

Veriopt

September 11, 2022

Abstract

The Veriopt project aims to prove the optimization pass of the GraalVM compiler. The GraalVM compiler includes a sophisticated Intermediate Representation (IR) in the form of a sea-of-nodes based graph structure. We first define the IR graph structure in the Isabelle/HOL interactive theorem prover. We subsequently give the evaluation of the structure a semantics based on the current understanding of the purpose of each IR graph node. Optimization phases are then encoded including the static analysis passes required for an optimization. Each optimization phase is proved to be correct by proving that a bisimulation exists between the unoptimized and optimized graphs. The following document has been automatically generated from the Isabelle/HOL source to provide a very comprehensive definition of the semantics and optimizations introduced by the Veriopt project.

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1 Runtime Values and Arithmetic

```

theory Values
  imports
    HOL-Library.Word
    HOL-Library.Signed-Division
    HOL-Library.Float
    HOL-Library.LaTeXsugar
  begin

  lemma  $-\!((x::\text{float})-y) = (y-x)$ 
    by simp

```

In order to properly implement the IR semantics we first introduce a type that represents runtime values. These runtime values represent the full range of primitive types currently allowed by our semantics, ranging from basic integer types to object references and arrays.

Note that Java supports 64, 32, 16, 8 signed ints, plus 1 bit (boolean) ints, and char is 16-bit unsigned. E.g. an 8-bit stamp has a default range of -128..+127. And a 1-bit stamp has a default range of -1..0, surprisingly.

During calculations the smaller sizes are sign-extended to 32 bits, but explicit widening nodes will do that, so most binary calculations should see equal input sizes.

An object reference is an option type where the *None* object reference points to the static fields. This is examined more closely in our definition of the heap.

```

type-synonym int64 = 64 word — long
type-synonym int32 = 32 word — int
type-synonym int16 = 16 word — short
type-synonym int8 = 8 word — char
type-synonym int1 = 1 word — boolean

```

```

abbreviation valid-int-widths :: nat set where
  valid-int-widths ≡ {1, 8, 16, 32, 64}

```

Option 2: explicit width stored with each integer value. However, this does not help us to distinguish between short (signed) and char (unsigned).

```

typedef IntWidth = { w :: nat . w=1 ∨ w=8 ∨ w=16 ∨ w=32 ∨ w=64 }
by blast

```

```

setup-lifting type-definition-IntWidth

```

```

lift-definition IntWidthBits :: IntWidth ⇒ nat
is λw. w .

```

Option 3: explicit type stored with each integer value.

```

datatype IntType = ILong | IInt | IShort | IChar | IByte | IBoolean

```

```

fun int-bits :: IntType  $\Rightarrow$  nat where
  int-bits ILong = 64 |
  int-bits IInt  = 32 |
  int-bits IShort = 16 |
  int-bits IChar  = 16 |
  int-bits IByte  = 8  |
  int-bits IBoolean = 1

```

```

fun int-signed :: IntType  $\Rightarrow$  bool where
  int-signed ILong = True |
  int-signed IInt  = True |
  int-signed IShort = True |
  int-signed IChar  = False |
  int-signed IByte  = True |
  int-signed IBoolean = True

```

Option 4: int64 with the number of significant bits.

```

type-synonym iwidth = nat
type-synonym objref = nat option

```

```

datatype (discs-sels) Value =
  UndefVal |

```

```

  IntVal iwidth int64 |

```

```

  ObjRef objref |
  ObjStr string

```

```

fun intval-bits :: Value  $\Rightarrow$  nat where
  intval-bits (IntVal b v) = b

```

```

fun intval-word :: Value  $\Rightarrow$  int64 where
  intval-word (IntVal b v) = v

```

```

fun bit-bounds :: nat  $\Rightarrow$  (int  $\times$  int) where
  bit-bounds bits = (((2  $\wedge$  bits) div 2) * -1, ((2  $\wedge$  bits) div 2) - 1)

```

```

definition logic-negate :: ('a::len) word  $\Rightarrow$  'a word where
  logic-negate x = (if x = 0 then 1 else 0)

```

```

fun int-signed-value :: iwidth  $\Rightarrow$  int64  $\Rightarrow$  int where
  int-signed-value b v = sint (signed-take-bit (b - 1) v)

```

```
fun int-unsigned-value :: iwidth  $\Rightarrow$  int64  $\Rightarrow$  int where
  int-unsigned-value b v = uint v
```

Converts an integer word into a Java value.

```
fun new-int :: iwidth  $\Rightarrow$  int64  $\Rightarrow$  Value where
  new-int b w = IntVal b (take-bit b w)
```

Converts an integer word into a Java value, iff the two types are equal.

```
fun new-int-bin :: iwidth  $\Rightarrow$  iwidth  $\Rightarrow$  int64  $\Rightarrow$  Value where
  new-int-bin b1 b2 w = (if b1=b2 then new-int b1 w else UndefVal)
```

```
fun wf-bool :: Value  $\Rightarrow$  bool where
  wf-bool (IntVal b w) = (b = 1) |
  wf-bool - = False
```

```
fun val-to-bool :: Value  $\Rightarrow$  bool where
  val-to-bool (IntVal b val) = (if val = 0 then False else True) |
  val-to-bool val = False
```

```
fun bool-to-val :: bool  $\Rightarrow$  Value where
  bool-to-val True = (IntVal 32 1) |
  bool-to-val False = (IntVal 32 0)
```

Converts an Isabelle bool into a Java value, iff the two types are equal.

```
fun bool-to-val-bin :: iwidth  $\Rightarrow$  iwidth  $\Rightarrow$  bool  $\Rightarrow$  Value where
  bool-to-val-bin t1 t2 b = (if t1 = t2 then bool-to-val b else UndefVal)
```

```
fun is-int-val :: Value  $\Rightarrow$  bool where
  is-int-val v = is-IntVal v
```

A convenience function for directly constructing -1 values of a given bit size.

```
fun neg-one :: iwidth  $\Rightarrow$  int64 where
  neg-one b = mask b
```

```
lemma neg-one-value[simp]: new-int b (neg-one b) = IntVal b (mask b)
by simp
```

```
lemma neg-one-signed[simp]:
  assumes 0 < b
  shows int-signed-value b (neg-one b) = -1
  by (smt (verit, best) assms diff-le-self diff-less int-signed-value.simps less-one
    mask-eq-take-bit-minus-one neg-one.simps nle-le signed-minus-1 signed-take-bit-of-minus-1
    signed-take-bit-take-bit verit-comp-simplify1(1))
```

1.1 Arithmetic Operators

We need to introduce arithmetic operations which agree with the JVM.

Within the JVM, bytecode arithmetic operations are performed on 32 or 64 bit integers, unboxing where appropriate.

The following collection of *intval* functions correspond to the JVM arithmetic operations. We merge the 32 and 64 bit operations into a single function, even though the stamp of each *IRNode* tells us exactly what the bit widths will be. These merged functions make it easier to do the instantiation of *Value* as 'plus', etc. It might be worse for reasoning, because it could cause more case analysis, but this does not seem to be a problem in practice.

```
fun intval-add :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-add (IntVal b1 v1) (IntVal b2 v2) =
    (if b1 = b2 then IntVal b1 (take-bit b1 (v1+v2)) else.UndefVal) |
  intval-add - - =.UndefVal
```

```
fun intval-sub :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-sub (IntVal b1 v1) (IntVal b2 v2) = new-int-bin b1 b2 (v1-v2) |
  intval-sub - - =.UndefVal
```

```
instantiation Value :: minus
begin
```

```
definition minus-Value :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  minus-Value = intval-sub
```

```
instance proof qed
end
```

```
fun intval-mul :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-mul (IntVal b1 v1) (IntVal b2 v2) = new-int-bin b1 b2 (v1*v2) |
  intval-mul - - =.UndefVal
```

```
instantiation Value :: times
begin
```

```
definition times-Value :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  times-Value = intval-mul
```

```
instance proof qed
```

end

```
fun intval-div :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-div (IntVal b1 v1) (IntVal b2 v2) =
    new-int-bin b1 b2 (word-of-int
      ((int-signed-value b1 v1) sdiv (int-signed-value b2 v2))) |
  intval-div - - =.UndefVal
```

```
instantiation Value :: divide
begin
```

```
definition divide-Value :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  divide-Value = intval-div
```

```
instance proof qed
end
```

```
fun intval-mod :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-mod (IntVal b1 v1) (IntVal b2 v2) =
    new-int-bin b1 b2 (word-of-int
      ((int-signed-value b1 v1) smod (int-signed-value b2 v2))) |
  intval-mod - - =.UndefVal
```

```
instantiation Value :: modulo
begin
```

```
definition modulo-Value :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  modulo-Value = intval-mod
```

```
instance proof qed
end
```

1.2 Bitwise Operators and Comparisons

```
context
  includes bit-operations-syntax
begin
```

```
fun intval-and :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-and (IntVal b1 v1) (IntVal b2 v2) = new-int-bin b1 b2 (v1 AND v2) |
  intval-and - - =.UndefVal
```

```
fun intval-or :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-or (IntVal b1 v1) (IntVal b2 v2) = new-int-bin b1 b2 (v1 OR v2) |
```


intval-or - - = *UndefVal*

fun *intval-xor* :: *Value* \Rightarrow *Value* \Rightarrow *Value* **where**
 intval-xor (*IntVal* *b1* *v1*) (*IntVal* *b2* *v2*) = *new-int-bin* *b1* *b2* (*v1 XOR v2*) |
 intval-xor - - = *UndefVal*

fun *intval-short-circuit-or* :: *Value* \Rightarrow *Value* \Rightarrow *Value* **where**
 intval-short-circuit-or (*IntVal* *b1* *v1*) (*IntVal* *b2* *v2*) = *bool-to-val-bin* *b1* *b2* (((*v1*
 $\neq 0$) \vee (*v2* $\neq 0$))) |
 intval-short-circuit-or - - = *UndefVal*

fun *intval-equals* :: *Value* \Rightarrow *Value* \Rightarrow *Value* **where**
 intval-equals (*IntVal* *b1* *v1*) (*IntVal* *b2* *v2*) = *bool-to-val-bin* *b1* *b2* (*v1* = *v2*) |
 intval-equals - - = *UndefVal*

fun *intval-less-than* :: *Value* \Rightarrow *Value* \Rightarrow *Value* **where**
 intval-less-than (*IntVal* *b1* *v1*) (*IntVal* *b2* *v2*) =
 bool-to-val-bin *b1* *b2* (*int-signed-value* *b1* *v1* < *int-signed-value* *b2* *v2*) |
 intval-less-than - - = *UndefVal*

fun *intval-below* :: *Value* \Rightarrow *Value* \Rightarrow *Value* **where**
 intval-below (*IntVal* *b1* *v1*) (*IntVal* *b2* *v2*) = *bool-to-val-bin* *b1* *b2* (*v1* < *v2*) |
 intval-below - - = *UndefVal*

fun *intval-not* :: *Value* \Rightarrow *Value* **where**
 intval-not (*IntVal* *t* *v*) = *new-int* *t* (*NOT v*) |
 intval-not - = *UndefVal*

fun *intval-negate* :: *Value* \Rightarrow *Value* **where**
 intval-negate (*IntVal* *t* *v*) = *new-int* *t* ($- v$) |
 intval-negate - = *UndefVal*

fun *intval-abs* :: *Value* \Rightarrow *Value* **where**
 intval-abs (*IntVal* *t* *v*) = *new-int* *t* (*if int-signed-value* *t* *v* < 0 *then* $- v$ *else* *v*) |
 intval-abs - = *UndefVal*

fun *intval-conditional* :: *Value* \Rightarrow *Value* \Rightarrow *Value* \Rightarrow *Value* **where**
 intval-conditional *cond* *tv* *fv* = (*if* (*val-to-bool* *cond*) *then* *tv* *else* *fv*)

TODO: clarify which widths this should work on: just 1-bit or all?

fun *intval-logic-negation* :: *Value* \Rightarrow *Value* **where**
 intval-logic-negation (*IntVal* *b* *v*) = *new-int* *b* (*logic-negate* *v*) |
 intval-logic-negation - = *UndefVal*

1.3 Narrowing and Widening Operators

Note: we allow these operators to have *inBits*=*outBits*, because the Graal compiler also seems to allow that case, even though it should rarely / never arise in practice.

```

value sint(signed-take-bit 0 (1 :: int32))

fun intval-narrow :: nat  $\Rightarrow$  nat  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-narrow inBits outBits (IntVal b v) =
    (if inBits = b  $\wedge$  0 < outBits  $\wedge$  outBits  $\leq$  inBits  $\wedge$  inBits  $\leq$  64
     then new-int outBits v
     else UndefVal) |
  intval-narrow - - - = UndefVal

value intval(intval-narrow 16 8 (IntVal32 (512 - 2)))

value sint (signed-take-bit 7 ((256 + 128) :: int64))

```

```

fun intval-sign-extend :: nat  $\Rightarrow$  nat  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-sign-extend inBits outBits (IntVal b v) =
    (if inBits = b  $\wedge$  0 < inBits  $\wedge$  inBits  $\leq$  outBits  $\wedge$  outBits  $\leq$  64
     then new-int outBits (signed-take-bit (inBits - 1) v)
     else UndefVal) |
  intval-sign-extend - - - = UndefVal

fun intval-zero-extend :: nat  $\Rightarrow$  nat  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-zero-extend inBits outBits (IntVal b v) =
    (if inBits = b  $\wedge$  0 < inBits  $\wedge$  inBits  $\leq$  outBits  $\wedge$  outBits  $\leq$  64
     then new-int outBits (take-bit inBits v)
     else UndefVal) |
  intval-zero-extend - - - = UndefVal

```

Some well-formedness results to help reasoning about narrowing and widening operators

```

lemma intval-narrow-ok:
  assumes intval-narrow inBits outBits val  $\neq$  UndefVal
  shows 0 < outBits  $\wedge$  outBits  $\leq$  inBits  $\wedge$  inBits  $\leq$  64  $\wedge$  outBits  $\leq$  64  $\wedge$ 
    is-IntVal val  $\wedge$ 
    intval-bits val = inBits
  using assms intval-narrow.simps neq0-conv intval-bits.simps
  by (metis Value.disc(2) intval-narrow.elims le-trans)

```

```

lemma intval-sign-extend-ok:
  assumes intval-sign-extend inBits outBits val  $\neq$  UndefVal
  shows 0 < inBits  $\wedge$ 
    inBits  $\leq$  outBits  $\wedge$  outBits  $\leq$  64  $\wedge$ 
    is-IntVal val  $\wedge$ 
    intval-bits val = inBits
  using assms intval-sign-extend.simps neq0-conv
  by (metis intval-bits.simps intval-sign-extend.elims is-IntVal-def)

```

```

lemma intval-zero-extend-ok:
  assumes intval-zero-extend inBits outBits val  $\neq$  UndefVal
  shows  $0 < \text{inBits} \wedge$ 
     $\text{inBits} \leq \text{outBits} \wedge \text{outBits} \leq 64 \wedge$ 
     $\text{is-IntVal } \text{val} \wedge$ 
     $\text{intval-bits } \text{val} = \text{inBits}$ 
  using assms intval-sign-extend.simps neq0-conv
  by (metis intval-bits.simps intval-zero-extend.elims is-IntVal-def)

```

1.4 Bit-Shifting Operators

```

definition shiffl (infix  $<<$  75) where
  shiffl w n = (push-bit n) w

```

```

lemma shiffl-power[simp]:  $(x::('a::\text{len}) \text{ word}) * (2^j) = x << j$ 
  unfolding shiffl-def apply (induction j)
  apply simp unfolding funpow-Suc-right
  by (metis (no-types, opaque-lifting) push-bit-eq-mult)

```

```

lemma  $(x::('a::\text{len}) \text{ word}) * ((2^j) + 1) = x << j + x$ 
  by (simp add: distrib-left)

```

```

lemma  $(x::('a::\text{len}) \text{ word}) * ((2^j) - 1) = x << j - x$ 
  by (simp add: right-diff-distrib)

```

```

lemma  $(x::('a::\text{len}) \text{ word}) * ((2^j) + (2^k)) = x << j + x << k$ 
  by (simp add: distrib-left)

```

```

lemma  $(x::('a::\text{len}) \text{ word}) * ((2^j) - (2^k)) = x << j - x << k$ 
  by (simp add: right-diff-distrib)

```

```

definition shiftr (infix  $>>$  75) where
  shiftr w n = (drop-bit n) w

```

```

value  $(255 :: 8 \text{ word}) >>> (2 :: \text{nat})$ 

```

```

definition sshiftr ::  $'a :: \text{len word} \Rightarrow \text{nat} \Rightarrow 'a :: \text{len word}$  (infix  $>>$  75) where
  sshiftr w n = word-of-int ((sint w) div (2^n))

```

```

value  $(128 :: 8 \text{ word}) >> 2$ 

```

Note that Java shift operators use unary numeric promotion, unlike other binary operators, which use binary numeric promotion (see the Java language reference manual). This means that the left-hand input determines the output size, while the right-hand input can be any size.

```

fun shift-amount ::  $iwidth \Rightarrow \text{int64} \Rightarrow \text{nat}$  where

```

```

    shift-amount b val = unat (val AND (if b = 64 then 0x3F else 0x1f))

fun intval-left-shift :: Value ⇒ Value ⇒ Value where
    intval-left-shift (IntVal b1 v1) (IntVal b2 v2) = new-int b1 (v1 << shift-amount
    b1 v2) |
    intval-left-shift - - = UndefVal

Signed shift is more complex, because we sometimes have to insert 1 bits at
the correct point, which is at b1 bits.

fun intval-right-shift :: Value ⇒ Value ⇒ Value where
    intval-right-shift (IntVal b1 v1) (IntVal b2 v2) =
        (let shift = shift-amount b1 v2 in
         let ones = mask b1 AND (NOT (mask (b1 - shift) :: int64)) in
         (if int-signed-value b1 v1 < 0
          then new-int b1 (ones OR (v1 >>> shift))
          else new-int b1 (v1 >>> shift))) |
    intval-right-shift - - = UndefVal

fun intval-uright-shift :: Value ⇒ Value ⇒ Value where
    intval-uright-shift (IntVal b1 v1) (IntVal b2 v2) = new-int b1 (v1 >>> shift-amount
    b1 v2) |
    intval-uright-shift - - = UndefVal

end

```

2 Examples of Narrowing / Widening Functions

experiment begin

corollary *intval-narrow 32 8 (IntVal 32 (256 + 128)) = IntVal 8 128 by simp*

corollary *intval-narrow 32 8 (IntVal 32 (-2)) = IntVal 8 254 by simp*

corollary *intval-narrow 32 1 (IntVal 32 (-2)) = IntVal 1 0 by simp*

corollary *intval-narrow 32 1 (IntVal 32 (-3)) = IntVal 1 1 by simp*

corollary *intval-narrow 32 8 (IntVal 64 (-2)) = UndefVal by simp*

corollary *intval-narrow 64 8 (IntVal 32 (-2)) = UndefVal by simp*

corollary *intval-narrow 64 8 (IntVal 64 254) = IntVal 8 254 by simp*

corollary *intval-narrow 64 8 (IntVal 64 (256+127)) = IntVal 8 127 by simp*

corollary *intval-narrow 64 64 (IntVal 64 (-2)) = IntVal 64 (-2) by simp*

end

experiment begin

corollary *intval-sign-extend 8 32 (IntVal 8 (256 + 128)) = IntVal 32 (2³² - 128) by simp*

corollary *intval-sign-extend 8 32 (IntVal 8 (-2)) = IntVal 32 (2³² - 2) by simp*

corollary *intval-sign-extend 1 32 (IntVal 1 (-2)) = IntVal 32 0 by simp*

corollary *intval-sign-extend 1 32 (IntVal 1 (-3)) = IntVal 32 (mask 32) by simp*

```

corollary intval-sign-extend 8 32 (IntVal 64 254) = UndefVal by simp
corollary intval-sign-extend 8 64 (IntVal 32 254) = UndefVal by simp
corollary intval-sign-extend 8 64 (IntVal 8 254) = IntVal 64 (-2) by simp
corollary intval-sign-extend 32 64 (IntVal 32 ( $2^{32} - 2$ )) = IntVal 64 (-2) by
simp
corollary intval-sign-extend 64 64 (IntVal 64 (-2)) = IntVal 64 (-2) by simp
end

```

experiment begin

```

corollary intval-zero-extend 8 32 (IntVal 8 (256 + 128)) = IntVal 32 128 by simp
corollary intval-zero-extend 8 32 (IntVal 8 (-2)) = IntVal 32 254 by simp
corollary intval-zero-extend 1 32 (IntVal 1 (-1)) = IntVal 32 1 by simp
corollary intval-zero-extend 1 32 (IntVal 1 (-2)) = IntVal 32 0 by simp

```

```

corollary intval-zero-extend 8 32 (IntVal 64 (-2)) = UndefVal by simp
corollary intval-zero-extend 8 64 (IntVal 64 (-2)) = UndefVal by simp
corollary intval-zero-extend 8 64 (IntVal 8 254) = IntVal 64 254 by simp
corollary intval-zero-extend 32 64 (IntVal 32 ( $2^{32} - 2$ )) = IntVal 64 ( $2^{32} - 2$ ) by simp
corollary intval-zero-extend 64 64 (IntVal 64 (-2)) = IntVal 64 (-2) by simp
end

```

experiment begin

```

corollary intval-right-shift (IntVal 8 128) (IntVal 8 0) = IntVal 8 128 by eval
corollary intval-right-shift (IntVal 8 128) (IntVal 8 1) = IntVal 8 192 by eval
corollary intval-right-shift (IntVal 8 128) (IntVal 8 2) = IntVal 8 224 by eval
corollary intval-right-shift (IntVal 8 128) (IntVal 8 8) = IntVal 8 255 by eval
corollary intval-right-shift (IntVal 8 128) (IntVal 8 31) = IntVal 8 255 by eval
end

```

lemma *intval-add-sym*:

shows *intval-add* a b = *intval-add* b a

by (*induction* a; *induction* b; *auto simp: add.commute*)

code-deps *intval-add*

code-thms *intval-add*

lemma *intval-add* (*IntVal* 32 ($2^{31}-1$)) (*IntVal* 32 ($2^{31}-1$)) = *IntVal* 32 ($2^{32} - 2$)

by *eval*

lemma *intval-add* (*IntVal* 64 ($2^{31}-1$)) (*IntVal* 64 ($2^{31}-1$)) = *IntVal* 64 4294967294

```

    by eval
end

```

3 Nodes

3.1 Types of Nodes

```

theory IRNodes
  imports
    Values
begin

```

The GraalVM IR is represented using a graph data structure. Here we define the nodes that are contained within the graph. Each node represents a Node subclass in the GraalVM compiler, the node classes have annotated fields to indicate input and successor edges.

We represent these classes with each IRNode constructor explicitly labelling a reference to the node IDs that it stores as inputs and successors.

The `inputs_of` and `successors_of` functions partition those labelled references into input edges and successor edges of a node.

To identify each Node, we use a simple natural number index. Zero is always the start node in a graph. For human readability, within nodes we write `INPUT` (or special case thereof) instead of `ID` for input edges, and `SUCC` instead of `ID` for control-flow successor edges. Optional edges are handled as "INPUT option" etc.

```

type-synonym ID = nat
type-synonym INPUT = ID
type-synonym INPUT-ASSOC = ID
type-synonym INPUT-STATE = ID
type-synonym INPUT-GUARD = ID
type-synonym INPUT-COND = ID
type-synonym INPUT-EXT = ID
type-synonym SUCC = ID

```

```

datatype (discs-sels) IRNode =
  AbsNode (ir-value: INPUT)
  | AddNode (ir-x: INPUT) (ir-y: INPUT)
  | AndNode (ir-x: INPUT) (ir-y: INPUT)
  | BeginNode (ir-next: SUCC)
  | BytecodeExceptionNode (ir-arguments: INPUT list) (ir-stateAfter-opt: INPUT-STATE option) (ir-next: SUCC)
  | ConditionalNode (ir-condition: INPUT-COND) (ir-trueValue: INPUT) (ir-falseValue: INPUT)
  | ConstantNode (ir-const: Value)

```

| *DynamicNewArrayNode* (*ir-elementType*: INPUT) (*ir-length*: INPUT) (*ir-voidClass-opt*: INPUT option) (*ir-stateBefore-opt*: INPUT-STATE option) (*ir-next*: SUCC)
 | *EndNode*
 | *ExceptionObjectNode* (*ir-stateAfter-opt*: INPUT-STATE option) (*ir-next*: SUCC)

 | *FrameState* (*ir-monitorIds*: INPUT-ASSOC list) (*ir-outerFrameState-opt*: INPUT-STATE option) (*ir-values-opt*: INPUT list option) (*ir-virtualObjectMappings-opt*: INPUT-STATE list option)
 | *IfNode* (*ir-condition*: INPUT-COND) (*ir-trueSuccessor*: SUCC) (*ir-falseSuccessor*: SUCC)
 | *IntegerBelowNode* (*ir-x*: INPUT) (*ir-y*: INPUT)
 | *IntegerEqualsNode* (*ir-x*: INPUT) (*ir-y*: INPUT)
 | *IntegerLessThanNode* (*ir-x*: INPUT) (*ir-y*: INPUT)
 | *InvokeNode* (*ir-nid*: ID) (*ir-callTarget*: INPUT-EXT) (*ir-classInit-opt*: INPUT option) (*ir-stateDuring-opt*: INPUT-STATE option) (*ir-stateAfter-opt*: INPUT-STATE option) (*ir-next*: SUCC)
 | *InvokeWithExceptionNode* (*ir-nid*: ID) (*ir-callTarget*: INPUT-EXT) (*ir-classInit-opt*: INPUT option) (*ir-stateDuring-opt*: INPUT-STATE option) (*ir-stateAfter-opt*: INPUT-STATE option) (*ir-next*: SUCC) (*ir-exceptionEdge*: SUCC)
 | *IsNullNode* (*ir-value*: INPUT)
 | *KillingBeginNode* (*ir-next*: SUCC)
 | *LeftShiftNode* (*ir-x*: INPUT) (*ir-y*: INPUT)
 | *LoadFieldNode* (*ir-nid*: ID) (*ir-field*: string) (*ir-object-opt*: INPUT option) (*ir-next*: SUCC)
 | *LogicNegationNode* (*ir-value*: INPUT-COND)
 | *LoopBeginNode* (*ir-ends*: INPUT-ASSOC list) (*ir-overflowGuard-opt*: INPUT-GUARD option) (*ir-stateAfter-opt*: INPUT-STATE option) (*ir-next*: SUCC)
 | *LoopEndNode* (*ir-loopBegin*: INPUT-ASSOC)
 | *LoopExitNode* (*ir-loopBegin*: INPUT-ASSOC) (*ir-stateAfter-opt*: INPUT-STATE option) (*ir-next*: SUCC)
 | *MergeNode* (*ir-ends*: INPUT-ASSOC list) (*ir-stateAfter-opt*: INPUT-STATE option) (*ir-next*: SUCC)
 | *MethodCallTargetNode* (*ir-targetMethod*: string) (*ir-arguments*: INPUT list)
 | *MulNode* (*ir-x*: INPUT) (*ir-y*: INPUT)
 | *NarrowNode* (*ir-inputBits*: nat) (*ir-resultBits*: nat) (*ir-value*: INPUT)
 | *NegateNode* (*ir-value*: INPUT)
 | *NewArrayNode* (*ir-length*: INPUT) (*ir-stateBefore-opt*: INPUT-STATE option) (*ir-next*: SUCC)
 | *NewInstanceNode* (*ir-nid*: ID) (*ir-instanceClass*: string) (*ir-stateBefore-opt*: INPUT-STATE option) (*ir-next*: SUCC)
 | *NotNode* (*ir-value*: INPUT)
 | *OrNode* (*ir-x*: INPUT) (*ir-y*: INPUT)
 | *ParameterNode* (*ir-index*: nat)
 | *PiNode* (*ir-object*: INPUT) (*ir-guard-opt*: INPUT-GUARD option)
 | *ReturnNode* (*ir-result-opt*: INPUT option) (*ir-memoryMap-opt*: INPUT-EXT option)
 | *RightShiftNode* (*ir-x*: INPUT) (*ir-y*: INPUT)
 | *ShortCircuitOrNode* (*ir-x*: INPUT-COND) (*ir-y*: INPUT-COND)
 | *SignExtendNode* (*ir-inputBits*: nat) (*ir-resultBits*: nat) (*ir-value*: INPUT)

```

| SignedDivNode (ir-nid: ID) (ir-x: INPUT) (ir-y: INPUT) (ir-zeroCheck-opt: INPUT-GUARD option) (ir-stateBefore-opt: INPUT-STATE option) (ir-next: SUCC)

| SignedRemNode (ir-nid: ID) (ir-x: INPUT) (ir-y: INPUT) (ir-zeroCheck-opt: INPUT-GUARD option) (ir-stateBefore-opt: INPUT-STATE option) (ir-next: SUCC)

| StartNode (ir-stateAfter-opt: INPUT-STATE option) (ir-next: SUCC)
| StoreFieldNode (ir-nid: ID) (ir-field: string) (ir-value: INPUT) (ir-stateAfter-opt: INPUT-STATE option) (ir-object-opt: INPUT option) (ir-next: SUCC)
| SubNode (ir-x: INPUT) (ir-y: INPUT)
| UnsignedRightShiftNode (ir-x: INPUT) (ir-y: INPUT)
| UnwindNode (ir-exception: INPUT)
| ValuePhiNode (ir-nid: ID) (ir-values: INPUT list) (ir-merge: INPUT-ASSOC)
| ValueProxyNode (ir-value: INPUT) (ir-loopExit: INPUT-ASSOC)
| XorNode (ir-x: INPUT) (ir-y: INPUT)
| ZeroExtendNode (ir-inputBits: nat) (ir-resultBits: nat) (ir-value: INPUT)
| NoNode

| RefNode (ir-ref:ID)

```

```

fun opt-to-list :: 'a option ⇒ 'a list where
  opt-to-list None = [] |
  opt-to-list (Some v) = [v]

```

```

fun opt-list-to-list :: 'a list option ⇒ 'a list where
  opt-list-to-list None = [] |
  opt-list-to-list (Some x) = x

```

The following functions, `inputs_of` and `successors_of`, are automatically generated from the GraalVM compiler. Their purpose is to partition the node edges into input or successor edges.

```

fun inputs-of :: IRNode ⇒ ID list where
  inputs-of-AbsNode:
  inputs-of (AbsNode value) = [value] |
  inputs-of-AddNode:
  inputs-of (AddNode x y) = [x, y] |
  inputs-of-AndNode:
  inputs-of (AndNode x y) = [x, y] |
  inputs-of-BEGINNode:
  inputs-of (BeginNode next) = [] |
  inputs-of-BytecodeExceptionNode:
  inputs-of (BytecodeExceptionNode arguments stateAfter next) = arguments @
  (opt-to-list stateAfter) |
  inputs-of-ConditionalNode:
  inputs-of (ConditionalNode condition trueValue falseValue) = [condition, true-

```



```

Value, falseValue] |
  inputs-of-ConstantNode:
    inputs-of (ConstantNode const) = [] |
  inputs-of-DynamicNewArrayNode:
    inputs-of (DynamicNewArrayNode elementType length0 voidClass stateBefore
next) = [elementType, length0] @ (opt-to-list voidClass) @ (opt-to-list stateBefore)
|
  inputs-of-EndNode:
    inputs-of (EndNode) = [] |
  inputs-of-ExceptionObjectNode:
    inputs-of (ExceptionObjectNode stateAfter next) = (opt-to-list stateAfter) |
  inputs-of-FrameState:
    inputs-of (FrameState monitorIds outerFrameState values virtualObjectMappings)
= monitorIds @ (opt-to-list outerFrameState) @ (opt-list-to-list values) @ (opt-list-to-list
virtualObjectMappings) |
  inputs-of-IfNode:
    inputs-of (IfNode condition trueSuccessor falseSuccessor) = [condition] |
  inputs-of-IntegerBelowNode:
    inputs-of (IntegerBelowNode x y) = [x, y] |
  inputs-of-IntegerEqualsNode:
    inputs-of (IntegerEqualsNode x y) = [x, y] |
  inputs-of-IntegerLessThanNode:
    inputs-of (IntegerLessThanNode x y) = [x, y] |
  inputs-of-InvokeNode:
    inputs-of (InvokeNode nid0 callTarget classInit stateDuring stateAfter next)
= callTarget # (opt-to-list classInit) @ (opt-to-list stateDuring) @ (opt-to-list
stateAfter) |
  inputs-of-InvokeWithExceptionNode:
    inputs-of (InvokeWithExceptionNode nid0 callTarget classInit stateDuring stateAfter
next exceptionEdge) = callTarget # (opt-to-list classInit) @ (opt-to-list stateDur-
ing) @ (opt-to-list stateAfter) |
  inputs-of-IsNullNode:
    inputs-of (IsNullNode value) = [value] |
  inputs-of-KillingBeginNode:
    inputs-of (KillingBeginNode next) = [] |
  inputs-of-LeftShiftNode:
    inputs-of (LeftShiftNode x y) = [x, y] |
  inputs-of-LoadFieldNode:
    inputs-of (LoadFieldNode nid0 field object next) = (opt-to-list object) |
  inputs-of-LogicNegationNode:
    inputs-of (LogicNegationNode value) = [value] |
  inputs-of-LoopBeginNode:
    inputs-of (LoopBeginNode ends overflowGuard stateAfter next) = ends @ (opt-to-list
overflowGuard) @ (opt-to-list stateAfter) |
  inputs-of-LoopEndNode:
    inputs-of (LoopEndNode loopBegin) = [loopBegin] |
  inputs-of-LoopExitNode:
    inputs-of (LoopExitNode loopBegin stateAfter next) = loopBegin # (opt-to-list
stateAfter) |

```

inputs-of-MergeNode:
inputs-of (MergeNode ends stateAfter next) = ends @ (opt-to-list stateAfter) |
inputs-of-MethodCallTargetNode:
inputs-of (MethodCallTargetNode targetMethod arguments) = arguments |
inputs-of-MulNode:
inputs-of (MulNode x y) = [x, y] |
inputs-of-NarrowNode:
inputs-of (NarrowNode inputBits resultBits value) = [value] |
inputs-of-NegateNode:
inputs-of (NegateNode value) = [value] |
inputs-of-NewArrayNode:
inputs-of (NewArrayNode length0 stateBefore next) = length0 # (opt-to-list state-Before) |
inputs-of-NewInstanceNode:
inputs-of (NewInstanceNode nid0 instanceClass stateBefore next) = (opt-to-list stateBefore) |
inputs-of-NotNode:
inputs-of (NotNode value) = [value] |
inputs-of-OrNode:
inputs-of (OrNode x y) = [x, y] |
inputs-of-ParameterNode:
inputs-of (ParameterNode index) = [] |
inputs-of-PiNode:
inputs-of (PiNode object guard) = object # (opt-to-list guard) |
inputs-of-ReturnNode:
inputs-of (ReturnNode result memoryMap) = (opt-to-list result) @ (opt-to-list memoryMap) |
inputs-of-RightShiftNode:
inputs-of (RightShiftNode x y) = [x, y] |
inputs-of-ShortCircuitOrNode:
inputs-of (ShortCircuitOrNode x y) = [x, y] |
inputs-of-SignExtendNode:
inputs-of (SignExtendNode inputBits resultBits value) = [value] |
inputs-of-SignedDivNode:
inputs-of (SignedDivNode nid0 x y zeroCheck stateBefore next) = [x, y] @ (opt-to-list zeroCheck) @ (opt-to-list stateBefore) |
inputs-of-SignedRemNode:
inputs-of (SignedRemNode nid0 x y zeroCheck stateBefore next) = [x, y] @ (opt-to-list zeroCheck) @ (opt-to-list stateBefore) |
inputs-of-StartNode:
inputs-of (StartNode stateAfter next) = (opt-to-list stateAfter) |
inputs-of-StoreFieldNode:
inputs-of (StoreFieldNode nid0 field value stateAfter object next) = value # (opt-to-list stateAfter) @ (opt-to-list object) |
inputs-of-SubNode:
inputs-of (SubNode x y) = [x, y] |
inputs-of-UnsignedRightShiftNode:
inputs-of (UnsignedRightShiftNode x y) = [x, y] |
inputs-of-UnwindNode:

inputs-of (*UnwindNode exception*) = [*exception*] |
inputs-of-ValuePhiNode:
inputs-of (*ValuePhiNode nid0 values merge*) = *merge* # *values* |
inputs-of-ValueProxyNode:
inputs-of (*ValueProxyNode value loopExit*) = [*value*, *loopExit*] |
inputs-of-XorNode:
inputs-of (*XorNode x y*) = [*x*, *y*] |
inputs-of-ZeroExtendNode:
inputs-of (*ZeroExtendNode inputBits resultBits value*) = [*value*] |
inputs-of-NoNode: *inputs-of* (*NoNode*) = [] |

inputs-of-RefNode: *inputs-of* (*RefNode ref*) = [*ref*]

fun *successors-of* :: *IRNode* \Rightarrow *ID list* **where**

successors-of-AbsNode:
successors-of (*AbsNode value*) = [] |
successors-of-AddNode:
successors-of (*AddNode x y*) = [] |
successors-of-AndNode:
successors-of (*AndNode x y*) = [] |
successors-of-BeginNode:
successors-of (*BeginNode next*) = [*next*] |
successors-of-BytecodeExceptionNode:
successors-of (*BytecodeExceptionNode arguments stateAfter next*) = [*next*] |
successors-of-ConditionalNode:
successors-of (*ConditionalNode condition trueValue falseValue*) = [] |
successors-of-ConstantNode:
successors-of (*ConstantNode const*) = [] |
successors-of-DynamicNewArrayNode:
successors-of (*DynamicNewArrayNode elementType length0 voidClass stateBefore next*) = [*next*] |
successors-of-EndNode:
successors-of (*EndNode*) = [] |
successors-of-ExceptionObjectNode:
successors-of (*ExceptionObjectNode stateAfter next*) = [*next*] |
successors-of-FrameState:
successors-of (*FrameState monitorIds outerFrameState values virtualObjectMappings*) = [] |
successors-of-IfNode:
successors-of (*IfNode condition trueSuccessor falseSuccessor*) = [*trueSuccessor*, *falseSuccessor*] |
successors-of-IntegerBelowNode:
successors-of (*IntegerBelowNode x y*) = [] |
successors-of-IntegerEqualsNode:
successors-of (*IntegerEqualsNode x y*) = [] |
successors-of-IntegerLessThanNode:
successors-of (*IntegerLessThanNode x y*) = [] |

successors-of-InvokeNode:
successors-of (InvokeNode nid0 callTarget classInit stateDuring stateAfter next)
 $= [next] \mid$
successors-of-InvokeWithExceptionNode:
successors-of (InvokeWithExceptionNode nid0 callTarget classInit stateDuring
stateAfter next exceptionEdge) = [next, exceptionEdge] \mid
successors-of-IsNullNode:
successors-of (IsNullNode value) = [] \mid
successors-of-KillingBeginNode:
successors-of (KillingBeginNode next) = [next] \mid
successors-of-LeftShiftNode:
successors-of (LeftShiftNode x y) = [] \mid
successors-of-LoadFieldNode:
successors-of (LoadFieldNode nid0 field object next) = [next] \mid
successors-of-LogicNegationNode:
successors-of (LogicNegationNode value) = [] \mid
successors-of-LoopBeginNode:
successors-of (LoopBeginNode ends overflowGuard stateAfter next) = [next] \mid
successors-of-LoopEndNode:
successors-of (LoopEndNode loopBegin) = [] \mid
successors-of-LoopExitNode:
successors-of (LoopExitNode loopBegin stateAfter next) = [next] \mid
successors-of-MergeNode:
successors-of (MergeNode ends stateAfter next) = [next] \mid
successors-of-MethodCallTargetNode:
successors-of (MethodCallTargetNode targetMethod arguments) = [] \mid
successors-of-MulNode:
successors-of (MulNode x y) = [] \mid
successors-of-NarrowNode:
successors-of (NarrowNode inputBits resultBits value) = [] \mid
successors-of-NegateNode:
successors-of (NegateNode value) = [] \mid
successors-of-NewArrayNode:
successors-of (NewArrayNode length0 stateBefore next) = [next] \mid
successors-of-NewInstanceNode:
successors-of (NewInstanceNode nid0 instanceClass stateBefore next) = [next] \mid
successors-of-NotNode:
successors-of (NotNode value) = [] \mid
successors-of-OrNode:
successors-of (OrNode x y) = [] \mid
successors-of-ParameterNode:
successors-of (ParameterNode index) = [] \mid
successors-of-PiNode:
successors-of (PiNode object guard) = [] \mid
successors-of-ReturnNode:
successors-of (ReturnNode result memoryMap) = [] \mid
successors-of-RightShiftNode:
successors-of (RightShiftNode x y) = [] \mid
successors-of-ShortCircuitOrNode:

successors-of (*ShortCircuitOrNode* $x\ y$) = [] |
successors-of-SignExtendNode:
successors-of (*SignExtendNode* $inputBits\ resultBits\ value$) = [] |
successors-of-SignedDivNode:
successors-of (*SignedDivNode* $nid0\ x\ y\ zeroCheck\ stateBefore\ next$) = [$next$] |
successors-of-SignedRemNode:
successors-of (*SignedRemNode* $nid0\ x\ y\ zeroCheck\ stateBefore\ next$) = [$next$] |
successors-of-StartNode:
successors-of (*StartNode* $stateAfter\ next$) = [$next$] |
successors-of-StoreFieldNode:
successors-of (*StoreFieldNode* $nid0\ field\ value\ stateAfter\ object\ next$) = [$next$] |
successors-of-SubNode:
successors-of (*SubNode* $x\ y$) = [] |
successors-of-UnsignedRightShiftNode:
successors-of (*UnsignedRightShiftNode* $x\ y$) = [] |
successors-of-UnwindNode:
successors-of (*UnwindNode* $exception$) = [] |
successors-of-ValuePhiNode:
successors-of (*ValuePhiNode* $nid0\ values\ merge$) = [] |
successors-of-ValueProxyNode:
successors-of (*ValueProxyNode* $value\ loopExit$) = [] |
successors-of-XorNode:
successors-of (*XorNode* $x\ y$) = [] |
successors-of-ZeroExtendNode:
successors-of (*ZeroExtendNode* $inputBits\ resultBits\ value$) = [] |
successors-of-NoNode: *successors-of* (*NoNode*) = [] |

successors-of-RefNode: *successors-of* (*RefNode* ref) = [ref]

lemma *inputs-of* (*FrameState* $x\ (Some\ y)\ (Some\ z)\ None$) = $x\ @\ [y]\ @\ z$

unfolding *inputs-of-FrameState* **by** *simp*

lemma *successors-of* (*FrameState* $x\ (Some\ y)\ (Some\ z)\ None$) = []

unfolding *inputs-of-FrameState* **by** *simp*

lemma *inputs-of* (*IfNode* $c\ t\ f$) = [c]

unfolding *inputs-of-IfNode* **by** *simp*

lemma *successors-of* (*IfNode* $c\ t\ f$) = [t, f]

unfolding *successors-of-IfNode* **by** *simp*

lemma *inputs-of* (*EndNode*) = [] \wedge *successors-of* (*EndNode*) = []

unfolding *inputs-of-EndNode* *successors-of-EndNode* **by** *simp*

end

3.2 Hierarchy of Nodes

```
theory IRNodeHierarchy
imports IRNodes
begin
```

It is helpful to introduce a node hierarchy into our formalization. Often the GraalVM compiler relies on explicit type checks to determine which operations to perform on a given node, we try to mimic the same functionality by using a suite of predicate functions over the IRNode class to determine inheritance.

