MAIWAR (Modelling Australian Industry with Weak-solutions, AMPL and Regions) [1]

By Patrick H. O'Callaghan [2]

Australian Institute for Business and Economics

University of Queensland

Policy makers make decisions regarding medium term policies where capital is evolving. They do so in the face of uncertainty. Existing Computable General Equilibrium models do not adequately model such intertemporal behaviour. Macroeconomic models typically address this concern, but rely on local approximations around the non-stochastic steady-state (where capital is no longer evolving). Ergodic-set methods are similarly suited to the long-term whereas grid-based value-function iteration is unstable or slow in multi-sectoral settings.

In this white paper, we extend a simple, yet powerful, method (of Cai and Judd) to the multisectoral setting. Along uncertain paths through time, agents look ahead multiple periods and choose policies that are optimal the certainty-equivalent sense. The model focuses on the medium term and the solution we derive is *weak*: characterised by an empirical distribution on the set of paths (unobservable strong solutions). Similar to structural econometric models, the key measure of accuracy is that the mean path satisfies the Intertemporal Euler equation to within a 1% margin of error.

[1] This work is part of the Sustainability, Modelling and Regional Transition (SMaRT) project. Thanks to John Mangan and Brent Ritchie for winning financial support for the SMaRT project and to the University of Queensland for this support (Research Support Package - Round 3). I thank Josh Aberdeen, Patrick Duenow, Cameron Gordon, John Mangan and Tina Rampino for many conversations and active contributions to the creation of the MAIWAR model. I thank Alex Yelkhovski for valuable conversations at an early stage. Last, but certainly not least, I thank the members of the SMaRT project panel: Hurriyet Babacan, Begoña Dominguez, Flavio Menezes, Alicia Rambaldi and Martie-Louise Verreynne. This version has time stamp 00:16 (AEST), 3rd July, 2022. The most recent version (as well as past versions) can be found at `https://github.com/uq-aibe/maiwar/blob/main/documentation/maiwar-white-paper.pdf` .

[2] Email address `p.ocallaghan@uq.edu.au` and ORCID iD 0000-0003-0606-5465

From the past, the present acts prudently, lest it spoil future action.

Titian: Allegory of Prudence

1. Introduction

Experience is the basis of prediction. Yet outside of stylised settings, even the most experienced forecasters do not claim access to "the full model". In the methodology below, we will argue that, for incomplete model specifications, weak solutions, that is ones that explicitly uncertainty of various kinds and are characterised by an empirical probability distribution are more suitable. Yet this kind of solution is absent from the literature: certainly from the Computational General Equilibrium (CGE) literature, and the related multi-sectoral macroeconomic literature. The weak solution we derive is implicit in Cai and Judd (2021): though they do not consider the case of multi-sectoral investment flows that is central to MAIWAR.

Beyond the short-term, economic theory provides the lens through which policy-makers peer into the future. In medium term, the economy is unlikely to reach its steady state. Moreover, in models with multiple sectors recent research has highlighted the importance of nonlinear effects. For such problems, first-order steady-state approximations are ill-suited. Yet this is the solution method that is most commonly applied today.

Computable General Equilibrium (CGE) models are often used to provide insights into the regional and sectoral impacts of current policies and shocks that propagate through the economy's production networks over time. Models that assume steady-state approximations are better suited to long-term policy questions precisely because steady-state equilibria are a special kind of equilibrium that may take many years to arrive. Yet even the impact of longer-term policies, such as 2050 net-zero emissions targets should involve a careful analysis of the short-to-medium term. For a policy that is unsustainable or sub-optimal in the interim is liable to fail altogether.

At this point, this white paper only contains a draft methodology.

2. Methodology

The goal of our method is to generate statistical information about the path that key regional variables such as gross value added, employment, consumption and investment will take over the next few years. In particular, we combine current and past data with the structure of economic theory to generate an *empirical likelihood* Owen (2001) on the set of uncertain paths through time. This is what we refer to as a *weak solution*. The weak solution can then be used not only for providing policy guidance as is the standard approach to CGE modelling, but also for quantifying the uncertainty and for measuring the accuracy of the model itself. For example, we can check that the mean of the empirical likelihood satisfies an optimality criteria such as the intertemporal Euler equation.

