

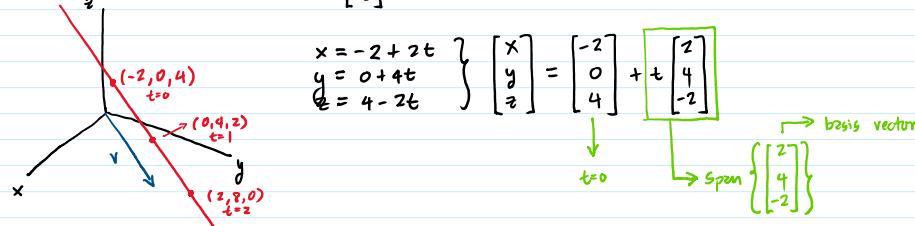
Objectives

1. Parametric equations for lines and planes
2. Define the spanning set
3. Define basis vectors

Parametric Equations for a line in space

The line parallel to $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ passing through point (x_0, y_0, z_0)
 is $x = x_0 + t v_1, \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad -\infty < t < \infty.$

→ Example:
 Find the line parallel to $\vec{v} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$ through the point $(-2, 0, 4)$.



$$\begin{aligned} x &= -2 + 2t \\ y &= 0 + 4t \\ z &= 4 - 2t \end{aligned} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} + t \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

↓
 $t=0$ $t=1$ $t=2$

basis vector
Span $\left\{ \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \right\}$

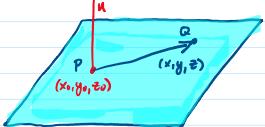
An Equation for a Plane in Space

The plane perpendicular to $\vec{n} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ passing through the point (x_0, y_0, z_0)

$$\text{is } A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

Simplified

$$Ax + By + Cz = D \text{ where } D = Ax_0 + By_0 + Cz_0 = \vec{n} \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$



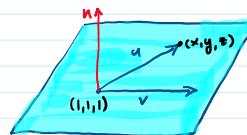
$$\begin{aligned} \vec{n} \cdot \vec{PQ} &= 0 \\ \vec{n} \cdot (\langle x_1, y_1, z_1 - x_0, y_0, z_0 \rangle) &= 0 \\ \vec{n} \cdot \langle x-x_0, y-y_0, z-z_0 \rangle &= 0 \\ A(x-x_0) + B(y-y_0) + C(z-z_0) &= 0 \end{aligned}$$

$$\begin{aligned} \vec{n} \cdot \vec{PQ} &= 0 \\ \vec{n} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \right) &= 0 \\ \vec{n} \cdot \begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix} &= 0 \end{aligned}$$

Parametric Equations for a Plane in Space

The plane parallel to $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ passing through the point (x_0, y_0, z_0)

$$\begin{aligned} \text{is } x &= tu_1 + sv_1 + x_0 \\ y &= tu_2 + sv_2 + y_0 \\ z &= tu_3 + sv_3 + z_0 \end{aligned} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + s \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + t \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad -\infty < t, s < \infty$$



Example:

Find the plane parallel to the vectors $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ passing through the point $(1, 1, 1)$.

$$\begin{aligned} x &= t - s + 1 \\ y &= t + s + 1 \\ z &= -t + s + 1 \end{aligned} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

↓
 $s=0, t=0$ $(1-1, 1+1)$ $(-1+1, 1+1)$

basis vectors

$$\begin{aligned} y &= t + s + 1 \\ z &= -t + s + 1 \end{aligned} \quad \left\{ \begin{array}{l} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} s \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \text{basis vectors} \end{array} \right.$$

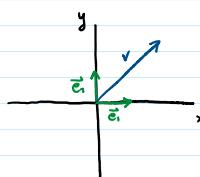
\downarrow
 $s=0, t=0$

$\rightarrow \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Basis Vectors

* Basis vectors is a set of independent vectors which span a vector space.

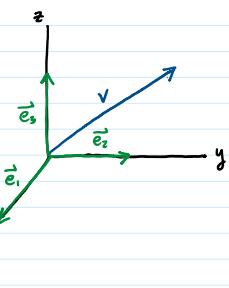
→ In \mathbb{R}^2



Basis vectors: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ → set $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$$\begin{aligned} \text{Vectors using } \vec{e}_1 \text{ & } \vec{e}_2 \text{ basis: } \vec{v} &= v_1 \vec{e}_1 + v_2 \vec{e}_2 \\ &= v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \vec{v} &= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{aligned}$$

→ In \mathbb{R}^3



Basis vectors: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ → set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\begin{aligned} \text{Vectors using the basis: } \vec{v} &= v_1 \vec{e}_1 + v_2 \vec{e}_2 + v_3 \vec{e}_3 \\ &= v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \vec{v} &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \end{aligned}$$

Basis Definition

* A basis has two conditions:

1. The basis set must be linearly independent.
2. The basis set must span the vector space.

Linear Independence - Basic definition

A sequence of vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ is linearly independent if and only if the only scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $\sum_{i=1}^n \lambda_i \vec{u}_i = 0$ are $\lambda_1 = \dots = \lambda_n = 0$.