As one would expect, the function `is<ClassName>Type` will be true if the node parameter is a subclass of the `ClassName` within the GraalVM compiler.

These functions have been automatically generated from the compiler.

```
fun is-EndNode :: IRNode  $\Rightarrow$  bool where
  is-EndNode EndNode = True |
  is-EndNode _ = False
```

```
fun is-VirtualState :: IRNode  $\Rightarrow$  bool where
  is-VirtualState n = ((is-FrameState n))
```

```
fun is-BinaryArithmeticNode :: IRNode  $\Rightarrow$  bool where
  is-BinaryArithmeticNode n = ((is-AddNode n)  $\vee$  (is-AndNode n)  $\vee$  (is-MulNode n)  $\vee$  (is-OrNode n)  $\vee$  (is-SubNode n)  $\vee$  (is-XorNode n))
```

```
fun is-ShiftNode :: IRNode  $\Rightarrow$  bool where
  is-ShiftNode n = ((is-LeftShiftNode n)  $\vee$  (is-RightShiftNode n)  $\vee$  (is-UnsignedRightShiftNode n))
```

```
fun is-BinaryNode :: IRNode  $\Rightarrow$  bool where
  is-BinaryNode n = ((is-BinaryArithmeticNode n)  $\vee$  (is-ShiftNode n))
```

```
fun is-AbstractLocalNode :: IRNode  $\Rightarrow$  bool where
  is-AbstractLocalNode n = ((is-ParameterNode n))
```

```
fun is-IntegerConvertNode :: IRNode  $\Rightarrow$  bool where
  is-IntegerConvertNode n = ((is-NarrowNode n)  $\vee$  (is-SignExtendNode n)  $\vee$  (is-ZeroExtendNode n))
```

```
fun is-UnaryArithmeticNode :: IRNode  $\Rightarrow$  bool where
  is-UnaryArithmeticNode n = ((is-AbsNode n)  $\vee$  (is-NegateNode n)  $\vee$  (is-NotNode n))
```

```
fun is-UnaryNode :: IRNode  $\Rightarrow$  bool where
  is-UnaryNode n = ((is-IntegerConvertNode n)  $\vee$  (is-UnaryArithmeticNode n))
```

```

fun is-PhiNode :: IRNode  $\Rightarrow$  bool where
  is-PhiNode n = ((is-ValuePhiNode n))

fun is-FloatingGuardedNode :: IRNode  $\Rightarrow$  bool where
  is-FloatingGuardedNode n = ((is-PiNode n))

fun is-UnaryOpLogicNode :: IRNode  $\Rightarrow$  bool where
  is-UnaryOpLogicNode n = ((is-IsNullNode n))

fun is-IntegerLowerThanNode :: IRNode  $\Rightarrow$  bool where
  is-IntegerLowerThanNode n = ((is-IntegerBelowNode n)  $\vee$  (is-IntegerLessThanNode n))

fun is-CompareNode :: IRNode  $\Rightarrow$  bool where
  is-CompareNode n = ((is-IntegerEqualsNode n)  $\vee$  (is-IntegerLowerThanNode n))

fun is-BinaryOpLogicNode :: IRNode  $\Rightarrow$  bool where
  is-BinaryOpLogicNode n = ((is-CompareNode n))

fun is-LogicNode :: IRNode  $\Rightarrow$  bool where
  is-LogicNode n = ((is-BinaryOpLogicNode n)  $\vee$  (is-LogicNegationNode n)  $\vee$ 
    (is-ShortCircuitOrNode n)  $\vee$  (is-UnaryOpLogicNode n))

fun is-ProxyNode :: IRNode  $\Rightarrow$  bool where
  is-ProxyNode n = ((is-ValueProxyNode n))

fun is-FloatingNode :: IRNode  $\Rightarrow$  bool where
  is-FloatingNode n = ((is-AbstractLocalNode n)  $\vee$  (is-BinaryNode n)  $\vee$  (is-ConditionalNode n)
     $\vee$  (is-ConstantNode n)  $\vee$  (is-FloatingGuardedNode n)  $\vee$  (is-LogicNode n)  $\vee$ 
    (is-PhiNode n)  $\vee$  (is-ProxyNode n)  $\vee$  (is-UnaryNode n))

fun is-AccessFieldNode :: IRNode  $\Rightarrow$  bool where
  is-AccessFieldNode n = ((is-LoadFieldNode n)  $\vee$  (is-StoreFieldNode n))

fun is-AbstractNewArrayNode :: IRNode  $\Rightarrow$  bool where
  is-AbstractNewArrayNode n = ((is-DynamicNewArrayNode n)  $\vee$  (is-NewArrayNode n))

fun is-AbstractNewObjectNode :: IRNode  $\Rightarrow$  bool where
  is-AbstractNewObjectNode n = ((is-AbstractNewArrayNode n)  $\vee$  (is-NewInstanceNode n))

fun is-IntegerDivRemNode :: IRNode  $\Rightarrow$  bool where
  is-IntegerDivRemNode n = ((is-SignedDivNode n)  $\vee$  (is-SignedRemNode n))

fun is-FixedBinaryNode :: IRNode  $\Rightarrow$  bool where
  is-FixedBinaryNode n = ((is-IntegerDivRemNode n))

fun is-DeoptimizingFixedWithNextNode :: IRNode  $\Rightarrow$  bool where

```

is-DeoptimizingFixedWithNextNode $n = ((\text{is-AbstractNewObjectNode } n) \vee (\text{is-FixedBinaryNode } n))$

fun *is-AbstractMemoryCheckpoint* :: *IRNode* \Rightarrow *bool* **where**
is-AbstractMemoryCheckpoint $n = ((\text{is-BytecodeExceptionNode } n) \vee (\text{is-InvokeNode } n))$

fun *is-AbstractStateSplit* :: *IRNode* \Rightarrow *bool* **where**
is-AbstractStateSplit $n = (\text{is-AbstractMemoryCheckpoint } n)$

fun *is-AbstractMergeNode* :: *IRNode* \Rightarrow *bool* **where**
is-AbstractMergeNode $n = ((\text{is-LoopBeginNode } n) \vee (\text{is-MergeNode } n))$

fun *is-BeginStateSplitNode* :: *IRNode* \Rightarrow *bool* **where**
is-BeginStateSplitNode $n = ((\text{is-AbstractMergeNode } n) \vee (\text{is-ExceptionObjectNode } n) \vee (\text{is-LoopExitNode } n) \vee (\text{is-StartNode } n))$

fun *is-AbstractBeginNode* :: *IRNode* \Rightarrow *bool* **where**
is-AbstractBeginNode $n = ((\text{is-BeginNode } n) \vee (\text{is-BeginStateSplitNode } n) \vee (\text{is-KillingBeginNode } n))$

fun *is-FixedWithNextNode* :: *IRNode* \Rightarrow *bool* **where**
is-FixedWithNextNode $n = ((\text{is-AbstractBeginNode } n) \vee (\text{is-AbstractStateSplit } n) \vee (\text{is-AccessFieldNode } n) \vee (\text{is-DeoptimizingFixedWithNextNode } n))$

fun *is-WithExceptionNode* :: *IRNode* \Rightarrow *bool* **where**
is-WithExceptionNode $n = (\text{is-InvokeWithExceptionNode } n)$

fun *is-ControlSplitNode* :: *IRNode* \Rightarrow *bool* **where**
is-ControlSplitNode $n = ((\text{is-IfNode } n) \vee (\text{is-WithExceptionNode } n))$

fun *is-ControlSinkNode* :: *IRNode* \Rightarrow *bool* **where**
is-ControlSinkNode $n = ((\text{is-ReturnNode } n) \vee (\text{is-UnwindNode } n))$

fun *is-AbstractEndNode* :: *IRNode* \Rightarrow *bool* **where**
is-AbstractEndNode $n = ((\text{is-EndNode } n) \vee (\text{is-LoopEndNode } n))$

fun *is-FixedNode* :: *IRNode* \Rightarrow *bool* **where**
is-FixedNode $n = ((\text{is-AbstractEndNode } n) \vee (\text{is-ControlSinkNode } n) \vee (\text{is-ControlSplitNode } n) \vee (\text{is-FixedWithNextNode } n))$

fun *is-CallTargetNode* :: *IRNode* \Rightarrow *bool* **where**
is-CallTargetNode $n = (\text{is-MethodCallTargetNode } n)$

fun *is-ValueNode* :: *IRNode* \Rightarrow *bool* **where**
is-ValueNode $n = ((\text{is-CallTargetNode } n) \vee (\text{is-FixedNode } n) \vee (\text{is-FloatingNode } n))$

fun *is-Node* :: *IRNode* \Rightarrow *bool* **where**


```

is-Node n = ((is-ValueNode n) ∨ (is-VirtualState n))

fun is-MemoryKill :: IRNode ⇒ bool where
  is-MemoryKill n = ((is-AbstractMemoryCheckpoint n))

fun is-NarrowableArithmeticNode :: IRNode ⇒ bool where
  is-NarrowableArithmeticNode n = ((is-AbsNode n) ∨ (is-AddNode n) ∨ (is-AndNode
n) ∨ (is-MulNode n) ∨ (is-NegateNode n) ∨ (is-NotNode n) ∨ (is-OrNode n) ∨
(is-ShiftNode n) ∨ (is-SubNode n) ∨ (is-XorNode n))

fun is-AnchoringNode :: IRNode ⇒ bool where
  is-AnchoringNode n = ((is-AbstractBeginNode n))

fun is-DeoptBefore :: IRNode ⇒ bool where
  is-DeoptBefore n = ((is-DeoptimizingFixedWithNextNode n))

fun is-IndirectCanonicalization :: IRNode ⇒ bool where
  is-IndirectCanonicalization n = ((is-LogicNode n))

fun is-IterableNodeType :: IRNode ⇒ bool where
  is-IterableNodeType n = ((is-AbstractBeginNode n) ∨ (is-AbstractMergeNode n) ∨
(is-FrameState n) ∨ (is-IfNode n) ∨ (is-IntegerDivRemNode n) ∨ (is-InvokeWithExceptionNode
n) ∨ (is-LoopBeginNode n) ∨ (is-LoopExitNode n) ∨ (is-MethodCallTargetNode n)
∨ (is-ParameterNode n) ∨ (is-ReturnNode n) ∨ (is-ShortCircuitOrNode n))

fun is-Invoke :: IRNode ⇒ bool where
  is-Invoke n = ((is-InvokeNode n) ∨ (is-InvokeWithExceptionNode n))

fun is-Proxy :: IRNode ⇒ bool where
  is-Proxy n = ((is-ProxyNode n))

fun is-ValueProxy :: IRNode ⇒ bool where
  is-ValueProxy n = ((is-PiNode n) ∨ (is-ValueProxyNode n))

fun is-ValueNodeInterface :: IRNode ⇒ bool where
  is-ValueNodeInterface n = ((is-ValueNode n))

fun is-ArrayLengthProvider :: IRNode ⇒ bool where
  is-ArrayLengthProvider n = ((is-AbstractNewArrayNode n) ∨ (is-ConstantNode
n))

fun is-StampInverter :: IRNode ⇒ bool where
  is-StampInverter n = ((is-IntegerConvertNode n) ∨ (is-NegateNode n) ∨ (is-NotNode
n))

fun is-GuardingNode :: IRNode ⇒ bool where
  is-GuardingNode n = ((is-AbstractBeginNode n))

fun is-SingleMemoryKill :: IRNode ⇒ bool where

```

is-SingleMemoryKill $n = ((\text{is-BytecodeExceptionNode } n) \vee (\text{is-ExceptionObjectNode } n) \vee (\text{is-InvokeNode } n) \vee (\text{is-InvokeWithExceptionNode } n) \vee (\text{is-KillingBeginNode } n) \vee (\text{is-StartNode } n))$

fun *is-LIRLowerable* :: *IRNode* \Rightarrow *bool* **where**

is-LIRLowerable $n = ((\text{is-AbstractBeginNode } n) \vee (\text{is-AbstractEndNode } n) \vee (\text{is-AbstractMergeNode } n) \vee (\text{is-BinaryOpLogicNode } n) \vee (\text{is-CallTargetNode } n) \vee (\text{is-ConditionalNode } n) \vee (\text{is-ConstantNode } n) \vee (\text{is-IfNode } n) \vee (\text{is-InvokeNode } n) \vee (\text{is-InvokeWithExceptionNode } n) \vee (\text{is-IsNullNode } n) \vee (\text{is-LoopBeginNode } n) \vee (\text{is-PiNode } n) \vee (\text{is-ReturnNode } n) \vee (\text{is-SignedDivNode } n) \vee (\text{is-SignedRemNode } n) \vee (\text{is-UnaryOpLogicNode } n) \vee (\text{is-UnwindNode } n))$

fun *is-GuardedNode* :: *IRNode* \Rightarrow *bool* **where**

is-GuardedNode $n = ((\text{is-FloatingGuardedNode } n))$

fun *is-ArithmeticLIRLowerable* :: *IRNode* \Rightarrow *bool* **where**

is-ArithmeticLIRLowerable $n = ((\text{is-AbsNode } n) \vee (\text{is-BinaryArithmeticNode } n) \vee (\text{is-IntegerConvertNode } n) \vee (\text{is-NotNode } n) \vee (\text{is-ShiftNode } n) \vee (\text{is-UnaryArithmeticNode } n))$

fun *is-SwitchFoldable* :: *IRNode* \Rightarrow *bool* **where**

is-SwitchFoldable $n = ((\text{is-IfNode } n))$

fun *is-VirtualizableAllocation* :: *IRNode* \Rightarrow *bool* **where**

is-VirtualizableAllocation $n = ((\text{is-NewArrayNode } n) \vee (\text{is-NewInstanceNode } n))$

fun *is-Unary* :: *IRNode* \Rightarrow *bool* **where**

is-Unary $n = ((\text{is-LoadFieldNode } n) \vee (\text{is-LogicNegationNode } n) \vee (\text{is-UnaryNode } n) \vee (\text{is-UnaryOpLogicNode } n))$

fun *is-FixedNodeInterface* :: *IRNode* \Rightarrow *bool* **where**

is-FixedNodeInterface $n = ((\text{is-FixedNode } n))$

fun *is-BinaryCommutative* :: *IRNode* \Rightarrow *bool* **where**

is-BinaryCommutative $n = ((\text{is-AddNode } n) \vee (\text{is-AndNode } n) \vee (\text{is-IntegerEqualsNode } n) \vee (\text{is-MulNode } n) \vee (\text{is-OrNode } n) \vee (\text{is-XorNode } n))$

fun *is-Canonicalizable* :: *IRNode* \Rightarrow *bool* **where**

is-Canonicalizable $n = ((\text{is-BytecodeExceptionNode } n) \vee (\text{is-ConditionalNode } n) \vee (\text{is-DynamicNewArrayNode } n) \vee (\text{is-PhiNode } n) \vee (\text{is-PiNode } n) \vee (\text{is-ProxyNode } n) \vee (\text{is-StoreFieldNode } n) \vee (\text{is-ValueProxyNode } n))$

fun *is-UncheckedInterfaceProvider* :: *IRNode* \Rightarrow *bool* **where**

is-UncheckedInterfaceProvider $n = ((\text{is-InvokeNode } n) \vee (\text{is-InvokeWithExceptionNode } n) \vee (\text{is-LoadFieldNode } n) \vee (\text{is-ParameterNode } n))$

fun *is-Binary* :: *IRNode* \Rightarrow *bool* **where**

is-Binary $n = ((\text{is-BinaryArithmeticNode } n) \vee (\text{is-BinaryNode } n) \vee (\text{is-BinaryOpLogicNode } n) \vee (\text{is-CompareNode } n) \vee (\text{is-FixedBinaryNode } n) \vee (\text{is-ShortCircuitOrNode } n))$

```

fun is-ArithmeticOperation :: IRNode  $\Rightarrow$  bool where
  is-ArithmeticOperation n = ((is-BinaryArithmeticNode n)  $\vee$  (is-IntegerConvertNode
n)  $\vee$  (is-ShiftNode n)  $\vee$  (is-UnaryArithmeticNode n))

fun is-ValueNumberable :: IRNode  $\Rightarrow$  bool where
  is-ValueNumberable n = ((is-FloatingNode n)  $\vee$  (is-ProxyNode n))

fun is-Lowerable :: IRNode  $\Rightarrow$  bool where
  is-Lowerable n = ((is-AbstractNewObjectNode n)  $\vee$  (is-AccessFieldNode n)  $\vee$ 
(is-BytecodeExceptionNode n)  $\vee$  (is-ExceptionObjectNode n)  $\vee$  (is-IntegerDivRemNode
n)  $\vee$  (is-UnwindNode n))

fun is-Virtualizable :: IRNode  $\Rightarrow$  bool where
  is-Virtualizable n = ((is-IsNullNode n)  $\vee$  (is-LoadFieldNode n)  $\vee$  (is-PiNode n)
 $\vee$  (is-StoreFieldNode n)  $\vee$  (is-ValueProxyNode n))

fun is-Simplifiable :: IRNode  $\Rightarrow$  bool where
  is-Simplifiable n = ((is-AbstractMergeNode n)  $\vee$  (is-BeginNode n)  $\vee$  (is-IfNode
n)  $\vee$  (is-LoopExitNode n)  $\vee$  (is-MethodCallTargetNode n)  $\vee$  (is-NewArrayNode n))

fun is-StateSplit :: IRNode  $\Rightarrow$  bool where
  is-StateSplit n = ((is-AbstractStateSplit n)  $\vee$  (is-BeginStateSplitNode n)  $\vee$  (is-StoreFieldNode
n))

fun is-ConvertNode :: IRNode  $\Rightarrow$  bool where
  is-ConvertNode n = ((is-IntegerConvertNode n))

fun is-sequential-node :: IRNode  $\Rightarrow$  bool where
  is-sequential-node (StartNode -) = True |
  is-sequential-node (BeginNode -) = True |
  is-sequential-node (KillingBeginNode -) = True |
  is-sequential-node (LoopBeginNode - - -) = True |
  is-sequential-node (LoopExitNode - -) = True |
  is-sequential-node (MergeNode - -) = True |
  is-sequential-node (RefNode -) = True |
  is-sequential-node - = False

```

The following convenience function is useful in determining if two IRNodes are of the same type irregardless of their edges. It will return true if both the node parameters are the same node class.

```

fun is-same-ir-node-type :: IRNode  $\Rightarrow$  IRNode  $\Rightarrow$  bool where
is-same-ir-node-type n1 n2 = (
  ((is-AbsNode n1)  $\wedge$  (is-AbsNode n2))  $\vee$ 
  ((is-AddNode n1)  $\wedge$  (is-AddNode n2))  $\vee$ 
  ((is-AndNode n1)  $\wedge$  (is-AndNode n2))  $\vee$ 
  ((is-BeginNode n1)  $\wedge$  (is-BeginNode n2))  $\vee$ 
  ((is-BytecodeExceptionNode n1)  $\wedge$  (is-BytecodeExceptionNode n2))  $\vee$ 

```

```

((is-ConditionalNode n1) ∧ (is-ConditionalNode n2)) ∨
((is-ConstantNode n1) ∧ (is-ConstantNode n2)) ∨
((is-DynamicNewArrayNode n1) ∧ (is-DynamicNewArrayNode n2)) ∨
((is-EndNode n1) ∧ (is-EndNode n2)) ∨
((is-ExceptionObjectNode n1) ∧ (is-ExceptionObjectNode n2)) ∨
((is-FrameState n1) ∧ (is-FrameState n2)) ∨
((is-IfNode n1) ∧ (is-IfNode n2)) ∨
((is-IntegerBelowNode n1) ∧ (is-IntegerBelowNode n2)) ∨
((is-IntegerEqualsNode n1) ∧ (is-IntegerEqualsNode n2)) ∨
((is-IntegerLessThanNode n1) ∧ (is-IntegerLessThanNode n2)) ∨
((is-InvokeNode n1) ∧ (is-InvokeNode n2)) ∨
((is-InvokeWithExceptionNode n1) ∧ (is-InvokeWithExceptionNode n2)) ∨
((is-IsNullNode n1) ∧ (is-IsNullNode n2)) ∨
((is-KillingBeginNode n1) ∧ (is-KillingBeginNode n2)) ∨
((is-LoadFieldNode n1) ∧ (is-LoadFieldNode n2)) ∨
((is-LogicNegationNode n1) ∧ (is-LogicNegationNode n2)) ∨
((is-LoopBeginNode n1) ∧ (is-LoopBeginNode n2)) ∨
((is-LoopEndNode n1) ∧ (is-LoopEndNode n2)) ∨
((is-LoopExitNode n1) ∧ (is-LoopExitNode n2)) ∨
((is-MergeNode n1) ∧ (is-MergeNode n2)) ∨
((is-MethodCallTargetNode n1) ∧ (is-MethodCallTargetNode n2)) ∨
((is-MulNode n1) ∧ (is-MulNode n2)) ∨
((is-NegateNode n1) ∧ (is-NegateNode n2)) ∨
((is-NewArrayNode n1) ∧ (is-NewArrayNode n2)) ∨
((is-NewInstanceNode n1) ∧ (is-NewInstanceNode n2)) ∨
((is-NotNode n1) ∧ (is-NotNode n2)) ∨
((is-OrNode n1) ∧ (is-OrNode n2)) ∨
((is-ParameterNode n1) ∧ (is-ParameterNode n2)) ∨
((is-PiNode n1) ∧ (is-PiNode n2)) ∨
((is-ReturnNode n1) ∧ (is-ReturnNode n2)) ∨
((is-ShortCircuitOrNode n1) ∧ (is-ShortCircuitOrNode n2)) ∨
((is-SignedDivNode n1) ∧ (is-SignedDivNode n2)) ∨
((is-StartNode n1) ∧ (is-StartNode n2)) ∨
((is-StoreFieldNode n1) ∧ (is-StoreFieldNode n2)) ∨
((is-SubNode n1) ∧ (is-SubNode n2)) ∨
((is-UnwindNode n1) ∧ (is-UnwindNode n2)) ∨
((is-ValuePhiNode n1) ∧ (is-ValuePhiNode n2)) ∨
((is-ValueProxyNode n1) ∧ (is-ValueProxyNode n2)) ∨
((is-XorNode n1) ∧ (is-XorNode n2))

```

end

4 Stamp Typing

```

theory Stamp
  imports Values
begin

```

The GraalVM compiler uses the Stamp class to store range and type infor-

mation for a given node in the IR graph. We model the Stamp class as a datatype, Stamp, and provide a number of functions on the datatype which correspond to the class methods within the compiler.

Stamp information is used in a variety of ways in optimizations, and so, we additionally provide a number of lemmas which help to prove future optimizations.

```
datatype Stamp =
  VoidStamp
| IntegerStamp (stp-bits: nat) (stp-lower: int) (stp-upper: int)

| KlassPointerStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
| MethodCountersPointerStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
| MethodPointersStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
| ObjectStamp (stp-type: string) (stp-exactType: bool) (stp-nonNull: bool) (stp-alwaysNull:
bool)
| RawPointerStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
| IllegalStamp
```

```
fun is-stamp-empty :: Stamp  $\Rightarrow$  bool where
  is-stamp-empty (IntegerStamp b lower upper) = (upper < lower) |

  is-stamp-empty x = False
```

Just like the IntegerStamp class, we need to know that our lo/hi bounds fit into the given number of bits (either signed or unsigned). Our integer stamps have infinite lo/hi bounds, so if the lower bound is non-negative, we can assume that all values are positive, and the integer bits of a related value can be interpreted as unsigned. This is similar (but slightly more general) to what IntegerStamp.java does with its test: if (sameSignBounds()) in the unsignedUpperBound() method.

Note that this is a bit different and more accurate than what StampFactory.forUnsignedInteger does (it widens large unsigned ranges to the max signed range to allow all bit patterns) because its lo/hi values are only 64-bit.

```
fun valid-stamp :: Stamp  $\Rightarrow$  bool where
  valid-stamp (IntegerStamp bits lo hi) =
    (0 < bits  $\wedge$  bits  $\leq$  64  $\wedge$ 
     fst (bit-bounds bits)  $\leq$  lo  $\wedge$  lo  $\leq$  snd (bit-bounds bits)  $\wedge$ 
     fst (bit-bounds bits)  $\leq$  hi  $\wedge$  hi  $\leq$  snd (bit-bounds bits)) |
  valid-stamp s = True
```

```
experiment begin
corollary bit-bounds 1 = (-1, 0) by simp
end
```

— A stamp which includes the full range of the type

```
fun unrestricted-stamp :: Stamp  $\Rightarrow$  Stamp where
  unrestricted-stamp VoidStamp = VoidStamp |
  unrestricted-stamp (IntegerStamp bits lower upper) = (IntegerStamp bits (fst
(bit-bounds bits)) (snd (bit-bounds bits))) |

  unrestricted-stamp (KlassPointerStamp nonNull alwaysNull) = (KlassPointerStamp
False False) |
  unrestricted-stamp (MethodCountersPointerStamp nonNull alwaysNull) = (MethodCountersPointerStamp
False False) |
  unrestricted-stamp (MethodPointersStamp nonNull alwaysNull) = (MethodPointersStamp
False False) |
  unrestricted-stamp (ObjectStamp type exactType nonNull alwaysNull) = (ObjectStamp
True True True) |
  unrestricted-stamp - = IllegalStamp
```

```
fun is-stamp-unrestricted :: Stamp  $\Rightarrow$  bool where
  is-stamp-unrestricted s = (s = unrestricted-stamp s)
```

— A stamp which provides type information but has an empty range of values

```
fun empty-stamp :: Stamp  $\Rightarrow$  Stamp where
  empty-stamp VoidStamp = VoidStamp |
  empty-stamp (IntegerStamp bits lower upper) = (IntegerStamp bits (snd (bit-bounds
bits)) (fst (bit-bounds bits))) |

  empty-stamp (KlassPointerStamp nonNull alwaysNull) = (KlassPointerStamp
nonNull alwaysNull) |
  empty-stamp (MethodCountersPointerStamp nonNull alwaysNull) = (MethodCountersPointerStamp
nonNull alwaysNull) |
  empty-stamp (MethodPointersStamp nonNull alwaysNull) = (MethodPointersStamp
nonNull alwaysNull) |
  empty-stamp (ObjectStamp type exactType nonNull alwaysNull) = (ObjectStamp
True True False) |
  empty-stamp stamp = IllegalStamp
```

— Calculate the meet stamp of two stamps

```
fun meet :: Stamp  $\Rightarrow$  Stamp  $\Rightarrow$  Stamp where
  meet VoidStamp VoidStamp = VoidStamp |
  meet (IntegerStamp b1 l1 u1) (IntegerStamp b2 l2 u2) = (
    if b1  $\neq$  b2 then IllegalStamp else
    (IntegerStamp b1 (min l1 l2) (max u1 u2))
  ) |
```

```

meet (KlassPointerStamp nn1 an1) (KlassPointerStamp nn2 an2) = (
  KlassPointerStamp (nn1  $\wedge$  nn2) (an1  $\wedge$  an2)
) |
meet (MethodCountersPointerStamp nn1 an1) (MethodCountersPointerStamp
nn2 an2) = (
  MethodCountersPointerStamp (nn1  $\wedge$  nn2) (an1  $\wedge$  an2)
) |
meet (MethodPointersStamp nn1 an1) (MethodPointersStamp nn2 an2) = (
  MethodPointersStamp (nn1  $\wedge$  nn2) (an1  $\wedge$  an2)
) |
meet s1 s2 = IllegalStamp

```

— Calculate the join stamp of two stamps

```

fun join :: Stamp  $\Rightarrow$  Stamp  $\Rightarrow$  Stamp where
  join VoidStamp VoidStamp = VoidStamp |
  join (IntegerStamp b1 l1 u1) (IntegerStamp b2 l2 u2) = (
    if b1  $\neq$  b2 then IllegalStamp else
    (IntegerStamp b1 (max l1 l2) (min u1 u2))
  ) |
  join (KlassPointerStamp nn1 an1) (KlassPointerStamp nn2 an2) = (
    if ((nn1  $\vee$  nn2)  $\wedge$  (an1  $\vee$  an2))
    then (empty-stamp (KlassPointerStamp nn1 an1))
    else (KlassPointerStamp (nn1  $\vee$  nn2) (an1  $\vee$  an2))
  ) |
  join (MethodCountersPointerStamp nn1 an1) (MethodCountersPointerStamp nn2
an2) = (
    if ((nn1  $\vee$  nn2)  $\wedge$  (an1  $\vee$  an2))
    then (empty-stamp (MethodCountersPointerStamp nn1 an1))
    else (MethodCountersPointerStamp (nn1  $\vee$  nn2) (an1  $\vee$  an2))
  ) |
  join (MethodPointersStamp nn1 an1) (MethodPointersStamp nn2 an2) = (
    if ((nn1  $\vee$  nn2)  $\wedge$  (an1  $\vee$  an2))
    then (empty-stamp (MethodPointersStamp nn1 an1))
    else (MethodPointersStamp (nn1  $\vee$  nn2) (an1  $\vee$  an2))
  ) |
  join s1 s2 = IllegalStamp

```

— In certain circumstances a stamp provides enough information to evaluate a value as a stamp, the asConstant function converts the stamp to a value where one can be inferred.

```

fun asConstant :: Stamp  $\Rightarrow$  Value where
  asConstant (IntegerStamp b l h) = (if l = h then IntVal b (word-of-int l) else
UndefVal) |
  asConstant - = UndefVal

```

— Determine if two stamps never have value overlaps i.e. their join is empty

```
fun alwaysDistinct :: Stamp  $\Rightarrow$  Stamp  $\Rightarrow$  bool where
  alwaysDistinct stamp1 stamp2 = is-stamp-empty (join stamp1 stamp2)
```

— Determine if two stamps must always be the same value i.e. two equal constants

```
fun neverDistinct :: Stamp  $\Rightarrow$  Stamp  $\Rightarrow$  bool where
  neverDistinct stamp1 stamp2 = (asConstant stamp1 = asConstant stamp2  $\wedge$ 
asConstant stamp1  $\neq$  UndefVal)
```

```
fun constantAsStamp :: Value  $\Rightarrow$  Stamp where
  constantAsStamp (IntVal b v) = (IntegerStamp b (int-signed-value b v) (int-signed-value
b v)) |
```

```
  constantAsStamp - = IllegalStamp
```

— Define when a runtime value is valid for a stamp. The stamp bounds must be valid, and val must be zero-extended.

```
fun valid-value :: Value  $\Rightarrow$  Stamp  $\Rightarrow$  bool where
  valid-value (IntVal b1 val) (IntegerStamp b l h) =
    (if b1 = b then
      valid-stamp (IntegerStamp b l h)  $\wedge$ 
      take-bit b val = val  $\wedge$ 
      l  $\leq$  int-signed-value b val  $\wedge$  int-signed-value b val  $\leq$  h
    else False) |
```

```
  valid-value (ObjRef ref) (ObjectStamp klass exact nonNull alwaysNull) =
    ((alwaysNull  $\longrightarrow$  ref = None)  $\wedge$  (ref=None  $\longrightarrow$   $\neg$  nonNull)) |
  valid-value stamp val = False
```

```
fun compatible :: Stamp  $\Rightarrow$  Stamp  $\Rightarrow$  bool where
  compatible (IntegerStamp b1 lo1 hi1) (IntegerStamp b2 lo2 hi2) =
    (b1 = b2  $\wedge$  valid-stamp (IntegerStamp b1 lo1 hi1)  $\wedge$  valid-stamp (IntegerStamp
b2 lo2 hi2)) |
  compatible (VoidStamp) (VoidStamp) = True |
  compatible - - = False
```

```
fun stamp-under :: Stamp  $\Rightarrow$  Stamp  $\Rightarrow$  bool where
  stamp-under x y = ((stpi-upper x) < (stpi-lower y))
```

— The most common type of stamp within the compiler (apart from the Void-Stamp) is a 32 bit integer stamp with an unrestricted range. We use *default-stamp* as it is a frequently used stamp.

```
definition default-stamp :: Stamp where
  default-stamp = (unrestricted-stamp (IntegerStamp 32 0 0))
```

```
value valid-value (IntVal 8 (255)) (IntegerStamp 8 (−128) 127)
```


end

5 Graph Representation

```

theory IRGraph
  imports
    IRNodeHierarchy
    Stamp
    HOL-Library.FSet
    HOL.Relation
  begin

```

This theory defines the main Graal data structure - an entire IR Graph.

IRGraph is defined as a partial map with a finite domain. The finite domain is required to be able to generate code and produce an interpreter.

```

typedef IRGraph = {g :: ID  $\rightarrow$  (IRNode  $\times$  Stamp) . finite (dom g)}
proof -
  have finite(dom(Map.empty))  $\wedge$  ran Map.empty = {} by auto
  then show ?thesis
    by fastforce
qed

```

setup-lifting type-definition-IRGraph

```

lift-definition ids :: IRGraph  $\Rightarrow$  ID set
  is  $\lambda g. \{nid \in \text{dom } g . \nexists s. g \text{ nid} = (\text{Some } (\text{NoNode}, s))\}$  .

```

```

fun with-default :: 'c  $\Rightarrow$  ('b  $\Rightarrow$  'c)  $\Rightarrow$  (('a  $\rightarrow$  'b)  $\Rightarrow$  'a  $\Rightarrow$  'c) where
  with-default def conv = ( $\lambda m k.$ 
    (case m k of None  $\Rightarrow$  def | Some v  $\Rightarrow$  conv v))

```

```

lift-definition kind :: IRGraph  $\Rightarrow$  (ID  $\Rightarrow$  IRNode)
  is with-default NoNode fst .

```

```

lift-definition stamp :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  Stamp
  is with-default IllegalStamp snd .

```

```

lift-definition add-node :: ID  $\Rightarrow$  (IRNode  $\times$  Stamp)  $\Rightarrow$  IRGraph  $\Rightarrow$  IRGraph
  is  $\lambda nid k g.$  if fst k = NoNode then g else g(nid  $\mapsto$  k) by simp

```

```

lift-definition remove-node :: ID  $\Rightarrow$  IRGraph  $\Rightarrow$  IRGraph
  is  $\lambda nid g.$  g(nid := None) by simp

```

```

lift-definition replace-node :: ID  $\Rightarrow$  (IRNode  $\times$  Stamp)  $\Rightarrow$  IRGraph  $\Rightarrow$  IRGraph
  is  $\lambda nid k g.$  if fst k = NoNode then g else g(nid  $\mapsto$  k) by simp

```

```

lift-definition as-list :: IRGraph  $\Rightarrow$  (ID  $\times$  IRNode  $\times$  Stamp) list
  is  $\lambda g.$  map ( $\lambda k. (k, \text{the } (g k))$ ) (sorted-list-of-set (dom g)) .

```

fun *no-node* :: (*ID* × (*IRNode* × *Stamp*)) *list* ⇒ (*ID* × (*IRNode* × *Stamp*)) *list*
where
no-node *g* = *filter* (λ*n*. *fst* (*snd* *n*) ≠ *NoNode*) *g*

lift-definition *irgraph* :: (*ID* × (*IRNode* × *Stamp*)) *list* ⇒ *IRGraph*
is *map-of* ∘ *no-node*
by (*simp* *add*: *finite-dom-map-of*)

definition *as-set* :: *IRGraph* ⇒ (*ID* × (*IRNode* × *Stamp*)) *set* **where**
as-set *g* = {(*n*, *kind* *g* *n*, *stamp* *g* *n*) | *n* . *n* ∈ *ids* *g*}

definition *true-ids* :: *IRGraph* ⇒ *ID* *set* **where**
true-ids *g* = *ids* *g* − {*n* ∈ *ids* *g*. ∃ *n'* . *kind* *g* *n* = *RefNode* *n'*}

definition *domain-subtraction* :: '*a* *set* ⇒ ('*a* × '*b*) *set* ⇒ ('*a* × '*b*) *set*
(**infix** ≤ 30) **where**
domain-subtraction *s* *r* = {(*x*, *y*) . (*x*, *y*) ∈ *r* ∧ *x* ∉ *s*}

notation (*latex*)
domain-subtraction (- ◀ -)

code-datatype *irgraph*

fun *filter-none* **where**
filter-none *g* = {*nid* ∈ *dom* *g* . ∄ *s*. *g* *nid* = (*Some* (*NoNode*, *s*))}

lemma *no-node-clears*:
res = *no-node* *xs* ⟶ (∀ *x* ∈ *set* *res*. *fst* (*snd* *x*) ≠ *NoNode*)
by *simp*

lemma *dom-eq*:
assumes ∀ *x* ∈ *set* *xs*. *fst* (*snd* *x*) ≠ *NoNode*
shows *filter-none* (*map-of* *xs*) = *dom* (*map-of* *xs*)
unfolding *filter-none.simps* **using** *assms* *map-of-SomeD*
by *fastforce*

lemma *fil-eq*:
filter-none (*map-of* (*no-node* *xs*)) = *set* (*map* *fst* (*no-node* *xs*))
using *no-node-clears*
by (*metis* *dom-eq* *dom-map-of-conv-image-fst* *list.set-map*)

lemma *irgraph[code]*: *ids* (*irgraph* *m*) = *set* (*map* *fst* (*no-node* *m*))
unfolding *irgraph-def* *ids-def* **using** *fil-eq*
by (*smt* *Rep-IRGraph* *comp-apply* *eq-onp-same-args* *filter-none.simps* *ids.abs-eq* *ids-def* *irgraph.abs-eq* *irgraph.rep-eq* *irgraph-def* *mem-Collect-eq*)

lemma [*code*]: *Rep-IRGraph* (*irgraph* *m*) = *map-of* (*no-node* *m*)

```

using Abs-IRGraph-inverse
by (simp add: irgraph.rep-eq)

— Get the inputs set of a given node ID
fun inputs :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID set where
  inputs g nid = set (inputs-of (kind g nid))
— Get the successor set of a given node ID
fun succ :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID set where
  succ g nid = set (successors-of (kind g nid))
— Gives a relation between node IDs - between a node and its input nodes
fun input-edges :: IRGraph  $\Rightarrow$  ID rel where
  input-edges g = ( $\bigcup$  i  $\in$  ids g. {(i,j)|j. j  $\in$  (inputs g i)})
— Find all the nodes in the graph that have nid as an input - the usages of nid
fun usages :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID set where
  usages g nid = {i. i  $\in$  ids g  $\wedge$  nid  $\in$  inputs g i}
fun successor-edges :: IRGraph  $\Rightarrow$  ID rel where
  successor-edges g = ( $\bigcup$  i  $\in$  ids g. {(i,j)|j. j  $\in$  (succ g i)})
fun predecessors :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID set where
  predecessors g nid = {i. i  $\in$  ids g  $\wedge$  nid  $\in$  succ g i}
fun nodes-of :: IRGraph  $\Rightarrow$  (IRNode  $\Rightarrow$  bool)  $\Rightarrow$  ID set where
  nodes-of g sel = {nid  $\in$  ids g . sel (kind g nid)}
fun edge :: (IRNode  $\Rightarrow$  'a)  $\Rightarrow$  ID  $\Rightarrow$  IRGraph  $\Rightarrow$  'a where
  edge sel nid g = sel (kind g nid)

fun filtered-inputs :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  (IRNode  $\Rightarrow$  bool)  $\Rightarrow$  ID list where
  filtered-inputs g nid f = filter (f  $\circ$  (kind g)) (inputs-of (kind g nid))
fun filtered-successors :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  (IRNode  $\Rightarrow$  bool)  $\Rightarrow$  ID list where
  filtered-successors g nid f = filter (f  $\circ$  (kind g)) (successors-of (kind g nid))
fun filtered-usages :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  (IRNode  $\Rightarrow$  bool)  $\Rightarrow$  ID set where
  filtered-usages g nid f = {n  $\in$  (usages g nid). f (kind g n)}

fun is-empty :: IRGraph  $\Rightarrow$  bool where
  is-empty g = (ids g = {})

fun any-usage :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID where
  any-usage g nid = hd (sorted-list-of-set (usages g nid))

lemma ids-some[simp]: x  $\in$  ids g  $\longleftrightarrow$  kind g x  $\neq$  NoNode
proof –
  have that: x  $\in$  ids g  $\longrightarrow$  kind g x  $\neq$  NoNode
  using ids.rep-eq kind.rep-eq by force
  have kind g x  $\neq$  NoNode  $\longrightarrow$  x  $\in$  ids g
  unfolding with-default.simps kind-def ids-def
  by (cases Rep-IRGraph g x = None; auto)
  from this that show ?thesis by auto
qed

lemma not-in-g:

```

```

assumes  $nid \notin ids\ g$ 
shows  $kind\ g\ nid = NoNode$ 
using  $assms\ ids\ some$  by  $blast$ 

lemma  $valid\_creation[simp]$ :
   $finite\ (dom\ g) \longleftrightarrow Rep\_IRGraph\ (Abs\_IRGraph\ g) = g$ 
using  $Abs\_IRGraph\_inverse$  by  $(metis\ Rep\_IRGraph\ mem\_Collect\_eq)$ 

lemma  $[simp]$ :  $finite\ (ids\ g)$ 
using  $Rep\_IRGraph\ ids.rep\_eq$  by  $simp$ 

lemma  $[simp]$ :  $finite\ (ids\ (irgraph\ g))$ 
by  $(simp\ add: finite\_dom\_map\_of)$ 

lemma  $[simp]$ :  $finite\ (dom\ g) \longrightarrow ids\ (Abs\_IRGraph\ g) = \{nid \in dom\ g . \nexists s. g\ nid = Some\ (NoNode, s)\}$ 
using  $ids.rep\_eq$  by  $simp$ 

lemma  $[simp]$ :  $finite\ (dom\ g) \longrightarrow kind\ (Abs\_IRGraph\ g) = (\lambda x . (case\ g\ x\ of\ None \Rightarrow NoNode \mid Some\ n \Rightarrow fst\ n))$ 
by  $(simp\ add: kind.rep\_eq)$ 

lemma  $[simp]$ :  $finite\ (dom\ g) \longrightarrow stamp\ (Abs\_IRGraph\ g) = (\lambda x . (case\ g\ x\ of\ None \Rightarrow IllegalStamp \mid Some\ n \Rightarrow snd\ n))$ 
using  $stamp.abs\_eq\ stamp.rep\_eq$  by  $auto$ 

lemma  $[simp]$ :  $ids\ (irgraph\ g) = set\ (map\ fst\ (no\_node\ g))$ 
using  $irgraph$  by  $auto$ 

lemma  $[simp]$ :  $kind\ (irgraph\ g) = (\lambda nid. (case\ (map\_of\ (no\_node\ g))\ nid\ of\ None \Rightarrow NoNode \mid Some\ n \Rightarrow fst\ n))$ 
using  $irgraph.rep\_eq\ kind.transfer\ kind.rep\_eq$  by  $auto$ 

lemma  $[simp]$ :  $stamp\ (irgraph\ g) = (\lambda nid. (case\ (map\_of\ (no\_node\ g))\ nid\ of\ None \Rightarrow IllegalStamp \mid Some\ n \Rightarrow snd\ n))$ 
using  $irgraph.rep\_eq\ stamp.transfer\ stamp.rep\_eq$  by  $auto$ 

lemma  $map\_of\_upd$ :  $(map\_of\ g)(k \mapsto v) = (map\_of\ ((k, v) \# g))$ 
by  $simp$ 

lemma  $[code]$ :  $replace\_node\ nid\ k\ (irgraph\ g) = (irgraph\ ((nid, k) \# g))$ 
proof  $(cases\ fst\ k = NoNode)$ 
  case  $True$ 
    then show  $?thesis$ 
    by  $(metis\ (mono\_tags, lifting)\ Rep\_IRGraph\_inject\ filter.simps(2)\ irgraph.abs\_eq\ no\_node.simps\ replace\_node.rep\_eq\ snd\_conv)$ 
  next
    case  $False$ 

```

then show *?thesis unfolding irgraph-def replace-node-def no-node.simps*
by (*smt (verit, best) Rep-IRGraph comp-apply eq-onp-same-args filter.simps(2)*
id-def irgraph.rep-eq map-fun-apply map-of-upd mem-Collect-eq no-node.elims re-
place-node.abs-eq replace-node-def snd-eqD)
qed

lemma [*code*]: *add-node nid k (irgraph g) = (irgraph (((nid, k) # g)))*
by (*smt (z3) Rep-IRGraph-inject add-node.rep-eq filter.simps(2) irgraph.rep-eq*
map-of-upd no-node.simps snd-conv)

lemma *add-node-lookup*:
gup = add-node nid (k, s) g \longrightarrow
(if k \neq NoNode then kind gup nid = k \wedge stamp gup nid = s else kind gup nid
= kind g nid)
proof (*cases k = NoNode*)
case *True*
then show *?thesis*
by (*simp add: add-node.rep-eq kind.rep-eq*)
next
case *False*
then show *?thesis*
by (*simp add: kind.rep-eq add-node.rep-eq stamp.rep-eq*)
qed

lemma *remove-node-lookup*:
gup = remove-node nid g \longrightarrow kind gup nid = NoNode \wedge stamp gup nid =
IllegalStamp
by (*simp add: kind.rep-eq remove-node.rep-eq stamp.rep-eq*)

lemma *replace-node-lookup[simp]*:
gup = replace-node nid (k, s) g \wedge k \neq NoNode \longrightarrow kind gup nid = k \wedge stamp
gup nid = s
by (*simp add: replace-node.rep-eq kind.rep-eq stamp.rep-eq*)

lemma *replace-node-unchanged*:
gup = replace-node nid (k, s) g \longrightarrow ($\forall n \in (ids\ g - \{nid\}) . n \in ids\ g \wedge n \in ids$
gup \wedge kind g n = kind gup n)
by (*simp add: kind.rep-eq replace-node.rep-eq*)

5.0.1 Example Graphs

Example 1: empty graph (just a start and end node)

definition *start-end-graph*:: *IRGraph where*
start-end-graph = irgraph [(0, StartNode None 1, VoidStamp), (1, ReturnNode
None None, VoidStamp)]

Example 2: public static int sq(int x) return x * x;
 [1 P(0)] / [0 Start] [4 *] | / V / [5 Return]

```

definition eg2-sq :: IRGraph where
  eg2-sq = irgraph [
    (0, StartNode None 5, VoidStamp),
    (1, ParameterNode 0, default-stamp),
    (4, MulNode 1 1, default-stamp),
    (5, ReturnNode (Some 4) None, default-stamp)
  ]

```

```

value input-edges eg2-sq
value usages eg2-sq 1

```

```

end

```

5.1 Control-flow Graph Traversal

```

theory
  Traversal
imports
  IRGraph
begin

```

```

type-synonym Seen = ID set

```

nextEdge helps determine which node to traverse next by returning the first successor edge that isn't in the set of already visited nodes. If there is not an appropriate successor, *None* is returned instead.

```

fun nextEdge :: Seen  $\Rightarrow$  ID  $\Rightarrow$  IRGraph  $\Rightarrow$  ID option where
  nextEdge seen nid g =
    (let nids = (filter ( $\lambda$ nid'. nid'  $\notin$  seen) (successors-of (kind g nid))) in
    (if length nids > 0 then Some (hd nids) else None))

```

pred determines which node, if any, acts as the predecessor of another.

