Exercises of Discrete Mathematics

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Note for the students: the proposed solutions of some exercises are quite lengthy. This does not necessarily mean that the exercise is difficult: in fact, in most cases, a very detailed (hence lengthy) solution to an exercise considered *important* has been preferred, in order to provide a discussion of the problem with full details which, in some cases, might be a little pedant. In many cases, a shorter solution (e.g. the same solution without the discussion of obvious details etc.) may suffice and may even be preferable: with the present choice, however, every student should be able to fully understand the techniques used in the solutions, and then find the appropriate detail level on writing his/her own solutions.

Basic Counting (product rule, and splitting technique)

1. (a) How many words of 10 letters can you form, using the Italian alphabet? (there are 21 different letters in the Italian alphabet, of which 5 are vowels).

(b) How many words of length 10 can you form, where no letter is repeated?

(c) How many words of length 8 can you form, where the first letter is the same as the last letter?

(d) How many words of length 8 can you form, which contain exactly one vowel?

Solution.

(a) You have 21 different choices for the first letter, 21 choices for the second, and so on. Moreover, all choices are indipendent. From the Product Rule, it follows that there are 21^{10} different words.

(b) You can construct the decision tree for this problem. You have 21 choices for the first letter, then (no matter how you have chosen the first letter) you have 20 choices for the second (since you must avoid repetitions) and so on. Finally, you have $21 - 10 + 1 = 12$ choices for the 10th letter. Counting the leaves of the decision trees, there are

$$21 \times 20 \times 19 \times \cdots \times 12$$

different words. Another approach is the following: first, choose the 10 different letters you want to use (you can do that in $\binom{21}{10}$ different ways), then, you have 10! different ways (permutations) to write down the 10 letters you have chosen. Finally, from the Product Rule, the number of words is given by

$$\binom{21}{10} 10!$$

which coincides with the previously found number (check it!).

(c) This is easy: the first 7 letters can be chosen freely, the 8th letter must be equal to the first. Therefore the answer is 21^7.

(d) Let S be the sef of all words of length 8 which contain exactly one vowel, and let $S_i \subset S$ be the sef of all words of length 8 which contain exactly one vowel at position i. Clearly,

$$S = S_1 \cup S_2 \cup \cdots \cup S_8,$$

and the sets S_i are pairwise disjoint. Therefore,

$$|S| = |S_1| + |S_2| + \cdots + |S_8|.$$

Moreover, from the product rule it is easy to see that, for all i, $|S_i| = 5 \times 16^7$, because there are 5 choices for the vowel at position i, and $21 - 5 = 16$ choices for the letter at any position other than i. As a consequence, we have that $|S| = 8 \times 5 \times 16^7$.

2. There are 5 chairs in a row, and three people. In how many ways can you place the people on the chairs? (each person must occupy exactly one chair, so that two chairs will remain free).

3. A password for the computer is a string of digits

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

and/or letters

a,b,c,...,x,y,z

(there are 26 different letters). The password must obey the following rules:

(a) the length of the password must be exactly 8 characters;

(b) the password must contain at least one digit, and no more than three digits.

What is the number of different passwords you can form?

4. On Sunday you can either go to the cinema or to the the theatre, on each of Monday, Tuesday and Wednesday you can study either maths, physics or chemisctry, on each of Thursday, Friday and Saturday you can visit one of your 5 best friends (you must choose exactly one thing for each day).

In how many different ways can you plan your week?

5. (a) In how many different ways can you put 20 equal balls into two numbered boxes?

(b) What if you have 20 equal balls, three numbered boxes, and you want to place at least 2 balls in the first box, and no more than 10 balls in the last box?

Solution.

(a) Call A and B the two boxes. Any configuration is completely determined by the number k of balls that go into box A. As k can take any value between 0 and n included, the are $n + 1$ possible choices for k, and hence $n + 1$ possible configurations.

(b) Call A, B, C the boxes and let a, b, c be the number of balls in each box. Since $b = 20 - a - c$, any configuration is completely determined by knowledge of a and c. The constrains are

$$2 \leq a \leq 20, \quad 0 \leq c \leq 10, \quad a + c \leq 20.$$

To satisfy the first condition, there are a priori 19 possible values for a, while for the second constraint there are a priori 11 possible values for c. But we cannot apply the product rule, because possible choices for a and c are not independent due to the third constraint (for instance, $a = 15$ is a possible choice for the first constraint, and $c = 8$ is also possible for the second constraint, but this pair would violate the third constraint).

Not that, if $2 \leq a \leq 10$ (the first 9 possible values of a), and $0 \leq c \leq 10$, then the third constraint is authomatically satisfied. For these cases the product rule gives $9 \cdot 11 = 99$ configurations.

Let's count the other possible configurations. If $a = 11$, then $c = 10$ is not admissible (it would violate the third constraint), but any $c \in \{0, \ldots, 9\}$ (10 possible values for c) would be fine: thus, if $a = 11$, there are 10 possible configurations.

Similarly, if $a = 12$, there are 9 possible values for c, if $a = 13$ there are 8 possible values for c, and so on. Finally, when $a = 20$, there is only one possible value for c (namely, $c = 0$).

Summing up, the total number of configurations is given by

$$99 + 10 + 9 + \cdots + 2 + 1 = 99 + \frac{10 \cdot 11}{2} = 99 + 55 = 154.$$

6. Consider, in the Cartesian plane, the two points $A = (0, 0)$ and $B = (m, n)$, where m and n are positive integers. You want to go from A to B along a *staircase path*, i.e. along a sequence of moves (steps) of length one, each move being either "go east" or "go north" (of course you can repeat the same move more than once, for instance "m steps east, n steps north" is a valid staircase, as well as "1 step east, $n - 1$ steps north, $m - 1$ steps east, 1 step north"). What is the number of different staircase paths?

Solution. Clearly, each staircase path consists of $m+n$ moves: m horizontal moves, and n vertical moves. The path is uniquely determined if we specify what are the m horizontal moves among the $m + n$ total moves, and this can be done in

$$\binom{m+n}{m}$$

different ways.

7. A flag consists of a 1×3 rectangle, divided into three equal squares, each square being painted in one of 10 possible colours.

