)\sin(2t) = \cos(2t).$$

Then matching the cosine terms on the left and right hand sides, we have

$$-6A + 2B = 1$$

while matching sine terms gives us

$$-6B - 6A = 0.$$

We then solve these two equations for the two unknowns and find that $A = -1/8$ and $B = 1/8$ so the particular solution is

$$y_p = -\frac{1}{8}\cos(2t) + \frac{1}{8}\sin(2t).$$

So far so good. This next example illustrates one of the ways in which an initial guess might fail to work out.

**Example 4.3.4.** Find a particular solution for the nonhomogeneous equation

$$y'' + y' - 2y = 2e^t.$$

As in example 4.3.1, the right hand side is an exponential, so we might guess

$$y_p = Ae^t$$

as our particular solution. However, upon substituting this into the equation, we find

$$\begin{aligned} Ae^t + Ae^t - 2Ae^t &= 2e^t \\ 0 &= 2e^t \\ 0 &= 2 \end{aligned}$$

Because we end up with a statement that is clearly not true, we can conclude that the function $y_p = Ae^t$ cannot be a solution to the differential equation (for any value of the coefficient $A$)! In other words, our guess did not work. Why not? If we look at the associated homogeneous equation

$$y'' + y' - 2y = 0$$

we will find that the roots to the characteristic equation are $r_1 = 1$ and $r_2 = -2$ so the exponential functions $y_1 = e^t$ and $y_2 = e^{-2t}$ are fundamental solutions. Notice then that our guess above for the particular solution is in fact a solution to the homogeneous equation! It cannot possible be a solution to the nonhomogeneous equation as well!

To resolve this difficulty, we can use a technique introduced when we were looking at first order equations. We will look for a particular solution of the form $y_p = v(t)e^t$ where $v(t)$ is an unknown function. Plugging this into the differential equation, we find

$$\begin{gathered} (v''e^t + 2v'e^t + ve^t) + (v'e^t + ve^t) - 2ve^t = 2e^t \\ v'' + 3v' = 2. \end{gathered}$$

Although this is a second order equation for $v$, we can turn it into a first order equation with the substitution $w = v'$. This gives us the equation

$$w' + 3w = 2$$

which is a linear first-order equation. Now solving *this* equation gives $w(t) = ce^{-3t} + 2/3$. Then rewriting in terms of $v$, we have the equation $v' = ce^{-3t} + 2/3$ which means

$$v = \widetilde{c}e^{-3t} + \frac{2}{3}t + d.$$

Finally, expressing the function $y_p$, we have

$$\begin{aligned} y_p &= \left(ce^{-3t} + \frac{2}{3}t + d\right)e^t \\ &= ce^{-2t} + \frac{2}{3}te^t + de^t. \end{aligned}$$

Since this is a solution for any choice of the constants $c$ and $d$, we may take the simplest case $y_p = \frac{2}{3}te^t$.

This last example illustrates an important point about making guesses for the particular solution. In cases where the function that we might otherwise guess (based on the form of the function $g(t)$) is *already* a solution to the associated homogeneous equation, the general principle is to make a guess that includes an extra factor of $t$. For instance, instead of the guess $y_p = Ae^t$, the correct guess is a particular solution of the form $y_p = Ate^t$. In some cases, it may be necessary to apply this principle more than once!

In general, we will be able to guess the particular solution as long as $g(t)$ is an exponential function, a polynomial, a sine or cosine function, or a sum or product of these kinds of functions. For any other type of $g$, it is usually too complicated to try to guess $y_p$. Here are some guidelines.

- If $g(t)$ is an exponential function, i.e. has the form $ae^{rt}$, then we should guess a particular solution
$$y_p = Ae^{rt}$$
for some unknown coefficient $A$.

- If $g(t)$ is a polynomial function, i.e. has the form $a_n t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$ then we should guess a particular solution
$$y_p = A_n t^n + A_{n-1}t^{n-1} + \ldots + A_1 t + A_0.$$

- If $g(t)$ is a sine or cosine function, i.e has the form $a\sin(\omega t)$ or $a\cos(\omega t)$, then we should guess a particular solution with the same form:
$$y_p = A\sin(\omega t) + B\cos(\omega t).$$

### 4.3.2 Variation of Parameters

In contrast to the method of undetermined coefficients described in the previous section, the method of **variation of parameters** requires no guesses. Instead, it provides a formula for the particular solution of the nonhomogeneous equation. The trade-off is that the solution formulas that are produced typically require more extensive calculations. To begin, suppose that $u_1(t)$ and $u_2(t)$ are two independent solutions to the *homogeneous* equation

$$u'' + pu' + qu = 0.$$

Then the general solution is the linear combination

$$u_h(t) = c_1u_1(t) + c_2u_2(t).$$

Just as we did in the case of first-order equations, the next step is to allow the parameters $c_1$ and $c_2$ to vary, i.e. we replace them with functions $c_1(t)$ and $c_2(t)$:

$$u_p(t) = c_1(t)u_1(t) + c_2(t)u_2(t). \tag{4.26}$$

We can now substitute this into the nonhomogeneous equation and try to solve for $c_1(t)$ and $c_2(t)$.

To substitute $u_p(t)$ into the differential equation, we will need to compute its first and second derivatives. In order to simplify the notation a bit, we will drop the explicit mention of the variable $t$. In other words, we will write $c_1$, $c_2$, $u_1$, and $u_2$ with the understanding that these are all functions of $t$. The first derivative is straightforward:

$$u_p' = c_1u_1' + c_1'u_1 + c_2u_2' + c_2'u_2. \tag{4.27}$$

At this point, we pause to make an important observation. Because we have *two* unknown functions ($c_1$ and $c_2$) but only *one* constraint (the differential equation (4.21)), the problem is actually underdetermined, meaning that we don't have enough information to completely specify $c_1$ and $c_2$. This gives us some freedom to impose an additional constraint which we can use to simplify our expression for $u_p'$. In particular, if we suppose that $c_1$ and $c_2$ satisfy the equation

$$c_1'u_1 + c_2'u_2 = 0$$

then (4.27) simplifies to

$$u_p' = c_1u_1' + c_2u_2'. \tag{4.28}$$

Now differentiating our particular solution a second time gives us

$$u_p'' = c_1u_1'' + c_1'u_1' + c_2u_2'' + c_2'c_2' \tag{4.29}$$

and substituting these expressions for the derivatives into the nonhomogeneous differential equation gives

$$c_1u_1'' + c_1'u_1' + c_2u_2'' + c_2'u_2' + p\left(c_1u_1' + c_2u_2'\right) + q\left(c_1u_1 + c_2u_2\right) = g(t). \tag{4.30}$$

We then rewrite by rearranging terms so that all of the terms with $c_1$ (and likewise $c_2$) are grouped together:

$$c_1'u_1' + c_1\left(u_1'' + pu_1' + qu_1\right) + c_2'u_2' + c_2\left(u_2'' + pu_2' + qu_2\right) = g(t). \tag{4.31}$$

Next we make use of the fact that $u_1$ and $u_2$ are in fact solutions to the homogeneous equation. This means that both expressions inside the parentheses are zero. The result is the much simpler

$$c_1' u_1' + c_2' u_2' = g(t). \tag{4.32}$$

We now have two equations for the two unknowns, $c_1'$ and $c_2'$, and we can solve these simultaneous equations algebraically in the usual way. Finally, we integrate to get

$$c_1(t) = -\int \frac{u_2(t)g(t)}{W(t)}\,dt \qquad c_2(t) = \int \frac{u_1(t)g(t)}{W(t)}\,dt \tag{4.33}$$

where $W(t) = u_1(t)u_2'(t) - u_1'(t)u_2(t)$ is shorthand introduced to simplify the notation. Now that we have solved for the the parameters $c_1$ and $c_2$, we can replace them in the original expression for the particular solution:

$$u_p(t) = -u_1(t)\int \frac{u_2(t)g(t)}{W(t)}\,dt + u_2(t)\int \frac{u_1(t)g(t)}{W(t)}\,dt. \tag{4.34}$$

The general solution of the nonhomogeneous equation (4.21) is then the sum of $u_p$ and $u_h$.

## 4.4 Oscillations and Vibrations

As a particular example of a nonhomogeneous equation, we will now return to the mass on a spring model introduced in Chapter 1. As we have seen, this same mathematical model also describes the basic RLC circuit; indeed, the universe is filled with examples of physical oscillators that are all modeled by this same equation. Furthermore, many other examples of oscillating systems can be described by modifications of this model so by understanding this equation and the behavior of its solutions, we will be well situated to understand a very wide range of other oscillating systems.

To review, the differential equation in question is

$$mx'' + \gamma x' + kx = F(t)$$

where $x(t)$ represents the displacement of a mass $m$ on a spring, $\gamma$ represents a damping coefficient, and $k$ is the spring constant. We note that all of these coefficients are positive parameters. The nonhomogeneous term $F(t)$ represents an external forcing applied to the system. In many practical applications, the applied force is periodic and so we will often focus on the special case $F(t) = F_0\cos(\omega t)$.

### 4.4.1 Unforced Oscillations

Before we consider the case of a forced oscillator, we will first describe the behavior of our system without any applied external force. In the special case of no damping, we have the homogeneous equation

$$mx'' + kx = 0$$

which has solutions

$$x = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \tag{4.35}$$

where we have introduced the notation $\omega_0 = \sqrt{k/m}$, known as the *natural frequency* (also called angular frequency). Given an initial value problem, we can then solve for the constants $c_1$ and $c_2$.

### Phase Amplitude Form

In many applications, it is helpful to rewrite this solution in so-called **phase-amplitude** form. Suppose that $R$ and $\delta$ are two constants, then applying a standard trigonometric identity we can write

$$R\cos(\omega_0 t - \delta) = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t).$$

Comparing the right hand side of this result with the homogeneous solution (4.35) we see that

$$x = R\cos(\omega_0 t - \delta) \tag{4.36}$$

is an equivalent form of the solution if we set $c_1 = R\cos(\delta)$ and $c_2 = R\sin(\delta)$. The constants $R$ and $\delta$ can be determined by observing that $R = \sqrt{c_1^2 + c_2^2}$ and $\delta$ is a solution to the equation $\tan(\delta) = c_2/c_1$. Note that there are multiple solutions to this last equation and care must be taken to determine the correct quadrant in which $\delta$ is located (see Example 4.4.1). The constant $R$ is the **amplitude** of this solution and the angle $\delta$ is referred to as the **phase shift**. The benefit to writing the solution in this form is that it make clear the amplitude of the oscillations exhibited by the solution, which in many applications is quantity of primary interest.

**Example 4.4.1.** Consider the initial value problem

$$x'' + x = 0 \qquad x(0) = -2, \quad x'(0) = 2.$$

Solving this equation with the given initial conditions yields

$$x(t) = -2\cos(t) + 2\sin(t).$$

If we look at the graph of this solution, it's clear that the oscillation has an amplitude greater than 2. To find the exact amplitude, we rewrite this solution in phase-amplitude form. From the formulas above, we have $R = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$ and $\tan(\delta) = -1$. The correct value of $\delta$, however, is *not* given by $\tan^{-1}(-1) = -\pi/4$. Note that from $c_1 = R\cos(\delta)$ we know that $\cos(\delta) < 0$ and therefore $\delta$ must lie in the second or third quadrant. Similarly, $c_2 = R\sin(\delta)$ which is positive in this example and we conclude that $\delta$ lies in second quadrant. The correct value of $\delta$ is $3\pi/4$:

$$x(t) = 2\sqrt{2}\cos(t - 3\pi/4).$$

Next we turn to the unforced equation with the addition of a damping term:

$$mx'' + \gamma x' + kx = 0.$$

As outlined in a previous sections, the general solution will take one of three forms, depending on the roots of the characteristic equation.

- In the case of two distinct real roots $\lambda_1$ and $\lambda_2$, the general solution will have the form

  $$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

  We can show that because the parameters $m$, $\gamma$, and $k$ are all positive, both roots $\lambda_1$ and $\lambda_2$ must be negative, giving solutions that decay to zero exponentially. In this scenario the solutions are said to be **overdamped**.

- In the case of a single (repeated) real root $\lambda$, the general solution is

  $$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}.$$

  Again, we can show that $\lambda$ is negative, resulting in solutions that decay to zero. These solutions are described as **critically damped**. This situation is of special interest because the solutions approach equilibrium as fast as possible without oscillation. If there is less damping, then oscillations are introduced which decay more slowly than the critically damped case and if there is greater damping, there is an exponential term that approaches zero slower than the critically damped case.

- In the case of complex roots to the characteristic equation, the general solution is

  $$x(t) = c_1 e^{\alpha t} \sin(\beta t) + c_2 e^{\alpha t} \cos(\beta t).$$

  where $\alpha = -\gamma/2m$. In order to have complex roots at all, the discriminant of the characteristic equation must be negative, which implies that $\gamma < 2\sqrt{mk}$. This in turn implies that the decay rate of the oscillations is less than the critically damped case. This situation is referred to as **underdamped**.

**Example 4.4.2.** A mass-spring system has a mass of 3 kg and spring constant of 12 N/m. Find the damping coefficient that makes the system critically damped.