Merge nodes represent a special case where-in the predecessor exists as an input edge of the merge node, to simplify the traversal we treat only the first input end node as the predecessor, ignoring that multiple nodes may act as a successor.

For all other nodes, the predecessor is the first element of the predecessors set. Note that in a well-formed graph there should only be one element in the predecessor set.

```

fun pred :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID option where
  pred g nid = (case kind g nid of
    (MergeNode ends -)  $\Rightarrow$  Some (hd ends) |
    -  $\Rightarrow$ 
    (if IRGraph.predecessors g nid = {}
      then None else

```

```

    Some (hd (sorted-list-of-set (IRGraph.predecessors g nid)))
  )
)

```

Here we try to implement a generic fork of the control-flow traversal algorithm that was initially implemented for the ConditionalElimination phase

type-synonym *'a TraversalState* = (*ID* × *Seen* × *'a*)

inductive Step

:: ('a TraversalState ⇒ *'a*) ⇒ *IRGraph* ⇒ *'a TraversalState* ⇒ *'a TraversalState option* ⇒ *bool*

for *sa g where*

— Hit a BeginNode with an IfNode predecessor which represents the start of a basic block for the IfNode. 1. *nid'* will be the successor of the begin node. 2. Find the first and only predecessor. 3. Extract condition from the preceding IfNode. 4. Negate condition if the begin node is second branch (we've taken the else branch of the condition) 5. Add the condition or the negated condition to stack 6. Perform any stamp updates based on the condition using the registerNewCondition function and place them on the top of the stack of stamp information

$\llbracket \text{kind } g \text{ nid} = \text{BeginNode } \text{nid}' ;$

$\text{nid} \notin \text{seen};$
 $\text{seen}' = \{\text{nid}\} \cup \text{seen};$

$\text{Some } \text{ifcond} = \text{pred } g \text{ nid};$
 $\text{kind } g \text{ ifcond} = \text{IfNode } \text{cond } t \text{ f};$

$\text{analysis}' = \text{sa } (\text{nid}, \text{seen}, \text{analysis}) \rrbracket$
 $\implies \text{Step } \text{sa } g \text{ (nid, seen, analysis) (Some (nid', seen', analysis')) } \mid$

— Hit an EndNode 1. *nid'* will be the usage of EndNode 2. pop the conditions and stamp stack

$\llbracket \text{kind } g \text{ nid} = \text{EndNode};$

$\text{nid} \notin \text{seen};$
 $\text{seen}' = \{\text{nid}\} \cup \text{seen};$

$\text{nid}' = \text{any-usage } g \text{ nid};$

$\text{analysis}' = \text{sa } (\text{nid}, \text{seen}, \text{analysis}) \rrbracket$
 $\implies \text{Step } \text{sa } g \text{ (nid, seen, analysis) (Some (nid', seen', analysis')) } \mid$

— We can find a successor edge that is not in seen, go there

$\llbracket \neg(\text{is-EndNode } (\text{kind } g \text{ nid}));$
 $\neg(\text{is-BeginNode } (\text{kind } g \text{ nid}));$

$\text{nid} \notin \text{seen};$
 $\text{seen}' = \{\text{nid}\} \cup \text{seen};$

```

    Some nid' = nextEdge seen' nid g;

    analysis' = sa (nid, seen, analysis)
  ==> Step sa g (nid, seen, analysis) (Some (nid', seen', analysis')) |

  — We cannot find a successor edge that is not in seen, give back None
  [[¬(is-EndNode (kind g nid));
    ¬(is-BEGINNode (kind g nid));

    nid ∉ seen;
    seen' = {nid} ∪ seen;

    None = nextEdge seen' nid g]]
  ==> Step sa g (nid, seen, analysis) None |

  — We've already seen this node, give back None
  [[nid ∈ seen]] ==> Step sa g (nid, seen, analysis) None

code-pred (modes: i ⇒ i ⇒ i ⇒ o ⇒ bool) Step .

end

```

5.2 Structural Graph Comparison

theory

Comparison

imports

IRGraph

begin

We introduce a form of structural graph comparison that is able to assert structural equivalence of graphs which differ in zero or more reference node chains for any given nodes.

```

fun find-ref-nodes :: IRGraph ⇒ (ID → ID) where
find-ref-nodes g = map-of
  (map (λn. (n, ir-ref (kind g n))) (filter (λid. is-RefNode (kind g id)) (sorted-list-of-set
    (ids g))))

```

```

fun replace-ref-nodes :: IRGraph ⇒ (ID → ID) ⇒ ID list ⇒ ID list where
replace-ref-nodes g m xs = map (λid. (case (m id) of Some other ⇒ other | None
  ⇒ id)) xs

```

```

fun find-next :: ID list ⇒ ID set ⇒ ID option where
find-next to-see seen = (let l = (filter (λnid. nid ∉ seen) to-see)
  in (case l of [] ⇒ None | xs ⇒ Some (hd xs)))

```

```

inductive reachables :: IRGraph ⇒ ID list ⇒ ID set ⇒ ID set ⇒ bool where
reachables g [] {} {} |
[[None = find-next to-see seen]] ==> reachables g to-see seen seen |

```



```

[[Some n = find-next to-see seen;
  node = kind g n;
  new = (inputs-of node) @ (successors-of node);
  reachables g (to-see @ new) ({n} ∪ seen) seen' ]] ⇒ reachables g to-see seen
seen'

```

```

code-pred (modes:  $i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$ ) [show-steps, show-mode-inference, show-intermediate-results]
reachables .

```

```

inductive nodeEq :: (ID → ID) ⇒ IRGraph ⇒ ID ⇒ IRGraph ⇒ ID ⇒ bool
where
  [[ kind g1 n1 = RefNode ref; nodeEq m g1 ref g2 n2 ]] ⇒ nodeEq m g1 n1 g2 n2 |
  [[ x = kind g1 n1;
    y = kind g2 n2;
    is-same-ir-node-type x y;
    replace-ref-nodes g1 m (successors-of x) = successors-of y;
    replace-ref-nodes g1 m (inputs-of x) = inputs-of y ]]
    ⇒ nodeEq m g1 n1 g2 n2

```

```

code-pred [show-modes] nodeEq .

```

```

fun diffNodesGraph :: IRGraph ⇒ IRGraph ⇒ ID set where
diffNodesGraph g1 g2 = (let refNodes = find-ref-nodes g1 in
  { n . n ∈ Predicate.the (reachables-i-i-i-o g1 [0] {}) ∧ (case refNodes n of Some
    - ⇒ False | - ⇒ True) ∧ ¬(nodeEq refNodes g1 n g2 n)})

```

```

fun diffNodesInfo :: IRGraph ⇒ IRGraph ⇒ (ID × IRNode × IRNode) set where
diffNodesInfo g1 g2 = {(nid, kind g1 nid, kind g2 nid) | nid . nid ∈ diffNodesGraph
g1 g2}

```

```

fun eqGraph :: IRGraph ⇒ IRGraph ⇒ bool where
eqGraph isabelle-graph graal-graph = ((diffNodesGraph isabelle-graph graal-graph)
= {})

```

```

end

```

6 Data-flow Semantics

```

theory IRTreeEval
imports
  Graph.Stamp
begin

```

We define a tree representation of data-flow nodes, as an abstraction of the graph view.

Data-flow trees are evaluated in the context of a method state (currently

called MapState in the theories for historical reasons).

The method state consists of the values for each method parameter, references to method parameters use an index of the parameter within the parameter list, as such we store a list of parameter values which are looked up at parameter references.

The method state also stores a mapping of node ids to values. The contents of this mapping is calculated during the traversal of the control flow graph. As a concrete example, as the *SignedDivNode* can have side-effects (during division by zero), it is treated as part of the control-flow, since the data-flow phase is specified to be side-effect free. As a result, the control-flow semantics for *SignedDivNode* calculates the value of a node and maps the node identifier to the value within the method state. The data-flow semantics then just reads the value stored in the method state for the node.

type-synonym *ID* = *nat*

type-synonym *MapState* = *ID* \Rightarrow *Value*

type-synonym *Params* = *Value list*

definition *new-map-state* :: *MapState* **where**

new-map-state = ($\lambda x.$ *UndefVal*)

6.1 Data-flow Tree Representation

datatype *IRUnaryOp* =

UnaryAbs
| *UnaryNeg*
| *UnaryNot*
| *UnaryLogicNegation*
| *UnaryNarrow* (*ir-inputBits*: *nat*) (*ir-resultBits*: *nat*)
| *UnarySignExtend* (*ir-inputBits*: *nat*) (*ir-resultBits*: *nat*)
| *UnaryZeroExtend* (*ir-inputBits*: *nat*) (*ir-resultBits*: *nat*)

datatype *IRBinaryOp* =

BinAdd
| *BinMul*
| *BinSub*
| *BinAnd*
| *BinOr*
| *BinXor*
| *BinShortCircuitOr*
| *BinLeftShift*
| *BinRightShift*
| *BinURightShift*
| *BinIntegerEquals*
| *BinIntegerLessThan*
| *BinIntegerBelow*

datatype (*discs-sels*) *IRExpr* =

```

    UnaryExpr (ir-uop: IRUnaryOp) (ir-value: IRExpr)
  | BinaryExpr (ir-op: IRBinaryOp) (ir-x: IRExpr) (ir-y: IRExpr)
  | ConditionalExpr (ir-condition: IRExpr) (ir-trueValue: IRExpr) (ir-falseValue:
IRExpr)

  | ParameterExpr (ir-index: nat) (ir-stamp: Stamp)

  | LeafExpr (ir-nid: ID) (ir-stamp: Stamp)

  | ConstantExpr (ir-const: Value)
  | ConstantVar (ir-name: string)
  | VariableExpr (ir-name: string) (ir-stamp: Stamp)

```

```

fun is-ground :: IRExpr ⇒ bool where
  is-ground (UnaryExpr op e) = is-ground e |
  is-ground (BinaryExpr op e1 e2) = (is-ground e1 ∧ is-ground e2) |
  is-ground (ConditionalExpr b e1 e2) = (is-ground b ∧ is-ground e1 ∧ is-ground
e2) |
  is-ground (ParameterExpr i s) = True |
  is-ground (LeafExpr n s) = True |
  is-ground (ConstantExpr v) = True |
  is-ground (ConstantVar name) = False |
  is-ground (VariableExpr name s) = False

```

```

typedef GroundExpr = { e :: IRExpr . is-ground e }
using is-ground.simps(6) by blast

```

6.2 Functions for re-calculating stamps

Note: in Java all integer calculations are done as 32 or 64 bit calculations. However, here we generalise the operators to allow any size calculations. Many operators have the same output bits as their inputs. However, the unary integer operators that are not *normal_unary* are narrowing or widening operators, so the result bits is specified by the operator. The binary integer operators are divided into three groups: (1) *binary_fixed_32* operators always output 32 bits, (2) *binary_shift_ops* operators output size is determined by their left argument, and (3) other operators output the same number of bits as both their inputs.

abbreviation *binary-fixed-32-ops* :: IRBinaryOp set **where**
binary-fixed-32-ops ≡ { BinShortCircuitOr, BinIntegerEquals, BinIntegerLessThan, BinIntegerBelow }

abbreviation *binary-shift-ops* :: IRBinaryOp set **where**
binary-shift-ops ≡ { BinLeftShift, BinRightShift, BinURightShift }

abbreviation *normal-unary* :: IRUnaryOp set **where**
normal-unary ≡ { UnaryAbs, UnaryNeg, UnaryNot, UnaryLogicNegation }

```

fun stamp-unary :: IRUnaryOp ⇒ Stamp ⇒ Stamp where

  stamp-unary op (IntegerStamp b lo hi) =
    unrestricted-stamp (IntegerStamp (if op ∈ normal-unary then b else (ir-resultBits
    op)) lo hi) |

  stamp-unary op - = IllegalStamp

fun stamp-binary :: IRBinaryOp ⇒ Stamp ⇒ Stamp ⇒ Stamp where
  stamp-binary op (IntegerStamp b1 lo1 hi1) (IntegerStamp b2 lo2 hi2) =
    (if op ∈ binary-shift-ops then unrestricted-stamp (IntegerStamp b1 lo1 hi1)
    else if b1 ≠ b2 then IllegalStamp else
    (if op ∈ binary-fixed-32-ops
    then unrestricted-stamp (IntegerStamp 32 lo1 hi1)
    else unrestricted-stamp (IntegerStamp b1 lo1 hi1))) |

  stamp-binary op - - = IllegalStamp

fun stamp-expr :: IRExpr ⇒ Stamp where
  stamp-expr (UnaryExpr op x) = stamp-unary op (stamp-expr x) |
  stamp-expr (BinaryExpr bop x y) = stamp-binary bop (stamp-expr x) (stamp-expr
  y) |
  stamp-expr (ConstantExpr val) = constantAsStamp val |
  stamp-expr (LeafExpr i s) = s |
  stamp-expr (ParameterExpr i s) = s |
  stamp-expr (ConditionalExpr c t f) = meet (stamp-expr t) (stamp-expr f)

export-code stamp-unary stamp-binary stamp-expr

```

6.3 Data-flow Tree Evaluation

```

fun unary-eval :: IRUnaryOp ⇒ Value ⇒ Value where
  unary-eval UnaryAbs v = intval-abs v |
  unary-eval UnaryNeg v = intval-negate v |
  unary-eval UnaryNot v = intval-not v |
  unary-eval UnaryLogicNegation v = intval-logic-negation v |
  unary-eval (UnaryNarrow inBits outBits) v = intval-narrow inBits outBits v |
  unary-eval (UnarySignExtend inBits outBits) v = intval-sign-extend inBits outBits
  v |
  unary-eval (UnaryZeroExtend inBits outBits) v = intval-zero-extend inBits outBits
  v

```

```

fun bin-eval :: IRBinaryOp ⇒ Value ⇒ Value ⇒ Value where
  bin-eval BinAdd v1 v2 = intval-add v1 v2 |
  bin-eval BinMul v1 v2 = intval-mul v1 v2 |
  bin-eval BinSub v1 v2 = intval-sub v1 v2 |
  bin-eval BinAnd v1 v2 = intval-and v1 v2 |

```

$\text{bin-eval BinOr } v1 \ v2 = \text{intval-or } v1 \ v2 \mid$
 $\text{bin-eval BinXor } v1 \ v2 = \text{intval-xor } v1 \ v2 \mid$
 $\text{bin-eval BinShortCircuitOr } v1 \ v2 = \text{intval-short-circuit-or } v1 \ v2 \mid$
 $\text{bin-eval BinLeftShift } v1 \ v2 = \text{intval-left-shift } v1 \ v2 \mid$
 $\text{bin-eval BinRightShift } v1 \ v2 = \text{intval-right-shift } v1 \ v2 \mid$
 $\text{bin-eval BinURightShift } v1 \ v2 = \text{intval-uright-shift } v1 \ v2 \mid$
 $\text{bin-eval BinIntegerEquals } v1 \ v2 = \text{intval-equals } v1 \ v2 \mid$
 $\text{bin-eval BinIntegerLessThan } v1 \ v2 = \text{intval-less-than } v1 \ v2 \mid$
 $\text{bin-eval BinIntegerBelow } v1 \ v2 = \text{intval-below } v1 \ v2$

lemmas *eval-thms* =

intval-abs.simps $\text{intval-negate.simps}$ intval-not.simps
 $\text{intval-logic-negation.simps}$ $\text{intval-narrow.simps}$
 $\text{intval-sign-extend.simps}$ $\text{intval-zero-extend.simps}$
 intval-add.simps intval-mul.simps intval-sub.simps
 intval-and.simps intval-or.simps intval-xor.simps
 $\text{intval-left-shift.simps}$ $\text{intval-right-shift.simps}$
 $\text{intval-uright-shift.simps}$ $\text{intval-equals.simps}$
 $\text{intval-less-than.simps}$ $\text{intval-below.simps}$

inductive *not-undef-or-fail* :: *Value* \Rightarrow *Value* \Rightarrow *bool* **where**

$\llbracket \text{value} \neq \text{UndefVal} \rrbracket \Longrightarrow \text{not-undef-or-fail value value}$

notation (*latex output*)

not-undef-or-fail (- = -)

inductive

evaltree :: *MapState* \Rightarrow *Params* \Rightarrow *IRExpr* \Rightarrow *Value* \Rightarrow *bool* ($\llbracket -, - \rrbracket \vdash - \mapsto -$ 55)

for *m p* **where**

ConstantExpr:

$\llbracket \text{valid-value } c \ (\text{constantAsStamp } c) \rrbracket$
 $\Longrightarrow [m, p] \vdash (\text{ConstantExpr } c) \mapsto c \mid$

ParameterExpr:

$\llbracket i < \text{length } p; \text{valid-value } (p!i) \ s \rrbracket$
 $\Longrightarrow [m, p] \vdash (\text{ParameterExpr } i \ s) \mapsto p!i \mid$

ConditionalExpr:

$\llbracket [m, p] \vdash ce \mapsto \text{cond};$
 $\text{branch} = (\text{if val-to-bool cond then te else fe});$
 $[m, p] \vdash \text{branch} \mapsto v;$
 $v \neq \text{UndefVal} \rrbracket$
 $\Longrightarrow [m, p] \vdash (\text{ConditionalExpr } ce \ te \ fe) \mapsto v \mid$

UnaryExpr:

$\llbracket [m, p] \vdash xe \mapsto v; \rrbracket$

$result = (unary\text{-}eval\ op\ v);$
 $result \neq UndefVal]$
 $\implies [m,p] \vdash (UnaryExpr\ op\ xe) \mapsto result \mid$

BinaryExpr:
 $[[m,p] \vdash xe \mapsto x;$
 $[m,p] \vdash ye \mapsto y;$
 $result = (bin\text{-}eval\ op\ x\ y);$
 $result \neq UndefVal]$
 $\implies [m,p] \vdash (BinaryExpr\ op\ xe\ ye) \mapsto result \mid$

LeafExpr:
 $[[val = m\ n;$
 $valid\text{-}value\ val\ s]]$
 $\implies [m,p] \vdash LeafExpr\ n\ s \mapsto val$

code-pred (*modes: $i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow bool$ as evalT*)
 $[show\text{-}steps, show\text{-}mode\text{-}inference, show\text{-}intermediate\text{-}results]$
 $evaltree .$

inductive

$evaltrees :: MapState \Rightarrow Params \Rightarrow IRExpr\ list \Rightarrow Value\ list \Rightarrow bool\ ([-,] \vdash - \mapsto_L$
 $- \ 55)$

for $m\ p$ **where**

EvalNil:
 $[m,p] \vdash [] \mapsto_L [] \mid$

EvalCons:
 $[[m,p] \vdash x \mapsto xval;$
 $[m,p] \vdash yy \mapsto_L yyval]$
 $\implies [m,p] \vdash (x\#yy) \mapsto_L (xval\#yyval)$

code-pred (*modes: $i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow bool$ as evalTs*)
 $evaltrees .$

definition $sq\text{-}param0 :: IRExpr$ **where**

$sq\text{-}param0 = BinaryExpr\ BinMul$
 $(ParameterExpr\ 0\ (IntegerStamp\ 32\ (-\ 2147483648)\ 2147483647))$
 $(ParameterExpr\ 0\ (IntegerStamp\ 32\ (-\ 2147483648)\ 2147483647))$

values $\{v.\ evaltree\ new\text{-}map\text{-}state\ [IntVal\ 32\ 5]\ sq\text{-}param0\ v\}$

declare $evaltree.intros\ [intro]$
declare $evaltrees.intros\ [intro]$

6.4 Data-flow Tree Refinement

We define the induced semantic equivalence relation between expressions. Note that syntactic equality implies semantic equivalence, but not vice versa.

definition *equiv-exprs* :: *IRExpr* \Rightarrow *IRExpr* \Rightarrow *bool* (*-* \doteq *-* 55) **where**
 $(e1 \doteq e2) = (\forall m p v. ([m,p] \vdash e1 \mapsto v) \longleftrightarrow ([m,p] \vdash e2 \mapsto v))$

We also prove that this is a total equivalence relation (*equivp equiv-exprs*) (HOL.Equiv_Relations), so that we can reuse standard results about equivalence relations.

lemma *equivp equiv-exprs*
apply (*auto simp add: equivp-def equiv-exprs-def*)
by (*metis equiv-exprs-def*)**+**

We define a refinement ordering over *IRExpr* and show that it is a preorder. Note that it is asymmetric because *e2* may refer to fewer variables than *e1*.

instantiation *IRExpr* :: *preorder* **begin**

notation *less-eq* (**infix** \sqsubseteq 65)

definition

le-expr-def [*simp*]:
 $(e_2 \leq e_1) \longleftrightarrow (\forall m p v. ([m,p] \vdash e_1 \mapsto v) \longrightarrow ([m,p] \vdash e_2 \mapsto v))$

definition

lt-expr-def [*simp*]:
 $(e_1 < e_2) \longleftrightarrow (e_1 \leq e_2 \wedge \neg (e_1 \doteq e_2))$

instance proof

fix *x y z* :: *IRExpr*
show $x < y \longleftrightarrow x \leq y \wedge \neg (y \leq x)$ **by** (*simp add: equiv-exprs-def; auto*)
show $x \leq x$ **by** *simp*
show $x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$ **by** *simp*
qed

end

abbreviation (**output**) *Refines* :: *IRExpr* \Rightarrow *IRExpr* \Rightarrow *bool* (**infix** \sqsupseteq 64)
where $e_1 \sqsupseteq e_2 \equiv (e_2 \leq e_1)$

6.5 Stamp Masks

A stamp can contain additional range information in the form of masks. A stamp has an up mask and a down mask, corresponding to the bits that may be set and the bits that must be set.

Examples: A stamp where no range information is known will have; an up mask of -1 as all bits may be set, and a down mask of 0 as no bits must be set.

A stamp known to be one should have; an up mask of 1 as only the first bit may be set, no others, and a down mask of 1 as the first bit must be set and no others.

We currently don't carry mask information in stamps, and instead assume correct masks to prove optimizations.

```

locale stamp-mask =
  fixes up :: IRExp ⇒ int64 (↑)
  fixes down :: IRExp ⇒ int64 (↓)
  assumes up-spec: [m, p] ⊢ e ↦ IntVal b v ⇒ (and v (not ((ucast (↑e))))) = 0
  and down-spec: [m, p] ⊢ e ↦ IntVal b v ⇒ (and (not v) (ucast (↓e))) = 0
begin

```

```

lemma may-implies-either:
  [m, p] ⊢ e ↦ IntVal b v ⇒ bit (↑e) n ⇒ bit v n = False ∨ bit v n = True
by simp

```

```

lemma not-may-implies-false:
  [m, p] ⊢ e ↦ IntVal b v ⇒ ¬(bit (↑e) n) ⇒ bit v n = False
using up-spec
using bit-and-iff bit-eq-iff bit-not-iff bit-unsigned-iff down-spec
by (smt (verit, best) bit.double-compl)

```

```

lemma must-implies-true:
  [m, p] ⊢ e ↦ IntVal b v ⇒ bit (↓e) n ⇒ bit v n = True
using down-spec
by (metis bit.compl-one bit-and-iff bit-minus-1-iff bit-not-iff impossible-bit ucast-id)

```

```

lemma not-must-implies-either:
  [m, p] ⊢ e ↦ IntVal b v ⇒ ¬(bit (↓e) n) ⇒ bit v n = False ∨ bit v n = True
by simp

```

```

lemma must-implies-may:
  [m, p] ⊢ e ↦ IntVal b v ⇒ n < 32 ⇒ bit (↓e) n ⇒ bit (↑e) n
by (meson must-implies-true not-may-implies-false)

```

```

lemma up-mask-and-zero-implies-zero:
  assumes and (↑x) (↑y) = 0
  assumes [m, p] ⊢ x ↦ IntVal b xv
  assumes [m, p] ⊢ y ↦ IntVal b yv
  shows and xv yv = 0
  using assms
  by (smt (z3) and.commute and.right-neutral and-zero-eq bit.compl-zero bit.conj-cancel-right
    bit.conj-disj-distrib(1) ucast-id up-spec word-bw-assocs(1) word-not-dist(2))

```

```

lemma not-down-up-mask-and-zero-implies-zero:
  assumes and (not (↓x)) (↑y) = 0
  assumes [m, p] ⊢ x ↦ IntVal b xv

```



```

assumes  $[m, p] \vdash y \mapsto \text{IntVal } b \ yv$ 
shows  $\text{and } xv \ yv = yv$ 
using assms
by (smt (z3) and-zero-eq bit.conj-cancel-left bit.conj-disj-distrib(1) bit.conj-disj-distrib(2)
bit.de-Morgan-disj down-spec or-eq-not-not-and ucast-id up-spec word-ao-absorbs(2)
word-ao-absorbs(8) word-bw-lcs(1) word-not-dist(2))

end

end

```

6.6 Data-flow Tree Theorems

```

theory IRTreeEvalThms
imports
  Graph.ValueThms
  IRTreeEval
begin

```

6.6.1 Deterministic Data-flow Evaluation

```

lemma evalDet:
 $[m, p] \vdash e \mapsto v_1 \implies$ 
 $[m, p] \vdash e \mapsto v_2 \implies$ 
 $v_1 = v_2$ 
apply (induction arbitrary:  $v_2$  rule: evaltree.induct)
by (elim EvalTreeE; auto)+

```

```

lemma evalAllDet:
 $[m, p] \vdash e \mapsto_L v1 \implies$ 
 $[m, p] \vdash e \mapsto_L v2 \implies$ 
 $v1 = v2$ 
apply (induction arbitrary:  $v2$  rule: evaltrees.induct)
apply (elim EvalTreeE; auto)
using evalDet by force

```

6.6.2 Typing Properties for Integer Evaluation Functions

We use three simple typing properties on integer values: *isIntVal32*, *isIntVal64* and the more general *isIntVal*.

```

lemma unary-eval-not-obj-ref:
shows unary-eval op  $x \neq \text{ObjRef } v$ 
by (cases op; cases  $x$ ; auto)

```

```

lemma unary-eval-not-obj-str:
shows unary-eval op  $x \neq \text{ObjStr } v$ 
by (cases op; cases  $x$ ; auto)

```

```

lemma unary-eval-int:
  assumes def: unary-eval op x  $\neq$  UndefVal
  shows is-IntVal (unary-eval op x)
  unfolding is-IntVal-def using def
  apply (cases unary-eval op x; auto)
  using unary-eval-not-obj-ref unary-eval-not-obj-str by simp+

lemma bin-eval-int:
  assumes def: bin-eval op x y  $\neq$  UndefVal
  shows is-IntVal (bin-eval op x y)
  apply (cases op; cases x; cases y)
  unfolding is-IntVal-def using def apply auto
    apply presburger+
    apply (meson bool-to-val.elims)
    apply (meson bool-to-val.elims)
    apply (smt (verit) new-int.simps)+
  by (meson bool-to-val.elims)+

lemma IntVal0:
  (IntVal 32 0) = (new-int 32 0)
  unfolding new-int.simps
  by auto

lemma IntVal1:
  (IntVal 32 1) = (new-int 32 1)
  unfolding new-int.simps
  by auto

lemma bin-eval-new-int:
  assumes def: bin-eval op x y  $\neq$  UndefVal
  shows  $\exists b\ v. (bin-eval\ op\ x\ y) = new-int\ b\ v \wedge$ 
     $b = (if\ op \in binary-fixed-32-ops\ then\ 32\ else\ intval-bits\ x)$ 
  apply (cases op; cases x; cases y)
  unfolding is-IntVal-def using def apply auto
    apply presburger+
    apply (metis take-bit-and)
    apply presburger
    apply (metis take-bit-or)
    apply presburger
    apply (metis take-bit-xor)
    apply presburger
  using IntVal0 IntVal1
  apply (metis bool-to-val.elims new-int.simps)

```

```

apply presburger
apply (smt (verit) new-int.elims)
apply (smt (verit, best) new-int.elims)
apply (metis IntVal0 IntVal1 bool-to-val.elims new-int.simps)
apply presburger
apply (metis IntVal0 IntVal1 bool-to-val.elims new-int.simps)
apply presburger
apply (metis IntVal0 IntVal1 bool-to-val.elims new-int.simps)
by meson

```

```

lemma int-stamp:
  assumes i: is-IntVal v
  shows is-IntegerStamp (constantAsStamp v)
  using i unfolding is-IntegerStamp-def is-IntVal-def by auto

```

```

lemma validStampIntConst:
  assumes v = IntVal b ival
  assumes  $0 < b \wedge b \leq 64$ 
  shows valid-stamp (constantAsStamp v)
proof –
  have bnds: fst (bit-bounds b) ≤ int-signed-value b ival ∧ int-signed-value b ival
   $\leq \text{snd (bit-bounds b)}$ 
  using assms int-signed-value-bounds
  by presburger
  have s: constantAsStamp v = IntegerStamp b (int-signed-value b ival) (int-signed-value
  b ival)
  using assms(1) constantAsStamp.simps(1) by blast
  then show ?thesis
  unfolding s valid-stamp.simps
  using assms(2) assms bnds by linarith
qed

```

```

lemma validDefIntConst:
  assumes v: v = IntVal b ival
  assumes  $0 < b \wedge b \leq 64$ 
  assumes take-bit b ival = ival
  shows valid-value v (constantAsStamp v)
proof –
  have bnds: fst (bit-bounds b) ≤ int-signed-value b ival ∧ int-signed-value b ival
   $\leq \text{snd (bit-bounds b)}$ 
  using assms int-signed-value-bounds
  by presburger
  have s: constantAsStamp v = IntegerStamp b (int-signed-value b ival) (int-signed-value
  b ival)
  using assms(1) constantAsStamp.simps(1) by blast
  then show ?thesis
  unfolding s unfolding v unfolding valid-value.simps

```

```

    using assms validStampIntConst
  by simp
qed

```

6.6.3 Evaluation Results are Valid

A valid value cannot be *UndefVal*.

```

lemma valid-not-undef:
  assumes a1: valid-value val s
  assumes a2: s ≠ VoidStamp
  shows val ≠ UndefVal
  apply (rule valid-value.elims(1)[of val s True])
  using a1 a2 by auto

```

```

lemma valid-VoidStamp[elim]:
  shows valid-value val VoidStamp ⇒
    val = UndefVal
  using valid-value.simps by metis

```

```

lemma valid-ObjStamp[elim]:
  shows valid-value val (ObjectStamp klass exact nonNull alwaysNull) ⇒
    (∃ v. val = ObjRef v)
  using valid-value.simps by (metis val-to-bool.cases)

```

```

lemma valid-int[elim]:
  shows valid-value val (IntegerStamp b lo hi) ⇒
    (∃ v. val = IntVal b v)
  using valid-value.elims(2) by fastforce

```

```

lemmas valid-value-elim =
  valid-VoidStamp
  valid-ObjStamp
  valid-int

```

```

lemma evaltree-not-undef:
  fixes m p e v
  shows ([m,p] ⊢ e ↦ v) ⇒ v ≠ UndefVal
  apply (induction rule: evaltree.induct)
  using valid-not-undef by auto

```

```

lemma leafint:
  assumes ev: [m,p] ⊢ LeafExpr i (IntegerStamp b lo hi) ↦ val
  shows ∃ b v. val = (IntVal b v)

```

```

proof –
  have valid-value val (IntegerStamp b lo hi)

```

```

    using ev by (rule LeafExprE; simp)
  then show ?thesis by auto
qed

```

```

lemma default-stamp [simp]: default-stamp = IntegerStamp 32 (-2147483648)
2147483647
  using default-stamp-def by auto

```

```

lemma valid-value-signed-int-range [simp]:
  assumes valid-value val (IntegerStamp b lo hi)
  assumes lo < 0
  shows  $\exists v. (val = \text{IntVal } b \ v \wedge$ 
     $lo \leq \text{int-signed-value } b \ v \wedge$ 
     $\text{int-signed-value } b \ v \leq hi)$ 
  using assms valid-int
  by (metis valid-value.simps(1))

```

6.6.4 Example Data-flow Optimisations

6.6.5 Monotonicity of Expression Refinement

We prove that each subexpression position is monotonic. That is, optimizing a subexpression anywhere deep inside a top-level expression also optimizes that top-level expression.

