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STABILITY OF LIMIT CYCLE SOLUTIONS OF REACTION-DIFFUSION EQUATIONS*

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Abstract. The kinetic equations of the two component reaction diffusion system

$$u_t = F(u, v) + (1+\alpha)\nabla^2 u + \delta_2 \nabla^2 v,$$
$$v_t = G(u, v) + \delta_1 \nabla^2 u + (1-\alpha)\nabla^2 v$$

are assumed to have a limit cycle solution, stable as a solution of the kinetic system, which provides a spatially homogeneous, oscillatory solution of the full system. It is well known that the limit cycle solution is then linearly stable to perturbations with wave number k^2 for sufficiently large k^2. Using perturbation methods, an exact condition is obtained for the limit cycle to be stable or unstable to small wave numbers k^2. Explicitly solvable examples of systems (with diagonal diffusion matrix) are constructed, whose limit cycles change stability with respect to small wave numbers k^2 as the diffusion parameter α passes a critical value. The results of this linearized analysis are compared with numerical solutions of the full system. The exact conditions for linear stability are shown equivalent to a multi-scaling approach based on a large spatial scale (corresponding to small wave number k^2). Perturbations with wave number k^2 for intermediate and large k^2 are also discussed.

1. Introduction. Consider the general reaction-diffusion system in two dependent variables, written in normalized form as

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} F(u,v) \\ G(u,v) \end{bmatrix} + \begin{bmatrix} 1+\alpha & \delta_2 \\ \delta_1 & 1-\alpha \end{bmatrix} \begin{bmatrix} \nabla^2 u \\ \nabla^2 v \end{bmatrix}, \qquad |\alpha| \leqq +1, \quad 0 \leqq \alpha^2 + \delta_1\delta_2 \leqq +1. \tag{1.1}$$

Any two-component system with constant diffusion matrix possessing real, nonnegative eigenvalues and nonnegative diagonal entries can be placed in the form of (1.1) by rescaling the space variables; the condition on $\alpha^2 + \delta_1\delta_2$ is equivalent to real, nonnegative eigenvalues for the matrix. The reason for this choice of diagonal coefficients is given below. The kinetic equations are assumed to possess an (exponentially) stable limit cycle $U(t)$, $V(t)$ with period T; the point $U(0)$, $V(0)$ is also assumed given so that $U(t)$, $V(t)$ has a unique meaning.

The limit cycle is a spatially homogeneous, oscillatory solution of the reaction-diffusion system (1.1). The linear stability problem for this solution is formulated by substituting $u = U(t) + \varepsilon\hat{u}$, $v = V(t) + \varepsilon\hat{v}$ into (1.1) to obtain the linear variational equation

$$\begin{bmatrix} \hat{u}_t \\ \hat{v}_t \end{bmatrix} = \begin{bmatrix} F_u(U(t), V(t)) & F_v(U(t), V(t)) \\ G_u(U(t), V(t)) & G_v(U(t), V(t)) \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} + \begin{bmatrix} (1+\alpha) & \delta_2 \\ \delta_1 & (1-\alpha) \end{bmatrix} \begin{bmatrix} \nabla^2 \hat{u} \\ \nabla^2 \hat{v} \end{bmatrix}. \tag{1.2}$$

Separating variables (or Fourier transforming) by $\hat{u} = p(t)\exp(+i\mathbf{k}\cdot\mathbf{x})$, $\hat{v} = q(t)\exp(+i\mathbf{k}\cdot\mathbf{x})$ yields the Floquet system (with the obvious definitions of $F_i(t)$, $G_i(t)$ as T-periodic functions)

$$\begin{bmatrix} p' \\ q' \end{bmatrix} = \begin{bmatrix} F_1(t) - k^2(1+\alpha) & F_2(t) - k^2\delta_2 \\ G_1(t) - k^2\delta_1 & G_2(t) - k^2(1-\alpha) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}, \qquad |\alpha| \leqq 1, \quad 0 \leqq \alpha^2 + \delta_1\delta_2 \leqq 1. \tag{1.3}$$

The limit cycle is linearly unstable for wave number k^2 as a solution of the original system (1.1) iff (1.3) has an exponentially growing solution for that value k^2.

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The purpose of this paper is to study the linear stability of the limit cycle solutions to (1.1). Kopell and Howard [12] have mentioned the scarcity of results in this area, which seems a natural next step after stability studies of the spatially homogeneous, stationary solutions corresponding to critical points of the kinetic equations.

The stability of these stationary solutions was first considered by Turing [20], who was the first to observe that a stable critical point of the kinetic system could be unstable when considered as a solution of the reaction-diffusion system. Specifically, he gave examples with the stationary solution linearly stable to perturbations with small and large wave numbers k^2, but unstable to intermediate k^2. Othmer and Scriven [17] have given an extensive analysis of the stability of stationary solutions (considering all types of critical points for the kinetic equations) for 2-component reaction-diffusion systems, with results for n-component systems. (A remarkable biological example of stationary state instabilities in a finite spatial domain is the explanation of the development of compartments of specialized cells—later to become wings, legs, etc.—in the *Drosophila* embryo, see Kauffman, Shymko, and Trabert [11].)

The stability of the limit cycle solutions to (1.1) has been considered by Othmer [16] for a finite spatial domain with Neumann boundary conditions. Interpreted in our context of a spatially periodic perturbation, he has shown the limit cycle to be stable to all large wave numbers k^2 and has given a sufficient condition for the linear stability of the limit cycle to perturbations for *all* wave numbers k^2 (see (2.1)). Conway, Hoff, and Smoller [5] have shown under a basic assumption of a positively-invariant region for the solutions of the reaction-diffusion system, that solutions on finite spatial regions with Neumann boundary conditions decay to spatially homogeneous solutions of the kinetic equations if the regions are sufficiently small. (Their work deals with the actual solutions of the system and is not a linearized simplification.) In the context of spatially periodic perturbations, their work also shows the limit cycle solution of (1.1) is stable to perturbations for all large wave numbers k^2 ($1/k^2$ corresponds to the size of the region). Cohen [2] gives a singular perturbation approach for a class of reaction-diffusion equations of the form (1.1) arising in chemical reactor theory. These equations are on a finite spatial domain with particular boundary conditions and with diffusion coefficients $O(1/\varepsilon)$, ε small. His calculations show that, in a time interval of $O(\varepsilon)$, solutions decay to the spatially homogeneous limit cycle. Roughly speaking, this result corresponds to saying the limit cycle is stable to perturbations with large wave number $k^2 \sim 1/\varepsilon$.

The linear stability problem for spatially homogeneous, stationary states is fully solvable for a *given* system because the linearized equations have constant coefficients, i.e. the system corresponding to (1.3) has constant terms in place of the $F_i(t)$, $G_i(t)$, and the full solution can be written in terms of the coefficient matrix. (The *general* classification of behavior, however, is still quite complicated—see Othmer and Scriven [17].) For spatially homogeneous, oscillatory states the linear stability analysis yields the Floquet system (1.3) and no such general solution is possible. Our approach has been to replace the exact calculations of the constant coefficient case with perturbation expansions. An exact characterization of the stability of the limit cycle solution of (1.1) for small wave numbers k^2 results (§ 3), complementing the known results above for large k^2. The situation for small k^2 is studied further in §§ 4, 5, and 8; other ranges of k^2 are considered in §§ 6 and 7. All perturbation approaches, except in § 4, are based on the Floquet theorem (Lefschitz [14]).

Specifically, we begin in § 2 with some basic lemmas and the reformulation of (1.3) as a Floquet system (2.4) depending on 3 parameters $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)$ with Floquet multipliers whose product is *independent* of $\boldsymbol{\beta}$. This fact permits a partial description of the behavior of the multipliers similar to that for Hill's equation.

Section 3 considers a perturbation expansion for small $|\boldsymbol{\beta}|$ (small k^2) and obtains a complete characterization of the limit cycle stability with respect to small k^2 perturbations—see Theorem 1.

Neu [15] has used a multi-scaling approach to obtain solutions to (1.1) of the form $U(t+\phi+\cdots)$, $V(t+\phi+\cdots)$ when the diffusion matrix is the identity. Here ϕ evolves by a form of Burgers' equation, and the spatial scaling corresponds to small wave number k^2. In § 4 we generalize this approach to arbitrary diffusion matrices and show that the condition for ϕ to remain bounded as $t\to+\infty$ is precisely the stability condition of Theorem 1.

Section 5 constructs explicitly solvable examples of systems (with diagonal diffusion coefficients) with limit cycles stable as a solution of the kinetic equations but unstable to perturbations for all small wave numbers k^2.

Section 6 considers intermediate values of $|\boldsymbol{\beta}|$ (that is, intermediate k^2). No general approach seems feasible here. Instead, the kinetic equations of (1.1) are assumed to contain a small parameter γ, which then occurs in the reformulated Floquet system (2.4). Then $\boldsymbol{\beta}$ is assumed fixed and the expansion is done in terms of γ.

Section 7 shows that the behavior of the reformulated system (2.4) for large $|\boldsymbol{\beta}|$ (large k^2) can be studied by standard methods for differential equations with a large parameter. The result, of course, is that the limit cycle is stable for all large wave numbers k^2 if the diffusion matrix is nonsingular. If the diffusion matrix is singular, however, instability may occur—a characterization is given in Theorem 2.

Section 8 gives numerical results, based on Lees' method for parabolic equations (Lees [13]), pertaining to the examples of unstable limit cycles constructed in § 5.

Section 9 mentions further problems of interest.

2. Formulation of the stability problem. Beginning with the formulation of the stability problem in terms of (1.3)—the limit cycle solution is stable as a solution of (1.1) iff (1.3) has no exponentially growing solutions—there are several observations to be made.

First, (1.3) is stable (that is, has no unbounded solution as $t\to+\infty$) if k^2 is sufficiently large and the diffusion matrix is nonsingular. Specifically, set

$$\sigma(t)=\text{spectral radius of}\begin{bmatrix} F_1(t) & F_2(t)\\ G_1(t) & G_2(t)\end{bmatrix}\quad\text{and}\quad \hat{\sigma}=\max_{[0,\,T]}\sigma(t),$$

$$\omega=\text{smaller eigenvalue of}\begin{bmatrix} 1+\alpha & \delta_2\\ \delta_1 & 1-\alpha\end{bmatrix}.$$

Then from (1.3) follows

$$\frac{1}{2}\frac{d}{dt}(p^2+q^2)\leqq(\hat{\sigma}-\omega k^2)(p^2+q^2).$$

This point has already been made in Othmer [16] and Conway, Hoff and Smoller [5], where it is shown that solutions of a reaction-diffusion system must die away to a spatially homogeneous solution satisfying the kinetic equations when the spatial domain (measured here by $1/k^2$) is sufficiently small.

