Report for exercise 4 from group B

Tasks addressed:	5
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Source code:	https://gitlab.com/turcumanhoria/crowd-modeling

The work on tasks was divided in the following way:

HORIA TURCUMAN (03752553)	Task 1	33.3%
	Task 2	33.3%
	Task 3	33.3%
	Task 4	33.3%
	Task 5	33.3%
GEORGI HRUSANOV (03714895)	Task 1	33.3%
	Task 2	33.3%
	Task 3	33.3%
	Task 4	33.3%
	Task 5	33.3%
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	Task 2	33.3%
	Task 3	33.3%
	Task 4	33.3%
	Task 5	33.3%

Report on task 1, Vector fields, orbits, and visualization

A dynamical system is comprised of a set X of states, denoted by the notation $x \in X$, as well as an evolution operator, denoted by the notation : $\phi : IxX \mapsto X$, which alters the state as a function of the parameter t, which is often referred to as "time" $t \in I$ in an index set I. In most cases, the $I \subseteq R$ choice is made for continuous dynamical systems, while the $I \subseteq N$ choice is used for discrete dynamical systems. The dynamical system is defined by the intersection of time, state space, and the evolution operator, and this intersection is often referred to as a triple (I, X, ϕ). For a system that is discrete, the evolution ϕ caused by beginning at a point x_0 and ending at X is often written out.

$$x_n = \phi(n, x_0), x_n \in X, n \in I. \tag{1}$$

The map ϕ itself is not explicitly specified in many formulations of continuous dynamical systems; rather, its derivative with respect to time is expressed as a function $v : X \mapsto TX$, where TX stands for the tangent bundle of X, and $v(x) \in T_xX$ for any values of x that are in X. The local tangent space at x is denoted by the symbol T_xX, and the equation for TX is: $TX = \cup_{x \in X} T_xX$. Because it assigns a vector to each point that falls within the range of X, the map denoted by the letter v is referred to as a vector field. The behavior of a dynamical system (that is, the map) may be altered in very arbitrary but generally smooth ways by adjusting its parameters. These parameters may be denoted by adding a subscript to the evolution operator. For example, the symbol shows that the operator ϕ_α relies on a certain number of parameters denoted by $\alpha \in R^k$. An examination of a dynamical system's bifurcation is concerned with the qualitative changes that take place in the system whenever the values of the parameters are altered. The concept of topological equivalence is used to define qualitative change. According to this definition, a system is qualitatively the same as another system if both systems have the same topological properties.

The scope of the first task within this exercise sheet was exactly that: Our goal was to visualize a dynamical system with a one- or two-dimensional state space using its *phase portrait.* We had to recreate and visualize plots from the book of Kuznetsov [1]. Within the examples that are shown in the book several different plots could be generated. They are specifically a **node** and a **focus** phase portrait, both **stable** and **unstable**, as well as an **unstable saddle** phase picture.

The linear dynamical system in object has a state space $X = \mathbb{R}^2$, $I = \mathbb{R}$ and parameter $\alpha \in \mathbb{R}$. Its flow is defined as follows:

$$\frac{\partial \phi_\alpha(t, x)}{\partial t}\Big|_{t=0} = A_\alpha x \tag{2}$$

where x, $t \in I$ and A is a 2x2 real matrix parametrized by α. Specifically, we obtained these portraits with the following configurations (where λ_1 and λ_2 indicate the 2 eigenvalues of A):

- **Unstable focus:** $\alpha = 0.1,\ A = \begin{bmatrix} \alpha & \alpha \\ -\frac{1}{4} & 0 \end{bmatrix},\ \lambda_1 = 0.05 + 0.15i,\ \lambda_2 = 0.05 - 0.15i$
- **Unstable saddle:** $\alpha = -0.3,\ A = \begin{bmatrix} \alpha & \alpha \\ -\frac{1}{4} & 0 \end{bmatrix},\ \lambda_1 = -0.462,\ \lambda_2 = 0.162$
- **Unstable node:** $\alpha = 1,\ A = \begin{bmatrix} \alpha & \alpha \\ -\frac{1}{4} & 0 \end{bmatrix},\ \lambda_1 = 0.5,\ \lambda_2 = 0.5$
- **Stable node:** $\alpha = -0.3,\ A = \begin{bmatrix} \alpha & 0 \\ 0 & -1 \end{bmatrix},\ \lambda_1 = -0.3,\ \lambda_2 = -1$
- **Stable focus:** $\alpha = 0.3,\ A = \begin{bmatrix} -\alpha & \alpha \\ -\frac{1}{4} & 0 \end{bmatrix},\ \lambda_1 = -0.15 + 0.229j,\ \lambda_2 = -0.15 - 0.229j$

Here you can observe the concrete configurations that we used in order to be able to recreate the pictures from the book and achieve different plots. The setup and also the parametrized matrix could be seen. Figures 1,2,3 represent the achieved results and plots.

Last but not least, we had to answer the question: *Are these systems topologically equivalent?*. Of course we can conclude that the achieved outcomes are not topologically equivalent, because they represent totally different things: some of them are stable, whereas some are not. Moreover we have plots with some specific characteristics like for example: `saddle`. This could be seen even from the generated figures that show the phase portrait of different dynamical systems.