The differential equation describing this system is $3x'' + \gamma x' + 12x = 0$ and so the characteristic equation is $3\lambda^2 + \gamma\lambda + 12 = 0$. To achieve critical damping, we need the discriminant to be zero: $\gamma^2 - 4(3)(12) = 0$ which gives $\gamma = 12$ Ns/m (taking the positive square root).

### 4.4.2 Forced Oscillations

We first consider the case of undamped motion:

$$mx'' + kx = F_0 \cos(\omega t).$$

As before, the solution to the associated homogeneous equation is

$$x_h = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t),$$

where $\omega_0 = \sqrt{k/m}$ is the natural frequency. Next, we find a particular solution using the method of undetermined coefficients. There are two possibilities to consider: first, suppose that $\omega_0 \neq \omega$, in which case we guess $x_p = A\cos(\omega t)$ and solve for $A$ to get

$$x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

Note that we do not need to include a sine term in the guess since the left hand side will generate only cosines anyway. The general solution is then

$$\boxed{x = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).}$$

or written using phase-amplitude formulation

$$x = C\cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

Although the solution is a sum of cosines, the behavior of the solution is not so clear since they have different frequencies. The following example explores this in a bit more detail.

**Example 4.4.3.** Consider the initial value problem

$$0.5x'' + 8x = 10\cos(\pi t), \qquad x(0) = 0, \qquad x'(0) = 0.$$

First we identify the parameters: $\omega = \pi$, $\omega_0 = \sqrt{8/0.5} = 4$, $F_0 = 10$, $m = 0.5$. The general solution is therefore

$$x = C_1 \cos(4t) + C_2 \sin(4t) + \frac{20}{16 - \pi^2} \cos(\pi t).$$

Now we solve for $C_1$ and $C_2$ using the initial conditions to find $C_1 = \frac{-20}{16-\pi^2}$ and $C_2 = 0$. Hence

$$x = \frac{20}{16 - \pi^2}(\cos(\pi t) - \cos(4t)).$$

The behavior of this solution is illuminated somewhat by using the trigonometric identity

$$2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right) = \cos B - \cos A$$

to rewrite the solution as

$$x = \frac{20}{16-\pi^2}\left(2\sin\left(\frac{4-\pi}{2}t\right)\sin\left(\frac{4+\pi}{2}t\right)\right).$$

This shows us that the solution $x$ can be thought of as a high frequency signal whose amplitude is modulated by a low frequency wave. The graph of the solution in Figure 4.1 shows this "beating" behavior.

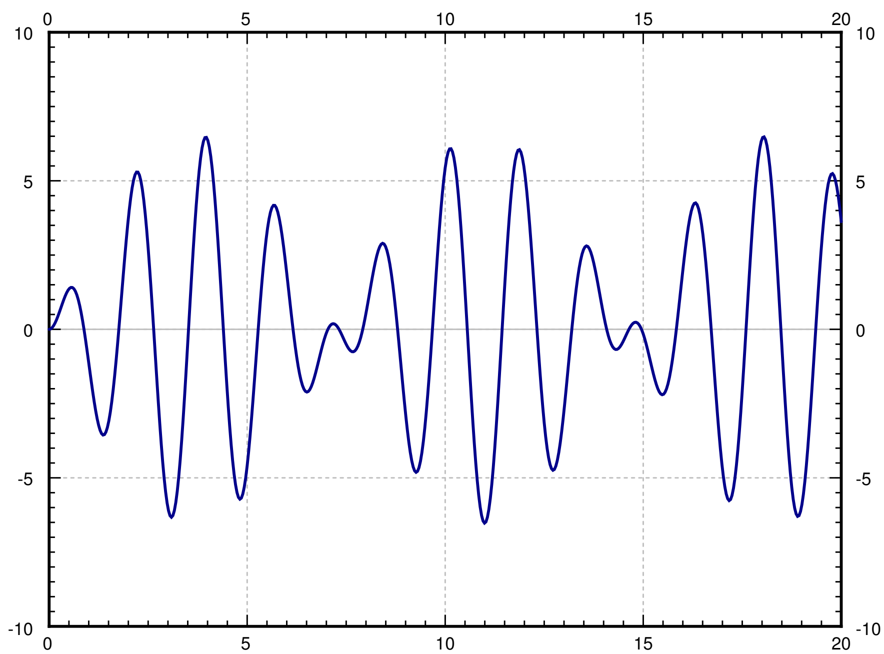



**Figure 4.1:** Graph of $\frac{20}{16-\pi^2}(\cos(\pi t) - \cos(4t))$.

Now we consider the second possibility for the nonhomogeneous term. Suppose that $\omega_0 = \omega$. In this case, the particular solution that we guessed previously cannot be used since it is already a solution to the associated homogeneous equation. We therefore use $x_p = At\cos(\omega t) + Bt\sin(\omega t)$. This time we do need the sine term since the second derivative of $t\cos(\omega t)$ generates sines. Plugging into the left hand side and solving for the undetermined coefficients, we get the particular solution $\frac{F_0}{2m\omega} t \sin(\omega t)$ and our general solution is therefore

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{F_0}{2m\omega} t \sin(\omega t).$$

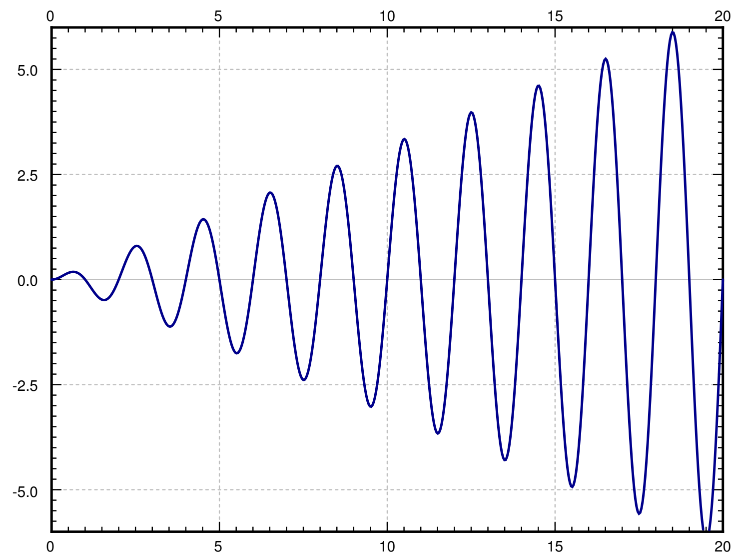



**Figure 4.2:** Graph of $\frac{1}{\pi} t \sin(\pi t)$.

The important term is the last one (i.e. the particular solution). Notice that as $t \to \infty$, this term grows without bound, oscillating between $\frac{F_0 t}{2m\omega}$ and $\frac{-F_0 t}{2m\omega}$. On the other hand, the first two terms contribute an oscillation with an amplitude $\sqrt{c_1^2 + c_2^2}$, which becomes smaller and smaller in relationship to the oscillations of the last term as $t$ gets larger. In Figure 4.2 we see the graph with $c_1 = c_2 = 0$, $F_0 = 2$, $m = 1$, $\omega = \pi$.

We've seen that by forcing the system at just the right frequency (i.e. at the natural frequency) we produce very large oscillations. This kind of behavior is called *resonance* or sometimes *pure resonance*.

In applications, resonance is sometimes desired. For example, while swinging on a swing set, you know that by leaning back and forth while kicking your legs you can get yourself swinging higher. The force of each kick is small, but after a while they produce large swings. On the other hand resonance can be destructive. In an earthquake some buildings collapse while others may be relatively undamaged. This is typically because different buildings have different resonance frequencies.

### 4.4.3 Damped Forced Motion

Of course, many real-world applications would also include friction. We say most since in some cases, the effect of friction is so small that we can consider it negligible and the methods of the previous section apply. If friction is not small, however, it may be important to include an additional damping term. The differential equation in this case becomes

$$mx'' + \gamma x' + kx = F_0 \cos(\omega t) \tag{4.37}$$

for some $\gamma > 0$.

The homogeneous part of this equation was addressed previously in section 4.2. Furthermore, it can be shown that in the case when all the coefficients are positive, then although the exact form of the homogeneous solution depends on the value of the eigenvalues, the solution must contain a decaying exponential. This means two things: first, the homogeneous part of the solution will decay to zero, leaving the particular solution behind, and second, we know that the particular solution can be expressed as $x_p = A\cos(\omega t) + B\sin(\omega t)$ and found using the method of undetermined coefficients.

Plugging in and solving for $A$ and $B$ (after some heroic algebra), we find the particular solution

$$x_p = \frac{m(\omega_0^2 - \omega^2)F_0}{\gamma^2\omega^2 + m^2{(\omega_0^2 - \omega^2)}^2} \cos(\omega t) + \frac{\gamma\omega F_0}{\gamma^2\omega^2 + m^2{(\omega_0^2 - \omega^2)}^2} \sin(\omega t)$$

where again, we use the notation $\omega_0 = \sqrt{k/m}$. In the case where $\omega \neq \omega_0$, it is usually more convenient to express this solution in phase-amplitude form:

$$\boxed{x_p = \frac{F_0}{\sqrt{\gamma^2\omega^2 + m^2(\omega_0^2 - \omega^2)^2}} \cos(\omega t - \delta)}$$

where the phase shift is given by

$$\tan\delta = \frac{\gamma\omega}{m(\omega_0^2 - \omega^2)}$$

As mentioned above, the terms in the general solution that contain information about the initial conditions contain decaying exponentials and therefore go to zero as $t \to \infty$, leaving only this periodic solution. For this reason, $x_h$ is called the **transient solution** and $x_p$ is called the **steady-state solution**. Figure 4.3 on the following page shows graphs for different initial conditions that all converge to the steady-state solution.

Notice that the speed at which the transient solution converges to the steady-state depends on the relationship between $\gamma$ and the other two parameters; larger values of $\gamma$ correspond to faster convergence, while for smaller values the "transient region" becomes larger. This makes sense if we recall that the parameter $\gamma$ represents damping. This also agrees with the observation that when $\gamma = 0$, the initial conditions affect the behavior for all time (i.e. we have an infinite "transient region").

In the case where damping is present, we no longer have resonance in the sense that solutions become unbounded. However, we can still investigate the effect that different values of the forcing frequency has on the amplitude of the steady state solution.

Letting $R$ denote the amplitude of $x_p$. Then plotting $R$ as a function of $\omega$ (with all other parameters fixed) we can look for a maximum. We call the $\omega$ that achieves this maximum the **practical resonance frequency**. We call the maximal amplitude $R(\omega)$ the **practical resonance amplitude**. Thus when damping is present we talk of practical resonance rather than pure resonance. A sample plot for three different values of $\gamma$ is given in Figure 4.4. As you can see the practical resonance amplitude grows as damping gets smaller, and any practical resonance can disappear when damping is large.

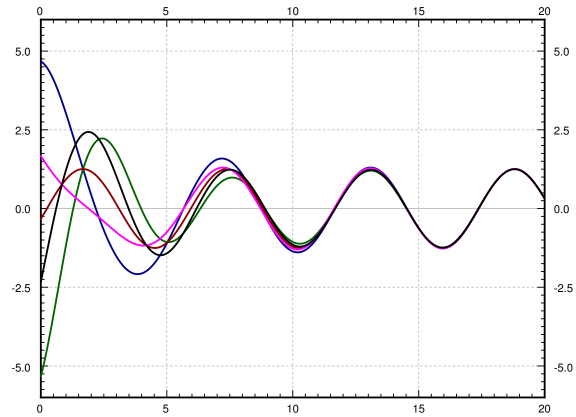


**Figure 4.3:** Solutions with different initial conditions for parameters $k = 1$, $m = 1$, $F_0 = 1$, $\gamma = 0.7$, and $\omega = 1.1$.

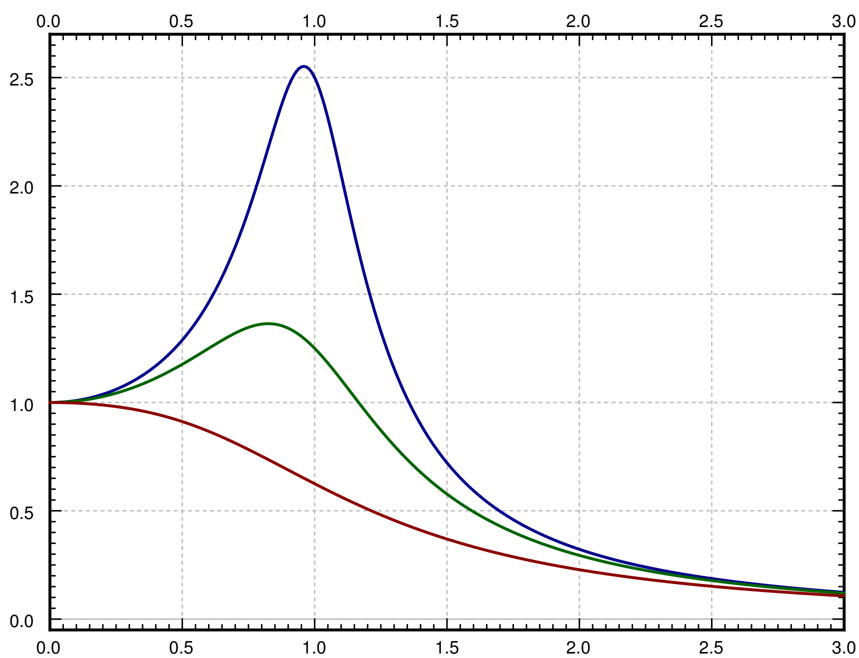


**Figure 4.4:** Graph of $R(\omega)$ showing practical resonance with parameters $k = 1$, $m = 1$, $F_0 = 1$. The top line is with $\gamma = 0.4$, the middle line with $\gamma = 0.8$, and the bottom line with $\gamma = 1.6$.