Moreover, two adjacent squares cannot be painted with the same colour.

Finally, a flag *may have* a coloured circle at the center of the middle square: in this case, the colour of the circle must be different from the other colours used to paint the squares.

How many different flags can be obtained, following the rules above?

Solution. We can suppose that the flagstaff is on the left. To count the possible flags, we split according to the presence of the circle.

Let's begin with the flags without the circle. There are 10 choices for the colour of the first square, then there are 9 choices for the colour of the middle square, and then 9 choices for the colour of the last square. The Product Rule gives

$$10 \times 9 \times 9 = 810$$

possible flags without the circle in the middle.

Now, let's consider flags with the circle in the middle. First, we paint the three squares, which can be done in 810 ways: for each of these flags, we can choose the colour of the circle in many ways. How many? It depends on how many colours (three or two) have already been used to paint the squares. So, we need to count how many flags, out of 810, have three colours, and how many have two colours. By the Product Rule, there are

$$10 \times 9 \times 8 = 720$$

flags with three colours, and hence there are

$$810 - 720 = 90$$

flags with two colours. This can be laso obtained in this way: first, choose two colours, out of 10, which can be done in

$$\binom{10}{2}$$

ways; then, for each pair of colours A and B, exactly two flags (ABA and BAB) can be done: this gives

$$2\binom{10}{2} = 90$$

flags with two colours.

Now, for each flag with three colours, the circle in the middle can be painted in 7 different ways, whereas for each flag with two colours, the circle in the middle can be painted in 8 different ways. Altogether, this gives

$$7 \times 720 + 8 \times 90 = 5760$$

different flags with the circle. Adding the 810 flags without circle, there are 6570 different flags.

Mathematical Induction

1. Prove the validity of the identity

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

for all $n \geq 1$.

2. Prove that

$$\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{(3n-2) \cdot (3n+1)} = \frac{n}{3n+1}$$

for every $n \geq 1$.

3. Prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$$

for every $n \geq 1$.

4. Determine, with proof, for what natural numbers n the inequality

$$n! \leq (n+1)2^n$$

is satisfied.

5. Prove that, for every $n \geq 1$,

$$\sum_{k=0}^{n} (2k+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}.$$

6. Prove that one can solve the Hanoi Towers game with n disks, in no more than $2^n - 1$ moves.

Solution. We prove the statement by induction on n. Clearly, we can solve the game in $2^1 - 1 = 1$ move if there is only one disk (induction basis). Now we prove that, if the game with n disks can be solved in $2^n - 1$ moves, then the game with $n+1$ disks can be solved in $2^{n+1} - 1$ moves, as follows:

(a) Forgetting for a while the largest disk at the bottom of the tower, in $2^n - 1$ moves we can move the n smaller disks from column 1 to say, column 2 (the largest disk will not be moved during this phase: being the largest, it will not prevent any move involving the smaller n disks, and can be ignored during this phase).

(b) Now the configuration is the following: the lasrgest disk in column 1, the n smaller disks in column 2, and no disk in column 3. Thus, now we can move the largest disk from column 1 to column 3 (which takes only one move).

(c) Again, we forget the largest disk in column three. In another $2^n - 1$ moves, we can move the whole tower made of the n smaller disks from column 2 to column 3 (as if column 3 were empty! Again, the largest disk can be ignored in this phase). The game is completed.

Observe that completing the game with this strategy takes $2^n - 1$ moves in (a), only 1 move in (b), and another $2^n - 1$ moves in (c). This means

$$(2^n - 1) + 1 + (2^n - 1) = 2 \times 2^n - 1 = 2^{n+1} - 1$$

moves, to solve the game with $n + 1$ disks.

7. Prove that the Hanoi Towers game with n disks cannot be solved in less than $2^n - 1$ moves (hence, the strategy which takes $2^n - 1$ moves is optimal).

Solution. Again we use induction. Supposing that the game with n disks cannot be solved in less than $2^n - 1$ moves (which is obvious when $n = 1$), we prove that the game with $n + 1$ disks cannot be solved in less than $2^{n+1} - 1$ moves.

For, suppose we solve the game with $n+1$ disks according to any sequence of moves M. We can split M as the union of the following sequences of moves:

1. From the beginning of the game, up to the move immediately before the first time we move the largest disk. Let's call M_1 this initial sequence of moves.

2. From the first time we move the largest disk, up to the last time we move it. There may be other moves in between, or maybe we move the larges disk only once: in any case, we must eventually move this block at least one time, so this sequence of moves (which we call M_2) is not empty.

3. From the move immediately after the last time the largest block has been moved, until the end. We call M_3 this sequence of moves.

Now we observe that, after the sequence M_1 has been played, the situation must be the following: one column is empty, the largest disk occupies one column, and the n smaller disks occupy the remaining column (this is so because, otherwise, it would be impossible to move the largest block, in the first move of the second sequence M_2). This means that, during M_1, the whole tower made of the n smaller disk has been moved from on top of the largest disk to a column that was initially empty. But by the inductive assumption, this means that the sequence M_1 consists of at least $2^n - 1$ moves. A similar argument shows that also M_3 consists of at least $2^n - 1$ moves and, as already observed, M_2 consists of at least one move. Therefore, the whole sequence M consists of at least

$$(2^n - 1) + 1 + (2^n - 1) = 2 \times 2^n - 1 = 2^{n+1} - 1$$

moves. Since M was arbitrary, the claim is proved.

The Pigeonhole Principle

1. Prove that, if you put 5 points inside a square of side-length 1, then you can always find two points whose distance is at most $\sqrt{2}/2$. Can you replace $\sqrt{2}/2$ by a smaller number?

   Solution. Divide the square into 4 smaller squares of side 1/2. From the Pigeonhole Principle, if we put 5 points (the pigeons) into the big square, then at least two of them will be pun in the same little square. But the small squares have diameter $\sqrt{2}/2$, so there are two points whose distance is at most $\sqrt{2}/2$. Finally, $\sqrt{2}/2$ cannot be replaced by a smaller number: indeed, if we place four points at the corners and the fifth point at the center of the big square, then each pair of points are a distance (no less than) $\sqrt{2}/2$ apart.