Note that we might also be able to do this via reusing Isabelle’s *mono* operator (HOL.Orderings theory), proving instantiations like *mono(UnaryExpr op)*, but it is not obvious how to do this for both arguments of the binary expressions.

```

lemma mono-unary:
  assumes  $e \geq e'$ 
  shows  $(\text{UnaryExpr } op \ e) \geq (\text{UnaryExpr } op \ e')$ 
  using UnaryExpr assms by auto

```

```

lemma mono-binary:
  assumes  $x \geq x'$ 
  assumes  $y \geq y'$ 
  shows  $(\text{BinaryExpr } op \ x \ y) \geq (\text{BinaryExpr } op \ x' \ y')$ 
  using BinaryExpr assms by auto

```

```

lemma never-void:
  assumes  $[m, p] \vdash x \mapsto xv$ 
  assumes valid-value xv (stamp-expr xe)
  shows stamp-expr xe  $\neq$  VoidStamp
  using valid-value.simps
  using assms(2) by force

```

lemma *compatible-trans*:
 $compatible\ x\ y \wedge compatible\ y\ z \implies compatible\ x\ z$
by (cases x; cases y; cases z; simp del: valid-stamp.simps)

lemma *compatible-refl*:
 $compatible\ x\ y \implies compatible\ y\ x$
using compatible.elims(2) **by** fastforce

lemma *mono-conditional*:
assumes $ce \geq ce'$
assumes $te \geq te'$
assumes $fe \geq fe'$
shows $(ConditionalExpr\ ce\ te\ fe) \geq (ConditionalExpr\ ce'\ te'\ fe')$
proof (simp only: le-expr-def; (rule allI)+; rule impI)
fix m p v
assume a: $[m, p] \vdash ConditionalExpr\ ce\ te\ fe \mapsto v$
then obtain cond **where** $ce: [m, p] \vdash ce \mapsto cond$ **by** auto
then have $ce': [m, p] \vdash ce' \mapsto cond$ **using** assms **by** auto

define branch **where** b: $branch = (if\ val\text{-}to\text{-}bool\ cond\ then\ te\ else\ fe)$
define branch' **where** b': $branch' = (if\ val\text{-}to\text{-}bool\ cond\ then\ te'\ else\ fe')$
then have beval: $[m, p] \vdash branch \mapsto v$ **using** a b ce evalDet **by** blast

from beval **have** $[m, p] \vdash branch' \mapsto v$ **using** assms b b' **by** auto
then show $[m, p] \vdash ConditionalExpr\ ce'\ te'\ fe' \mapsto v$
using ConditionalExpr ce' b'
using a **by** blast
qed

6.7 Unfolding rules for evaltree quadruples down to bin-eval level

These rewrite rules can be useful when proving optimizations. They support top-down rewriting of each level of the tree into the lower-level bin_eval / $unary_eval$ level, simply by saying *unfoldingunfold_evaltree*.

lemma *unfold-const*:
shows $([m, p] \vdash ConstantExpr\ c \mapsto v) = (valid_value\ v\ (constantAsStamp\ c) \wedge v = c)$
by blast

lemma *unfold-binary*:

```

shows ( $[m,p] \vdash \text{BinaryExpr } op \ x \ y \mapsto val$ ) = ( $\exists \ x \ y.$ 
  ( $[m,p] \vdash x \mapsto x$ )  $\wedge$ 
  ( $[m,p] \vdash y \mapsto y$ )  $\wedge$ 
  ( $val = \text{bin-eval } op \ x \ y$ )  $\wedge$ 
  ( $val \neq \text{UndefVal}$ )
  )) (is ?L = ?R)
proof (intro iffI)
  assume  $\mathcal{B}$ : ?L
  show ?R by (rule evaltree.cases[OF  $\mathcal{B}$ ]; blast+)
next
  assume ?R
  then obtain  $x \ y$  where  $[m,p] \vdash x \mapsto x$ 
    and  $[m,p] \vdash y \mapsto y$ 
    and  $val = \text{bin-eval } op \ x \ y$ 
    and  $val \neq \text{UndefVal}$ 
    by auto
  then show ?L
    by (rule BinaryExpr)
qed

```

```

lemma unfold-unary:
shows ( $[m,p] \vdash \text{UnaryExpr } op \ x \mapsto val$ )
  = ( $\exists \ x.$ 
    ( $[m,p] \vdash x \mapsto x$ )  $\wedge$ 
    ( $val = \text{unary-eval } op \ x$ )  $\wedge$ 
    ( $val \neq \text{UndefVal}$ )
    ) (is ?L = ?R)
by auto

```

```

lemmas unfold-evaltree =
  unfold-binary
  unfold-unary

```

6.8 Lemmas about *new__int* and integer eval results.

```

lemma unary-eval-new-int:
assumes def:  $\text{unary-eval } op \ x \neq \text{UndefVal}$ 
shows  $\exists \ b \ v. \text{unary-eval } op \ x = \text{new-int } b \ v \wedge$ 
   $b = (\text{if } op \in \text{normal-unary} \text{ then } \text{intval-bits } x \text{ else } \text{ir-resultBits } op)$ 
proof (cases op  $\in$  normal-unary)
  case True
  then show ?thesis
    by (metis def empty-iff insert-iff intval-abs.elims intval-bits.simps intval-logic-negation.elims
      intval-negate.elims intval-not.elims unary-eval.simps(1) unary-eval.simps(2) unary-eval.simps(3)
      unary-eval.simps(4))
  next
  case False

```

```

consider ib ob where op = UnaryNarrow ib ob |
             ib ob where op = UnaryZeroExtend ib ob |
             ib ob where op = UnarySignExtend ib ob
by (metis False IRUnaryOp.exhaust insert-iff)
then show ?thesis
proof (cases)
  case 1
    then show ?thesis
    by (metis False IRUnaryOp.sel(4) def intval-narrow.elims unary-eval.simps(5))
  next
    case 2
      then show ?thesis
      by (metis False IRUnaryOp.sel(6) def intval-zero-extend.elims unary-eval.simps(7))
    next
      case 3
        then show ?thesis
        by (metis False IRUnaryOp.sel(5) def intval-sign-extend.elims unary-eval.simps(6))
      qed
    qed

```

```

lemma new-int-unused-bits-zero:
  assumes IntVal b ival = new-int b ival0
  shows take-bit b ival = ival
  using assms(1) new-int-take-bits by blast

```

```

lemma unary-eval-unused-bits-zero:
  assumes unary-eval op x = IntVal b ival
  shows take-bit b ival = ival
  using assms unary-eval-new-int
  by (metis Value.inject(1) Value.simps(5) new-int.elims new-int-unused-bits-zero)

```

```

lemma bin-eval-unused-bits-zero:
  assumes bin-eval op x y = (IntVal b ival)
  shows take-bit b ival = ival
  using assms bin-eval-new-int
  by (metis Value.distinct(1) Value.inject(1) new-int.elims new-int-take-bits)

```

```

lemma eval-unused-bits-zero:
   $[m,p] \vdash xe \mapsto (IntVal\ b\ ix) \implies take-bit\ b\ ix = ix$ 
proof (induction xe)
  case (UnaryExpr x1 xe)
    then show ?case
    using unary-eval-unused-bits-zero by force
  next
    case (BinaryExpr x1 xe1 xe2)
    then show ?case
    using bin-eval-unused-bits-zero by force
  next
    case (ConditionalExpr xe1 xe2 xe3)

```



```

    then show ?case
      by (metis (full-types) EvalTreeE(3))
next
  case (ParameterExpr i s)
  then have valid-value (p!i) s
    by fastforce
  then show ?case
    by (metis ParameterExprE Value.distinct(7) intval-bits.simps intval-word.simps
      local.ParameterExpr valid-value.elims(2))
next
  case (LeafExpr x1 x2)
  then show ?case
    by (smt (z3) EvalTreeE(6) Value.simps(11) valid-value.elims(1) valid-value.simps(1))

next
  case (ConstantExpr x)
  then show ?case
    by (metis EvalTreeE(1) constantAsStamp.simps(1) valid-value.simps(1))
next
  case (ConstantVar x)
  then show ?case
    by fastforce
next
  case (VariableExpr x1 x2)
  then show ?case
    by fastforce
qed

```

```

lemma unary-normal-bitsize:
  assumes unary-eval op x = IntVal b ival
  assumes op ∈ normal-unary
  shows ∃ ix. x = IntVal b ix
  apply (cases op)
    prefer 7 using assms apply blast
    prefer 6 using assms apply blast
    prefer 5 using assms apply blast
  using Value.distinct(1) Value.sel(1) assms(1) new-int.simps unary-eval.simps
    intval-abs.elims intval-negate.elims intval-not.elims intval-logic-negation.elims
  apply metis+
done

```

```

lemma unary-not-normal-bitsize:
  assumes unary-eval op x = IntVal b ival
  assumes op ∉ normal-unary
  shows b = ir-resultBits op ∧ 0 < b ∧ b ≤ 64
  apply (cases op)
  using assms apply blast+
  apply (metis IRUnaryOp.sel(4) Value.distinct(1) Value.sel(1) assms(1) intval-narrow.elims

```

```

intval-narrow-ok new-int.simps unary-eval.simps(5))
  apply (smt (verit) IRUnaryOp.sel(5) Value.distinct(1) Value.sel(1) assms(1)
intval-sign-extend.elims new-int.simps order-less-le-trans unary-eval.simps(6))
  apply (metis IRUnaryOp.sel(6) Value.distinct(1) assms(1) intval-bits.simps intval-zero-extend.elims linorder-not-less neq0-conv new-int.simps unary-eval.simps(7))
done

```

```

lemma unary-eval-bitsize:
  assumes unary-eval op x = IntVal b ival
  assumes 2: x = IntVal bx ix
  assumes 0 < bx ∧ bx ≤ 64
  shows 0 < b ∧ b ≤ 64
proof (cases op ∈ normal-unary)
  case True
  then obtain tmp where unary-eval op x = new-int bx tmp
    by (cases op; simp; auto simp: 2)
  then show ?thesis
    using assms by simp
next
  case False
  then obtain tmp where unary-eval op x = new-int b tmp ∧ 0 < b ∧ b ≤ 64
    apply (cases op; simp; auto simp: 2)
    apply (metis 2 Value.inject(1) Value.simps(5) assms(1) intval-narrow.simps(1)
intval-narrow-ok new-int.simps unary-eval.simps(5))
    apply (metis 2 Value.distinct(1) Value.inject(1) assms(1) bot-nat-0.not-eq-extremum
diff-is-0-eq intval-sign-extend.elims new-int.simps unary-eval.simps(6) zero-less-diff)
    by (smt (verit, del-insts) 2 Value.simps(5) assms(1) intval-bits.simps intval-zero-extend.simps(1) new-int.simps order-less-le-trans unary-eval.simps(7))
  then show ?thesis
    by blast
qed

```

```

lemma bin-eval-inputs-are-ints:
  assumes bin-eval op x y = IntVal b ix
  obtains xb yb xi yi where x = IntVal xb xi ∧ y = IntVal yb yi
proof –
  have bin-eval op x y ≠ UndefVal
    by (simp add: assms)
  then show ?thesis
    using assms apply (cases op; cases x; cases y; simp)
    using that by blast+
qed

```

```

lemma eval-bits-1-64:

```

```

[m,p] ⊢ xe ↦ (IntVal b ix) ⇒ 0 < b ∧ b ≤ 64
proof (induction xe arbitrary: b ix)
  case (UnaryExpr op x2)
  then obtain xv where
    xv: ([m,p] ⊢ x2 ↦ xv) ∧
        IntVal b ix = unary-eval op xv
    using unfold-binary by auto
  then have b = (if op ∈ normal-unary then intval-bits xv else ir-resultBits op)
    using unary-eval-new-int
    by (metis Value.disc(1) Value.discI(1) Value.sel(1) new-int.simps)
  then show ?case
    by (metis xv UnaryExpr.IH unary-normal-bitsize unary-not-normal-bitsize)
next
  case (BinaryExpr op x y)
  then obtain xv yv where
    xy: ([m,p] ⊢ x ↦ xv) ∧
        ([m,p] ⊢ y ↦ yv) ∧
        IntVal b ix = bin-eval op xv yv
    using unfold-binary by auto
  then have def: bin-eval op xv yv ≠ UndefVal and xv: xv ≠ UndefVal and yv ≠
  UndefVal
    using evaltree-not-undef xy by (force, blast, blast)
  then have b = (if op ∈ binary-fixed-32-ops then 32 else intval-bits xv)
    by (metis xy intval-bits.simps new-int.simps bin-eval-new-int)
  then show ?case
    by (metis BinaryExpr.IH(1) Value.distinct(7) Value.distinct(9) xv bin-eval-inputs-are-ints
  intval-bits.elims le-add-same-cancel1 less-or-eq-imp-le numeral-Bit0 xy zero-less-numeral)
next
  case (ConditionalExpr xe1 xe2 xe3)
  then show ?case
    by (metis (full-types) EvalTreeE(3))
next
  case (ParameterExpr x1 x2)
  then show ?case
    using ParameterExprE intval-bits.simps valid-stamp.simps(1) valid-value.elims(2)
  valid-value.simps(17)
    by (metis (no-types, lifting))
next
  case (LeafExpr x1 x2)
  then show ?case
    by (smt (z3) EvalTreeE(6) Value.distinct(7) Value.inject(1) valid-stamp.simps(1)
  valid-value.elims(1))
next
  case (ConstantExpr x)
  then show ?case
    by (metis EvalTreeE(1) constantAsStamp.simps(1) valid-stamp.simps(1) valid-value.simps(1))
next
  case (ConstantVar x)

```

```

    then show ?case
      by blast
  next
    case (VariableExpr x1 x2)
    then show ?case
      by blast
  qed

end

```

7 Tree to Graph

```

theory TreeToGraph
  imports
    Semantics.IRTreeEval
    Graph.IRGraph
begin

```

7.1 Subgraph to Data-flow Tree

```

fun find-node-and-stamp :: IRGraph  $\Rightarrow$  (IRNode  $\times$  Stamp)  $\Rightarrow$  ID option where
  find-node-and-stamp g (n,s) =
    find ( $\lambda i.$  kind g i = n  $\wedge$  stamp g i = s) (sorted-list-of-set(ids g))

export-code find-node-and-stamp

```

```

fun is-preevaluated :: IRNode  $\Rightarrow$  bool where
  is-preevaluated (InvokeNode n - - - -) = True |
  is-preevaluated (InvokeWithExceptionNode n - - - -) = True |
  is-preevaluated (NewInstanceNode n - -) = True |
  is-preevaluated (LoadFieldNode n - -) = True |
  is-preevaluated (SignedDivNode n - - - -) = True |
  is-preevaluated (SignedRemNode n - - - -) = True |
  is-preevaluated (ValuePhiNode n -) = True |
  is-preevaluated - = False

```

```

inductive
  rep :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  IRExpr  $\Rightarrow$  bool (-  $\vdash$  -  $\simeq$  - 55)
  for g where

```

```

  ConstantNode:
     $\llbracket \text{kind } g \text{ } n = \text{ConstantNode } c \rrbracket$ 
     $\implies g \vdash n \simeq (\text{ConstantExpr } c)$  |

```

```

  ParameterNode:
     $\llbracket \text{kind } g \text{ } n = \text{ParameterNode } i; \rrbracket$ 

```

$stamp\ g\ n = s$
 $\implies g \vdash n \simeq (ParameterExpr\ i\ s) \mid$

ConditionalNode:

$\llbracket kind\ g\ n = ConditionalNode\ c\ t\ f;$
 $g \vdash c \simeq ce;$
 $g \vdash t \simeq te;$
 $g \vdash f \simeq fe \rrbracket$
 $\implies g \vdash n \simeq (ConditionalExpr\ ce\ te\ fe) \mid$

AbsNode:

$\llbracket kind\ g\ n = AbsNode\ x;$
 $g \vdash x \simeq xe \rrbracket$
 $\implies g \vdash n \simeq (UnaryExpr\ UnaryAbs\ xe) \mid$

NotNode:

$\llbracket kind\ g\ n = NotNode\ x;$
 $g \vdash x \simeq xe \rrbracket$
 $\implies g \vdash n \simeq (UnaryExpr\ UnaryNot\ xe) \mid$

NegateNode:

$\llbracket kind\ g\ n = NegateNode\ x;$
 $g \vdash x \simeq xe \rrbracket$
 $\implies g \vdash n \simeq (UnaryExpr\ UnaryNeg\ xe) \mid$

LogicNegationNode:

$\llbracket kind\ g\ n = LogicNegationNode\ x;$
 $g \vdash x \simeq xe \rrbracket$
 $\implies g \vdash n \simeq (UnaryExpr\ UnaryLogicNegation\ xe) \mid$

AddNode:

$\llbracket kind\ g\ n = AddNode\ x\ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye \rrbracket$
 $\implies g \vdash n \simeq (BinaryExpr\ BinAdd\ xe\ ye) \mid$

MulNode:

$\llbracket kind\ g\ n = MulNode\ x\ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye \rrbracket$
 $\implies g \vdash n \simeq (BinaryExpr\ BinMul\ xe\ ye) \mid$

SubNode:

$\llbracket kind\ g\ n = SubNode\ x\ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye \rrbracket$
 $\implies g \vdash n \simeq (BinaryExpr\ BinSub\ xe\ ye) \mid$

AndNode:

$\llbracket \text{kind } g \ n = \text{AndNode } x \ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye \rrbracket$
 $\implies g \vdash n \simeq (\text{BinaryExpr BinAnd } xe \ ye) \mid$

OrNode:

$\llbracket \text{kind } g \ n = \text{OrNode } x \ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye \rrbracket$
 $\implies g \vdash n \simeq (\text{BinaryExpr BinOr } xe \ ye) \mid$

XorNode:

$\llbracket \text{kind } g \ n = \text{XorNode } x \ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye \rrbracket$
 $\implies g \vdash n \simeq (\text{BinaryExpr BinXor } xe \ ye) \mid$

ShortCircuitOrNode:

$\llbracket \text{kind } g \ n = \text{ShortCircuitOrNode } x \ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye \rrbracket$
 $\implies g \vdash n \simeq (\text{BinaryExpr BinShortCircuitOr } xe \ ye) \mid$

LeftShiftNode:

$\llbracket \text{kind } g \ n = \text{LeftShiftNode } x \ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye \rrbracket$
 $\implies g \vdash n \simeq (\text{BinaryExpr BinLeftShift } xe \ ye) \mid$

RightShiftNode:

$\llbracket \text{kind } g \ n = \text{RightShiftNode } x \ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye \rrbracket$
 $\implies g \vdash n \simeq (\text{BinaryExpr BinRightShift } xe \ ye) \mid$

UnsignedRightShiftNode:

$\llbracket \text{kind } g \ n = \text{UnsignedRightShiftNode } x \ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye \rrbracket$
 $\implies g \vdash n \simeq (\text{BinaryExpr BinURightShift } xe \ ye) \mid$

IntegerBelowNode:

$\llbracket \text{kind } g \ n = \text{IntegerBelowNode } x \ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye \rrbracket$
 $\implies g \vdash n \simeq (\text{BinaryExpr BinIntegerBelow } xe \ ye) \mid$

IntegerEqualsNode:

$\llbracket \text{kind } g \ n = \text{IntegerEqualsNode } x \ y; \\ g \vdash x \simeq xe; \\ g \vdash y \simeq ye \rrbracket \\ \implies g \vdash n \simeq (\text{BinaryExpr } \text{BinIntegerEquals } xe \ ye) \mid$

IntegerLessThanNode:

$\llbracket \text{kind } g \ n = \text{IntegerLessThanNode } x \ y; \\ g \vdash x \simeq xe; \\ g \vdash y \simeq ye \rrbracket \\ \implies g \vdash n \simeq (\text{BinaryExpr } \text{BinIntegerLessThan } xe \ ye) \mid$

NarrowNode:

$\llbracket \text{kind } g \ n = \text{NarrowNode } \text{inputBits } \text{resultBits } x; \\ g \vdash x \simeq xe \rrbracket \\ \implies g \vdash n \simeq (\text{UnaryExpr } (\text{UnaryNarrow } \text{inputBits } \text{resultBits}) \ xe) \mid$

SignExtendNode:

$\llbracket \text{kind } g \ n = \text{SignExtendNode } \text{inputBits } \text{resultBits } x; \\ g \vdash x \simeq xe \rrbracket \\ \implies g \vdash n \simeq (\text{UnaryExpr } (\text{UnarySignExtend } \text{inputBits } \text{resultBits}) \ xe) \mid$

ZeroExtendNode:

$\llbracket \text{kind } g \ n = \text{ZeroExtendNode } \text{inputBits } \text{resultBits } x; \\ g \vdash x \simeq xe \rrbracket \\ \implies g \vdash n \simeq (\text{UnaryExpr } (\text{UnaryZeroExtend } \text{inputBits } \text{resultBits}) \ xe) \mid$

LeafNode:

$\llbracket \text{is-preevaluated } (\text{kind } g \ n); \\ \text{stamp } g \ n = s \rrbracket \\ \implies g \vdash n \simeq (\text{LeafExpr } n \ s) \mid$

RefNode:

$\llbracket \text{kind } g \ n = \text{RefNode } n'; \\ g \vdash n' \simeq e \rrbracket \\ \implies g \vdash n \simeq e$

code-pred (*modes*: $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool as exprE}$) *rep* .

inductive

replist :: *IRGraph* \Rightarrow *ID list* \Rightarrow *IRExpr list* \Rightarrow *bool* ($- \vdash - \simeq_L -$ 55)
for *g* **where**

RepNil:

$g \vdash [] \simeq_L [] \mid$

RepCons:
 $\llbracket g \vdash x \simeq xe; \quad g \vdash xs \simeq_L xse \rrbracket$
 $\implies g \vdash x \# xs \simeq_L xe \# xse$

code-pred (*modes*: $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$ as *exprListE*) *replist* .

definition *wf-term-graph* :: *MapState* \Rightarrow *Params* \Rightarrow *IRGraph* \Rightarrow *ID* \Rightarrow *bool* **where**
wf-term-graph *m p g n* = $(\exists e. (g \vdash n \simeq e) \wedge (\exists v. ([m, p] \vdash e \mapsto v)))$

values {*t*. *eg2-sq* $\vdash 4 \simeq t$ }

7.2 Data-flow Tree to Subgraph

fun *unary-node* :: *IRUnaryOp* \Rightarrow *ID* \Rightarrow *IRNode* **where**
unary-node *UnaryAbs* *v* = *AbsNode* *v* |
unary-node *UnaryNot* *v* = *NotNode* *v* |
unary-node *UnaryNeg* *v* = *NegateNode* *v* |
unary-node *UnaryLogicNegation* *v* = *LogicNegationNode* *v* |
unary-node (*UnaryNarrow* *ib* *rb*) *v* = *NarrowNode* *ib* *rb* *v* |
unary-node (*UnarySignExtend* *ib* *rb*) *v* = *SignExtendNode* *ib* *rb* *v* |
unary-node (*UnaryZeroExtend* *ib* *rb*) *v* = *ZeroExtendNode* *ib* *rb* *v*

fun *bin-node* :: *IRBinaryOp* \Rightarrow *ID* \Rightarrow *ID* \Rightarrow *IRNode* **where**
bin-node *BinAdd* *x* *y* = *AddNode* *x* *y* |
bin-node *BinMul* *x* *y* = *MulNode* *x* *y* |
bin-node *BinSub* *x* *y* = *SubNode* *x* *y* |
bin-node *BinAnd* *x* *y* = *AndNode* *x* *y* |
bin-node *BinOr* *x* *y* = *OrNode* *x* *y* |
bin-node *BinXor* *x* *y* = *XorNode* *x* *y* |
bin-node *BinShortCircuitOr* *x* *y* = *ShortCircuitOrNode* *x* *y* |
bin-node *BinLeftShift* *x* *y* = *LeftShiftNode* *x* *y* |
bin-node *BinRightShift* *x* *y* = *RightShiftNode* *x* *y* |
bin-node *BinURightShift* *x* *y* = *UnsignedRightShiftNode* *x* *y* |
bin-node *BinIntegerEquals* *x* *y* = *IntegerEqualsNode* *x* *y* |
bin-node *BinIntegerLessThan* *x* *y* = *IntegerLessThanNode* *x* *y* |
bin-node *BinIntegerBelow* *x* *y* = *IntegerBelowNode* *x* *y*

inductive *fresh-id* :: *IRGraph* \Rightarrow *ID* \Rightarrow *bool* **where**
 $n \notin \text{ids } g \implies \text{fresh-id } g \ n$

code-pred *fresh-id* .


```

fun get-fresh-id :: IRGraph  $\Rightarrow$  ID where

    get-fresh-id g = last(sorted-list-of-set(ids g)) + 1

export-code get-fresh-id

value get-fresh-id eg2-sq
value get-fresh-id (add-node 6 (ParameterNode 2, default-stamp) eg2-sq)

inductive
  unrep :: IRGraph  $\Rightarrow$  IRExpr  $\Rightarrow$  (IRGraph  $\times$  ID)  $\Rightarrow$  bool (-  $\oplus$  -  $\rightsquigarrow$  - 55)
  where

    ConstantNodeSame:
     $\llbracket \text{find-node-and-stamp } g \text{ (ConstantNode } c, \text{ constantAsStamp } c) = \text{Some } n \rrbracket$ 
       $\implies g \oplus (\text{ConstantExpr } c) \rightsquigarrow (g, n) \mid$ 

    ConstantNodeNew:
     $\llbracket \text{find-node-and-stamp } g \text{ (ConstantNode } c, \text{ constantAsStamp } c) = \text{None};$ 
       $n = \text{get-fresh-id } g;$ 
       $g' = \text{add-node } n \text{ (ConstantNode } c, \text{ constantAsStamp } c) \text{ } g \rrbracket$ 
       $\implies g \oplus (\text{ConstantExpr } c) \rightsquigarrow (g', n) \mid$ 

    ParameterNodeSame:
     $\llbracket \text{find-node-and-stamp } g \text{ (ParameterNode } i, s) = \text{Some } n \rrbracket$ 
       $\implies g \oplus (\text{ParameterExpr } i \text{ } s) \rightsquigarrow (g, n) \mid$ 

    ParameterNodeNew:
     $\llbracket \text{find-node-and-stamp } g \text{ (ParameterNode } i, s) = \text{None};$ 
       $n = \text{get-fresh-id } g;$ 
       $g' = \text{add-node } n \text{ (ParameterNode } i, s) \text{ } g \rrbracket$ 
       $\implies g \oplus (\text{ParameterExpr } i \text{ } s) \rightsquigarrow (g', n) \mid$ 

    ConditionalNodeSame:
     $\llbracket g \oplus ce \rightsquigarrow (g2, c);$ 
       $g2 \oplus te \rightsquigarrow (g3, t);$ 
       $g3 \oplus fe \rightsquigarrow (g4, f);$ 
       $s' = \text{meet (stamp } g4 \text{ } t) \text{ (stamp } g4 \text{ } f);$ 
       $\text{find-node-and-stamp } g4 \text{ (ConditionalNode } c \text{ } t \text{ } f, s') = \text{Some } n \rrbracket$ 
       $\implies g \oplus (\text{ConditionalExpr } ce \text{ } te \text{ } fe) \rightsquigarrow (g4, n) \mid$ 

    ConditionalNodeNew:
     $\llbracket g \oplus ce \rightsquigarrow (g2, c);$ 
       $g2 \oplus te \rightsquigarrow (g3, t);$ 
       $g3 \oplus fe \rightsquigarrow (g4, f);$ 
       $s' = \text{meet (stamp } g4 \text{ } t) \text{ (stamp } g4 \text{ } f);$ 
       $\text{find-node-and-stamp } g4 \text{ (ConditionalNode } c \text{ } t \text{ } f, s') = \text{None};$ 

```

$n = \text{get-fresh-id } g4;$
 $g' = \text{add-node } n \text{ (ConditionalNode } c \text{ t } f, s') \text{ } g4$
 $\implies g \oplus (\text{ConditionalExpr } ce \text{ te } fe) \rightsquigarrow (g', n) \mid$

UnaryNodeSame:

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$
 $s' = \text{stamp-unary op (stamp } g2 \text{ } x);$
 $\text{find-node-and-stamp } g2 \text{ (unary-node op } x, s') = \text{Some } n \rrbracket$
 $\implies g \oplus (\text{UnaryExpr op } xe) \rightsquigarrow (g2, n) \mid$

UnaryNodeNew:

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$
 $s' = \text{stamp-unary op (stamp } g2 \text{ } x);$
 $\text{find-node-and-stamp } g2 \text{ (unary-node op } x, s') = \text{None};$
 $n = \text{get-fresh-id } g2;$
 $g' = \text{add-node } n \text{ (unary-node op } x, s') \text{ } g2$
 $\implies g \oplus (\text{UnaryExpr op } xe) \rightsquigarrow (g', n) \mid$

BinaryNodeSame:

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$
 $g2 \oplus ye \rightsquigarrow (g3, y);$
 $s' = \text{stamp-binary op (stamp } g3 \text{ } x) \text{ (stamp } g3 \text{ } y);$
 $\text{find-node-and-stamp } g3 \text{ (bin-node op } x \text{ } y, s') = \text{Some } n \rrbracket$
 $\implies g \oplus (\text{BinaryExpr op } xe \text{ } ye) \rightsquigarrow (g3, n) \mid$

BinaryNodeNew:

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$
 $g2 \oplus ye \rightsquigarrow (g3, y);$
 $s' = \text{stamp-binary op (stamp } g3 \text{ } x) \text{ (stamp } g3 \text{ } y);$
 $\text{find-node-and-stamp } g3 \text{ (bin-node op } x \text{ } y, s') = \text{None};$
 $n = \text{get-fresh-id } g3;$
 $g' = \text{add-node } n \text{ (bin-node op } x \text{ } y, s') \text{ } g3$
 $\implies g \oplus (\text{BinaryExpr op } xe \text{ } ye) \rightsquigarrow (g', n) \mid$

AllLeafNodes:

$\llbracket \text{stamp } g \text{ } n = s;$
 $\text{is-preevaluated (kind } g \text{ } n) \rrbracket$
 $\implies g \oplus (\text{LeafExpr } n \text{ } s) \rightsquigarrow (g, n)$

code-pred (*modes: $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$ as unrepE*)
 unrep .

$$\begin{array}{c}
\frac{\text{find-node-and-stamp } g \text{ (ConstantNode } c, \text{ constantAsStamp } c) = \text{Some } n}{g \oplus \text{ConstantExpr } c \rightsquigarrow (g, n)} \\
\\
\frac{\begin{array}{c} \text{find-node-and-stamp } g \text{ (ConstantNode } c, \text{ constantAsStamp } c) = \text{None} \\ n = \text{get-fresh-id } g \\ g' = \text{add-node } n \text{ (ConstantNode } c, \text{ constantAsStamp } c) \text{ } g \end{array}}{g \oplus \text{ConstantExpr } c \rightsquigarrow (g', n)} \\
\\
\frac{\text{find-node-and-stamp } g \text{ (ParameterNode } i, s) = \text{Some } n}{g \oplus \text{ParameterExpr } i \text{ } s \rightsquigarrow (g, n)} \\
\\
\frac{\begin{array}{c} \text{find-node-and-stamp } g \text{ (ParameterNode } i, s) = \text{None} \\ n = \text{get-fresh-id } g \quad g' = \text{add-node } n \text{ (ParameterNode } i, s) \text{ } g \end{array}}{g \oplus \text{ParameterExpr } i \text{ } s \rightsquigarrow (g', n)} \\
\\
\frac{\begin{array}{c} g \oplus ce \rightsquigarrow (g2, c) \quad g2 \oplus te \rightsquigarrow (g3, t) \\ g3 \oplus fe \rightsquigarrow (g4, f) \quad s' = \text{meet (stamp } g4 \text{ } t) \text{ (stamp } g4 \text{ } f) \\ \text{find-node-and-stamp } g4 \text{ (ConditionalNode } c \text{ } t \text{ } f, s') = \text{Some } n \end{array}}{g \oplus \text{ConditionalExpr } ce \text{ } te \text{ } fe \rightsquigarrow (g4, n)} \\
\\
\frac{\begin{array}{c} g \oplus ce \rightsquigarrow (g2, c) \quad g2 \oplus te \rightsquigarrow (g3, t) \\ g3 \oplus fe \rightsquigarrow (g4, f) \quad s' = \text{meet (stamp } g4 \text{ } t) \text{ (stamp } g4 \text{ } f) \\ \text{find-node-and-stamp } g4 \text{ (ConditionalNode } c \text{ } t \text{ } f, s') = \text{None} \\ n = \text{get-fresh-id } g4 \quad g' = \text{add-node } n \text{ (ConditionalNode } c \text{ } t \text{ } f, s') \text{ } g4 \end{array}}{g \oplus \text{ConditionalExpr } ce \text{ } te \text{ } fe \rightsquigarrow (g', n)} \\
\\
\frac{\begin{array}{c} g \oplus xe \rightsquigarrow (g2, x) \\ g2 \oplus ye \rightsquigarrow (g3, y) \quad s' = \text{stamp-binary op (stamp } g3 \text{ } x) \text{ (stamp } g3 \text{ } y) \\ \text{find-node-and-stamp } g3 \text{ (bin-node op } x \text{ } y, s') = \text{Some } n \end{array}}{g \oplus \text{BinaryExpr op } xe \text{ } ye \rightsquigarrow (g3, n)} \\
\\
\frac{\begin{array}{c} g \oplus xe \rightsquigarrow (g2, x) \\ g2 \oplus ye \rightsquigarrow (g3, y) \quad s' = \text{stamp-binary op (stamp } g3 \text{ } x) \text{ (stamp } g3 \text{ } y) \\ \text{find-node-and-stamp } g3 \text{ (bin-node op } x \text{ } y, s') = \text{None} \\ n = \text{get-fresh-id } g3 \quad g' = \text{add-node } n \text{ (bin-node op } x \text{ } y, s') \text{ } g3 \end{array}}{g \oplus \text{BinaryExpr op } xe \text{ } ye \rightsquigarrow (g', n)} \\
\\
\frac{\begin{array}{c} g \oplus xe \rightsquigarrow (g2, x) \quad s' = \text{stamp-unary op (stamp } g2 \text{ } x) \\ \text{find-node-and-stamp } g2 \text{ (unary-node op } x, s') = \text{Some } n \end{array}}{g \oplus \text{UnaryExpr op } xe \rightsquigarrow (g2, n)} \\
\\
\frac{\begin{array}{c} g \oplus xe \rightsquigarrow (g2, x) \quad s' = \text{stamp-unary op (stamp } g2 \text{ } x) \\ \text{find-node-and-stamp } g2 \text{ (unary-node op } x, s') = \text{None} \\ n = \text{get-fresh-id } g2 \quad g' = \text{add-node } n \text{ (unary-node op } x, s') \text{ } g2 \end{array}}{g \oplus \text{UnaryExpr op } xe \rightsquigarrow (g', n)} \\
\\
\frac{\text{stamp } g \text{ } n = s \quad \text{is-preevaluated (kind } g \text{ } n)}{g \oplus \text{LeafExpr } n \text{ } s \rightsquigarrow (g, n)}
\end{array}$$

values $\{(n, g) . (eg2\text{-}sq \oplus sq\text{-}param0 \rightsquigarrow (g, n))\}$

7.3 Lift Data-flow Tree Semantics

definition *encodeeval* :: *IRGraph* \Rightarrow *MapState* \Rightarrow *Params* \Rightarrow *ID* \Rightarrow *Value* \Rightarrow *bool*
 $([\cdot, \cdot, \cdot] \vdash \cdot \mapsto \cdot \text{ } 50)$
where
encodeeval *g m p n v* = $(\exists e. (g \vdash n \simeq e) \wedge ([m, p] \vdash e \mapsto v))$

7.4 Graph Refinement

definition *graph-represents-expression* :: *IRGraph* \Rightarrow *ID* \Rightarrow *IRExpr* \Rightarrow *bool*
 $(\vdash \cdot \leq \cdot \text{ } 50)$
where
 $(g \vdash n \leq e) = (\exists e'. (g \vdash n \simeq e') \wedge (e' \leq e))$

definition *graph-refinement* :: *IRGraph* \Rightarrow *IRGraph* \Rightarrow *bool* **where**
graph-refinement *g1 g2* =
 $((ids\ g_1 \subseteq ids\ g_2) \wedge$
 $(\forall n. n \in ids\ g_1 \longrightarrow (\forall e. (g_1 \vdash n \simeq e) \longrightarrow (g_2 \vdash n \leq e))))$

lemma *graph-refinement*:

graph-refinement *g1 g2* $\implies (\forall n\ m\ p\ v. n \in ids\ g1 \longrightarrow ([g1, m, p] \vdash n \mapsto v) \longrightarrow$
 $([g2, m, p] \vdash n \mapsto v))$
by (*meson encodeeval-def graph-refinement-def graph-represents-expression-def le-expr-def*)

7.5 Maximal Sharing

definition *maximal-sharing*:

maximal-sharing *g* = $(\forall n_1\ n_2. n_1 \in true\text{-}ids\ g \wedge n_2 \in true\text{-}ids\ g \longrightarrow$
 $(\forall e. (g \vdash n_1 \simeq e) \wedge (g \vdash n_2 \simeq e) \wedge (stamp\ g\ n_1 = stamp\ g\ n_2) \longrightarrow n_1 =$
 $n_2))$

end

7.6 Formedness Properties

theory *Form*

imports

Semantics.TreeToGraph

begin

definition *wf-start* **where**

wf-start *g* = $(0 \in ids\ g \wedge$
 $is\text{-}StartNode\ (kind\ g\ 0))$

definition *wf-closed* **where**

wf-closed *g* =
 $(\forall n \in ids\ g .$

$$\begin{aligned} &inputs\ g\ n \subseteq ids\ g \wedge \\ &succ\ g\ n \subseteq ids\ g \wedge \\ &kind\ g\ n \neq NoNode) \end{aligned}$$

definition *wf-phs* **where**

$$\begin{aligned} wf-phs\ g = & \\ &(\forall\ n \in ids\ g. \\ &\quad is-PhiNode\ (kind\ g\ n) \longrightarrow \\ &\quad length\ (ir-values\ (kind\ g\ n)) \\ &= length\ (ir-ends \\ &\quad (kind\ g\ (ir-merge\ (kind\ g\ n)))))) \end{aligned}$$

definition *wf-ends* **where**

$$\begin{aligned} wf-ends\ g = & \\ &(\forall\ n \in ids\ g . \\ &\quad is-AbstractEndNode\ (kind\ g\ n) \longrightarrow \\ &\quad card\ (usages\ g\ n) > 0) \end{aligned}$$

fun *wf-graph* :: *IRGraph* \Rightarrow *bool* **where**

$$wf-graph\ g = (wf-start\ g \wedge wf-closed\ g \wedge wf-phs\ g \wedge wf-ends\ g)$$

lemmas *wf-folds* =

$$\begin{aligned} &wf-graph.simps \\ &wf-start-def \\ &wf-closed-def \\ &wf-phs-def \\ &wf-ends-def \end{aligned}$$

fun *wf-stamps* :: *IRGraph* \Rightarrow *bool* **where**

$$\begin{aligned} wf-stamps\ g = &(\forall\ n \in ids\ g . \\ &(\forall\ v\ m\ p\ e . (g \vdash n \simeq e) \wedge ([m, p] \vdash e \mapsto v) \longrightarrow valid-value\ v\ (stamp-expr\ e))) \end{aligned}$$

fun *wf-stamp* :: *IRGraph* \Rightarrow (*ID* \Rightarrow *Stamp*) \Rightarrow *bool* **where**

$$\begin{aligned} wf-stamp\ g\ s = &(\forall\ n \in ids\ g . \\ &(\forall\ v\ m\ p\ e . (g \vdash n \simeq e) \wedge ([m, p] \vdash e \mapsto v) \longrightarrow valid-value\ v\ (s\ n))) \end{aligned}$$

lemma *wf-empty*: *wf-graph start-end-graph*

unfolding *start-end-graph-def wf-folds by simp*

lemma *wf-eg2-sq*: *wf-graph eg2-sq*

unfolding *eg2-sq-def wf-folds by simp*

fun *wf-logic-node-inputs* :: *IRGraph* \Rightarrow *ID* \Rightarrow *bool* **where**

$$\begin{aligned} wf-logic-node-inputs\ g\ n = & \\ &(\forall\ inp \in set\ (inputs-of\ (kind\ g\ n)) . (\forall\ v\ m\ p . ([g, m, p] \vdash inp \mapsto v) \longrightarrow wf-bool \\ &v)) \end{aligned}$$

fun *wf-values* :: *IRGraph* \Rightarrow *bool* **where**

$$wf-values\ g = (\forall\ n \in ids\ g .$$

$$\begin{aligned}
& (\forall v m p . ([g, m, p] \vdash n \mapsto v) \longrightarrow \\
& \quad (is-LogicNode (kind g n) \longrightarrow \\
& \quad \quad wf-bool v \wedge wf-logic-node-inputs g n)))
\end{aligned}$$

end

7.7 Dynamic Frames

This theory defines two operators, 'unchanged' and 'changeonly', that are useful for specifying which nodes in an IRGraph can change. The dynamic framing idea originates from 'Dynamic Frames' in software verification, started by Ioannis T. Kassios in "Dynamic frames: Support for framing, dependencies and sharing without restrictions", In FM 2006.

theory *IRGraphFrames*

imports

Form

begin

fun *unchanged* :: *ID set* \Rightarrow *IRGraph* \Rightarrow *IRGraph* \Rightarrow *bool* **where**

unchanged ns g1 g2 = $(\forall n . n \in ns \longrightarrow$
 $(n \in ids\ g1 \wedge n \in ids\ g2 \wedge kind\ g1\ n = kind\ g2\ n \wedge stamp\ g1\ n = stamp\ g2\ n))$

fun *changeonly* :: *ID set* \Rightarrow *IRGraph* \Rightarrow *IRGraph* \Rightarrow *bool* **where**

changeonly ns g1 g2 = $(\forall n . n \in ids\ g1 \wedge n \notin ns \longrightarrow$
 $(n \in ids\ g1 \wedge n \in ids\ g2 \wedge kind\ g1\ n = kind\ g2\ n \wedge stamp\ g1\ n = stamp\ g2\ n))$

lemma *node-unchanged*:

assumes *unchanged ns g1 g2*

assumes *nid* \in *ns*

shows *kind g1 nid* = *kind g2 nid*

using *assms* **by** *auto*

lemma *other-node-unchanged*:

assumes *changeonly ns g1 g2*

assumes *nid* \in *ids g1*

assumes *nid* \notin *ns*

shows *kind g1 nid* = *kind g2 nid*

using *assms*

using *changeonly.simps* **by** *blast*

Some notation for input nodes used

inductive *eval-uses*:: *IRGraph* \Rightarrow *ID* \Rightarrow *ID* \Rightarrow *bool*

for *g* **where**

use0: *nid* \in *ids g*

$\implies eval-uses\ g\ nid\ nid$ |

```

use-inp:  $nid' \in inputs\ g\ n$ 
 $\implies eval\text{-}uses\ g\ nid\ nid' \mid$ 

use-trans:  $\llbracket eval\text{-}uses\ g\ nid\ nid';$ 
 $eval\text{-}uses\ g\ nid'\ nid'' \rrbracket$ 
 $\implies eval\text{-}uses\ g\ nid\ nid''$ 

fun eval-usages :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID set where
  eval-usages g nid = {n  $\in$  ids g . eval-usages g nid n}

lemma eval-usages-self:
  assumes nid  $\in$  ids g
  shows nid  $\in$  eval-usages g nid
  using assms eval-usages.simps eval-uses.intros(1)
  by (simp add: ids.rep-eq)

lemma not-in-g-inputs:
  assumes nid  $\notin$  ids g
  shows inputs g nid = {}
proof –
  have k: kind g nid = NoNode using assms not-in-g by blast
  then show ?thesis by (simp add: k)
qed

lemma child-member:
  assumes n = kind g nid
  assumes n  $\neq$  NoNode
  assumes List.member (inputs-of n) child
  shows child  $\in$  inputs g nid
  unfolding inputs.simps using assms
  by (metis in-set-member)

lemma child-member-in:
  assumes nid  $\in$  ids g
  assumes List.member (inputs-of (kind g nid)) child
  shows child  $\in$  inputs g nid
  unfolding inputs.simps using assms
  by (metis child-member ids-some inputs.elims)

lemma inp-in-g:
  assumes n  $\in$  inputs g nid
  shows nid  $\in$  ids g
proof –
  have inputs g nid  $\neq$  {}
  using assms
  by (metis empty-iff empty-set)

```

```

then have kind g nid  $\neq$  NoNode
  using not-in-g-inputs
  using ids-some by blast
then show ?thesis
  using not-in-g
  by metis
qed

```

```

lemma inp-in-g-wf:
  assumes wf-graph g
  assumes  $n \in \text{inputs } g \text{ nid}$ 
  shows  $n \in \text{ids } g$ 
  using assms unfolding wf-folds
  using inp-in-g by blast

```

```

lemma kind-unchanged:
  assumes nid  $\in \text{ids } g1$ 
  assumes unchanged (eval-usages g1 nid) g1 g2
  shows kind g1 nid = kind g2 nid
proof -
  show ?thesis
    using assms eval-usages-self
    using unchanged.simps by blast
qed

```

```

lemma stamp-unchanged:
  assumes nid  $\in \text{ids } g1$ 
  assumes unchanged (eval-usages g1 nid) g1 g2
  shows stamp g1 nid = stamp g2 nid
  by (meson assms(1) assms(2) eval-usages-self unchanged.elims(2))

```

```

lemma child-unchanged:
  assumes child  $\in \text{inputs } g1 \text{ nid}$ 
  assumes unchanged (eval-usages g1 nid) g1 g2
  shows unchanged (eval-usages g1 child) g1 g2
  by (smt assms(1) assms(2) eval-usages.simps mem-Collect-eq
      unchanged.simps use-inp use-trans)

```

```

lemma eval-usages:
  assumes us = eval-usages g nid
  assumes nid'  $\in \text{ids } g$ 
  shows eval-uses g nid nid'  $\longleftrightarrow$  nid'  $\in \text{us}$  (is ?P  $\longleftrightarrow$  ?Q)
  using assms eval-usages.simps
  by (simp add: ids.rep-eq)

```

```

lemma inputs-are-uses:
  assumes nid'  $\in \text{inputs } g \text{ nid}$ 

