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# MAIWAR (MODELLING AUSTRALIAN INDUSTRY WITH WEAK-SOLUTIONS, AMPL AND REGIONS) <sup>1</sup>

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Policy makers make decisions regarding medium term policies where capital is evolving. They do so in the face of uncertainty. Existing Computable General Equilibrium models do not adequately model such intertemporal behaviour. Macroeconomic models typically address this concern, but rely on local approximations around the non-stochastic steady-state (where capital is no longer evolving). Ergodic-set methods are similarly suited to the long-term whereas grid-based value-function iteration is unstable or slow in multi-sectoral settings.

In this white paper, we extend a simple, yet powerful, method (of Cai and Judd) to the multisectoral setting. Along uncertain paths through time, agents look ahead multiple periods and choose policies that are optimal the certainty-equivalent sense. The model focuses on the medium term and the solution we derive is *weak*: characterised by an empirical distribution on the set of paths (unobservable strong solutions). Similar to structural econometric models, the key measure of accuracy is that the mean path satisfies the Intertemporal Euler equation to within a 1% margin of error.

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From the past, the present acts prudently, lest it spoil future action.

*Titian: Allegory of Prudence*

## 1. INTRODUCTION

Experience is the basis of prediction. Yet outside of stylised settings, even the most experienced forecasters do not claim access to “the full model”. In the methodology below, we will argue that, for incomplete model specifications, weak solutions, that is ones that explicitly uncertainty of various kinds and are characterised by an empirical probability distribution are more suitable. Yet this kind of solution is absent from the literature: certainly from the Computational General Equilibrium (CGE) literature, and the related multi-sectoral macroeconomic literature. The weak solution we derive is implicit in Cai and Judd (2021): though they do not consider the case of multi-sectoral investment flows that is central to MAIWAR.

Beyond the short-term, economic theory provides the lens through which policy-makers peer into the future. In medium term, the economy is unlikely to reach its steady state. Moreover, recent research has highlighted the importance of nonlinear effects. For such problems, first-order steady-state approximations are ill-suited. Yet this is the solution method that is most commonly applied today.

Computable General Equilibrium (CGE) models are often used to provide insights into the regional and sectoral impacts of current policies and shocks that propagate through the economy’s production networks over time. Models that assume steady-state approximations are better suited to long-term policy questions precisely because steady-state equilibria are a special kind of equilibrium that may take many years to arrive. Yet even the impact of longer-term policies, such as 2050 net-zero emissions targets should involve a careful analysis of the short-to-medium term. For a policy that is unsustainable or sub-optimal in the interim is liable to fail altogether.

*At this point, this white paper only contains a draft methodology.*

## 2. METHODOLOGY

The goal of our method is to generate statistical information about the path that key regional variables such as gross value added, employment, consumption and investment will take over the next few years. In particular, we combine current and past data with the structure of economic theory to generate an *empirical likelihood* Owen (2001) on the set of uncertain paths through time. This is what we refer to as a *weak solution*. The weak solution can then be used not only for providing policy guidance as is the standard approach to CGE modelling, but also for quantifying the uncertainty and for measuring the accuracy of the model itself. For example, we can check that the mean of the empirical likelihood satisfies an optimality criteria such as the intertemporal Euler equation.

Before turning to a more detailed example with uncertainty, we explain the model under certainty.

## 3. THE MODEL UNDER CERTAINTY

Certainty is characterised by a unique path through time. Each step along this path is a node where economic decisions are made. Let  $\mathcal{S}$  denote the set of decision nodes or *starting* nodes. Under certainty, the unique path induces a natural ordering on  $\mathcal{S}$ .<sup>3</sup> At each  $s$  in  $\mathcal{S}$ , the economic agent(s) look ahead and make a plan that extends into the future indexed by the set  $T$  of look-ahead times.<sup>4</sup> The agent(s) formulate plans that ensure resources are optimally allocated throughout  $T$ .

**Example 1** (2050 Net-zero carbon emissions target). *Consider a government that wishes to achieve a net-zero carbon emissions target by 2050. Under certainty and some form of adjustment costs, it is natural to assume a gradual transition is optimal: one that spreads the discounted cost of transition to clean energy evenly across time. Our goal is to derive an estimate of key policy and state variables along this optimal path. That is to say, we seek a suitably accurate policy approximation.*

<sup>3</sup>Formally, for each  $s$  in  $\mathcal{T}$ , there is a unique  $s^+$  such that the edge  $\mathfrak{e} = (s, s^+)$  belongs to the set  $\mathcal{E}$  of edges formed with members of  $\mathcal{S}$ .

<sup>4</sup>Cai and Judd (2021) allow the set  $T$  to vary across  $\mathcal{S}$  whereas we assume it is constant to simplify the exposition.