2. There are 12 chairs in a row, and 9 people sitting (so that 9 chairs are occupied, and 3 chairs are free). Prove that there are 3 consecutive chairs occupied.

Solution. Number the chairs consecutively 1 through 12, and partition them into four groups (pigeon-holes) as follows:

$$\{1, 2, 3\}, \quad \{4, 5, 6\}, \quad \{7, 8, 9\}, \quad \{10, 11, 12\}.$$

When the 9 people sit down, by the pigeoholes principle there will be at least one group of chairs containing at least $\lceil 9/4 \rceil = 3$ people, which means that the three consecutive chairs in that group will be occupied.

3. Consider the set of numbers

$$A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}.$$

Prove that, if you pick 7 numbers from A, then you can always find two of them whose sum is exactly 15.

Solution. Partition the given numbers into six groups (the pigeon-holes) as follows:

$$\{2, 13\}, \quad \{3, 12\}, \quad \{4, 11\}, \quad \{5, 10\}, \quad \{6, 9\}, \quad \{7, 8\}.$$

If you pick 7 numbers, there will be at least one of the previous groups from which you have picked at least $\lceil 7/6 \rceil = 2$ numbers. But each pair of numbers in the same group have sum equal to 15, which proves the claim.

4. In your shoe rack there are n pairs of shoes, of n different colours. Prove that, if you pick $n + 1$ shoes at random, then you can find at least one pair of the same colour.

What is the least number of shoes that you should pick, to be sure that you can find (at least) two pairs of the same colours? And three pairs? Generalize.

5. There are 8 different courses available, and each student must choose 5 courses to put in his/her plan of studies. What is the minimum number of students such that, no matter what they choose, there will be at least 10 studentws with the same plan?

Solution. There are $\binom{8}{5} = 56$ possible different plans (the pigeon-holes). If there are n students, then from the pigeon-holes principle there will be at least $\lceil n/56 \rceil$ students with the same plan, and the least n such that $\lceil n/56 \rceil \geq 10$ is $n = 9 * 56 + 1 = 505$.

6. Prove that, in any group of 900 people, at least three have the same birthday.

7. If the score of each exam is A, B or C, what is the least number of exams you have to take such that, no matter what your scores are, there will be at least 10 exams with the same score?

Binomial coefficients

1. Let A be a finite set with cardinality $|A| = n$. Prove that A has exactly $\binom{n}{k}$ subsets of cardinality k (here, n and k are arbitrary integers such that $0 \leq k \leq n$).

2. Compute the following expressions:

(a) $\displaystyle\sum_{k=0}^{27} \binom{27}{k} (-3)^{2k+1}$.

(b) $\displaystyle\sum_{k=0}^{27} k \binom{27}{k} 3^{2k}$.

(hint: use Newton's binomial and its derivative.)

Solution. From Newton's binomial expansion we have that

$$(1+x)^{27} = \sum_{k=0}^{27} \binom{27}{k} x^k. \tag{1}$$

Now, observe that

$$\sum_{k=0}^{27} \binom{27}{k} (-3)^{2k+1} = -3 \sum_{k=0}^{27} \binom{27}{k} (-3)^{2k} = -3 \sum_{k=0}^{27} \binom{27}{k} 9^k.$$

Now, letting $x = 9$ in (1), we find that

$$10^{27} = \sum_{k=0}^{27} \binom{27}{k} 9^k,$$

and therefore

$$\sum_{k=0}^{27} \binom{27}{k} (-3)^{2k+1} = -3 \times 10^{27}.$$

To compute the expression in (b), we differentiate (1) with respect to x, obtaining

$$27(1+x)^{26} = \sum_{k=0}^{27} \binom{27}{k} k x^{k-1}$$

(this has the advantage of having k multiply the binomial coefficient within the summation symbol). Again, letting $x = 9$, we have that

$$27 \times 10^{26} = \sum_{k=0}^{27} \binom{27}{k} k 9^{k-1},$$

and hence

$$\sum_{k=0}^{27} k \binom{27}{k} 3^{2k} = \sum_{k=0}^{27} k \binom{27}{k} 9^k = 9 \sum_{k=0}^{27} k \binom{27}{k} 9^{k-1} = 9 \times 27 \times 10^{26} = 243 \times 10^{26}.$$

3. Compute the following expressions:

(a) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} (-1)^k \ (n \geq 1).$

(b) $\displaystyle\sum_{k=0}^{n} k \binom{n}{k} (-1)^k \ (n \geq 2).$

(hint: use Newton's binomial and its derivative.)

Solution. Letting $x = -1$ in the

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \tag{2}$$

one immediately obtains that

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0 \quad \forall n \geq 1.$$

To compute the expression in (b), we differentiate (2) with respect to x, obtaining

$$n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}$$

(this has the advantage of having k multiply the binomial coefficient within the summation symbol). Letting $x = -1$ in the last equation, we obtain that

$$\sum_{k=0}^{n} \binom{n}{k} k(-1)^{k-1} = 0$$

and hence

$$\sum_{k=0}^{n} k \binom{n}{k} (-1)^k = -\sum_{k=0}^{n} k \binom{n}{k} (-1)^{k-1} = 0.$$

Recurrence relations

1. Determine the solution of the linear recurrence relation:

$$a_{n+2} = 5a_{n+1} - 6a_n + n\,, \quad a_0 = a_1 = 1.$$

2. Determine the solution of the linear recurrence relation:

$$a_{n+2} = 5a_{n+1} - 6a_n + 2\,, \quad a_0 = a_1 = 1.$$

3. Find the solution of the recurrence relation

$$\begin{cases} a_{n+1} = a_{n-1} + 2^n \\ a_0 = a_1 = 0 \end{cases}$$

4. Find the solution of the recurrence relation

$$\begin{cases} a_{n+1} = -2a_n + 3a_{n-1} + n - 1 \\ a_0 = a_1 = 0 \end{cases}$$

Counting using recurrence relations

1. Let a_n denote the number of all possible words of length n, on the alphabet $\{a, b, c\}$, which do *not* contain the substring "aa".

(a) Prove that the numbers a_n satisfy the recurrence relation

$$a_{n+1} = 2a_n + 2a_{n-1}, \quad n = 2, 3, \ldots$$

(b) Using a), find an explicit formula for a_n.

