**Unit 2: Numerical Approaches**

Goals/Rationale

The goals of this unit are for students to develop the conceptual foundation for Euler’s method and to reinvent the Euler method algorithm. Only at the end of the unit should students be told that this unit is about Euler’s method, otherwise they might google it and spoil the fun. By the end of the unit the class should have a generalizable algorithm for approximating solutions to any differential equation for any given initial condition. The problems are sequenced so that students first informally and intuitively graphically stitch together tangent vectors. Experimenting by hand and with a Geogebra app facilitates student graphical reinvention of Euler’s method, which is then leads to a table of values that reflect the graphical tip to tail approach. The unit ends with students creating their own symbolic expression for their graphical/numerical approach.

**Page 2.1 – A Rate of Change Equation for Limited Resources**

Implementation Notes and Student Thinking

*Problem 1* – The unit starts by introducing a rate of change equation for limited resources, which builds on and extends their work in Unit 1 with an unlimited growth situation. The instructor may want to tell students that this modification is an example of the logistic model. Parts (a) and (b) should be completed (either in small groups or as a whole class discussion) and discussed before work on part (c) because part (c) is the first time that students use one of the many apps and hence the instructor will need to orient students to the app.

* Why, algebraically, does the modified rate of change look like the unlimited growth model dP/dt = 0.3P if the value of P is really small? Why does this makes sense?
* Suppose the initial population is 20. What does the modified rate of change equation predict will happen to the population over time? Why does this make sense?
* How many equilibrium solutions are there?

*Problem 2* – This is a super important problem and sets up the remaining problems in the unit. Students’ first response to this problem is usually to visually estimate the population value from the slope field. Encourage them to go further and try to figure out a way to use the rate of change equation and the initial condition to numerically compute the number of fish at time t=2. To do this, we’ve found that it is necessary for students to articulate the meaning of the value of dP/dt at the initial condition. In particular, they should realize that .48 is the amount of fish (scaled appropriately) that would be added after one year. Do not proceed to the next problem until this way of thinking about .48 makes sense to students and that they can use it to estimate the future number of fish.

* At time t=0 there are 2 fish in the lake (where 2 is a scaled value), which means that the initial rate of change is .48. What does the value of .48 mean?
* How can you use the meaning of .48 to estimate the future population?

With these guiding questions students will make progress. Here are some typical ways that they make progress.

* If the population increases by .48 in one year then it will increase .96 in two years.
* After one year the population will be 2.48. But then the rate of change after one year is different and so we’re stuck.
* After one year the population will be 2.48 and so we plugged this back into the rate of change equation to find a new rate of change.

Students may or may not get to the point where they realize that they have to recalculate the rate of change. This is fine and not necessary. In fact, the next problem will graphically make apparent the fact that the value of the rate of change changes with each step.

**Page 2.2 – Using a Slope Field to Predict Future Fish Populations**

Implementation Notes and Student Thinking

*Problems 3& 4* – In part (a) students stitch together by hand several vectors to produce a piece wise linear graph. They may do this without consciously thinking about what is being held constant in their process. You may find that some students use the slope field in such a way that the change in P is constant rather than the change in t. If so, this could lead to an interesting whole class discussion about what the two different approaches are predicting (population or time) and what the advantages and disadvantages are of the two approaches.

In part (b) students will use a new Geogebra app to stitch together vectors to graphically produce a population versus time graph. Students can decide for themselves the length of the vector that they want to iterate. Consider demonstrating at some point what the tip to tail looks like if you use a fairly large step size (such as 2). This will help covney the fact that the approximation method keeps constant the rate of change when in fact it changes continuously. The class can also compare iterations with different vector lengths by using the Freeze and Restart option. Some possible discussion questions:

* Why is the graph you created an approximation?
* What different quantities are held constant to create the graph?
* What does the tip to tail method produce if you start exactly at P=10? What if your initial condition is slightly less or more than 10?
* What does the tip to tail method produce if you start at or near P=0? What’s different in this case?

In Unit 6 students will develop terminology and distinctions between stable and unstable equilibrium solutions and hence there is no need here to make these distinctions. The above suggested discussion questions can foreshadow what will come.

**Pages 2.3 & 2.4**

*Problems 5-7* – These problems use a similar differential equation (but one that lends itself to simpler calculations) to revisit and solidify the tip to tail method. It is necessary for students to be able to correctly complete each table, otherwise they will not likely be able to generalize the approach. Suggested discussion question:

* On the same set of axes, show how the graph of one step of Jose and Julie’s approach compares to a graph of the first two steps of Derrick and Delores’s approach. Explain.

*Problem 8* – Here students need to generalize the approach to a differential equation that does not have a context and which depends explicitly on both y and t.

*Problem 9* – In this problem students symbolize and coalesce all their previous work into one algorithm. It might be helpful to start by asking students what they understand the *algorithm* to mean.

You may need to tell students that they should use the symbol dy/dt to represent any differential equation. This is similar to how they have used the letter *f* to represent any number of functions (e.g., let *f* be a continuous function). In such a statement we don’t care if *f* is *f*(x)=sin(x) or *f*(x)=x^2. Same is true with the symbol dy/dt. The conceptual difficulty for students is that they may not readily think of dy/dt as a function that represents any differential equation.

Students will not reinvent the canonical expression of Euler’s method, but they will come close. Ask students to share the different ways that they expressed the method and then conclude with an agreed upon way to write the algorithm. We have found that the following way to write the algorithm connects very well with student thinking: ynext = ynow + (dy/dt)now · Δt

Finally, it is empowering if you now tell students that they have reinvented what the mathematics community calls Euler’s method.

**Notes for Personal Reflections on Unit 2**