Before turning to a more detailed example with uncertainty, we explain the model under certainty.

3. A canonical example without uncertainty

Under certainty the solution is a unique path through time. In other words, the decision tree $\mathcal{T}$ simplifies to a chain or sequence of decision nodes.[3]

4. The normalised Euler error

We now derive the normalised Euler error for the example in section 3.

5. Uncertainty

As in structural econometric modelling (Reiss and Wolak, 2007) it is important to distinguish the following kinds of uncertainty:

I agent uncertainty about the economic environment;

II optimization errors / behaviour of economic agents;

III researcher measurement and approximation errors.

[3]Formally, for each n in $\mathcal{T}$, there is a unique n^+ such that (n, n^+) belongs to $\mathcal{E}$.

5.1. Type I uncertainty

In MAIWAR, type I uncertainty is captured using a decision tree. A decision tree $\mathcal{T}$ is a pair $(\mathcal{N}, \mathcal{E})$ of nodes and directed arcs/edges connecting future nodes to preceding ones. A *node* n is where agents make decisions. An *edge* e is a pair (n, n^+) of nodes that are connected in the sense that n immediately precedes n^+. Uncertainty arises when there is more than one edge emanating from some node n.[4] Uncertainty at node n unfolds along edge e to reveal the subsequent decision node. A *path* through time consists of a maximal sequence of connected nodes. That is sequences that start at the (unique) root node o of the tree and end on a leaf (a node with no explicitly modelled future times). Let $\mathcal{P}$ denote the set of paths.

Example 1 (Irreversible risk)**.** *In our canonical example, where the focus is on net-zero emissions in the year 2050, we identify the root with the current year, e.g. 2022. A leaf is then a state of the world at time 2050. Each year between now and 2049, there is a small but positive probability of a one-off, permanent productivity shock to output. The following paths characterise the decision tree:*

1 *the path where the shock happens between 2022 and 2023;*

2 *the path where the shock happens between 2023 and 2024;*

...

28 *the path where the shock happens between 2049 and 2050.*

Each path identifies a distinct state of the world. Although this stochastic process is as simple as flipping a coin each year and the resulting state space consists of just 28 paths, the irreversible nature of the shock yields a nonstationary *(i.e. time inhomogeneous) economy with a nonunique steady state.*

The objective function at each node At each decision node n in $\mathcal{N}$, agents solve a forward-looking optimization problem of the form that we now describe. Let $T = 0, 1, \ldots$ denote the set of look-ahead times and, as a matter

[4] Since $\mathcal{T}$ is a tree, for every $e = (n, n^+) \in \mathcal{E}$ if $e' = (n', n^+) \in \mathcal{E}$, then $n' = n$.

of expedience, let $\sharp T \equiv T$. The larger the set T the more rational the agents in the economy: a point we discuss further in section 5.2. In our canonical example and algorithm, we assume a representative agent maximises the social welfare of the economy subject to constraints. (Recasting the model in terms of equilibrium equations is straightforward: see Cai and Judd (2021, Algorithm 2).)

The objective of the social planner at node n is to choose the action a that maximises the following forward-looking expression:

$$\mathrm{obj}_n(a_n, \omega_n) \stackrel{\mathrm{def}}{=} \mathrm{u}_{n+0}(a_{n+0}, \omega_{n+0}) + \mathbb{E}_n \left\{ \mathrm{u}_{n+1}(a_{n+1}, \omega_{n+1}) + \cdots + \mathrm{V}_{n+T}(\omega_{n+T}) \right\} \quad (1)$$