Solution. It is convenient to draw the decision tree for the task "write an admissible word of length $n + 1$". The first decision concerns the choice of the first letter, which can be "a", "b" or "c", therefore below the root the decision tree splits into three subtrees:

(a) "a" is the first letter. In this case the second letter can be either "b" or "c" (not "a"), hence this subtree will in turn split into two subtrees, corresponding to the possible ways you can complete the word (with $n - 1$ more letters) having started, respectively, with "ab" or "ac"; it is clear that, having started with "ab" or "ac", the task "complete the word with the remaining $n - 1$ letters" is equivalent to the task "write an admissible word of $n - 1$ letters". Therefore, each of this subtrees will have a_{n-1} leaves. To sum up, if the first letter is "a", then you can complete the word in $a_{n-1} + a_{n-1} = 2a_{n-1}$ different ways.

(b) "b" is the first letter. In this case, the task "complete the word with the remaining n letters" is equivalent to the task "write an admissible word of n letters". Hence this subtree will have a_n leaves.

(c) "c" is the first letter. Exactly as in the previous case, this subtree will have a_n leaves.

Thus the decision tree for the task "write an admissible word of length $n+1$" will have $2a_{n-1}+2a_n$ leaves, which proves the recurrence relation

$$a_{n+1} = 2a_n + 2a_{n-1}, \quad n = 2, 3, \ldots$$

To find a_n, it suffices to find the initial conditions. There are 3 admissible words of length 1, namely "a", "b" and "c", and 8 admissible words of length 2, namely the $3^2 = 9$ possible words minus the one forbidden word "aa", and hence the initial conditions are $a_1 = 3$, $a_2 = 8$. However, it is more convenient to set $a_0 = 1$ and $a_1 = 3$ (note that in this case $a_2 = 2a_1 + 2a_0 = 8$ follows authomatically from the recurrence relation), because this will lead to much simpler computations when solving the recurrence relation.

2. Describe the solution of the previous exercise without drawing the decision tree (perform a "splitting according to the first/second letter").

3. Let w_n denote the number of all possible words of length n, on the alphabet $\{D, S, U\}$, which do *not* contain the substrings "SD" and "DS".

(a) Prove that the numbers w_n satisfy the recurrence relation

$$w_{n+1} = 2w_n + w_{n-1}, \quad n = 2, 3, \ldots$$

(b) Using a), find an explicit formula for w_n.

Solution. For brevity, we will call an *n-word* any word of length n which is admissible.

Considering the task "write an $(n+1)$-word", we perform a splitting according to the choice of the first letter:

(a) If the first letter is U, then the remaining part of the word can be any n-word, i.e. there are w_n different ways to complete an $(n+1)$-word which begins with U.

(b) If the first letter is D, then the word can be completed by any n-word *with the exception* of those n-words which begin with S.

(c) If the first letter is S, then the word can be completed by any n-word *with the exception* of those n-words which begin with D.

Combining these cases with the Splitting Rule, we obtain the recursion

$$\begin{aligned} w_{n+1} &= w_n + (w_n - \#\text{ of } n\text{-words which begin with } S) + \\ &\qquad (w_n - \#\text{ of } n\text{-words which begin with } D) \\ &= 3w_n - (\#\text{ of } n\text{-words which begin with } S \text{ or } D) \\ &= 3w_n - (w_n - \#\text{ of } n\text{-words which begin with } U) \\ &= 2w_n + (\#\text{ of } n\text{-words which begin with } U). \end{aligned}$$

But the number of n-words which begin with U is exactly w_{n-1} (why?), hence we find the recursion

$$w_{n+1} = 2w_n + w_{n-1}$$

as required.

To find the initial conditions, observe that $w_1 = 3$ (the possible 1-words being "D", "S" and "U") and that $w_2 = 7$ (there are $3^2 = 9$ possible words of length 2, all of which are admissible except the two forbidden words "SD" and "DS"). However, it is convenient to set $w_0 = 1$ and $w_1 = 3$ (note that, with this choice, $w_2 = 7$ follows authomatically from the recurrence relation) to simplify the computations, when we solve the linear recurrence.

4. Consider a rectangle R of size $2 \times n$, where $n \geq 1$ is a natural number. You must *tile* (that is, fill without overlapping) R using rectangular bricks of size 2×1: each brick can be placed with the long side either vertically or horizontally, and the bricks cannot overlap or get out of R. In how many different ways can you fill R? (Hint: call a_n the number of ways you can tile R, and find a recurrence relation for a_n.)

Solution. Draw R as a rectangle with base n and height 2, suppose $n \geq 3$, and consider the task "fill a rectangle $2 \times n$". We perform a splitting according to the way we place the brick which covers the lower-left corner of the rectangle. This brick can be put either vertically or horizontally:

(a) The brick which covers the lower-left corner is placed vertically. In this case, the brick covers the lower-left and the upper-left corner of R. To complete the task, it remains to cover (in some way) a rectangle of size $2 \times (n-1)$, and this can be done in exactly a_{n-1} different ways.

(b) The brick which covers the lower-left corner is placed horizontally. In this case, in order to cover the upper-left corner of R, it is necessary to place another brick horizontally, just above the first brick. This two bricks cover a 2×2 square on the left, and to complete the task it remains to cover (in some way) a rectangle of size $2 \times (n-2)$, and this can be done in a_{n-2} different ways.

To sum up, we have obtained the recurrence relation $a_n = a_{n-1} + a_{n-2}$, for every $n > 2$ (note that this is the same recurrence relations satisfied by the Fibonacci numbers). Now, when $n = 1$, there is only one way to tile the 2×1 rectangle (with just one brick!), hence $a_1 = 1$; on the other hand, if $n = 2$, one can fill the 2×2 rectangle in two different ways (with two bricks both placed vertically, or both placed horizontally), hence $a_2 = 2$. Now observe that, if we set $a_0 = 1$ for convenience, then $a_2 = 2$ follows from the recurrence relation, and we obtain that a_n coincides with the n-th Fibonacci number (because the recurrence relation as well as the initial conditions are the same as for the Fibonacci numbers).