We can use standard calculus techniques to find the maximum of $R(\omega)$. Differentiating, we find that $R'(\omega) = 0$ when either

$$\boxed{\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{2m^2}} \quad \text{or} \quad \omega = 0.}$$

When $\gamma < \sqrt{2}m\omega_0$, then we can say that $\sqrt{\omega_0^2 - \gamma^2/2m^2}$ is the practical resonance frequency, i.e. it is the value of $\omega$ for which $R(\omega)$ is maximal. Note that in the case when $R(\omega)$ achieves it's maximal value for $\omega = 0$ then essentially there is no practical resonance since $\omega$ is generally assumed to be a positive parameter. In this case the amplitude gets larger as the forcing frequency gets smaller.

If practical resonance does occurs, it occurs at a frequency that is smaller than $\omega_0$. As damping $\gamma$ becomes smaller, the closer the practical resonance frequency gets to $\omega_0$. So when damping is very small, $\omega_0$ is a good estimate of the resonance frequency. This behavior agrees with the observation that when $\gamma = 0$, then $\omega_0$ is the resonance frequency.

Finally, we note that the behavior of steady-state solutions behavior will be more complicated if the forcing function is not an exact cosine wave.

## 4.5 Variable Coefficients

Up until this point, we have focused on second-order linear equations with constant coefficients. In this section, we look at a few techniques that can be applied when the coefficients are variable:

$$y'' + p(t)y' + q(t)y = 0. \tag{4.38}$$

Compared to the constant coefficient case, however, variable coefficients present a much greater challenge and only in few special cases is it possible to find exact solutions. Although the general strategy outlined in section 4.1 still applies, the first step of finding two fundamental solutions is much more difficult. Even an equation like

$$u'' - tu = 0$$

which looks quite simple does not have solutions in terms of familiar functions.

### 4.5.1 Cauchy-Euler Equations

One special case that *can* be solved analytically is the second-order differential equation that can be written in the form

$$y'' + \frac{b}{t}y' + \frac{c}{t^2}y = 0 \tag{4.39}$$

where $b$ and $c$ are constants. An equation of this form is called a **Cauchy-Euler** equation. Notice that a Cauchy-Euler equation may also be written in the form

$$t^2y'' + bty' + cy = 0. \tag{4.40}$$

As we can see from writing the Cachy-Euler equation in the first form, the interval of existence cannot include the point $t = 0$. Typically, we will make the assumption when dealing with Cauchy-Euler equations that $t > 0$.

Our approach to equations of this form is similar to our approach to the constant coefficient case: we will make a guess for which each of the three terms in (4.39) yields the same *type* of expression. For the constant coefficient case, we guessed exponential solutions because differentiating exponentials gives us exponentials. In this case, we must try something different because of the extra factors of $t$. What would happen if we guessed a polynomial solution? In particular, consider a solution of the form $t^m$, a $m$-th degree polynomial. We know that differentiating a $m$-th degree polynomial gives us a $m-1$-th degree polynomial, but then if we multiply by a factor of $t$, we get back a $m$-th degree polynomial. Let's write this out in more detail. If we substitute the guess $y = t^m$ into (4.40), we get

$$m(m-1)t^m + bmt^m + ct^m = 0 \tag{4.41}$$

$$\Rightarrow \qquad m(m-1) + bm + c = 0 \tag{4.42}$$

$$\Rightarrow \qquad m^2 + (b-1)m + c = 0 \tag{4.43}$$

Just as in the constant coefficient case, we now have a quadratic equation which we can solve to get values of $m$ for which $t^m$ is a solution. And just as in the constant coefficient case, we will refer to this quadratic equation as the **characteristic equation**.

There are three cases to consider. If there are two real roots $m_1$ and $m_2$ to (4.43) then we have our two independent solutions: $t^{m_1}$ and $t^{m_2}$ and so the general solution to the Cauchy-Euler equation is

$$y = c_1 t^{m_1} + c_2 t^{m_2}. \tag{4.44}$$

In the second case, the characteristic equation has two equal roots: $m_1 = m_2$ (and since they're the same, we'll just refer to the root as $m$). Now we have one solution $t^m$, but still need a second independent solution to form the general solution. We can find our second solution by looking for a solution of the form $y = v(t)t^m$. Substituting this guess into the Cauchy-Euler equation, we get

$$v''t^{m+2} + 2v'mt^{m+1} + vm(m-1)t^m + b[v't^{m+1} + vmt^m] + cvt^m = 0. \tag{4.45}$$

Remember that $b$ and $c$ are given constants and that $m$ is the root of the characteristic equation. Our goal is to determine the function $v$. If we collect the terms with the same powers of $t$, we have

$$v''t^{m+2} + [2v'm + bv']t^{m+1} + [vm(m-1) + bvm + cv]t^m = 0. \tag{4.46}$$

Now, if we recall that $m$ is a root of the characteristic equation, then we see that the third term is zero. This then simplifies our equation to

$$v''t^{m+2} + [2v'm + bv']t^{m+1} = 0 \tag{4.47}$$

and if we now divide by $t^{m+1}$, we get

$$v''t + v'(2m + b) = 0. \tag{4.48}$$

If we also remember that in the second case (repeated roots) the solution to the quadratic is in fact $m = (1 - b)/2$. Plugging this into the equation for $v$ gives

$$v''t + v' = 0. \tag{4.49}$$

Now although this is a second-order equation with variable coefficients, it can be simplified because there is no $v$ term. If we let $w = v'$ then the equation becomes

$$w't + w = 0 \tag{4.50}$$

Which is a separable first-order equation. Solving this for $w$ gives

$$w = \frac{d}{t}.$$

Then remembering that $w = v'$, we integrate to get

$$v = d_1 \ln t + d_2$$

where $d_1$ and $d_2$ are arbitrary constants. At this point, we can go back and replace $v$ in our guess and get $y = (d_1 \ln t + d_2)t^m$. Notice that although we were looking for the second of our independent solutions, we actually ended up with a solution that includes the first independent solution. Since our equation is still linear, we may subtract the solution $d_2 t^m$ to get a simpler solution. To summarize, the general solution in the case of repeated roots is

$$y = c_1 t^m + c_2 t^m \ln t. \tag{4.51}$$

Notice that our assumption that $t > 0$ (as discussed above) has the added benefit that it simplifies the expression of the solution since we do not have to worry about including an absolute value sign in the logarithm.

In the third and final case, the roots of the characteristic equation are complex valued: $m = \alpha \pm i\beta$. As we did in the constant coefficient case, we can rewrite the solutions $t^{\alpha \pm i\beta}$ using Euler's formula so that

$$t^{\alpha \pm i\beta} = t^\alpha t^{\pm i\beta} = t^\alpha e^{\ln t^{i\beta}} = t^\alpha e^{i\beta \ln t} = t^\alpha[\cos(\beta \ln t) \pm i \sin(\beta \ln t)].$$

This gives us two solutions (one each for the two signs). If we add these two solutions together, the result is still a solution. Similarly, subtracting these two solutions gives a solution. This gives us the two independent solutions that we need to form the general solution, which can now be written as

$$y = c_1 t^\alpha \cos(\beta \ln t) + c_2 t^\alpha \sin(\beta \ln t).$$

**Example 4.5.1.** Find the general solution to the equation

$$t^2y'' + 3ty' + y = 0.$$

Plugging in the guess $y = t^m$ gives the characteristic equation $m(m-1) + 3m + 1 = 0$ which simplifies to $m^2 + 2m + 1 = 0$. There is only one root to this quadratic: $m = -1$ and therefore the solution is of the form

$$y = c_1t^{-1} + c_2t^{-1}\ln t.$$

## 4.6 Power Series Solutions

What about more general examples of linear equations with variable coefficients? As we have already mentioned, most equations cannot be solved in terms of the simple, elementary functions that we are most familiar with. However, there are other ways to describe solutions. As we know from calculus, many functions that we're familiar can be represented with a power series. If we think about how those power series representations are generated, we realize that each coefficient in the power series is given in terms of derivatives. This suggests that even if we don't know any thing else about the function, as long as we have some information about the derivatives, we may be able to generate a power series. But information about the derivatives is precisely what a differential equation gives us!

Historically, the use of power series has been very important in the study of differential equations. Prior to the availability of computers and the ease with which they can provide an approximation of solutions, a series solution was often used to generate approximations to solutions, particularly in those cases for which an exact solution could not be found. Additionally, there many important equations for which precise series representations can be found, even though they don't correspond to familiar functions.

The basic idea is to guess that a solution to our differential equation has a power series representation

$$u = \sum_{i=0}^{\infty} a_i t^i. \tag{4.52}$$

Our job is to determine the coefficients by plugging this series into the differential equation. In this section, we will look at a couple of examples of how to find series solutions. There is much more to this topic that we will not get into, primarily because of time constraints. For a more thorough treatment of this topic the reader may wish to consult the text by Boyce & DiPrima.

### Brief Review of Series Notation

The next step is to substitute the power series representation for $u$ into the differential equation. To accomplish this, we will need to differentiate the series term-by-term. The only thing tricky about this is how to represent the results using summation notation. To illustrate, we will first consider the series written out term by term:

$$u(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \dots$$

and then differentiate term by term:

$$u'(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + \dots$$

The issue is how to express $u'$ using summation notation. If you're familiar with series notation, you will realize that there are different ways that we could choose to represent this last sum. The key is deciding whether we want the index $i$ to correspond to the unknown coefficients $a_i$ or to the powers of $t$. In the expansion of $u$ they're the same, but in $u'$ they not, and so we can make a choice. For instance, if we decide that we want the index $i$ to correspond to the coefficients, then series expansion for $u'$ would begin with $i = 1$ and each term includes $t$ raised to the power $i - 1$:

$$u'(t) = \sum_{i=1}^{\infty} i a_i t^{i-1} \tag{4.53}$$

Notice that this expression is exactly what we get if we simply apply the power rule to the original function written in summation notation (4.52), with the starting value of $i$ shifted from 0 to 1.

On the other hand, we could write our sum so that the index $i$ corresponds to the power of $t$ in each term. In this case, we would write

$$u'(t) = \sum_{i=0}^{\infty} (i+1) a_{i+1} t^i \tag{4.54}$$

shifting the index in order to get the correct values for the coefficients. To summarize, the sums given in (4.53) and (4.54) are the same series, they're just written differently.

The second derivative gets handled the same way and as you can verify, $u''$ can be expressed in the following equivalent ways:

$$u''(t) = \sum_{i=2}^{\infty} i(i-1) a_i t^{i-2} = \sum_{i=0}^{\infty} (i+2)(i+1) a_{i+2} t^i$$

**An Example**

To illustrate how this method works, we will use the differential equation mentioned above, namely

$$u'' - tu = 0.$$

If we suppose that the solutions have a power series representation, then we can express the solution as

$$u = \sum_{i=0}^{\infty} a_i t^i \tag{4.55}$$

and plugging this series into the differential equation gives

$$\left(\sum_{i=2}^{\infty} i(i-1)a_i t^{i-2}\right) - t\left(\sum_{i=0}^{\infty} a_i t^i\right) = 0. \tag{4.56}$$

In the second term, the series is multiplied by a factor of $t$, so we can incorporate that into the series by adding one to the exponent:

$$\sum_{i=2}^{\infty} i(i-1)a_i t^{i-2} - \sum_{i=0}^{\infty} a_i t^{i+1} = 0. \tag{4.57}$$

We would like to combine these two series into a single sum, but to do so we need to rewrite them so that the powers of $t$ can be compared:

$$\sum_{i=0}^{\infty}(i+2)(i+1)a_{i+2}t^i - \sum_{i=1}^{\infty} a_{i-1}t^i = 0. \tag{4.58}$$

At this point, we can combine the two summations into a single sum, except for the fact that the first sum starts at $i = 0$ while the second starts at $i = 1$. We simply stripping off the first term of the first series, and include it as a separate term:

$$2a_2 + \sum_{i=1}^{\infty} \left((i+2)(i+1)a_{i+2} - a_{i-1}\right) t^i = 0. \tag{4.59}$$

Because the right hand side of this equation is zero, and the left hand side is a now expressed as a single power series, we can say that each coefficient is zero:

1. $2a_2 = 0$. This determines the value of a single coefficient, $a_2$.
2. $(i+2)(i+1)a_{i+2} - a_{i-1} = 0$. This gives us a relationship between coefficients. In particular, if we solve for $a_{i+2}$, we have
$$a_{i+2} = \frac{a_{i-1}}{(i+1)(i+2)}$$
which gives us a way of determining coefficients, provided we know the values of coefficients with lower indices. This formula is known as a **recursion relation**.