We now explain the terms in expression (1) in detail. First, V_n is the value function at node n and V_{n+T} is the *terminal* or *continuation value* of the optimization problem. The *reward/utility* function u_{n+t} extends $0 \leqslant t < T$ periods ahead from node n. The conditional expectation $\mathbb{E}_n$ is defined on the set of functions that are measurable with respect to the information available to the agent at node n. [5] The state ω_{n+t} is a pair of variables consisting of capital $\mathbf{kap}_{n+t}$ and the productivity shock term

$$\mathrm{shock}_{n+t} = \begin{cases} 0.9 & \text{if the shock has happened} \\ 1 & \text{otherwise.} \end{cases} \quad (2)$$

We return to discuss capital in more detail below. The shock shock is therefore macroeconomic in the sense that its effect is uniform across regions R and sectors J in the economy. A more general shock will yield a larger state space and corresponding set of paths.[6]

Finally, in (1), action a_{n+t} is often referred to as the (set of) control variable(s). In our canonical example, it is a quadruple consisting of future consumption $\mathbf{con}_{n+t}$, labour $\mathbf{lab}_{n+t}$, investment inv_{n+t} and savings $\mathbf{sav}_{n+t}$. Note that $\mathbf{con}_{r,n+t}$, $\mathbf{lab}_{r,n+t}$ and $\mathrm{inv}_{r,n+t}$ are vectors on J indexed by region. Similarly, we view $\mathbf{sav}_{r,n+t}$ as the $J \times J$ matrix-valued function of flows between regions that we define in (4) (also indexed by region). It is this latter matrix of *dynamic* flows that significantly increases the dimension of the problem: for 20 sectors, there are 400 flows; for 100 sectors, there are 10,000.

[5] All these functions are scalar-valued.

[6] This simplification is purely for expositional purposes. In our code, this and other parameters that appear below are allowed to vary across regions and sectors.

We now turn to the constraints, beginning with the transition laws/dynamics for capital before turning to feasibility constraints such as market clearing.

Constraints at each node: transition dynamics Capital $\mathbf{kap}_{n+t} : R \times J \to \mathbb{R}_+$ is a vector that evolves according to the standard equation for each time t in T:

$$\mathbf{kap}_{r,j,n+t+1} = (1 - \delta_{r,j}) \cdot \mathbf{kap}_{r,j,n+t} + \mathrm{inv}_{r,j,n+t}. \tag{3}$$

Note that (3) is a stochastic difference equation. This is because, for every t in T, the set N_t of nodes that are reachable in t steps typically contains more than one node. In (3), δ is the depreciation rate and inv_{n+t} is a vector of constant elasticity of substitution (CES) functions of intermediate gross fixed capital formation. In the following equation, we suppress regional subscripts. Thus, for every j in J and $n + t$, we have

$$\mathrm{inv}_{j,n+t} = A_j^{\mathrm{inv}} \cdot \left(\sum \sigma_{ij}^{\mathrm{inv}} \cdot \mathbf{sav}_{ij,n+t}^{\rho_j^{\mathrm{inv}}} : i \in J \right)^{\kappa_j^{\mathrm{inv}}/\rho_j^{\mathrm{inv}}}. \tag{4}$$

As usual, $\rho_{r,j}^{\mathrm{inv}} = \frac{\epsilon_{r,j}^{\mathrm{inv}} - 1}{\epsilon_{r,j}^{\mathrm{inv}}}$ is a parameter that captures the role of the elasticity of substitution $\epsilon_{r,j}^{\mathrm{inv}} \approx 0.1$ across sectors of production. Finally, to ensure strict concavity of $\mathrm{inv}_{j,n+t}$ we assume that κ is slightly less than 1. In Kojić (2021, Theorem 2) it is shown that a necessary and sufficient condition for strict concavity of the fuction $(\mathbf{sav}_{r,ij,n+t} : i \in J) \mapsto \mathrm{inv}_{r,j,n+t}$ is that $0 < \kappa_{r,j}^{\mathrm{inv}} < 1$.

The production function In addition to capital, labour is the other *primary factor of production.* Labour resources also vary across regions, sectors and time. To simplify this exposition, we avoid notions of human capital.