```


shows *eval-uses* *g nid nid'*
by (*metis* *assms* *use-inp*)

lemma *inputs-are-usages*:
assumes *nid' ∈ inputs g nid*
assumes *nid' ∈ ids g*
shows *nid' ∈ eval-usages g nid*
using *assms(1) assms(2) eval-usages inputs-are-uses* **by** *blast*

lemma *inputs-of-are-usages*:
assumes *List.member (inputs-of (kind g nid)) nid'*
assumes *nid' ∈ ids g*
shows *nid' ∈ eval-usages g nid*
by (*metis* *assms(1) assms(2) in-set-member inputs.elims inputs-are-usages*)

lemma *usage-includes-inputs*:
assumes *us = eval-usages g nid*
assumes *ls = inputs g nid*
assumes *ls ⊆ ids g*
shows *ls ⊆ us*
using *inputs-are-usages eval-usages*
using *assms(1) assms(2) assms(3)* **by** *blast*

lemma *elim-inp-set*:
assumes *k = kind g nid*
assumes *k ≠ NoNode*
assumes *child ∈ set (inputs-of k)*
shows *child ∈ inputs g nid*
using *assms* **by** *auto*

lemma *encode-in-ids*:
assumes *g ⊢ nid ≃ e*
shows *nid ∈ ids g*
using *assms*
apply (*induction* *rule: rep.induct*)
apply *simp+*
by *fastforce+*

lemma *eval-in-ids*:
assumes *[g, m, p] ⊢ nid ↦ v*
shows *nid ∈ ids g*
using *assms* **using** *encodeeval-def encode-in-ids*
by *auto*

lemma *transitive-kind-same*:
assumes *unchanged (eval-usages g1 nid) g1 g2*
shows $\forall \text{nid}' \in (\text{eval-usages } g1 \text{ nid}) . \text{kind } g1 \text{ nid}' = \text{kind } g2 \text{ nid}'$
using *assms*
by (*meson* *unchanged.elims(1)*)

```

theorem stay-same-encoding:
  assumes nc: unchanged (eval-usages g1 nid) g1 g2
  assumes g1: g1  $\vdash$  nid  $\simeq$  e
  assumes wf: wf-graph g1
  shows g2  $\vdash$  nid  $\simeq$  e
proof –
  have dom: nid  $\in$  ids g1
  using g1 encode-in-ids by simp
  show ?thesis
using g1 nc wf dom proof (induction e rule: rep.induct)
  case (ConstantNode n c)
  then have kind g2 n = ConstantNode c
  using dom nc kind-unchanged
  by metis
  then show ?case using rep.ConstantNode
  by presburger
next
  case (ParameterNode n i s)
  then have kind g2 n = ParameterNode i
  by (metis kind-unchanged)
  then show ?case
  by (metis ParameterNode.hyps(2) ParameterNode.premis(1) ParameterNode.premis(3)
  rep.ParameterNode stamp-unchanged)
next
  case (ConditionalNode n c t f ce te fe)
  then have kind g2 n = ConditionalNode c t f
  by (metis kind-unchanged)
  have c  $\in$  eval-usages g1 n  $\wedge$  t  $\in$  eval-usages g1 n  $\wedge$  f  $\in$  eval-usages g1 n
  using inputs-of-ConditionalNode
  by (metis ConditionalNode.hyps(1) ConditionalNode.hyps(2) ConditionalNode.hyps(3)
  ConditionalNode.hyps(4) encode-in-ids inputs.simps inputs-are-usages
  list.set-intros(1) set-subset-Cons subset-code(1))
  then show ?case using transitive-kind-same
  by (metis ConditionalNode.hyps(1) ConditionalNode.premis(1) IRNodes.inputs-of-ConditionalNode
   $\langle$ kind g2 n = ConditionalNode c t f $\rangle$  child-unchanged inputs.simps list.set-intros(1)
  local.ConditionalNode(5) local.ConditionalNode(6) local.ConditionalNode(7) local.ConditionalNode(9)
  rep.ConditionalNode set-subset-Cons subset-code(1) unchanged.elims(2))
next
  case (AbsNode n x xe)
  then have kind g2 n = AbsNode x
  using kind-unchanged
  by metis
  then have x  $\in$  eval-usages g1 n
  using inputs-of-AbsNode
  by (metis AbsNode.hyps(1) AbsNode.hyps(2) encode-in-ids inputs.simps inputs-are-usages
  list.set-intros(1))
  then show ?case
  by (metis AbsNode.IH AbsNode.hyps(1) AbsNode.premis(1) AbsNode.premis(3))

```

```

IRNodes.inputs-of-AbsNode ⟨kind g2 n = AbsNode x⟩ child-member-in child-unchanged
local.wf member-rec(1) rep.AbsNode unchanged.simps)
next
  case (NotNode n x xe)
  then have kind g2 n = NotNode x
    using kind-unchanged
    by metis
  then have x ∈ eval-usages g1 n
    using inputs-of-NotNode
    by (metis NotNode.hyps(1) NotNode.hyps(2) encode-in-ids inputs.simps in-
      puts-are-usages list.set-intros(1))
  then show ?case
    by (metis NotNode.IH NotNode.hyps(1) NotNode.prem(1) NotNode.prem(3)
      IRNodes.inputs-of-NotNode ⟨kind g2 n = NotNode x⟩ child-member-in child-unchanged
      local.wf member-rec(1) rep.NotNode unchanged.simps)
next
  case (NegateNode n x xe)
  then have kind g2 n = NegateNode x
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n
    using inputs-of-NegateNode
    by (metis NegateNode.hyps(1) NegateNode.hyps(2) encode-in-ids inputs.simps
      inputs-are-usages list.set-intros(1))
  then show ?case
    by (metis IRNodes.inputs-of-NegateNode NegateNode.IH NegateNode.hyps(1)
      NegateNode.prem(1) NegateNode.prem(3) ⟨kind g2 n = NegateNode x⟩ child-member-in
      child-unchanged local.wf member-rec(1) rep.NegateNode unchanged.elims(1))
next
  case (LogicNegationNode n x xe)
  then have kind g2 n = LogicNegationNode x
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
    by (metis LogicNegationNode.hyps(1) LogicNegationNode.hyps(2) encode-in-ids
      member-rec(1))
  then show ?case
    by (metis IRNodes.inputs-of-LogicNegationNode LogicNegationNode.IH Logic-
      NegationNode.hyps(1) LogicNegationNode.hyps(2) LogicNegationNode.prem(1) ⟨kind
      g2 n = LogicNegationNode x⟩ child-unchanged encode-in-ids inputs.simps list.set-intros(1)
      local.wf rep.LogicNegationNode)
next
  case (AddNode n x y xe ye)
  then have kind g2 n = AddNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
    by (metis AddNode.hyps(1) AddNode.hyps(2) AddNode.hyps(3) IRNodes.inputs-of-AddNode
      encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case

```

```

    by (metis AddNode.IH(1) AddNode.IH(2) AddNode.hyps(1) AddNode.hyps(2)
AddNode.hyps(3) AddNode.premis(1) IRNodes.inputs-of-AddNode ⟨kind g2 n = AddNode
x y⟩ child-unchanged encode-in-ids in-set-member inputs.simps local.wf member-rec(1)
rep.AddNode)
next
  case (MulNode n x y xe ye)
  then have kind g2 n = MulNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
  by (metis MulNode.hyps(1) MulNode.hyps(2) MulNode.hyps(3) IRNodes.inputs-of-MulNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using MulNode inputs-of-MulNode
  by (metis ⟨kind g2 n = MulNode x y⟩ child-unchanged inputs.simps list.set-intros(1)
rep.MulNode set-subset-Cons subset-iff unchanged.elims(2))
next
  case (SubNode n x y xe ye)
  then have kind g2 n = SubNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
  by (metis SubNode.hyps(1) SubNode.hyps(2) SubNode.hyps(3) IRNodes.inputs-of-SubNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using SubNode inputs-of-SubNode
  by (metis ⟨kind g2 n = SubNode x y⟩ child-member child-unchanged encode-in-ids
ids-some member-rec(1) rep.SubNode)
next
  case (AndNode n x y xe ye)
  then have kind g2 n = AndNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
  by (metis AndNode.hyps(1) AndNode.hyps(2) AndNode.hyps(3) IRNodes.inputs-of-AndNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using AndNode inputs-of-AndNode
  by (metis ⟨kind g2 n = AndNode x y⟩ child-unchanged inputs.simps list.set-intros(1)
rep.AndNode set-subset-Cons subset-iff unchanged.elims(2))
next
  case (OrNode n x y xe ye)
  then have kind g2 n = OrNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-OrNode inputs-of-are-usages
  by (metis OrNode.hyps(1) OrNode.hyps(2) OrNode.hyps(3) IRNodes.inputs-of-OrNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using OrNode inputs-of-OrNode
  by (metis ⟨kind g2 n = OrNode x y⟩ child-member child-unchanged encode-in-ids
ids-some member-rec(1) rep.OrNode)
next

```

```

case (XorNode n x y xe ye)
then have kind g2 n = XorNode x y
  using kind-unchanged by metis
then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
  using inputs-of-XorNode inputs-of-are-usages
  by (metis XorNode.hyps(1) XorNode.hyps(2) XorNode.hyps(3) IRNodes.inputs-of-XorNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using XorNode inputs-of-XorNode
  by (metis ⟨kind g2 n = XorNode x y⟩ child-member child-unchanged encode-in-ids
ids-some member-rec(1) rep.XorNode)
next
case (ShortCircuitOrNode n x y xe ye)
then have kind g2 n = ShortCircuitOrNode x y
  using kind-unchanged by metis
then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
  using inputs-of-XorNode inputs-of-are-usages
  by (metis ShortCircuitOrNode.hyps(1) ShortCircuitOrNode.hyps(2) ShortCircuitOrNode.hyps(3) IRNodes.inputs-of-ShortCircuitOrNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using ShortCircuitOrNode inputs-of-ShortCircuitOrNode
  by (metis ⟨kind g2 n = ShortCircuitOrNode x y⟩ child-member child-unchanged
encode-in-ids ids-some member-rec(1) rep.ShortCircuitOrNode)
next
case (LeftShiftNode n x y xe ye)
then have kind g2 n = LeftShiftNode x y
  using kind-unchanged by metis
then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
  using inputs-of-XorNode inputs-of-are-usages
  by (metis LeftShiftNode.hyps(1) LeftShiftNode.hyps(2) LeftShiftNode.hyps(3) IRNodes.inputs-of-LeftShiftNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using LeftShiftNode inputs-of-LeftShiftNode
  by (metis ⟨kind g2 n = LeftShiftNode x y⟩ child-member child-unchanged
encode-in-ids ids-some member-rec(1) rep.LeftShiftNode)
next
case (RightShiftNode n x y xe ye)
then have kind g2 n = RightShiftNode x y
  using kind-unchanged by metis
then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
  using inputs-of-RightShiftNode inputs-of-are-usages
  by (metis RightShiftNode.hyps(1) RightShiftNode.hyps(2) RightShiftNode.hyps(3) IRNodes.inputs-of-RightShiftNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using RightShiftNode inputs-of-RightShiftNode
  by (metis ⟨kind g2 n = RightShiftNode x y⟩ child-member child-unchanged
encode-in-ids ids-some member-rec(1) rep.RightShiftNode)
next
case (UnsignedRightShiftNode n x y xe ye)
then have kind g2 n = UnsignedRightShiftNode x y

```

```

    using kind-unchanged by metis
  then have  $x \in \text{eval-usages } g1 \ n \wedge y \in \text{eval-usages } g1 \ n$ 
    using inputs-of-UnsignedRightShiftNode inputs-of-are-usages
    by (metis UnsignedRightShiftNode.hyps(1) UnsignedRightShiftNode.hyps(2) UnsignedRightShiftNode.hyps(3) IRNodes.inputs-of-UnsignedRightShiftNode encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using UnsignedRightShiftNode inputs-of-UnsignedRightShiftNode
    by (metis  $\langle \text{kind } g2 \ n = \text{UnsignedRightShiftNode } x \ y \rangle$  child-member child-unchanged encode-in-ids ids-some member-rec(1) rep.UnsignedRightShiftNode)
next
  case (IntegerBelowNode  $n \ x \ y \ xe \ ye$ )
  then have  $\text{kind } g2 \ n = \text{IntegerBelowNode } x \ y$ 
    using kind-unchanged by metis
  then have  $x \in \text{eval-usages } g1 \ n \wedge y \in \text{eval-usages } g1 \ n$ 
    using inputs-of-IntegerBelowNode inputs-of-are-usages
    by (metis IntegerBelowNode.hyps(1) IntegerBelowNode.hyps(2) IntegerBelowNode.hyps(3) IRNodes.inputs-of-IntegerBelowNode encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using IntegerBelowNode inputs-of-IntegerBelowNode
    by (metis  $\langle \text{kind } g2 \ n = \text{IntegerBelowNode } x \ y \rangle$  child-member child-unchanged encode-in-ids ids-some member-rec(1) rep.IntegerBelowNode)
next
  case (IntegerEqualsNode  $n \ x \ y \ xe \ ye$ )
  then have  $\text{kind } g2 \ n = \text{IntegerEqualsNode } x \ y$ 
    using kind-unchanged by metis
  then have  $x \in \text{eval-usages } g1 \ n \wedge y \in \text{eval-usages } g1 \ n$ 
    using inputs-of-IntegerEqualsNode inputs-of-are-usages
    by (metis IntegerEqualsNode.hyps(1) IntegerEqualsNode.hyps(2) IntegerEqualsNode.hyps(3) IRNodes.inputs-of-IntegerEqualsNode encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using IntegerEqualsNode inputs-of-IntegerEqualsNode
    by (metis  $\langle \text{kind } g2 \ n = \text{IntegerEqualsNode } x \ y \rangle$  child-member child-unchanged encode-in-ids ids-some member-rec(1) rep.IntegerEqualsNode)
next
  case (IntegerLessThanNode  $n \ x \ y \ xe \ ye$ )
  then have  $\text{kind } g2 \ n = \text{IntegerLessThanNode } x \ y$ 
    using kind-unchanged by metis
  then have  $x \in \text{eval-usages } g1 \ n \wedge y \in \text{eval-usages } g1 \ n$ 
    using inputs-of-IntegerLessThanNode inputs-of-are-usages
    by (metis IntegerLessThanNode.hyps(1) IntegerLessThanNode.hyps(2) IntegerLessThanNode.hyps(3) IRNodes.inputs-of-IntegerLessThanNode encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using IntegerLessThanNode inputs-of-IntegerLessThanNode
    by (metis  $\langle \text{kind } g2 \ n = \text{IntegerLessThanNode } x \ y \rangle$  child-member child-unchanged encode-in-ids ids-some member-rec(1) rep.IntegerLessThanNode)
next
  case (NarrowNode  $n \ ib \ rb \ x \ xe$ )
  then have  $\text{kind } g2 \ n = \text{NarrowNode } ib \ rb \ x$ 
    using kind-unchanged by metis

```

```

then have  $x \in \text{eval-usages } g1 \ n$ 
  using inputs-of-NarrowNode inputs-of-are-usages
  by (metis NarrowNode.hyps(1) NarrowNode.hyps(2) IRNodes.inputs-of-NarrowNode
encode-in-ids inputs.simps inputs-are-usages list.set-intros(1))
  then show ?case using NarrowNode inputs-of-NarrowNode
    by (metis  $\langle \text{kind } g2 \ n = \text{NarrowNode } ib \ rb \ x \rangle$  child-unchanged inputs.elims
list.set-intros(1) rep.NarrowNode unchanged.simps)
next
  case (SignExtendNode n ib rb x xe)
  then have  $\text{kind } g2 \ n = \text{SignExtendNode } ib \ rb \ x$ 
    using kind-unchanged by metis
  then have  $x \in \text{eval-usages } g1 \ n$ 
    using inputs-of-SignExtendNode inputs-of-are-usages
    by (metis SignExtendNode.hyps(1) SignExtendNode.hyps(2) encode-in-ids in-
puts.simps inputs-are-usages list.set-intros(1))
    then show ?case using SignExtendNode inputs-of-SignExtendNode
      by (metis  $\langle \text{kind } g2 \ n = \text{SignExtendNode } ib \ rb \ x \rangle$  child-member-in child-unchanged
in-set-member list.set-intros(1) rep.SignExtendNode unchanged.elims(2))
  next
  case (ZeroExtendNode n ib rb x xe)
  then have  $\text{kind } g2 \ n = \text{ZeroExtendNode } ib \ rb \ x$ 
    using kind-unchanged by metis
  then have  $x \in \text{eval-usages } g1 \ n$ 
    using inputs-of-ZeroExtendNode inputs-of-are-usages
    by (metis ZeroExtendNode.hyps(1) ZeroExtendNode.hyps(2) IRNodes.inputs-of-ZeroExtendNode
encode-in-ids inputs.simps inputs-are-usages list.set-intros(1))
    then show ?case using ZeroExtendNode inputs-of-ZeroExtendNode
      by (metis  $\langle \text{kind } g2 \ n = \text{ZeroExtendNode } ib \ rb \ x \rangle$  child-member-in child-unchanged
member-rec(1) rep.ZeroExtendNode unchanged.simps)
  next
  case (LeafNode n s)
  then show ?case
    by (metis kind-unchanged rep.LeafNode stamp-unchanged)
  next
  case (RefNode n n')
  then have  $\text{kind } g2 \ n = \text{RefNode } n'$ 
    using kind-unchanged by metis
  then have  $n' \in \text{eval-usages } g1 \ n$ 
    by (metis IRNodes.inputs-of-RefNode RefNode.hyps(1) RefNode.hyps(2) en-
code-in-ids inputs.elims inputs-are-usages list.set-intros(1))
    then show ?case
      by (metis IRNodes.inputs-of-RefNode RefNode.IH RefNode.hyps(1) RefNode.hyps(2)
RefNode.prem(1)  $\langle \text{kind } g2 \ n = \text{RefNode } n' \rangle$  child-unchanged encode-in-ids in-
puts.elims list.set-intros(1) local.wf rep.RefNode)
qed
qed

```

```

theorem stay-same:
  assumes nc: unchanged (eval-usages g1 nid) g1 g2
  assumes g1:  $[g1, m, p] \vdash \text{id} \mapsto v1$ 
  assumes wf: wf-graph g1
  shows  $[g2, m, p] \vdash \text{id} \mapsto v1$ 
proof –
  have nid:  $\text{id} \in \text{ids } g1$ 
    using g1 eval-in-ids by simp
  then have  $\text{id} \in \text{eval-usages } g1 \text{ nid}$ 
    using eval-usages-self by blast
  then have kind-same:  $\text{kind } g1 \text{ nid} = \text{kind } g2 \text{ nid}$ 
    using nc node-unchanged by blast
  obtain e where  $e: (g1 \vdash \text{id} \simeq e) \wedge ([m, p] \vdash e \mapsto v1)$ 
    using encodeeval-def g1
    by auto
  then have val:  $[m, p] \vdash e \mapsto v1$ 
    using g1 encodeeval-def
    by simp
  then show ?thesis using e nid nc
    unfolding encodeeval-def
proof (induct e v1 arbitrary: nid rule: evaltree.induct)
  case (ConstantExpr c)
    then show ?case
      by (meson local.wf stay-same-encoding)
  next
    case (ParameterExpr i s)
    have  $g2 \vdash \text{id} \simeq \text{ParameterExpr } i \text{ s}$ 
      using stay-same-encoding ParameterExpr
      by (meson local.wf)
    then show ?case using evaltree.ParameterExpr
      by (meson ParameterExpr.hyps)
  next
    case (ConditionalExpr ce cond branch te fe v)
    then have  $g2 \vdash \text{id} \simeq \text{ConditionalExpr } ce \text{ te } fe$ 
      using ConditionalExpr.prem1 ConditionalExpr.prem3 local.wf stay-same-encoding
      by presburger
    then show ?case
      by (meson ConditionalExpr.prem1 ConditionalExpr.prem3 local.wf
stay-same-encoding)
  next
    case (UnaryExpr xe v op)
    then show ?case
      using local.wf stay-same-encoding by blast
  next
    case (BinaryExpr xe x ye y op)
    then show ?case
      using local.wf stay-same-encoding by blast
  next
    case (LeafExpr val nid s)

```



```

    then show ?case
    by (metis local.wf stay-same-encoding)
qed
qed

```

```

lemma add-changed:
  assumes gup = add-node new k g
  shows changeonly {new} g gup
  using assms unfolding add-node-def changeonly.simps
  using add-node.rep-eq add-node-def kind.rep-eq stamp.rep-eq by simp

```

```

lemma disjoint-change:
  assumes changeonly change g gup
  assumes nochange = ids g - change
  shows unchanged nochange g gup
  using assms unfolding changeonly.simps unchanged.simps
  by blast

```

```

lemma add-node-unchanged:
  assumes new  $\notin$  ids g
  assumes nid  $\in$  ids g
  assumes gup = add-node new k g
  assumes wf-graph g
  shows unchanged (eval-usages g nid) g gup
proof -
  have new  $\notin$  (eval-usages g nid) using assms
  using eval-usages.simps by blast
  then have changeonly {new} g gup
  using assms add-changed by blast
  then show ?thesis using assms add-node-def disjoint-change
  using Diff-insert-absorb by auto
qed

```

```

lemma eval-uses-imp:
  ((nid'  $\in$  ids g  $\wedge$  nid = nid')
   $\vee$  nid'  $\in$  inputs g nid
   $\vee$  ( $\exists$  nid'' . eval-uses g nid nid''  $\wedge$  eval-uses g nid'' nid'))
 $\longleftrightarrow$  eval-uses g nid nid'
  using use0 use-inp use-trans
  by (meson eval-uses.simps)

```

```

lemma wf-use-ids:
  assumes wf-graph g
  assumes nid  $\in$  ids g
  assumes eval-uses g nid nid'
  shows nid'  $\in$  ids g
  using assms(3)
proof (induction rule: eval-uses.induct)

```

```

    case use0
    then show ?case by simp
next
    case use-inp
    then show ?case
        using assms(1) inp-in-g-wf by blast
next
    case use-trans
    then show ?case by blast
qed

lemma no-external-use:
  assumes wf-graph g
  assumes nid'  $\notin$  ids g
  assumes nid  $\in$  ids g
  shows  $\neg$ (eval-uses g nid nid')
proof -
  have 0: nid  $\neq$  nid'
    using assms by blast
  have inp: nid'  $\notin$  inputs g nid
    using assms
    using inp-in-g-wf by blast
  have rec-0:  $\nexists n . n \in$  ids g  $\wedge$  n = nid'
    using assms by blast
  have rec-inp:  $\nexists n . n \in$  ids g  $\wedge$  n  $\in$  inputs g nid'
    using assms(2) inp-in-g by blast
  have rec:  $\nexists$  nid'' . eval-uses g nid nid''  $\wedge$  eval-uses g nid'' nid'
    using wf-use-ids assms(1) assms(2) assms(3) by blast
  from inp 0 rec show ?thesis
    using eval-uses-imp by blast
qed

end

```

7.8 Tree to Graph Theorems

```

theory TreeToGraphThms
imports
  IRTreeEvalThms
  IRGraphFrames
  HOL-Eisbach.Eisbach
  HOL-Eisbach.Eisbach-Tools
begin

```

7.8.1 Extraction and Evaluation of Expression Trees is Deterministic.

First, we prove some extra rules that relate each type of IRNode to the corresponding IRExpr type that 'rep' will produce. These are very helpful

for proving that 'rep' is deterministic.

named-theorems *rep*

lemma *rep-constant* [*rep*]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = ConstantNode\ c \implies$
 $e = ConstantExpr\ c$
by (*induction rule: rep.induct; auto*)

lemma *rep-parameter* [*rep*]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = ParameterNode\ i \implies$
 $(\exists\ s.\ e = ParameterExpr\ i\ s)$
by (*induction rule: rep.induct; auto*)

lemma *rep-conditional* [*rep*]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = ConditionalNode\ c\ t\ f \implies$
 $(\exists\ ce\ te\ fe.\ e = ConditionalExpr\ ce\ te\ fe)$
by (*induction rule: rep.induct; auto*)

lemma *rep-abs* [*rep*]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = AbsNode\ x \implies$
 $(\exists\ xe.\ e = UnaryExpr\ UnaryAbs\ xe)$
by (*induction rule: rep.induct; auto*)

lemma *rep-not* [*rep*]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = NotNode\ x \implies$
 $(\exists\ xe.\ e = UnaryExpr\ UnaryNot\ xe)$
by (*induction rule: rep.induct; auto*)

lemma *rep-negate* [*rep*]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = NegateNode\ x \implies$
 $(\exists\ xe.\ e = UnaryExpr\ UnaryNeg\ xe)$
by (*induction rule: rep.induct; auto*)

lemma *rep-logicnegation* [*rep*]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = LogicNegationNode\ x \implies$
 $(\exists\ xe.\ e = UnaryExpr\ UnaryLogicNegation\ xe)$
by (*induction rule: rep.induct; auto*)

lemma *rep-add* [*rep*]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = AddNode\ x\ y \implies$
 $(\exists\ xe\ ye.\ e = BinaryExpr\ BinAdd\ xe\ ye)$

by (*induction rule: rep.induct; auto*)

lemma *rep-sub* [*rep*]:
 $g \vdash n \simeq e \implies$
 $\text{kind } g \ n = \text{SubNode } x \ y \implies$
 $(\exists xe \ ye. \ e = \text{BinaryExpr BinSub } xe \ ye)$
by (*induction rule: rep.induct; auto*)

lemma *rep-mul* [*rep*]:
 $g \vdash n \simeq e \implies$
 $\text{kind } g \ n = \text{MulNode } x \ y \implies$
 $(\exists xe \ ye. \ e = \text{BinaryExpr BinMul } xe \ ye)$
by (*induction rule: rep.induct; auto*)

lemma *rep-and* [*rep*]:
 $g \vdash n \simeq e \implies$
 $\text{kind } g \ n = \text{AndNode } x \ y \implies$
 $(\exists xe \ ye. \ e = \text{BinaryExpr BinAnd } xe \ ye)$
by (*induction rule: rep.induct; auto*)

lemma *rep-or* [*rep*]:
 $g \vdash n \simeq e \implies$
 $\text{kind } g \ n = \text{OrNode } x \ y \implies$
 $(\exists xe \ ye. \ e = \text{BinaryExpr BinOr } xe \ ye)$
by (*induction rule: rep.induct; auto*)

lemma *rep-xor* [*rep*]:
 $g \vdash n \simeq e \implies$
 $\text{kind } g \ n = \text{XorNode } x \ y \implies$
 $(\exists xe \ ye. \ e = \text{BinaryExpr BinXor } xe \ ye)$
by (*induction rule: rep.induct; auto*)

lemma *rep-short-circuit-or* [*rep*]:
 $g \vdash n \simeq e \implies$
 $\text{kind } g \ n = \text{ShortCircuitOrNode } x \ y \implies$
 $(\exists xe \ ye. \ e = \text{BinaryExpr BinShortCircuitOr } xe \ ye)$
by (*induction rule: rep.induct; auto*)

lemma *rep-left-shift* [*rep*]:
 $g \vdash n \simeq e \implies$
 $\text{kind } g \ n = \text{LeftShiftNode } x \ y \implies$
 $(\exists xe \ ye. \ e = \text{BinaryExpr BinLeftShift } xe \ ye)$
by (*induction rule: rep.induct; auto*)

lemma *rep-right-shift* [*rep*]:
 $g \vdash n \simeq e \implies$
 $\text{kind } g \ n = \text{RightShiftNode } x \ y \implies$
 $(\exists xe \ ye. \ e = \text{BinaryExpr BinRightShift } xe \ ye)$
by (*induction rule: rep.induct; auto*)

lemma *rep-unsigned-right-shift* [rep]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = UnsignedRightShiftNode\ x\ y \implies$
 $(\exists\ xe\ ye. e = BinaryExpr\ BinURightShift\ xe\ ye)$
by (induction rule: *rep.induct*; auto)

lemma *rep-integer-below* [rep]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = IntegerBelowNode\ x\ y \implies$
 $(\exists\ xe\ ye. e = BinaryExpr\ BinIntegerBelow\ xe\ ye)$
by (induction rule: *rep.induct*; auto)

lemma *rep-integer-equals* [rep]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = IntegerEqualsNode\ x\ y \implies$
 $(\exists\ xe\ ye. e = BinaryExpr\ BinIntegerEquals\ xe\ ye)$
by (induction rule: *rep.induct*; auto)

lemma *rep-integer-less-than* [rep]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = IntegerLessThanNode\ x\ y \implies$
 $(\exists\ xe\ ye. e = BinaryExpr\ BinIntegerLessThan\ xe\ ye)$
by (induction rule: *rep.induct*; auto)

lemma *rep-narrow* [rep]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = NarrowNode\ ib\ rb\ x \implies$
 $(\exists\ x. e = UnaryExpr\ (UnaryNarrow\ ib\ rb)\ x)$
by (induction rule: *rep.induct*; auto)

lemma *rep-sign-extend* [rep]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = SignExtendNode\ ib\ rb\ x \implies$
 $(\exists\ x. e = UnaryExpr\ (UnarySignExtend\ ib\ rb)\ x)$
by (induction rule: *rep.induct*; auto)

lemma *rep-zero-extend* [rep]:

$g \vdash n \simeq e \implies$
 $kind\ g\ n = ZeroExtendNode\ ib\ rb\ x \implies$
 $(\exists\ x. e = UnaryExpr\ (UnaryZeroExtend\ ib\ rb)\ x)$
by (induction rule: *rep.induct*; auto)

lemma *rep-load-field* [rep]:

$g \vdash n \simeq e \implies$
 $is\ preevaluated\ (kind\ g\ n) \implies$
 $(\exists\ s. e = LeafExpr\ n\ s)$
by (induction rule: *rep.induct*; auto)

```

lemma rep-ref [rep]:
  g ⊢ n ≃ e ⇒
    kind g n = RefNode n' ⇒
      g ⊢ n' ≃ e
by (induction rule: rep.induct; auto)

```

```

method solve-det uses node =
  (match node in kind - - = node - for node ⇒
    ⟨match rep in r: - ⇒ - = node - ⇒ - ⇒
      ⟨match IRNode.inject in i: (node - = node -) = - ⇒
        ⟨match RepE in e: - ⇒ (∧x. - = node x ⇒ -) ⇒ - ⇒
          ⟨match IRNode.distinct in d: node - ≠ RefNode - ⇒
            ⟨metis i e r d⟩⟩⟩⟩ |
    match node in kind - - = node - - for node ⇒
      ⟨match rep in r: - ⇒ - = node - - ⇒ - ⇒
        ⟨match IRNode.inject in i: (node - - = node - -) = - ⇒
          ⟨match RepE in e: - ⇒ (∧x y. - = node x y ⇒ -) ⇒ - ⇒
            ⟨match IRNode.distinct in d: node - - ≠ RefNode - ⇒
              ⟨metis i e r d⟩⟩⟩⟩ |
    match node in kind - - = node - - - for node ⇒
      ⟨match rep in r: - ⇒ - = node - - - ⇒ - ⇒
        ⟨match IRNode.inject in i: (node - - - = node - - -) = - ⇒
          ⟨match RepE in e: - ⇒ (∧x y z. - = node x y z ⇒ -) ⇒ - ⇒
            ⟨match IRNode.distinct in d: node - - - ≠ RefNode - ⇒
              ⟨metis i e r d⟩⟩⟩⟩ |
    match node in kind - - = node - - - for node ⇒
      ⟨match rep in r: - ⇒ - = node - - - ⇒ - ⇒
        ⟨match IRNode.inject in i: (node - - - = node - - -) = - ⇒
          ⟨match RepE in e: - ⇒ (∧x. - = node - - x ⇒ -) ⇒ - ⇒
            ⟨match IRNode.distinct in d: node - - - ≠ RefNode - ⇒
              ⟨metis i e r d⟩⟩⟩⟩)

```

Now we can prove that 'rep' and 'eval', and their list versions, are deterministic.

```

lemma repDet:
  shows (g ⊢ n ≃ e1) ⇒ (g ⊢ n ≃ e2) ⇒ e1 = e2
proof (induction arbitrary: e2 rule: rep.induct)
  case (ConstantNode n c)
  then show ?case using rep-constant by auto
next
  case (ParameterNode n i s)
  then show ?case
    by (metis IRNode.disc(2685) ParameterNodeE is-RefNode-def rep-parameter)
next
  case (ConditionalNode n c t f ce te fe)
  then show ?case
    using IRNode.distinct(593)
    using IRNode.inject(6) ConditionalNodeE rep-conditional

```

```

    by metis
next
  case (AbsNode n x xe)
  then show ?case
    by (solve-det node: AbsNode)
next
  case (NotNode n x xe)
  then show ?case
    by (solve-det node: NotNode)
next
  case (NegateNode n x xe)
  then show ?case
    by (solve-det node: NegateNode)
next
  case (LogicNegationNode n x xe)
  then show ?case
    by (solve-det node: LogicNegationNode)
next
  case (AddNode n x y xe ye)
  then show ?case
    by (solve-det node: AddNode)
next
  case (MulNode n x y xe ye)
  then show ?case
    by (solve-det node: MulNode)
next
  case (SubNode n x y xe ye)
  then show ?case
    by (solve-det node: SubNode)
next
  case (AndNode n x y xe ye)
  then show ?case
    by (solve-det node: AndNode)
next
  case (OrNode n x y xe ye)
  then show ?case
    by (solve-det node: OrNode)
next
  case (XorNode n x y xe ye)
  then show ?case
    by (solve-det node: XorNode)
next
  case (ShortCircuitOrNode n x y xe ye)
  then show ?case
    by (solve-det node: ShortCircuitOrNode)
next
  case (LeftShiftNode n x y xe ye)
  then show ?case
    by (solve-det node: LeftShiftNode)

```

```

next
  case (RightShiftNode n x y xe ye)
  then show ?case
    by (solve-det node: RightShiftNode)
next
  case (UnsignedRightShiftNode n x y xe ye)
  then show ?case
    by (solve-det node: UnsignedRightShiftNode)
next
  case (IntegerBelowNode n x y xe ye)
  then show ?case
    by (solve-det node: IntegerBelowNode)
next
  case (IntegerEqualsNode n x y xe ye)
  then show ?case
    by (solve-det node: IntegerEqualsNode)
next
  case (IntegerLessThanNode n x y xe ye)
  then show ?case
    by (solve-det node: IntegerLessThanNode)
next
  case (NarrowNode n x xe)
  then show ?case
    by (metis IRNode.distinct(2203) IRNode.inject(28) NarrowNodeE rep-narrow)
next
  case (SignExtendNode n x xe)
  then show ?case
    by (metis IRNode.distinct(2599) IRNode.inject(39) SignExtendNodeE rep-sign-extend)
next
  case (ZeroExtendNode n x xe)
  then show ?case
    by (metis IRNode.distinct(2753) IRNode.inject(50) ZeroExtendNodeE rep-zero-extend)
next
  case (LeafNode n s)
  then show ?case using rep-load-field LeafNodeE
    by (metis is-preevaluated.simps(53))
next
  case (RefNode n')
  then show ?case
    using rep-ref by blast
qed

lemma repAllDet:
   $g \vdash xs \simeq_L e1 \implies$ 
   $g \vdash xs \simeq_L e2 \implies$ 
   $e1 = e2$ 
proof (induction arbitrary: e2 rule: replist.induct)
  case RepNil
  then show ?case

```



```

    using replist.cases by auto
next
  case (RepCons x xe xs xse)
  then show ?case
    by (metis list.distinct(1) list.sel(1) list.sel(3) repDet replist.cases)
qed

```

```

lemma encodeEvalDet:
  [g,m,p] ⊢ e ↦ v1 ⟹
  [g,m,p] ⊢ e ↦ v2 ⟹
  v1 = v2
by (metis encodeeval-def evalDet repDet)

```

```

lemma graphDet: ([g,m,p] ⊢ n ↦ v1) ∧ ([g,m,p] ⊢ n ↦ v2) ⟹ v1 = v2
using encodeEvalDet by blast

```

7.8.2 Monotonicity of Graph Refinement

Lift refinement monotonicity to graph level. Hopefully these shouldn't really be required.

```

lemma mono-abs:
  assumes kind g1 n = AbsNode x ∧ kind g2 n = AbsNode x
  assumes (g1 ⊢ x ≃ xe1) ∧ (g2 ⊢ x ≃ xe2)
  assumes xe1 ≥ xe2
  assumes (g1 ⊢ n ≃ e1) ∧ (g2 ⊢ n ≃ e2)
  shows e1 ≥ e2
by (metis AbsNode assms(1) assms(2) assms(3) assms(4) mono-unary repDet)

```

```

lemma mono-not:
  assumes kind g1 n = NotNode x ∧ kind g2 n = NotNode x
  assumes (g1 ⊢ x ≃ xe1) ∧ (g2 ⊢ x ≃ xe2)
  assumes xe1 ≥ xe2
  assumes (g1 ⊢ n ≃ e1) ∧ (g2 ⊢ n ≃ e2)
  shows e1 ≥ e2
by (metis NotNode assms(1) assms(2) assms(3) assms(4) mono-unary repDet)

```

```

lemma mono-negate:
  assumes kind g1 n = NegateNode x ∧ kind g2 n = NegateNode x
  assumes (g1 ⊢ x ≃ xe1) ∧ (g2 ⊢ x ≃ xe2)
  assumes xe1 ≥ xe2
  assumes (g1 ⊢ n ≃ e1) ∧ (g2 ⊢ n ≃ e2)
  shows e1 ≥ e2
by (metis NegateNode assms(1) assms(2) assms(3) assms(4) mono-unary repDet)

```

```

lemma mono-logic-negation:
  assumes kind g1 n = LogicNegationNode x ∧ kind g2 n = LogicNegationNode x
  assumes (g1 ⊢ x ≃ xe1) ∧ (g2 ⊢ x ≃ xe2)
  assumes xe1 ≥ xe2
  assumes (g1 ⊢ n ≃ e1) ∧ (g2 ⊢ n ≃ e2)

```

shows $e1 \geq e2$
by (*metis* *LogicNegationNode* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *mono-unary* *repDet*)

lemma *mono-narrow*:

assumes $\text{kind } g1 \ n = \text{NarrowNode } ib \ rb \ x \wedge \text{kind } g2 \ n = \text{NarrowNode } ib \ rb \ x$
assumes $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$
assumes $xe1 \geq xe2$
assumes $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$
shows $e1 \geq e2$
using *assms* *mono-unary* *repDet* *NarrowNode*
by *metis*

lemma *mono-sign-extend*:

assumes $\text{kind } g1 \ n = \text{SignExtendNode } ib \ rb \ x \wedge \text{kind } g2 \ n = \text{SignExtendNode } ib \ rb \ x$
assumes $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$
assumes $xe1 \geq xe2$
assumes $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$
shows $e1 \geq e2$
by (*metis* *SignExtendNode* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *mono-unary* *repDet*)

lemma *mono-zero-extend*:

assumes $\text{kind } g1 \ n = \text{ZeroExtendNode } ib \ rb \ x \wedge \text{kind } g2 \ n = \text{ZeroExtendNode } ib \ rb \ x$
assumes $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$
assumes $xe1 \geq xe2$
assumes $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$
shows $e1 \geq e2$
using *assms* *mono-unary* *repDet* *ZeroExtendNode*
by *metis*

lemma *mono-conditional-graph*:

assumes $\text{kind } g1 \ n = \text{ConditionalNode } c \ t \ f \wedge \text{kind } g2 \ n = \text{ConditionalNode } c \ t \ f$
assumes $(g1 \vdash c \simeq ce1) \wedge (g2 \vdash c \simeq ce2)$
assumes $(g1 \vdash t \simeq te1) \wedge (g2 \vdash t \simeq te2)$
assumes $(g1 \vdash f \simeq fe1) \wedge (g2 \vdash f \simeq fe2)$
assumes $ce1 \geq ce2 \wedge te1 \geq te2 \wedge fe1 \geq fe2$
assumes $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$
shows $e1 \geq e2$
using *ConditionalNodeE* *IRNode.inject*(6) *assms*(1) *assms*(2) *assms*(3) *assms*(4) *assms*(5) *assms*(6) *mono-conditional* *repDet* *rep-conditional*
by (*smt* (*verit*, *best*) *ConditionalNode*)

lemma *mono-add*:

assumes $\text{kind } g1 \ n = \text{AddNode } x \ y \wedge \text{kind } g2 \ n = \text{AddNode } x \ y$
assumes $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$
assumes $(g1 \vdash y \simeq ye1) \wedge (g2 \vdash y \simeq ye2)$

```

assumes  $xe1 \geq xe2 \wedge ye1 \geq ye2$ 
assumes  $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$ 
shows  $e1 \geq e2$ 
using mono-binary assms AddNodeE IRNode.inject(2) repDet rep-add
by (metis IRNode.distinct(205))

```

lemma *mono-mul*:

```

assumes  $kind\ g1\ n = MulNode\ x\ y \wedge kind\ g2\ n = MulNode\ x\ y$ 
assumes  $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$ 
assumes  $(g1 \vdash y \simeq ye1) \wedge (g2 \vdash y \simeq ye2)$ 
assumes  $xe1 \geq xe2 \wedge ye1 \geq ye2$ 
assumes  $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$ 
shows  $e1 \geq e2$ 
using mono-binary assms IRNode.inject(27) MulNodeE repDet rep-mul
by (smt (verit, best) MulNode)

```

lemma *term-graph-evaluation*:

```

 $(g \vdash n \sqsubseteq e) \implies (\forall\ m\ p\ v.\ ([m,p] \vdash e \mapsto v) \longrightarrow ([g,m,p] \vdash n \mapsto v))$ 
unfolding graph-represents-expression-def apply auto
by (meson encodeeval-def)

```

lemma *encodes-contains*:

```

 $g \vdash n \simeq e \implies$ 
 $kind\ g\ n \neq NoNode$ 
apply (induction rule: rep.induct)
apply (match IRNode.distinct in e: ?n ≠ NoNode ⇒
 $\langle presburger\ add: e \rangle +$ 
apply force
by fastforce)

```

lemma *no-encoding*:

```

assumes  $n \notin ids\ g$ 
shows  $\neg(g \vdash n \simeq e)$ 
using assms apply simp apply (rule notI) by (induction e; simp add: en-
codes-contains)