*In this setting,  $\mathcal{S}$  consists only of the years 2022, ..., 2050 : when the target is met. In contrast,  $T$  might extend much further into the future e.g.  $\sup T = \infty$ . Or, since planning and computation is costly, agents might adopt some suitably small, finite horizon. For  $T < \infty$ , it is necessary to approximate the continuation value of the agents' optimisation problem. The most interesting case is where  $T$  is equal to the number of years it takes to approximately reach the steady state of the model.*

**THE OBJECTIVE FUNCTION AT EACH NODE** As a minor expedience, let  $T$  not only denote the set of look-ahead times, let it also also denote its own cardinality  $\#T$ . The larger the set  $T$ , the more rational the agents in the economy: a point we discuss further in section 5.2. Under certainty, the agents at node  $s$  take the state  $\omega_s$  as given and choose the action  $\mathbf{a}_s$  that maximises the following forward-looking scalar-valued objective function:

$$F_s(\mathbf{a}_s, \omega_s) \stackrel{\text{def}}{=} \sum \{u_{s,t}(\mathbf{a}_{s,t}, \omega_{s,t}) : 0 \leq t < T\} + V_{s,T}(\omega_{s,T}) \quad (1)$$

We now explain the terms in expression (1) in detail. The *reward/utility* function  $u_{s,t}$  is, in general, time-varying. That is to say, time-homogeneity which is required for standard recursive models and steady-state approximation models is not a requirement in the present model. Under certainty, the state  $\omega_{s,t}$  is simply the vector  $\mathbf{K}_{s,t}$  of capital endowments for each region and sector. The action or control variable  $\mathbf{a}_{s,t}$  is a quadruple consisting of consumption  $\mathbf{C}_{s,t}$ , labour  $\mathbf{L}_{s,t}$ , intermediate input flows  $\mathbf{M}_{s,t}$ , and fixed capital formation or investment flows  $\mathbf{N}_{s,t}$ . Note that fixed capital formation includes, for instance, land reclaimed from forest or sea.

We discuss these variables in more detail below. As we will see, there are numerous other variables that are vital to the model. In the language of nonlinear programming, the state-action pair  $(\omega_{s,t}, \mathbf{a}_{s,t})$  are *basic variables*. To emphasize this property, we write all such variables in **bold** font. In the present model, all variables that are not basic are *intermediate*. By basic we mean that the variable cannot be explicitly written as a function of the other variables without recourse to first-order or second-order conditions of the optimization (or equilibrium) problem.

Turning now to the constraints, we begin with the transition laws/dynamics

for capital followed by the intratemporal feasibility constraints such as production technology and market clearing.

**CONSTRAINTS AT EACH NODE: TRANSITION DYNAMICS** Let  $I$  and  $J$  both denote the set of sectors and let  $R$  denote the set of regions. We use  $I$  to index the rows of an input-output matrix and  $J$  to index the columns. Capital  $\mathbf{K}_{s,t}$  is a vector with values  $\mathbb{R}_+^{J \times R}$  that evolves according to the standard equation for each node  $s$  and time  $t$ . Since the depreciation rate  $0 < \delta_j < 1$  may vary across sectors, we allow the following generalised forward difference operator  $\Delta_j$  to vary across sectors:

$$\Delta_j(\mathbf{K}_{j,s,t}) = \mathbf{K}_{j,s,t+1} - (1 - \delta_j) \cdot \mathbf{K}_{j,s,t}. \quad (2)$$

Note that (2) allows us to implicitly model the component of capital (such as land) that does not change between periods and is not subject to depreciation. For each  $j, s$  and  $t$ , the agent(s) are constrained in their ability to modify the transition law for capital. Letting  $\Delta$  denote the function with component functions  $\Delta_j$ , the transition law holds when the constraint function

$$G_{s,t}^{\text{tran}}(\mathbf{a}_{s,t}, \boldsymbol{\omega}_{s,t}) = \text{inv}_{s,t} - \Delta(\mathbf{K}_{s,t}) \quad (3)$$

is equal to zero (in  $\mathbb{R}^{J \times R}$ ). We now break the formation of new capital down into its scalar components. Letting the indices vary across their corresponding sets, investment  $\text{inv}_{j,r,s,t}$  is the scalar-valued variable defined by aggregating over the *basic* fixed capital formation vector  $\mathbf{N}_{j,r,s,t} = (\mathbf{N}_{i,j,r,s,t} : i \in I)$ . In particular the constant-elasticity-of-substitution (CES) aggregation

$$\text{inv}_{j,r,s,t} = A_{j,r}^{\text{inv}} \cdot \left( \sum \left\{ \phi_{i,j,r}^{\text{inv}} \cdot (\mathbf{N}_{i,j,r,s,t})^{\rho_{j,r}^{\text{inv}}} : i \in I \right\} \right)^{\kappa_{j,r}^{\text{inv}} / \rho_{j,r}^{\text{inv}}}. \quad (4)$$