5. You want to form a colourful rod of length n, by placing side by side segments of length 1, whose colour can be red or greeen, and segments of length 2, whose colour can be blue, white or black (you can use segments of any kind in any order, with no rule, except for the total length which must be n). Prove that, if a_n denotes the number of different rods of length n that you can compose, then the recurrence relation

$$a_n = 2a_{n-1} + 3a_{n-2}$$

holds for all $n > 2$. Can you determine a_n explicitly?

Solution. There are a_n rods of length n, and we count them by splitting according to the length of the first segment.

(a) If the first segment has length 1, the remaining part of the rod is a rod of length $n - 1$, and there are a_{n-1} such rods. But the first segment can be red or green, thus there are $2a_{n-1}$ rods of length n whose first segment has length 1.

(b) If the first segment has length 2, the remaining part of the rod is a rod of length $n - 2$, and there are a_{n-2} such rods. But the first segment can be blue, white or black, thus there are $3a_{n-2}$ rods of length n whose first segment has length 2.

From the Splitting Rule, we find that $a_n = 2a_{n-1} + 3a_{n-2}$ as required. Then a_n can be computed as usual, solving the recurrence relation.

6. Using rectangular tiles of size 3×1 and 3×2, it is clearly possible to tile a rectangle of size $3 \times n$ for arbitrary $n \geq 1$, in several ways (each rectangular tile can be placed with the long side either vertical or horizontal). Let a_n denote the number of different ways you can tile a $3 \times n$ rectangle.

(a) Prove that

$$a_n = a_{n-1} + a_{n-2} + 3a_{n-3} \quad \forall n \geq 4.$$

(Hint: split, according to how you cover the lower-left corner of the rectangle.)

(b) Determine explicitly (by considering all possible cases) the initial conditions a_1, a_2 and a_3.

7. Let w_n denote the number of words of length n, formed using the Italian alphabet $\{a, b, c, \ldots, u, v, z\}$, which do *not* contain the substrings "aa", "ee", "ii", "oo", "uu". Find a recursion and an explicit formula for w_n.

Solution. It is convenient to call n-word any valid word of length n, and let c_n denote the number of n-words which begin with a consonant. Similarly, we let a_n, e_n, i_n, o_n, u_n denote the number of n-words which begin, respectively, with "a", "e" etc.

Looking for a recursion, note that we clearly have

$$w_{n+1} = c_{n+1} + \big(a_{n+1} + e_{n+1} + i_{n+1} + o_{n+1} + u_{n+1}\big). \tag{3}$$

To eliminate c_{n+1}, note that

$$c_{n+1} = 16w_n \tag{4}$$

(because there are 16 consonants and, having written any consonant, we can complete the $n+1$-word by writing any n-word after it). Similarly, we have

$$a_{n+1} = w_n - a_n$$

because, having written "a", we can complete the $n + 1$-word using any n-word, except those n-words which begin with "a". Similar recurrences hold for e_{n+1} etc.; substituting into (3), we obtain

$$\begin{aligned} w_{n+1} &= 16w_n + \big(w_n - a_n + w_n - e_n + \cdots + w_n - u_n\big) \\ &= 21w_n - \big(a_n + e_n + \cdots + u_n\big) \\ &= 21w_n - \big(w_n - c_n\big) \\ &= 20w_n + c_n. \end{aligned}$$

To eliminate c_n, we use (4) with $n-1$ in place of n, thus obtaining

$$w_{n+1} = 20w_n + 16w_{n-1}$$

which is the required recurrence relation. The initial conditions are $w_1 = 21$ (any word of length 1 is valid) and $w_2 = 21^2 - 5$ (why?). Alternatively, one can work with $w_0 = 1$ (there is only one word of length zero: the empty string!) and $w_1 = 21$. In either case, the explicit formula for w_n can be found in the usual way.

Algorithms on graphs and trees

1. Draw a "reasonable" connected, weighted graph with 7 vertices, and simulate the Prim's algorithm on it, to find the cheapest spanning tree.

2. As before, for the Kruskal's algorithm.

3. As before, for the Reverse-Delete algorithm.

4. As before, for the Dijkstra's algorithm (choose an arbitrary source).

5. In the previous graphs, adjust the weights in such a way that they satisfy the triangle inequality, and construct a closed tour of the graph, whose cost is less than twice the cost of the cheapest hamiltonian tour (Hint: find the cheapest spanning tree, build a tour from it, and optimize using the triangle inequality).

6. Draw some "reasonable" graphs with, say, 6 or 7 vertices. For each of them, answer the following questions:

(a) What are the degrees of the vertices?

(b) Is there an Eulerian path?

(c) Is there an Eulerian circuit?

(d) When the answer is affirmative, construct an Eulerian path/circuit using the Fleury's algorithm.

7. (a) Draw some trees with vertices labelled $1, 2, \ldots, n$, and for each of them find its Prüfer's code.

(b) Choose $n > 2$, and write down some sequences of length $n-2$, using the numbers $1, 2, \ldots, n$ as you like (repetitions are allowed). Then interpret each sequence as the Prüfer's code of a tree, and reconstruct the tree.

Graph theory

1. Is there a graph with 102 vertices, such that exactly 49 vertices have degree 5, and the remaining 53 vertices have degree 6?

Solution. No, because we know that for any graph with n vertices and e edges,

$$\deg(1) + \cdots + \deg(n) = 2e.$$

In our case this formula would reduce to

$$49 \cdot 5 + 53 \cdot 6 = 2e,$$

which is impossible since the sum on the left is an odd number, whereas $2e$ is in any case an even number.

2. Let K_n be the complete graph with n vertices. For what values of n does K_n admit an Eulerian circuit?

Solution. Clearly, every vertex of K_n has degree equal to $n-1$. Since K_n is connected, we know that it has an Eulerian circuit *if and only if* each of its vertices has even degree. Therefore, K_n has an Eulerian circuit if and only if n is odd.

