Name:

Collaborators:

Instructions: Worksheets are graded mostly on completion, and partially on correctness. Please write complete solutions showing explanations and key steps to the following problems, unless it says otherwise.

Superposition of Vectors

This worksheet has a supplementary Jupyter Notebook (Python) in Posit Cloud. Please log-in into this course's Posit Cloud Workspace and navigate to the appropriate environment.

1. Linear Combination

Given vectors $\vec{v}_1$, $\vec{v}_2$, $\cdots$, $\vec{v}_k$ and scalars c_1, c_2, $\cdots$, c_k, a *linear combination* is any vector of the form:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k.$$

Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

Express the vector $\vec{b} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ as linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. That is, find scalars c_1, c_2, and c_3 so that $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{b}$.

Note: In the Posit Cloud Workspace, open the file `row-reduction.ipynb` under the "Supplementary: Superposition of Vectors" project. Use it to complete the tasks.

2. Linear Combinations and Solving Linear Systems

Consider the linear system,

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ -2x_1 - 3x_2 + x_3 &= 1 \\ -3x_1 + x_2 - 3x_3 &= 3 \end{aligned}$$

a. Write the linear system in matrix-vector form. Then, express the augmented matrix in REF and RREF. What does each row in the REF and RREF tell you about the variables?

b. Write the solution x_1, x_2, and x_3 as a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then, write it as a linear combination of the standard basis vectors. Compare this represntation to the structure of the RREF in Part (a). What do you notice?

c. Given the linear system, set the each equation to zero. Then, write the system as a linear combination. Is $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ the only solution? Explain your reasoning.

d. Let $A\vec{x} = \vec{b}$ and A be a square matrix. Based on your observations in Parts (a)-(c):

   i. What seems to determine whether the solution is unique?

   ii. Formulate a conjecture connecting the structure of A (and its pivot positions) with the way unique solutions are expressed as linear combinations.