**Remark** (on CES aggregators). *CES functions are our default, flexible functional form. Let  $f_{j,r}^{\text{inv}}$  denote the scalar-valued function that maps the fixed capital formation vector  $\mathbf{N}_{j,r,s,t} \mapsto \text{inv}_{j,r,s,t} / A_{j,r}^{\text{inv}}$  and let  $f^{\text{inv}} = (f_{j,r}^{\text{inv}} : j \in J, r \in R)$  denote the corresponding vector-valued function that allows us to rewrite the family of relations (4) more concisely as*

$$\text{inv}_{s,t} = A^{\text{inv}} \circ f^{\text{inv}}(\mathbf{N}_{s,t}), \quad (5)$$

where  $\circ$  denotes the Hadamard (component-wise) product. The reason for leaving the productivity parameter  $A^{\text{var}}$  out of  $f^{\text{var}}$  is that, for some variables, we will allow  $A^{\text{var}}$  to evolve over time whereas the aggregator  $f^{\text{var}}$  is always homogenous across time. We return to this point in our discussion of balanced growth paths.

The scalar  $A_{j,r}^{\text{inv}}$  is the productivity parameter that captures how efficient agents are at transforming other goods into new type- $(j, r)$  capital. Next, the modified share parameters

$$\sigma_{i,j,r}^{\text{inv}} = (\sigma_{i,j,r}^{\text{inv}})^{1/\varepsilon_{j,r}^{\text{inv}}}$$

belongs to the closed interval  $[0, 1]$  where  $\sigma^{\text{inv}}$  in  $[0, 1]^{I \times J \times R}$  is the standard, normalised vector of *share* parameters with elements representing the importance of good  $i$  in forming new capital of each region-sector pair  $(j, r) \in J \times R$ .<sup>5</sup> As usual, the elasticity enters the CES function via the parameter  $\rho_{j,r}^{\text{inv}} = 1 - 1/\varepsilon_{j,r}^{\text{inv}}$ . It is decreasing in the elasticity of substitution  $\varepsilon_{j,r}^{\text{inv}}$  which lies between 0 and 1.<sup>6</sup> Finally, a necessary and sufficient condition (Kojić, 2021, Theorem 2) for strict concavity of the components of  $f^{\text{inv}}$  is that the *scale* parameter  $\kappa^{\text{inv}}$  has elements strictly between 0 and 1.

We capture the inertia in adjusting capital via the cost

$$\text{adj}_{j,r,s,t} = A_{j,r}^{\text{adj}} \cdot \frac{(\mathbf{K}_{j,r,s,t+1} - \mathbf{K}_{j,r,s,t})^2}{\mathbf{K}_{j,r,s,t}}. \quad (6)$$

A useful property of this adjustment cost is that it makes for more realistic rates of convergence between regions. It also allows us to capture the potential costs of closing down a factory or mine and repurposing the land to other uses. For each region and each sector, this cost is deducted from time  $s, t$  output which we describe below.

**THE PRODUCTION FUNCTION** In addition to capital, labour is the other *primary factor of production*. Labour resources also vary across regions, sectors and time.

<sup>5</sup>We derive  $\sigma_{j,r}^{\text{inv}}$  by taking a column of fixed capital formation flows between sectors for region  $r$  and normalising so that the column sum is one. In the language of input-output analysis,  $\sigma_r^{\text{inv}}$  is akin to the matrix of technical coefficients for region  $r$ , but for new capital formation instead of output.

<sup>6</sup>Recall that CES aggregator is Leontief when its elasticity parameter tends to zero and “linear” (i.e. Cobb–Douglas) when its elasticity parameter tends to one.

To simplify the exposition we avoid explicit constraints on labour supply and instead have a disutility of labour supply similar to Atalay (2017) and Cai and Judd (2021).

Beyond primary factors, we have intermediate inputs. As with investment, we specify intermediate inputs to production via a CES aggregator  $f^{\text{m}\bar{\text{e}}\text{d}} : \mathbb{R}_+^{I \times J \times R} \rightarrow \mathbb{R}_+^{J \times R}$ . Thus,

$$\text{m}\bar{\text{e}}\text{d}_{s,t} = A^{\text{m}\bar{\text{e}}\text{d}} \circ f^{\text{m}\bar{\text{e}}\text{d}}(\mathbf{M}_{s,t}) \quad (7)$$

where, for each  $r, n$  and  $t$ ,  $\mathbf{M}_{r,s,t}$  corresponds to a standard input-output matrix of intermediate flows for a given region.