```

lemma *not-excluded-keep-type*:

```

assumes  $n \in ids\ g1$ 
assumes  $n \notin excluded$ 
assumes  $(excluded \sqsubseteq as-set\ g1) \subseteq as-set\ g2$ 
shows  $kind\ g1\ n = kind\ g2\ n \wedge stamp\ g1\ n = stamp\ g2\ n$ 
using assms unfolding as-set-def domain-subtraction-def by blast

```

method *metis-node-eq-unary* **for** *node :: 'a ⇒ IRNode =*

```

(match IRNode.inject in i: (node - = node -) = - ⇒
 $\langle metis\ i \rangle$ )

```

method *metis-node-eq-binary* **for** *node :: 'a ⇒ 'a ⇒ IRNode =*

```

(match IRNode.inject in i: (node - - = node - -) = - ⇒)

```

```

    ⟨metis i⟩
method metis-node-eq-ternary for node :: 'a ⇒ 'a ⇒ 'a ⇒ IRNode =
  (match IRNode.inject in i: (node - - - = node - - -) = - ⇒
    ⟨metis i⟩)

```

7.8.3 Lift Data-flow Tree Refinement to Graph Refinement

```

theorem graph-semantic-preservation:
  assumes a:  $e1' \geq e2'$ 
  assumes b:  $(\{n'\} \trianglelefteq \text{as-set } g1) \subseteq \text{as-set } g2$ 
  assumes c:  $g1 \vdash n' \simeq e1'$ 
  assumes d:  $g2 \vdash n' \simeq e2'$ 
  shows graph-refinement g1 g2
  unfolding graph-refinement-def apply rule
  apply (metis b d ids-some no-encoding not-excluded-keep-type singleton-iff sub-
    setI)
  apply (rule allI) apply (rule impI) apply (rule allI) apply (rule impI)
  unfolding graph-represents-expression-def
proof –
  fix n e1
  assume e:  $n \in \text{ids } g1$ 
  assume f:  $(g1 \vdash n \simeq e1)$ 

  show  $\exists e2. (g2 \vdash n \simeq e2) \wedge e1 \geq e2$ 
proof (cases  $n = n'$ )
  case True
  have g:  $e1 = e1'$  using c f True repDet by simp
  have h:  $(g2 \vdash n \simeq e2') \wedge e1' \geq e2'$ 
    using True a d by blast
  then show ?thesis
    using g by blast
  next
  case False
  have n  $\notin \{n'\}$ 
    using False by simp
  then have i:  $\text{kind } g1 \ n = \text{kind } g2 \ n \wedge \text{stamp } g1 \ n = \text{stamp } g2 \ n$ 
    using not-excluded-keep-type
    using b e by presburger
  show ?thesis using f i
proof (induction e1)
  case (ConstantNode n c)
  then show ?case
    by (metis eq-refl rep.ConstantNode)
  next
  case (ParameterNode n i s)
  then show ?case
    by (metis eq-refl rep.ParameterNode)
  next
  case (ConditionalNode n c t f ce1 te1 fe1)

```

```

have k: g1 ⊢ n ≃ ConditionalExpr ce1 te1 fe1 using f ConditionalNode
  by (simp add: ConditionalNode.hyps(2) rep.ConditionalNode)
obtain cn tn fn where l: kind g1 n = ConditionalNode cn tn fn
  using ConditionalNode.hyps(1) by blast
then have mc: g1 ⊢ cn ≃ ce1
  using ConditionalNode.hyps(1) ConditionalNode.hyps(2) by fastforce
from l have mt: g1 ⊢ tn ≃ te1
  using ConditionalNode.hyps(1) ConditionalNode.hyps(3) by fastforce
from l have mf: g1 ⊢ fn ≃ fe1
  using ConditionalNode.hyps(1) ConditionalNode.hyps(4) by fastforce
then show ?case
proof -
  have g1 ⊢ cn ≃ ce1 using mc by simp
  have g1 ⊢ tn ≃ te1 using mt by simp
  have g1 ⊢ fn ≃ fe1 using mf by simp
  have cer: ∃ ce2. (g2 ⊢ cn ≃ ce2) ∧ ce1 ≥ ce2
    using ConditionalNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-ternary ConditionalNode)
  have ter: ∃ te2. (g2 ⊢ tn ≃ te2) ∧ te1 ≥ te2
    using ConditionalNode a b c d l no-encoding not-excluded-keep-type repDet
    singletonD
    by (metis-node-eq-ternary ConditionalNode)
  have ∃ fe2. (g2 ⊢ fn ≃ fe2) ∧ fe1 ≥ fe2
    using ConditionalNode a b c d l no-encoding not-excluded-keep-type repDet
    singletonD
    by (metis-node-eq-ternary ConditionalNode)
  then have ∃ ce2 te2 fe2. (g2 ⊢ n ≃ ConditionalExpr ce2 te2 fe2) ∧
    ConditionalExpr ce1 te1 fe1 ≥ ConditionalExpr ce2 te2 fe2
    using ConditionalNode.premis l rep.ConditionalNode cer ter
    by (smt (verit) mono-conditional)
  then show ?thesis
    by meson
qed
next
case (AbsNode n x xe1)
have k: g1 ⊢ n ≃ UnaryExpr UnaryAbs xe1 using f AbsNode
  by (simp add: AbsNode.hyps(2) rep.AbsNode)
obtain xn where l: kind g1 n = AbsNode xn
  using AbsNode.hyps(1) by blast
then have m: g1 ⊢ xn ≃ xe1
  using AbsNode.hyps(1) AbsNode.hyps(2) by fastforce
then show ?case
proof (cases xn = n')
case True
  then have n: xe1 = e1' using c m repDet by simp
  then have ev: g2 ⊢ n ≃ UnaryExpr UnaryAbs e2' using AbsNode.hyps(1)
    l m n
    using AbsNode.premis True d rep.AbsNode by simp

```

```

    then have r: UnaryExpr UnaryAbs e1' ≥ UnaryExpr UnaryAbs e2'
      by (meson a mono-unary)
    then show ?thesis using ev r
      by (metis n)
  next
    case False
    have g1 ⊢ xn ≃ xe1 using m by simp
    have ∃ xe2. (g2 ⊢ xn ≃ xe2) ∧ xe1 ≥ xe2
      using AbsNode
    using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
      by (metis-node-eq-unary AbsNode)
    then have ∃ xe2. (g2 ⊢ n ≃ UnaryExpr UnaryAbs xe2) ∧ UnaryExpr
UnaryAbs xe1 ≥ UnaryExpr UnaryAbs xe2
      by (metis AbsNode.premis l mono-unary rep.AbsNode)
    then show ?thesis
      by meson
  qed
next
  case (NotNode n x xe1)
  have k: g1 ⊢ n ≃ UnaryExpr UnaryNot xe1 using f NotNode
    by (simp add: NotNode.hyps(2) rep.NotNode)
  obtain xn where l: kind g1 n = NotNode xn
    using NotNode.hyps(1) by blast
  then have m: g1 ⊢ xn ≃ xe1
    using NotNode.hyps(1) NotNode.hyps(2) by fastforce
  then show ?case
  proof (cases xn = n')
    case True
    then have n: xe1 = e1' using c m repDet by simp
    then have ev: g2 ⊢ n ≃ UnaryExpr UnaryNot e2' using NotNode.hyps(1)
l m n
      using NotNode.premis True d rep.NotNode by simp
    then have r: UnaryExpr UnaryNot e1' ≥ UnaryExpr UnaryNot e2'
      by (meson a mono-unary)
    then show ?thesis using ev r
      by (metis n)
  next
    case False
    have g1 ⊢ xn ≃ xe1 using m by simp
    have ∃ xe2. (g2 ⊢ xn ≃ xe2) ∧ xe1 ≥ xe2
      using NotNode
    using False i b l not-excluded-keep-type singletonD no-encoding
      by (metis-node-eq-unary NotNode)
    then have ∃ xe2. (g2 ⊢ n ≃ UnaryExpr UnaryNot xe2) ∧ UnaryExpr
UnaryNot xe1 ≥ UnaryExpr UnaryNot xe2
      by (metis NotNode.premis l mono-unary rep.NotNode)
    then show ?thesis
      by meson
  qed

```

```

next
  case (NegateNode n x xe1)
  have k:  $g1 \vdash n \simeq \text{UnaryExpr UnaryNeg xe1}$  using f NegateNode
    by (simp add: NegateNode.hyps(2) rep.NegateNode)
  obtain xn where l:  $\text{kind } g1 \ n = \text{NegateNode } xn$ 
    using NegateNode.hyps(1) by blast
  then have m:  $g1 \vdash xn \simeq xe1$ 
    using NegateNode.hyps(1) NegateNode.hyps(2) by fastforce
  then show ?case
  proof (cases  $xn = n'$ )
    case True
    then have n:  $xe1 = e1'$  using c m repDet by simp
    then have ev:  $g2 \vdash n \simeq \text{UnaryExpr UnaryNeg } e2'$  using NegateNode.hyps(1)
  l m n
    using NegateNode.premis True d rep.NegateNode by simp
  then have r:  $\text{UnaryExpr UnaryNeg } e1' \geq \text{UnaryExpr UnaryNeg } e2'$ 
    by (meson a mono-unary)
  then show ?thesis using ev r
    by (metis n)
  next
  case False
  have  $g1 \vdash xn \simeq xe1$  using m by simp
  have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using NegateNode
    using False i b l not-excluded-keep-type singletonD no-encoding
    by (metis node-eq-unary NegateNode)
  then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr UnaryNeg } xe2) \wedge \text{UnaryExpr}$ 
  UnaryNeg  $xe1 \geq \text{UnaryExpr UnaryNeg } xe2$ 
    by (metis NegateNode.premis l mono-unary rep.NegateNode)
  then show ?thesis
    by meson
  qed
  next
  case (LogicNegationNode n x xe1)
  have k:  $g1 \vdash n \simeq \text{UnaryExpr UnaryLogicNegation xe1}$  using f LogicNegationNode
  by (simp add: LogicNegationNode.hyps(2) rep.LogicNegationNode)
  obtain xn where l:  $\text{kind } g1 \ n = \text{LogicNegationNode } xn$ 
    using LogicNegationNode.hyps(1) by blast
  then have m:  $g1 \vdash xn \simeq xe1$ 
    using LogicNegationNode.hyps(1) LogicNegationNode.hyps(2) by fastforce
  then show ?case
  proof (cases  $xn = n'$ )
    case True
    then have n:  $xe1 = e1'$  using c m repDet by simp
    then have ev:  $g2 \vdash n \simeq \text{UnaryExpr UnaryLogicNegation } e2'$  using
  LogicNegationNode.hyps(1) l m n
    using LogicNegationNode.premis True d rep.LogicNegationNode by simp
  then have r:  $\text{UnaryExpr UnaryLogicNegation } e1' \geq \text{UnaryExpr UnaryLog-}$ 

```

```

icNegation e2'
  by (meson a mono-unary)
  then show ?thesis using ev r
  by (metis n)
next
case False
have g1 ⊢ xn ≃ xe1 using m by simp
have ∃ xe2. (g2 ⊢ xn ≃ xe2) ∧ xe1 ≥ xe2
  using LogicNegationNode
  using False i b l not-excluded-keep-type singletonD no-encoding
  by (metis-node-eq-unary LogicNegationNode)
  then have ∃ xe2. (g2 ⊢ n ≃ UnaryExpr UnaryLogicNegation xe2) ∧
UnaryExpr UnaryLogicNegation xe1 ≥ UnaryExpr UnaryLogicNegation xe2
  by (metis LogicNegationNode.prem1 mono-unary rep.LogicNegationNode)
  then show ?thesis
  by meson
qed
next
case (AddNode n x y xe1 ye1)
have k: g1 ⊢ n ≃ BinaryExpr BinAdd xe1 ye1 using f AddNode
  by (simp add: AddNode.hyps(2) rep.AddNode)
obtain xn yn where l: kind g1 n = AddNode xn yn
  using AddNode.hyps(1) by blast
then have mx: g1 ⊢ xn ≃ xe1
  using AddNode.hyps(1) AddNode.hyps(2) by fastforce
from l have my: g1 ⊢ yn ≃ ye1
  using AddNode.hyps(1) AddNode.hyps(3) by fastforce
then show ?case
proof -
  have g1 ⊢ xn ≃ xe1 using mx by simp
  have g1 ⊢ yn ≃ ye1 using my by simp
  have xer: ∃ xe2. (g2 ⊢ xn ≃ xe2) ∧ xe1 ≥ xe2
    using AddNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary AddNode)
  have ∃ ye2. (g2 ⊢ yn ≃ ye2) ∧ ye1 ≥ ye2
    using AddNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary AddNode)
  then have ∃ xe2 ye2. (g2 ⊢ n ≃ BinaryExpr BinAdd xe2 ye2) ∧ BinaryExpr
BinAdd xe1 ye1 ≥ BinaryExpr BinAdd xe2 ye2
  by (metis AddNode.prem1 mono-binary rep.AddNode xer)
  then show ?thesis
  by meson
qed
next
case (MulNode n x y xe1 ye1)
have k: g1 ⊢ n ≃ BinaryExpr BinMul xe1 ye1 using f MulNode
  by (simp add: MulNode.hyps(2) rep.MulNode)

```



```

obtain  $xn\ yn$  where  $l$ :  $kind\ g1\ n = MulNode\ xn\ yn$ 
  using  $MulNode.hyps(1)$  by blast
then have  $mx$ :  $g1 \vdash xn \simeq xe1$ 
  using  $MulNode.hyps(1)\ MulNode.hyps(2)$  by fastforce
from  $l$  have  $my$ :  $g1 \vdash yn \simeq ye1$ 
  using  $MulNode.hyps(1)\ MulNode.hyps(3)$  by fastforce
then show ?case
proof –
  have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
  have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
  have  $xer$ :  $\exists\ xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using  $MulNode$ 
    using  $a\ b\ c\ d\ l\ no\_encoding\ not\_excluded\_keep\_type\ repDet\ singletonD$ 
    by (metis-node-eq-binary  $MulNode$ )
  have  $\exists\ ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
    using  $MulNode$ 
    using  $a\ b\ c\ d\ l\ no\_encoding\ not\_excluded\_keep\_type\ repDet\ singletonD$ 
    by (metis-node-eq-binary  $MulNode$ )
  then have  $\exists\ xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinMul\ xe2\ ye2) \wedge BinaryExpr$ 
     $BinMul\ xe1\ ye1 \geq BinaryExpr\ BinMul\ xe2\ ye2$ 
    by (metis  $MulNode.premis\ l\ mono\_binary\ rep.MulNode\ xer$ )
  then show ?thesis
    by meson
qed
next
case ( $SubNode\ n\ x\ y\ xe1\ ye1$ )
have  $k$ :  $g1 \vdash n \simeq BinaryExpr\ BinSub\ xe1\ ye1$  using  $f\ SubNode$ 
  by (simp  $add$ :  $SubNode.hyps(2)\ rep.SubNode$ )
obtain  $xn\ yn$  where  $l$ :  $kind\ g1\ n = SubNode\ xn\ yn$ 
  using  $SubNode.hyps(1)$  by blast
then have  $mx$ :  $g1 \vdash xn \simeq xe1$ 
  using  $SubNode.hyps(1)\ SubNode.hyps(2)$  by fastforce
from  $l$  have  $my$ :  $g1 \vdash yn \simeq ye1$ 
  using  $SubNode.hyps(1)\ SubNode.hyps(3)$  by fastforce
then show ?case
proof –
  have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
  have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
  have  $xer$ :  $\exists\ xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using  $SubNode$ 
    using  $a\ b\ c\ d\ l\ no\_encoding\ not\_excluded\_keep\_type\ repDet\ singletonD$ 
    by (metis-node-eq-binary  $SubNode$ )
  have  $\exists\ ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
using  $SubNode\ a\ b\ c\ d\ l\ no\_encoding\ not\_excluded\_keep\_type\ repDet\ singletonD$ 
  by (metis-node-eq-binary  $SubNode$ )
  then have  $\exists\ xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinSub\ xe2\ ye2) \wedge BinaryExpr$ 
     $BinSub\ xe1\ ye1 \geq BinaryExpr\ BinSub\ xe2\ ye2$ 
    by (metis  $SubNode.premis\ l\ mono\_binary\ rep.SubNode\ xer$ )
  then show ?thesis

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      by meson
    qed
  next
    case (AndNode n x y xe1 ye1)
    have k:  $g1 \vdash n \simeq \text{BinaryExpr BinAnd } xe1 \ ye1$  using f AndNode
      by (simp add: AndNode.hyps(2) rep.AndNode)
    obtain xn yn where l: kind g1 n = AndNode xn yn
      using AndNode.hyps(1) by blast
    then have mx:  $g1 \vdash xn \simeq xe1$ 
      using AndNode.hyps(1) AndNode.hyps(2) by fastforce
    from l have my:  $g1 \vdash yn \simeq ye1$ 
      using AndNode.hyps(1) AndNode.hyps(3) by fastforce
    then show ?case
    proof -
      have  $g1 \vdash xn \simeq xe1$  using mx by simp
      have  $g1 \vdash yn \simeq ye1$  using my by simp
      have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
        using AndNode
        using a b c d l no-encoding not-excluded-keep-type repDet singletonD
        by (metis-node-eq-binary AndNode)
      have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
        using AndNode a b c d l no-encoding not-excluded-keep-type repDet
        singletonD
        by (metis-node-eq-binary AndNode)
      then have  $\exists xe2 \ ye2. (g2 \vdash n \simeq \text{BinaryExpr BinAnd } xe2 \ ye2) \wedge \text{BinaryExpr BinAnd } xe1 \ ye1 \geq \text{BinaryExpr BinAnd } xe2 \ ye2$ 
        by (metis AndNode.premis l mono-binary rep.AndNode xer)
      then show ?thesis
        by meson
    qed
  next
    case (OrNode n x y xe1 ye1)
    have k:  $g1 \vdash n \simeq \text{BinaryExpr BinOr } xe1 \ ye1$  using f OrNode
      by (simp add: OrNode.hyps(2) rep.OrNode)
    obtain xn yn where l: kind g1 n = OrNode xn yn
      using OrNode.hyps(1) by blast
    then have mx:  $g1 \vdash xn \simeq xe1$ 
      using OrNode.hyps(1) OrNode.hyps(2) by fastforce
    from l have my:  $g1 \vdash yn \simeq ye1$ 
      using OrNode.hyps(1) OrNode.hyps(3) by fastforce
    then show ?case
    proof -
      have  $g1 \vdash xn \simeq xe1$  using mx by simp
      have  $g1 \vdash yn \simeq ye1$  using my by simp
      have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
        using OrNode
        using a b c d l no-encoding not-excluded-keep-type repDet singletonD
        by (metis-node-eq-binary OrNode)
      have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 

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    using OrNode a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary OrNode)
  then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr BinOr xe2 ye2) \wedge BinaryExpr$ 
     $BinOr xe1 ye1 \geq BinaryExpr BinOr xe2 ye2$ 
    by (metis OrNode.premis l mono-binary rep.OrNode xer)
  then show ?thesis
    by meson
qed
next
case (XorNode n x y xe1 ye1)
have k:  $g1 \vdash n \simeq BinaryExpr BinXor xe1 ye1$  using f XorNode
  by (simp add: XorNode.hyps(2) rep.XorNode)
obtain xn yn where l: kind g1 n = XorNode xn yn
  using XorNode.hyps(1) by blast
then have mx:  $g1 \vdash xn \simeq xe1$ 
  using XorNode.hyps(1) XorNode.hyps(2) by fastforce
from l have my:  $g1 \vdash yn \simeq ye1$ 
  using XorNode.hyps(1) XorNode.hyps(3) by fastforce
then show ?case
proof -
  have  $g1 \vdash xn \simeq xe1$  using mx by simp
  have  $g1 \vdash yn \simeq ye1$  using my by simp
  have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using XorNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary XorNode)
  have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
    using XorNode a b c d l no-encoding not-excluded-keep-type repDet
    singletonD
    by (metis-node-eq-binary XorNode)
  then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr BinXor xe2 ye2) \wedge BinaryExpr$ 
     $BinXor xe1 ye1 \geq BinaryExpr BinXor xe2 ye2$ 
    by (metis XorNode.premis l mono-binary rep.XorNode xer)
  then show ?thesis
    by meson
qed
next
case (ShortCircuitOrNode n x y xe1 ye1)
have k:  $g1 \vdash n \simeq BinaryExpr BinShortCircuitOr xe1 ye1$  using f ShortCir-
  cuitOrNode
  by (simp add: ShortCircuitOrNode.hyps(2) rep.ShortCircuitOrNode)
obtain xn yn where l: kind g1 n = ShortCircuitOrNode xn yn
  using ShortCircuitOrNode.hyps(1) by blast
then have mx:  $g1 \vdash xn \simeq xe1$ 
  using ShortCircuitOrNode.hyps(1) ShortCircuitOrNode.hyps(2) by fastforce
from l have my:  $g1 \vdash yn \simeq ye1$ 
  using ShortCircuitOrNode.hyps(1) ShortCircuitOrNode.hyps(3) by fastforce
then show ?case
proof -

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    have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
    have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
    have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using ShortCircuitOrNode
      using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary ShortCircuitOrNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using ShortCircuitOrNode a b c d l no-encoding not-excluded-keep-type
repDet singletonD
      by (metis-node-eq-binary ShortCircuitOrNode)
    then have  $\exists xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinShortCircuitOr\ xe2\ ye2) \wedge$ 
BinaryExpr BinShortCircuitOr xe1 ye1  $\geq BinaryExpr BinShortCircuitOr xe2 ye2$ 
      by (metis ShortCircuitOrNode.premis l mono-binary rep.ShortCircuitOrNode xer)
    then show ?thesis
      by meson
  qed
next
case (LeftShiftNode n x y xe1 ye1)
have  $k: g1 \vdash n \simeq BinaryExpr\ BinLeftShift\ xe1\ ye1$  using  $f$  LeftShiftNode
  by (simp add: LeftShiftNode.hyps(2) rep.LeftShiftNode)
obtain  $xn\ yn$  where  $l: kind\ g1\ n = LeftShiftNode\ xn\ yn$ 
  using LeftShiftNode.hyps(1) by blast
then have  $mx: g1 \vdash xn \simeq xe1$ 
  using LeftShiftNode.hyps(1) LeftShiftNode.hyps(2) by fastforce
from  $l$  have  $my: g1 \vdash yn \simeq ye1$ 
  using LeftShiftNode.hyps(1) LeftShiftNode.hyps(3) by fastforce
then show ?case
proof -
  have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
  have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
  have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using LeftShiftNode
    using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary LeftShiftNode)
  have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
    using LeftShiftNode a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary LeftShiftNode)
  then have  $\exists xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinLeftShift\ xe2\ ye2) \wedge$ 
BinaryExpr BinLeftShift xe1 ye1  $\geq BinaryExpr BinLeftShift xe2 ye2$ 
    by (metis LeftShiftNode.premis l mono-binary rep.LeftShiftNode xer)
  then show ?thesis
    by meson
qed
next
case (RightShiftNode n x y xe1 ye1)
have  $k: g1 \vdash n \simeq BinaryExpr\ BinRightShift\ xe1\ ye1$  using  $f$  RightShiftNode
  by (simp add: RightShiftNode.hyps(2) rep.RightShiftNode)

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obtain  $xn\ yn$  where  $l$ :  $\text{kind } g1\ n = \text{RightShiftNode } xn\ yn$ 
  using  $\text{RightShiftNode.hyps}(1)$  by blast
then have  $mx$ :  $g1 \vdash xn \simeq xe1$ 
  using  $\text{RightShiftNode.hyps}(1)\ \text{RightShiftNode.hyps}(2)$  by fastforce
from  $l$  have  $my$ :  $g1 \vdash yn \simeq ye1$ 
  using  $\text{RightShiftNode.hyps}(1)\ \text{RightShiftNode.hyps}(3)$  by fastforce
then show ?case
proof –
  have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
  have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
  have  $xer$ :  $\exists\ xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using  $\text{RightShiftNode}$ 
    using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary RightShiftNode)
  have  $\exists\ ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
    using  $\text{RightShiftNode } a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet
singletonD
    by (metis-node-eq-binary RightShiftNode)
  then have  $\exists\ xe2\ ye2. (g2 \vdash n \simeq \text{BinaryExpr } \text{BinRightShift } xe2\ ye2) \wedge$ 
BinaryExpr BinRightShift xe1 ye1  $\geq \text{BinaryExpr } \text{BinRightShift } xe2\ ye2$ 
    by (metis RightShiftNode.premis l mono-binary rep.RightShiftNode xer)
  then show ?thesis
    by meson
qed
next
  case ( $\text{UnsignedRightShiftNode } n\ x\ y\ xe1\ ye1$ )
  have  $k$ :  $g1 \vdash n \simeq \text{BinaryExpr } \text{BinURightShift } xe1\ ye1$  using  $f$   $\text{UnsignedRightShiftNode}$ 
    by (simp add: UnsignedRightShiftNode.hyps(2) rep.UnsignedRightShiftNode)
  obtain  $xn\ yn$  where  $l$ :  $\text{kind } g1\ n = \text{UnsignedRightShiftNode } xn\ yn$ 
    using  $\text{UnsignedRightShiftNode.hyps}(1)$  by blast
  then have  $mx$ :  $g1 \vdash xn \simeq xe1$ 
    using  $\text{UnsignedRightShiftNode.hyps}(1)\ \text{UnsignedRightShiftNode.hyps}(2)$  by
fastforce
  from  $l$  have  $my$ :  $g1 \vdash yn \simeq ye1$ 
    using  $\text{UnsignedRightShiftNode.hyps}(1)\ \text{UnsignedRightShiftNode.hyps}(3)$  by
fastforce
  then show ?case
  proof –
    have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
    have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
    have  $xer$ :  $\exists\ xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using  $\text{UnsignedRightShiftNode}$ 
      using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary UnsignedRightShiftNode)
    have  $\exists\ ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using  $\text{UnsignedRightShiftNode } a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type
repDet singletonD
      by (metis-node-eq-binary UnsignedRightShiftNode)

```

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      then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr\ BinURightShift\ xe2\ ye2) \wedge$ 
BinaryExpr BinURightShift xe1 ye1  $\geq BinaryExpr BinURightShift xe2 ye2$ 
      by (metis UnsignedRightShiftNode.premis l mono-binary rep.UnsignedRightShiftNode
xer)
      then show ?thesis
      by meson
    qed
  next
  case (IntegerBelowNode n x y xe1 ye1)
  have k:  $g1 \vdash n \simeq BinaryExpr\ BinIntegerBelow\ xe1\ ye1$  using f IntegerBe-
lowNode
  by (simp add: IntegerBelowNode.hyps(2) rep.IntegerBelowNode)
  obtain xn yn where l: kind g1 n = IntegerBelowNode xn yn
  using IntegerBelowNode.hyps(1) by blast
  then have mx:  $g1 \vdash xn \simeq xe1$ 
  using IntegerBelowNode.hyps(1) IntegerBelowNode.hyps(2) by fastforce
  from l have my:  $g1 \vdash yn \simeq ye1$ 
  using IntegerBelowNode.hyps(1) IntegerBelowNode.hyps(3) by fastforce
  then show ?case
  proof -
    have  $g1 \vdash xn \simeq xe1$  using mx by simp
    have  $g1 \vdash yn \simeq ye1$  using my by simp
    have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using IntegerBelowNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary IntegerBelowNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
    using IntegerBelowNode a b c d l no-encoding not-excluded-keep-type repDet
singletonD
    by (metis-node-eq-binary IntegerBelowNode)
    then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr\ BinIntegerBelow\ xe2\ ye2) \wedge$ 
BinaryExpr BinIntegerBelow xe1 ye1  $\geq BinaryExpr BinIntegerBelow xe2 ye2$ 
    by (metis IntegerBelowNode.premis l mono-binary rep.IntegerBelowNode
xer)
    then show ?thesis
    by meson
  qed
next
case (IntegerEqualsNode n x y xe1 ye1)
have k:  $g1 \vdash n \simeq BinaryExpr\ BinIntegerEquals\ xe1\ ye1$  using f IntegerEqual-
sNode
by (simp add: IntegerEqualsNode.hyps(2) rep.IntegerEqualsNode)
obtain xn yn where l: kind g1 n = IntegerEqualsNode xn yn
using IntegerEqualsNode.hyps(1) by blast
then have mx:  $g1 \vdash xn \simeq xe1$ 
using IntegerEqualsNode.hyps(1) IntegerEqualsNode.hyps(2) by fastforce
from l have my:  $g1 \vdash yn \simeq ye1$ 
using IntegerEqualsNode.hyps(1) IntegerEqualsNode.hyps(3) by fastforce
then show ?case

```

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proof –
  have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
  have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
  have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using IntegerEqualsNode
    using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary IntegerEqualsNode)
  have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
    using IntegerEqualsNode  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type
repDet singletonD
    by (metis-node-eq-binary IntegerEqualsNode)
  then have  $\exists xe2\ ye2. (g2 \vdash n \simeq \text{BinaryExpr BinIntegerEquals } xe2\ ye2) \wedge$ 
BinaryExpr BinIntegerEquals  $xe1\ ye1 \geq \text{BinaryExpr BinIntegerEquals } xe2\ ye2$ 
    by (metis IntegerEqualsNode.premis l mono-binary rep.IntegerEqualsNode
xer)
    then show ?thesis
      by meson
  qed
next
  case (IntegerLessThanNode  $n\ x\ y\ xe1\ ye1$ )
    have  $k: g1 \vdash n \simeq \text{BinaryExpr BinIntegerLessThan } xe1\ ye1$  using  $f$  IntegerLessThanNode
    by (simp add: IntegerLessThanNode.hyps(2) rep.IntegerLessThanNode)
    obtain  $xn\ yn$  where  $l: \text{kind } g1\ n = \text{IntegerLessThanNode } xn\ yn$ 
    using IntegerLessThanNode.hyps(1) by blast
    then have  $mx: g1 \vdash xn \simeq xe1$ 
    using IntegerLessThanNode.hyps(1) IntegerLessThanNode.hyps(2) by fast-force
    from  $l$  have  $my: g1 \vdash yn \simeq ye1$ 
    using IntegerLessThanNode.hyps(1) IntegerLessThanNode.hyps(3) by fast-force
    then show ?case
      proof –
        have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
        have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
        have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
          using IntegerLessThanNode
          using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
          by (metis-node-eq-binary IntegerLessThanNode)
        have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
          using IntegerLessThanNode  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type
repDet singletonD
          by (metis-node-eq-binary IntegerLessThanNode)
        then have  $\exists xe2\ ye2. (g2 \vdash n \simeq \text{BinaryExpr BinIntegerLessThan } xe2\ ye2) \wedge$ 
BinaryExpr BinIntegerLessThan  $xe1\ ye1 \geq \text{BinaryExpr BinIntegerLessThan } xe2\ ye2$ 
          by (metis IntegerLessThanNode.premis l mono-binary rep.IntegerLessThanNode
xer)
        then show ?thesis

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      by meson
    qed
  next
    case (NarrowNode n inputBits resultBits x xe1)
    have k:  $g1 \vdash n \simeq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe1$  using
    f NarrowNode
      by (simp add: NarrowNode.hyps(2) rep.NarrowNode)
    obtain xn where l: kind g1 n = NarrowNode inputBits resultBits xn
      using NarrowNode.hyps(1) by blast
    then have m:  $g1 \vdash xn \simeq xe1$ 
      using NarrowNode.hyps(1) NarrowNode.hyps(2)
      by auto
    then show ?case
    proof (cases xn = n')
      case True
        then have n:  $xe1 = e1'$  using c m repDet by simp
        then have ev:  $g2 \vdash n \simeq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) e2'$ 
        using NarrowNode.hyps(1) l m n
          using NarrowNode.premis True d rep.NarrowNode by simp
        then have r:  $\text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) e1' \geq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) e2'$ 
          by (meson a mono-unary)
        then show ?thesis using ev r
          by (metis n)
      next
        case False
        have  $g1 \vdash xn \simeq xe1$  using m by simp
        have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
          using NarrowNode
          using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
          by (metis node-eq-ternary NarrowNode)
        then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe2) \wedge \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe1 \geq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe2$ 
          by (metis NarrowNode.premis l mono-unary rep.NarrowNode)
        then show ?thesis
          by meson
    qed
  next
    case (SignExtendNode n inputBits resultBits x xe1)
    have k:  $g1 \vdash n \simeq \text{UnaryExpr } (\text{UnarySignExtend inputBits resultBits}) xe1$ 
    using f SignExtendNode
      by (simp add: SignExtendNode.hyps(2) rep.SignExtendNode)
    obtain xn where l: kind g1 n = SignExtendNode inputBits resultBits xn
      using SignExtendNode.hyps(1) by blast
    then have m:  $g1 \vdash xn \simeq xe1$ 
      using SignExtendNode.hyps(1) SignExtendNode.hyps(2)
      by auto
    then show ?case

```



```

proof (cases  $xn = n'$ )
  case True
    then have  $n: xe1 = e1'$  using  $c\ m\ repDet$  by simp
    then have  $ev: g2 \vdash n \simeq UnaryExpr\ (UnarySignExtend\ inputBits\ resultBits)$ 
     $e2'$  using  $SignExtendNode.hyps(1)\ l\ m\ n$ 
      using  $SignExtendNode.premis\ True\ d\ rep.SignExtendNode$  by simp
      then have  $r: UnaryExpr\ (UnarySignExtend\ inputBits\ resultBits)\ e1' \geq$ 
 $UnaryExpr\ (UnarySignExtend\ inputBits\ resultBits)\ e2'$ 
        by (meson a mono-unary)
      then show ?thesis using  $ev\ r$ 
        by (metis n)
    next
      case False
        have  $g1 \vdash xn \simeq xe1$  using  $m$  by simp
        have  $\exists\ xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
          using  $SignExtendNode$ 
          using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
          by (metis-node-eq-ternary SignExtendNode)
        then have  $\exists\ xe2. (g2 \vdash n \simeq UnaryExpr\ (UnarySignExtend\ inputBits\ result-$ 
 $Bits)\ xe2) \wedge UnaryExpr\ (UnarySignExtend\ inputBits\ resultBits)\ xe1 \geq UnaryExpr$ 
 $(UnarySignExtend\ inputBits\ resultBits)\ xe2$ 
          by (metis SignExtendNode.premis l mono-unary rep.SignExtendNode)
        then show ?thesis
          by meson
      qed
    next
      case ( $ZeroExtendNode\ n\ inputBits\ resultBits\ x\ xe1$ )
        have  $k: g1 \vdash n \simeq UnaryExpr\ (UnaryZeroExtend\ inputBits\ resultBits)\ xe1$ 
using  $f\ ZeroExtendNode$ 
          by (simp add: ZeroExtendNode.hyps(2) rep.ZeroExtendNode)
        obtain  $xn$  where  $l: kind\ g1\ n = ZeroExtendNode\ inputBits\ resultBits\ xn$ 
          using  $ZeroExtendNode.hyps(1)$  by blast
        then have  $m: g1 \vdash xn \simeq xe1$ 
          using  $ZeroExtendNode.hyps(1)\ ZeroExtendNode.hyps(2)$ 
          by auto
        then show ?case
          proof (cases  $xn = n'$ )
            case True
              then have  $n: xe1 = e1'$  using  $c\ m\ repDet$  by simp
              then have  $ev: g2 \vdash n \simeq UnaryExpr\ (UnaryZeroExtend\ inputBits\ resultBits)$ 
               $e2'$  using  $ZeroExtendNode.hyps(1)\ l\ m\ n$ 
                using  $ZeroExtendNode.premis\ True\ d\ rep.ZeroExtendNode$  by simp
                then have  $r: UnaryExpr\ (UnaryZeroExtend\ inputBits\ resultBits)\ e1' \geq$ 
 $UnaryExpr\ (UnaryZeroExtend\ inputBits\ resultBits)\ e2'$ 
                  by (meson a mono-unary)
                then show ?thesis using  $ev\ r$ 
                  by (metis n)
            next
              case False

```

```

    have  $g1 \vdash xn \simeq xe1$  using  $m$  by simp
    have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using ZeroExtendNode
      using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
      by (metis-node-eq-ternary ZeroExtendNode)
    then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) xe2) \wedge \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) xe1 \geq \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) xe2$ 
      by (metis ZeroExtendNode.premis l mono-unary rep.ZeroExtendNode)
    then show ?thesis
      by meson
  qed
next
  case (LeafNode n s)
  then show ?case
    by (metis eq-refl rep.LeafNode)
next
  case (RefNode n')
  then show ?case
    by (metis a b c d no-encoding not-excluded-keep-type rep.RefNode repDet singletonD)
  qed
qed
qed

```

lemma *graph-antics-preservation-subscript*:

```

  assumes  $a: e_1' \geq e_2'$ 
  assumes  $b: (\{n\} \trianglelefteq \text{as-set } g_1) \subseteq \text{as-set } g_2$ 
  assumes  $c: g_1 \vdash n \simeq e_1'$ 
  assumes  $d: g_2 \vdash n \simeq e_2'$ 
  shows graph-refinement  $g_1 g_2$ 
  using graph-antics-preservation assms by simp

```

lemma *tree-to-graph-rewriting*:

```

   $e_1 \geq e_2$ 
   $\wedge (g_1 \vdash n \simeq e_1) \wedge \text{maximal-sharing } g_1$ 
   $\wedge (\{n\} \trianglelefteq \text{as-set } g_1) \subseteq \text{as-set } g_2$ 
   $\wedge (g_2 \vdash n \simeq e_2) \wedge \text{maximal-sharing } g_2$ 
 $\implies \text{graph-refinement } g_1 g_2$ 
  using graph-antics-preservation
  by auto

```

```

declare [[simp-trace]]
lemma equal-refines:
  fixes  $e1 e2 :: \text{IRExpr}$ 
  assumes  $e1 = e2$ 
  shows  $e1 \geq e2$ 
  using assms

```

```

  by simp
declare [[simp-trace=false]]

```

```

lemma eval-contains-id[simp]:  $g1 \vdash n \simeq e \implies n \in \text{ids } g1$ 
  using no-encoding by blast

```

```

lemma subset-kind[simp]:  $\text{as-set } g1 \subseteq \text{as-set } g2 \implies g1 \vdash n \simeq e \implies \text{kind } g1 \ n = \text{kind } g2 \ n$ 
  using eval-contains-id unfolding as-set-def
  by blast

```

```

lemma subset-stamp[simp]:  $\text{as-set } g1 \subseteq \text{as-set } g2 \implies g1 \vdash n \simeq e \implies \text{stamp } g1 \ n = \text{stamp } g2 \ n$ 
  using eval-contains-id unfolding as-set-def
  by blast

```

```

method solve-subset-eval uses as-set eval =
  (metis eval as-set subset-kind subset-stamp |
   metis eval as-set subset-kind)

```

```

lemma subset-implies-evals:
  assumes  $\text{as-set } g1 \subseteq \text{as-set } g2$ 
  assumes  $(g1 \vdash n \simeq e)$ 
  shows  $(g2 \vdash n \simeq e)$ 
  using assms(2)
  apply (induction e)
    apply (solve-subset-eval as-set: assms(1) eval: ConstantNode)
    apply (solve-subset-eval as-set: assms(1) eval: ParameterNode)
    apply (solve-subset-eval as-set: assms(1) eval: ConditionalNode)
    apply (solve-subset-eval as-set: assms(1) eval: AbsNode)
    apply (solve-subset-eval as-set: assms(1) eval: NotNode)
    apply (solve-subset-eval as-set: assms(1) eval: NegateNode)
    apply (solve-subset-eval as-set: assms(1) eval: LogicNegationNode)
    apply (solve-subset-eval as-set: assms(1) eval: AddNode)
    apply (solve-subset-eval as-set: assms(1) eval: MulNode)
    apply (solve-subset-eval as-set: assms(1) eval: SubNode)
    apply (solve-subset-eval as-set: assms(1) eval: AndNode)
    apply (solve-subset-eval as-set: assms(1) eval: OrNode)
    apply (solve-subset-eval as-set: assms(1) eval: XorNode)
    apply (solve-subset-eval as-set: assms(1) eval: ShortCircuitOrNode)
    apply (solve-subset-eval as-set: assms(1) eval: LeftShiftNode)
    apply (solve-subset-eval as-set: assms(1) eval: RightShiftNode)
    apply (solve-subset-eval as-set: assms(1) eval: UnsignedRightShiftNode)
    apply (solve-subset-eval as-set: assms(1) eval: IntegerBelowNode)
    apply (solve-subset-eval as-set: assms(1) eval: IntegerEqualsNode)
    apply (solve-subset-eval as-set: assms(1) eval: IntegerLessThanNode)

```

```

    apply (solve-subset-eval as-set: assms(1) eval: NarrowNode)
    apply (solve-subset-eval as-set: assms(1) eval: SignExtendNode)
    apply (solve-subset-eval as-set: assms(1) eval: ZeroExtendNode)
    apply (solve-subset-eval as-set: assms(1) eval: LeafNode)
    by (solve-subset-eval as-set: assms(1) eval: RefNode)

lemma subset-refines:
  assumes as-set g1  $\subseteq$  as-set g2
  shows graph-refinement g1 g2
proof -
  have ids g1  $\subseteq$  ids g2 using assms unfolding as-set-def
  by blast
  then show ?thesis unfolding graph-refinement-def apply rule
  apply (rule allI) apply (rule impI) apply (rule allI) apply (rule impI)
  unfolding graph-represents-expression-def
  proof -
    fix n e1
    assume 1:n  $\in$  ids g1
    assume 2:g1  $\vdash$  n  $\simeq$  e1

    show  $\exists e2. (g2 \vdash n \simeq e2) \wedge e1 \geq e2$ 
    using assms 1 2 using subset-implies-evals
    by (meson equal-refines)
  qed
qed

```

```

lemma graph-construction:
  e1  $\geq$  e2
   $\wedge$  as-set g1  $\subseteq$  as-set g2
   $\wedge$  (g2  $\vdash$  n  $\simeq$  e2)
   $\implies$  (g2  $\vdash$  n  $\trianglelefteq$  e1)  $\wedge$  graph-refinement g1 g2
  using subset-refines
  by (meson encodeeval-def graph-represents-expression-def le-expr-def)

```

7.8.4 Term Graph Reconstruction

```

lemma find-exists-kind:
  assumes find-node-and-stamp g (node, s) = Some nid
  shows kind g nid = node
  using assms unfolding find-node-and-stamp.simps
  by (metis (mono-tags, lifting) find-Some-iff)