The next step is to use the recursion relation to rewrite each coefficient in terms of *previous* coefficients. Starting with $i = 1$, the recursion relation tells us:

$$\begin{aligned} a_3 &= \frac{a_0}{2 \cdot 3} \\ a_4 &= \frac{a_1}{3 \cdot 4} \\ a_5 &= \frac{a_2}{4 \cdot 5} \\ a_6 &= \frac{a_3}{5 \cdot 6} \\ a_7 &= \frac{a_4}{6 \cdot 7} \\ &\vdots \end{aligned}$$

But notice that $a_6$ is expressed in terms of $a_3$ and we already have $a_3$ expressed in terms of $a_0$, so we can write

$$a_6 = \frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}.$$

The next coefficient that is expressed in terms of $a_6$ is $a_9$:

$$a_9 = \frac{a_6}{8 \cdot 9} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}.$$

And so on. The recursion formula tells us that every third coefficient will be similarly expressed in terms of $a_0$. The same thing happens if we look at $a_7$, which can be expressed in terms of $a_4$ (and therefore $a_1$):

$$a_7 = \frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}$$

and then

$$a_{10} = \frac{a_7}{9 \cdot 10} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}.$$

The pattern continues with every third coefficient expressed in terms of $a_1$. Finally, we make use of the fact that $a_2 = 0$ which tells us that $a_5$, $a_8$, and every third coefficients after will also be zero.

We can now substitute our expressions for the coefficients into the general series representation of our solution:

$$\begin{aligned} u = \sum_{i=0}^{\infty} a_i t^i &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots \\ &= a_0 + a_1 t + \left(\frac{a_0}{2 \cdot 3}\right) t^3 + \left(\frac{a_1}{3 \cdot 4}\right) t^4 + \left(\frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}\right) t^6 + \left(\frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}\right) t^7 \ldots \end{aligned}$$

and then group all the $a_0$ and $a_1$ terms together:

$$= a_0\left(1 + \frac{1}{2\cdot 3}t^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}t^6 \dots\right) + a_1\left(t + \frac{1}{3\cdot 4}t^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}t^7 \dots\right)$$

At this point, we have done something remarkable – we have rewritten our solution $u$ in terms of two arbitrary coefficients, each of which is multiplied by a different power series. If we take these power series to *define* two functions, that is if we define

$$u_1 = 1 + \frac{1}{2\cdot 3}t^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}t^6 + \dots \tag{4.60}$$

$$u_2 = t + \frac{1}{3\cdot 4}t^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}t^7 + \dots \tag{4.61}$$

then we can write our solution as

$$u = a_0u_1 + a_1u_2. \tag{4.62}$$

Furthermore, notice that if we plug in the value $t = 0$, we get $u_1(0) = 1$ and $u_2(0) = 0$. This means that if we have the initial conditions

$$u(0) = u_0$$
$$u'(0) = u_0'$$

then the coefficient $a_0$ is precisely the initial condition $u_0$. If we differentiate the functions $u_1$ and $u_2$ and then plug in the value $t = 0$, we get $u_1'(0) = 0$ and $u_2'(0) = 1$. This implies that the coefficient $a_1$ is precisely the initial condition $u_0'$. Equation (4.62) represents the general solution to our differential equation and the two functions $u_1$ and $u_2$ are in fact solutions themselves.

In most cases, it will not be possible to simplify the series beyond what we have written in (4.60)-(4.61), although in some special cases we may be able to find a pattern in the coefficients that allows us to simplify our expressions for the coefficients. In our particular example, for instance, the denominators in our coefficients look very much like factorials. In fact, they are factorials with every third factor missing. So if $n$ is a multiple of 3, then we can write $n$ as $3k$ and then the $k$th coefficient in (4.60) could be written as

$$\frac{1}{2\cdot 3\cdot 5\cdot 6\cdots(n-1)n} = \frac{4\cdot 7\cdot 10\cdots(n-2)}{n!} = \frac{\prod_{j=1}^{k-1}(3j+1)}{(3k)!}$$

where $n$ is a multiple of 3.

## 4.7 Additional Exercises

**4.1:** Consider the second order equation

$$ay'' + by' + cy = 0$$

where $a$, $b$, and $c$ are constants. Show that if $y_1$ and $y_2$ are two solutions to this equation and $c_1$ and $c_2$ are any constants then the function

$$y = c_1 y_1 + c_2 y_2$$

is also a solution.

**4.2:** By guessing functions of the form $y = e^{\lambda t}$, find two solutions to the equation

$$y'' + 7y' + 10y = 0.$$

Then find another solution that satisfies the conditions $y(0) = 1$ and $y'(0) = 0$ (hint: do the previous problem first).

**4.3:** In order to measure the body mass of an astronaut in space, a special chair attached to springs is used. The frequency of the oscillation of the astronaut is measured and from this, the mass is computed. Suppose that the chair has been calibrated as follows: the chair itself has mass of 20kg and an additional 25kg is placed in the chair. The resulting oscillation has period of 1.30 seconds. What is the mass of an astronaut if the period of oscillation is 1.75 seconds? (Assume that the damping is so small that we can ignore it.)

**4.4:** A mass-spring system is described by the equation

$$y'' + y' + 2y = 2\cos(t)$$

with initial conditions $y(0) = a$, $y'(0) = 0$. Find and plot on the same axes the solutions for $a = 5, 1, -2$ on the interval $0 \le t \le 15$.

**4.5:** A mass-spring system is described by the equation

$$y'' + y' + 2y = 2\cos(\omega t).$$

(a) Find the steady-state part of the solution.

(b) Find the amplitude of the steady-state and graph it as a function of $\omega$.

(c) Find the maximum value of the amplitude and the frequency $\omega$ at which it occurs.

**4.6:** For each of the following second order equations, find the solution by using a substitution to reduce them to first order equations.

(a) $x'' = -\frac{2}{t}x'$

(b) $x'' = -4x$

**4.7:** For the following second order equations, find the solution that satisfies the initial conditions $u(0) = 1$ and $u'(0) = 0$.

(a) $u'' - 4u' + 4u = 0$

(b) $u'' + u' + 4u = 0$

**4.8:** Find the general solution to the following nonhomogeneous equations.

(a) $y'' + 2y' = 3 + 4\sin(2t)$

(b) $y'' - 4y' + y = 5te^{2t}$

**4.9:** Show that if $y_1$ and $y_2$ are two solutions to the 2nd order linear differential equation

$$y'' + py' + qy = 0$$

then the linear combination $y = c_1y_1 + c_2y_2$ is also a solution for any constants $c_1$ and $c_2$.

**4.10:** Consider the second order equation

$$y'' + py' + qy = 0.$$

If we are somehow able to find a solution, $y_1(t)$, to this equation we can often find a second solution by looking for a function of the form $y_2 = v(t)y_1(t)$. By plugging this into the DE, we end up with a first order equation for $v$. This method is called **reduction of order**.

(a) Show that in the case of repeated roots ($p^2 - 4q = 0$) this method produces the solution $y_2 = te^{\lambda t}$.

(b) Show that $y_1 = e^t$ is a solution to

$$(t-1)y'' - ty' + y = 0$$

and then use this method to find a second solution.

**4.11:** Consider the initial value problem

$$u'' + u = \begin{cases} t & 0 \le t < \pi \\ 2\pi - t & \pi \le t < 2\pi \\ 0 & 2\pi \le t \end{cases}$$

with the initial conditions $u(0) = u'(0) = 0$. Find the solution to this equation by considering each of the three time intervals separately. In other words, find the solution on the first interval. Then find the solution on the second interval that picks up where the first solution leaves off. Repeat for the third interval. Sketch a graph of your solution.

# Chapter 5

# Laplace Transforms

In this chapter, we introduce the Laplace transform and investigate how it can be used to solve certain types of differential equations. As we shall see, the Laplace transform technique is particularly well suited to linear equations that involve discontinuous or piecewise functions. The Laplace transform also gives us a convenient way to solve differential equations involving impulse forcing, something that would be quite cumbersome to do otherwise. Both of these types of of equations are common in applications.

## 5.1 Definitions and Basic Properties

To begin, let's discuss the term *transform*. We'll start with an analogy. If you think back to the first week of calculus, you may have spent some time discussing the definition of a *function*. One way to think about a function is as a black-box: given some "input" number, a function produces some other "output" number according to some rule. For example, given any number $x$ as input, the "squaring" function produces the number $x^2$ as output; we summarize this rule with the formula $f(x) = x^2$. The $x$ on the left hand side of the equation represents the "input", the $x^2$ on the right-hand side is the "output" (and the name "f" is simply a shorthand name we've given to the function).

A transform is a similar relationship between an input and an output, except now the input and output aren't numbers, they're functions. We've seen this idea before: when we differentiate a function ($f(t) = t^2$ for example), the result is a new function ($f'(t) = 2t$). The operation "differentiate" can be thought of as a transform that turns one function into a new one.

The Laplace transform is an example of what we call an **integral transform**, meaning that it is defined in terms of doing a integral.

**Definition.** The **Laplace transform** of the function $y(t)$ is the function

$$Y(s) = \int_0^\infty e^{-st} y(t)\, dt \tag{5.1}$$

(provided the integral exists). Note that since the definite integral is done with respect to $t$, the right hand side depends only upon $s$. We will also denote the Laplace transform using the notation

$$\mathcal{L}\{y(t)\}(s) = Y(s) \qquad \text{or simply} \qquad \mathcal{L}\{y(t)\} = Y(s).$$

Note: the use of the capital letter $Y$ here should not be confused with the common practice in calculus of using capital letters to denote an antiderivative!

Before looking at examples, a brief note about the existence of the transform. Simply put, the Laplace transform exists so long as the integral in the definition exists. So when does that integral exist? Since it's an improper integral, the main thing we have to worry about is that the function $y(t)$ does not grow too fast as $t$ goes to infinity. Since it's the product $e^{-st}y(t)$ that we're integrating, $y(t)$ shouldn't grow faster than any exponential function. More precisely, we require that there exist constants $M > 0$, $\alpha$, and $T$ for which

$$|y(t)| \leq Me^{\alpha t}$$

for all $t > T$, in which case we say that $y$ is of **exponential order**. We will also require $y$ to be at least piecewise continuous so that the integral doesn't fail to exist for reasons related to discontinuities in the integrand.

### 5.1.1 Brief and Optional Aside: Where Does This Come From?

Before moving on to investigate the properties and applications, here is a little motivation about where the Laplace transform comes from. If you're content to simply use the definition, you may safely skip this section.

Think about power series for a moment. We know that many functions can be expressed in terms of a power series. Here are two familiar examples:

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad |x| < 1.$$

In calculus, we typically start with a known function and then work out what the coefficients of the power series should be. But we can also think about power series the other way around; if we start with a set of coefficients, the resulting power series determines some function (provided it converges). This suggests that we can think about power series as a relationship between a sequence of coefficients $a_n$ on one side and a function $A(t)$ on the other:

$$\sum_{n=0}^{\infty} a_n x^n = A(x).$$

Next, we make the observation that the coefficients themselves can be thought of as functions over the non-negative integers; in other words, we can think of $a_0$ as $a(0)$, $a_1$ as $a(1)$, and so on. Rather than a sequence of coefficients, then, we have a coefficient *function* $a(n)$ which is defined for $n = 0, 1, 2, \ldots$. Although this change in the way we express the coefficients is really nothing more than different notation, it sets us up for the big idea!

Here is the big idea: what happens if we allow the coefficient function to be defined for *all* positive real numbers instead of just integers? In other words, we suppose the coefficient function $a(t)$ be defined for all $t$ on the interval $[0, \infty)$ instead of just the discrete indices $n$. Since we now want to sum over a continuous index, we need to replace the summation with an integral, which gives us:

$$\int_0^{\infty} a(t) x^t \, dt = A(x).$$

Just as with the original power series, this defines a relationship between a set of coefficients (or rather the coefficient *function* $a(t)$) on the left and the function $A(x)$ on the right.

Now, let's rewrite this integral slightly. First of all, it's more conventional when working with exponential functions to use $e$ as a base. Therefore we rewrite

$$x^t = e^{\ln x^t} = e^{t \ln x}.$$

Since we want to make sure that the integral converges, we would like $\ln x$ to be negative. To emphasize this, we can introduce $s = -\ln x$ and write

$$\int_0^{\infty} a(t) e^{-st} \, dt = A(e^{-s}).$$

Finally, we adopt the more traditional name $f$ for the input function and rename the output function to match and we get

$$\int_0^{\infty} f(t) e^{-st} \, dt = F(s).$$

This is precisely the definition of the Laplace transform, obtained by generalizing the standard power series to a sum over a continuous index.

### 5.1.2 Transforms of Common Functions

Before we can use the Laplace transform to help us solve differential equations, we will need to know how to apply it to a variety of common functions. We will also need to learn some of the basic properties of the transform. As we will see, applying the definition can be cumbersome and we certainly don't want to repeat these calculations every time we go to solve a differential equation. In this section we will begin building up a library of facts about the Laplace transform which we can later use. Once we have derived the transform of a particular function, we will make a note of it and simply use that result in subsequent problems (rather than re-derive the transforms each time we need them). For your convenience, a table of these results is given in Appendix C.