3. For $n \geq 1$, the *hypercube* H_n is a graph (V_n, E_n) with 2^n vertices, constructed as follows. Label the vertices with the integers from 0 to $2^n - 1$; then, if $i, j \in V_n$, then $(i, j) \in E_n$ if and only if the expansions of i and j in base two differ exactly in one single bit (note that each $i \in V$ can be written in base two using exactly n bits). For example, if $n = 3$, then $V = \{0, 1, 2, 3, 4, 5, 6, 7\}$, and $(5, 7)$ is an edge of H_3, because $5 = (101)_2$ and $7 = (111)_2$ differ exactly in one bit; similarly, $(5, 6)$ is not an edge of H_3, because $5 = (101)_2$ and $6 = (110)_2$ differ in two bits.

(a) Write explicitly the sets (V_n, E_n) for $n = 1, 2, 3, 4$, and represent graphically H_n for $n = 1, 2, 3$.

(b) What are the degrees of the vertices of H_n?

(c) How many edges are there in H_n?

(d) For what values of n does H_n admit an Eulerian path? And an Eulerian circuit?

(e) Can H_n have a cycle of length 3?

(f) For what values of n is H_n a planar graph?

Solution.

(a) (V_1, E_1) consists of two vertices joined by one edge, (V_2, E_2) coincides with C_4 (the standard cycle of length 4), and (V_3, E_3) is the graph obtained from the eight vertices and the twelve edges of a cube.

(b) Each vertex i of H_n is an integer such that $0 \leq i \leq 2^n - 1$, which can be uniquely represented in base two by a sequence $(i)_2$ of n bits. There is an edge from i to j if and only if the base-two representation of j differs from the base-two representation of i in exactly one bit: this bit can be any of the n bits of $(i)_2$, hence there are n possible choices for j. This proves that $\deg(i) = n$ for all $i \in V_n$.

(c) We know that, in any graph (V, E),

$$\sum_{i \in V} \deg(i) = 2|E|.$$

In our case (V_n, E_n), $|V_n| = 2^n$ and we have seen that $\deg(i) = n$ for all $i \in V_n$, therefore the identity above reduces to

$$n2^n = 2|E_n|$$

which gives

$$|E_n| = n2^{n-1}.$$

(d) Since H_n is connected, the existence of an Eulerian circuit is equivalent to all degrees being even, thus there is an Eulerian circuit if and only if n is even.

Similarly, the existence of an Eulerian path is equivalent to all degrees being even (in which case the Eulerian path is an Eulerian circuit), or all degrees being even except for two degrees being odd. Since in our case all the degrees are equal to n and there are 2^n vertices, the latter case occurs only when $n = 1$.

(e) The answer is no. Indeed, consider three distinct vertices i, j, k such that $(i, j) \in E_n$ and $(j, k) \in E_n$. From $(i, j) \in E_n$ we deduce that the two binaries representations $(i)_2$ and $(j)_2$ differ exactly in one bit, say, in position p_1. Similarly, $(j, k) \in E_n$ means that the representations $(j)_2$ and $(k)_2$ differ exactly in one bit, say, in position p_2. Then $p_1 \neq p_2$, otherwise we would have that $i = k$. But then $(i, k) \notin E_n$, because $(i)_2$ and $(k)_2$ differ at least in two bits (one in position p_1, the other in position p_2). So, three distinct vertices i, j, k cannot form a triangle.

(f) One can easily check that H_1, H_2 and H_3 are planar, so we consider H_n with $n \geq 4$. Note that H_n is clearly connected, and for co-nencted planar graphs we know that necessarily $|E| <= 3|V| - 6$. In our case, since $|E_n| = n2^{n-1}$ and $|V_n| = 2^n$, this inequality becomes

$$n2^{n-1} \leq 3 \cdot 2^n - 6$$

and, dividing by 2^{n-1}, we would have that

$$n \leq 3 \cdot 2 - \frac{6}{2^{n-1}} < 3 \cdot 2 = 6,$$

which means that H_n is not planar if $n \geq 6$.

It remains to study the planarity of H_4 and H_5, for which the previous inequality (which is only *necessary* for planarity) is true.

But (see the previous question) H_n has no cycle of length 3, and we know that such graphs *must* satisfy the inequality $|E| <= 2|V| - 4$ *if* they are planar and connected. In our case this inequality becomes

$$n2^{n-1} \leq 2 \cdot 2^n - 4$$

and, after simplification,

$$n \leq 2 \times 2 - \frac{4}{2^{n-1}} < 4.$$

This means that H_n cannot be planar if $n \geq 4$. Hence, H_n is planar if and only if $n \in \{1, 2, 3\}$.

4. Let G be the graph obtained from the sides of a regular tetrahedron, in the obvious way. Does G admit an Eulerian circuit?

Solution. A tetrahedron has 6 edges and 4 vertices, each of multiplicity 3 which is odd. Therefore, there is no Eulerian circuit.

5. As before, with the octahedron.

Solution. An octahedron has 12 edges and 6 vertices, each of multiplicity 4 which is even. Therefore, it has an Eulerian circuit.

6. Let $V = \{1, 2, 3, 4, 5\}$.

(a) How many different graphs can you form, with V as set of vertices?

(b) How does the answer to the previous question change, if you allow self-loops? And if you consider oriented graphs, with or without self loops?

(c) How many different graphs can you form, with V as set of vertices, such that $\deg(1) = 2$?

(d) Consider the complete graph with V as set of vertices. How many different hamiltonian cycles are there? (two cycles must be considered equal if they coincide after being closed, e.g. the two cycles 1-2-3-4-5-1 and 3-4-5-1-2-3 are equal).

(e) * Consider again the graph in (d). Can you count the number of different Eulerian circuits?

Solution.

(a) The set of vertices V is given; to form a graph, one has to specify what are the vertices of the graph. Let E_p be the set of all *possible* edges (note that the graph (V, E_p) is the complete graph with 5 vertices). Specifying the set of edges in the graph is equivalent to choosing a subset of E_p, and this can be done in

$$2^{|E_p|}$$

different ways. But E_p is the set of all unordered pairs of distinct vertices, and hence

$$|E_p| = \binom{5}{2} = 10.$$

Therefore, the number of graphs that you can form is 2^{10}.

(b) If we admit self loops, the answer is still $2^{|E_p|}$, but now E_p must also contain all possible self loops. There are five possible self loops (one for each vertex), and hence

$$|E_p| = \binom{5}{2} + 5 = 15.$$

The number of different graphs, possibly with self loops, is therefore 2^{15}.