For every  $r$  in  $R$ ,  $j$  in  $J$ , let  $f_{i,r}^{\text{out}} : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$  be a standard CES function of capital, labour and the intermediate-input bundle respectively:

$$\text{out}_{s,t} = A^{\text{out}} \circ f^{\text{out}}(\mathbf{K}_{s,t}, \bar{\text{l}}\text{ab}_{s,t}, \text{m}\bar{\text{e}}\text{d}_{s,t}) \quad (8)$$

where  $\bar{\text{l}}\text{ab}_{s,t} = A_{s,t}^{\text{tek}} \circ \mathbf{L}_{s,t}$  is labour  $\mathbf{L}_{s,t}$  augmented by the deterministic technological progress factor  $A_{s,t}^{\text{tek}}$ .<sup>7</sup>

**CONSTRAINTS AT EACH NODE: MARKET CLEARING** In our canonical model, we assume free trade between regions. That is, we have a single market that clears for each sector, node and time in  $I \times \mathcal{S} \times T$ . Formally, market clearing holds for action and state vectors  $\mathbf{a}$  and  $\boldsymbol{\omega}$  such that the constraint function

$$G^{\text{clear}}(\mathbf{a}, \boldsymbol{\omega}) = \sum_{r \in R} \left\{ \text{out}_r - \mathbf{C}_r - \sum_{j \in J} \{ \mathbf{N}_{j,r} + \mathbf{M}_{j,r} \} - \text{adj}_r \right\} \quad (9)$$

is equal to zero in  $\mathbb{R}^{I \times \mathcal{S} \times T}$ .

**REWARD AND TERMINAL VALUE FUNCTIONS AT EACH NODE** Beyond strictly increasing, for each region, node and time, we take the utility function

$$u_{r,s,t} : \mathbb{R}_+^{I \times J} \rightarrow \mathbb{R}, \quad (\mathbf{C}_{r,s,t}, \mathbf{L}_{r,s,t}) \mapsto \text{utl}_{r,s,t},$$

<sup>7</sup>Explicitly, on  $J \times R \times \mathcal{S} \times T$ , we have

$$\text{out}_{j,r,s,t} = A_{j,r}^{\text{out}} \cdot (\sigma_{1,j,r}^{\text{out}} \cdot \mathbf{K}_{j,r,s,t}^{\rho_{j,r}} + \sigma_{2,j,r}^{\text{out}} \cdot \bar{\text{l}}\text{ab}_{j,r,s,t}^{\rho_{j,r}} + \sigma_{3,j,r}^{\text{out}} \cdot \text{m}\bar{\text{e}}\text{d}_{j,r,s,t}^{\rho_{j,r}})^{\kappa_{j,r}/\rho_{j,r}}$$

to be strictly concave in consumption and labour. To ensure Similar to both Atalay (2017) and Cai and Judd (2021), we have

$$\text{utl}_{r,s,t} = \beta^t \cdot (\text{con}_{r,s,t} + \bar{\text{lab}}_{r,s,t})$$

where  $0 < \beta < 1$  is the one-period discount factor and  $\text{con}_{r,s,t} = A_r^{\text{con}} \cdot f_r^{\text{con}}(\mathbf{C}_{r,s,t})$  and  $\bar{\text{lab}}_{r,s,t} = A_r^{\bar{\text{lab}}} \cdot f_r^{\bar{\text{lab}}}(\mathbf{L}_{r,s,t})$ . The utility weight  $A_r^{\bar{\text{lab}}}$  is distinguished by the property that it is typically negative to reflect the disutility of labour. To ensure concavity of utility in its arguments, a feature of the sectoral aggregator for labour is that  $\rho_r^{\bar{\text{lab}}} = 1 + 1/\varepsilon_r^{\bar{\text{lab}}}$ . Thus, unlike every other  $\rho$  parameters of the model, it is always positive, greater than one, and decreasing in  $\varepsilon_r^{\bar{\text{lab}}}$ . We recover the Atalay functional for the labour aggregate in utility by taking  $\varepsilon_r^{\bar{\text{lab}}} = \infty$  (so that  $\rho_r^{\bar{\text{lab}}} = 1$ ) and by taking the scale parameter  $\kappa_r^{\text{con}} = 3$ . In this case,  $\bar{\text{lab}}$  satisfies perfect substitutability across labour from different sectors. For strict convexity of  $f_r^{\bar{\text{lab}}}$  take  $\kappa_r^{\bar{\text{lab}}} > \rho_r^{\bar{\text{lab}}}$ . This, together with  $0 < \kappa_r^{\text{con}} < 1$ , suffices for strict concavity of  $u_{r,s,t}$ .