In other words, $\lambda_1 \vec{u}_1 + \lambda_2 \vec{u}_2 + \dots + \lambda_n \vec{u}_n = 0$ is true only if $\lambda_1 = \dots = \lambda_n = 0$.

If any $\lambda_1, \lambda_2, \dots, \lambda_n$ is non-zero, then it is linearly dependent.

→ Example: Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\downarrow \quad \downarrow$$

• Is \vec{u} & \vec{v} linearly independent?

$$\begin{aligned} \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \lambda_1(1) + \lambda_2(0) &= 0 \quad \rightarrow \lambda_1 = 0 \\ \lambda_1(0) + \lambda_2(1) &= 0 \quad \rightarrow \lambda_2 = 0 \quad \checkmark \text{ yes.} \end{aligned}$$

• Is \vec{u}, \vec{v} , & \vec{w} linearly independent?

$$\begin{aligned} \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \lambda_1(1) + \lambda_2(0) + \lambda_3(1) &= 0 \\ \lambda_1(0) + \lambda_2(1) + \lambda_3(1) &= 0 \\ \lambda_1 + \lambda_3 &= 0 \\ \lambda_2 + \lambda_3 &= 0 \end{aligned}$$

$\rightarrow \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} \text{ is a free variable} \quad \times \text{ no.}$

→ Example: Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in \mathbb{R}^2 .

$$\lambda_2 + \lambda_3 = 0 \quad | \quad \boxed{1} \boxed{0} \boxed{1} \boxed{1} \boxed{1} \boxed{0} \quad \dots$$

→ Example: Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in \mathbb{R}^2 .

• Is \vec{u} & \vec{v} linearly independent?

$$\lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \end{array} \right]$$

RREF $\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right] \quad \checkmark \text{ yes.}$

$\lambda_1 = 0, \lambda_2 = 0$

Span

* If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are elements of a vector space V , then $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a subspace of V .

• Graphical representation of span

→ Span of one vector in 2D

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\text{span}(v) \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \rightarrow x = tv_1, y = tv_2$

parametrized line in 2D

→ Span of two linearly dependent vectors in 2D:

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\text{span}(u, v) \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

parametrized line in 2D

→ Span of two linearly independent vectors in 3D:

$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\text{span}(u, v) \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow x = t, y = s, z = 0$

parametrized plane in 3D

If we include $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, which is linearly independent to u and v , then $\text{span}(u, v, w)$ is a cube.

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + q \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

parametrized cube in 3D

• Spanning sequence / spanning set is equivalent to system of linear equations.

→ Example: Check if $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are a spanning sequence for \mathbb{R}^2 .

$$\lambda \vec{u} + \beta \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$\lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ system of linear equations $\rightarrow \begin{cases} \lambda - \beta = x \\ \lambda + \beta = y \end{cases}$ two equations
two unknowns

\downarrow two lines in \mathbb{R}^2
with an intersection

$\lambda = \underline{x+y}$ and $\beta = \underline{y-x}$

$\lambda + \mu = \beta$ two unknowns
 ↓
 two lines in \mathbb{R}^2
 with an intersection
 $\lambda = \frac{x+y}{2}$ and $\beta = \frac{y-x}{2}$

Check: • $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow x=1, y=1 \rightarrow \lambda = 1, \beta = 0 \rightarrow \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{u}$ ✓

• $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow x=-1, y=1 \rightarrow \lambda = 0, \beta = 1 \rightarrow \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (1) \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \vec{v}$ ✓

In the span of \vec{u} & \vec{v} ✓

• $\vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow x=1, y=-1 \rightarrow \lambda = 0, \beta = -1 \rightarrow \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{w}$ ✓

• $\vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \rightarrow x=2, y=3 \rightarrow \lambda = \frac{1}{2}, \beta = \frac{1}{2} \rightarrow \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \vec{b}$ ✓

→ Example: Check if $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are spanning sequence for \mathbb{R}^3 .

$$\lambda \vec{v}_1 + \beta \vec{v}_2 + \gamma \vec{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\lambda \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{\text{System of linear equations}} \begin{aligned} \lambda + \gamma &= x \\ -\lambda + \beta &= y \\ -\beta - \gamma &= z \end{aligned} \quad \left. \begin{array}{l} \text{three equations} \\ \text{three unknowns} \end{array} \right\}$$

Check: $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \rightarrow 1-1+0=0$ ✓

↓
 three planes in \mathbb{R}^3
 with an intersection
 $\lambda = x - \gamma$
 $-(x-\gamma) + \beta = y \rightarrow -x + \gamma + \beta = y \rightarrow \beta = y + x - \gamma$
 $-(y+x-\gamma) - \gamma = z \rightarrow -y - x = z \rightarrow x + y + z = 0$.