Second, there is the sufficient condition for stability given by Othmer [16]. His result is for general n-component systems; interpreted for the case of (1.1) with $\delta_1=\delta_2=0$, it says the limit cycle is stable as a solution of (1.1) if

$$\frac{M-1}{M}<\frac{1+\alpha}{1-\alpha}<\frac{M}{M-1},\qquad M=\max_{[0,\,T]}\{\|\theta^{-1}(t)\|\cdot\|\theta(t)\|\}.$$

Here $\theta(t)$ is any fundamental matrix for (1.3) with $k^2=0$ (note Lemma 2 below); $\|\theta\|$ means Euclidean norm. The result basically says that if the diffusion matrix is sufficiently close to the identity, the limit cycle is stable. (Kopell and Howard [12] have pointed out the linear stability of the limit cycle solution when the diffusion matrix is the identity.)

Third, if $k^2=0$ in (1.3), we can assume the solutions to be known exactly. Since the limit cycle is assumed stable, there is no essential difficulty in calculating it numerically (at worst, a relaxation oscillation may show "stiff" behavior as it develops discontinuities). An immediate solution of (1.3) (for $k^2=0$) is $U'(t)$, $V'(t)$, easily obtained by substituting the limit cycle in the original kinetic equations. With one solution known, the calculation of the second reduces to solving a first order system. This procedure is summarized in the following two lemmas; the first gives the relevant results for first-order systems and the second describes the full solution of (1.3) with $k^2=0$. Both lemmas will be used repeatedly over the next 3 sections.

LEMMA 1. *Let $f(t)$ be C^1 with period T and mean $\bar{f}=(1/T)\int_0^T f(s)\,ds$; define $F(t)=\int_0^t \exp(-\mu s)f(s)\,ds$. Then:*

(a) *if $\mu=0$, then $F(t)=\bar{f}t+g(t)$, where g has period T;*

(b) *if $\mu\neq 0$, then $F(t)=C+\exp(-\mu t)g(t)$, where g has period T, $\bar{g}=-(1/\mu)\bar{f}$, and*

$$C=\frac{1}{1-\exp(-\mu T)}\int_0^T \exp(-\mu s)f(s)\,ds; \tag{2.1}$$

(c) *the equation $x'-\mu x=f(t)$ has a unique T-periodic solution given by $x=g(t)$ defined in* (b).

Proof. (a) is trivial; (b) follows by integrating the (absolutely convergent) Fourier series for $f(t)$ termwise, and the expression for C comes from manipulating the periodicity property; (c) is a direct calculation.

LEMMA 2. *The system* (1.3) *with $k^2=0$ has a fundamental matrix*

$$\begin{bmatrix} U'(t) & \exp(-\mu t)\hat{U}(t) \\ V'(t) & \exp(-\mu t)\hat{V}(t) \end{bmatrix}, \tag{2.2a}$$

where U', V', $\hat{U}$, $\hat{V}$ are real T-periodic functions; *U', V' is the derivative of the limit cycle solution*;

$$-\mu=\frac{1}{T}\int_0^T (F_1(s)+G_2(s))\,ds \quad \text{and} \tag{2.2b}$$

$$\begin{bmatrix} \hat{U}(t) \\ \hat{V}(t) \end{bmatrix}=A(t)\begin{bmatrix} -V'(t) \\ U'(t) \end{bmatrix}+B(t)\begin{bmatrix} U'(t) \\ V'(t) \end{bmatrix}, \tag{2.2c}$$

with $A(t)$ a T-periodic function determined (*up to multiplication by a constant*) *by*

$$\begin{aligned} A(t)\exp(-\mu t)[(U'(t))^2+(V'(t))^2]&=A(0)[(U'(0))^2+(V'(0))^2] \\ &\quad\cdot\exp\left(+\int_0^t F_1(s)+G_2(s)\,ds\right), \end{aligned} \tag{2.2d}$$

and $B(t)$ is the unique T-periodic solution to

$$B'-\mu B=\frac{A}{(U')^2+(V')^2}[(G_1+F_2)[(U')^2-(V')^2]+2U'V'(G_2-F_1)]. \tag{2.2e}$$

Proof. It is immediate that U', V' is one solution of (1.3) with $k^2=0$. By Floquet's theorem (Coddington and Levinson [1, Chap. 3]) and the assumed exponential stability of the limit cycle, the second solution can be written as an exponential multiplying a periodic function. Abel's identity gives the determinant of the Wronskian in terms of the trace of the coefficient matrix; this identity gives (b). The idea of splitting the second solution into the form (c) is from Halanay [10, Chap. 3]. Abel's identity gives (d); substitution of (c) into (1.3) with $k^2=0$ gives (e).

It is important to note that the fundamental matrix (2.2)(a) can be calculated numerically with little trouble, since many of the quantities below will be definite integrals involving these functions. In practice, the most awkward calculation seems to be the determination of $g(t)$ in Lemma 1(b):

$$g(t)=\exp(\mu t)\left[-C+\int_0^t \exp(-\mu s)f(s)\,ds\right]. \tag{2.3a}$$

(Here $f(t)$ will be a T-periodic function known, usually, only in tabulated form from other numerical work.) It is simple to calculate C by the expression in Lemma 1(b), but (2.3) as it stands involves multiplying exponentially large and exponentially small quantities as t varies from 0 to T; this can lead to serious error even for moderate values of μT (say, $\mu T\sim 20$). Integration backwards (possible by the periodicity of g) overcomes this difficulty: set $g_n=g(-nh)$ for some step size h and use

$$\begin{aligned} &g_0=C,\\ &g_n=\exp(-\mu h)g_{n-1}-\int_{(n-1)h}^{nh}\exp(\mu(s-nh))f(-s)\,ds. \end{aligned} \tag{2.3b}$$

The stability problem will now be restated in the form used in the remainder of the paper. Introducing new variables into (1.3) by $x, y=\exp(+k^2t)p, \exp(+k^2t)q$, and setting $\boldsymbol{\beta}=(\beta_0,\beta_1,\beta_2)=(\alpha k^2, \delta_1k^2, \delta_2k^2)$ gives the equation

$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}F_1(t)-\beta_0 & F_2(t)-\beta_2\\ G_1(t)-\beta_1 & G_2(t)+\beta_0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}. \tag{2.4}$$

In analyzing (2.4), $\boldsymbol{\beta}$ will be considered as an arbitrary vector; when the results are applied to (1.3), constraints on $\boldsymbol{\beta}$ similar to those on α, δ_1, δ_2 must be imposed. Let B be a Floquet exponent for (2.4), so some solution grows like $\exp(Bt)$. Then the corresponding solution of (1.3) grows like $\exp((-k^2+B)t)$. The limit cycle is linearly unstable as a solution of (1.1) iff (2.4) has a Floquet exponent B with $\mathrm{Re}(B)>k^2$ for $\boldsymbol{\beta}=(\alpha k^2, \delta_1k^2, \delta_2k^2)$, $|\alpha|\leqq 1$.

In (2.4) the trace of the coefficient matrix, and therefore the product of the Floquet multipliers, is *independent of* $\boldsymbol{\beta}$. This result is the reason for choosing the normalized form of the diffusion matrix in (1.1). A classification of the behavior of the Floquet exponents can now be given similar to the classification for Hill's equation (see Eastham [7, Chap. 1]).

Fix a class of fundamental matrices for (2.4), say

$$\begin{bmatrix}x_1(t) & x_2(t)\\ y_1(t) & y_2(t)\end{bmatrix} \quad\text{with}\quad \begin{bmatrix}x_1(0) & x_2(0)\\ y_1(0) & y_2(0)\end{bmatrix}\doteq\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix};$$

the solutions, of course, depend implicitly on $\boldsymbol{\beta}$. The Floquet multipliers are the roots ρ_1, ρ_2 of

$$\rho^2-[X_1(T)+y_2(T)]\rho+[X_1(T)y_2(T)-X_2(T)y_1(T)]=0, \tag{2.5a}$$

that is,

$$\rho^2 - D(\beta_0, \beta_1, \beta_2)\rho + \exp(-\mu T) = 0. \tag{2.5b}$$

The value of the determinant follows from the value at $k^2 = 0$ (Lemma 2), Abel's identity, and the trace's independence of $\boldsymbol{\beta}$. Using the terminology of Hill's equation, $D(\beta_0, \beta_1, \beta_2)$ is the *discriminant* and it alone determines the roots. The roots $\rho_1, \rho_2 = \exp(-\mu_1 T), \exp(-\mu_2 T)$ yield solutions growing like $\exp(-\mu_1 t), \exp(-\mu_2 t)$ (unless $\rho_1 = \rho_2$, in which case the growth is $\exp(-\mu_1 t), t\exp(-\mu_1 t)$); here $\mu_1 + \mu_2 \equiv \mu \pmod{2\pi i}$.

The possible combinations of roots can be classified as:

(2.6a) $|D| > 2\exp(-(\mu/2)T)$. The roots are real and both have the same sign as D.

(2.6b) $|D| = 2\exp(-(\mu/2)T)$. Both roots $= \exp(-(\mu/2)T)$.

(2.6c) $|D| < 2\exp(-(\mu/2)T)$. The roots are complex conjugate and have modulus $\exp(-(\mu/2)T)$.

For $\boldsymbol{\beta} = 0$, the roots are $\rho_1, \rho_2 = 1, \exp(-\mu T)$, so case (2.6a) with $D > 0$ occurs for small $\boldsymbol{\beta}$ (analyzed in § 3). The stability condition, as stated for (2.4), says that a Floquet multiplier ρ with $|\rho| > 1$ must exist for the limit cycle to be unstable—this can only happen in case (2.6a). As $|\boldsymbol{\beta}|$ becomes large (corresponding to large k^2), case (2.6a) occurs again (analyzed in § 7). Cases (2.6b) and (2.6c) can only occur for intermediate $|\boldsymbol{\beta}|$ (they do occur in the analysis of § 6). Summarizing:

LEMMA 3. (a) *The product of the Floquet multipliers of* (2.4) *is* $\exp(-\mu T)$, *independent of* $\boldsymbol{\beta}$.

(b) *The limit cycle is unstable as a solution of* (1.1) *iff there exists* $k^2 > 0$ *such that* $\boldsymbol{\beta} = (\alpha k^2, \delta_1 k^2, \delta_2 k^2)$ *in* (2.4) *with* $|\beta_0| \leqq k^2$, $0 \leqq \beta_0^2 + \beta_1\beta_2 \leqq k^4$; *the Floquet multipliers* ρ_1, ρ_2 *are real, and*

$$\log(\max(|\rho_1|, |\rho_2|)) > k^2 T.$$

3. Behavior for small $|\boldsymbol{\beta}|$. Since the solution of (2.4) is known for $|\boldsymbol{\beta}| = 0$, it is natural to try a perturbation expansion for the solutions for small $|\boldsymbol{\beta}|$ (which corresponds to small k^2 in (1.3)). Floquet's theorem suggests an expansion of the form

$$\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix} = \exp(\mathbf{A}\cdot\boldsymbol{\beta} + O(|\boldsymbol{\beta}|^2)t)\Bigg(\begin{bmatrix} U'(t) \\ V'(t) \end{bmatrix} + \beta_0\begin{bmatrix} x_0(t) \\ y_1(t) \end{bmatrix} + \beta_1\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} \\ + \beta_2\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} + O(|\boldsymbol{\beta}|^2)\Bigg),\end{aligned} \tag{3.1}$$

where $\mathbf{A} = (A_0, A_1, A_2)$ and the vector terms are required to be T-periodic functions. By Lemma 3, stability is determined by the larger Floquet multiplier and as $|\boldsymbol{\beta}| \to 0$, (3.1) reduces to the solution U', V' corresponding to the larger Floquet multiplier 1, that is, (3.1) represents an expansion for the solution corresponding to the larger Floquet multiplier.

