Geometric Distribution Expected Value

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$P(x) = p(1-p)^{x-1}$ → pmf for the geometric r.v.

Show that $E[X] = \frac{1}{p}$.

$E[X] = \sum_{x=1}^{\infty} x P(x)$ → expected value definition of discrete r.v.

$= \sum_{x=1}^{\infty} x\, p(1-p)^{x-1}$

$= p \boxed{\sum_{x=1}^{\infty} x(1-p)^{x-1}}$

* we know that $\sum_{x=0}^{\infty} r^x = \frac{1}{1-r}$ if $|r|<1$; geometric series convergence

So, $\frac{d}{dr} \sum_{x=0}^{\infty} r^x = \sum_{x=1}^{\infty} x r^{x-1}$

$\frac{d}{dr}\left(\frac{1}{1-r}\right) = \sum_{x=1}^{\infty} x r^{x-1}$

$\boxed{-\frac{1}{(1-r)^2} = \sum_{x=1}^{\infty} x r^{x-1}}$

$= p \sum_{x=1}^{\infty} x r^{x-1}$, $r = 1-p$, $|r|<1$

$= p \sum_{x=1}^{\infty} x(1-p)^{x-1}$

$= -p \frac{d}{dp}\left(\sum_{x=0}^{\infty} (1-p)^x\right)$

$= -p \frac{d}{dp}\left(\frac{1}{1-(1-p)}\right)$

$= -p \frac{d}{dp}\left(\frac{1}{p}\right)$

$= \cancel{-}\cancel{p}\left(\cancel{-}\frac{1}{p^{\cancel{2}}}\right)$

$E[X] = \frac{1}{p}$