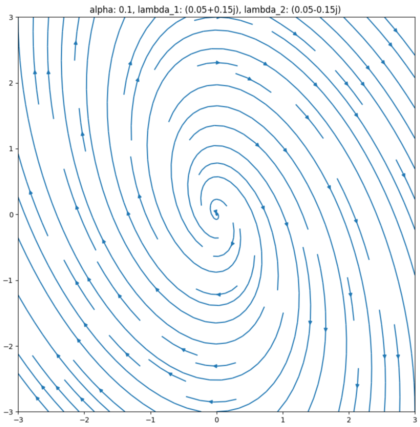

Figure 1: Bifurcation phase portrait: Unstable focus

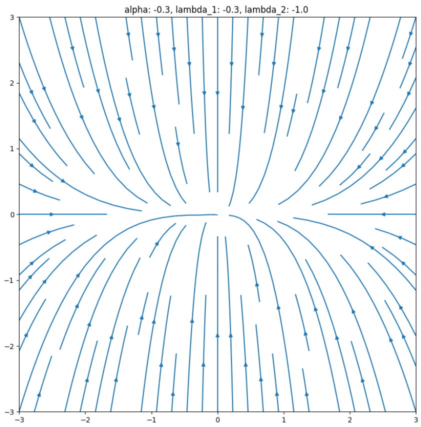

Figure 2: Bifurcation phase portrait: Stable node

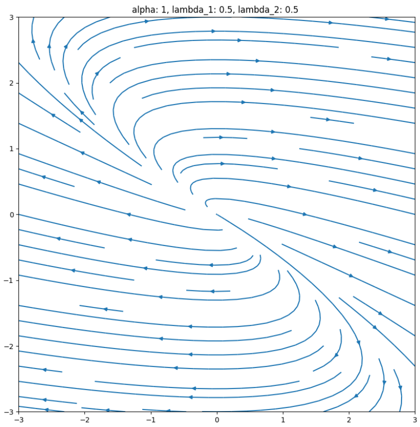

Figure 3: Bifurcation phase portrait: Unstable node

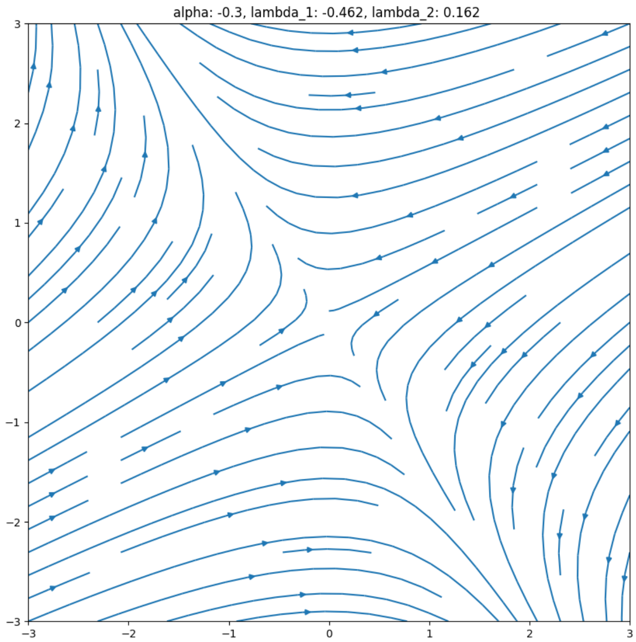

Figure 4: Bifurcation phase portrait: Unstable saddle

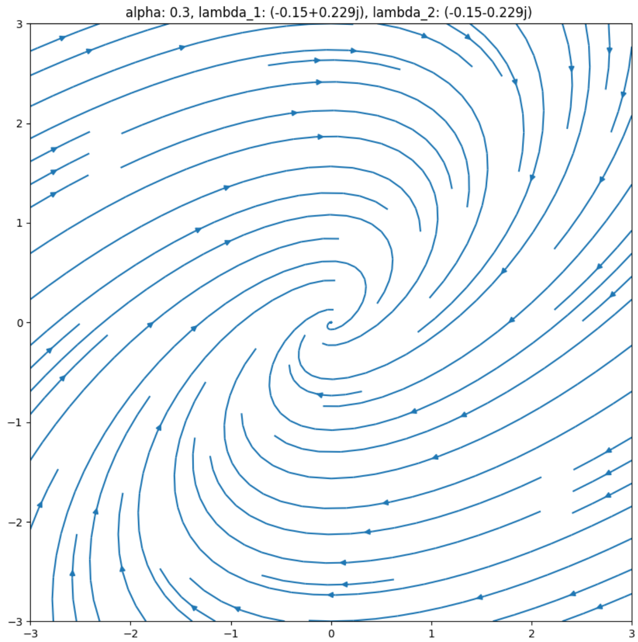

Figure 5: Bifurcation phase portrait: Stable focus

Report on task 2, Common bifurcations in nonlinear systems

In the following task we had to describe, visualize and analyse two different dynamical systems that could be described with the following equations:

$$\dot{x} = \alpha - x^2$$

and

$$\dot{x} = \alpha - 2x^2 - 3$$

The very first question that we had to answer within this subtask is to examine what type of bifurcation happens when the value for $\alpha = 0$. Both of these models display analogous behavior in that neither can reach a steady state until a specific parameter value corresponds to the bifurcation point (for example, if $\alpha = 0$ for the left model). Following that, the system will always be in one of two separate stable states. A saddle-node bifurcation occurs between these two systems, and one of the two stable states, shown in Figure 6 and Figure 7 by the colors blue and red, is repellent while the other is appealing. The blue color represents the states and points where we have stable equilibrium, whereas the red color represents the unstable equilibrium.