In addition to primary factors, we have intermediate inputs. As with investment, we specify intermediate-input bundles via a CES aggregator $f_{r,j}^{\mathrm{int}}$ with its corresponding set of parameters $A_{r,j}^{\mathrm{int}}, \{\sigma_{r,ij}^{\mathrm{int}} : i \in J\}, \rho_{r,j}^{\mathrm{int}}, \kappa_{r,j}^{\mathrm{int}}$ and $\epsilon_{r,j}^{\mathrm{int}}$. That is,

$$\mathrm{int}_{r,j,n+t} = A_{r,j,n+t}^{\mathrm{int}} \cdot f_{r,j}^{\mathrm{int}}(\mathbf{med}_{r,ij,n+t} : i \in J) \tag{5}$$

where, for each r, j, n and t, $(\mathbf{med}_{r,ij,n+t} : i \in J)$ corresponds to the jth column of a standard input-output matrix of intermediate of intermediate flows.

Constraints at each node: market clearing In our canonical model, there is a single market for each sector. This implies free trade between regions.

For every r in R, i in J, let $\mathrm{out}_{r,i,n+t} : \mathbb{R}^3_+ \to \mathbb{R}_+$ be a standard CES function of capital, labour and the intermediate-input bundle respectively:

$$\mathrm{out}_{r,i,n+t} = A^{\mathrm{out}}_{r,i,n+t} \cdot f^{\mathrm{out}}_{r,i}(\mathbf{kap}_{r,i,n+t}, \mathbf{lab}_{r,i,n+t}, \mathrm{int}_{r,i,n+t}) \tag{6}$$

Let $\mathrm{adj}_{r,i,n+t} : \mathbb{R}^2_+ \to \mathbb{R}$ be the quadratic cost of adjusting capital:

$$\mathrm{adj}_{r,i,n+t} = A^{\mathrm{adj}}_{r,i} \cdot \frac{(\mathbf{kap}_{r,i,n+t+1} - \mathbf{kap}_{r,i,n+t})^2}{\mathbf{kap}_{r,i}}. \tag{7}$$

Finally, the key feasibility condition is that the following expression (where we have suppressed reference to $n + t$), for total production minus total usage, is nonnegative for every i in J:

$$\sum \left\{ \mathrm{out}_{r,i} - \mathbf{con}_{r,i} - \sum \{\mathbf{sav}_{r,ij} + \mathbf{med}_{r,ij} : j \in J\} - \mathrm{adj}_{r,i} : r \in R \right\}. \tag{8}$$

The market clearing property itself then follows by virtue of the fact that the reward function u_{n+t} is strictly increasing in the consumption of every good.

Reward and Terminal Value functions at each node Beyond strictly increasing, we take the reward function u_{n+t} to be strictly concave in consumption and labour for each node n and time t. For the consumption and labour components of utility such as those we find in Atalay (2017), for each region r:

$$\mathrm{utl}_{r,n+t} = \beta^t \cdot \left(f^{\mathrm{utl}}_r(\mathbf{con}_{r,i,n+t} : i \in J) - \sum \left\{ \lambda_{r,i} \cdot \mathbf{lab}^2_{r,i,n+t} : i \in J \right\} \right) \tag{9}$$

where $0 < \beta < 1$ is the parameter that describes how agents discount future consumption and work; the consumption shares $\gamma_{r,i}$ reflect the amount spent on sector-i goods in region r; the labour shares $\lambda_{r,i}$ reflect the proportion of sector-i workers in region r. For $0 < \kappa < 1$, and ρ defined as for eq. (4), $\mathrm{u}_{r,n+t}$ is strictly concave. The weighted sum across regions (with weights determined by population) of such regional reward functions yields the "global"/"whole-of-economy" reward functions u_{n+t} we find in the objective function (1) each node. This, together with standard assumptions (Stachurski, 2009, Theorem 12.2.12) are sufficient for strict concavity of the terminal value function V_{n+T} in the state vector