**Example 5.1.1.** We'll start with a very simple function, $f(t) = c$. From the definition, we compute

$$F(s) = \int_0^\infty e^{-st} c\, dt = \frac{-c}{s} e^{-st}\Big|_0^\infty .$$

Now, in order for this integral to exist, we need the quantity $-st$ to go to zero as $t \to \infty$. This means that $s$ must be positive so that we have a decaying exponential. Then we have $F(s) = \frac{-c}{s}(0-1) = \frac{c}{s}$ for $s > 0$. We have just shown that

$$\boxed{\mathcal{L}\{c\} = \frac{c}{s} \qquad s > 0} \tag{5.2}$$

The condition that $s$ must be positive is simply specifying the domain of the the function $\mathcal{L}\{c\}(s)$

**Example 5.1.2.** Consider the function, $f(t) = t$. Using the definition and integrating by parts, we get

$$F(s) = \int_0^\infty e^{-st} t\, dt = \frac{-t}{s} e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt = \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s^2}.$$

Just as in the previous example, we need to impose the condition $s > 0$ so that the integral exists. Notice also that we have used the result from the previous example. We have shown that

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2} \qquad s > 0} \tag{5.3}$$

**Example 5.1.3.** Consider the function $f(t) = e^{at}$. From the definition, we compute

$$F(s) = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \frac{1}{(a-s)}e^{(a-s)t}\Big|_0^\infty = \frac{1}{(a-s)}(0-1) = \frac{1}{s-a}.$$

Again, in order for the integral to exist, we need to impose the condition $s > a$ so that the exponential function decays to 0 as $t \to \infty$. We have shown that

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad s > a} \tag{5.4}$$

**Example 5.1.4.** Compute the Laplace transform of $\sin(at)$. From the definition, we can integrate by parts twice:

$$\begin{aligned}
\mathcal{L}\{\sin(at)\} = \int_0^\infty e^{-st}\sin(at)\,dt &= -\frac{1}{s}e^{-st}\sin(at)\Big|_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)\,dt \\
&= \frac{a}{s}\left[-\frac{1}{s}e^{-st}\cos(at)\Big|_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)\,dt\right] \\
&= \frac{a}{s^2} - \frac{a^2}{s^2}\mathcal{L}\{\sin(at)\}
\end{aligned}$$

Then solving for $\mathcal{L}\{\sin(at)\}$ gives

$$\boxed{\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}} \tag{5.5}$$

A similar calculation shows that

$$\boxed{\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}} \tag{5.6}$$

At this point, we have seen how to use the definition to compute the Laplace transform of a variety of common functions. As this last example illustrates, however, these computations can sometimes be a bit cumbersome. In the next section, we will investigate some basic properties of the transform, which will make computing transforms much simpler in many cases.

### 5.1.3 Basic Properties of the Transform

In calculus, one of the first things that we did after learning the definition of the derivative was to look at some basic properties and rules that made it much easier to actually compute derivatives. For instance, given the function $f(t) = 3t^2 - 2t + 6$, we know that we can differentiate each term in the polynomial separately. We also use the power rule to differentiate the second order term. We certainly do *not* want to use the definition of the derivative! Below we look at some similar properties for the Laplace transform.

#### Linearity

First, let's consider the sum of two functions. It would be nice if we could take the transform of the two functions and add the result together. Of course, simply wishing something is true doesn't make it true! Using the definition of the transform as well as properties of integrals, we have

$$\begin{aligned}\mathcal{L}\{f(t)+g(t)\} &= \int_0^\infty e^{-st}(f(t)+g(t))\,dt \\ &= \int_0^\infty e^{-st}f(t)\,dt + \int_0^\infty e^{-st}g(t)\,dt = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}\end{aligned}$$

at least if the integrals in the second line exist. Notice that we can do something similar if we have a constant times a function:

$$\mathcal{L}\{cf(t)\} = \int_0^\infty e^{-st}cf(t)\,dt = c\int_0^\infty e^{-st}f(t)\,dt = c\mathcal{L}\{f(t)\}$$

We have just proved the following theorem.

**Theorem 5.1.1** (Linearity)**.** *Given two functions $f(t)$ and $g(t)$ and any two constants $c_1$ and $c_2$, the Laplace transform satisfies*

$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 F(s) + c_2 G(s) \tag{5.7}$$

*provided that the transforms $F(s)$ and $G(s)$ exist.*

#### Shift Property

In the previous section, we looked at what happens when we take the transform of a sum of two functions. Of course, there are many other ways of combining two functions. Here, we look at the special case of the product of a function and an exponential.

Suppose we take the transform of an exponential function times a function $u(t)$. From the definition, we have

$$\mathcal{L}\{u(t)e^{at}\} = \int_0^\infty e^{-st}u(t)e^{at}\,dt = \int_0^\infty e^{-(s-a)t}u(t)\,dt.$$

If we compare this with the definition of the Laplace transform, this is precisely the transform $U$ evaluated at the point $s-a$ rather than $s$. We will refer to this rule as the **shift property**:

$$\boxed{\mathcal{L}\{u(t)e^{at}\} = U(s-a)} \tag{5.8}$$

### Other Useful Properties

There are many other useful properties that will help us compute Laplace transforms. Here is one that we will use often. Consider the transform $U(s)$:

$$U(s) = \int_0^\infty e^{-st}u(t)\,dt.$$

If we differentiate this with respect to $s$, we get

$$U'(s) = \int_0^\infty (-t)e^{-st}u(t)\,dt = -\mathcal{L}\{tu(t)\}.$$

Moving the negative sign to the other side gives this expression for computing the transform of the product $tu(t)$:

$$\boxed{\mathcal{L}\{tu(t)\} = -U'(s)} \tag{5.9}$$

**Example 5.1.5.** Compute the transform of $f(t) = t^2$. Since we already have shown that

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

we can apply new useful property (5.9):

$$\mathcal{L}\{t^2\} = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}.$$

**Example 5.1.6.** By repeated use of the above useful property, we obtain the general formula

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad n = 0, 1, 2, \ldots}$$

Note: to be completely rigorous here, we need an induction argument. For hints about how to generalize this formula to non-integer exponents, see problems 5.10 and 5.11.

**Example 5.1.7.** Find the transform of $y(t) = te^{2t}$.

To use the shift property, we note that $a = 2$ and $f(t) = t$, in which case, $F(s) = 1/s^2$. Therefore,

$$Y(s) = \mathcal{L}\{e^{2t}t\} = F(s-2) = \frac{1}{(s-2)^2}.$$

Alternatively, we can find this transform using useful property (5.9). Since we know that $\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$, we have

$$Y(s) = -\frac{d}{ds}\left(\frac{1}{s-2}\right) = \frac{1}{(s-2)^2}.$$

**Example 5.1.8.** Find the transform of $e^{-3t}\cos(2t)$.

We use the shift property with $f(t) = \cos(2t)$, so that $F(s) = s/(s^2+4)$. Then

$$\mathcal{L}\{e^{-3t}\cos(2t)\} = F(s+3) = \frac{s+3}{(s+3)^2+4}.$$

### Inverse Transforms

The usefulness of the Laplace transform as a method for solving differential equations depends not only on our ability to compute the transforms of a variety of functions but also on our ability to go the other direction. In other words, we need to be able to recognize when an expression is the transform of some function. The process of recovering a function $f(t)$ given its transform $F(s)$ is described as finding the *inverse transform* of $F$. We use the notation

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

**Example 5.1.9.** Find the inverse Laplace transform of $F(s) = \dfrac{1}{s+2}$.

Reviewing the list of transforms that we have computed, we see that $F(s)$ has the form $\dfrac{1}{s-a}$ with $a = -2$ and therefore $f(t) = e^{-2t}$.

We're actually making a fairly big assumption here about inverse transforms, namely that inverse transforms are unique. In this last example, for instance, how do we know that the function $f(t) = e^{-2t}$ is the *only* function whose Laplace transform is $1/(s+2)$? As it happens, there is a formula for computing inverse transforms, but it involves contour integrals in the complex plane and it therefore beyond the scope of this course. In this course, we will simply take the uniqueness of the inverse transform for granted.

**Example 5.1.10.** Find the inverse Laplace transform of $F(s) = \dfrac{1}{s^2-1}$.

Reviewing the list of transforms that we have computed, we don't see anything that looks like $F(s)$. The transforms for sine and cosine do have an $s^2$ in the denominator, but there is no (real) value of $a$ that would make $s^2 + a^2$ equal to $s^2 - 1$. Instead, we factor the denominator and then use a partial fraction decomposition:

$$F(s) = \frac{1}{s^2-1} = \frac{1/2}{s-1} - \frac{1/2}{s+1} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

and therefore $f(t) = (e^t - e^{-t})/2$.

This last example is typical of many of the problems that we'll be seeing in that we will usually need to rewrite an expression before we recognize it as the transform of something. Here is another example in which we need to rewrite before we can use the shift property.

**Example 5.1.11.** Find the inverse transform of $\dfrac{1}{s^2+2s+5}$.

To use the shift property, we need to rewrite the given expression so that it looks like $F(s-a)$ for some $a$. Completing the square in the denominator, we have

$$\frac{1}{s^2+2s+5} = \frac{1}{(s^2+2s+1)+4} = \frac{1}{(s+1)^2+4}.$$

We can now call this expression $F(s+1)$ where $F(s) = \dfrac{1}{s^2+4}$, and the inverse transform is then

$$\mathcal{L}^{-1}\{F(s+1)\} = e^{-t}f(t) = \frac{1}{2}e^{-t}\sin(2t).$$

**Example 5.1.12.** Find the inverse transform of $\dfrac{s}{s^2+2s+5}$.

As in the previous example, we rewrite the denominator to get

$$\frac{s}{(s+1)^2+4}.$$

We want to call this $F(s+1)$, but in order to figure out what $F(s)$ is, we need to see a "$s+1$" in the numerator to match the denominator. We do this by adding and subtracting:

$$F(s+1) = \frac{s}{(s+1)^2+4} = \frac{(s+1)-1}{(s+1)^2+4} = \frac{(s+1)}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}$$

Now we can see that

$$F(s) = \frac{s}{s^2+4} - \frac{1}{s^2+4}$$

and therefore $f(t) = \cos(2t) - \frac{1}{2}\sin(2t)$. Putting it all together,

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\{F(s+1)\} = e^{-t}f(t) = e^{-t}\left[\cos(2t) - \frac{1}{2}\sin(2t)\right].$$

### Sine and Cosine Revisited

As further examples of how the basic properties of the Laplace transform can simplify the process of computing transforms, we return to the sine and cosine functions. Here are two other ways of finding their transforms.

Instead of dealing directly with the sine and cosine function, we can make use of the complex exponential. Recall Euler's formula

$$e^{iat} = \cos(at) + i\sin(at).$$

Now take the transform of both sides of this equation, making use of the linearity property on the right hand side. On the left hand side, we will apply our previously derived result for the transform of an exponential. Although our proof for that formula was done with the assumption that the exponent was real, it doesn't take much work to show that it also applies to complex exponents.

$$\frac{1}{s-(ia)} = \mathcal{L}\{\cos(at)\} + i\mathcal{L}\{\sin(at)\}$$

Now under the not unreasonable assumption that the Laplace transforms of sine and cosine are themselves real valued functions, we observe that the right hand side of this equation is a complex-valued expression separated into real and imaginary parts. Rewriting the left hand side into real and imaginary parts by multiplying the numerator and the denominator by the complex conjugate:

$$\frac{1}{s-ia} = \frac{1}{s-ia}\cdot\frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}.$$

Now equating real and imaginary parts, we have the identities:

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2} \quad \text{and} \quad \mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$$

Here is yet another way to compute the Laplace transform of $\sin(at)$ and $\cos(at)$ using their Taylor series expressions. Recall that

$$\sin(at) = at - \frac{(at)^3}{3!} + \frac{(at)^5}{5!} + \dots$$

Taking the Laplace transform of both sides,

$$\begin{aligned}
\mathcal{L}\{\sin(at)\} &= \frac{a}{s^2} - \frac{a^3}{3!}\frac{3!}{s^4} + \frac{a^5}{5!}\frac{5!}{s^6}\dots \\
&= \frac{a}{s^2} - \frac{a^2}{s^2}\left(\frac{a}{s^2} - \frac{a^3}{s^4} + \frac{a^5}{s^6}\dots\right) \\
&= \frac{a}{s^2} - \frac{a^2}{s^2}\mathcal{L}\{\sin(at)\}
\end{aligned}$$

Now solving for $\mathcal{L}\{\sin(at)\}$ gives the desired result. A similar argument works for cosine.

## 5.2 Initial Value Problems

Before we can begin using the Laplace transform to help us solve differential equations, we need to investigate what happens when we apply the Laplace transform to the derivative of a function. From the definition, we can write

$$\mathcal{L}\{y'(t)\} = \int_0^\infty e^{-st} y'(t)\, dt.$$

What we would like to do is express the right hand side in terms of the function $y$ – not its derivative – and so we integrate by parts to obtain

$$\int_0^\infty e^{-st} y'(t)\, dt = y(t)e^{st}\Big|_0^\infty + \int_0^\infty se^{-st} y(t)\, dt.$$

Provided that the transform of $y(t)$ exists, this simplifies to give the property

$$\boxed{\mathcal{L}\{y'(t)\} = sY(s) - y(0)} \tag{5.10}$$

which tells us how to take the transform of the derivative of a function. By applying this property twice, we can also compute the transform of the second derivative of a function:

$$\boxed{\mathcal{L}\{y''(t)\} = s^2 Y(s) - sy(0) - y'(0)} \tag{5.11}$$

We are now ready to use the Laplace transform to solve a differential equation. The plan is as follows:

1. Apply the Laplace transform to the initial value problem. The unknown function $u(t)$ is replaced with the unknown function $U(s)$.
2. Solve for $U(s)$. Because of property (5.10), this step requires nothing more than algebra.
3. Apply the inverse transform to $U(s)$ and thereby recover the function $u(t)$.