Substitution of (3.1) into (2.4) yields the $O(|\boldsymbol{\beta}|)$-term

$$\begin{aligned}\sum_{i=0}^{2} \beta_i \begin{bmatrix} x_i' \\ y_i' \end{bmatrix} - \begin{bmatrix} F_1(t) & F_2(t) \\ G_1(t) & G_2(t) \end{bmatrix}\begin{bmatrix} x_i \\ y_i \end{bmatrix} \\ = \beta_0\begin{bmatrix} -U' \\ V' \end{bmatrix} + \beta_1\begin{bmatrix} 0 \\ -U' \end{bmatrix} + \beta_2\begin{bmatrix} -V' \\ 0 \end{bmatrix} - (A_0\beta_0 + A_1\beta_1 + A_2\beta_2)\begin{bmatrix} U' \\ V' \end{bmatrix}.\end{aligned} \tag{3.2}$$

We solve the $O(\beta_0)$-term in detail and give the results for the $O(\beta_1)$-, $O(\beta_2)$-terms.

The coefficient of β_0 in (3.2) gives an ODE for x_0, y_0. Using the fundamental matrix from Lemma 2 gives the general solution $(W(s)=U'(s)\hat{V}(s)-V'(s)\hat{V}(s))$;

(3.3)
$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} U' & \exp(-\mu t)\hat{U} \\ V' & \exp(-\mu t)\hat{V} \end{bmatrix} \cdot \left(\begin{bmatrix} C_0 \\ D_0 \end{bmatrix} + \int_0^t \frac{1}{W(s)} \begin{bmatrix} \hat{V}(s) & -\hat{U}(s) \\ -e^{\mu s}V'(s) & e^{\mu s}U'(s) \end{bmatrix} \begin{bmatrix} -U'(s)(1+A_0) \\ V'(s)(1-A_0) \end{bmatrix} ds \right).$$

The first component of the integral will grow like $O(t)$ unless a special value for A_0 is chosen; referring to Lemma 1(a), this value (with corresponding values for A_1, A_2 from the β_1-, β_2-terms) is:

(3.4a)
$$A_0 = -\frac{1}{T}\int_0^T \frac{U'(s)\hat{V}(s)+V'(s)\hat{U}(s)}{W(s)}\,ds,$$

(3.4b)
$$A_1 = +\frac{1}{T}\int_0^T \frac{U'(s)\hat{U}(s)}{W(s)}\,ds,$$

(3.4c)
$$A_2 = -\frac{1}{T}\int_0^T \frac{V'(s)\hat{V}(s)}{W(s)}\,ds.$$

We can then set $C_0=0$ with no loss of generality since its inclusion merely amounts to rescaling the first term U', V'. The constant D_0 is determined by the requirement that exponentially decaying terms must be eliminated, retaining only periodic ones; D_0 is then found using Lemma 1(b). (Constants C_1, C_2, D_1, D_2 are determined in the same way for the β_1-, β_2-terms.)

Consequently, the larger Floquet multiplier has the Floquet exponent $B = \mathbf{A}\cdot\boldsymbol{\beta}+O(|\boldsymbol{\beta}|^2)=(A_0\alpha+A_1\delta_1+A_2\delta_2)k^2+O(k^4)$. As noted in § 2, the limit cycle is unstable iff $B>k^2$. This gives

THEOREM 1. (a) *The limit cycle is linearly stable as a solution of* (1.1) *for all sufficiently small wave numbers* k^2 *if* $A_0\alpha+A_1\delta_1+A_2\delta_2<1$.

(b) *The limit cycle is linearly unstable as a solution of* (1.1) *for all sufficiently small wave numbers* k^2 *if* $A_0\alpha+A_1\delta_1+A_2\delta_2>1$.

In particular, if $\delta_1=\delta_2=0$, the limit cycle is unstable for all small k^2 if $A_0\alpha>1$. It may seem to violate continuity for the limit cycle to be stable at $k^2=0$ and unstable for all small $k^2>0$. However, an analogous case occurs for critical points: if the (kinetically) stable critical point has eigenvalues $0, -\mu$ with respect to the kinetic equations, then the constant solution corresponding to the 0-eigenvalue can be made unstable by the addition of diffusion terms. The (kinetically) stable limit cycle has Floquet exponents $0, -\mu$ with respect to the kinetic equations and the periodic solution corresponding to the 0-Floquet exponent can be made unstable by the diffusion terms.

As examples, we consider two classes of equations for which the limit cycle and related functions can be calculated explicitly.

Let (1.1) be a λ–ω system with full diffusion matrix $(R^2=u^2+v^2)$,

(3.5)
$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} \lambda(R) & -\omega(R) \\ \omega(R) & \lambda(R) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} (1+\alpha) & \delta_2 \\ \delta_1 & (1-\alpha) \end{bmatrix}\begin{bmatrix} \nabla^2 u \\ \nabla^2 v \end{bmatrix},$$
$$|\alpha|\leqq 1, \quad 0\leqq \alpha^2+\delta_1\delta_2\leqq 1.$$

The kinetic equations have the limit cycle solution $U(t)=R_0\cos(\omega_0 t)$, $V(t)=$

$R_0 \sin(\omega_0 t)$, with $\lambda(R_0)=0$, $\omega_0=\omega(R_0)$. The analogue of (1.3) is

$$(3.6)\quad \begin{bmatrix} p' \\ q' \end{bmatrix} = \begin{bmatrix} R_0S_0\cos(\omega_0 t)\cos(\omega_0 t+\sigma_0)-k^2(1+\alpha) & R_0S_0\sin(\omega_0 t)\cos(\omega_0 t+\sigma_0)-\omega_0-k^2\delta_2 \\ R_0S_0\cos(\omega_0 t)\sin(\omega_0 t+\sigma_0)+\omega_0-k^2\delta_1 & R_0S_0\sin(\omega_0 t)\sin(\omega_0 t+\sigma_0)-k^2(1-\alpha) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix},$$

where $S_0\cos\sigma_0=\lambda'(R_0)$, $S_0\sin\sigma_0=\omega'(R_0)$, $S_0>0$, and $-\mu=R_0\lambda'(R_0)$ with the assumption $-\mu<0$ to insure stability of the limit cycle. The fundamental matrix, found by Lemma 2, is

$$(3.7)\quad \begin{bmatrix} U'(t) & \exp(-\mu t)\hat{U}(t) \\ V'(t) & \exp(-\mu t)\hat{V}(t) \end{bmatrix} = \begin{bmatrix} -R_0\omega_0\sin(\omega_0 t) & -\exp(-\mu t)\dfrac{\cos(\omega_0 t+\sigma_0)}{R_0\omega_0\cos\sigma_0} \\ R_0\omega_0\cos(\omega_0 t) & -\exp(-\mu t)\dfrac{\sin(\omega_0 t+\sigma_0)}{R_0\omega_0\cos\sigma_0} \end{bmatrix}.$$

Substitution into (3.4) gives

$$(3.8a)\qquad A_0=0,$$

$$(3.8b)\qquad A_1=-\tfrac{1}{2}\tan\sigma_0,$$

$$(3.8c)\qquad A_2=\tfrac{1}{2}\tan\sigma_0.$$

In Theorem 1, therefore, $A_0\alpha+A_1\delta_1+A_2\delta_2=\frac{1}{2}(\delta_2-\delta_1)\tan\sigma_0$. In particular, for $\delta_1=\delta_2=0$, the limit cycle of *any* $\lambda-\omega$ system (3.5) is stable to perturbations for small wave numbers k^2 since $|A_0|=0<1$.

Next, consider the system

$$(3.9)\quad \begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} a(1-u^2)u-v \\ u+a(1-bu^2)v \end{bmatrix} + \begin{bmatrix} (1+\alpha)\nabla^2 u \\ (1-\alpha)\nabla^2 v \end{bmatrix}, \qquad |\alpha|\leqq 1, \quad a,b>0.$$

This system occurs as a model in the study of chemical reactors (Cohen [2] and Cohen, Hoppensteadt and Miura [4]—(3.9) is a rescaled form of equations in these papers). We have found an exact solution in the case $b=+1$, used here as a second example.

The substitution $u=R\cos\psi$, $v=R\sin\psi$ changes the kinetic equations of (3.9) to

$$(3.10)\quad \begin{aligned} R' &= aR(1-(R\cos\psi)^2)+a(1-b)R^3(\cos\psi\sin\psi)^2, \\ \psi' &= 1+a(1-b)R^2(\cos\psi)^3\sin\psi. \end{aligned}$$

For $b=+1$, these equations reduce to a Bernoulli equation for R with solution

$$(3.11)\quad \frac{1}{R^2}=\frac{1}{2}+\frac{1}{2}\frac{a^2\cos 2\psi+a\sin 2\psi}{a^2+1}+C_2\exp(-2a\psi), \qquad \psi=t+C_1.$$

Setting $C_1, C_2=0$ gives the limit cycle $U(t)$, $V(t)=R_0(t)\cos t$, $R_0(t)\sin t$, and the analogue of (1.3) is

$$(3.12)\quad \begin{bmatrix} p' \\ q' \end{bmatrix} = \begin{bmatrix} a(1-3U^2)-k^2(1+\alpha) & -1 \\ 1-2aUV & a(1-U^2)-k^2(1-\alpha) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

The fundamental matrix for $k^2=0$ is

$$(3.13)\quad \begin{bmatrix} U'(t) & \exp(-\mu t)U(t) \\ V'(t) & \exp(-\mu t)V(t) \end{bmatrix} = \begin{bmatrix} R_0'\cos t-R_0\sin t & \exp(-2at)R_0^3\cos t \\ R_0'\sin t+R_0\cos t & \exp(-2at)R_0^3\sin t \end{bmatrix}.$$

(Incidentally, since the full solution is known, the easiest way to obtain the second solution is to differentiate (3.11) with respect to C_2 at $C_2=0$, as suggested in Lefschitz [14, Chap. 3].)