ω_{n+t}. Together with the structure we have imposed on the transition laws and feasibility constraints, the social welfare Lagrangian $\mathcal{L}_n$ is strictly concave function of the control vector a_n for every n in $\mathcal{N}$. [7]

Remark. *As usual, the Lagrange multipliers of market clearing constraints are also the competitive market prices. The AMPL language has features for constraining prices where necessary.*

WELL-POSEDNESS OF THE OPTIMIZATION PROBLEM AT EACH NODE The above conditions ensure that the solving for the general equilibrium of the economy is equivalent to solving a strictly concave program at each node. This implies that the equilibrium at each node is unique. The first-order (necessary) conditions for are also sufficient and we can tackle the problem with local (nonlinear optimization) solvers such as Conopt or Ipopt which are many times faster than global solvers such as Baron.

CONSTRUCTING THE EMPIRICAL DISTRIBUTION ON PATHS Let us first define the strong solution we obtain from the above node-specific optimization problems. For a given decision tree $\mathcal{T} = (\mathcal{N}, \mathcal{E})$ and corresponding set of paths $\mathcal{P}$, let

$$\mathrm{StrongSol}(\mathcal{P}) \stackrel{\mathrm{def}}{=} \left\{(\omega_p^*, a_p^*) : p \in \mathcal{P}\right\}$$

where the superscript $*$ denotes the optimal state-policy values for each decision node n along a given path p. Note that $\mathrm{StrongSol}(\mathcal{P})$ only contains the realised initial state-policy values since, for each n, the look-ahead values $\left\{(\omega_{n+t}^*, a_{n+t}^*) : t > 0\right\}$ are only used to formulate the plan at node n. This notion of solution is strong in the sense that, assuming the model is fully-specified, it tells us precisely what will happen once the economy reaches a given node. Moreover, in simple cases, such as example 1 where there are just 28 paths, $\mathrm{StrongSol}(\mathcal{P})$ can also be viewed as a complete contingent plan of action and future capital. Another sense in which the solution is strong when $\sharp\mathcal{P}$ is small is that we are able to observe the entire sample space: we are able to observe the true distribution of optimal paths through time.

[7] Recall that the Lagrangian is the sum of the objective and each of the constraints scaled by the *dual* variables (also known as Lagrange multipliers).

Weak solutions Beyond simple examples such as example 1, we can only hope to partially observe the sample space. The set of optimal solution paths that we have derived is then a subsample and, since our collection of nodes is incomplete, we can no longer refer to our solution as strong. The only sense in which we can speak of a solution is in terms of the empirical distribution over paths. From this we can estimate the expected path, the variance, and so on. We now formalise this weaker notion of solution.

Let $\mathfrak{o}$ denote the root node of the decision tree. Recalling that we assume the decision tree has paths of uniform length, let S denote the set of *path times*: for each s in S the node p_s is s steps into the future from $\mathfrak{o}$ along path p. We begin by constructing the empirical distribution associated with $\mathfrak{o} + s$ for each $s \in S$. Let N_s denote the set of nodes that are s steps into the future from $\mathfrak{o}$.[8]

For any s in S note that $\mathbf{con}_{r,i,\mathfrak{o}+s}$ is a scalar-valued random variable on N_s. Moreover, for each n in N_s, let $\mathrm{CON}_{r,i,n}$ denote a realisation of this random variable. By realisation, we mean a strong solution to this particular variable. For any of our policy or state variables $x_{\mathfrak{o}+s}$ let X_n denote the realisation associated with node n. Then the empirical distribution generated by $\{X_n : n \in N_s\}$ is the function $F_{x_s} : \mathbb{R} \to [0, 1]$,

$$F_{x_s}(z) = \frac{1}{\sharp N_s} \cdot \sum_{n \in N_s} 1_{X_n \leqslant z}. \tag{10}$$

The weak solution for a given policy or state variable x is simply the sequence $F_x = \{F_{x_s} : s \in S\}$ of empirical distributions. F_x is the empirical distribution of the path-valued random variable x. For instance, $F_{\mathbf{con}_{r,i}}$ is the empirical distribution for the path of consumption of sector-i goods in region r.