```

```

lemma find-exists-stamp:
  assumes find-node-and-stamp g (node, s) = Some nid
  shows stamp g nid = s
  using assms unfolding find-node-and-stamp.simps
  by (metis (mono-tags, lifting) find-Some-iff)

```

```

lemma find-new-kind:

```

```

assumes  $g' = \text{add-node } \text{nid} \ (\text{node}, s) \ g$ 
assumes  $\text{node} \neq \text{NoNode}$ 
shows  $\text{kind } g' \ \text{nid} = \text{node}$ 
using assms
using add-node-lookup by presburger

lemma find-new-stamp:
assumes  $g' = \text{add-node } \text{nid} \ (\text{node}, s) \ g$ 
assumes  $\text{node} \neq \text{NoNode}$ 
shows  $\text{stamp } g' \ \text{nid} = s$ 
using assms
using add-node-lookup by presburger

lemma sorted-bottom:
assumes finite xs
assumes  $x \in xs$ 
shows  $x \leq \text{last}(\text{sorted-list-of-set}(xs::\text{nat set}))$ 
using assms
using sorted2-simps(2) sorted-list-of-set(2)
by (smt (verit, del-insts) Diff-iff Max-ge Max-in empty-iff list.set(1) snoc-eq-iff-butlast
sorted-insort-is-snoc sorted-list-of-set(1) sorted-list-of-set.fold-insort-key.infinite sorted-list-of-set.fold-insort-ke)

lemma fresh:  $\text{finite } xs \implies \text{last}(\text{sorted-list-of-set}(xs::\text{nat set})) + 1 \notin xs$ 
using sorted-bottom
using not-le by auto

lemma fresh-ids:
assumes  $n = \text{get-fresh-id } g$ 
shows  $n \notin \text{ids } g$ 
proof –
have finite (ids g) using Rep-IRGraph by auto
then show ?thesis
using assms fresh unfolding get-fresh-id.simps
by blast
qed

lemma graph-unchanged-rep-unchanged:
assumes  $\forall n \in \text{ids } g. \text{kind } g \ n = \text{kind } g' \ n$ 
assumes  $\forall n \in \text{ids } g. \text{stamp } g \ n = \text{stamp } g' \ n$ 
shows  $(g \vdash n \simeq e) \longrightarrow (g' \vdash n \simeq e)$ 
apply (rule impI) subgoal premises e using e assms
apply (induction  $n \ e$ )
apply (metis no-encoding rep.ConstantNode)
apply (metis no-encoding rep.ParameterNode)
apply (metis no-encoding rep.ConditionalNode)
apply (metis no-encoding rep.AbsNode)
apply (metis no-encoding rep.NotNode)
apply (metis no-encoding rep.NegateNode)
apply (metis no-encoding rep.LogicNegationNode)

```

```

    apply (metis no-encoding rep.AddNode)
    apply (metis no-encoding rep.MulNode)
    apply (metis no-encoding rep.SubNode)
    apply (metis no-encoding rep.AndNode)
    apply (metis no-encoding rep.OrNode)
    apply (metis no-encoding rep.XorNode)
    apply (metis no-encoding rep.ShortCircuitOrNode)
    apply (metis no-encoding rep.LeftShiftNode)
    apply (metis no-encoding rep.RightShiftNode)
    apply (metis no-encoding rep.UnsignedRightShiftNode)
    apply (metis no-encoding rep.IntegerBelowNode)
    apply (metis no-encoding rep.IntegerEqualsNode)
    apply (metis no-encoding rep.IntegerLessThanNode)
    apply (metis no-encoding rep.NarrowNode)
    apply (metis no-encoding rep.SignExtendNode)
    apply (metis no-encoding rep.ZeroExtendNode)
    apply (metis no-encoding rep.LeafNode)
  by (metis no-encoding rep.RefNode)
done

```

lemma *fresh-node-subset*:

```

  assumes  $n \notin \text{ids } g$ 
  assumes  $g' = \text{add-node } n (k, s) g$ 
  shows  $\text{as-set } g \subseteq \text{as-set } g'$ 
  using assms
  by (smt (verit, del-Insts) Collect-mono-iff Diff-idemp Diff-insert-absorb add-changed
    as-set-def disjoint-change unchanged.simps)

```

lemma *unrep-subset*:

```

  assumes  $(g \oplus e \rightsquigarrow (g', n))$ 
  shows  $\text{as-set } g \subseteq \text{as-set } g'$ 
  using assms proof (induction  $g \ e \ (g', n)$  arbitrary:  $g' \ n$ )
  case (ConstantNodeSame  $g \ c \ n$ )
  then show ?case by blast
next
  case (ConstantNodeNew  $g \ c \ n \ g'$ )
  then show ?case using fresh-ids fresh-node-subset
    by presburger
next
  case (ParameterNodeSame  $g \ i \ s \ n$ )
  then show ?case by blast
next
  case (ParameterNodeNew  $g \ i \ s \ n \ g'$ )
  then show ?case using fresh-ids fresh-node-subset
    by presburger
next
  case (ConditionalNodeSame  $g \ ce \ g2 \ c \ te \ g3 \ t \ fe \ g4 \ f \ s' \ n$ )
  then show ?case by blast
next

```

```

    case (ConditionalNodeNew g ce g2 c te g3 t fe g4 f s' n g')
    then show ?case using fresh-ids fresh-node-subset
      by (meson subset-trans)
next
    case (UnaryNodeSame g xe g2 x s' op n)
    then show ?case by blast
next
    case (UnaryNodeNew g xe g2 x s' op n g')
    then show ?case using fresh-ids fresh-node-subset
      by (meson subset-trans)
next
    case (BinaryNodeSame g xe g2 x ye g3 y s' op n)
    then show ?case by blast
next
    case (BinaryNodeNew g xe g2 x ye g3 y s' op n g')
    then show ?case using fresh-ids fresh-node-subset
      by (meson subset-trans)
next
    case (AllLeafNodes g n s)
    then show ?case by blast
qed

```

lemma *fresh-node-preserves-other-nodes*:

```

  assumes n' = get-fresh-id g
  assumes g' = add-node n' (k, s) g
  shows  $\forall n \in \text{ids } g. (g \vdash n \simeq e) \longrightarrow (g' \vdash n \simeq e)$ 
  using assms
  by (smt (verit, ccfv-SIG) Diff-idemp Diff-insert-absorb add-changed disjoint-change
    fresh-ids graph-unchanged-rep-unchanged unchanged.elims(2))

```

lemma *found-node-preserves-other-nodes*:

```

  assumes find-node-and-stamp g (k, s) = Some n
  shows  $\forall n \in \text{ids } g. (g \vdash n \simeq e) \longleftrightarrow (g \vdash n \simeq e)$ 
  using assms
  by blast

```

lemma *unrep-ids-subset[simp]*:

```

  assumes  $g \oplus e \rightsquigarrow (g', n)$ 
  shows  $\text{ids } g \subseteq \text{ids } g'$ 
  using assms unrep-subset
  by (meson graph-refinement-def subset-refines)

```

lemma *unrep-unchanged*:

```

  assumes  $g \oplus e \rightsquigarrow (g', n)$ 
  shows  $\forall n \in \text{ids } g. \forall e. (g \vdash n \simeq e) \longrightarrow (g' \vdash n \simeq e)$ 
  using assms unrep-subset fresh-node-preserves-other-nodes
  by (meson subset-implies-evals)

```

theorem *term-graph-reconstruction*:

$g \oplus e \rightsquigarrow (g', n) \implies (g' \vdash n \simeq e) \wedge \text{as-set } g \subseteq \text{as-set } g'$
subgoal premises e **apply** (rule *conjI*) **defer**
 using e *unrep-subset* **apply** *blast* **using** e
proof (induction g e (g', n) arbitrary: $g' n$)
 case (*ConstantNodeSame* $g' c n$)
 then have $\text{kind } g' n = \text{ConstantNode } c$
 using *find-exists-kind* *local.ConstantNodeSame* **by** *blast*
 then show ?*case* **using** *ConstantNode* **by** *blast*
next
 case (*ConstantNodeNew* $g c$)
 then show ?*case*
 using *ConstantNode* *IRNode.distinct(683)* *add-node-lookup* **by** *presburger*
next
 case (*ParameterNodeSame* $i s$)
 then show ?*case*
 by (*metis* *ParameterNode* *find-exists-kind* *find-exists-stamp*)
next
 case (*ParameterNodeNew* $g i s$)
 then show ?*case*
 by (*metis* *IRNode.distinct(2447)* *ParameterNode* *add-node-lookup*)
next
 case (*ConditionalNodeSame* $g ce g2 c te g3 t fe g4 f s' n$)
 then have k : $\text{kind } g4 n = \text{ConditionalNode } c t f$
 using *find-exists-kind* **by** *blast*
 have c : $g4 \vdash c \simeq ce$ **using** *local.ConditionalNodeSame* *unrep-unchanged*
 using *no-encoding* **by** *blast*
 have t : $g4 \vdash t \simeq te$ **using** *local.ConditionalNodeSame* *unrep-unchanged*
 using *no-encoding* **by** *blast*
 have f : $g4 \vdash f \simeq fe$ **using** *local.ConditionalNodeSame* *unrep-unchanged*
 using *no-encoding* **by** *blast*
 then show ?*case* **using** $c t f$
 using *ConditionalNode* k **by** *blast*
next
 case (*ConditionalNodeNew* $g ce g2 c te g3 t fe g4 f s' n g'$)
 moreover have *ConditionalNode* $c t f \neq \text{NoNode}$
 using *unary-node.elims* **by** *blast*
 ultimately have k : $\text{kind } g' n = \text{ConditionalNode } c t f$
 using *find-new-kind* *local.ConditionalNodeNew*
 by *presburger*
 then have c : $g' \vdash c \simeq ce$ **using** *local.ConditionalNodeNew* *unrep-unchanged*
 using *no-encoding*
 by (*metis* *ConditionalNodeNew.hyps(9)* *fresh-node-preserves-other-nodes*)
 then have t : $g' \vdash t \simeq te$ **using** *local.ConditionalNodeNew* *unrep-unchanged*
 using *no-encoding* *fresh-node-preserves-other-nodes*
 by *metis*
 then have f : $g' \vdash f \simeq fe$ **using** *local.ConditionalNodeNew* *unrep-unchanged*
 using *no-encoding* *fresh-node-preserves-other-nodes*
 by *metis*
 then show ?*case* **using** $c t f$


```

    using ConditionalNode k by blast
next
case (UnaryNodeSame g xe g' x s' op n)
then have k: kind g' n = unary-node op x
    using find-exists-kind local.UnaryNodeSame by blast
then have g' ⊢ x ≃ xe using local.UnaryNodeSame by blast
then show ?case using k
    apply (cases op)
    using AbsNode unary-node.simps(1) apply presburger
    using NegateNode unary-node.simps(3) apply presburger
    using NotNode unary-node.simps(2) apply presburger
    using LogicNegationNode unary-node.simps(4) apply presburger
    using NarrowNode unary-node.simps(5) apply presburger
    using SignExtendNode unary-node.simps(6) apply presburger
    using ZeroExtendNode unary-node.simps(7) by presburger
next
case (UnaryNodeNew g xe g2 x s' op n g')
moreover have unary-node op x ≠ NoNode
    using unary-node.elims by blast
ultimately have k: kind g' n = unary-node op x
    using find-new-kind local.UnaryNodeNew
    by presburger
have x ∈ ids g2 using local.UnaryNodeNew
    using eval-contains-id by blast
then have x ≠ n using local.UnaryNodeNew(5) fresh-ids by blast
have g' ⊢ x ≃ xe using local.UnaryNodeNew fresh-node-preserves-other-nodes
    using ⟨x ∈ ids g2⟩ by blast
then show ?case using k
    apply (cases op)
    using AbsNode unary-node.simps(1) apply presburger
    using NegateNode unary-node.simps(3) apply presburger
    using NotNode unary-node.simps(2) apply presburger
    using LogicNegationNode unary-node.simps(4) apply presburger
    using NarrowNode unary-node.simps(5) apply presburger
    using SignExtendNode unary-node.simps(6) apply presburger
    using ZeroExtendNode unary-node.simps(7) by presburger
next
case (BinaryNodeSame g xe g2 x ye g3 y s' op n)
then have k: kind g3 n = bin-node op x y
    using find-exists-kind by blast
have x: g3 ⊢ x ≃ xe using local.BinaryNodeSame unrep-unchanged
    using no-encoding by blast
have y: g3 ⊢ y ≃ ye using local.BinaryNodeSame unrep-unchanged
    using no-encoding by blast
then show ?case using x y k apply (cases op)
    using AddNode bin-node.simps(1) apply presburger
    using MulNode bin-node.simps(2) apply presburger
    using SubNode bin-node.simps(3) apply presburger
    using AndNode bin-node.simps(4) apply presburger

```

```

    using OrNode bin-node.simps(5) apply presburger
    using XorNode bin-node.simps(6) apply presburger
    using ShortCircuitOrNode bin-node.simps(7) apply presburger
    using LeftShiftNode bin-node.simps(8) apply presburger
    using RightShiftNode bin-node.simps(9) apply presburger
    using UnsignedRightShiftNode bin-node.simps(10) apply presburger
    using IntegerEqualsNode bin-node.simps(11) apply presburger
    using IntegerLessThanNode bin-node.simps(12) apply presburger
    using IntegerBelowNode bin-node.simps(13) by presburger
next
case (BinaryNodeNew g xe g2 x ye g3 y s' op n g')
moreover have bin-node op x y  $\neq$  NoNode
  using bin-node.elims by blast
ultimately have k: kind g' n = bin-node op x y
  using find-new-kind local.BinaryNodeNew
  by presburger
then have k: kind g' n = bin-node op x y
  using find-exists-kind by blast
have x: g'  $\vdash$  x  $\simeq$  xe using local.BinaryNodeNew unrep-unchanged
  using no-encoding
  by (meson fresh-node-preserves-other-nodes)
have y: g'  $\vdash$  y  $\simeq$  ye using local.BinaryNodeNew unrep-unchanged
  using no-encoding
  by (meson fresh-node-preserves-other-nodes)
then show ?case using x y k apply (cases op)
  using AddNode bin-node.simps(1) apply presburger
  using MulNode bin-node.simps(2) apply presburger
  using SubNode bin-node.simps(3) apply presburger
  using AndNode bin-node.simps(4) apply presburger
  using OrNode bin-node.simps(5) apply presburger
  using XorNode bin-node.simps(6) apply presburger
  using ShortCircuitOrNode bin-node.simps(7) apply presburger
  using LeftShiftNode bin-node.simps(8) apply presburger
  using RightShiftNode bin-node.simps(9) apply presburger
  using UnsignedRightShiftNode bin-node.simps(10) apply presburger
  using IntegerEqualsNode bin-node.simps(11) apply presburger
  using IntegerLessThanNode bin-node.simps(12) apply presburger
  using IntegerBelowNode bin-node.simps(13) by presburger
next
case (AllLeafNodes g n s)
then show ?case using rep.LeafNode by blast
qed
done

```

lemma ref-refinement:
 assumes g \vdash n \simeq e₁
 assumes kind g n' = RefNode n
 shows g \vdash n' \trianglelefteq e₁
 using assms RefNode

```

by (meson equal-refines graph-represents-expression-def)

lemma unrep-refines:
  assumes  $g \oplus e \rightsquigarrow (g', n)$ 
  shows graph-refinement  $g$   $g'$ 
  using assms
  using graph-refinement-def subset-refines unrep-subset by blast

lemma add-new-node-refines:
  assumes  $n \notin \text{ids } g$ 
  assumes  $g' = \text{add-node } n \ (k, s) \ g$ 
  shows graph-refinement  $g$   $g'$ 
  using assms unfolding graph-refinement
  using fresh-node-subset subset-refines by presburger

lemma add-node-as-set:
  assumes  $g' = \text{add-node } n \ (k, s) \ g$ 
  shows  $(\{n\} \sqsubseteq \text{as-set } g) \subseteq \text{as-set } g'$ 
  using assms unfolding as-set-def domain-subtraction-def
  using add-changed
  by (smt (z3) case-prodE changeonly.simps mem-Collect-eq prod.sel(1) subsetI)

theorem refined-insert:
  assumes  $e_1 \geq e_2$ 
  assumes  $g_1 \oplus e_2 \rightsquigarrow (g_2, n')$ 
  shows  $(g_2 \vdash n' \sqsubseteq e_1) \wedge \text{graph-refinement } g_1 \ g_2$ 
  using assms
  using graph-construction term-graph-reconstruction by blast

lemma ids-finite: finite (ids  $g$ )
  using Rep-IRGraph ids.rep-eq by simp

lemma unwrap-sorted: set (sorted-list-of-set (ids  $g$ )) = ids  $g$ 
  using Rep-IRGraph set-sorted-list-of-set ids-finite
  by blast

lemma find-none:
  assumes find-node-and-stamp  $g \ (k, s) = \text{None}$ 
  shows  $\forall n \in \text{ids } g. \text{kind } g \ n \neq k \vee \text{stamp } g \ n \neq s$ 
proof -
  have  $(\nexists n. n \in \text{ids } g \wedge (\text{kind } g \ n = k \wedge \text{stamp } g \ n = s))$ 
    using assms unfolding find-node-and-stamp.simps using find-None-iff un-
  wrap-sorted
  by (metis (mono-tags, lifting))
  then show ?thesis
    by blast
qed

```

```

method ref-represents uses node =
  (metis IRNode.distinct(2755) RefNode dual-order.refl find-new-kind fresh-node-subset
node subset-implies-evals)

```

7.8.5 Data-flow Tree to Subgraph Preserves Maximal Sharing

```

lemma same-kind-stamp-encodes-equal:
  assumes kind g n = kind g n'
  assumes stamp g n = stamp g n'
  assumes  $\neg(\text{is-preevaluated } (\text{kind } g \ n))$ 
  shows  $\forall \ e. (g \vdash n \simeq e) \longrightarrow (g \vdash n' \simeq e)$ 
  apply (rule allI)
  subgoal for e
    apply (rule impI)
    subgoal premises eval using eval assms
      apply (induction e)
    using ConstantNode apply presburger
    using ParameterNode apply presburger
      apply (metis ConditionalNode)
      apply (metis AbsNode)
      apply (metis NotNode)
      apply (metis NegateNode)
      apply (metis LogicNegationNode)
      apply (metis AddNode)
      apply (metis MulNode)
      apply (metis SubNode)
      apply (metis AndNode)
      apply (metis OrNode)
      apply (metis XorNode)
      apply (metis ShortCircuitOrNode)
      apply (metis LeftShiftNode)
      apply (metis RightShiftNode)
      apply (metis UnsignedRightShiftNode)
      apply (metis IntegerBelowNode)
      apply (metis IntegerEqualsNode)
      apply (metis IntegerLessThanNode)
      apply (metis NarrowNode)

```

```

    apply (metis SignExtendNode)
    apply (metis ZeroExtendNode)
  defer
    apply (metis RefNode)
  by blast
done
done

```

lemma *new-node-not-present*:

```

  assumes find-node-and-stamp  $g$  (node, s) = None
  assumes  $n = \text{get-fresh-id } g$ 
  assumes  $g' = \text{add-node } n$  (node, s)  $g$ 
  shows  $\forall n' \in \text{true-ids } g. (\forall e. ((g \vdash n \simeq e) \wedge (g \vdash n' \simeq e)) \longrightarrow n = n')$ 
  using assms
  using encode-in-ids fresh-ids by blast

```

lemma *true-ids-def*:

```

  true-ids  $g = \{n \in \text{ids } g. \neg(\text{is-RefNode } (\text{kind } g \ n)) \wedge ((\text{kind } g \ n) \neq \text{NoNode})\}$ 
  unfolding true-ids-def ids-def
  using ids-def is-RefNode-def by fastforce

```

lemma *add-node-some-node-def*:

```

  assumes  $k \neq \text{NoNode}$ 
  assumes  $g' = \text{add-node } \text{nid}$  (k, s)  $g$ 
  shows  $g' = \text{Abs-IRGraph } ((\text{Rep-IRGraph } g)(\text{nid} \mapsto (k, s)))$ 
  using assms
  by (metis Rep-IRGraph-inverse add-node.rep-eq fst-conv)

```

lemma *ids-add-update-v1*:

```

  assumes  $g' = \text{add-node } \text{nid}$  (k, s)  $g$ 
  assumes  $k \neq \text{NoNode}$ 
  shows  $\text{dom } (\text{Rep-IRGraph } g') = \text{dom } (\text{Rep-IRGraph } g) \cup \{\text{nid}\}$ 
  using assms ids.rep-eq add-node-some-node-def
  by (simp add: add-node.rep-eq)

```

lemma *ids-add-update-v2*:

```

  assumes  $g' = \text{add-node } \text{nid}$  (k, s)  $g$ 
  assumes  $k \neq \text{NoNode}$ 
  shows  $\text{nid} \in \text{ids } g'$ 
  using assms
  using find-new-kind ids-some by presburger

```

lemma *add-node-ids-subset*:

```

  assumes  $n \in \text{ids } g$ 
  assumes  $g' = \text{add-node } n$  node  $g$ 
  shows  $\text{ids } g' = \text{ids } g \cup \{n\}$ 
  using assms unfolding add-node-def
  apply (cases fst node = NoNode)
  using ids.rep-eq replace-node.rep-eq replace-node-def apply auto[1]

```

unfolding *ids-def*
by (*smt* (*verit*, *best*) *Collect-cong Un-insert-right dom-fun-upd fst-conv fun-upd-apply*
ids.rep-eq ids-def insert-absorb mem-Collect-eq option.inject option.simps(3) re-
place-node.rep-eq replace-node-def sup-bot.right-neutral)

lemma *convert-maximal:*

assumes $\forall n\ n'.\ n \in \text{true-ids } g \wedge n' \in \text{true-ids } g \longrightarrow (\forall e\ e'.\ (g \vdash n \simeq e) \wedge (g \vdash n' \simeq e') \longrightarrow e \neq e')$
shows *maximal-sharing* *g*
using *assms*
using *maximal-sharing* **by** *blast*

lemma *add-node-set-eq:*

assumes $k \neq \text{NoNode}$
assumes $n \notin \text{ids } g$
shows $\text{as-set } (\text{add-node } n\ (k, s)\ g) = \text{as-set } g \cup \{(n, (k, s))\}$
using *assms* **unfolding** *as-set-def add-node-def* **apply** *transfer* **apply** *simp*
by *blast*

lemma *add-node-as-set-eq:*

assumes $g' = \text{add-node } n\ (k, s)\ g$
assumes $n \notin \text{ids } g$
shows $\{n\} \sqsubseteq \text{as-set } g' = \text{as-set } g$
using *assms* **unfolding** *domain-subtraction-def*
using *add-node-set-eq*
by (*smt* (*z3*) *Collect-cong Rep-IRGraph-inverse UnCI UnE add-node.rep-eq as-set-def*
case-prodE2 case-prodI2 le-boolE le-boolI' mem-Collect-eq prod.sel(1) singletonD
singletonI)

lemma *true-ids:*

$\text{true-ids } g = \text{ids } g - \{n \in \text{ids } g.\ \text{is-RefNode } (\text{kind } g\ n)\}$
unfolding *true-ids-def*
by *fastforce*

lemma *as-set-ids:*

assumes $\text{as-set } g = \text{as-set } g'$
shows $\text{ids } g = \text{ids } g'$
using *assms*
by (*metis antisym equalityD1 graph-refinement-def subset-refines*)

lemma *ids-add-update:*

assumes $k \neq \text{NoNode}$
assumes $n \notin \text{ids } g$
assumes $g' = \text{add-node } n\ (k, s)\ g$
shows $\text{ids } g' = \text{ids } g \cup \{n\}$
using *assms* **apply** (*subst* *assms(3)*) **using** *add-node-set-eq as-set-ids*
by (*smt* (*verit*, *del-insts*) *Collect-cong Diff-idemp Diff-insert-absorb Un-commute*
add-node.rep-eq add-node-def ids.rep-eq ids-add-update-v1 ids-add-update-v2 insertE
insert-Collect insert-is-Un map-upd-Some-unfold mem-Collect-eq replace-node-def)

replace-node-unchanged)

lemma *true-ids-add-update*:

assumes $k \neq \text{NoNode}$
assumes $n \notin \text{ids } g$
assumes $g' = \text{add-node } n \ (k, s) \ g$
assumes $\neg(\text{is-RefNode } k)$
shows $\text{true-ids } g' = \text{true-ids } g \cup \{n\}$
using *assms* **using** *true-ids* *ids-add-update*
by (*smt* (*z3*) *Collect-cong* *Diff-iff* *Diff-insert-absorb* *Un-commute* *add-node-def* *find-new-kind* *insert-Diff-if* *insert-is-Un* *mem-Collect-eq* *replace-node-def* *replace-node-unchanged*)

lemma *new-def*:

assumes $(\text{new} \sqsubseteq \text{as-set } g') = \text{as-set } g$
shows $n \in \text{ids } g \longrightarrow n \notin \text{new}$
using *assms*
by (*smt* (*z3*) *as-set-def* *case-prodD* *domain-subtraction-def* *mem-Collect-eq*)

lemma *add-preserves-rep*:

assumes *unchanged*: $(\text{new} \sqsubseteq \text{as-set } g') = \text{as-set } g$
assumes *closed*: *wf-closed* *g*
assumes *existed*: $n \in \text{ids } g$
assumes $g' \vdash n \simeq e$
shows $g \vdash n \simeq e$
proof (*cases* $n \in \text{new}$)
case *True*
have $n \notin \text{ids } g$
using *unchanged* *True* **unfolding** *as-set-def* *domain-subtraction-def*
by *blast*
then show *?thesis* **using** *existed* **by** *simp*
next
case *False*
then have *kind-eq*: $\forall n'. n' \notin \text{new} \longrightarrow \text{kind } g \ n' = \text{kind } g' \ n'$
— can be more general than *stamp_eq* because *NoNode* default is equal
using *unchanged* *not-excluded-keep-type*
by (*smt* (*z3*) *case-prodE* *domain-subtraction-def* *ids-some* *mem-Collect-eq* *subsetI*)
from *False* **have** *stamp-eq*: $\forall n' \in \text{ids } g'. n' \notin \text{new} \longrightarrow \text{stamp } g \ n' = \text{stamp } g' \ n'$
using *unchanged* *not-excluded-keep-type*
by (*metis* *equalityE*)
show *?thesis* **using** *assms*(4) *kind-eq* *stamp-eq* *False*
proof (*induction* *n e* *rule*: *rep.induct*)
case (*ConstantNode* *n c*)
then show *?case*
using *rep.ConstantNode* *kind-eq* **by** *presburger*
next

```

    case (ParameterNode n i s)
    then show ?case
      using rep.ParameterNode
      by (metis no-encoding)
  next
    case (ConditionalNode n c t f ce te fe)
    have kind: kind g n = ConditionalNode c t f
      using ConditionalNode.hyps(1) ConditionalNode.prem(3) kind-eq by pres-
    burger
    then have isin: n ∈ ids g
      by simp
    have inputs: {c, t, f} = inputs g n
      using kind unfolding inputs.simps using inputs-of-ConditionalNode by simp
    have c ∈ ids g ∧ t ∈ ids g ∧ f ∈ ids g
      using closed unfolding wf-closed-def
      using isin inputs by blast
    then have c ∉ new ∧ t ∉ new ∧ f ∉ new
      using new-def unchanged by blast
    then show ?case using ConditionalNode apply simp
      using rep.ConditionalNode by presburger
  next
    case (AbsNode n x xe)
    then have kind: kind g n = AbsNode x
      by simp
    then have isin: n ∈ ids g
      by simp
    have inputs: {x} = inputs g n
      using kind unfolding inputs.simps by simp
    have x ∈ ids g
      using closed unfolding wf-closed-def
      using isin inputs by blast
    then have x ∉ new
      using new-def unchanged by blast
    then show ?case
      using AbsNode
      using rep.AbsNode by presburger
  next
    case (NotNode n x xe)
    then have kind: kind g n = NotNode x
      by simp
    then have isin: n ∈ ids g
      by simp
    have inputs: {x} = inputs g n
      using kind unfolding inputs.simps by simp
    have x ∈ ids g
      using closed unfolding wf-closed-def
      using isin inputs by blast
    then have x ∉ new
      using new-def unchanged by blast

```



```

    then show ?case using NotNode
      using rep.NotNode by presburger
next
  case (NegateNode n x xe)
  then have kind: kind g n = NegateNode x
    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{x\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $x \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $x \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using NegateNode
    using rep.NegateNode by presburger
next
  case (LogicNegationNode n x xe)
  then have kind: kind g n = LogicNegationNode x
    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{x\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $x \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $x \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using LogicNegationNode
    using rep.LogicNegationNode by presburger
next
  case (AddNode n x y xe ye)
  then have kind: kind g n = AddNode x y
    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{x, y\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using AddNode
    using rep.AddNode by presburger
next
  case (MulNode n x y xe ye)

```

```

then have kind: kind g n = MulNode x y
  by simp
then have isin: n ∈ ids g
  by simp
have inputs: {x, y} = inputs g n
  using kind unfolding inputs.simps by simp
have x ∈ ids g ∧ y ∈ ids g
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have x ∉ new ∧ y ∉ new
  using new-def unchanged by blast
then show ?case using MulNode
  using rep.MulNode by presburger
next
case (SubNode n x y xe ye)
then have kind: kind g n = SubNode x y
  by simp
then have isin: n ∈ ids g
  by simp
have inputs: {x, y} = inputs g n
  using kind unfolding inputs.simps by simp
have x ∈ ids g ∧ y ∈ ids g
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have x ∉ new ∧ y ∉ new
  using new-def unchanged by blast
then show ?case using SubNode
  using rep.SubNode by presburger
next
case (AndNode n x y xe ye)
then have kind: kind g n = AndNode x y
  by simp
then have isin: n ∈ ids g
  by simp
have inputs: {x, y} = inputs g n
  using kind unfolding inputs.simps by simp
have x ∈ ids g ∧ y ∈ ids g
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have x ∉ new ∧ y ∉ new
  using new-def unchanged by blast
then show ?case using AndNode
  using rep.AndNode by presburger
next
case (OrNode n x y xe ye)
then have kind: kind g n = OrNode x y
  by simp
then have isin: n ∈ ids g
  by simp

```

```

have inputs:  $\{x, y\} = \text{inputs } g \ n$ 
  using kind unfolding inputs.simps by simp
have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
  using new-def unchanged by blast
then show ?case using OrNode
  using rep.OrNode by presburger
next
case (XorNode n x y xe ye)
then have kind:  $\text{kind } g \ n = \text{XorNode } x \ y$ 
  by simp
then have isin:  $n \in \text{ids } g$ 
  by simp
have inputs:  $\{x, y\} = \text{inputs } g \ n$ 
  using kind unfolding inputs.simps by simp
have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
  using new-def unchanged by blast
then show ?case using XorNode
  using rep.XorNode by presburger
next
case (ShortCircuitOrNode n x y xe ye)
then have kind:  $\text{kind } g \ n = \text{ShortCircuitOrNode } x \ y$ 
  by simp
then have isin:  $n \in \text{ids } g$ 
  by simp
have inputs:  $\{x, y\} = \text{inputs } g \ n$ 
  using kind unfolding inputs.simps by simp
have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
  using new-def unchanged by blast
then show ?case using ShortCircuitOrNode
  using rep.ShortCircuitOrNode by presburger
next
case (LeftShiftNode n x y xe ye)
then have kind:  $\text{kind } g \ n = \text{LeftShiftNode } x \ y$ 
  by simp
then have isin:  $n \in \text{ids } g$ 
  by simp
have inputs:  $\{x, y\} = \text{inputs } g \ n$ 
  using kind unfolding inputs.simps by simp
have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
  using closed unfolding wf-closed-def

```

```

    using isin inputs by blast
  then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using LeftShiftNode
    using rep.LeftShiftNode by presburger
next
case (RightShiftNode  $n\ x\ y\ xe\ ye$ )
then have kind:  $\text{kind } g\ n = \text{RightShiftNode } x\ y$ 
  by simp
then have isin:  $n \in \text{ids } g$ 
  by simp
have inputs:  $\{x, y\} = \text{inputs } g\ n$ 
  using kind unfolding inputs.simps by simp
have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
  using new-def unchanged by blast
then show ?case using RightShiftNode
  using rep.RightShiftNode by presburger
next
case (UnsignedRightShiftNode  $n\ x\ y\ xe\ ye$ )
then have kind:  $\text{kind } g\ n = \text{UnsignedRightShiftNode } x\ y$ 
  by simp
then have isin:  $n \in \text{ids } g$ 
  by simp
have inputs:  $\{x, y\} = \text{inputs } g\ n$ 
  using kind unfolding inputs.simps by simp
have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
  using new-def unchanged by blast
then show ?case using UnsignedRightShiftNode
  using rep.UnsignedRightShiftNode by presburger
next
case (IntegerBelowNode  $n\ x\ y\ xe\ ye$ )
then have kind:  $\text{kind } g\ n = \text{IntegerBelowNode } x\ y$ 
  by simp
then have isin:  $n \in \text{ids } g$ 
  by simp
have inputs:  $\{x, y\} = \text{inputs } g\ n$ 
  using kind unfolding inputs.simps by simp
have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
  using new-def unchanged by blast
then show ?case using IntegerBelowNode

```

```

    using rep.IntegerBelowNode by presburger
next
  case (IntegerEqualsNode n x y xe ye)
  then have kind: kind g n = IntegerEqualsNode x y
    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{x, y\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using IntegerEqualsNode
    using rep.IntegerEqualsNode by presburger
next
  case (IntegerLessThanNode n x y xe ye)
  then have kind: kind g n = IntegerLessThanNode x y
    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{x, y\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using IntegerLessThanNode
    using rep.IntegerLessThanNode by presburger
next
  case (NarrowNode n inputBits resultBits x xe)
  then have kind: kind g n = NarrowNode inputBits resultBits x
    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{x\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $x \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $x \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using NarrowNode
    using rep.NarrowNode by presburger
next
  case (SignExtendNode n inputBits resultBits x xe)
  then have kind: kind g n = SignExtendNode inputBits resultBits x

```

```

    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{x\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $x \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $x \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using SignExtendNode
    using rep.SignExtendNode by presburger
next
case (ZeroExtendNode n inputBits resultBits x xe)
  then have kind:  $\text{kind } g \ n = \text{ZeroExtendNode inputBits resultBits } x$ 
    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{x\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $x \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $x \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using ZeroExtendNode
    using rep.ZeroExtendNode by presburger
next
case (LeafNode n s)
  then show ?case
    by (metis no-encoding rep.LeanNode)
next
case (RefNode n n' e)
  then have kind:  $\text{kind } g \ n = \text{RefNode } n'$ 
    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{n'\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $n' \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $n' \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case
    using RefNode
    using rep.RefNode by presburger
qed
qed

```

```

lemma not-in-no-rep:
   $n \notin \text{ids } g \implies \forall e. \neg(g \vdash n \simeq e)$ 
  using eval-contains-id by blast

lemma unary-inputs:
  assumes  $\text{kind } g \ n = \text{unary-node } op \ x$ 
  shows  $\text{inputs } g \ n = \{x\}$ 
  using assms by (cases op; auto)

lemma unary-succ:
  assumes  $\text{kind } g \ n = \text{unary-node } op \ x$ 
  shows  $\text{succ } g \ n = \{\}$ 
  using assms by (cases op; auto)

lemma binary-inputs:
  assumes  $\text{kind } g \ n = \text{bin-node } op \ x \ y$ 
  shows  $\text{inputs } g \ n = \{x, y\}$ 
  using assms by (cases op; auto)

lemma binary-succ:
  assumes  $\text{kind } g \ n = \text{bin-node } op \ x \ y$ 
  shows  $\text{succ } g \ n = \{\}$ 
  using assms by (cases op; auto)

lemma unrep-contains:
  assumes  $g \oplus e \rightsquigarrow (g', n)$ 
  shows  $n \in \text{ids } g'$ 
  using assms
  using not-in-no-rep term-graph-reconstruction by blast

lemma unrep-preserves-contains:
  assumes  $n \in \text{ids } g$ 
  assumes  $g \oplus e \rightsquigarrow (g', n')$ 
  shows  $n \in \text{ids } g'$ 
  using assms
  by (meson subsetD unrep-ids-subset)

lemma unrep-preserves-closure:
  assumes wf-closed  $g$ 
  assumes  $g \oplus e \rightsquigarrow (g', n)$ 
  shows wf-closed  $g'$ 
  using assms(2,1) unfolding wf-closed-def
  proof (induction  $g \ e \ (g', n)$  arbitrary:  $g' \ n$ )
    case (ConstantNodeSame  $g \ c \ n$ )
    then show ?case
      by blast

```

```

next
  case (ConstantNodeNew g c n g')
  then have dom:  $ids\ g' = ids\ g \cup \{n\}$ 
    by (meson IRNode.distinct(683) add-node-ids-subset ids-add-update)
  have k:  $kind\ g'\ n = ConstantNode\ c$ 
    using ConstantNodeNew add-node-lookup by simp
  then have inp:  $\{\} = inputs\ g'\ n$ 
    unfolding inputs.simps by simp
  from k have suc:  $\{\} = succ\ g'\ n$ 
    unfolding succ.simps by simp
  have  $inputs\ g'\ n \subseteq ids\ g' \wedge succ\ g'\ n \subseteq ids\ g' \wedge kind\ g'\ n \neq NoNode$ 
    using inp suc k by simp
  then show ?case
    by (smt (verit) ConstantNodeNew.hyps(3) ConstantNodeNew.premis Un-insert-right
    add-changed changeonly.elims(2) dom inputs.simps insert-iff singleton-iff subset-insertI
    subset-trans succ.simps sup-bot-right)
  next
    case (ParameterNodeSame g i s n)
    then show ?case by blast
  next
    case (ParameterNodeNew g i s n g')
    then have dom:  $ids\ g' = ids\ g \cup \{n\}$ 
      using IRNode.distinct(2447) fresh-ids ids-add-update by presburger
    have k:  $kind\ g'\ n = ParameterNode\ i$ 
      using ParameterNodeNew add-node-lookup by simp
    then have inp:  $\{\} = inputs\ g'\ n$ 
      unfolding inputs.simps by simp
    from k have suc:  $\{\} = succ\ g'\ n$ 
      unfolding succ.simps by simp
    have  $inputs\ g'\ n \subseteq ids\ g' \wedge succ\ g'\ n \subseteq ids\ g' \wedge kind\ g'\ n \neq NoNode$ 
      using k inp suc by simp
    then show ?case
      by (smt (verit) ParameterNodeNew.hyps(3) ParameterNodeNew.premis Un-insert-right
      add-node-as-set dom inputs.elims insertE not-excluded-keep-type order-trans single-
      tonD subset-insertI succ.elims sup-bot-right)
    next
      case (ConditionalNodeSame g ce g2 c te g3 t fe g4 f s' n)
      then show ?case by blast
    next
      case (ConditionalNodeNew g ce g2 c te g3 t fe g4 f s' n g')
      then have dom:  $ids\ g' = ids\ g4 \cup \{n\}$ 
        by (meson IRNode.distinct(591) add-node-ids-subset ids-add-update)
      have k:  $kind\ g'\ n = ConditionalNode\ c\ t\ f$ 
        using ConditionalNodeNew add-node-lookup by simp
      then have inp:  $\{c, t, f\} = inputs\ g'\ n$ 
        unfolding inputs.simps by simp
      from k have suc:  $\{\} = succ\ g'\ n$ 
        unfolding succ.simps by simp
      have  $inputs\ g'\ n \subseteq ids\ g' \wedge succ\ g'\ n \subseteq ids\ g' \wedge kind\ g'\ n \neq NoNode$ 

```



```

    using k inp suc unrep-contains unrep-preserves-contains
    using ConditionalNodeNew(1,3,5,10)
    by (smt (verit) IRNode.simps(643) Un-insert-right bot.extremum dom in-
sert-absorb insert-subset subset-insertI sup-bot-right)
    then show ?case using dom
    by (smt (z3) ConditionalNodeNew.hyps(10) ConditionalNodeNew.hyps(2) Con-
ditionalNodeNew.hyps(4) ConditionalNodeNew.hyps(6) ConditionalNodeNew.prem
Diff-eq-empty-iff Diff-iff Un-insert-right Un-upper1 add-node-def inputs.simps in-
sertE replace-node-def replace-node-unchanged subset-trans succ.simps sup-bot-right)
  next
    case (UnaryNodeSame g xe g2 x s' op n)
    then show ?case by blast
  next
    case (UnaryNodeNew g xe g2 x s' op n g')
    then have dom:  $ids\ g' = ids\ g2 \cup \{n\}$ 
    by (metis add-node-ids-subset add-node-lookup ids-add-update ids-some un-
rep.UnaryNodeNew unrep-contains)
    have k:  $kind\ g'\ n = unary-node\ op\ x$ 
    using UnaryNodeNew add-node-lookup
    by (metis fresh-ids ids-some)
    then have inp:  $\{x\} = inputs\ g'\ n$ 
    using unary-inputs by simp
    from k have suc:  $\{\} = succ\ g'\ n$ 
    using unary-succ by simp
    have  $inputs\ g'\ n \subseteq ids\ g' \wedge succ\ g'\ n \subseteq ids\ g' \wedge kind\ g'\ n \neq NoNode$ 
    using k inp suc unrep-contains unrep-preserves-contains
    using UnaryNodeNew(1,6)
    by (metis Un-upper1 dom empty-subsetI ids-some insert-not-empty in-
sert-subsetI not-in-g-inputs subset-iff)
    then show ?case
    by (smt (verit) Un-insert-right UnaryNodeNew.hyps(2) UnaryNodeNew.hyps(6)
UnaryNodeNew.prem
add-changed changeonly.elims(2) dom inputs.simps insert-iff
singleton-iff subset-insertI subset-trans succ.simps sup-bot-right)
  next
    case (BinaryNodeSame g xe g2 x ye g3 y s' op n)
    then show ?case by blast
  next
    case (BinaryNodeNew g xe g2 x ye g3 y s' op n g')
    then have dom:  $ids\ g' = ids\ g3 \cup \{n\}$ 
    by (metis binary-inputs fresh-ids ids-add-update ids-some insert-not-empty
not-in-g-inputs)
    have k:  $kind\ g'\ n = bin-node\ op\ x\ y$ 
    using BinaryNodeNew add-node-lookup
    by (metis fresh-ids ids-some)
    then have inp:  $\{x, y\} = inputs\ g'\ n$ 
    using binary-inputs by simp
    from k have suc:  $\{\} = succ\ g'\ n$ 
    using binary-succ by simp
    have  $inputs\ g'\ n \subseteq ids\ g' \wedge succ\ g'\ n \subseteq ids\ g' \wedge kind\ g'\ n \neq NoNode$ 