The Wronksian of (3.13) is $-\exp(-2at)R_0^4$, and (3.4a) becomes

$$A_0=\frac{1}{2\pi}\int_0^{2\pi}\frac{a^2+(a^2+1)\cos 2t}{a^2+1+a^2\cos 2t+a\sin 2t}\,dt. \tag{3.14}$$

The integral can be evaluated by the substitution $z=\exp(+it)$, giving

$$A_0=\frac{1}{2\pi}\int_{|z|=1}\frac{(a^2+1)z^4+2a^2z^2+(a^2+1)}{(a^2-ai)z^4+2(a^2+1)z^2(a^2+ia)}\frac{dx}{iz}.$$

The denominator has roots at

$$z^2=\frac{1}{a^2-ia}[-(a^2+1)\pm(a^2+1)^{1/2}],$$

with one pair inside the unit circle and one pair outside for all $0<a<+\infty$. Evaluating the residues at $z=0$ and at the pair of roots inside the unit circle gives

$$A_0=\frac{(a^2+1)^{1/2}-1}{(a^2+1)^{1/2}},\qquad 0<a<+\infty. \tag{3.15}$$

Since $|A_0|<1$ for all $a>0$, Theorem 1 shows the limit cycles of (3.9) (with $b=1$) to be stable to perturbations of small wave numbers k^2 for all α, $|\alpha|\leqq 1$.

These examples show only stable behavior when the diffusion matrix is diagonal. Explicitly solvable examples of systems with diagonal diffusion matrices showing unstable behavior will be constructed in § 5.

4. A nonlinear approach. In this section we consider a nonlinear approach to the stability problem by assuming the limit cycle solution of (1.1) has a perturbation varying slowly in space. It then makes sense to introduce a new space variable $\boldsymbol{\xi}=\varepsilon^{1/2}\mathbf{x}$ and a formal multi-scaling expansion.

$$\boldsymbol{\xi}=\varepsilon^{1/2}\mathbf{x},\qquad \tau=\varepsilon t, \tag{4.1a}$$

$$\theta=t+\phi(\varepsilon,\tau,\boldsymbol{\xi})=t+\phi_0(\tau,\boldsymbol{\xi})+\varepsilon\phi_1(\tau,\boldsymbol{\xi})+\cdots, \tag{4.1b}$$

$$\begin{aligned} u&=u(\varepsilon,\theta,\tau,\boldsymbol{\xi})=u_0(\theta,\tau,\boldsymbol{\xi})+\varepsilon u_1(\theta,\tau,\boldsymbol{\xi})+\cdots,\\ v&=v(\varepsilon,\theta,\tau,\boldsymbol{\xi})=v_0(\theta,\tau,\boldsymbol{\xi})+\varepsilon v_1(\theta,\tau,\boldsymbol{\xi})+\cdots. \end{aligned} \tag{4.1c}$$

In the new variables $\theta,\tau,\boldsymbol{\xi}$, (1.1) becomes (setting the cross-terms $\delta_1=\delta_2=0$ for simplicity—the full result is given at the end of this section)

$$\begin{aligned} (1+\varepsilon\phi_\tau)u_\theta+\varepsilon u_\tau&=F(u,v)+\varepsilon(1+\alpha)[|\hat\nabla\phi|^2\theta_{\theta\theta}+2\hat\nabla\phi\cdot\hat\nabla u_\theta+\hat\nabla u_\theta+\hat\nabla^2u+\hat\nabla^2\phi u_\theta]\\ (1+\varepsilon\phi_\tau)v_\theta+\varepsilon v_\tau&=G(u,v)+\varepsilon(1-\alpha)[|\hat\nabla\phi|^2v_{\theta\theta}+2\hat\nabla\phi\cdot\hat\nabla u_\theta+\hat\nabla^2v+\hat\nabla^2\phi v_\theta]. \end{aligned} \tag{4.2}$$

Here $\hat\nabla$ means differentiation with respect to $\boldsymbol{\xi}$. We proceed by equating coefficients of ε, taking the first term to be the limit cycle.

This multi-scaling approach has been used by Neu [15] (with $\alpha=0$), who obtained (4.7) below and studied its traveling wave solutions in connection with waves propagating between two oscillating chemical reactions. Neu's motivation for his multi-scaling approach was a geometric perturbation scheme for studying the solutions of ODE's near a limit cycle (Halanay [10, Chap. 3]). Howard and Kopell [22] have used multi-scaling methods in the general setting of periodic traveling waves and, in particular, have derived equation (4.7a) below to describe the propagation of "weak shocks." Here its relation to the linear stability results of Theorem 1 is shown. Cohen and Rosenblat [3] have used the approach for reaction-diffusion systems with variables representing populations and additional integral terms representing hereditary effects (they also keep the diffusion matrix equal to the identity).

In this section we first extend this analysis to arbitrary diffusion matrices. Next, the assumption that the behavior varies slowly on the spatial scale corresponds to long wavelength, i.e. small k^2, behavior, and the results of such an assumption must be related to the stability results for small k^2 in the last section. Our second purpose is to state this relation precisely.

The $O(1)$-terms in (4.2) give

$$u_{0\theta}=F(u_0, v_0),\qquad v_{0\theta}=G(u_0, v_0). \tag{4.3a}$$

The solution is taken to be the limit cycle

$$u_0=U(\theta),\qquad v_0=V(\theta). \tag{4.3b}$$

The $O(\varepsilon)$-term then becomes

$$\phi_{0\tau}\begin{bmatrix}U'(\theta)\\ V'(\theta)\end{bmatrix}+\begin{bmatrix}u_{1\theta}\\ v_{1\theta}\end{bmatrix}=\begin{bmatrix}F_u(U(\theta), V(\theta)) & F_v(U(\theta), V(\theta))\\ G_u(U(\theta), V(\theta)) & G_v(U(\theta), V(\theta))\end{bmatrix}\begin{bmatrix}u_1\\ v_1\end{bmatrix}+\begin{bmatrix}(1+\alpha)(|\hat\nabla\phi_0|^2U''(\theta)+\hat\nabla^2\phi_0U'(\theta))\\ (1-\alpha)(|\hat\nabla\phi_0|^2V''(\theta)+\hat\nabla^2\phi_0V'(\theta))\end{bmatrix}, \tag{4.4}$$

and the solution—using the fundamental matrix of Lemma 2—can be written as (keeping $W(s)=U'(s)\hat V(s)-V'(s)\hat U(s)$):

$$\begin{bmatrix}u_1\\ v_1\end{bmatrix}=\begin{bmatrix}U'(\theta) & e^{-\mu\theta}\hat U(\theta)\\ V'(\theta) & e^{-\mu\theta}\hat V(\theta)\end{bmatrix}\left(\begin{bmatrix}C(\tau,\boldsymbol{\xi})\\ D(\tau,\boldsymbol{\xi})\end{bmatrix}+\int_{\theta_0}^{\theta}\frac{1}{W(s)}\begin{bmatrix}\hat V(s) & -\hat U(s)\\ -e^{\mu s}V'(s) & e^{\mu s}U'(s)\end{bmatrix}\cdot\begin{bmatrix}-\phi_{0\tau}U'(s)+(1+\alpha)(\hat\nabla^2\phi_0\cdot U'(s)+|\hat\nabla\phi_0|^2U''(s))\\ -\phi_{0\tau}V'(s)+(1-\alpha)(\hat\nabla^2\phi_0\cdot V'(s)+|\hat\nabla\phi_0|^2V''(s))\end{bmatrix}ds\right). \tag{4.5}$$

Here $\theta_0=\phi_0(0,\boldsymbol{\xi})$ evaluated at $t=0$. To obtain a bounded solution as $\theta\to+\infty$, Lemma 1(a) shows that a secularity condition must be satisfied by the first component under the integral. Defining

$$B_1=\frac{1}{T}\int_0^T\frac{U''(s)\hat V(s)}{W(s)}ds,\qquad B_2=\frac{1}{T}\int_0^T\frac{-V''(s)\hat U(s)}{W(s)}ds,$$
$$C_1=\frac{1}{T}\int_0^T\frac{-U''(s)\hat U(s)}{W(s)}ds,\qquad C_2=\frac{1}{T}\int_0^T\frac{V''(s)\hat V(s)}{W(s)}ds, \tag{4.6}$$

and keeping the definitions of A_0, A_1, A_2 from (3.4), the secularity condition becomes

$$\phi_{0\tau}=(1-A_0\alpha)\hat\nabla^2\phi_0-[B_1(1+\alpha)+B_2(1-\alpha)]|\hat\nabla\phi_0|^2. \tag{4.7a}$$

This equation is a form of Burgers' equation and reduces to the heat equation by the

Hopf–Cole transformation (assuming the coefficient of $|\hat{\nabla}\phi_0|^2$ is nonzero),

$$\psi = \exp\left(\frac{1-\alpha A_0}{(1+\alpha)B_1+(1-\alpha)B_2}\phi_0\right), \tag{4.7b}$$

$$\psi_\tau = (1-\alpha A_0)\hat{\nabla}^2\psi. \tag{4.7c}$$

The same calculations with cross-terms δ_1, δ_2 present give

$$\begin{aligned}\phi_{0\tau} &= [1-A_0\alpha - A_1\delta_1 - A_2\delta_2]\nabla^2\phi_0\\ &\quad -[B_1(1+\alpha)+B_2(1-\alpha)+C_1\delta_1+C_2\delta_2]|\hat{\nabla}\phi_0|^2,\end{aligned} \tag{4.8a}$$

$$\psi = \exp\left(\frac{1-A_0\alpha - A_1\delta_1 - A_2\delta_2}{B_1(1+\alpha)+B_2(1-\alpha)+C_1\delta_1+C_2\delta_2}\phi_0\right), \tag{4.8b}$$

$$\psi_\tau = (1-A_0\alpha - A_1\delta_1 - A_2\delta_2)\hat{\nabla}^2\psi. \tag{4.8c}$$

The heat equation in (4.8c) is stable iff the coefficient of $\hat{\nabla}^2\psi$ is nonnegative and this is *exactly* the linear stability condition of Theorem 1! This fact connects the linear stability analysis for small wave number and the multi-scaling expansion (4.1).

It should be mentioned that, at least for small amplitude perturbations of the limit cycle, $u, v \sim U(t+\phi_0), V(t+\phi_0)$ as $t\to\infty$. This is rather surprising since a multi-scaling argument introducing a new variable $\tau = \varepsilon t$ is typically only valid for times like $O(1/\varepsilon)$. For a detailed discussion and comparison with numerical results, see Cope [6].

5. Instability examples for small $|\beta|$. In this section we construct examples of explicitly solvable systems of the form (1.1), with cross terms $\delta_1=\delta_2=0$ for which $|A_0|>1$ (A_0 defined in (3.4)). By Theorem 1, it follows that as the parameter α in the diffusion matrix varies past $1/A_0$, the limit cycle suddenly switches from being stable to perturbations for all small wave numbers k^2 to being unstable for all small k^2. A numerical examination of this behavior is given in § 8.

