**Unit 9: Introduction to Systems**

Goals/Rationale

The mathematical goals of Unit 9 are to introduce students to the concept of systems, particularly systems of two differential equations. There is a focus on autonomous differential equations. In this first systems unit, the goals for the student are

1. Students will understand and use a system of two nonlinear differential equations.
2. Students will use the context of predator prey systems to begin analyzing systems
3. Students will recreate Euler’s method for systems
4. Students will visualize a solution for a system in 3-D space and connect to contexts.
5. Students will build intuition about solutions of systems.
6. Students will use an app to begin describing and analyzing solutions to systems of two differential equations
7. Students will recreate phase plane vector field representations and representations of solutions to a system in the phase plane

Technology

This is the general vector field app. <https://ggbm.at/kkNXUVds> for use in this unit.

Implementation Notes

This is one of the longer units in the materials. We recommend that you take at least two weeks (for a 3 hour class). Also, the homework is interesting here but long, and complex. Several of the problems in homework would work well if more is needed to help students understanding of phase planes and vector fields.

**Page 9.1 – Rabbits and Foxes**

Implementation Notes and Student Thinking

*Problem 1* -New differential equations can be created by modifying earlier assumptions and the DEs associated with these assumptions (this is one way in which scientists create DEs). For the given DEs, the only assumption that has changed is the introduction of a second species. Continuous reproduction is still assumed, as is unlimited resources for the rabbits. This latter conclusion can be justified by examining what the DEs predict will happen when there are no foxes. Thus, the 3R in the dR/dt equation is expected to be interpreted as unlimited growth. Students should also be able to interpret the –F term in dF/dt as exponential decay.

The –1.4RF and 0.8RF terms should be interpreted as “interaction terms” that make the expected impact on the respective rate of change. For example, students should be able to argue why it makes sense that it is –1.4RF instead of +1.4RF in terms of the meaning of dR/dt, R,F. In the subsequent problems students will be using this and other similar systems of differential equations to work numerically and graphically and therefore it is important for students to understand that the symbolic equations can be meaningfully interpreted.

This problem may go many different directions and teachers need to be ready to go “with the students”. For example, in attempting to answer this question, some students ended up finding equilibrium points, and making phase plane styled arguments... they called it the phase quadrant.

Note how the units are "hidden," but the students don't always take this line of inquiry. For example the 3 in the first DE actually carries the units of *1/time*. This means that -1.4 needs units of *1/(time \* foxes)*, which is a strange unit indeed! It is best to think of it as "per fox, per time."

Chemistry and the law of mass action also connects to this problem

For connecting to standard language and notation, the teacher could highlight that our Rabbit Fox system is an instance of the Lotka-Volterra equations.

Notes on Student Thinking

* Students often use the change in population when technically they mean rate of change (for some students this may actually be a conceptual problem rather than being careful with their language). The teacher can help students be more careful in their talk, thereby helping them become more explicitly aware that the rate of change in a quantity is not the same as the quantity or change in the quantity.
* One of the issues students often discuss is the “unlimited resources” for the foxes. This is a worthwhile discussion to have- are the rabbits resources for the foxes? As long as the students understand what is going on, then it is ok to consider rabbits as resources or not as resources. But the “unlimited resources” other than the prey is consistent here as the beginning of ODE solutions.

*Problem* 2 - This is to allow students to figure out a way to adapt Euler’s method to systems of two DEs by assuming a constant rate of change over a specified time interval.

Students do not seem to need much support on this question. In small groups, they “just do it”" This is a good chance to reinforce how much they are learning. They struggled with Euler's method 7 units ago, and as a result of that struggle, they were able to generalize the method to systems of two equations. That early struggle paid off!

The instructor might want to bring out the fact that there are actually five quantities that are continuously changing, dR/dt, dF/dt, the number of rabbits, the number of foxes, and time. Euler’s method assumes constant rate of change over a specified time interval.

The instructor show note that there are many ways to explore the graphical representations of the (t, R, F) data. Students will not likely use a 3-dimensional display, so the teacher will have to push/encourage students to try this.

There are Geogebra Applets to help with this in the student materials. The instructor needs to be sure s/he has made sure s/he understands what is happening.

This problem could be done as homework. Or just the part about using Excel could also be done as homework.

**Page 9.2 – Three Dimensional Visualization**

Implementation Notes and Student Thinking

*Problem 3* – Many of us who have used this material have explained up front that the airplane problem is for *building up intuition about 3D and related graph*s that will be used in subsequent problems. This usually heads off such comments about not connecting to Rabbits and Foxes (yet!)

This problem should be started by briefly revisiting the need to represent three bits of information, such as (t, R, F) from the previous problem. Explain that this problem is intended to promote three-dimensional visualization.

