Name:

Collaborators:

Instructions: Worksheets are graded mostly on completion, and partially on correctness. Please write complete solutions showing explanations and key steps to the following problems, unless it says otherwise.

Testing a Vector Space

1. Vector Space Definition

A *vector space* over $\mathbb{R}$ is a set V with two operations:

- **Vector addition** $(+)$
- **Scalar multiplication** $(\cdot)$

such that the following properties hold (closure under addition, associativity, commutativity, identities, additive inverses, distributive properties, scalar rules, etc.).

a. Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^n$. Identify whether each property belongs to addition, scalar multiplication, or both so that $\vec{u}, \vec{v}, \vec{w} \in V$ and scalars $c, d \in \mathbb{R}$:

(1) $\vec{u} + \vec{v} \in V$

(2) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(3) $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$

(4) $\vec{v} + \vec{0} = \vec{v}$

(5) $\vec{v} + (-\vec{v}) = \vec{0}$

(6) $c \cdot \vec{v} \in V$

(7) $(c + d) \cdot \vec{v} = c \cdot \vec{v} + d \cdot \vec{v}$

(8) $c \cdot (\vec{v} + \vec{w}) = c \cdot \vec{v} + c \cdot \vec{w}$

(9) $(cd) \cdot \vec{v} = c \cdot (s \cdot \vec{v})$

(10) $1 \cdot \vec{v} = \vec{v}$

b. For each case below, decide: Is this a vector space over $\mathbb{R}$? If yes, write a short explanation. If no, give a counterexample showing which property fails.

i. $\mathbb{R}^2$ with usual addition and usual scalar multiplication.

ii. $\mathbb{R}^2$ with addition defined by

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x + a \\ y + b + 1 \end{bmatrix}$$

and usual scalar multiplication.

2. More Examples and Reflections

a. For each case below, decide: Is this a vector space over $\mathbb{R}$? If yes, write a short explanation. If no, give a counterexample showing which property fails.

i. The set of all polynomials of degree ≤ 3, with usual operations.

ii. The set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$, with usual operations.

b. Come up with your own example of a set with operations that almost looks like a vector space, but fails one property.

i. Define your set and operations.

ii. Which property fails?

c. Why does changing just one operation often break the vector space structure?

d. Which examples felt "obviously" vector spaces? Which required careful checking?