The weighted sum across regions (with weights determined by population) of regional reward functions yields the “global”/“whole-of-economy” reward functions  $u_{s,t}$  of (1) for each node. This, together with standard assumptions (Stachurski, 2009, Theorem 12.2.12), is sufficient for strict concavity of the terminal value function  $V_{n+T}$  in the state vector  $\boldsymbol{\omega}_{s,t}$ . Together with the structure we have imposed on the transition laws and feasibility constraints, the social welfare Lagrangian  $\mathfrak{L}_s$  is strictly concave function of the control vector  $\mathbf{a}_s$  for every  $s$  in  $\mathcal{N}$ .<sup>8</sup>

**Remark.** *As usual, the Lagrange multipliers of market clearing constraints are also the competitive market prices. The AMPL language has features for constraining prices where necessary.*

**WELL-POSEDNESS OF THE OPTIMIZATION PROBLEM AT EACH NODE** The above conditions ensure that the solving for the general equilibrium of the economy is equivalent to solving a strictly concave program at each node. This implies that the equilibrium at each node is unique. The first-order (necessary) conditions for are

<sup>8</sup>Recall that the Lagrangian is the sum of the objective and each of the constraints scaled by the *dual* variables (also known as Lagrange multipliers).



also sufficient and we can tackle the problem with local (nonlinear optimization) solvers such as Conopt or Ipopt which are many times faster than global solvers such as Baron.

#### 4. THE NORMALISED EULER ERROR

We now derive the normalised Euler error for the example in section 3.

#### 5. UNCERTAINTY

As in structural econometric modelling (Reiss and Wolak, 2007) it is important to distinguish the following kinds of uncertainty:

- I agent uncertainty about the economic environment;
- II optimization errors / behaviour of economic agents;
- III researcher measurement and approximation errors.

##### 5.1. Type I uncertainty

In MAIWAR, type I uncertainty is captured using a decision tree. A decision tree  $\mathcal{T}$  is a pair  $(\mathcal{S}, \mathcal{E})$  of nodes and directed arcs/edges connecting future nodes to preceding ones. A *node*  $s$  is where agents make decisions. An *edge*  $\mathfrak{e}$  is a pair  $(s, s^+)$  of nodes that are connected in the sense that  $s$  immediately precedes  $s^+$ . Uncertainty arises when there is more than one edge emanating from some node  $s$ .<sup>9</sup> Uncertainty at node  $s$  unfolds along edge  $e$  to reveal the subsequent decision node. A *path* through time consists of a maximal sequence of connected nodes. That is sequences that start at the (unique) root node  $\mathfrak{o}$  of the tree and end on a leaf (a node with no explicitly modelled future times). Let  $\mathcal{P}$  denote the set of paths.

**Example 2** (Irreversible risk). *In our canonical example, where the focus is on net-zero emissions in the year 2050, we identify the root with the current year, e.g. 2022. A leaf is then a state of the world at time 2050. Each year between*

<sup>9</sup>Since  $\mathcal{T}$  is a tree, for every  $\mathfrak{e} = (s, s^+) \in \mathcal{E}$  if  $e' = (s', s^+) \in \mathcal{E}$ , then  $s' = s$ .

now and 2049, there is a small but positive probability of a one-off, permanent productivity shock to output. The following paths characterise the decision tree:

1 the path where the shock happens between 2022 and 2023;

2 the path where the shock happens between 2023 and 2024;

...

28 the path where the shock happens between 2049 and 2050.

Each path identifies a distinct state of the world. Although this stochastic process is as simple as flipping a coin each year and the resulting state space consists of just 28 paths, the irreversible nature of the shock yields a nonstationary (i.e. time inhomogeneous) economy with a nonunique steady state.

a pair of variables consisting of capital  $\mathbf{K}_{s,t}$  and the productivity shock term

$$\text{shock}_{s,t} = \begin{cases} 0.9 & \text{if the shock has happened} \\ 1 & \text{otherwise.} \end{cases} \quad (10)$$

We return to discuss capital in more detail below. The shock shock is therefore macroeconomic in the sense that its effect is uniform across regions  $R$  and sectors  $J$  in the economy. A more general shock will yield a larger state space and corresponding set of paths.<sup>10</sup> The conditional expectation  $\mathbb{E}_s$  is defined on the set of functions that are measurable with respect to the information available to the agent at node  $s$ .<sup>11</sup>

CONSTRUCTING THE EMPIRICAL DISTRIBUTION ON PATHS Let us first define the strong solution we obtain from the above node-specific optimization problems. For a given decision tree  $\mathcal{T} = (\mathcal{N}, \mathcal{E})$  and corresponding set of paths  $\mathcal{P}$ , let

$$\text{StrongSol}(\mathcal{P}) \stackrel{\text{def}}{=} \{(\omega_p^*, a_p^*) : p \in \mathcal{P}\}$$

where the superscript  $*$  denotes the optimal state-policy values for each decision node  $s$  along a given path  $p$ . Note that  $\text{StrongSol}(\mathcal{P})$  only contains the realised initial state-policy values since, for each  $s$ , the look-ahead values  $\{(\omega_{s,t}^*, a_{s,t}^*) : t > 0\}$

<sup>10</sup>This simplification is purely for expositional purposes. In our code, this and other parameters that appear below are allowed to vary across regions and sectors.