Pipe cleaners are excellent physical representations of the trace the airplane leaves. Give each student a pipe cleaner and they use it to explore what is going on. They can turn their pipe cleaner in space to see the different perspectives.

Teacher might add that thinking about it as if it were a flare attached to the blade and leaving a trail of smoke, seems to help.

It also helps to tell the students that in seeking the “ideal” perspective, we are assuming that we can see through the plane to see the mark at all times, etc.

**Page 9.3-9.5 - Three Dimensional Visualization (Continued)**

*Problem 4-7.*

This sequence of problems revisits and extends the ideas from problems 1-3. The GeoGebra applet <https://ggbm.at/U3U6MsyA> is used to generate numerical estimates and graphical representations. To use the applet, the user must first click “Generate Runge-Kutta Approximation” and subsequently click “Animate Solutions.” The user can click and drag in the 3d window to rotate the view. The advanced controls checkbox allows for snapping to the specified viewpoints and to provide an alternative initial condition to explore the system (required to answer 4d).

*Problem 4:* Students will be able to use the ideas from the airplane problem and connect to the Rabbit/Fox system; students will be able to use the GeoGebra applet for the Rabbit and Fox system

*Problem 4c:* One possible teacher move could be to ask students to draw their proposed solution to R-F small F medium on the board and then overlay the numerical solution.

*Problem 4d* - If students do not already experiment with the initial condition near an axis or on an axis, or even at or near equilibrium, teachers could encourage them to do so.

*Problem 5a* – Students will understand that when F=0, the system degenerates into an exponential growth model for the rabbit population.

*Problem 5b* – Students will interpret all the different graphical representations, be able to make explicit connections between the RF plane and earlier work with the flow line for first order autonomous differential equations.

*Problem 6a* – Students will relate the meaning of equilibrium solution to first order differential equations (constant function that satisfies the differential equation). In terms of equations, an equilibrium solution to a system of differential equations is a pair of constant functions that satisfy the differential equations. Graphically, an equilibrium solution is a straight line in 3-space, a pair of horizontal straight lines in the R-t, F-t planes, or a point in the R-F plane. (use DEExplorer in conjunction with algebra to figure out the exact equilibrium values). Be prepared to deal with the question of whether or not both differential equations have to be zero in order to have an equilibrium solution.

*Problem 6b* – This problem will connect students’ imagery to conventional terms of saddle and center (the expectation is that they will make a conceptual distinction between the different equilibrium solutions, the conventional terms can be then be labels for their distinction).

*Problem 7* – responses will vary, but students should begin to see the value in the R-F view.

**Page 9.6 - Three Dimensional Visualization (Continued)**

*Problem 8* - The goal of this problem is to introduce students to a tool (phase plane vector field) that can be used to represent graphs of solutions to differential equations. Building the vector (direction) field by hand is tedious and labor intensive.

*Problem 8b -* One suggestion to situation this problem might be: “How many pieces of information do you need to determine the orientation? One or two points? And for each point, how many of dR/dt, dF/dt, dF/dR do you need? Assume that you know the solution forms a particular closed curve.” - It is possible the students settle on only needing dR/dt until someone pointed out the places where dR/dt = 0!

**Page 9.7-9.9 - Vector Fields**

Implementation Notes and Student Thinking

## In the Vector field section, students will develop the following understanding:

* Equilibrium solutions can be determined graphically by finding where the nullclines intersect. Connections between graphical and algebraic approaches for finding equilibrium solutions should be made explicit by the instructor at the end of the exploration.

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* Nullclines also offer a way to create graphs for solutions in the phase plane, although these graphs they are less accurate but still helpful for a general, overall flow of curves in the phase plane.

Note the change to x and y instead of R and F.

*Problem 9* - The instructor might suggest that the groups share the calculations and then combine results into a class “graph”. One teacher tried it and it took about 20 minutes.

*Problem 10* - Using nullclines to find equilibrium comes up again in Systems 4 and so skipping nullclines is not a good idea. The Rabbit-Fox system is used again deliberately so students can remember what they had done with it before

*Problem 11 -* This problem is to connect the ideas of nullclines and vector fields back into a (new) context. It might work as homework as well.

Suggested questions if you wish to expand the conversation:

Give a physical interpretation of each equilibrium point of the system.

Do the rabbits and sheep have limited resources?

Can you determine if the rabbits and sheep have a cooperative or competitive relationship by only looking at the phase plane?

**Homework Problems**

*Problems 1-3* are carryovers directly from the unit.

*Problem 4* is pretty complex and if time, would work well in class.

**Notes for Personal Reflections on Unit 9**