Bayes' Theorem

Friday, February 10, 2023

Previously...

Conditional Probability

Definition: If B is dependent on A, then

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

(| → given)

i. If A and B are independent, then $P(B|A) = P(B)$

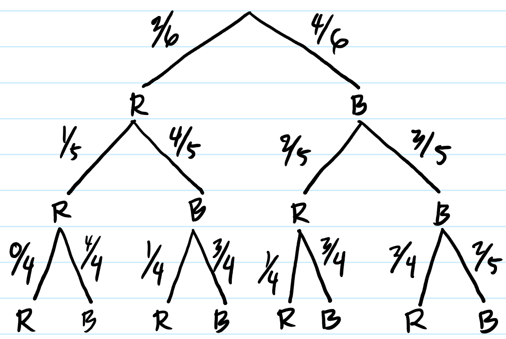
ii. If B is dependent on A, then $P(A \cap B) = P(A)P(B|A)$

iii. $P(B|A) \neq P(A|B)$

Example: Balls in 1 Urn

Suppose an Urn has 6 balls in total; 2 reds and 4 blacks.
Consider you draw three balls from the urn without replacement.
What is the probability that at least two black balls are drawn?

Way 1: Using the probability tree.



$$\begin{aligned}
P(\text{black balls} \geq 2) &= P(B_1)P(B_2|B_1)P(B_3|B_1 \cap B_2) \rightarrow \{B_1, B_2, B_3\} \\
&+ P(B_1)P(B_2|B_1)P(R_3|B_1 \cap B_2) \rightarrow \{B_1, B_2, R_3\} \\
&+ P(B_1)P(R_2|B_1)P(B_3|B_1 \cap B_2) \rightarrow \{B_1, R_2, B_3\} \\
&+ P(R_1)P(B_2|R_1)P(B_3|R_1 \cap B_2) \rightarrow \{R_1, B_2, B_3\} \\
&= 4(1/5) \\
&= 4/5
\end{aligned}$$

Way 2:

$$P(\text{draw at least 2 black balls}) = \frac{\binom{4}{2}\binom{2}{1} + \binom{4}{3}\binom{2}{0}}{\binom{6}{3}}$$

→ ways to draw exactly 2 blacks

→ ways to draw exactly 3 blacks

↳ all ways to draw 3 balls out of 6

$$= \frac{6(2) + 4(1)}{20}$$

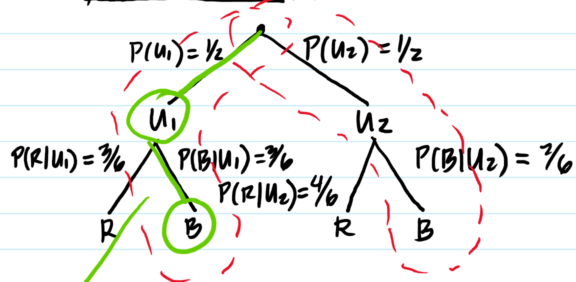
$$= 4/5$$

Example: Balls in 2 Urns

	Red (R)	Black (B)
Urn 1 (U_1)	3	3
Urn 2 (U_2)	4	2

$P(U_1) = 1/2$
$P(U_2) = 1/2$

forward tree



1. What is the probability that a randomly chosen ball is black and it came from Urn 1?

$$P(U_1 \cap B) = \boxed{P(U_1)P(B|U_1)} = (1/2)(3/6) = 3/12$$

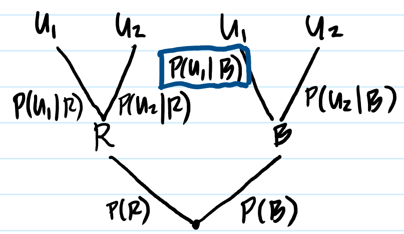
intersection (and)

2. One jar is chosen at random and a single ball is selected. If the ball is black, what is the probability that it came from Urn 1?

$$\boxed{P(U_1|B)} = \frac{\boxed{P(U_1)P(B|U_1)}}{\boxed{P(B)}} = \frac{P(U_1)P(B|U_1)}{P(U_1)P(B|U_1) + P(U_2)P(B|U_2)} = \frac{(1/2)(3/6)}{(1/2)(3/6)+(1/2)(2/6)} = \frac{1/4}{5/12} = \frac{5}{48}$$

The probability that the ball came from Urn 1 given that we observed that the ball is black.

reverse tree



The Law of total Probability

Let A and B be events in the sample space S. Then,

$$P(A) = P(A \cap B) + P(A \cap B^c) = P(B)P(A|B) + P(B^c)P(A|B^c)$$

The law allows the computation of the probability of an event "conditioning" on another event.

Bayes' Theorem

Let A and B be events in the sample space S.

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)} = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}$$

In general,

Let $A_1, \dots, A_K$ be a partition of the sample space S and let B be an event.

That is, $A_1, \dots A_K$, is a partition of the sample space if $\bigcup_{i=1}^{K} A_i = S$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

That is, $A_1, \cdots A_K$, is a partition of the sample space if $\bigcup_{i=1}^{K} A_i = S$ and $A_i \cap A_j = \phi$ whenever $i \neq j$.

Then,

$$P(B) = \sum_{i=1}^{K} P(B \cap A_i) = \sum_{i=1}^{K} P(A_i) P(B | A_i)$$

$$P(A_j | B) = \frac{P(A_j) P(B | A_j)}{\sum_{i=1}^{K} P(A_i) P(B | A_i)} .$$