<sup>11</sup>All these functions are scalar-valued.

are only used to formulate the plan at node  $s$ . This notion of solution is strong in the sense that, assuming the model is fully-specified, it tells us precisely what will happen once the economy reaches a given node. Moreover, in simple cases, such as example 2 where there are just 28 paths,  $\text{StrongSol}(\mathcal{P})$  can also be viewed as a complete contingent plan of action and future capital. Another sense in which the solution is strong when  $\#\mathcal{P}$  is small is that we are able to observe the entire sample space: we are able to observe the true distribution of optimal paths through time.

**WEAK SOLUTIONS** Beyond simple examples such as example 2, we can only hope to partially observe the sample space. The set of optimal solution paths that we have derived is then a subsample and, since our collection of nodes is incomplete, we can no longer refer to our solution as strong. The only sense in which we can speak of a solution is in terms of the empirical distribution over paths. From this we can estimate the expected path, the variance, and so on. We now formalise this weaker notion of solution.

Let  $\mathbf{o}$  denote the root node of the decision tree. Recalling that we assume the decision tree has paths of uniform length, let  $S$  denote the set of *path times*: for each  $s$  in  $S$  the node  $p_s$  is  $s$  steps into the future from  $\mathbf{o}$  along path  $p$ . We begin by constructing the empirical distribution associated with  $\mathbf{o} + s$  for each  $s \in S$ . Let  $D_s$  denote the set of nodes that are  $s$  steps into the future from  $\mathbf{o}$ .<sup>12</sup>

For any  $s$  in  $S$  note that  $\mathbf{C}_{i,r,\mathbf{o}+s}$  is a scalar-valued random variable on  $D_s$ . Moreover, for each  $s$  in  $D_s$ , let  $\text{CON}_{i,r,n}$  denote a realisation of this random variable. By realisation, we mean a strong solution to this particular variable. For any of our policy or state variables  $x_{\mathbf{o}+s}$  let  $X_d$  denote the realisation associated with node  $s$ . Then the empirical distribution generated by  $\{X_{d'} : d' \in D_s\}$  is the function  $F_{x_s} : \mathbb{R} \rightarrow [0, 1]$ ,

$$F_{x_s}(z) = \frac{1}{\#N_s} \cdot \sum_{n \in N_s} 1_{X_d \leq z}. \quad (11)$$

The weak solution for a given policy or state variable  $x$  is simply the sequence  $F_x = \{F_{x_s} : s \in S\}$  of empirical distributions.  $F_x$  is the empirical distribution of the path-valued random variable  $x$ . For instance,  $F_{\mathbf{C}_{i,r}}$  is the empirical distribution for the path of consumption of sector- $i$  goods in region  $r$ .

<sup>12</sup>That is  $N_s = \{p_s \in \mathcal{N} : p \in \mathcal{P}\}$ .

In turn, the weak solution for the entire problem is the joint empirical distribution  $F_0$  we obtain by taking the product of the marginals. (This is the case because all these probability measures are push-forwards that are dependent on the same source of uncertainty.)

**MEASURING ACCURACY IN TERMS OF THE EULER ERROR** One normative measure of the accuracy of our weak solutions is the Euler error that is associated with the (intertemporal) Euler equation. This equation is a necessary condition for optimality of the stochastic dynamic program Stokey et al. (1989). In the present setting the derivation of Euler equation is complicated by two factors: the number of control variables exceeds the number of state variables; and there is no explicit form for To address this issue, for each  $s$ , we appeal to the envelope theorem. In particular, we follow González-Sánchez and Hernández-Lerma (2014) and derive an indirect utility function  $v_s$  as a function of current and next-period states  $\omega_s$  and  $\omega_{s+}$ .

$$v_s(\omega_s, \omega_{s+}) \stackrel{\text{def}}{=} \max_{\mathbf{a}_s} u_s(\omega_s, \mathbf{a}_s) \quad \text{subject to} \\ \omega_{s+} = \text{tran}_s(\omega_s, a_s) \quad \text{and} \quad \text{feas}_s(\omega_s, \mathbf{a}_s) \geq 0$$

where the transition and feasibility constraints are those we have described above. Then, via the envelope theorem, for models with exogenous uncertainty (where actions do not affect transition probabilities), the Euler error associated with step  $s$  is

$$\frac{\beta \cdot \mathbb{E}_0 \{ \nabla_{\mathbf{K}_{s+1}} v_{s+1}(\omega_{s+1}, \omega_{s+2}) \}}{\mathbb{E}_0 \{ \nabla_{\mathbf{K}_{s+1}} v_s(\omega_s, \omega_{s+1}) \}} + 1 \quad (12)$$

It requires that the marginal loss (the denominator is negative) associated with foregoing today's consumption (by increasing tomorrow's capital) is balanced against the marginal benefit of increasing consumption tomorrow. Note that the expectation  $\mathbb{E}_0$  in the expression for Euler error is the empirical one we have derived as part of our weak solution. In other words, our condition for accuracy demands that the empirical Euler error is small. As Cai and Judd (2021) show for a wide variety of settings, the worst case of this error across  $s$  in  $S$  is less than 1%.