The system has solutions if $x+y+z=0$.
 If $x+y+z \neq 0$, then it has no solution.

$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \rightarrow 0+1-1=0$ ✓

• $\vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in the span
 of \vec{v}_1, \vec{v}_2 , and \vec{v}_3 because $1+0+0=1 \neq 0$.

$\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rightarrow 1+0-1=0$ ✓

• $\vec{v}_5 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ is in the span
 of \vec{v}_1, \vec{v}_2 , and \vec{v}_3 because $1+1-2=0$.

Subspace

A subspace is a subset of a vector space that is also a vector space under the same operations of vector addition and scalar multiplication.

That is a subset W of a vector space V is a subset if:

1. The zero vector of V is in W .
2. W is closed under addition: if $\vec{u}, \vec{v} \in W$, then $\vec{u} + \vec{v} \in W$.
3. W is closed under scalar multiplication: if $\vec{v} \in W$ and c is any scalar, then $c\vec{v} \in W$.

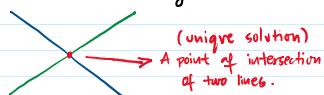
Subspace Dimensions

The dimension of a subspace is the number of vectors in a basis for that subspace.

In other words, dimension counts how many directions are needed to describe all vectors in the subspace.

Points of Intersections

- Two lines intersecting



- Parallel lines



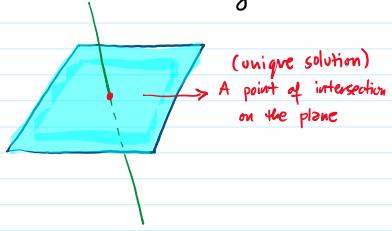
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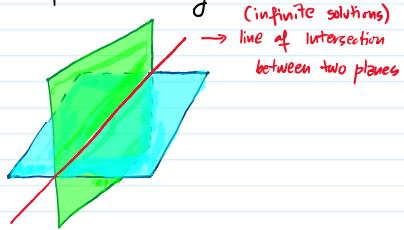
- A line and a plane intersecting



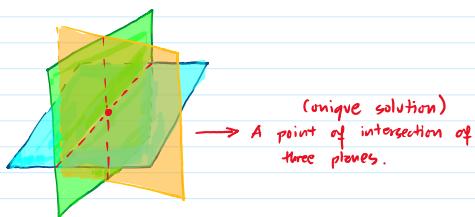
- A line and a plane intersecting



- Two planes intersecting



- Three planes intersecting



→ Example:

Find vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

$$\left[\begin{array}{ccc|c} 3 & 6 & -2 & 15 \\ 2 & 1 & -2 & 5 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -1/4 & 5/3 \\ 0 & 1 & 3/4 & 5/3 \end{array} \right]$$

*z is a free variable
Let $z=t$*

$$\text{Solution: } \left. \begin{array}{l} x - 1/4t = 5/3 \\ y + 3/4t = 5/3 \\ z = t \end{array} \right\} \quad \left. \begin{array}{l} x = 5/3 + 1/4t \\ y = 5/3 - 3/4t \\ z = t \end{array} \right\} \quad \left[\begin{array}{c} x \\ y \\ z \end{array} \right] = \left[\begin{array}{c} 5/3 \\ 5/3 \\ 0 \end{array} \right] + t \left[\begin{array}{c} 1/4 \\ -3/4 \\ 1 \end{array} \right] \rightarrow \text{Span} \left\{ \left[\begin{array}{c} 1/4 \\ -3/4 \\ 1 \end{array} \right] \right\}$$

↓
A line with basis vector $\left[\begin{array}{c} 1/4 \\ -3/4 \\ 1 \end{array} \right]$

So, the solution set is $\left\{ \left[\begin{array}{c} 5/3 \\ 5/3 \\ 0 \end{array} \right] + t \left[\begin{array}{c} 1/4 \\ -3/4 \\ 1 \end{array} \right] : t \in \mathbb{R} \right\}$. The basis vector $\left[\begin{array}{c} 1/4 \\ -3/4 \\ 1 \end{array} \right]$ is parallel to the line of intersection.

Since there is only one vector, the dimension of subspace is 1, basically a line.
The line of intersection does not pass through $\vec{0}$. So, S is not a subspace of \mathbb{R}^3 .