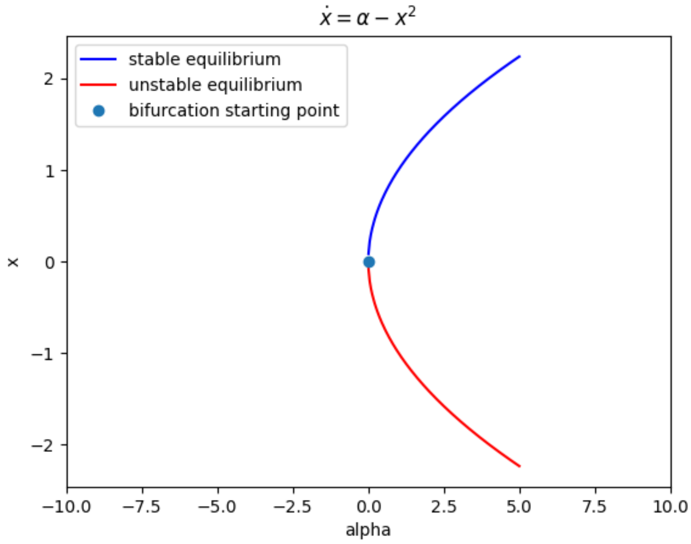

Figure 6: Bifurcation Diagram of the system:

$$\dot{x} = \alpha - x^2$$

Any comparison of objects is conducted on the basis of an equivalence relation, which enables us to define classes of objects that are equivalent to one another and study the transitions between these classes. As a result, we need to be more specific when defining when two dynamical systems are "qualitatively similar" to one another or equivalent to one another. A definition of this kind needs to accommodate a number of general intuitive requirements. For instance, it is reasonable to anticipate that two equivalent systems will have the same number of equilibria and cycles of the same types of stability. This is because equivalent systems are designed to produce the same results. It is expected that the "relative location" of these invariant sets and the geometry of their areas of attraction will be comparable for equal system configurations. In other words, we regard two dynamical systems to be equal if their phase pictures are "qualitatively similar." This means that if one portrait can be produced from another portrait by a continuous transformation, then the two portraits are qualitatively similar.[1]

Since it is observable that the system on the left has two steady states, whereas the system on the right does not have any steady states, it is safe to say that the two systems are not topologically equivalent at $\alpha = 1$. This is because the system on the left has two steady states. On the other hand, the two systems are topologically equivalent at the point where $\alpha = -1$ because they both have the same number of steady states and actually have the same normal form. This indicates that they are topologically equivalent. When looking at the graphics, it is possible to form the hypothesis that the two systems share the same normal form due to the fact that the bifurcation diagrams of both are qualitatively equivalent.[1]

Report on task 3, Bifurcations in higher dimensions

The next task we had to solve and tackle wants to address some other functionality and characteristic of Bifurcations - namely the ability to take a form also of higher dimension. Bifurcations are able to take place in dynamical systems with state spaces that have any dimension, and they may also take place in more than one parameter. If the state space is just one dimension (and the system in question is continuous), then certain bifurcations will not take place. For two-dimensional state spaces, there is a bifurcation that is significant for one-parameter systems called the *Andronov-Hopf* bifurcation [1][page 57]. This bifurcation occurs when the vector field is in normal form.

$$\dot{x}_1 = \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2)$$

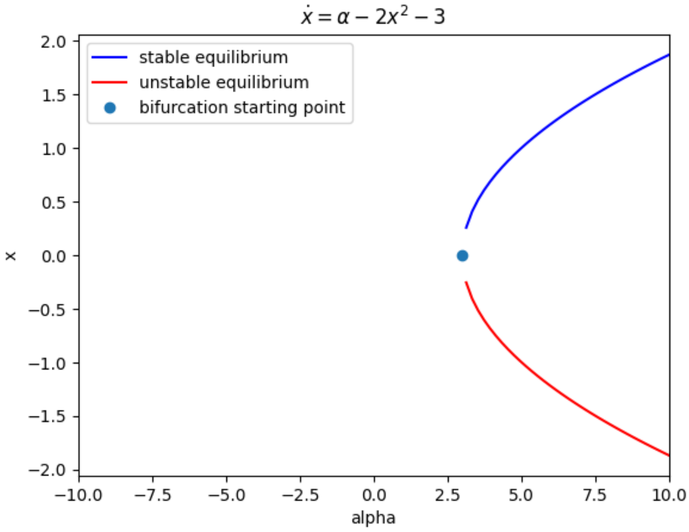

Figure 7: Bifurcation Diagram of the system:

$$\dot{x} = \alpha - 2x^2 - 3$$

In Figures 8,9,10,11,12 we could see the phase diagrams that visualize the bifurcation of the system. It is important to mention that for generating these plots, different values for alpha were examined in order to be able to see the behaviour of the system with different alpha values.