We note that, since there is no explicit form for the indirect utility function  $v$ , we build on Scheidegger and Bilonis (2019) to use gaussian processes to generate

an approximation. That is, we introduce uncertainty of type III. This brings us to the next section.

## 5.2. Types II and III uncertainty: approximations

In MAIWAR, weak solutions allow us to accommodate uncertainty of type II and III without sacrificing theoretically appealing solution properties such as the *intertemporal Euler equation* a feature that is absent from CGE models: see Dixon and R. (2020) for a recent attempt to introduce this concept into the CoPS methodology.<sup>13</sup> Our agents (and modellers) strive to ensure that the marginal benefits of current consumption against the marginal benefits of uncertain future production.

Yet in order to arrive at a tractable and fast method for deriving a weak solution, we will need to approximate. By this we do not only mean the inevitable partial sampling of the set of paths we have already highlighted in the previous section, we mean:

1. approximating the terminal value function (see Bertsekas (2019) for a detailed exposition of this matter in the setting of reinforcement learning);
2. by using certainty-equivalent projections (see Cai and Judd (2021) and Cai et al. (2017) for a detailed exposition of this concept).

Our first departure from canonical expected utility maximisation, where agents form a complete contingent plans of action, is as follows. At each decision node, agents solve a nonlinear program where errors associated with future type I uncertain variables are replaced with some convex combination of the best and worst case scenario. Abundant evidence from the literature on behavioural economics shows that agents adopt similar heuristic approaches to dealing with complex decisions in the face of uncertainty. The “*certainty equivalent*” projection of Cai and Judd (2021) is one such heuristic. Aside from tractability, the inbuilt flexibility and scope for improvement in aspect of the model is one reason for adopting the methodology of Cai and Judd (2021). That is say, we view this, not only as an approximation, but also a reasonable behavioural feature.

<sup>13</sup>This paper provided inspiration for our efforts.

A second form of type II uncertainty arises because economic agents struggle with long time horizons: indeed, isn't this why policy makers appeal to CGE models in the first place? Evidence shows that agents focus their attention on the next few periods. This motivates the finite look-ahead feature of our model. Our agents focus on the near future as opposed to the far-flung steady states at the heart of other approaches.

In MAIWAR there is a useful alignment between modeller's interests and the type II uncertainties we have just outlined. For although there are well-established methods for implementing large-scale linear stochastic programs, the tools for large-scale nonlinear stochastic programming are less evolved Rehfeldt et al. (2022). Furthermore, although the time horizon of our agents is indefinite (and typically modelled as infinite), most methods for dealing with such time horizons assume stationarity or provide a linear approximation around the nonstochastic steady state. Nor is there an obvious way to model 2050 net-zero emissions without proper modelling of the medium term. Constraints that will bind in 28 years time, but have implications today are poorly suited to steady-state approximations and standard policy/value function iteration methods.<sup>14</sup>

In summary, our agents solve finite sequences of nonlinear programs each with a finite horizon and an approximation for the terminal value function. This is an approximation in the sense that normative agents solve nonlinear stochastic programs each with an infinite horizon or the true terminal value function. We argue that our approximations are well-aligned with behaviour of agents in practice. Moreover, the weak solution that we derive explicitly accommodates our approximation methods. Finally, since this weak solution is an empirical distribution we can verify that the solution is accurate to within a 1% Euler error approximation. The same measure is typical of "global approaches". For instance, in the deep neural network approach of Azinovic et al. (2019) the policy iteration process is successful once the Euler error term converges to a sufficiently small number. Our investigations of this approach yielded slow and unstable results. The notion of a weak solution recognizes the fact that our modeller is typically facing substan-

<sup>14</sup>In our experience, "global" value and policy function iteration methods are also slow and unstable in the setting of multi-sectoral models with medium-term constraints. See Cai and Judd (2021) for a related discussion.

tial type III uncertainty. Parametric uncertainty is a feature of the model and a central part of the solution.

### 5.3. Computation and software

The uncertainty we model is facilitated and expedited by the fact that we have in place the software infrastructure to ensure that simple model sweeps are easy to conduct on the local HPC cluster. The unit for parallelisation is paths: each path is passed to a single core. This means that most desktop computers can easily parallelise. For example, consider our canonical example of irreversible risk: where there are 28 paths (each of length 28 years), and assume 7 regions and 20 sectors. For the functions we specify above, each step along a path takes 20 seconds on average. Thus a path takes  $28 \times \frac{1}{3} \approx 9$  minutes to solve. For a desktop with 4 available cores, we are able to assign 7 paths to each core and complete the generate the full empirical distribution in  $7 \times 9 \approx 1$  hour.