We wish to construct examples of (1.1) with diagonal diffusion matrix such that (a) the limit cycle and associated functions can be found explicitly, (b) A_0 can be evaluated exactly, and (c) $|A_0|>1$. As shown in § 3, all λ–ω systems have $A_0=0$. Also, the explicitly solvable case of (3.10) has $0<A_0<1$ for all values of the parameter a. This difficulty in finding suitable examples eventually led to the following systematic approach.

(1) We first restrict ourselves to systems (1.1) with *almost* solvable kinetic equations: (5.1);

(2) for kinetic equations of the form (5.1), the value of A_0 can be written in a simplified form: Lemma 4;

(3) to make the equations more nearly solvable a further restriction is made: (5.6);

(4) under the new restriction, Lemma 4 is simplified further: (5.9);

(5) finally, the expression for A_0 in (5.9) is sufficiently simple that explicitly solvable examples with $|A_0|>1$ can be constructed by guessing.

To begin, notice that the kinetic equations of (1.1) are reduced by the substitution $u=R\cos\psi$, $v=R\sin\psi$ to

$$\begin{bmatrix} R' \\ \psi' \end{bmatrix} = \begin{bmatrix} u/R & v/R \\ -v/R^2 & u/R^2 \end{bmatrix}\begin{bmatrix} F(u,v) \\ G(u,v)\end{bmatrix} = \begin{bmatrix} R\lambda(R,\psi) \\ \omega(R,\psi)\end{bmatrix},$$

where λ, ω are 2π-periodic functions of ψ. We first restrict attention to kinetic equations with the polar form

(5.1a) $$R' = R\lambda(R), \qquad \psi' = \omega(R, \psi),$$

(5.1b) $$\lambda(R_0) = 0 \quad \text{for some } R_0 > 0 \text{ and } \lambda_R(R_0) < 0,$$

(5.1c) $$\omega(R_0, \psi) > 0 \quad \text{for all } \psi,$$

(5.1d) $$\omega(R, \psi + 2\pi) = \omega(R, \psi).$$

(Subscripts represent the corresponding derivatives; prime $'$ always means derivative with respect to t.) Conditions (5.1a)–(5.1c) say that $R = R_0$ is a stable limit cycle; (5.1d) is the natural consequence of a polar transformation. This is step (1).

To calculate A_0, the limit cycle and associated quantities in (3.4a) must be known. The limit cycle is $U, V = R_0 \cos(\psi_0(t)), R_0 \sin(\psi_0(t))$, where

(5.2a) $$\psi_0' = \omega(R_0, \psi_0) \quad \text{with } \psi_0(0) = 0,$$

(5.2b) $$T = \int_0^{2\pi} \frac{d\psi}{\omega(R_0, \psi)}.$$

It is convenient to retain the polar form for finding solutions of the variational equations. Setting $R = R_0 + \varepsilon\rho$, $\psi = \psi_0 + \varepsilon\phi$ in (5.1a) leads to the variational equations

(5.3) $$\rho' = R_0\lambda_R(R_0)\rho, \qquad \phi' = \omega_R(R_0, \psi_0)\rho + \omega_\psi(R_0, \psi_0)\phi.$$

One solution is clearly $\rho = 0$, $\phi = \psi_0'$, corresponding to the periodic solution U', V'. Setting $-\mu = R_0\lambda_R(R_0)$ and $\rho(0) = 1$, a second solution for ρ is $\rho = \exp(-\mu t)$. To obtain ϕ, a convenient substitution is $\phi = \exp(-\mu t)\psi_0'\theta$; using (5.2a), the equation for θ reduces to

(5.4a) $$\theta' - \mu\theta = \frac{\omega_R(R_0, \psi_0)}{\omega(R_0, \psi_0)},$$

and the desired solution is the unique T-periodic solution (note Lemma 1(c)). Actually, θ as a function of ψ will be more useful, so that $\theta(\psi)$ is the unique 2π-periodic solution to

(5.4b) $$\frac{d\theta}{d\psi} - \frac{\mu}{\omega(R_0, \psi)}\theta = \frac{\omega_R(R_0, \psi)}{(\omega(R_0, \psi))^2}.$$

The solutions to the variational equation can now be written as

(5.5) $$\begin{bmatrix} U'(t) & \exp(-\mu t)\hat{U}(t) \\ V'(t) & \exp(-\mu t)\hat{V}(t) \end{bmatrix} = \begin{bmatrix} -R_0 \sin\psi_0\psi_0' & \exp(-\mu t)(\cos\psi_0 - R_0 \sin\psi_0\psi_0'\theta) \\ R_0 \cos\psi_0\psi_0' & \exp(-\mu t)(\sin\psi_0 + R_0\cos\psi_0\psi_0'\theta) \end{bmatrix}.$$

Substitution of these expressions into (3.4a) yields an expression for A_0. Summarizing to finish step (2) we have

LEMMA 4. *For the system* (1.1) *with kinetic equations* (*in polar form*) *given by* (5.1),

$$A_0 = \frac{1}{T}\int_0^T [\cos(2\psi_0(t)) - R_0 \sin(2\psi_0(t))\psi_0'(t)\theta(t)]\,dt$$
$$= \frac{1}{T}\int_0^{2\pi} \left[\frac{\cos(2\psi)}{\omega(R_0, \psi)} - R_0 \sin(2\psi)\theta(\psi)\right] d\psi,$$

where $\psi_0(t)$, T *are given by* (5.2) *and* $\theta(t)$ (*or* $\theta(\psi)$) *is the unique T-periodic* (*or* 2π*-periodic*) *solution to* (5.4).

We now try to pick $\omega(R, \psi)$ so that ψ_0 and θ can be found. Some experimentation suggests the additional constraint (n = arbitrary constant)

(5.6a) $$\frac{1}{\omega(R, \psi)} = \frac{h(R)}{\mu}\left[-n\frac{f'(\psi)}{f(\psi)} + g(R)\right],$$

(5.6b) $f(\psi)$ is positive and 2π-periodic,

(5.6c) $h(R_0) = +1$ and $g(R)$ is such that $\omega(R, \psi) > 0$ for all $R > 0$ and all ψ.

(Although $\omega(R_0, \psi) > 0$ is sufficient, (5.6c) is chosen to give a simpler phase plane—the origin is the only possible critical point.) Such an ω is still sufficiently general that some choice of f, g, h can be expected to force $|A_0| > 1$. This ends step (3).

The restriction (5.6) gives simple expressions for ψ_0, θ. Substitution into (5.2) gives

(5.7a) $$t(\psi) = \frac{1}{\mu}\left[g(R_0)\psi - n \ln\frac{f(\psi)}{f(0)}\right] \quad \text{defining } \psi_0(t),$$

(5.7b) $$T = 2\pi\frac{g(R_0)}{\mu}.$$

Substitution into (5.4b) gives

(5.8a) $$\theta(\psi) = -(f(\psi))^{-n}\frac{g'(R_0)}{\mu}F(R_0, \psi) + \frac{h'(R_0)}{\mu},$$

where $F(R_0, \psi)$ is the unique periodic function (Lemma 1(b)) defined by

(5.8b) $$F(R_0, \psi) = \exp(+g(R_0)\psi)\left[C(R_0) + \int_0^{\psi} \exp(-g(R_0)s)(f(s))^n\, ds\right].$$

Therefore, assuming $\omega(R, \psi)$ is given by (5.6) and substituting (5.6), (5.7), (5.8) into Lemma 4 yields

(5.9) $$A_0 = \frac{1}{2\pi g(R_0)}\int_0^{2\pi}\left[-n\cos(2\psi)\frac{f'(\psi)}{f(\psi)} + R_0 g'(R_0)\sin(2\psi)(f(\psi))^{-n}F(R_0, \psi)\right] d\psi.$$

This finishes step (4).

The expression for A_0 would considerably simplify if we require $g'(R_0) = 0$; curiously enough, this is inadequate because $|A_0| < 1$ is forced: since $\omega(R_0, \psi) > 0$, clearly

$$\int_0^{2\pi}\frac{d\psi}{\omega(R_0, \psi)} > \left|\int_0^{2\pi}\cos(2\psi)\frac{d\psi}{\omega(R_0, \psi)}\right|.$$

The integral on the left is $2\pi g(R_0)/\mu$. Substituting (5.6a) into the integral on the right and assuming $g'(R_0) = 0$ in (5.9), the right side becomes $(2\pi g(R_0)/\mu)|A_0|$, or $1 > |A_0|$.

We are now ready to carry out step (5). Explicit choices will be made for n, $h(R)$, $f(\psi)$ and the appropriate conditions on $g(R)$ deduced.

(5.10a) $$n = +1, \qquad h(R) \equiv +1, \qquad f(\psi) = 1 + \varepsilon\cos 2\psi \quad \text{with } |\varepsilon| < 1.$$

Since

$$\max\left|\frac{f'(\psi)}{f(\psi)}\right| = \max\left|\frac{2\varepsilon\sin 2\psi}{1 + \varepsilon\cos 2\psi}\right| = \frac{2|\varepsilon|}{(1-\varepsilon^2)^{1/2}},$$

(5.6c) can be satisfied by

(5.10b) $$g(R) > \frac{2|\varepsilon|}{(1-\varepsilon^2)^{1/2}} \quad \text{for all } R.$$

Calculation of $F(R_0, \psi)$ and substitution into (5.9) gives

$$A_0 = \frac{1}{2\pi g(R_0)} \int_0^{2\pi} \left(\frac{-2\varepsilon \cos 2\psi \sin 2\psi}{1+\varepsilon \cos 2\psi} + \frac{R_0 g'(R_0) \sin 2\psi}{1+\varepsilon \cos 2\psi} \left[-\frac{1}{g(R_0)} + \frac{\varepsilon(-g(R_0)\cos 2\psi + 2 \sin 2\psi)}{(g(R_0))^2+4} \right] \right) d\psi.$$

This integral simplifies, due to perfect derivatives, to

$$A_0 = \frac{R_0 g'(R_0)\varepsilon}{\pi g(R_0)[(g(R_0))^2+4]} \int_0^{2\pi} \frac{(\sin 2\psi)^2 \, d\psi}{1+\varepsilon \cos 2\psi}.$$

The substitution $z = \exp(+i\psi)$ and use of the residue theorem gives

(5.10c) $$A_0 = \frac{2R_0}{\varepsilon}[1-(1-\varepsilon^2)^{1/2}] \frac{g'(R_0)}{g(R_0)[g(R_0)^2+4]}.$$

Therefore, *for each* $\varepsilon, 1 > |\varepsilon| > 0$, *any function* $g(R)$ *with* $|A_0| > 1$ *in* (5.10c) *and satisfying* (5.10b) *can be used to construct an example of* (1.1) *with limit cycle unstable to small wave numbers* k^2. The kinetic equations are constructed using (5.1a), (5.6a), and (5.10a, b, c), in which case all other conditions (5.1b, c, d), (5.6b, c) are automatically satisfied. Instability occurs as α varies past $1/A_0$, $|\alpha|$ increasing.