In turn, the weak solution for the entire problem is the joint empirical distribution $F_{\mathfrak{o}}$ we obtain by taking the product of the marginals. (This is the case because all these probability measures are push-forwards that are dependent on the same source of uncertainty.)

Measuring accuracy in terms of the Euler error One normative measure of the accuracy of our weak solutions is the Euler error that is associated with

[8] That is $N_s = \{p_s \in \mathcal{N} : p \in \mathcal{P}\}$.

the (intertemporal) Euler equation. This equation is a necessary condition for optimality of the stochastic dynamic program Stokey et al. (1989). In the present setting the derivation of Euler equation is complicated by two factors: the number of control variables exceeds the number of state variables; and there is no explicit form for To address this issue, for each n, we appeal to the envelope theorem. In particular, we follow González-Sánchez and Hernández-Lerma (2014) and derive an indirect utility function v_n as a function of current and next-period states ω_n and ω_{n^+}.

$$\mathrm{v}_n(\omega_n, \omega_{n^+}) \stackrel{\text{def}}{=} \max_{a_n} \quad \mathrm{u}_n(\omega_n, a_n) \quad \text{subject to}$$
$$\omega_{n^+} = \mathrm{tran}_n(\omega_n, a_n) \quad \text{and} \quad \mathrm{feas}_n(\omega_n, a_n) \geqslant 0$$

where the transition and feasibility constraints are those we have described above. Then, via the envelope theorem, for models with exogenous uncertainty (where actions do not affect transition probabilities), the Euler error associated with step s is

$$\frac{\beta \cdot \mathbb{E}_{\mathsf{o}} \left\{ \nabla_{\mathbf{kap}_{s+1}} \mathrm{v}_{s+1}(\omega_{s+1}, \omega_{s+2}) \right\}}{\mathbb{E}_{\mathsf{o}} \left\{ \nabla_{\mathbf{kap}_{s+1}} \mathrm{v}_{s}(\omega_{s}, \omega_{s+1}) \right\}} + 1 \tag{11}$$

It requires that the marginal loss (the denominator is negative) associated with foregoing today's consumption (by increasing tomorrow's capital) is balanced against the marginal benefit of increasing consumption tomorrow. Note that the expectation $\mathbb{E}_{\mathsf{o}}$ in the expression for Euler error is the empirical one we have derived as part of our weak solution. In other words, our condition for accuracy demands that the empirical Euler error is small. As Cai and Judd (2021) show for a wide variety of settings, the worst case of this error across s in S is less than 1%.

We note that, since there is no explicit form for the indirect utility function v, we build on Scheidegger and Bilionis (2019) to use gaussian processes to generate an approximation. That is, we introduce uncertainty of type III. This brings us to the next section.

5.2. Types II and III uncertainty: approximations

In MAIWAR, weak solutions allow us to accommodate uncertainty of type II and III without sacrificing theoretically appealing solution properties such as the *in-*

tertemporal Euler equation a feature that is absent from CGE models: see Dixon and R. (2020) for a recent attempt to introduce this concept into the CoPS methodology.[9] Our agents (and modellers) strive to ensure that the marginal benefits of current consumption against the marginal benefits of uncertain future production.

Yet in order to arrive at a tractable and fast method for deriving a weak solution, we will need to approximate. By this we do not only mean the inevitable partial sampling of the set of paths we have already highighted in the previous section, we mean:

1. approximating the terminal value function (see Bertsekas (2019) for a detailed exposition of this matter in the setting of reinforcement learning);
2. by using certainty-equivalent projections (see Cai and Judd (2021) and Cai et al. (2017) for a detailed exposition of this concept).