If we consider *oriented* graphs (without self loops), then

$$|E_p| = 2\binom{5}{2} = 20$$

(because each unoriented edge corresponds to two possible oriented edges), hence the answer is 2^{15} in this case.

Finally, considering *oriented* graphs *with* possible self loops, we have

$$|E_p| = 2\binom{5}{2} + 5 = 25$$

(because there are five possible self loops), and the answer is 2^{25} in this case.

7. A dodecahedron is a regular solid with 12 faces, each face being a regular pentagon. How many edges are there? And how many vertices? (Hint: use combinatorics to count edges, and the Euler formula to count vertices)

Solution. Let e, v denote the number of edges and vertices. Each edge is shared by two pentagonal faces, and each face has 5 edges. Thus, if we sum the number of edges for each face, we get a total of $5 \cdot 12 = 60$. But in this sum, each edge has been counted twice (overcounting), that is, $2e = 60$ and hence $e = 30$. Now the Euler formula tells us that

$$v - e + f = 2.$$

Since $e = 30$ and $f = 12$, we conclude that $v = 20$.

8. The shape of a soccer ball can be represented by a polyhedron, called the *truncated icosahedron*, whose faces are 12 regular pentagons and

Figure 1: A soccer ball and the corresponding polyhedron.

20 regular hexagons, having the same side length. Each pentagon is surrounded by 5 hexagons, whereas each hexagon is surrounded by 3 pentagons and 3 hexagons (see Figure 1).

How many sides are there? And how many vertices? (Hint: use combinatorics to count edges, and the Euler formula to count vertices)

Solution. Each side is shared by either two hexagons, or by one hexagon and one pentagon. There are 12 pentagons, each with 5 sides, and 20 hexagons, each with 6 sides: therefore, summing the number of sides of each face, we get a total of

$$12 \cdot 5 + 20 \cdot 6 = 180$$

edges, but each edge is being counted twice (overcounting), so the actual number of edges is $e = 180/2 = 90$. Now, from the Euler formula, we have that

$$v - e + f = 2.$$

Since $e = 90$ and $f = 12 + 20 = 32$, we conclude that there are $v = 60$ vertices.

9. A number $n \geq 3$ of persons meet at a party, and some of them shake hands. It is known that at least one person does *not* shake hands with everybody (that is, maybe he/she shakes hands with someone, but not with everyone). What is the maximum number of persons that may have shaken hands with everyone?

Solution. We can consider the graph of handshakings (V, E), with set of vertices $V = \{1, 2, \ldots, n\}$ corresponding to the n persons, and edges $(i, j) \in E$ if and only if person i has shaken hands with person j. Then, the degree of a node $\deg(i)$ corresponds to the number of persons, that i has shaken hands with.

The problem, from the point of view of graph theory, becomes the following: a graph (V, E) with $V = \{1, 2, \ldots, n\}$ is *not* the complete graph (that is, there exists at least one pair $i \neq j$ such that $(i, j) \notin E$). What is the largest possible number k of vertices of G, having degree equal to $n - 1$?

It is clear that the answer is obtained when G is the complete graph K_n with one edge, say $(1, 2)$, *removed.* For this graph we have

$$\deg(1) = \deg(2) = n - 2, \quad \deg(i) = n - 1 \quad i = 3, \ldots, n,$$

and there are $n - 2$ vertices with maximum degree. The anser is therefore $k = n - 2$.

(Note that $k = n - 1$ is impossible: explain why.)

10. Given a set V of l elements ($l \geq 3$), find the number of different graphs which have the structure of a cycle of length l, and have V as set of vertices.

Solution. We only have to decide in what order the l vertices appear in the cycle. Any permutation of the l elements corresponds to exactly one possible cycle: but the number $l!$ of permutations *overcounts* the number of cycles, because each cycle corresponds to several permutations. More precisely, given a cycle, we can associate to it a unique permutation by choosing one vertex of the cycle (which we consider the *first* element of the permutation), and then choosing an orientation (clockwise or couterclockwise) to enumerate the vertices starting from the first. There are l possible choices for the first element, and 2 possible choices for the orientation: thus, any cycle corresponds to exactly $2l$ permutations. In this way, we can compensate the overcounting, and we deduce that the number of different cycles over l given vertices is given by

$$\#\text{ of different cycles over } l \text{ given vertices} = \frac{l!}{2l} = \frac{(l-1)!}{2}.$$

11. Given $n \geq 4$, find the number of graphs $G = (V, E)$ with set of vertices $V = \{1, 2, \ldots\}$ having the following structure: a cycle (whose length is *not* prescribed), with a nonempty chain (the "tail") attached to the cycle at one endpoit.

Solution. Given $n \geq 4$, the process of assembling a graph with the required properties can be described as follows:

(a) We fix the length l of the cycle. Clearly, $l \geq 3$ and $l \geq n-1$, because at least one vertex must belong to the tail.

(b) We choose the l vertices which form the cycle. This can be done in $\binom{n}{l}$ ways.

(c) We decide in what order the l chosen vertices appear in the cycle. From the previous exercise, we know that the number of choices is given by $l!/(2l)$.

(d) At this stage, the cycle of length l has been fixed, and the remaining $n-l$ vertices will form the tail. We have to decide *to what vertex of the cycle* the tail will be attached, and the number of possible choices is l.

(e) It remains to choose the order in which the $n-l$ elements of the tail will be attached to the cycle. Clearly, the number of possible choices is given by $(n-l)!$.

Note that step (a) corresponds to a splitting according to the length l of the cycle. After l has been chosen, we can apply the product rule to steps (b) through (e). Summing up, the number of graphs is given by

$$\sum_{l=3}^{n-1} \binom{n}{l} \cdot \frac{l!}{2l} \cdot l \cdot (n-l)!.$$

Expanding the binomial coefficient and simplifying, this expressione reduces to

$$\sum_{l=3}^{n-1} \frac{n!}{2} = \frac{(n-3)n!}{2}, \tag{5}$$

which suggests another possible solution to the problem.

Alternative solution. Instead of first constructing the cycle of length l and the attaching to it the tail of length $n-l$, we can construct the graph starting from a chain of length n, and then closing the chain (i.e., attaching the end of the chain, by one extra edge, to some vertex in the middle of the chain). Try to do that, in a concrete case.