A specific system, constructed according to this prescription, is studied numerically in § 8.

6. Intermediate values of $|\boldsymbol{\beta}|$. In the preceding 3 sections, (2.3)—the basic equation for studying the linear stability of the limit cycle solution of (1.1)—has been analyzed in detail for small $|\boldsymbol{\beta}|$, corresponding to perturbations with small wave number k^2. In § 7, a general analysis for large $|\boldsymbol{\beta}|$, corresponding to large k^2, will be given. These two general results depend on perturbation approaches, which in turn depend on the knowledge of the exact solution as $|\boldsymbol{\beta}| \to 0$ or $\to +\infty$. For intermediate values of $|\boldsymbol{\beta}|$, no such general results seem possible. There are no general formulas giving the Floquet exponents in terms of the coefficients of a Floquet system; and there are no intermediate values of $|\boldsymbol{\beta}|$ for which an exact solution is known, so there is no way to start a perturbation expansion. In this case, therefore, we introduce further structure into the situation by assuming a small parameter in the kinetic equations; specifically, consider the systems of the form

(6.1) $$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} -\omega_0 v + \gamma F(u, v) \\ \omega_0 u + \gamma G(u, v) \end{bmatrix} + \begin{bmatrix} (1+\alpha)\nabla^2 u \\ (1-\alpha)\nabla^2 v \end{bmatrix}, \qquad 0 < \gamma \ll 1, \quad \omega_0 > 0,$$

where the kinetic equations are assumed to possess, for $0 < \gamma$, a stable limit cycle $U(\gamma; t)$, $V(\gamma; t)$ with period $T(\gamma)$ such that $T(0) = 2\pi/\omega_0$ and $U(0; t)$, $V(0; t) = A \cos(\omega_0 t)$, $A \sin(\omega_0 t)$, $A > 0$. For example, if $F = u(1-u^2)$ and $G = 0$, the kinetic system would correspond to the van der Pol oscillator and $T(\gamma) =$

$2\pi[1+(\gamma^2/16\omega_0^2)+\cdots]$ with

$$U(\gamma,t),\ V(\gamma,t)=2(3^{-1/2})\cos\left(\left(1-\frac{\gamma^2}{16\omega_0^2}+\cdots\right)t\right),2(3^{-1/2})\sin\left(\left(1-\frac{\gamma^2}{16\omega_0^2}+\cdots\right)t\right).$$

For a discussion of the calculation of $U(\gamma, t)$, $V(\gamma, t)T(\gamma)$ as power series in γ, see Lefshetz [14, Chap. 12]. Such expansions will not be explicitly used in the approach below; U, V, T are assumed known for each γ. Cross terms corresponding to δ_1, δ_2 in (1.1) are omitted for simplicity.

Linearizing (6.1) about the limit cycle and separating variables by $u, v = U+\varepsilon\hat{u}$, $V+\varepsilon\hat{v}$; $\hat{u}, \hat{v} = q\exp(+i\mathbf{k}\cdot\mathbf{x})$, $q\exp(+i\mathbf{k}\cdot\mathbf{x})$ gives the analog of (1.3),

$$\text{(6.2)}\qquad \begin{bmatrix} p' \\ q' \end{bmatrix} = \begin{bmatrix} -k^2(1+\alpha)+\gamma F_1(\gamma;t) & -\omega_0+\gamma F_2(\gamma;t) \\ \omega_0+G_1(\gamma;t)\gamma & -k^2(1-\alpha)+\gamma G_2(\gamma;t) \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix},$$

where $F_1(\gamma;t)=F_u(U(\gamma;t), V(\gamma;t))$ with similar definitions for F_2, G_1, G_2, and F_i, G_i have period $T(\gamma)$ in t. Continuing as in § 2, we set $p, q = x\exp(-k^2t)$, $y\exp(-k^2t)$ and $\beta=\alpha k^2$ to obtain the analog of (2.3)

$$\text{(6.3)}\qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -\beta+\gamma F_1(\gamma;t) & -\omega_0+|F_2(\gamma;t) \\ \omega_0+\gamma G_1(\gamma;t) & \beta+\gamma G_2(\gamma;t) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},\qquad 0<\gamma\ll 1.$$

The limit cycle is linearly unstable as a solution of (6.1) iff (6.3) has a solution with growth rate greater than $O(\exp(+k^2t))$ for some $|\beta|<k^2$. As in § 2, the product of the Floquet multipliers is independent of β and instability can only occur for real, distinct multipliers.

We shall now determine, for fixed β, ω_0 which are $O(1)$, the larger Floquet exponent, through $O(\gamma)$.

A series expansion for solutions of (6.3) is assumed in the form

$$\text{(6.4a)}\qquad \begin{bmatrix} x \\ y \end{bmatrix} = \exp(B(\gamma)t)\sum_{n=0}^{\infty}\begin{bmatrix} x_n(t) \\ y_n(t) \end{bmatrix}\gamma^n,$$

$$\text{(6.4b)}\qquad B(\gamma)=B_0+B_1\gamma+B_2\gamma^2+\cdots,$$

(6.4c) $\qquad x_n(t)$, $y_n(t)$ are to be periodic with period $T(\gamma)$.

The expansion is suggested by the Floquet theorem, which says the general solutions have the form of an exponential multiplying a $T(\gamma)$-periodic function. The expansion is not a strict power series in γ, because x_n, y_n are required to have period $T(\gamma)$ and therefore depend on γ implicitly.

Substitution of (6.4) into (6.3) and equating powers of γ in the obvious way yields

$$\text{(6.5a)}\qquad \begin{bmatrix} x_0' \\ y_0' \end{bmatrix} = \begin{bmatrix} -\beta-B_0 & -\omega_0 \\ \omega_0 & \beta-B_0 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix};$$

for $n\geqq 1$,

$$\text{(6.5b)}\ \begin{bmatrix} x_n' \\ y_n' \end{bmatrix} = \begin{bmatrix} -\beta-B_0 & -\omega_0 \\ \omega_0 & \beta-B_0 \end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix} - \sum_{m=1}^{n} B_m\begin{bmatrix} x_{n-m} \\ y_{n-m} \end{bmatrix} + \begin{bmatrix} F_1(\gamma;t) & F_2(\gamma;t) \\ G_1(\gamma;t) & G_2(\gamma;t) \end{bmatrix}\begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}.$$

For $n=0$, B_0 is determined by the requirement that (6.5a) have a $T(\gamma)$-periodic solution, so the characteristic equation $(\lambda+B_0)^2-\beta^2+\omega_0^2=0$ must have a 0-eigenvalue, and $B_0=\pm(\beta^2-\omega_0^2)^{1/2}$.

If $\beta^2 < \omega_0^2$, the Floquet exponents will be complex-conjugate, and instability cannot occur by Lemma 3(b). If $\beta^2 > \omega_0^2$ and β_0 is chosen as $-(\beta^2-\omega_0^2)^{1/2}$, the expansion (6.4a) corresponds to the solution for the smaller Floquet multiplier. We shall choose $B_0 = +(\beta^2-\omega_0^2)^{1/2}$, the expansions in the other cases being exactly similar. (Incidentally, the fact that the Floquet exponents are complex-conjugate for $\beta^2 < \omega_0^2$ may seem to contradict the fact that the exponents are real and distinct at $\beta = 0$. However, the expansion is based on the assumption that γ is small compared with $|\beta|$, ω_0, so if $|\beta| \to 0$, then $\gamma \to 0$ also and the whole expansion is no longer valid.)

For the above choice of B_0, a constant (hence $T(\gamma)$-periodic for *all* γ) solution occurs for x_0, y_0, and a fundamental matrix for the linear system in (6.5a) can be given,

$$(6.6a) \qquad B_0 = +(\beta^2-\omega_0^2)^{1/2}, \qquad \beta^2 > \omega_0^2,$$

$$(6.6b) \qquad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \omega_0 \\ -\beta - B_0 \end{bmatrix},$$

$$(6.6c) \qquad \text{fundamental matrix for (6.5a)} = \begin{bmatrix} \omega_0 & \omega_0 \exp(-2B_0 t) \\ -\beta - B_0 & (-\beta + B_0)\exp(-2B_0 t) \end{bmatrix}.$$

Using this fundamental matrix and Lemma 2, the general solution x_n, y_n can be written as

$$(6.7) \qquad \begin{aligned} \begin{bmatrix} x_n \\ y_n \end{bmatrix} &= \begin{bmatrix} \omega_0 & \omega_0 \exp(-2B_0 t) \\ -\beta - B_0 & (-\beta + B_0)\exp(-2B_0 t) \end{bmatrix} \\ &\quad \left\{ \begin{bmatrix} C_n \\ D_n \end{bmatrix} + \int_0^t \frac{1}{2\omega_0 B_0} \begin{bmatrix} -\beta + B_0 & -\omega_0 \\ (\beta + B_0)\exp(+2B_0 s) & \omega_0 \exp(+2B_0 s) \end{bmatrix} \right. \\ &\quad \left. \cdot \left(-\sum_{m=1}^{n} B_m \begin{bmatrix} x_{n-m}(s) \\ y_{n-m}(s) \end{bmatrix} + \begin{bmatrix} F_1(\gamma; s) & F_2(\gamma; s) \\ G_1(\gamma; s) & G_2(\gamma; s) \end{bmatrix} \begin{bmatrix} x_{n-1}(s) \\ y_{n-1}(s) \end{bmatrix} \right) ds \right\}. \end{aligned}$$

This procedure can be shown well-defined (that is, B_n, C_n, D_n determined and a $T(\gamma)$-periodic solution x_n, y_n existing for all n) by the following induction argument: assuming B_k, C_k, D_k, x_k, y_k have been determined and x_k, y_k are $T(\gamma)$-periodic for each $k < n$,

(a) B_n is determined uniquely by the requirement that the first component of the integral be $T(\gamma)$-periodic (i.e., apply Lemma 1(a)); notice the coefficient of B_n is always -1, never 0;

(b) C_n is chosen arbitrarily since it only adds in a multiple of x_0, y_0, and therefore merely corresponds to a rescaling of the first term—to be specific, take $C_n = 0$;

(c) D_n is then uniquely determined, using Lemma 2(b), by the requirement that

$$\exp(-2B_0 t)\left[D_n + \int_0^t \exp(+2B_0 S)[\cdots T(\gamma)\text{-periodic functions} \cdots]\, ds \right]$$
$$= T(\gamma)\text{-periodic function.}$$

In particular,

$$(6.8) \quad B_1 = \frac{1}{2B_0 T(\gamma)} \int_0^{T(\gamma)} [(-\beta + B_0)F_1(s) + (\beta + B_0)G_2(s) + \omega_0(F_2(s) - G_1(s))]\, ds,$$

where $F_1, F_2, G_1, G_2 = O(1)$ and $T(\gamma) = (2\pi/\omega_0) + O(\gamma)$ as $\gamma \to 0$.

