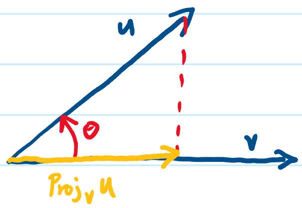
Cross Product & Planes

Wednesday, January 24, 2024

Objectives:

1. The cross product

Recall:



$u = \langle u_1, u_2, u_3 \rangle = u_1 i + u_2 j + u_3 k$

$v = \langle v_1, v_2, v_3 \rangle = v_1 i + v_2 j + v_3 k$

where i, j, & k are standard normal vectors

Dot Product: $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$

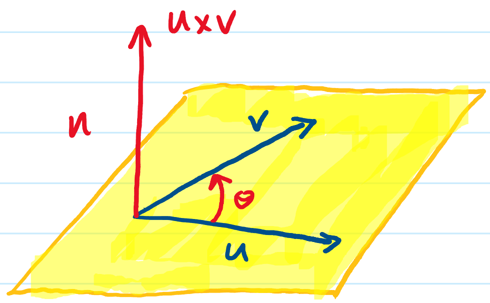
$\rightarrow$ u & v are <u>orthogonal</u> if and only if $u \cdot v = 0$.

Direction: $\theta = \arccos\left(\dfrac{u \cdot v}{\|u\|\|v\|}\right)$

Projection: $\text{Proj}_v u = \left(\dfrac{u \cdot v}{\|v\|^2}\right) v$

<u>The Cross Product</u>

- Positive n:

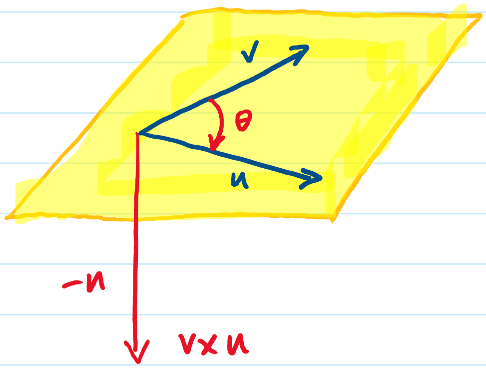


$$u \times v = \underbrace{\left(\|u\| \, \|v\| \sin(\theta)\right)}_{\text{scalar multiple of the } n \text{ unit vector}} n$$

n is also called the normal vector.

$u \times v$ is orthogonal to u and v.

- Negative n:



$$v \times u = \underbrace{\left(\|u\|\,\|v\|\sin(\theta)\right)}_{\text{scalar multiple of the } n \text{ unit vector}}(-n)$$

Properties of the Cross Product:

1. $(ru)\times(sv) = (rs)(u\times v)$
2. $u\times(v+w) = u\times v + u\times w$
3. $v\times u = -(u\times v)$
4. $(v+w)\times u = v\times u + w\times u$
5. $0\times u = 0$
6. $u\times(v\times w) = (u\cdot w)v - (u\cdot v)w$

Parallel Vectors

u and v are parallel if and only if $u\times v = 0$.

Explicit Formula for $u\times v$

$$u\times v = (u_1 i + u_2 j + u_3 k)\times(v_1 i + v_2 j + v_3 k)$$

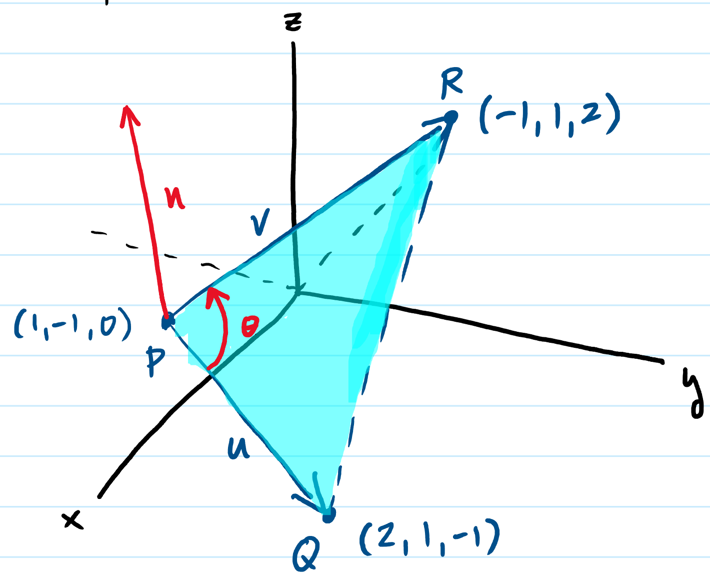
$$\begin{aligned}
&= u_1\cancel{v_1}\, i\times i + \boxed{u_1 v_2\, i\times j} + \boxed{u_1 v_3\, i\times k} \\
&\quad + \boxed{u_2 v_1\, j\times i} + u_2\cancel{v_2}\, j\times j + \boxed{u_2 v_3\, j\times k} \\
&\quad + \boxed{u_3 v_1\, k\times i} + \boxed{u_3 v_2\, k\times j} + u_3\cancel{v_3}\, k\times k
\end{aligned}$$

Facts:
$$\begin{aligned}
i\times j &= -(j\times i) = k \\
j\times k &= -(k\times j) = i \\
k\times i &= -(i\times k) = j \\
i\times i &= j\times j = k\times k = 0
\end{aligned}$$

$$\begin{aligned}
&= u_2 v_3\, j\times k + u_3 v_2\, k\times j \longrightarrow u_2 v_3 j - u_3 v_2 j = (u_2 v_3 - u_3 v_2) i \\
&\quad + u_1 v_3\, i\times k + u_3 v_1\, k\times i \longrightarrow -u_1 v_3 k + u_3 v_1 j = (u_3 v_1 - u_1 v_3) j \\
&\quad + u_1 v_2\, i\times j + u_2 v_1\, j\times i \longrightarrow u_1 v_2 k - u_2 v_1 k = (u_1 v_2 - u_2 v_1) k
\end{aligned}$$

$$u \times v = (u_2v_3 - u_3v_2)i + (u_3v_1 - u_1v_3)j + (u_1v_2 - u_2v_1)k$$

Example:



Triangular Plane

→ Find u & v given points

$$u = \overrightarrow{PQ} = (2-1)i + (1+1)j + (-1-0)k = i + 2j - k$$
$$v = \overrightarrow{PR} = (-1-1)i + (1+1)j + (2-0)k = -2i + 2j + 2k$$

→ the vector perpendicular to the plane.

$$u \times v = \big(2(2) - (-1)(2)\big)i + \big((-1)(-2) - 1(2)\big)j + \big(1(2) - 2(-2)\big)k$$

$$= 6i + 0j + 6k$$

$$u \times v = 6i + 6k$$

→ Area of the Triangle

$$\frac{1}{2}\|u \times v\| = \frac{1}{2}\|6i + 6k\|$$

$$= \frac{1}{2}\sqrt{6^2 + 6^2}$$

$$= \frac{1}{2}\sqrt{2(36)}$$

$$= \frac{6\sqrt{2}}{2}$$

$$\text{Area} = 3\sqrt{2}$$