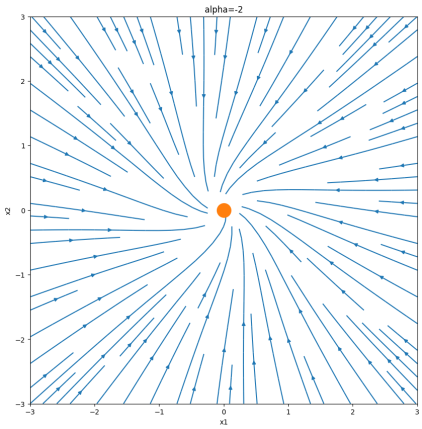

Figure 8: Bifurcation phase portrait Andronov-Hopf: $\alpha = -2$

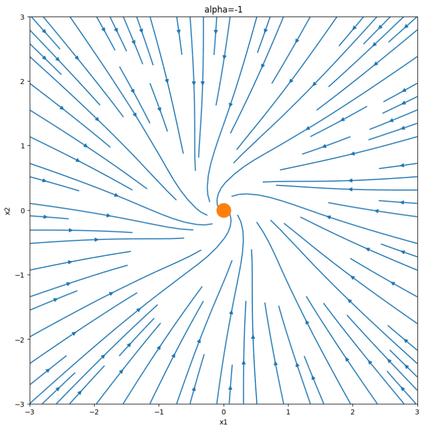

Figure 9: Bifurcation phase portrait Andronov-Hopf: $\alpha = -1$

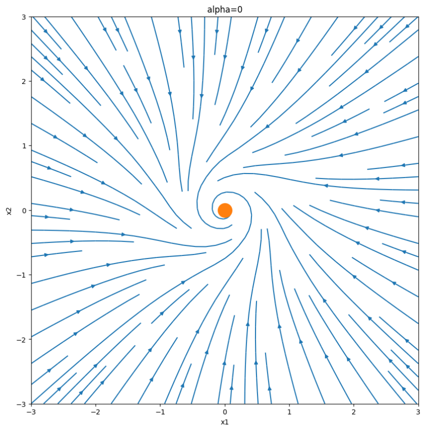

Figure 10: Bifurcation phase portrait Andronov-Hopf: $\alpha = 0$

Task 3.2: The second portion of the task will need you to do a numerical calculation and produce a graphic depiction of two circles that begin at **[2, 0]** and **[0.5, 0]**, respectively, given a certain number for (in the example supplied, $\alpha = 1$). The Euler's approach was utilized in the computation, and the desired two orbits were created. Given the precise value of, one should expect that any point that does not begin in the origin will eventually converge to the limit cycle, either because the origin repels them or because they are drawn to the limit cycle from outside the origin.

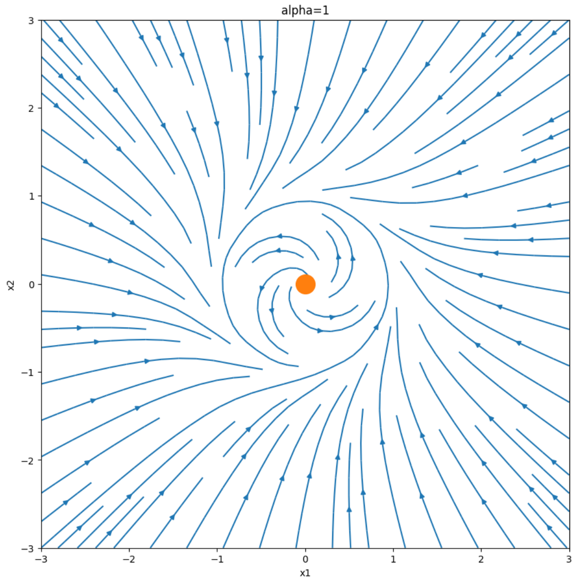

Figure 11: Bifurcation phase portrait Andronov-Hopf: $\alpha = 1$

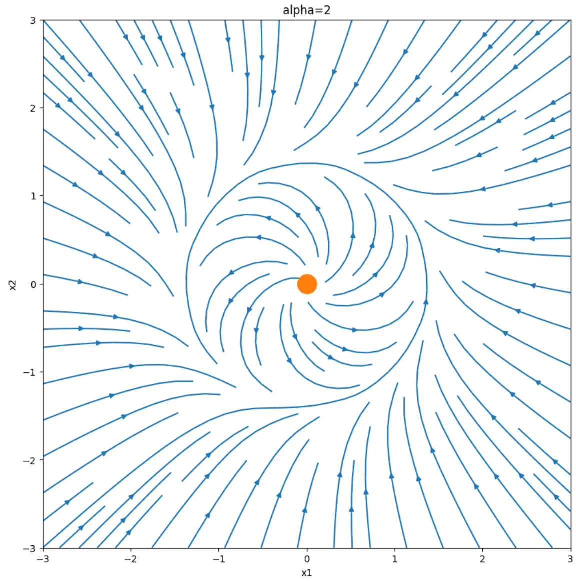

Figure 12: Bifurcation phase portrait Andronov-Hopf: $\alpha = 2$

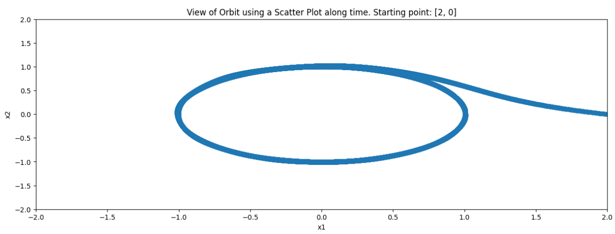

Figure 13: View of Orbit using a Scatter Plot along time. Starting point: [2, 0]

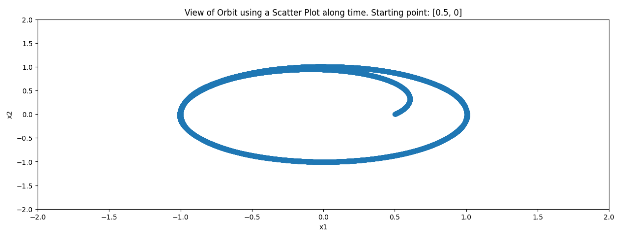

Figure 14: View of Orbit using a Scatter Plot along time. Starting point: [0.5, 0]

Task 3.3: The final subtask requires the study of a double parametered bifurcation, the cusp bifurcation, with the normal form shown below.

$$\dot{x} = \alpha_1 + \alpha_2 x - x^3$$

The cusp bifurcation is a type of equilibria bifurcation that occurs in a two-parameter family of autonomous `ODEs` at the point where the critical equilibrium has one zero eigenvalue and the quadratic coefficient for the saddle-node bifurcation disappears. This is the point at which the cusp bifurcation occurs.

When two branches of a saddle-node bifurcation curve meet tangentially at the cusp bifurcation point, the result is a semicubic parabola. The system is capable of having three equilibria for parameter values that are close together, but these equilibria will eventually clash and disappear pairwise due to saddle-node bifurcations. The existence of a hysteresis phenomena may be inferred from the bifurcation of the cusp.