After an exhaustive search, we chose AMPL (A modelling language for mathematical programming) Fourer et al. (1990) as the language and ecosystem for communicating with solvers. AMPL is used for pre-processing/pre-solving: the stage before the nonlinear optimization problem is passed to the solver. The goal of pre-solving is to ensure the solver stage is faster and more reliable. Pre-solving includes the generation of automatic, sparse derivatives as well as domain/variable reduction. AMPL syntax is much closer to standard mathematics relative to GAMS and we have found it to produce faster total solve times than GAMS.<sup>15</sup> We spent a lot of time trying purely open source options in Python, Julia and to lesser extent C++. This experimentation also involved machine learning tools (based on Scheidegger’s work). We found that, overall, combining the Cai and Judd (2021) methodology with AMPL pre-solving yields unrivalled modelling performance, flexibility and reliability.

We have found three nonlinear solvers to be most useful. Conopt which is proprietary and based on the reduced-gradient method (itself based on the robust simplex method). The open source Ipopt (pronounced “eye-pea-opt”). This is celebrated application of the interior point by Wächter and Biegler (2006). Knitro

<sup>15</sup>Cai and Judd (2021) find that GAMS is substantially faster than Matlab.

which is tries a number of approaches and, unlike Conopt and Ipopt can handle integer-valued variables. We have tried other solvers as well, but the above are outstanding, with Conopt being the most robust.

Future improvements to our methodology relate to tuning MAIWAR model parameters and improving on our existing framework for parallel computing. In particular, to enable simple automated parallelisation even desktop computers, we will use the Python API for AMPL along with nextflow. Currently this process is manual and automatic tuning of parameters using machine learning is step in the development of MAIWAR. Many of the tools that we worked on in the second half of last year (first six months of SMaRT) will serve us well in this respect.

#### 5.4. Nonlinearity and (non)steady-state methods

Atalay (2017) estimates to be approximately 0.1 so that  $\rho \approx -9$ .

Such approximations are even less appealing for nonlinear production networks that arise when elasticities of substitution are below one.

Impressive recent work using Hamiltonian methods still fails to capture the central economic theme of uncertainty.

#### REFERENCES

- Atalay, E. (2017). How important are sectoral shocks? *American Economic Journal: Macroeconomics*, 9(4):254–80.
- Azinovic, M., Gaegauf, L., and Scheidegger, S. (2019). Deep equilibrium nets. *Available at SSRN 3393482*.
- Bertsekas, D. (2019). *Reinforcement learning and optimal control*. Athena Scientific.
- Cai, Y., Judd, K., and Steinbuks, J. (2017). A nonlinear certainty equivalent approximation method for dynamic stochastic problems. *Quantitative Economics*, 8(1):117–147.
- Cai, Y. and Judd, K. L. (2021). A simple but powerful simulated certainty equivalent



- lent approximation method for dynamic stochastic problems. Technical report, National Bureau of Economic Research.
- Dixon, P. B. and R., R. M. (2020). Developing a dsge consumption function for a cge model.
- Fourer, R., Gay, D. M., and Kernighan, B. W. (1990). A modeling language for mathematical programming. *Management Science*, 36(5):519–554.
- González-Sánchez, D. and Hernández-Lerma, O. (2014). On the euler equation approach to discrete-time nonstationary optimal control problems. *Journal of Dynamics & Games*, 1(1):57.
- Hosoe, N., Gasawa, K., and Hashimoto, H. (2010). *Textbook of computable general equilibrium modeling: programming and simulations*. Springer.
- Kojić, V. (2021). A note on concavity conditions of cobb–douglas and ces production function with at least two inputs. *Studies in Microeconomics*, 9(1):1–10.
- Long, J. B. and Plosser, C. I. (1983). Real business cycles. *Journal of political Economy*, 91(1):39–69.
- Owen, A. B. (2001). *Empirical likelihood*. Chapman and Hall/CRC.
- Raymond, H., Elazaar, D., and Norden, I. (2022). Modelling production networks and supply chain disruption.
- Rehfeldt, D., Hobbie, H., Schönheit, D., Koch, T., Möst, D., and Gleixner, A. (2022). A massively parallel interior-point solver for lps with generalized arrow-head structure, and applications to energy system models. *European Journal of Operational Research*, 296(1):60–71.
- Reiss, P. C. and Wolak, F. A. (2007). Structural econometric modeling: Rationales and examples from industrial organization. *Handbook of econometrics*, 6:4277–4415.
- Scheidegger, S. and Bilonis, I. (2019). Machine learning for high-dimensional dynamic stochastic economies. *Journal of Computational Science*, 33:68–82.

- Stachurski, J. (2009). *Economic dynamics: theory and computation*. MIT Press.
- Stokey, N. L., Lucas, R. E., and C., P. E. (1989). *Recursive methods in economic dynamics*. Harvard University Press.
- Wächter, A. and Biegler, L. T. (2006). On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Mathematical programming*, 106(1):25–57.
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