Our first departure from canonical expected utility maximisation, where agents form a complete contingent plans of action, is as follows. At each decision node, agents solve a nonlinear program where errors associated with future type I uncertain variables are replaced with some convex combination of the best and worst case scenario. Abundant evidence from the literature on behavioural economics shows that agents adopt similar heuristic approaches to dealing with complex decisions in the face of uncertainty. The "*certainty equivalent*" projection of Cai and Judd (2021) is one such heuristic. Aside from tractability, the inbuilt flexibility and scope for improvement in aspect of the model is one reason for adopting the methodology of Cai and Judd (2021). That is say, we view this, not only as an approximation, but also a reasonable behavioural feature.

A second form of type II uncertainty arises because economic agents struggle with long time horizons: indeed, isn't this why policy makers appeal to CGE models in the first place? Evidence shows that agents focus their attention on the next few periods. This motivates the finite look-ahead feature of our model. Our agents focus on the near future as opposed to the far-flung steady states at the heart of other approaches.

[9] This paper provided inspiration for our efforts.

In MAIWAR there is a useful alignment between modeller's interests and the type II uncertainties we have just outlined. For although there are well-established methods for implementing large-scale linear stochastic programs, the tools for large-scale nonlinear stochastic programming are less evolved Rehfeldt et al. (2022). Furthermore, although the time horizon of our agents is indefinite (and typically modelled as infinite), most methods for dealing with such time horizons assume stationarity or provide a linear approximation around the nonstochastic steady state. Nor is there an obvious way to model 2050 net-zero emissions without proper modelling of the medium term. Constraints that will bind in 28 years time, but have implications today are poorly suited to steady-state approximations and standard policy/value function iteration methods. [10]

In summary, our agents solve finite sequences of nonlinear programs each with a finite horizon and an approximation for the terminal value function. This is an approximation in the sense that normative agents solve nonlinear stochastic programs each with an infinite horizon or the true terminal value function. We argue that our approximations are well-aligned with behaviour or agents in practice. Moreover, the weak solution that we derive explicitly accommodates our approximation methods. Finally, since this weak solution is an empirical distribution we can verify that the solution is accurate to within a 1% Euler error approximation. The same measure is typical of "global approaches". For instance, in the deep neural network approach of Azinovic et al. (2019) the policy iteration process is successful once the Euler error term converges to a sufficiently small number. Our investigations of this approach yielded slow and unstable results. The notion of a weak solution recognizes the fact that our modeller is typically facing substantial type III uncertainty. Parametric uncertainty is a feature of the model and a central part of the solution.

[10] In our experience, "global" value and policy function iteration methods are also slow and unstable in the setting of multi-sectoral models with medium-term constraints. See Cai and Judd (2021) for a related discussion.

5.3. Computation and software

The uncertainty we model is facilitated and expedited by the fact that we have in place the software infrastructure to ensure that simple model sweeps are easy to conduct on the local HPC cluster. The unit for parallelisation is paths: each path is passed to a single core. This means that most desktop computers can easily parallelise. For example, consider our canonical example of irreversible risk: where there are 28 paths (each of length 28 years), and assume 7 regions and 20 sectors. For the functions we specify above, each step along a path takes 20 seconds on average. Thus a path takes $28 \times \frac{1}{3} \approx 9$ minutes to solve. For a desktop with 4 available cores, we are able to assign 7 paths to each core and complete the generate the full empirical distribution in $7 \times 9 \approx 1$ hour.

After an exhaustive search, we chose AMPL (A modelling language for mathematical programming) Fourer et al. (1990) as the language and ecosystem for communicating with solvers. AMPL is used for pre-processing/pre-solving: the stage before the nonlinear optimization problem is passed to the solver. The goal of pre-solving is to ensure the solver stage is faster and more reliable. Pre-solving includes the generation of automatic, sparse derivatives as well as domain/variable reduction. AMPL syntax is much closer to standard mathematics relative to GAMS and we have found it to produce faster total solve times than GAMS.[11] We spent a lot of time trying purely open source options in Python, Julia and to lesser extent C++. This experimentation also involved machine learning tools (based on Scheidegger's work). We found that, overall, combining the Cai and Judd (2021) methodology with AMPL pre-solving yields unrivalled modelling performance, flexibility and reliability.