This bifurcation is distinguished by its two bifurcation conditions, $\lambda_1 = 0$ and $a(0) = 0$, and it appears generically in two-parameter families of smooth `ODEs`. Its codimension is two. In a general sense, the critical equilibrium, denoted by x^0, is the triple root of the equation $f(x, 0) = 0$ and the origin of the two branches of the saddle-node bifurcation curve in the parameter plane is located at the point at $\alpha = 0$. When you go from one branch to another, you cause a pairwise collision, which leads to the loss of equilibrium. These bifurcations are nondegenerate, and in the neighborhood of x^0, there can be no more than three equilibria at any one time. These phenomena could be observed in the generated cusp bifurcation plot in Figure 15

Report on task 4, Chaotic dynamics

Dynamical systems can behave in very irregular ways, and changes in their parameters can lead to very drastic changes in their behavior. In this task we analyze some aspects of the logistic map system, expressed

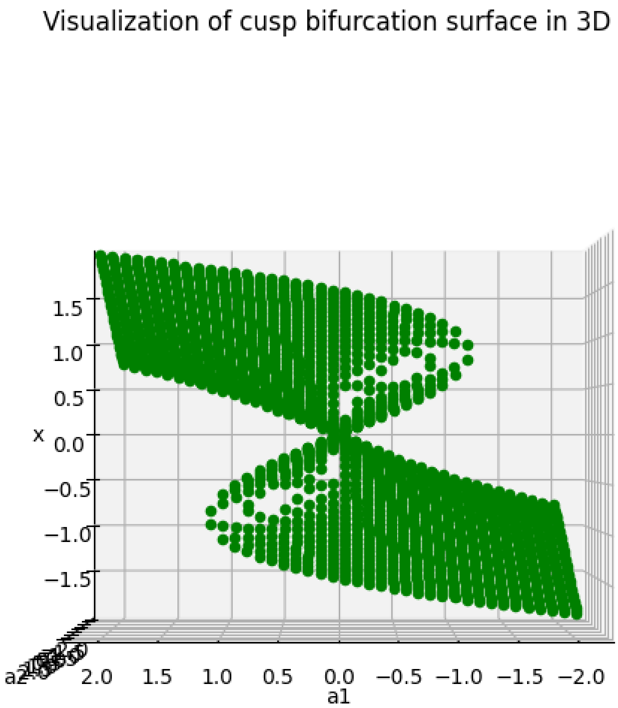

Figure 15: Visualization of cusp bifurcation surface in 3D

by the following discrete map:
$x_{n+1} = rx_n(1 - x_n)$
with $x_n \in R, n \in N$ and $r \in (0, 4]$
To analyze this system, we use the functions logisticmap and iterate as mentioned in file task4_part1.ipynb.

Creating discrete map function

```
def logisticmap(x, r):
    """
    Function computes the logistic map
    :param x: x_n value
    :param r: growth rate
    :return: x_n+1 value
    """
    return x * r * (1 - x)
```

Figure 16: Logistic Map Function

Defining Iteration function for logistic map

```
def iterate(n, x, r):
    """
    This function computes nth iteration of the logistic map
    :param n: iteration/index
    :param x: x_n value
    :param r: growth rate
    :return: x_n+1 value
    """
    for i in range(n):
        x = logisticmap(x, r)
    return x
```

Figure 17: Iterate Function to plot bifurcation graph

```python
# Generate list values (iterate for each value of r)
for idx, ri in enumerate(rs):
    rlist.append(ri)
    xlist.append(iterate(idx, seed, ri))

plt.scatter(rlist, xlist, s = .02)
plt.xlabel("r")
plt.ylabel("x")
plt.savefig("task_4_bifurcation_logistic_map_0_4.png")
show()
```

Figure 18: Application of functions for bifurcation graph

- Logisticmap: takes in 2 parameters - x_n value and growth rate r. This functions returns the x_{n+1} value.
- Iterate: takes in 3 parameters - the iteration index, x_n value and growth rate r. This function returns the value of the output of logisticmap function at the current value.
- Bifurcation plot: In here we use a for loop to return all the values of x_{n+1} for different r values.

Task 4.1: By varying r from 0 to 2, the bifurcation diagram obtained is as shown below:

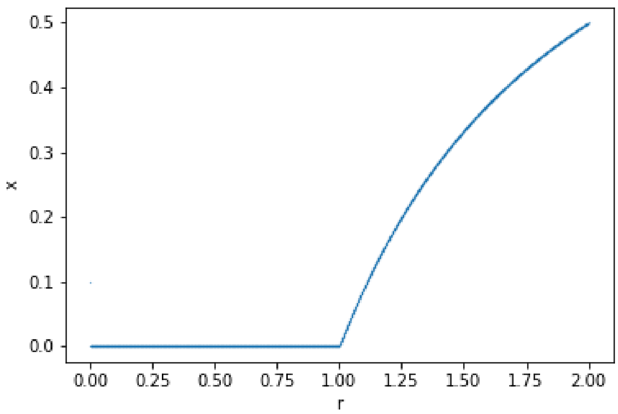

Figure 19: Bifurcation plot when $r \in (0, 2]$

It is possible to see that there is one steady state for each value of r, in particular:

- $r \in (0, 1] \implies x = 0$
- $r \in (1, 2] \implies x = \frac{r-1}{r}$

In this case, there is no bifurcation and the steady state of the system is always one. This fixed point is attractive, because, since each point in the bifurcation diagram is obtained plotting the last p states assumed by the system for a certain value of r, if the steady state in object was repulsive, we would have multiple points per value of r, instead this is not the case.