Consider the following simple example. Although we don't actually need to use the Laplace transform method to solve this problem, it will serve to illustrate the steps involved.

**Example 5.2.1.** Solve the initial value problem

$$y' = y - 4e^{-t} \qquad y(0) = 1.$$

We begin by taking the Laplace transform of both sides of the equation:

$$sY(s) - y(0) = Y(s) - 4\left(\frac{1}{s+1}\right).$$

Notice that we used the linearity property on the right hand side. Then we plug in the initial value $y(0) = 1$

$$sY - 1 = Y - \frac{4}{s+1}$$

and solve for $Y$

$$Y = \frac{1}{s-1} - \frac{4}{(s+1)(s-1)}.$$

At this point, we need to rewrite the right hand side so that it looks like the transform of something. The key is using a partial fraction decomposition:

$$\frac{4}{(s+1)(s-1)} = \frac{2}{s-1} - \frac{2}{s+1}.$$

Then we have

$$Y = \frac{1}{s-1} - \frac{2}{s-1} + \frac{2}{s+1} = -\frac{1}{s-1} + \frac{2}{s+1}$$

and we recognize the right hand side as the transform of two exponential functions. Taking the inverse transform we get

$$y(t) = -e^t + 2e^{-t}.$$

**More Sine and Cosine**

If we know that a particular function satisfies a particular initial value problem, we can use that fact to help us determine its transform. For instance, consider the function $y(t) = \sin(at)$. We know that this is the unique solution to the initial value problem

$$y'' + a^2 y = 0 \qquad y(0) = 0, \quad y'(0) = a.$$

Now if we take the Laplace transform of this differential equation, and use the given initial values, we have

$$s^2 Y - sy(0) - y'(0) + a^2 Y = 0.$$

Solving for $Y$, we get

$$Y = \frac{a}{s^2 + a^2}.$$

Since we already know that $y(t) = \sin(at)$, this gives us the formula

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}.$$

A similar argument can be made to show that

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2 + a^2}.$$

## 5.3 Discontinuous and Piecewise Sources

As mentioned in the introduction to this chapter, the Laplace transform is particularly useful for differential equations that involve piecewise defined functions. For example, consider the function

$$f(t) = \begin{cases} t & 0 \le t < 3 \\ 3 & 3 \le t < 6 \\ 9 - t & 6 \le t < 9 \\ 0 & 9 \le t \end{cases} \tag{5.12}$$

A function like this might represent a voltage that is applied to a circuit. Initially, the applied voltage ramps up linearly from 0 to a value of 3, at which point it is held constant for three seconds. It is then ramped back down to zero.

To help us deal with these types of functions, we will introduce a method for rewriting them as a collection of terms that "switch on" as $t$ increases from smaller to larger values. By adding these terms together, we get a function that takes on a new definition on different time intervals.

### 5.3.1 Heaviside Function

The key to rewriting functions like the one above is the **step function**, also known as the **Heaviside function**, defined as

$$h_a(t) = \begin{cases} 0 & t < a \\ 1 & a \le t. \end{cases} \tag{5.13}$$

Figure 5.1 shows a graph of the Heaviside function.

It is helpful to think of the Heaviside function as a switch that is off for values of $t < a$ and on for $t \ge a$. By multiplying a function by $h_a(t)$, we get a new function that "turns on" at $t = a$. Adding together a string of such terms, we get a function that takes on a new definition each time a switch is turned on. For example, a function that is written piecewise as

$$f(t) = \begin{cases} f_1(t) & t < a \\ f_1(t) + f_2(t) & a \le t \end{cases}$$

is equivalent to

$$f(t) = f_1(t) + h_a(t) \cdot f_2(t). \tag{5.14}$$

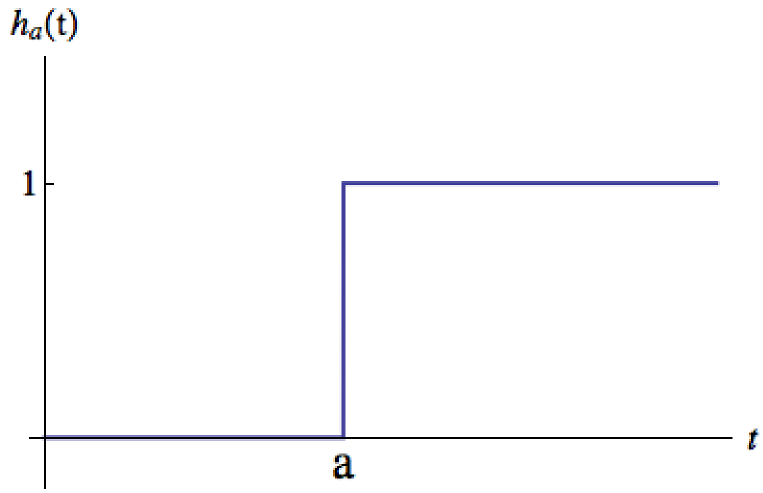



**Figure 5.1:** The Heaviside function $h_a(t)$.

The advantage in using the Heaviside function to express this function is that we do not have to break the definition up into different time intervals. This will make it easier to calculuate the Laplace transform of $h_a(t)$ (and expressions involving $h_a$).

**Example 5.3.1.** Consider the function

$$f(t) = \begin{cases} t & 0 \le t < 3 \\ 3 & 3 \le t < 6 \\ 9 - t & 6 \le t < 9 \\ 0 & 9 \le t \end{cases} \tag{5.15}$$

defined in the introduction of this section. To rewrite this function in terms of the Heaviside, we will look at the definition of $f(t)$ on each of the indicated time intervals. As we go from left to right on the $t$-axis, we will add a new term to redefine the function each time the function changes its definition (i.e. at $t = 3, 6, 9$):

$$f(t) = t + h_3(t) \cdot \boxed{?} + h_6(t) \cdot \boxed{?} + h_9(t) \cdot \boxed{?}$$

Our next task it to determine what should replace each of the question marks. Now let's consider this expression on each of the four different time intervals.

- For $\mathbf{t < 3}$, each of the three Heaviside functions is equal to zero and therefore $f(t) = t$.
- On the interval $\mathbf{3 \le t < 6}$ the first Heaviside function is non-zero and therefore the function looks like this:
$$f(t) = t + h_3(t) \cdot \boxed{?} = t + \boxed{?}$$

From (5.15) we know that we want $f(t)$ to equal 3 on this interval, so the question is what function do we need to add to $t$ in order to get 3. The answer is $3-t$ and therefore we replace the question mark with this expression:

$$f(t) = t + h_3(t)(3-t).$$

- For $\mathbf{6 \le t < 9}$, the first two Heaviside functions are non-zero and therefore the function looks like this:

$$f(t) = \underbrace{t + h_3(t)(3-t)}_{=3} + h_6(t) \cdot \boxed{?} = 3 + \boxed{?}$$

  From (5.15) we know that we want $f(t)$ to equal $9-t$ on this interval, so we must determine what function we need to add to 3 in order to get $9-t$. The answer is $6-t$ and therefore we replace the question mark with this expression:

$$f(t) = t + h_3(t)(3-t) + h_6(t)(6-t)$$

  For reasons that will be made clear later, we will rewrite this slightly as

$$f(t) = t + h_3(t)(3-t) - h_6(t)(t-6)$$

- Finally, if $\mathbf{9 \le t}$ then all three Heaviside functions are non-zero and therefore the function looks like this:

$$f(t) = \underbrace{t + h_3(t)(3-t) - h_6(t)(t-6)}_{=9-t} + h_9(t) \cdot \boxed{?} = 9 - t + \boxed{?}$$

  From (5.15) we know that we want $f(t)$ to equal 0 on this interval, so we must determine what function we need to add to $9-t$ in order to get 0. The answer is $t-9$ and therefore we replace the question mark with this expression:

$$f(t) = t + h_3(t)(3-t) - h_6(t)(t-6) + h_9(t)(t-9)$$

We have now rewritten the original function (5.15) in terms of the Heaviside function.

Note about notation: the Heaviside function is widely used in physics and engineering and there are many variations on the notation used to describe it. One common variation is to denote the location of the jump via a translation instead of a subscript. In other words, instead of writing $h_a(t)$, the same function would be written $h(t-a)$. Another common variation is to use the letter $u$ instead of $h$ to denote the Heaviside function.

### 5.3.2 Heaviside Function and the Laplace Transform

First, we compute the Laplace transform of the Heaviside function. From the definition of the transform, we have

$$\mathcal{L}\{h_a(t)\} = \int_0^\infty e^{-st} h_a(t)\,dt$$

and by the definition of the Heaviside function,

$$= \int_a^\infty e^{-st}\,dt$$
$$= \frac{e^{-as}}{s}.$$

Note that we are assuming that $a > 0$ so that the Heaviside function "switches on" at some positive time $t$.

As the previous example suggests, however, the Heaviside function seldom appears by itself. What is much more common is to have $h_a(t)$ multiplied by some other function. Notice also in the example that in each of the three terms involving a Heaviside function $h_a$, there appears not just $t$, but $t-a$. This is not an accident, suggesting that what we need is the Laplace transform of $h_a$ multiplied by a function *shifted* a distance $a$:

$$h_a(t) \cdot f(t-a).$$

As before, we use the definition of the Laplace transform:

$$\mathcal{L}\{h_a(t) f(t-a)\} = \int_0^\infty e^{-st} h_a(t) f(t-a)\,dt$$

and the definition of the Heaviside function,

$$= \int_a^\infty e^{-st} f(t-a)\,dt.$$

At this point, we perform a change of variables and let $u = t-a$. This gives

$$\begin{aligned}\int_a^\infty e^{-st} f(t-a)\,dt &= \int_0^\infty e^{-s(u+a)} f(u)\,du \\ &= e^{-sa} \int_0^\infty e^{-su} f(u)\,du \\ &= e^{-sa} F(s).\end{aligned}$$

We have just proved the following theorem.

**Theorem 5.3.1.** *If $F(s) = \mathcal{L}\{f(t)\}$ exists and $a$ is a positive constant, then*

$$\mathcal{L}\{h_a(t)f(t-a)\} = e^{-as}F(s) \qquad s > a. \tag{5.16}$$

**Example 5.3.2.** Find the Laplace transform of the expression $h_3(t) \cdot (3-t)$.

In order to use the shift property, we need to think of the expression as $h_a(t) \cdot f(t-a)$:

$$h_3(t) \cdot \underbrace{(3-t)}_{f(t-a)}.$$

The subscript on the Heaviside function tells us that $a = 3$ and therefore we know that $f(t-3) = 3-t$. So what is $f(t)$? If we think of the number $t-3$ as the input to this function, what is the output? It's just the opposite, $3-t$. In other words, this function is simply changing the sign of the input. So if the input was simply the number $t$, the output would be $-t$. Another way to say the same thing is

$$f(t) = -t.$$

The Laplace transform of this function is $F(s) = -1/s^2$ and so the shift property tells us that

$$\mathcal{L}\{h_3(t)(3-t)\} = e^{-3s}\left(\frac{-1}{s^2}\right).$$

## 5.4 Impulse Forcing

Another type of problem for which the Laplace transform is particularly well suited are those involving impulse forces or point sources. The term *impulse force* refers to a force that only acts in a single instant of time (or over a such a short time interval that we can essentially consider it to be instantaneous). For example, a bell that receives a single blow from a hammer, or the response in an electric circuit in response to a voltage spike can both be modeled using impulse forces. The term *point source* comes from examples such as a drop of dye dropped in glass of water. Over time, the dye spreads out, but initially we can think of it as being localized at a single point.

To handle this type of forcing function mathematically, we will start by approximating an impulse with a function that acts over a very short time interval. Consider the function

$$d_\epsilon(t) = \begin{cases} 1/\epsilon & -\frac{\epsilon}{2} < t < \frac{\epsilon}{2} \\ 0 & \text{otherwise} \end{cases}. \tag{5.17}$$

Think of $\epsilon$ as a small positive value, in which case $d_\epsilon$ describes a function which is nonzero only over an interval of length $\epsilon$. Notice that the height of the function is related to $\epsilon$ as well so that the total area under the curve is 1. The precise value doesn't really matter, the important part is that it's a constant, and 1 is a convenient constant. If we want our step to be centered somewhere else on the time axis, we can simply shift it: $d_\epsilon(t-a)$.