This corresponds to a complete description of the graph, based on its *Eulerian paths* which start at the tail endpoint. More precisely, take a graph with the given structure, and try to move along an Eulerian path which starts at the tail endpoint. Clearly, all the steps are forced until we touch the cycle: at this stage, one must *choose* how to walk along the cycle (clockwise, or counterclockwise). Once this decision has been made, the Eulerian path is again forced, until the cycle has been walked along completely (and the Eulerian path ends, at the junction between the cycle and the tail: note that the junction and the tail endpoint are the only two vertices with odd degree). Thus, there are exactly *two Eulerian paths* which start at the tail endpoint, and knowledge of *one* of them completely determines the graph. The reader ir suggested to check this in some concrete examples.

Summing up, we can first count all possible Eulerian paths (that is, all possible chains closed onto themselves somewhere in the middle), and then divide by two to compensate overcounting (because each graph with the required structure has exactly two such Eulerian paths).

Thus, we proceed as follows:

(a) We form an oriented chain $v_1 \to v_2 \to \cdots \to v_n$ using all n vertices (and $n-1$ edges in between). Clearly, this can be done in $n!$ different ways. The idea is that vertex v_1 will be the end of the tail, whereas vertex v_n will be used to "close the chain".

(b) Now we close the chain. More precisely, by adding one more edge we connect vertex v_n to one of the previous vertices, say v_j, in the chain. Clearly, $j \neq 1$ (because we would obtain a cycle of length n, without the tail), $j \neq n$ (because this would produce a self loop), and $j \neq n-1$ (because the edge (v_n, v_{n-1}) already belongs to the chain). Thus, we must have $j \in \{2, n-2\}$, and hence there are $n-3$ possible choices for j (that is, $n-3$ possible ways to close the chain). Note that $j-1$ is the length of the resulting tail.

Using the product rule, we see that there are $n! \cdot (n-3)$ ways to perform steps (a) and (b). This is the number of possible Eulerian paths (in all possible graphs with the given structure) which start at the tail endpoint: as discussed above, this number is *twice* the number of graphs, hence dividing by 2 we obtain the answer to the question, in accordance with the expression (5) previously found.

Although the solution is complete, we spend a few more words about the *overcounting factor* 2 which appeared in this alternative solution. As we have already discussed, by performing steps (a) and (b) one obtains a unique graph G with the required structure. On the other hand, the same graph G can be obtained in exactly *two different ways* as the result of steps (a)(b) (i.e., overcounting by a factor 2).

This is best seen by an example. Take $n = 6$ and choose the permutation $1, 2, 3, 4, 5, 6$ at step (a); then, choose $j = 3$ at step (b), that is, close the chain by adding the edge $(6, 3)$. This sequence (a)(b) leads to a graph with cycle

$$3 \to 4 \to 5 \to 6 \to 3,$$

and tail 1–2–3 attached to it. But the *same graph* is obtained if we choose the permutation $1, 2, 3, 6, 5, 4$ at step (a) (and we keep $j = 3$ at step (b), that is, we close the chain by adding the edge $(4, 3)$).

More generally, any permutation $v_1, v_2, \ldots, v_n$ chosen at step (a) and any choice of $j \in \{2, \ldots, n-2\}$ at (b) will produce the *same result* as the permutation

$$v_1, v_2, \ldots, v_j, v_n, v_{n-1}, \ldots v_{j+1}$$

chosen at step (a) followed by the same choice of j at step (b).

Note that the second permutation is obtained from the first in this way: the first j elements are the same, whereas the last $n - j$ elements appear in reversed order. These two permutations correspond to the two possible ways (clockwise, counterclockwise) we can walk along the cycle, when we follow one of the two Eulerian paths which start at the tail endpoint.

Complexity

1. The following procedure `Procedure Asterisk(n:integer)` will print a sequence of asterisks

$$* * * \cdots\cdots\cdots * **$$

on the screen:

```
Procedure Asterisk(n:integer);
begin
  if n=1 then write(``*'');
  if n=2 then write(``***'');
  if n>2 then
    begin
      write(``**'');
      Asterisk(n-1);
      Asterisk(n-1);
      Asterisk(n-2);
    end;
end.
```

Let a_n denote the number of asterisks printed by `Asterisk(n)`, when n is a positive integer.

(a) Prove that

$$a_n = 2a_{n-1} + a_{n-2} + 2 \quad \forall n > 2.$$

(b) Find a_n explicitly.

Solution. If `Asterisk(n)` is executed with $n > 2$, then the first two `if` tests will fail, whereas the third will be successful. As a consequence, the instructions

```
      write(``**'');
      Asterisk(n-1);
      Asterisk(n-1);
      Asterisk(n-2);
```

will be executed, thus printing $2 + a_{n-1} + a_{n-1} + a_{n-2}$ asterisks. As a consequence, we find that

$$a_n = 2a_{n-1} + a_{n-2} + 2 \quad \forall n > 2$$

as required. This is a nonhomogeneous linear recurrence with two terms and, since the nonhomogeneous term is a constant, we look for a particular solution which is constant:

$$a_n = C \quad \forall n.$$

The linear recurrence reduces to $C = 3C + 2$, i.e. $C = -1$, and indeed, one can check that $a_n = -1$ is a particular solution of the linear recurrence.

Now the characteristic equation of the linear recurrence is $x^2 = 2x + 1$, which has two distinct roots $1 \pm \sqrt{2}$. Thus the general solution of the homogeneous equation is

$$A(1 + \sqrt{2})^n + B(1 - \sqrt{2})^n.$$

Since $a_n = -1$ is a particular solution of the nonhomogeneous recurrence, we obtain that

$$a_n = -1 + A(1 + \sqrt{2})^n + B(1 - \sqrt{2})^n,$$

where A and B must be determined in order to match the initial conditions. Note that `Asterisk(1)` will print one asterisk, whereas `Asterisk(2)` will print three asterisks, thus the two initial conditions are $a_1 = 1$, $a_2 = 3$. Finally, an easy computation shows that $A = 1/\sqrt{2}$ and $B = -1/\sqrt{2}$ (the details are left to the reader).