The above expansion can be used to calculate the Floquet exponents for γ small in comparison with $|\beta|$, ω_0. We have presented the procedure in some detail because in other treatments of Floquet systems with a small parameter the coefficient matrix has a period independent of the parameter, while (6.3) necessarily requires an approach in which the period depends on the parameter. (As an example of other treatments, note Stevens [19], where the coefficient matrix has a constant $O(1)$-component and an $O(\varepsilon)$-component of finite trigonometric polynomials of fixed period.)

From (6.6a), it is clear that the solution corresponding to the larger Floquet exponent grows like $\exp((\alpha^2 k^2 - \omega_0^2)t)$, $|\alpha| \leqq 1$. Since the growth rate of these solutions is strictly less than $O(\exp(+k^2 t))$, the limit cycle is linearly stable as a solution of (6.1) for small γ (with β, ω_0 held finite).

7. Behavior for large $|\boldsymbol{\beta}|$. To obtain the Floquet multipliers as $|\boldsymbol{\beta}| \to +\infty$ in (2.4), we set $\boldsymbol{\beta} = \rho(\cos\gamma_0, \cos\gamma_1, \cos\gamma_2)$, where $\rho = |\boldsymbol{\beta}|$ and $(\cos\gamma_0)^2 + (\cos\gamma_1)^2 + (\cos\gamma_2)^2 = +1$. Then (2.4) becomes

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \left(\rho\begin{bmatrix} -\cos\gamma_0 & -\cos\gamma_2 \\ -\cos\gamma_1 & \cos\gamma_0 \end{bmatrix} + \begin{bmatrix} F_1(t) & F_2(t) \\ G_1(t) & G_2(t) \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}, \qquad \rho \gg 0. \tag{7.1}$$

The result is a system of differential equations containing a charge parameter and a well-developed asymptotic theory exists. For instance, systems of the form (7.1) are treated in Chapter 6 of Coddington and Levinson [1]. Theorem 2.1 of that chapter states that a fundamental matrix $P(\rho, t)\exp[\rho Q_0(t) + Q_1(t)]$ can be formally constructed such that ($\delta^2 = (\cos\gamma_0)^2 + \cos\gamma_1\cos\gamma_2$ gives the eigenvalues $\pm\delta$ of the coefficient of ρ; $\delta^2 \neq 0$ assumed)

$$P(\rho, t) = \sum_{n=0}^{\infty} \rho^{-n} P_n(t) \quad \text{with each matrix } P_n(t) \text{ independent of } \rho, \tag{7.2a}$$

$$Q_0'(t) = \begin{bmatrix} \delta & 0 \\ 0 & -\delta \end{bmatrix} \quad \text{and} \quad Q_0(t), Q_1(t) \text{ are diagonal matrices.} \tag{7.2b}$$

Furthermore, Theorem 3.1 of that chapter also holds and the formal solution given by (7.2) on $[0, T]$ is asymptotic to the real solution on $[0, T]$ as $\rho \to +\infty$.

As it stands, (7.2) is sufficient to give the desired leading order behavior. However, since it was derived for quite general systems, it does not represent the solution in Floquet normal form ($P(\rho, t)$ is not a periodic matrix in general). We give here an alternate expansion, based on the Floquet representation, which yields another form of the solution.

Define the following eigendata:

$$\delta \equiv [(\cos\gamma_0)^2 + \cos\gamma_1\cos\gamma_2]^{1/2} \neq 0; \tag{7.3a}$$

$$\begin{bmatrix} -\cos\gamma_0 & -\cos\gamma_2 \\ -\cos\gamma_1 & \cos\gamma_0 \end{bmatrix} \quad \text{has eigenvalues } \pm\delta; \tag{7.3b}$$

$$\delta \text{ has eigenvector } \begin{bmatrix} a \\ b \end{bmatrix}, \qquad -\delta \text{ has eigenvector } \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}. \tag{7.3c}$$

We show that the solution corresponding to the larger Floquet multiplier can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \exp\left(\delta\rho t + \sum_{n=0}^{\infty} \rho^{-n} C_n(t)\right)\left(\begin{bmatrix} a \\ b \end{bmatrix} + \sum_{n=0}^{\infty} \rho^{-n-1} B_n(t)\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}\right); \tag{7.4a}$$

$$C_n'(t), B_n(t) \text{ are } T\text{-periodic functions.} \tag{7.4b}$$

The $C_n(t)$, of course, grow like $O(t)$ and provide the exponential growth of the solution; $C_n(0)$ is arbitrary since it merely scales the solution, so we pick $C_n(0)=0$. The coefficients $P_n(t)$ in (7.2) will generally contain polynomial terms in t as a consequence of restricting the exponential part to a finite series in ρ; the restriction of the solution to $[0, T]$ is therefore essential. In contrast, (7.4) retains the form of an exponential multiplying a periodic function, and it should be valid over large t-regions—for instance, the periodic part should be asymptotic in ρ to the real solution for all t.

To obtain the C_n, B_n, first define the T-periodic functions H, K, $\hat{H}$, $\hat{K}$,

(7.5)
$$\begin{bmatrix} F_1(t) & F_2(t) \\ G_1(t) & G_2(t) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \equiv H(t)\begin{bmatrix} a \\ b \end{bmatrix} + K(t)\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix},$$
$$\begin{bmatrix} F_1(t) & F_2(t) \\ G_1(t) & G_2(t) \end{bmatrix}\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} \equiv \hat{H}(t)\begin{bmatrix} a \\ b \end{bmatrix} + \hat{K}(t)\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}.$$

Substitution of (6.4a) into (7.1) gives $O(\rho)$-terms which cancel, $O(1)$-terms implying

(7.6a)
$$C_0'(t) = H(t), \qquad C_0(0) = 0,$$
$$B_0(t) = \frac{1}{2\delta}K(t),$$

and $O(\rho^{-n})$-terms, $n \geqq 1$, with

(7.6b)
$$C_n'(t) = \hat{H}B_{n-1}, \qquad C_n(0) = 0,$$
$$B_n(t) = \frac{1}{2\delta}\left[\hat{K}B_{n-1} - B_{n-1}' - \sum_{m=0}^{n-1} C_m' B_{n-m-1}\right].$$

All quantities in (7.4) are uniquely determined with the correct properties, so (7.4) is well-defined.

The larger Floquet multiplier is $\exp[(\rho\delta + O(1))t] = \exp[((\alpha^2+\delta_1\delta_2)^{1/2}k^2 + O(1))t]$ and the limit cycle is unstable as a solution of (1.1) if $\exp[(-k^2 + (\alpha^2+\delta_1\delta_2)^{1/2}k^2 + O(1))t]$ is an increasing function. But the initial hypothesis on the diffusion matrix of (1.1) is $0 \leqq \alpha^2 + \delta_1\delta_2 \leqq 1$, and therefore the limit cycle is stable unless $\alpha^2 + \delta_1\delta_2 = +1$, in which case the next order term is needed.

Since the known results for limit cycle stability, due to Othmer [16], Conway, Hoff, and Smoller [5], and Cohen [2], require the diffusion matrix to be nonsingular (i.e. $\alpha^2+\delta_1\delta_2 \neq +1$), we assume $\alpha^2+\delta_1\delta_2 = +1$ and consider this case in more detail. The growth rate of the solution corresponding to the larger Floquet multiplier is then $\exp[C_0(t) + O(1/\rho)t]$. From (6.7a), $C_0(t) = \bar{h}t +$ (periodic function of t), where

$$\bar{h} = \frac{1}{T}\int_0^T H(t)\,dt.$$

Therefore, if $\alpha^2+\delta_1\delta_2 = +1$ and $\bar{h}$ is negative (positive), the limit cycle is linearly stable (unstable) as a solution of (1.1) to all sufficiently large wave numbers k^2.

In particular, consider (1.1) with $\delta_1 = \delta_2 = 0$ and $\alpha = \pm 1$. With the identification $\boldsymbol{\beta} = (\alpha k^2, \delta_1 k^2, \delta_2 k^2)$, we have $\rho = k^2$, $\cos\gamma_0 = \alpha = \pm 1$, $\cos\gamma_1 = \cos\gamma_2 = 0$ for the linearized system (7.1). These values in (7.3) show the eigenvalue $\delta = +1$ has the eigenvectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ if } \alpha = +1 \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ if } \alpha = -1.$$

Consequently,

(7.7) $$H(t)=F_1(t) \quad \text{if } \alpha=+1 \quad \text{and} \quad H(t)=G_2(t) \quad \text{if } \alpha=-1.$$

The stability implications are summarized as

THEOREM 2. *Define, for* $F_1(t)$ *and* $G_2(t)$ *as given in* (1.3),

$$\bar{F}_1=\frac{1}{T}\int_0^T F_1(t)\,dt \quad \text{and} \quad \bar{G}_2=\frac{1}{T}\int_0^T G_2(t)\,dt.$$

Then (a) *if* $\alpha=+1$, $\delta_1=\delta_2=0$ *in* (1.1) *and* $\bar{F}_1$ *is negative* (*positive*), *then the limit cycle is linearly stable* (*unstable*) *as a solution of* (1.1) *with respect to all sufficiently large wave numbers* k^2;

(b) *if* $\alpha=-1$, $\delta_1=\delta_2=0$ *in* (1.1) *and* $\bar{G}_2$ *is negative* (*positive*), *then the limit cycle is linearly stable* (*unstable*) *as a solution of* (1.1) *with respect to all sufficiently large wave numbers* k^2.

As noted in Lemma 2(b), $\bar{F}_1+\bar{G}_2=-\mu$, the negative Floquet exponent, so at most one of $\bar{F}_1$, $\bar{G}_2$ is positive.

For example, if $\alpha=\pm 1$ and $\delta_1=\delta_2=0$ in the general λ–ω system (3.5), we obtain (referring to (3.6) with $k^2=0$)

(7.8)
$$\bar{F}_1=\frac{1}{T}\int_0^T R_0S_0\cos(\omega_0 t)\cos(\omega_0 t+\sigma_0)\,dt=\frac{1}{2}R_0\lambda'(R_0),$$
$$\bar{G}_2=\frac{1}{T}\int_0^T R_0S_0\sin(\omega_0 t)\sin(\omega_0 t+\sigma_0)\,dt=\frac{1}{2}R_0\lambda'(R_0).$$

Kinetic stability of the limit cycle required $\lambda'(R_0)<0$, so the limit cycle is linearly stable as a solution of the reaction-diffusion equations to all large wave numbers k^2.