Note that we can also write this rectangular function in terms of the Heaviside function:

$$d_\epsilon(t) = \frac{1}{\epsilon}(h(t-(a-\epsilon/2)) - h(t-(a+\epsilon/2))). \tag{5.18}$$

We can now define our impulse function as the limit of $d_{a,\epsilon}$ as $\epsilon \to 0$ with the added condition that the area under the curve, i.e. the integral, should remain constant. The resulting impulse function is also known as the **delta function**, denoted $\delta_a(t)$. It has the properties that $\delta_a(t) = 0$ for any $t \neq a$ and it's integral over any interval that contains $a$ is equal to 1. Although it doesn't really make sense as a function in the usual sense (notice that $\delta_a(a)$ isn't really defined), it is nevertheless quite useful for modeling impulse forces.

In order to use the delta function as a forcing term in a differential equation, we will need to compute its Laplace transform. From the definition of the delta function, we have

$$\mathcal{L}\{\delta_a(t)\} = \mathcal{L}\left\{\lim_{\epsilon\to 0} d_{a,\epsilon}(t)\right\} = \lim_{\epsilon\to 0}\mathcal{L}\left\{d_{a,\epsilon}(t)\right\}. \tag{5.19}$$

Note: if you're unsure about whether bringing the limit outside of the Laplace transform is a valid move, you can rewrite the transform in terms of an integral and go from there. Note also that we are assuming that the impulse is located at a point $a > 0$, otherwise this integral is zero. At this point, we can rewrite the function $d_{a,\epsilon}$ in terms of Heaviside functions using (5.18)

$$\begin{aligned}\mathcal{L}\left\{d_{a,\epsilon}(t)\right\} = \mathcal{L}\left\{\frac{1}{\epsilon}(h_{a-\epsilon/2}(t) - h_{a+\epsilon/2}(t))\right\} &= \frac{1}{\epsilon}\left(\frac{e^{-(a-\epsilon/2)s}}{s} - \frac{e^{-(a+\epsilon/2)s}}{s}\right)\\ &= \frac{e^{-as}}{s}\left(\frac{e^{\epsilon s/2} - e^{-\epsilon s/2}}{\epsilon}\right)\end{aligned}$$

In order to take the limit as $\epsilon \to 0$, focus on the expression inside the parentheses. As $\epsilon$ goes to zero, both the numerator and the denominator go to zero as well. We therefore need to evaluate the limit using L'Hopital's rule.

$$\lim_{\epsilon\to 0}\frac{e^{-as}}{s}\left(\frac{e^{\epsilon s/2} - e^{-\epsilon s/2}}{\epsilon}\right) = \frac{e^{-as}}{s}\lim_{\epsilon\to 0}\left(\frac{\frac{s}{2}e^{\epsilon s/2} + \frac{s}{2}e^{-\epsilon s/2}}{1}\right) = e^{-as}.$$

We have just shown that the Laplace transform of the delta (or impulse) function is

$$\boxed{\mathcal{L}\{\delta_a(t)\} = e^{-as}} \tag{5.20}$$

where $a > 0$.

**Example 5.4.1.** Solve the initial value problem

$$u'' - u = \delta_2(t) \qquad u(0) = u'(0) = 0.$$

Taking the transform,

$$s^2 U - U = e^{-2s}$$

and solving for $U$ gives

$$U = \frac{e^{-2s}}{s^2-1} = \frac{1}{2}\frac{e^{-2s}}{s-1} - \frac{1}{2}\frac{e^{-2s}}{s+1}.$$

Using the shift property to find the inverse transform,

$$u(t) = \frac{1}{2}h_2(t)e^{t-2} - \frac{1}{2}h_2(t)e^{-(t-2)}.$$

### A Mathematical Aside

Although the delta function cannot be considered an function in the ordinary sense, it can be described as an example of a so-called **generalized function** which basically means that we should think about the delta function not in terms of inputs and outputs (like an ordinary function), but in terms of the integrals in which it appears. For example, in the case of the Laplace transform, the important thing about the delta function is that it satisfies the identity

$$\mathcal{L}\{\delta_a(t)\} = \int_0^\infty e^{-st}\delta_a(t)\,dt = e^{-as}. \tag{5.21}$$

In general, the defining characteristic of the delta function is the fact that it satisfies the identity

$$\int_0^\infty \delta_a(t)\phi(t)\,dt = \phi(a) \tag{5.22}$$

for any "nice" function $\phi(t)$.

This property can be proved using an argument similar to what we did above. Rewriting $\delta_a$ in terms of $d_{a,\epsilon}$ and then moving the limit outside the integral (this is why $\phi$ needs to be "nice") we have

$$\begin{aligned}\int_0^\infty \delta_a(t)\phi(t)\,dt &= \lim_{\epsilon\to 0}\int_0^\infty d_{a,\epsilon}\phi(t)\,dt \\ &= \lim_{\epsilon\to 0}\int_{a-\epsilon/2}^{a+\epsilon/2}\frac{1}{\epsilon}\phi(t)\,dt = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{a-\epsilon/2}^{a+\epsilon/2}\phi(t)\,dt\end{aligned}$$

Now, although we cannot evaluate this integral (since we don't know what $\phi$ is), we can apply the Fundamental Theorem of Calculus. If we let $\Phi$ be an antiderivative of $\phi$ (in other words, $\Phi$ is a function with the property that $\Phi'(t) = \phi(t)$) then we can write

$$\frac{1}{\epsilon}\int_{a-\epsilon/2}^{a+\epsilon/2} \phi(t)\,dt = \frac{\Phi(a+\epsilon/2) - \Phi(a-\epsilon/2)}{\epsilon}$$

and in the limit of $\epsilon \to 0$, the right-hand side is precisely the derivative of $\Phi$ evaluated at $a$. This establishes equation (5.22).

## 5.5 Convolutions

As we have seen in many applications, the nonhomogeneous terms on the right hand side of the differential equation often represent some external forcing that is applied to a system, while the terms on the left hand side represent known physical characteristics of the system. For instance, in the mass-spring system

$$my'' + \gamma y' + by = g(t)$$

the constants $m$, $\gamma$, and $b$ are typically fixed and we are interested in how the system responds when different forcing functions $g(t)$ are applied. Another example is the RCL circuit with an applied electromotive force:

$$Lq'' + Rq' + \frac{1}{C}q = E(t).$$

We might be interested in investigating how a given circuit, with known values of $L$, $R$, and $C$, responds to various input signals (represented by $E(t)$). Note that mathematically, these two examples are exactly the same. In general, the situation that we're describing is one where we have a system whose physical characteristics are known and we want to see how the system responds to various inputs. Our goal in this section is to describe a method for representing the solution in terms of an arbitrary input signal.

Let's begin by looking at a specific example. Consider the following nonhomogeneous differential equation, which might represent a mass-spring system without damping subject to an external force:

$$y'' + 4y = g(t)$$

where $g(t)$ is a yet to be determined input function. Let's also assume that $y(0) = y'(0) = 0$ for simplicity. Then taking the Laplace transform, we have

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = G(s).$$

Note that even though we have not specified a particular function for $g(t)$, we can still represent its Laplace transform by $G(s)$. Now using the given initial conditions and solving for $Y(s)$, we get

$$Y(s) = \frac{1}{s^2+4}G(s). \tag{5.23}$$

If we let $F(s) = \frac{1}{s^2+4}$ then we know that $f(t) = \frac{1}{2}\sin(2t)$ so at this point, we have a product of two quantities for which we know the inverse transforms. Of course we know that the inverse of an expression like $F(s)G(s)$ is **not** the product $f(t)g(t)$, but we can ask the question: just what *is* its inverse? Or asked another way, what combination of $f(t)$ and $g(t)$ has a Laplace transform equal to $F(s)G(s)$? To try to answer this question, we can appeal to the definition of the transform and write

$$F(s)G(s) = \left(\int_0^\infty e^{-st} f(t)\,dt\right)\left(\int_0^\infty e^{-st} g(t)\,dt\right).$$

Now we would like to rewrite this in some helpful way. For instance, we can combine the two integrals into a single double integral. In order to help keep things straight, it's helpful if we don't use the same variable of integration in both integrals.

$$\begin{aligned}\left(\int_0^\infty e^{-st} f(t)\,dt\right)\left(\int_0^\infty e^{-s\tau} g(\tau)\,d\tau\right) &= \int_0^\infty \left(\int_0^\infty e^{-s\tau} g(\tau)\,d\tau\right) e^{-st} f(t)\,dt \\ &= \int_0^\infty \int_0^\infty e^{-(t+\tau)s} g(\tau) f(t)\,d\tau\,dt\end{aligned}$$

Now we use a change of variables to simplify the exponential in the integrand. Let $r = t + \tau$, in which case $\tau = r - t$ and $d\tau = dr$. Then we have

$$= \int_0^\infty \int_t^\infty e^{-rs} g(r-t) f(t)\,dr\,dt$$

Then, changing the order of integration, we have

$$\begin{aligned} &= \int_0^\infty \int_0^r e^{-rs} f(t) g(r-t)\,dt\,dr \\ &= \int_0^\infty \left(\int_0^r f(t) g(r-t)\,dt\right) e^{-rs}\,dr.\end{aligned}$$

Finally, if we realize that the expression inside the parentheses is a function of $r$, then what we have is the Laplace transform of that function. This function is known as the **convolution** of $f$ and $g$.

**Definition.** The **convolution** of two functions $f(t)$ and $g(t)$ is denoted $f * g$ and is defined by

$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau)\,d\tau. \tag{5.24}$$

Using this notation, we have established the identity

$$\mathcal{L}\{f * g\} = F(s)G(s) \tag{5.25}$$

**Example 5.5.1.** Find the inverse transform of $F(s) = \frac{1}{s(s^2+4)}$. Although we can do this using partial fractions, here we can also use the convolution property by rewriting $F(s)$ as a product:

$$F(s) = \frac{1}{s(s^2+4)} = \frac{1}{s} \cdot \frac{1}{s^2+4}.$$

If we let $G(s) = 1/s$ and $H(s) = 1/(s^2+4)$ then $g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$ and $h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin(2t)$, and then by the convolution property we have

$$\begin{aligned} f(t) = g(t) * h(t) = h(t) * g(t) &= \frac{1}{2}\int_0^t \sin(2\tau)\,d\tau \\ &= \frac{1}{4}(1 - \cos(2t)). \end{aligned}$$

**Example 5.5.2.** Use the convolution property to solve the nonhomogeneous differential equation

$$y'' + 4y = g(t)$$

with initial conditions $y(0) = y'(0) = 0$ and where $g(t)$ is a yet to be determined input function. Taking the Laplace transform, we have

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = G(s).$$

and solving for $Y(s)$, we get

$$Y(s) = \frac{1}{s^2+4}G(s). \tag{5.26}$$

and we recognize the right hand side as the product of two transforms. If we let $F(s) = \frac{1}{s^2+4}$ then $f(t) = \frac{1}{2}\sin(2t)$ and the solution $y(t)$ can be written as

$$y(t) = g(t) * f(t) = \frac{1}{2}\int_0^t g(\tau)\sin(2(t-\tau))\,d\tau.$$

At this point, we have a formula for the solution $y(t)$ in terms of the input function $f(t)$. We can simply plug in $f(t)$ and do the integral.

This last example illustrates the power of the Laplace transform to provide a very simple relationship between the solution (i.e. the output) and the forcing function (i.e. the input). In particular, if we consider the problem expressed in terms of the transform variable (that is, in terms of $Y(s)$ and $G(s)$ rather than $y(t)$ and $g(t)$), then (5.26) says that the output is given by simply multiplying the input $G(s)$ by the function $F(s) = \frac{1}{s^2+4}$. This function is known as the **transfer function** and it depends only on the characteristics of the system (i.e. the the coefficients on the left hand side of the original differential equation).

## 5.6 Exercises

**5.1:** Find the Laplace transform of the following functions.

(a) $f(t) = \pi^2 t^3 - 42$

(b) $g(t) = e^{2t} + 4\sin(\pi t)$

(c) $h(t) = t\sin(3t)$

(d) $f(t) = t^2 e^{-3t}$

**5.2:** Find the inverse transforms of the following.

(a) $U(s) = \dfrac{2s+1}{s^2+4}$

(b) $U(s) = \dfrac{2}{s^2 - s - 2}$

**5.3:** Find the transform of the function

$$f(t) = \begin{cases} 0 & t < 2 \\ e^{-t} & 2 \leq t \end{cases}.$$

**5.4:** Find the Laplace transforms of the following functions:

(a) $w(t) = (t-\pi)^3$

(b) $g(t) = h_2(t)e^{-t}$

**5.5:** Recall that $\cosh(at) = (e^{at} + e^{-at})/2$ and $\sinh(at) = (e^{at} - e^{-at})/2$. Find the Laplace transform of the following.

(a) $\cosh(at)$

(b) $\sinh(at)$

(c) $e^{at}\cosh(bt)$

(d) $e^{at}\sinh(bt)$

**5.6:** Find the following Laplace transforms

(a) $\mathcal{L}\{e^{at}\sin(bt)\}$

(b) $\mathcal{L}\{e^{at}\cos(bt)\}$

**5.7:** Find the inverse transforms of the following.

(a) $U(s) = \dfrac{s}{s^2 + 7s - 8}$

(b) $U(s) = \dfrac{3 - 2s}{s^2 + 2s + 10}$

(c) $F(s) = \dfrac{1}{s^2(s - 1)}$

(d) $G(s) = \dfrac{2}{(s - 5)^4}$

**5.8:** Find the inverse transforms of the following.

(a) $Y(s) = \dfrac{e^{-2s}}{s^2 + 2s + 2}$

(b) $F(s) = \dfrac{2(s - 3)}{s^2 + 2s - 5}$

**5.9:** Solve the following initial value problems using the Laplace transform.

(a) $y' + 5y = \sin(2t), \quad y(0) = 0.$

(b) $y'' - y' - 6y = 0, \quad y(0) = 2, \quad y'(0) = 0.$

(c) $y'' + 2y' + y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$

**5.10:** Use the definition of the Laplace transform to find $\mathcal{L}\{1/\sqrt{t}\}$. Hint: use the substitution $r = \sqrt{st}$ to rewrite the integral and then review section 2.1.1 to evaluate the integral.