Task 4.2: Varying r from 2 to 4, the number of steady states continue to double more and more often, with the exceptions of few stability islands. This behavior is shown below:

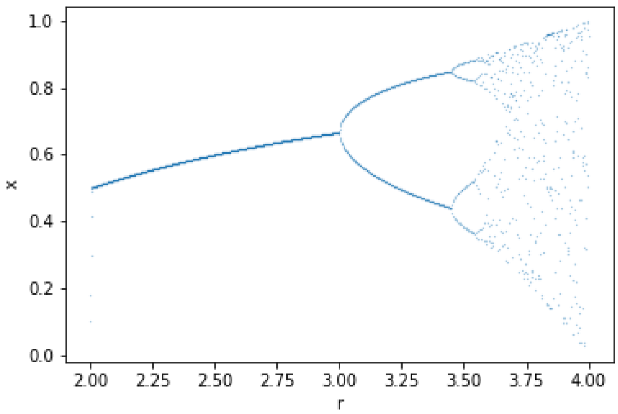

Figure 20: Bifurcation plot when $r \in (2, 4]$

Task 4.3: The bifurcation map when $r \in (0, 4]$ is as shown below:

- $r \in (0, 1] \implies x = 0$ is a steady state.
- $r \in (1, 3] \implies x = \frac{r-1}{r}$ is a steady state.
- $r \geq 3$ there are limit cycles that include a different number of points, initially an oscillation between two states, then at roughly 3.4 the oscillation becomes between four states. This number tend to increase (doubling) as r increases, until at approximately $r = 3.5$ the behaviour becomes chaotic with some exceptions being sporadic stability islands where once again oscillation between a little number of states is present.

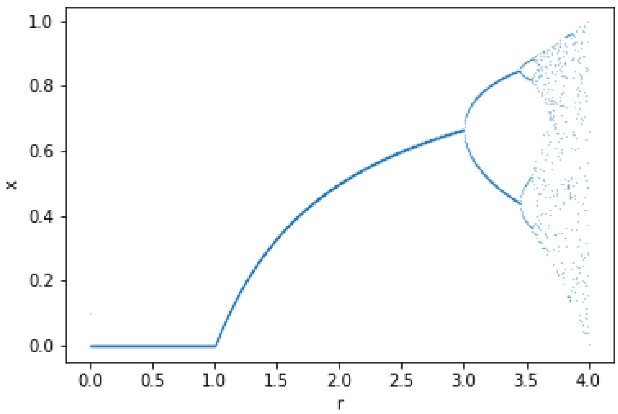

Figure 21: Bifurcation plot when $r \in (0, 4]$

The following figure shows the logistic map for different values of r.

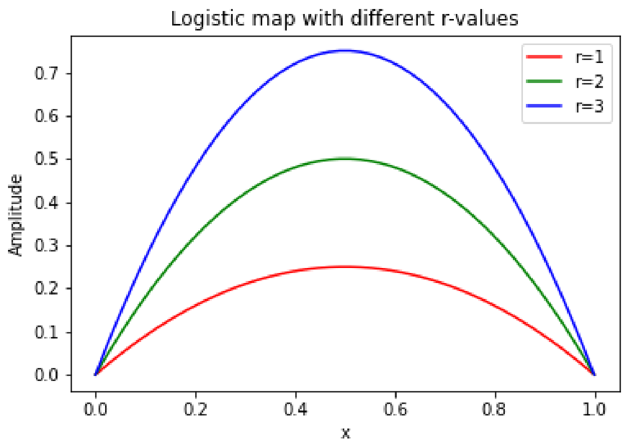

Figure 22: Logistic map for differnt values of r

Following figure shows the logistic map orbits for differnt values of r (x vs iterations for specific value of r):

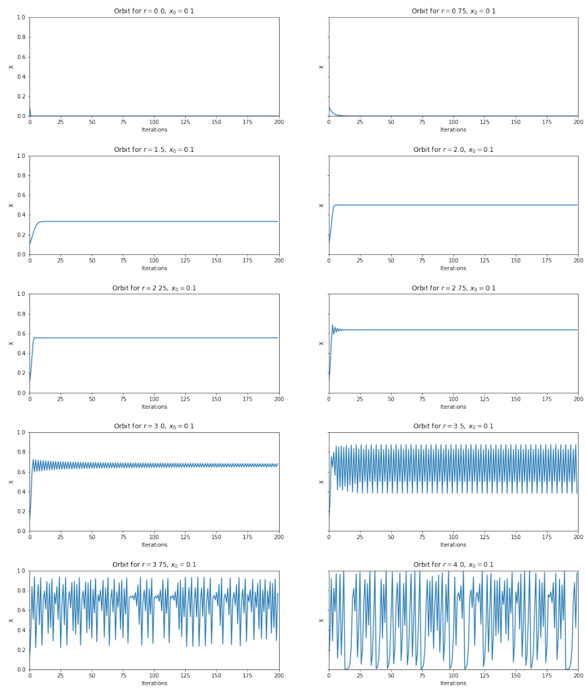

Figure 23: Logistic map orbit for different values of r

Part 2: Lorenz Attractor
The focus of this part is to analyze the Lorenz attractor. The system is defined as:

$$\frac{\partial x}{\partial t} = \sigma(y - x)$$

$$\frac{\partial y}{\partial t} = x(\rho - z) - y$$

$$\frac{\partial z}{\partial t} = xy - \beta z$$

where x, y and z represent the coordinates of a point/state of the system, while σ, ρ, β are parameters. The system has been simulated for 1000 seconds and its trajectory has been plotted. Specifically, starting from $x_0 = (10, 10, 10)$ we obtain the plot as below:

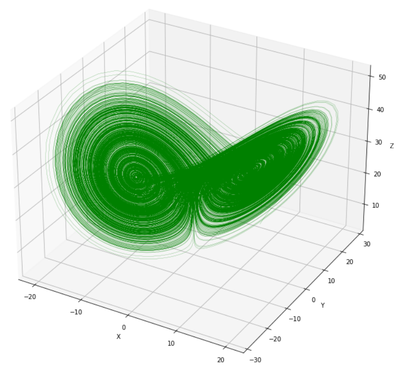

Figure 24: Lorenz Attractor plot when $x_0 = (10, 10, 10), \rho = 28$

While the following plot is for when $x_0 = (10 + 10^{-8}, 10, 10)$:

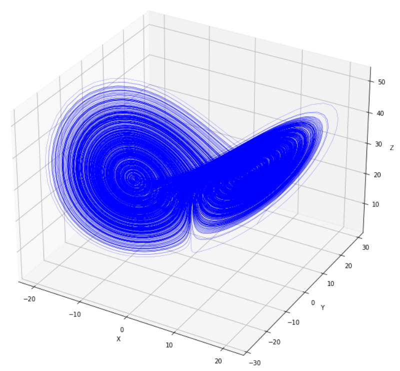

Figure 25: Lorenz Attractor plot when $x_0 = (10 + 10^{-8}, 10, 10), \rho = 28$

In both cases the parameters were set to $\sigma = 10, \beta = \frac{8}{3}, \rho = 28$
Since the system is chaotic, the minimal perturbation in the starting point is visible on a macroscopic scale when plotting the trajectory of the system starting from that point. The difference between the two trajectories is expressed as $||x(t) - \hat{x}(t)||^2$. This value assumes large values (maximum being 56.76070625268944). Specifically, this distance becomes larger than 1 at $2071^{th} iteration$.

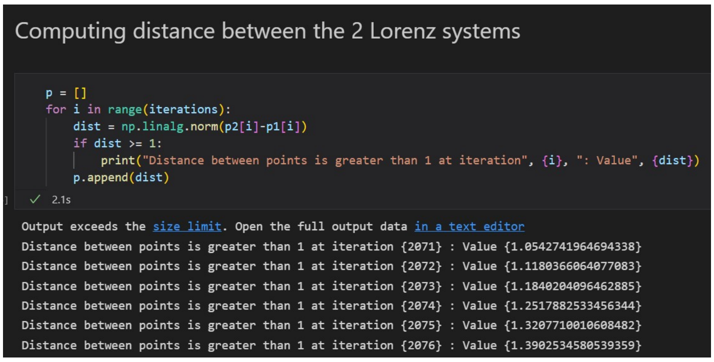

Figure 26: Code and result for comparing 2 Lorenz functions

Now, setting $\rho = 0.5$ and repeating these tests, it is possible to observe in below figure that the trajectories are identical to each other, but very different from before, indicating that a bifurcation occurs for a certain value of ρ between 0.5 and 28.

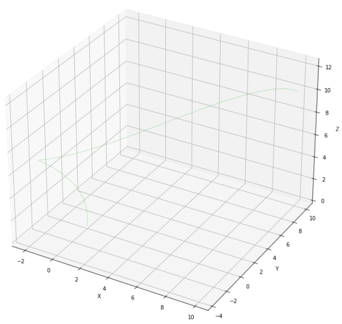

Figure 27: Lorenz Attractor plot when $x_0 = (10, 10, 10), \rho = 0.5$

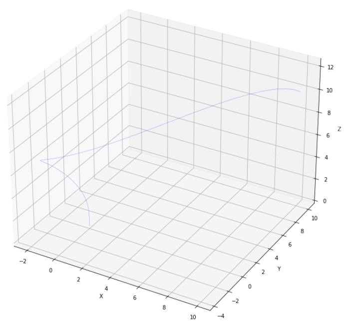

Figure 28: Lorenz Attractor plot when $x_0 = (10 + 10^{-8}, 10, 10), \rho = 0.5$

Report on task 5, Bifurcations in crowd dynamics

The evolution of the system for different parameters b is shown in figure 29. It is clear that there is a qualitative difference between the depicted systems. Varying the initial state changes the qualitative picture in some occasions as the one shown in **??**. The bifurcation happening at $b = 0.22$ when starting at the initial state $(195.7, 0.03, 3.92)$ is a Andronov-Hopf bifurcation since the steady state turns into a limit cycle. It's normal form is the one given in task 3 which we copy here for convenience:

$$\dot{x}_1 = \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2)$$

The basic reproduction number is defined as:

$$\mathbf{R_0} = \frac{\beta}{\delta + \nu + \mu_1}$$

The β is the average number of contacts per unit of time with infectious people, δ is the the natural death rate per capita, ν is the disease- induced death rate per capita and μ_1 is, as described in the paper, the maximum per capita recovery rate due to the sufficient health care resource and few infectious individuals as well as the inherent property of a specific disease. The equation is linear in β hence $\mathbf{R_0}$ increases with the same rate as β

and decreases with the same as β. Therefore, when β increases, people recover faster so there are less infectious people at any time, when β decreases there will be more infectious persons.

An attracting node is a state that is the limit point of all initial states in a sufficiently close neighborhood of it. Since it is an equilibrium point, once this state is reached, the system will not be able to reach any other state ever. Hence, starting sufficiently close to this state, the number of infectious people will go to 0 as time increases and if it reaches 0 there will be no more infection and the disease is extinguished.