For the system (3.9) with $\alpha=\pm 1$ and $b=+1$, reference to (3.12) gives

(7.9a)
$$\bar{F}_1=\frac{1}{T}\int_0^T a(1-3(U(t))^2)\,dt,$$
$$\bar{G}_2=\frac{1}{T}\int_0^T a(1-(U(t))^2)\,dt.$$

Here $T=2\pi$, $U(t)=$ first component of the limit cycle $=R_0(t)\cos t$, so from (3.11),

$$(U(t))^2=\frac{(a^2+1)(1+\cos 2t)}{a^2+1+a^2\cos 2t+a\sin 2t}$$

and

$$\int_0^{2\pi}(U(t))^2\,dt=2\pi, \text{ independent of } a!$$

Therefore,

(7.9b) $$\bar{F}_1=-2a, \qquad \bar{G}_2=0.$$

For $\alpha=+1$, Theorem 2 shows the limit cycle is linearly stable as a solution of (3.9), but for $\alpha=-1$, the exponential growth is given by

$$\exp\left[C_0(t)+\frac{1}{\rho}C_1(t)+O\left(\frac{1}{\rho^2}\right)\right],$$

where $C_0(t)$ is periodic; the limit cycle is linearly stable (unstable) if the mean value of $C_1'(t)$ is negative (positive). Using (7.6), $C_1'(t)=\hat{H}(t)B_0(t)$ and $\hat{H}(t)=G_1(t)$, $B_0(t)=\frac{1}{2}F_2(t)$. Using (3.12) (with $k^2=0$) for F_2, G_1 gives

$$\begin{aligned}\text{mean value of } C_1'(t) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}(-1+2aU(t)V(t))\,dt\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(-\frac{1}{2}+\frac{a(a^2+1)\sin 2t}{(a^2+1)+a^2\cos 2t+a\sin 2t}\right)dt \qquad (7.10)\\ &= +\frac{1}{2}-(a^2+1)^{1/2}.\end{aligned}$$

This quantity is always negative and the limit cycle is linearly stable as a solution of (3.9) with $\alpha=-1$.

8. Numerical results for an unstable limit cycle. In § 5 examples of (1.1) with diagonal diffusion matrices were constructed whose limit cycles become linearly unstable to all small wave numbers k^2 as the diffusion parameter α passes a critical value α_0. The linear analysis gives $\alpha_0=(1/A_0)+O(k^2)$. In this section a specific example from § 5 is selected and small wave number perturbations of the limit cycle are examined numerically for various values of α.

First, following the instructions at the end of § 5, the kinetic system of (1.1) is assumed to have the polar form

$$\begin{aligned}&R'=R(1-R^2),\\ &\psi'=\omega(R,\psi)\quad\text{with}\quad\frac{1}{\omega(R,\psi)}=\frac{.3\sin 2\psi}{5+3\cos 2\psi}+\frac{1}{2}g(R).\end{aligned}\qquad (8.1\text{a})$$

This choice satisfies (5.1a) with limit cycle radius $R_0=+1$ and Floquet exponent $-\mu=+R_0\lambda'(R_0)=-2$, and satifies (5.6a), (5.10a) with $\varepsilon=\frac{3}{5}$. We choose

$$g(R)=\tfrac{11}{4}+\tanh(a(R^2-1)),\qquad (8.1\text{b})$$

which satisfies the lower bound of (5.10b) with $\varepsilon=\frac{3}{5}$. From (5.10c) follows

$$A_0=\frac{256a}{6105}=.0419\text{a}.\qquad (8.1\text{c})$$

The limit cycle period is (from (5.1b))

$$T=\frac{11\pi}{4}=8.64.\qquad (8.1\text{d})$$

In terms of the original variables, the system under consideration is

$$\begin{aligned}&u_t=u(1-u^2-v^2)-v\omega+(1+\alpha)u_{xx}\\ &v_t=v(1-u^2-v^2)+u\omega+(1-\alpha)v_{xx},\quad\text{where}\\ &\omega=\frac{8u^2+2v^2}{6uv+(4u^2+v^2)(\frac{11}{4}+\tanh(a(u^2+v^2-1)))}.\end{aligned}\qquad (8.2)$$

For $a>0$, the limit cycle should go unstable to all small wave numbers k^2 as α increases past $\alpha_0\sim 1/A_0=23.9/a$. Notice the large value of a required to make $\alpha_0<1$ makes ω nearly discontinuous across the limit cycle.

Lees' method (Lees [13], Varah [21]) was used for the numerical solution of (8.2). This difference scheme is an extrapolated variation of the Crank–Nicholson method. For a scalar equation

$$u_t = \delta u_{xx} + f(u),$$

one substitutes the difference approximation ($u_m^n = u(m\,\Delta x, n\,\Delta t)$)

$$\frac{1}{\Delta t}(u_m^{n+1} - u_m^n) = \frac{\delta}{2(\Delta x)^2}[(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}) + (u_{m+1}^n - 2u_m^n + u_{m-1}^n)] + f(\tfrac{3}{2}u_m^n - \tfrac{1}{2}u_m^{n-1}).$$

A similar formula is used for two or more dependent variables. For constant boundary conditions, the difference scheme requires only the solution of a tridiagonal system at each time-step. Periodic boundary conditions are used here and the linear system is not quite tridiagonal, but a straightforward Gaussian elimination can be arranged; using only the three main diagonals, the last row, and the last column, which is almost as efficient as solving a tridiagonal system. Lees' method is easily programmed, has accuracy $O((\Delta x)^2 + (\Delta t)^2)$, and is stable. (In the actual implementation of the program, initial data and the diffusion coefficients were rescaled to obtain the equivalent system for the fixed interval $0 \leqq x \leqq 1$; the step sizes were $\Delta x = .02$ and $\Delta t = T/500 \sim .017$.)

It is useful to consider the solutions as curves in the phase plane; for periodic boundary conditions, $u(x, t)$, $v(x, t)$ is a closed curve with x as parameter. A perturbation of the limit cycle is then a small closed curve near some point on the limit cycle—for instance, the initial data of our computer runs (before scaling to $0 \leqq x \leqq 1$):

$$u(x, 0) = 1, \qquad v(x, 0) = .1 \cos(.2x), \tag{8.3}$$

corresponding to wave number $k^2 = .04$. To represent the results of the calculations, polar coordinates are especially useful: for $u = R \cos \psi$, $v = R \sin \psi$, define

$$\begin{aligned} \Delta R &= \max_x R(x, t) - \min_x R(x, t), \\ \Delta\psi &= \max_x \psi(x, t) - \min_x \psi(x, t). \end{aligned} \tag{8.4}$$

These two quantities give a t-dependent annular segment in which the solution lies. For example, the initial data (8.3) gives $\Delta R \sim .010$ and $\Delta\psi \sim .2$ at $t = 0$.

One expects that $\Delta R, \Delta\psi \to 0$ as $t \to +\infty$ for a stable limit cycle and other behavior for an unstable one, but difficulties arise in attempting to observe this behavior. First, ΔR becomes very small (that is, $R(x, t) \sim 1$) in all computer runs, both for stable and unstable cases, and growth or decay is most easily observed in $\Delta\psi$. Two problems occur in observing $\Delta\psi$. From § 3, the growth (or decay) rate for the larger Floquet exponent is approximately $\exp((-1 + A_0\alpha)k^2t)$. Using $k^2 = .04$ and (8.1c) with $a > 0$, the maximum growth rate occurs for $\alpha = +1$. This maximum is $\exp(.044t)$ for $a = 50$ (the value used in the calculations), a rather mild growth rate. To observe growth, then, one must integrate the equations over quite long time intervals. Furthermore, $\Delta\psi$ undergoes oscillations over each period of the limit cycle, typically varying by a factor of about 3 (for instance, for (8.2) with $a = 50$, initial data (8.3), and $\alpha = .9$, $\Delta\psi$ varies between .085 and .230 on $(0, T]$, between .079 and .215 on $(T, 2T]$, etc.). These fluctuations mean growth or decay cannot be determined by observing $\Delta\psi$ at some arbitrary sequence of times. As a measure of growth or decay, we give *maximum values of* $\Delta\psi$ over the intervals $(0, T]$, $(T, 2T]$, $(2T, 3T]$, etc. For $a = 50$ and various values of α, the successive maxima of $\Delta\psi$ are

$\alpha = .70$	.256, .233, .214, .197,
	.182, .168, .156, .145,
	.134, .125, .116, .108,
$\alpha = .80$	.243, .225, .211, .197,
	.184, .173, .162, .151,
	.142, .133, .125, .117,
$\alpha = .90$	.230, .215, .202, .190,
	.176, .169, .427, .598,
$\alpha = .99$	.218, .204, .330, .527,
	.613, .676, .674, .674.

Experimentation with different step sizes suggests the above values for $\Delta\psi$ are accurate. The linearized analysis shows instability for $.478 \lesssim \alpha \leqq 1$. For $\alpha = .99$, growth begins to show at $t \sim 3T$; for $\alpha = .90$, at $t \sim 7T$. Presumably growth would appear for $\alpha = .80$ and $\alpha = .70$ if the calculations had been extended beyond $12T$. (As mentioned above, the maximum growth rate here is $\exp(+.044t)$ for $\alpha = 1$; the exponent decreases with α.)

Of course, the data for $\alpha = .90$ and $\alpha = .99$ confirm that a (kinetically) stable limit cycle can become unstable as a solution of the fully nonlinear system.

The system (8.2) was also considered with $a = 30$, in which case the linearized result gives instability occurring for $\alpha_0 \sim 1/A_0 = .796$ (using (8.1c)). However, numerical solutions for $\alpha = .999$, initial data (8.3), and $k^2 = .04$ show no instability. In this case the full nonlinearity has completely damped out the linear growth.

9. Remarks. Of the many remaining questions on the stability of limit cycle solutions, we wish to mention two in particular.

Turing [20] gave examples of systems (1.1) with critical points stable as solutions of the kinetic equations, linearly stable to perturbations with small and large wave number k^2 but unstable to intermediate values of k^2. Here we have considered (1.1) with limit cycle solutions stable as solutions of the kinetic equations and have studied their linear stability with respect to small and large wave numbers k^2. Almost nothing is known for the case of intermediate k^2. (Kopell and Howard [12] apparently have an example of instability for intermediate wave numbers, shown by calculation.)

This paper has been entirely concerned with the linearized stability of the limit cycle solution of (1.1); a question of equal importance is: when does linear stability imply stability of the limit cycle as a solution of the original nonlinear system? Results of this type have been obtained for traveling wave-solutions $\phi(x+ct)$ in reaction-diffusion systems by Evans [8], [9] and in great generality by Sattinger [18]. However, their approaches do not seem to carry over to the limit cycle $\Phi(t)$. For instance, the standard initial step for the linearized analysis of the traveling wave $\phi(x+ct)$ is to switch to moving coordinates $y = x + ct$—the resulting linearized system then has coefficients depending only on y. For the limit cycle $\Phi(t)$, no such transformation exists and the linearized coefficients are necessarily functions of t.

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