We have found three nonlinear solvers to be most useful. Conopt which is proprietary and based on the reduced-gradient method (itself based on the robust simplex method). The open source Ipopt (pronounced "eye-pea-opt"). This is celebrated application of the interior point by Wächter and Biegler (2006). Knitro which is tries a number of approaches and, unlike Conopt and Ipopt can handle integer-valued variables. We have tried other solvers as well, but the above are outstanding, with Conopt being the most robust.

[11]Cai and Judd (2021) find that GAMS is substantially faster than Matlab.

Future improvements to our methodology relate to tuning MAIWAR model parameters and improving on our existing framework for parallel computing. In particular, to enable simple automated parallelisation even desktop computers, we will use the Python API for AMPL along with nextflow. Currently this process is manual and automatic tuning of parameters using machine learning is step in the development of MAIWAR. Many of the tools that we worked on in the second half of last year (first six months of SMaRT) will serve us well in this respect.

5.4. Nonlinearity and (non)steady-state methods

Atalay (2017) estimates to be approximately 0.1 so that $\rho \approx -9$.
Such approximations are even less appealing for nonlinear production networks that arise when elasticities of substition are below one.
Impressive recent work using Hamiltonian methods still fails to capture the central economic theme of uncertainty.

References

Atalay, E. (2017). How important are sectoral shocks? *American Economic Journal: Macroeconomics*, 9(4):254–80.

Azinovic, M., Gaegauf, L., and Scheidegger, S. (2019). Deep equilibrium nets. *Available at SSRN 3393482*.

Bertsekas, D. (2019). *Reinforcement learning and optimal control*. Athena Scientific.

Cai, Y., Judd, K., and Steinbuks, J. (2017). A nonlinear certainty equivalent approximation method for dynamic stochastic problems. *Quantitative Economics*, 8(1):117–147.

Cai, Y. and Judd, K. L. (2021). A simple but powerful simulated certainty equivalent approximation method for dynamic stochastic problems. Technical report, National Bureau of Economic Research.

Dixon, P. B. and R., R. M. (2020). Developing a dsge consumption function for a cge model.

Fourer, R., Gay, D. M., and Kernighan, B. W. (1990). A modeling language for mathematical programming. *Management Science*, 36(5):519–554.

González-Sánchez, D. and Hernández-Lerma, O. (2014). On the euler equation approach to discrete–time nonstationary optimal control problems. *Journal of Dynamics & Games*, 1(1):57.

Hosoe, N., Gasawa, K., and Hashimoto, H. (2010). *Textbook of computable general equilibrium modeling: programming and simulations*. Springer.

Kojić, V. (2021). A note on concavity conditions of cobb–douglas and ces production function with at least two inputs. *Studies in Microeconomics*, 9(1):1–10.

Long, J. B. and Plosser, C. I. (1983). Real business cycles. *Journal of political Economy*, 91(1):39–69.

Owen, A. B. (2001). *Empirical likelihood*. Chapman and Hall/CRC.

Raymond, H., Elazaar, D., and Norden, I. (2022). Modelling production networks and supply chain disruption.

Rehfeldt, D., Hobbie, H., Schönheit, D., Koch, T., Möst, D., and Gleixner, A. (2022). A massively parallel interior-point solver for lps with generalized arrowhead structure, and applications to energy system models. *European Journal of Operational Research*, 296(1):60–71.

Reiss, P. C. and Wolak, F. A. (2007). Structural econometric modeling: Rationales and examples from industrial organization. *Handbook of econometrics*, 6:4277–4415.

Scheidegger, S. and Bilionis, I. (2019). Machine learning for high-dimensional dynamic stochastic economies. *Journal of Computational Science*, 33:68–82.

Stachurski, J. (2009). *Economic dynamics: theory and computation*. MIT Press.

Stokey, N. L., Lucas, R. E., and C., P. E. (1989). *Recursive methods in economic dynamics*. Harvard University Press.

Wächter, A. and Biegler, L. T. (2006). On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Mathematical programming*, 106(1):25–57.