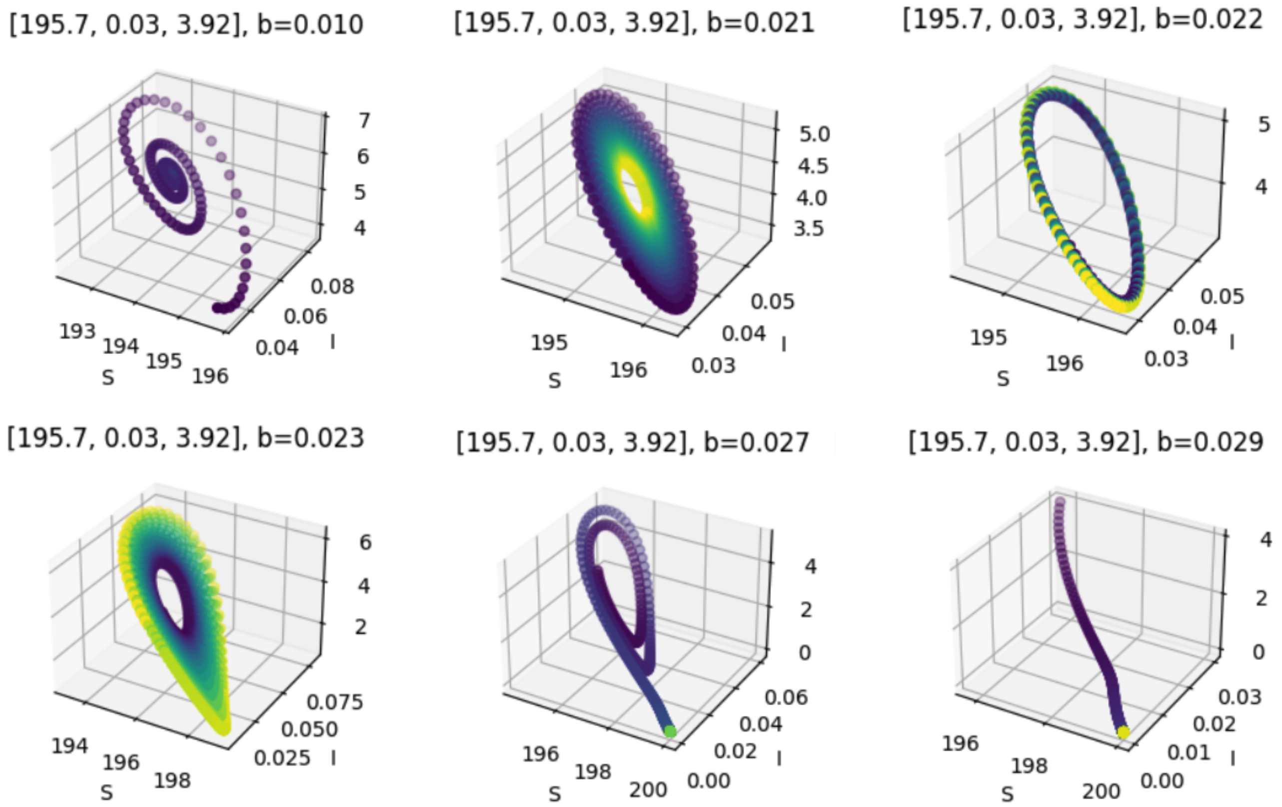

Figure 29: Evolution of the system with initial state $(195.7, 0.03, 3.92)$ for various parameters b.

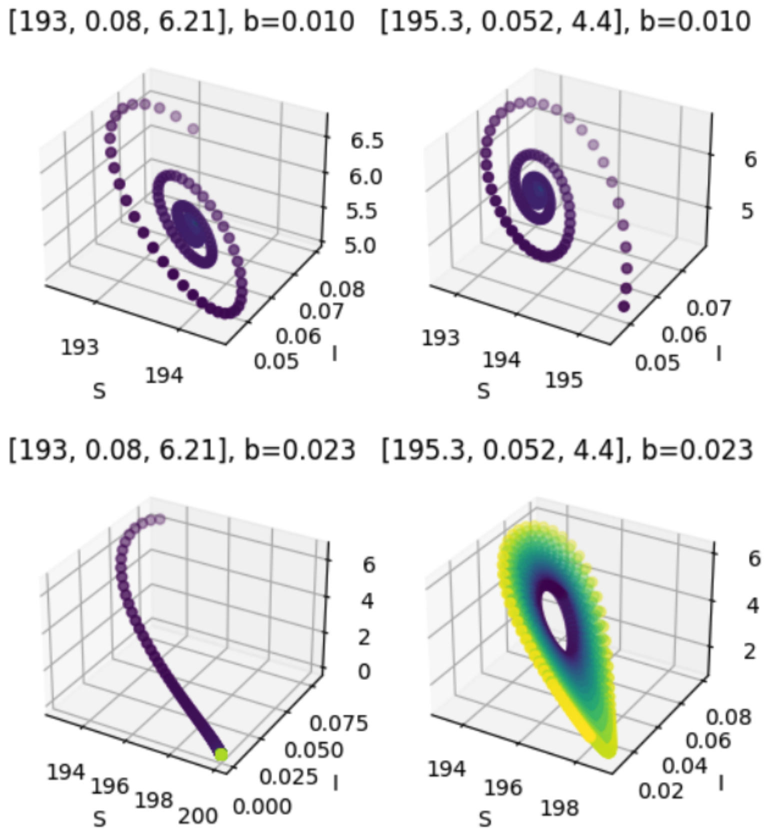

Figure 30: Evolution of the systems starting in two different initial states for two values of the parameter b.

References

[1] Yuri A. Kuznetsov. *Elements of applied bifurcation theory.* Number 112 in Applied Mathematical Sciences. Springer, Berlin, 2 edition, 1998.