**5.11:** The *gamma function* is defined as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt.$$

We will assume $x > 0$ so that the integral converges. The question of whether this this integral converges for other values of $x$ is, to put it mildly, interesting.

(a) Show that $\Gamma(n + 1) = n\Gamma(n)$.

(b) Show that $\Gamma(1) = 1$.

(c) Show that if $n$ is a positive integer then $\Gamma(n + 1) = n!$.

(d) Show that $\Gamma(1/2) = \sqrt{\pi}$. (Hint: do problem 5.10 first.)

(e) Show that

$$\mathcal{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}.$$

**5.12:** Consider the initial value problem

$$y'' + 3y = w(t) \quad y(0) = 2,\ y'(0) = 0$$

where $w(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$

(a) Rewrite the differential equation using the Heaviside function.

(b) Solve the initial value problem using the Laplace transform and graph the solution.

**5.13:** Use the definition (5.25) to show that the convolution satisfies the property

$$(f * g)(t) = (g * f)(t). \tag{5.27}$$

**5.14:** Solve the initial value problems and graph the solutions.

(a) $y'' + 3y = 5\delta_2(t) \qquad y(0) = y'(0) = 0.$

(b) $y'' + 2y' + 2y = 2\delta_2(t) \qquad y(0) = 2 \quad y'(0) = 0.$

(c) $y'' + 2y' + 5y = \delta_3(t), \quad y(0) = 1, \quad y'(0) = 0.$

(d) $y'' + y = \sum_{n=1}^{\infty} \delta_{2n\pi}(t)$ with initial conditions $y(0) = 0$ and $y'(0) = 0$.

**5.15:** Find the Laplace transform of the function

$$g(t) = \sum_{n=1}^{\infty} \delta_{na}(t)$$

and simplify as much as possible.

**5.16:** Consider the initial value problem

$$2y'' + y' + 2y = \delta_5(t) + k\delta_a(t) \qquad y(0) = y'(0) = 0$$

Find values of $k$ and $a$ so that the system is brought to rest after exactly one complete cycle. In other words, $y(t) = 0$ for all $t > a$ where $a$ is equal to $5 + T_0$ and $T_0$ is the period of the oscillatory part of the solution.

**5.17:** A 100-gallon tank is initially full of pure water. We then begin adding salt water at a rate of 2 gallons per minute. This salt water has a concentration of 1/2 lbs. of salt per gallon. At the same time, the water in the tank is allowed to drain at the rate of 2 gallons per minute (so the volume remains constant). After 10 minutes, the salt water being added to the tank suddenly switches to fresh water. Use Laplace transforms to find the amount of salt in the tank after 10 additional minutes.

**5.18:** Consider the initial value problem

$$y'' + y = f(t) \qquad y(0) = y'(0) = 0$$

where

$$f(t) = h_0(t) + 2\sum_{k=1}^{15}(-1)^k h_{k\pi}(t)$$

(a) Draw a graph of $f(t)$ on the interval $[0, 6\pi]$.

(b) Find the solution to the initial value problem and plot the graph for $0 \le t \le 60$.

**5.19:** Use the convolution property to find the inverse transform of $\dfrac{1}{s^2(s^2+1)}$.

**5.20:** Use the definition of the convolution to show that

$$(f * g)(t) = (g * f)(t).$$

**5.21:** (a) If $f(t) = t^m$ and $g(t) = t^n$, where $m$ and $n$ are positive integers, show that

$$(f * g)(t) = t^{m+n+1}\int_0^1 \tau^m(1-\tau)^n \, d\tau.$$

(b) Use the convolution property to show that

$$\int_0^1 \tau^m(1-\tau)^n \, d\tau = \frac{m!n!}{(m+n+1)!}$$

(c) Extend the result of part (b) to the case where $m$ and $n$ are positive numbers but not necessarily integers.

# Appendix A

# Euler's Formula

The key to working with complex valued exponentials is figuring out how to handle imaginary numbers in the exponent. Essentially we need to define the function $e^{i\theta}$. There are several possible approaches, but one simple approach is to use a Taylor series expansion for the real-valued exponential to *define* the exponential function in the case of imaginary-valued exponents. Recall the series expansion

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \ldots$$

Then for $x = i\theta$ we get

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \ldots + \frac{(i\theta)^n}{n!} + \ldots$$

We now make use of the properties of the number $i$. By definition, $i^2 = -1$, and therefore the Taylor series expansion above simplifies to

$$e^{i\theta} = 1 + (i\theta) - \frac{(\theta)^2}{2} - \frac{i(\theta)^3}{3!} + \ldots + \frac{(i\theta)^n}{n!} + \ldots$$

If we collect all the real terms together and all of the imaginary terms together, we can rewrite the series as

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots\right)$$

The first series we recognize as the series expansion for the cosine function while the second series is the sine function. We have just derived **Euler's formula**, which relates complex exponential functions with the more familiar sine and cosine functions:

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)} \tag{A.1}$$

Euler's Formula can also be used to express the usual sine and cosine functions as complex exponentials. Because sine is an odd function and cosine is an even function, we have

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta).$$

Therefore if we add this to (A.1) and divide by two, we have the identity

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}. \tag{A.2}$$

Similarly, subtracting and dividing by two, we have

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2}. \tag{A.3}$$

These identities are useful because they allow us to work with exponential functions instead of sines and cosines.

# Appendix B

# Vectors and Matrices

Working with systems of equations is made much easier by using vector and matrix notation, which not only makes it possible to express ourselves in a very concise way, but also allows us to make use of some powerful concepts from linear algebra. Since this is not a course in linear algebra we will only be scratching the surface of this subject, introducing only the tools that we need to continue our study of differential equations.

We begin by introducing the concept of a **matrix**. Chances are, you have encountered matrices before, but just to make sure that we've covered all the bases, we will introduce the basic definitions and properties.

**Definition.** An $m \times n$ **matrix** is a rectangular array of numbers arranged in $m$ rows and $n$ columns:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

It is common to use capital letters to denote matrices. The number, or **element**, in the $i$th row and $j$th column is denoted by $a_{ij}$ while the whole matrix $\mathbf{A}$ is sometimes denoted $(a_{ij})$. The **size** of a matrix refers to its dimensions, $m$ and $n$. If the number of rows and columns is the same, the matrix is said to be **square**. In this course, we will be focusing primarily on $2 \times 2$ or $3 \times 3$ square matrices:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

If $\mathbf{A}$ is a square matrix, the **main diagonal**, or just simply the **diagonal**, refers to the elements $a_{11}, a_{22}, \ldots, a_{nn}$. A **vector** (also called a column vector) is a special matrix with a single column ($n = 1$). The elements of a vector are also called its **components**.

In order to work with vectors and matrices, we need to describe a few of their properties.

1. Two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are equal if all the corresponding elements are equal – i.e. if $a_{ij} = b_{ij}$ for all $i$ and $j$.

2. The **zero matrix** (or vector), denoted $\mathbf{0}$, is the matrix with every element equal to zero.

3. [Addition of Matrices] The sum of two $m \times n$ matrices is the matrix given by adding each corresponding entry:
$$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$$
Note that this means that $\mathbf{A} + \mathbf{B}$ is only defined for two matrices of the same size.

4. [Scalar Multiplication] The product of a matrix $\mathbf{A}$ and a scalar $c$ (either real or complex) is given by multiplying each entry of the matrix by $c$:
$$c\mathbf{A} = (ca_{ij})$$

5. [Matrix Multiplication] If $\mathbf{A}$ is a $m \times n$ matrix and $\mathbf{B}$ is a $n \times r$ matrix then the product $\mathbf{AB}$ is a $m \times r$ matrix whose elements are defined by the rule
$$(\mathbf{AB})_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$
Note that this implies that the product $\mathbf{AB}$ is only defined if the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$.

   Another consequence of this definition is that matrix multiplication is not commutative: if $\mathbf{A}$ and $\mathbf{B}$ are both square matrices of the same size (so that $\mathbf{AB}$ and $\mathbf{BA}$ are both defined), it is not true in general that these products are equal. Matrix multiplication is, however, associative, so that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

6. The **identity matrix**, denoted $\mathbf{I}$, is the square matrix that has ones on the main diagonal and zeros everywhere else. For instance, the $2 \times 2$ identity is
$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

As we can easily verify, this matrix has the property that for any other matrix $\mathbf{A}$ of the same size,

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}.$$

This property is what gives the identity matrix its name – it acts as the multiplicative identity.

**Example B.0.1.** Let

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -2 & 2 \\ 1 & 3 \end{pmatrix}.$$

Then by the definition of matrix addition,

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1-2 & 3+2 \\ 0+1 & -2+3 \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 1 & 1 \end{pmatrix};$$

by the definition of scalar multiplication,

$$-2\mathbf{A} = \begin{pmatrix} -2 & -6 \\ 0 & 4 \end{pmatrix};$$

and by the definition of matrix multiplication,

$$\mathbf{AB} = \begin{pmatrix} -2+3 & 2+9 \\ 0+1 & 0-6 \end{pmatrix} = \begin{pmatrix} 1 & 11 \\ 1 & -6 \end{pmatrix}.$$

Note that

$$\mathbf{BA} = \begin{pmatrix} -2+0 & -6-4 \\ 1+0 & 3-6 \end{pmatrix} = \begin{pmatrix} -2 & -10 \\ 1 & -3 \end{pmatrix}.$$

If $\mathbf{A}$ is an $n \times n$ square matrix and there exists a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ then $\mathbf{B}$ is called the **inverse** of $\mathbf{A}$ and is denoted $\mathbf{A}^{-1}$. If the inverse $\mathbf{A}^{-1}$ does exist, then $\mathbf{A}$ is said to be **nonsingular**, otherwise we say $\mathbf{A}$ is singular.

**Example B.0.2.** Given the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$$

we can show that this is invertible and that its inverse is

$$\mathbf{B} = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}$$

by computing the product **AB** (or **BA**).

$$\mathbf{AB} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Every square matrix $\mathbf{A}$ has a unique number associated to it called its **determinant**, denoted $\det \mathbf{A}$ or $|\mathbf{A}|$. The determinant tells us many things about a matrix, but in the present context we are interested in determinants because of their role in determining the inverse of a matrix. More precisely we can say that the matrix $\mathbf{A}$ is nonsingular if and only if $\det \mathbf{A} \neq 0$. For a general square matrix of size $n$, calculating a determinant can be quite complicated. Since we are only focused on systems of only two or three variables, we will refrain from offering a general definition and simply present formulas for the determinants for the cases $n = 2, 3$.

**Definition.** The $2 \times 2$ matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

has a determinant

$$\det \mathbf{A} = a_{11}a_{22} - a_{21}a_{12}.$$

The $3 \times 3$ matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

has a determinant

$$\det \mathbf{A} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

**Example B.0.3.** Given the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$$

we can show that this is invertible by computing its determinant.

$$\det \mathbf{A} = 3(-1) - (-4)1 = 1.$$

Since this is nonzero, the matrix $\mathbf{A}$ has an inverse.

For invertible $2 \times 2$ matrices, we can write down a formula for the inverse of a matrix. Given

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the inverse is

$$\mathbf{A}^{-1} = \frac{1}{ad - bd} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Although it's possible to write down a similar formula for larger matrices, this is beyond the scope of this text.

# Appendix C

# Table of Laplace Transforms

| $u(t)$ | $U(s)$ |
|---|---|
| $c$ | $\frac{c}{s}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $t^a$ | $\frac{\Gamma(a+1)}{s^{a+1}}$ |
| $\sin(at)$ | $\frac{a}{s^2+a^2}$ |
| $\cos(at)$ | $\frac{s}{s^2+a^2}$ |

| | |
|---|---|
| $e^{at}\sin(bt)$ | $\dfrac{b}{(s-a)^2+b^2}$ |
| $e^{at}\cos(bt)$ | $\dfrac{s-a}{(s-a)^2+b^2}$ |
| $e^{at}t^n$ | $\dfrac{n!}{(s-a)^{n+1}}$ ($n$ a positive integer) |
| $u'(t)$ | $sU(s)-u(0)$ |
| $u''(t)$ | $s^2U(s)-su(0)-u'(0)$ |
| $h_a(t)$ | $\dfrac{e^{-as}}{s}$ |
| $h_a(t)u(t-a)$ | $e^{-as}U(s)$ |
| $e^{at}u(t)$ | $U(s-a)$ |
| $tu(t)$ | $-U'(s)$ |
| $\delta_c(t)$ | $e^{-cs}$ |
| $u(t)*v(t)$ | $U(s)V(s)$ |

# Further Reading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