```

```

    using k inp suc unrep-contains unrep-preserves-contains
    using BinaryNodeNew(1,3,6)
    by (metis Un-upper1 dom empty-subsetI ids-some insert-not-empty in-
sert-subsetI not-in-g-inputs subset-iff)
    then show ?case using dom BinaryNodeNew
    by (smt (verit, del-Insts) Diff-eq-empty-iff Diff-iff Un-insert-right Un-upper1
add-node-def inputs.simps insertE replace-node-def replace-node-unchanged subset-trans
succ.simps sup-bot-right)
  next
  case (AllLeafNodes g n s)
  then show ?case
  by blast
qed

```

inductive-cases *ConstUnrepE*: $g \oplus (\text{ConstantExpr } x) \rightsquigarrow (g', n)$

definition *constant-value* **where**

constant-value = (IntVal 32 0)

definition *bad-graph* **where**

```

bad-graph = irgraph [
  (0, AbsNode 1, constantAsStamp constant-value),
  (1, RefNode 2, constantAsStamp constant-value),
  (2, ConstantNode constant-value, constantAsStamp constant-value)
]

```

experiment begin

lemma

```

  assumes maximal-sharing g
  assumes wf-closed g
  assumes kind g y = AbsNode y'
  assumes kind g y' = RefNode y''
  assumes kind g y'' = ConstantNode v
  assumes stamp g y'' = constantAsStamp v
  assumes  $g \oplus (\text{UnaryExpr UnaryAbs } (\text{ConstantExpr } v)) \rightsquigarrow (g', n)$  (is  $g \oplus ?e \rightsquigarrow$ 
 $(g', n)$ )
  shows  $\neg(\text{maximal-sharing } g')$ 
  using assms(3,2,1)
proof –
  have  $y'' \in \text{ids } g$ 
  using assms(5) by simp
  then have List.member (sorted-list-of-set (ids g))  $y''$ 
  by (metis member-def unwrap-sorted)
  then have find  $(\lambda i. \text{kind } g i = \text{ConstantNode } v \wedge \text{stamp } g i = \text{constantAsStamp } v)$ 
(sorted-list-of-set (ids g)) = Some  $y''$ 
  using assms(5,6) find-Some-iff sorry
  then have  $g \oplus \text{ConstantExpr } v \rightsquigarrow (g, y'')$ 
  using assms(5) ConstUnrepE sorry
  then show ?thesis sorry
qed

```

end

lemma *conditional-rep-kind*:

assumes $g \vdash n \simeq \text{ConditionalExpr } ce \ te \ fe$
assumes $g \vdash c \simeq ce$
assumes $g \vdash t \simeq te$
assumes $g \vdash f \simeq fe$
assumes $\neg(\exists n'. \text{kind } g \ n = \text{RefNode } n')$
shows $\text{kind } g \ n = \text{ConditionalNode } c \ t \ f$
using *assms* **apply** (*induction* n *ConditionalExpr* $ce \ te \ fe$ *rule*: *rep.induct*) **defer**
apply *meson* **using** *repDet* **sorry**

lemma *unary-rep-kind*:

assumes $g \vdash n \simeq \text{UnaryExpr } op \ xe$
assumes $g \vdash x \simeq xe$
assumes $\neg(\exists n'. \text{kind } g \ n = \text{RefNode } n')$
shows $\text{kind } g \ n = \text{unary-node } op \ x$
using *assms* **apply** (*cases* op) **using** *AbsNodeE* **sorry**

lemma *binary-rep-kind*:

assumes $g \vdash n \simeq \text{BinaryExpr } op \ xe \ ye$
assumes $g \vdash x \simeq xe$
assumes $g \vdash y \simeq ye$
assumes $\neg(\exists n'. \text{kind } g \ n = \text{RefNode } n')$
shows $\text{kind } g \ n = \text{bin-node } op \ x \ y$
using *assms* **sorry**

theorem *unrep-maximal-sharing*:

assumes *maximal-sharing* g
assumes *wf-closed* g
assumes $g \oplus e \rightsquigarrow (g', n)$
shows *maximal-sharing* g'
using *assms*(3,2,1)
proof (*induction* $g \ e \ (g', n)$ *arbitrary*: $g' \ n$)
 case (*ConstantNodeSame* $g \ c \ n$)
 then show *?case* **by** *blast*
next
 case (*ConstantNodeNew* $g \ c \ n \ g'$)
 then have $\text{kind } g' \ n = \text{ConstantNode } c$
 using *find-new-kind* **by** *blast*
 then have *repn*: $g' \vdash n \simeq \text{ConstantExpr } c$
 using *rep.ConstantNode* **by** *simp*
 from *ConstantNodeNew* **have** *real-node*: $\neg(\text{is-RefNode } (\text{ConstantNode } c)) \wedge$
ConstantNode $c \neq \text{NoNode}$
 by *simp*
 then have *dom*: $\text{true-ids } g' = \text{true-ids } g \cup \{n\}$
 using *ConstantNodeNew.hyps*(2) *ConstantNodeNew.hyps*(3) *fresh-ids*
 by (*meson* *true-ids-add-update*)
 have *new*: $n \notin \text{ids } g$

```

    using fresh-ids
    using ConstantNodeNew.hyps(2) by blast
    obtain new where new = true-ids g' - true-ids g
    by simp
    then have new-def: new = {n}
    by (metis (no-types, lifting) DiffE Diff-cancel IRGraph.true-ids-def Un-insert-right
    dom insert-Diff-if new sup-bot-right)
    then have unchanged: (new  $\subseteq$  as-set g') = as-set g
    using ConstantNodeNew(3) new add-node-as-set-eq
    by presburger
    then have kind-eq:  $\forall n'. n' \notin \text{new} \longrightarrow \text{kind } g \ n' = \text{kind } g' \ n'$ 
    by (metis ConstantNodeNew.hyps(3)  $\langle \text{new} = \{n\} \rangle$  add-node-as-set dual-order.eq-iff
    not-excluded-keep-type not-in-g)
    from unchanged have stamp-eq:  $\forall n' \in \text{ids } g. n' \notin \text{new} \longrightarrow \text{stamp } g \ n' =$ 
    stamp g' n'
    using not-excluded-keep-type new-def new
    by (metis ConstantNodeNew.hyps(3) add-node-as-set)
    show ?case unfolding maximal-sharing apply (rule allI; rule allI; rule impI)
    using ConstantNodeNew(5) unfolding maximal-sharing apply auto
    proof -
    fix n1 n2 e
    assume 1:  $\forall n_1 \ n_2.$ 
     $n_1 \in \text{true-ids } g \wedge n_2 \in \text{true-ids } g \longrightarrow$ 
     $(\exists e. (g \vdash n_1 \simeq e) \wedge (g \vdash n_2 \simeq e) \wedge \text{stamp } g \ n_1 = \text{stamp } g \ n_2) \longrightarrow n_1 = n_2$ 
    assume n1  $\in \text{true-ids } g'$ 
    assume n2  $\in \text{true-ids } g'$ 
    show  $g' \vdash n_1 \simeq e \implies g' \vdash n_2 \simeq e \implies \text{stamp } g' \ n_1 = \text{stamp } g' \ n_2 \implies n_1 =$ 
    n2
    proof (cases n1  $\in \text{true-ids } g$ )
    case n1: True
    then show  $g' \vdash n_1 \simeq e \implies g' \vdash n_2 \simeq e \implies \text{stamp } g' \ n_1 = \text{stamp } g' \ n_2$ 
 $\implies n_1 = n_2$ 
    proof (cases n2  $\in \text{true-ids } g$ )
    case n2: True
    assume n1rep':  $g' \vdash n_1 \simeq e$ 
    assume n2rep':  $g' \vdash n_2 \simeq e$ 
    assume stmp: stamp g' n1 = stamp g' n2
    have n1rep:  $g \vdash n_1 \simeq e$ 
    using n1rep' kind-eq stamp-eq new-def add-preserves-rep
    using ConstantNodeNew.prem(1) IRGraph.true-ids-def n1 unchanged
    by auto
    have n2rep:  $g \vdash n_2 \simeq e$ 
    using n2rep' kind-eq stamp-eq new-def add-preserves-rep
    using ConstantNodeNew.prem(1) IRGraph.true-ids-def n2 unchanged
    by auto
    have stamp g n1 = stamp g n2
    by (metis ConstantNodeNew.hyps(3) stmp fresh-node-subset n1rep n2rep
    new subset-stamp)
    then show ?thesis using 1

```

```

    using n1 n2
    using n1rep n2rep by blast
next
case n2: False
assume n1rep':  $g' \vdash n_1 \simeq e$ 
assume n2rep':  $g' \vdash n_2 \simeq e$ 
assume stmp:  $\text{stamp } g' n_1 = \text{stamp } g' n_2$ 
have n2-def:  $n_2 = n$ 
  using  $\langle n_2 \in \text{true-ids } g' \rangle \text{ dom } n2$  by auto
have n1rep:  $g \vdash n_1 \simeq \text{ConstantExpr } c$ 
  by (metis (no-types, lifting) ConstantNodeNew.prem(1) DiffE IR-
    Graph.true-ids-def add-preserves-rep n1 n1rep' n2-def n2rep' repDet repn unchanged)
then have n1in:  $n_1 \in \text{ids } g$ 
  using no-encoding by metis
have k:  $\text{kind } g n_1 = \text{ConstantNode } c$ 
  using TreeToGraphThms.true-ids-def n1 n1rep by force
have s:  $\text{stamp } g n_1 = \text{constantAsStamp } c$ 
by (metis ConstantNodeNew.hyps(3) real-node n2-def stmp find-new-stamp
  fresh-node-subset n1rep new subset-stamp)
from k s show ?thesis
  using find-none ConstantNodeNew.hyps(1) n1in by blast
qed
next
case n1: False
then show  $g' \vdash n_1 \simeq e \implies g' \vdash n_2 \simeq e \implies \text{stamp } g' n_1 = \text{stamp } g' n_2$ 
 $\implies n_1 = n_2$ 
proof (cases  $n_2 \in \text{true-ids } g$ )
case n2: True
assume n1rep':  $g' \vdash n_1 \simeq e$ 
assume n2rep':  $g' \vdash n_2 \simeq e$ 
assume stmp:  $\text{stamp } g' n_1 = \text{stamp } g' n_2$ 
have n1-def:  $n_1 = n$ 
  using  $\langle n_1 \in \text{true-ids } g' \rangle \text{ dom } n1$  by auto
have n2in:  $n_2 \in \text{ids } g$ 
  using IRGraph.true-ids-def n2 by auto
have k:  $\text{kind } g n_2 = \text{ConstantNode } c$ 
  by (metis (mono-tags, lifting) ConstantNodeE ConstantNodeNew.prem(1)
    DiffE IRGraph.true-ids-def add-preserves-rep mem-Collect-eq n1-def n1rep' n2 n2rep'
    repDet repn unchanged)
have s:  $\text{stamp } g n_2 = \text{constantAsStamp } c$ 
  by (metis ConstantNodeNew.hyps(3) TreeToGraphThms.new-def
    add-node-lookup n1-def n2in real-node stamp-eq stmp unchanged)
from k s show ?thesis
  using find-none ConstantNodeNew.hyps(1) n2in by blast
next
case n2: False
assume n1rep':  $g' \vdash n_1 \simeq e$ 
assume n2rep':  $g' \vdash n_2 \simeq e$ 
assume stmp:  $\text{stamp } g' n_1 = \text{stamp } g' n_2$ 

```

```

    have  $n_1 = n \wedge n_2 = n$ 
      using  $\langle n_1 \in \text{true-ids } g' \rangle \text{ dom } n1$ 
      using  $\langle n_2 \in \text{true-ids } g' \rangle n2$  by blast
    then show ?thesis
      by simp
  qed
qed
qed
next
  case (ParameterNodeSame  $g \ i \ s \ n$ )
  then show ?case by blast
next
  case (ParameterNodeNew  $g \ i \ s \ n \ g'$ )
  then have  $k: \text{kind } g' \ n = \text{ParameterNode } i$ 
    using find-new-kind by blast
  have  $\text{stamp } g' \ n = s$ 
    using ParameterNodeNew.hyps(3) find-new-stamp by blast
  then have  $\text{repn}: g' \vdash n \simeq \text{ParameterExpr } i \ s$ 
    using rep.ParameterNode  $k$  by simp
  from ConstantNodeNew have  $\neg(\text{is-RefNode } (\text{ParameterNode } i)) \wedge \text{ParameterNode } i \neq \text{NoNode}$ 
    by simp
  then have  $\text{dom}: \text{true-ids } g' = \text{true-ids } g \cup \{n\}$ 
    using ParameterNodeNew.hyps(2) ParameterNodeNew.hyps(3) fresh-ids
    by (meson true-ids-add-update)
  have  $\text{new}: n \notin \text{ids } g$ 
    using fresh-ids
    using ParameterNodeNew.hyps(2) by blast
  obtain new where  $\text{new} = \text{true-ids } g' - \text{true-ids } g$ 
    by simp
  then have new-def:  $\text{new} = \{n\}$ 
    by (metis (no-types, lifting) DiffE Diff-cancel IRGraph.true-ids-def Un-insert-right
    dom insert-Diff-if new sup-bot-right)
  then have unchanged:  $(\text{new} \sqsubseteq \text{as-set } g') = \text{as-set } g$ 
    using ParameterNodeNew(3) new add-node-as-set-eq
    by presburger
  then have kind-eq:  $\forall n'. n' \notin \text{new} \longrightarrow \text{kind } g \ n' = \text{kind } g' \ n'$ 
    by (metis ParameterNodeNew.hyps(3)  $\langle \text{new} = \{n\} \rangle$  add-node-as-set dual-order.eq-iff
    not-excluded-keep-type not-in-g)
  from unchanged have stamp-eq:  $\forall n' \in \text{ids } g. n' \notin \text{new} \longrightarrow \text{stamp } g \ n' = \text{stamp } g' \ n'$ 
    using not-excluded-keep-type new-def new
    by (metis ParameterNodeNew.hyps(3) add-node-as-set)
  show ?case unfolding maximal-sharing apply (rule allI; rule allI; rule impI)
    using ParameterNodeNew(5) unfolding maximal-sharing apply auto
  proof -
    fix  $n_1 \ n_2 \ e$ 
    assume 1:  $\forall n_1 \ n_2. n_1 \in \text{true-ids } g \wedge n_2 \in \text{true-ids } g \longrightarrow$ 

```

```

      (∃ e. (g ⊢ n₁ ≃ e) ∧ (g ⊢ n₂ ≃ e) ∧ stamp g n₁ = stamp g n₂) ⟶ n₁ = n₂
assume n₁ ∈ true-ids g'
assume n₂ ∈ true-ids g'
show g' ⊢ n₁ ≃ e ⟹ g' ⊢ n₂ ≃ e ⟹ stamp g' n₁ = stamp g' n₂ ⟹ n₁ =
n₂
proof (cases n₁ ∈ true-ids g)
  case n1: True
    then show g' ⊢ n₁ ≃ e ⟹ g' ⊢ n₂ ≃ e ⟹ stamp g' n₁ = stamp g' n₂
⟹ n₁ = n₂
  proof (cases n₂ ∈ true-ids g)
    case n2: True
      assume n1rep': g' ⊢ n₁ ≃ e
      assume n2rep': g' ⊢ n₂ ≃ e
      assume stamp g' n₁ = stamp g' n₂
      have n1rep: g ⊢ n₁ ≃ e
        using n1rep' kind-eq stamp-eq new-def add-preserves-rep
        using ParameterNodeNew.premis(1) IRGraph.true-ids-def n1 unchanged
by auto
      have n2rep: g ⊢ n₂ ≃ e
        using n2rep' kind-eq stamp-eq new-def add-preserves-rep
        using ParameterNodeNew.premis(1) IRGraph.true-ids-def n2 unchanged
by auto
      have stamp g n₁ = stamp g n₂
        by (metis ParameterNodeNew.hyps(3) ⟨stamp g' n₁ = stamp g' n₂⟩
fresh-node-subset n1rep n2rep new subset-stamp)
      then show ?thesis using 1
        using n1 n2
        using n1rep n2rep by blast
    next
      case n2: False
      assume n1rep': g' ⊢ n₁ ≃ e
      assume n2rep': g' ⊢ n₂ ≃ e
      assume stamp g' n₁ = stamp g' n₂
      have n₂ = n
        using ⟨n₂ ∈ true-ids g'⟩ dom n2 by auto
      then have ne: n₂ ∉ ids g
        using new n2 by blast
      have n1rep: g ⊢ n₁ ≃ e
        using n1rep' kind-eq stamp-eq new-def add-preserves-rep
        using ParameterNodeNew.premis(1) IRGraph.true-ids-def n1 unchanged
by auto
      have n2rep: g ⊢ n₂ ≃ e
        using n2rep' kind-eq stamp-eq new-def add-preserves-rep
        using ParameterNodeNew.premis(1) IRGraph.true-ids-def unchanged
        by (metis (no-types, lifting) IRNode.disc(2703) ParameterNodeE
ParameterNodeNew.hyps(1) TreeToGraphThms.true-ids-def ⟨n₂ = n⟩ find-none
mem-Collect-eq n1 n1rep' repDet repn)
      then show ?thesis
        using n2rep not-in-no-rep ne by blast

```

```

    qed
  next
    case n1: False
    then show  $g' \vdash n_1 \simeq e \implies g' \vdash n_2 \simeq e \implies \text{stamp } g' n_1 = \text{stamp } g' n_2$ 
 $\implies n_1 = n_2$ 
    proof (cases  $n_2 \in \text{true-ids } g$ )
    case n2: True
    assume n1rep':  $g' \vdash n_1 \simeq e$ 
    assume n2rep':  $g' \vdash n_2 \simeq e$ 
    assume stamp  $g' n_1 = \text{stamp } g' n_2$ 
    have  $n_1 = n$ 
    using  $\langle n_1 \in \text{true-ids } g' \rangle \text{ dom } n1$  by auto
    then have ne:  $n_1 \notin \text{ids } g$ 
    using new n2 by blast
    have n1rep:  $g \vdash n_1 \simeq e$ 
    using n1rep' kind-eq stamp-eq new-def add-preserves-rep
    using ParameterNodeNew.premis(1) IRGraph.true-ids-def n1 unchanged
    by (metis (no-types, lifting) IRNode.disc(2703) ParameterNodeE
    ParameterNodeNew.hyps(1) TreeToGraphThms.true-ids-def  $\langle n_1 = n \rangle \text{ find-none}$ 
    mem-Collect-eq n2 n2rep' repDet repn)
    then show ?thesis
    using n1rep not-in-no-rep ne by blast
  next
    case n2: False
    assume n1rep':  $g' \vdash n_1 \simeq e$ 
    assume n2rep':  $g' \vdash n_2 \simeq e$ 
    assume stamp  $g' n_1 = \text{stamp } g' n_2$ 
    have  $n_1 = n \wedge n_2 = n$ 
    using  $\langle n_1 \in \text{true-ids } g' \rangle \text{ dom } n1$ 
    using  $\langle n_2 \in \text{true-ids } g' \rangle n2$  by blast
    then show ?thesis
    by simp
  qed
qed
qed
next
  case (ConditionalNodeSame g ce g2 c te g3 t fe g4 f s' n)
  then show ?case
  using unrep-preserves-closure by blast
next
  case (ConditionalNodeNew g ce g2 c te g3 t fe g4 f s' n g')
  then have k: kind  $g' n = \text{ConditionalNode } c t f$ 
  using find-new-kind by blast
  have stamp  $g' n = s'$ 
  using ConditionalNodeNew.hyps(10) IRNode.distinct(591) find-new-stamp by
blast
  then have repn:  $g' \vdash n \simeq \text{ConditionalExpr } ce te fe$ 
  using rep.ConditionalNode k
  by (metis ConditionalNodeNew.hyps(1) ConditionalNodeNew.hyps(10) Condi-

```


tionalNodeNew.hyps(3) ConditionalNodeNew.hyps(5) ConditionalNodeNew.hyps(9)
fresh-ids fresh-node-subset subset-implies-evals term-graph-reconstruction)
from *ConstantNodeNew* **have** $\neg(\text{is-RefNode } (\text{ConditionalNode } c \ t \ f)) \wedge \text{ConditionalNode } c \ t \ f \neq \text{NoNode}$
by *simp*
then have *dom: true-ids $g' = \text{true-ids } g_4 \cup \{n\}$*
using *ConditionalNodeNew.hyps(10) ConditionalNodeNew.hyps(9) fresh-ids*
true-ids-add-update **by** *presburger*
have *new: $n \notin \text{ids } g$*
using *fresh-ids*
by (*meson ConditionalNodeNew.hyps(1) ConditionalNodeNew.hyps(3) ConditionalNodeNew.hyps(5) ConditionalNodeNew.hyps(9) unrep-preserves-contains*)
obtain *new where $\text{new} = \text{true-ids } g' - \text{true-ids } g_4$*
by *simp*
then have *new-def: $\text{new} = \{n\}$*
using *dom*
by (*metis ConditionalNodeNew.hyps(9) DiffD1 DiffI Diff-cancel Diff-insert*
Un-insert-right boolean-algebra.disj-zero-right fresh-ids insertCI insert-Diff true-ids)
then have *unchanged: $(\text{new} \triangleleft \text{as-set } g^\wedge) = \text{as-set } g_4$*
using *new add-node-as-set-eq*
using *ConditionalNodeNew.hyps(10) ConditionalNodeNew.hyps(9) fresh-ids*
by *presburger*
then have *kind-eq: $\forall n'. n' \notin \text{new} \longrightarrow \text{kind } g_4 \ n' = \text{kind } g' \ n'$*
by (*metis ConditionalNodeNew.hyps(10) add-node-as-set equalityE local.new-def*
not-excluded-keep-type not-in-g)
from *unchanged* **have** *stamp-eq: $\forall n' \in \text{ids } g. n' \notin \text{new} \longrightarrow \text{stamp } g_4 \ n' = \text{stamp } g' \ n'$*
using *not-excluded-keep-type new-def new*
by (*metis ConditionalNodeNew.hyps(1) ConditionalNodeNew.hyps(10) ConditionalNodeNew.hyps(3) ConditionalNodeNew.hyps(5) add-node-as-set unrep-preserves-contains*)
have *max-g4: maximal-sharing g_4*
using *ConditionalNodeNew.hyps(1) ConditionalNodeNew.hyps(2) ConditionalNodeNew.hyps(3) ConditionalNodeNew.hyps(4) ConditionalNodeNew.hyps(6) ConditionalNodeNew.premis(1) ConditionalNodeNew.premis(2) unrep-preserves-closure*
by *blast*
show *?case unfolding maximal-sharing* **apply** (*rule allI; rule allI; rule impI*)
using *max-g4 unfolding maximal-sharing* **apply** *auto*
proof –
fix *$n_1 \ n_2 \ e$*
assume *1: $\forall n_1 \ n_2.$*
 $n_1 \in \text{true-ids } g_4 \wedge n_2 \in \text{true-ids } g_4 \longrightarrow$
 $(\exists e. (g_4 \vdash n_1 \simeq e) \wedge (g_4 \vdash n_2 \simeq e) \wedge \text{stamp } g_4 \ n_1 = \text{stamp } g_4 \ n_2) \longrightarrow$
 $n_1 = n_2$
assume *$n_1 \in \text{true-ids } g'$*
assume *$n_2 \in \text{true-ids } g'$*
show *$g' \vdash n_1 \simeq e \implies g' \vdash n_2 \simeq e \implies \text{stamp } g' \ n_1 = \text{stamp } g' \ n_2 \implies n_1 =$*
 n_2
proof (*cases $n_1 \in \text{true-ids } g_4$*)
case *n1: True*

```

then show  $g' \vdash n_1 \simeq e \implies g' \vdash n_2 \simeq e \implies \text{stamp } g' \ n_1 = \text{stamp } g' \ n_2$ 
 $\implies n_1 = n_2$ 
proof (cases  $n_2 \in \text{true-ids } g_4$ )
  case  $n_2$ : True
    assume  $n1rep'$ :  $g' \vdash n_1 \simeq e$ 
    assume  $n2rep'$ :  $g' \vdash n_2 \simeq e$ 
    assume  $\text{stamp } g' \ n_1 = \text{stamp } g' \ n_2$ 
    have  $n1rep$ :  $g_4 \vdash n_1 \simeq e$ 
      using  $n1rep'$  kind-eq stamp-eq new-def add-preserves-rep
    using ConditionalNodeNew.premis(1) IRGraph.true-ids-def n1 unchanged
      by (metis (mono-tags, lifting) ConditionalNodeNew.hyps(1) ConditionalNodeNew.hyps(3) ConditionalNodeNew.hyps(5) DiffE unrep-preserves-closure)
    have  $n2rep$ :  $g_4 \vdash n_2 \simeq e$ 
      using  $n2rep'$  kind-eq stamp-eq new-def add-preserves-rep
    using ConditionalNodeNew.premis(1) IRGraph.true-ids-def n2 unchanged
      by (metis (no-types, lifting) ConditionalNodeNew.hyps(1) ConditionalNodeNew.hyps(3) ConditionalNodeNew.hyps(5) DiffE unrep-preserves-closure)
    have  $\text{stamp } g_4 \ n_1 = \text{stamp } g_4 \ n_2$ 
      by (metis ConditionalNodeNew.hyps(10) ConditionalNodeNew.hyps(9)
         $\langle \text{stamp } g' \ n_1 = \text{stamp } g' \ n_2 \rangle$  fresh-ids fresh-node-subset n1rep n2rep subset-stamp)
    then show ?thesis using 1
      using  $n1 \ n2$ 
      using  $n1rep \ n2rep$  by blast
  next
    case  $n_2$ : False
      assume  $n1rep'$ :  $g' \vdash n_1 \simeq e$ 
      assume  $n2rep'$ :  $g' \vdash n_2 \simeq e$ 
      assume  $\text{stamp}$ :  $\text{stamp } g' \ n_1 = \text{stamp } g' \ n_2$ 
      have  $n2\text{-def}$ :  $n_2 = n$ 
        using  $\langle n_2 \in \text{true-ids } g' \rangle$  dom n2 by auto
      have  $n1rep$ :  $g_4 \vdash n_1 \simeq \text{ConditionalExpr } ce \ te \ fe$ 
        by (metis (no-types, lifting) ConditionalNodeNew.hyps(1) ConditionalNodeNew.hyps(3) ConditionalNodeNew.hyps(5) ConditionalNodeNew.premis(1) Diff-iff
          IRGraph.true-ids-def add-preserves-rep n1 n1rep' n2-def n2rep' repDet repn unchanged unrep-preserves-closure)
      then have  $n1in$ :  $n_1 \in \text{ids } g_4$ 
        using no-encoding by metis

      have  $\text{rep}$ :  $(g_4 \vdash c \simeq ce) \wedge (g_4 \vdash t \simeq te) \wedge (g_4 \vdash f \simeq fe)$ 
        by (meson ConditionalNodeNew.hyps(1) ConditionalNodeNew.hyps(3) ConditionalNodeNew.hyps(5) subset-implies-evals term-graph-reconstruction)
      have not-ref:  $\neg(\exists n'. \text{kind } g_4 \ n_1 = \text{RefNode } n')$ 
        using TreeToGraphThms.true-ids-def n1 by fastforce
      then have  $\text{kind } g_4 \ n_1 = \text{ConditionalNode } c \ t \ f$ 
        using conditional-rep-kind
      using local.rep n1rep by presburger
      then show ?thesis
        using find-none ConditionalNodeNew.hyps(8) n1in
        by (metis ConditionalNodeNew.hyps(10) ConditionalNodeNew.hyps(9))

```

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  ⟨stamp g' n = s'⟩ fresh-ids fresh-node-subset n1rep n2-def stmp subset-stamp)
  qed
next
  case n1: False
  then show g' ⊢ n1 ≃ e ⇒ g' ⊢ n2 ≃ e ⇒ stamp g' n1 = stamp g' n2
⇒ n1 = n2
  proof (cases n2 ∈ true-ids g4)
  case n2: True
  assume n1rep': g' ⊢ n1 ≃ e
  assume n2rep': g' ⊢ n2 ≃ e
  assume stamp g' n1 = stamp g' n2
  have new-n1: n1 = n
  using ⟨n1 ∈ true-ids g'⟩ dom n1 by auto
  then have ne: n1 ∉ ids g4
  using new n1
  using ConditionalNodeNew.hyps(9) fresh-ids by blast
  have unrep-cond: g4 ⊢ n2 ≃ ConditionalExpr ce te fe
  using n1rep' kind-eq stamp-eq new-def add-preserves-rep
  using ConditionalNodeNew.premis(1) IRGraph.true-ids-def n2 unchanged
  by (metis (no-types, lifting) ConditionalNodeNew.hyps(1) ConditionalNodeNew.hyps(3) ConditionalNodeNew.hyps(5) DiffD1 n2rep' new-n1 repDet repn unrep-preserves-closure)
  have rep: (g4 ⊢ c ≃ ce) ∧ (g4 ⊢ t ≃ te) ∧ (g4 ⊢ f ≃ fe)
  by (meson ConditionalNodeNew.hyps(1) ConditionalNodeNew.hyps(3) ConditionalNodeNew.hyps(5) subset-implies-evals term-graph-reconstruction)
  have not-ref: ¬(∃ n'. kind g4 n2 = RefNode n')
  using TreeToGraphThms.true-ids-def n2 by fastforce
  then have kind g4 n2 = ConditionalNode c t f
  using conditional-rep-kind
  using local.rep unrep-cond by presburger
  then show ?thesis using find-none ConditionalNodeNew.hyps(8)
  by (metis ConditionalNodeNew.hyps(10) ⟨stamp g' n = s'⟩ ⟨stamp g' n1 = stamp g' n2⟩ encodes-contains fresh-node-subset ne new-n1 not-in-g subset-stamp unrep-cond)
  next
  case n2: False
  assume n1rep': g' ⊢ n1 ≃ e
  assume n2rep': g' ⊢ n2 ≃ e
  assume stamp g' n1 = stamp g' n2
  have n1 = n ∧ n2 = n
  using ⟨n1 ∈ true-ids g'⟩ dom n1
  using ⟨n2 ∈ true-ids g'⟩ n2
  by simp
  then show ?thesis
  by simp
  qed
qed
qed
next

```

```

    case (UnaryNodeSame g xe g2 x s' op n)
    then show ?case by blast
next
case (UnaryNodeNew g xe g2 x s' op n g')
then have k: kind g' n = unary-node op x
  using find-new-kind
  by (metis add-node-lookup fresh-ids ids-some)
have stamp g' n = s'
  by (metis UnaryNodeNew.hyps(6) empty-iff find-new-stamp ids-some insertI1
    k not-in-g-inputs unary-inputs)
then have repn: g' ⊢ n ≃ UnaryExpr op xe
  using k
  using UnaryNodeNew.hyps(1) UnaryNodeNew.hyps(3) UnaryNodeNew.hyps(4)
  UnaryNodeNew.hyps(5) UnaryNodeNew.hyps(6) term-graph-reconstruction unrep.UnaryNodeNew
by blast
  from ConstantNodeNew have ¬(is-RefNode (unary-node op x)) ∧ unary-node
    op x ≠ NoNode
  by (cases op; auto)
  then have dom: true-ids g' = true-ids g2 ∪ {n}
  using UnaryNodeNew.hyps(5) UnaryNodeNew.hyps(6) fresh-ids true-ids-add-update
by presburger
  have new: n ∉ ids g
  using fresh-ids
  by (meson UnaryNodeNew.hyps(1) UnaryNodeNew.hyps(5) unrep-preserves-contains)
  obtain new where new = true-ids g' − true-ids g2
  by simp
  then have new-def: new = {n}
  using dom
  by (metis Diff-cancel Diff-iff Un-insert-right UnaryNodeNew.hyps(5) fresh-ids
    insert-Diff-if sup-bot.right-neutral true-ids)
  then have unchanged: (new ⊆ as-set g') = as-set g2
  using new add-node-as-set-eq
  using UnaryNodeNew.hyps(5) UnaryNodeNew.hyps(6) fresh-ids by presburger
  then have kind-eq: ∀ n'. n' ∉ new ⟶ kind g2 n' = kind g' n'
  by (metis UnaryNodeNew.hyps(6) add-node-as-set equalityD1 local.new-def
    not-excluded-keep-type not-in-g)
  from unchanged have stamp-eq: ∀ n' ∈ ids g . n' ∉ new ⟶ stamp g2 n' =
    stamp g' n'
  using not-excluded-keep-type new-def new
  by (metis UnaryNodeNew.hyps(1) UnaryNodeNew.hyps(6) add-node-as-set
    unrep-preserves-contains)
  have max-g2: maximal-sharing g2
  by (simp add: UnaryNodeNew.hyps(2) UnaryNodeNew.prem(1) UnaryNode-
    New.prem(2))
  show ?case unfolding maximal-sharing apply (rule allI; rule allI; rule impI)
  using max-g2 unfolding maximal-sharing apply auto
  proof −
  fix n1 n2 e
  assume 1: ∀ n1 n2.

```

$$\begin{array}{l}
n_1 \in \text{true-ids } g2 \wedge n_2 \in \text{true-ids } g2 \longrightarrow \\
(\exists e. (g2 \vdash n_1 \simeq e) \wedge (g2 \vdash n_2 \simeq e) \wedge \text{stamp } g2 \ n_1 = \text{stamp } g2 \ n_2) \longrightarrow \\
n_1 = n_2 \\
\text{assume } n_1 \in \text{true-ids } g' \\
\text{assume } n_2 \in \text{true-ids } g' \\
\text{show } g' \vdash n_1 \simeq e \implies g' \vdash n_2 \simeq e \implies \text{stamp } g' \ n_1 = \text{stamp } g' \ n_2 \implies n_1 = \\
n_2 \\
\text{proof (cases } n_1 \in \text{true-ids } g2) \\
\text{case } n1: \text{True} \\
\text{then show } g' \vdash n_1 \simeq e \implies g' \vdash n_2 \simeq e \implies \text{stamp } g' \ n_1 = \text{stamp } g' \ n_2 \\
\implies n_1 = n_2 \\
\text{proof (cases } n_2 \in \text{true-ids } g2) \\
\text{case } n2: \text{True} \\
\text{assume } n1rep': g' \vdash n_1 \simeq e \\
\text{assume } n2rep': g' \vdash n_2 \simeq e \\
\text{assume } \text{stamp } g' \ n_1 = \text{stamp } g' \ n_2 \\
\text{have } n1rep: g2 \vdash n_1 \simeq e \\
\text{using } n1rep' \text{ kind-eq stamp-eq new-def add-preserves-rep} \\
\text{using } \text{Diff-iff IRGraph.true-ids-def UnaryNodeNew.hyps(1) UnaryNode-} \\
\text{New.prem(1) } n1 \text{ unchanged unrep-preserves-closure by auto} \\
\text{have } n2rep: g2 \vdash n_2 \simeq e \\
\text{using } n2rep' \text{ kind-eq stamp-eq new-def add-preserves-rep} \\
\text{by (metis (no-types, lifting) Diff-iff IRGraph.true-ids-def UnaryNode-} \\
\text{New.hyps(1) UnaryNodeNew.prem(1) } n2 \text{ unchanged unrep-preserves-closure)} \\
\text{have } \text{stamp } g2 \ n_1 = \text{stamp } g2 \ n_2 \\
\text{by (metis UnaryNodeNew.hyps(5) UnaryNodeNew.hyps(6) } \langle \text{stamp } g' \ n_1 \\
= \text{stamp } g' \ n_2 \rangle \text{ fresh-ids fresh-node-subset n1rep n2rep subset-stamp)} \\
\text{then show ?thesis using 1} \\
\text{using } n1 \ n2 \\
\text{using } n1rep \ n2rep \text{ by blast} \\
\text{next} \\
\text{case } n2: \text{False} \\
\text{assume } n1rep': g' \vdash n_1 \simeq e \\
\text{assume } n2rep': g' \vdash n_2 \simeq e \\
\text{assume } \text{stamp } g' \ n_1 = \text{stamp } g' \ n_2 \\
\text{have } new-n2: n_2 = n \\
\text{using } \langle n_2 \in \text{true-ids } g' \rangle \text{ dom } n2 \text{ by auto} \\
\text{then have } ne: n_2 \notin \text{ids } g2 \\
\text{using } new \ n2 \\
\text{using } \text{UnaryNodeNew.hyps(5) fresh-ids by blast} \\
\text{have } unrep-un: g2 \vdash n_1 \simeq \text{UnaryExpr op } xe \\
\text{using } n1rep' \text{ kind-eq stamp-eq new-def add-preserves-rep} \\
\text{by (metis (no-types, lifting) Diff-iff IRGraph.true-ids-def UnaryNode-} \\
\text{New.hyps(1) UnaryNodeNew.prem(1) } n1 \ n2rep' \text{ new-n2 repDet repn unchanged} \\
\text{unrep-preserves-closure)} \\
\text{have } rep: (g2 \vdash x \simeq xe) \\
\text{using } \text{UnaryNodeNew.hyps(1) term-graph-reconstruction by auto} \\
\text{have } not-ref: \neg(\exists n'. \text{kind } g2 \ n_1 = \text{RefNode } n') \\
\text{using } \text{TreeToGraphThms.true-ids-def } n1 \text{ by force}
\end{array}$$

```

then have kind g2 n1 = unary-node op x
  using unrep-un unary-rep-kind rep by simp

then show ?thesis using find-none UnaryNodeNew.hyps(4)
  by (metis UnaryNodeNew.hyps(6) ⟨stamp g' n = s'⟩ ⟨stamp g' n1 =
stamp g' n2⟩ fresh-node-subset ne new-n2 no-encoding subset-stamp unrep-un)
qed
next
case n1: False
  then show g' ⊢ n1 ≃ e ⇒ g' ⊢ n2 ≃ e ⇒ stamp g' n1 = stamp g' n2
⇒ n1 = n2
  proof (cases n2 ∈ true-ids g2)
    case n2: True
      assume n1rep': g' ⊢ n1 ≃ e
      assume n2rep': g' ⊢ n2 ≃ e
      assume stamp g' n1 = stamp g' n2
      have new-n1: n1 = n
        using ⟨n1 ∈ true-ids g'⟩ dom n1 by auto
      then have ne: n1 ∉ ids g2
        using new n1
      using UnaryNodeNew.hyps(5) fresh-ids by blast
      have unrep-un: g2 ⊢ n2 ≃ UnaryExpr op xe
        using n1rep' kind-eq stamp-eq new-def add-preserves-rep
        by (metis (no-types, lifting) Diff-iff IRGraph.true-ids-def UnaryNode-
New.hyps(1) UnaryNodeNew.prem(1) n2 n2rep' new-n1 repDet repn unchanged
unrep-preserves-closure)
      have rep: (g2 ⊢ x ≃ xe)
        using UnaryNodeNew.hyps(1) term-graph-reconstruction by presburger
      have not-ref: ¬(∃ n'. kind g2 n2 = RefNode n')
        using TreeToGraphThms.true-ids-def n2 by fastforce
      then have kind g2 n2 = unary-node op x
        using unary-rep-kind
        using local.rep unrep-un by presburger
      then show ?thesis using find-none UnaryNodeNew.hyps(4)
        by (metis UnaryNodeNew.hyps(6) ⟨stamp g' n = s'⟩ ⟨stamp g' n1 =
stamp g' n2⟩ fresh-node-subset ne new-n1 no-encoding subset-stamp unrep-un)
    next
    case n2: False
      assume n1rep': g' ⊢ n1 ≃ e
      assume n2rep': g' ⊢ n2 ≃ e
      assume stamp g' n1 = stamp g' n2
      have n1 = n ∧ n2 = n
        using ⟨n1 ∈ true-ids g'⟩ dom n1
        using ⟨n2 ∈ true-ids g'⟩ n2
        by simp
      then show ?thesis
        by simp
    qed
  qed

```

```

qed
next
  case (BinaryNodeSame g xe g2 x ye g3 y s' op n)
  then show ?case
    using unrep-preserves-closure by blast
next
  case (BinaryNodeNew g xe g2 x ye g3 y s' op n g')
  then have k: kind g' n = bin-node op x y
    using find-new-kind
    by (metis add-node-lookup fresh-ids ids-some)
  have stamp g' n = s'
    by (metis BinaryNodeNew.hyps(1) BinaryNodeNew.hyps(3) BinaryNode-
New.hyps(5) BinaryNodeNew.hyps(6) BinaryNodeNew.hyps(7) BinaryNodeNew.hyps(8)
find-new-stamp ids-some k unrep.BinaryNodeNew unrep-contains)
  then have repn: g' ⊢ n ≃ BinaryExpr op xe ye
    using k
    using BinaryNodeNew.hyps(1) BinaryNodeNew.hyps(3) BinaryNodeNew.hyps(5)
BinaryNodeNew.hyps(6) BinaryNodeNew.hyps(7) BinaryNodeNew.hyps(8) term-graph-reconstruction
unrep.BinaryNodeNew by blast
  from BinaryNodeNew have ¬(is-RefNode (bin-node op x y)) ∧ bin-node op x
y ≠ NoNode
    by (cases op; auto)
  then have dom: true-ids g' = true-ids g3 ∪ {n}
    using BinaryNodeNew.hyps(7) BinaryNodeNew.hyps(8) fresh-ids true-ids-add-update
by presburger
  have new: n ∉ ids g
    using fresh-ids
    by (meson BinaryNodeNew.hyps(1) BinaryNodeNew.hyps(3) BinaryNode-
New.hyps(7) unrep-preserves-contains)
  obtain new where new = true-ids g' − true-ids g3
    by simp
  then have new-def: new = {n}
    using dom
    by (metis BinaryNodeNew.hyps(7) Diff-cancel Diff-iff Un-insert-right fresh-ids
insert-Diff-if sup-bot.right-neutral true-ids)
  then have unchanged: (new ⊆ as-set g') = as-set g3
    using new add-node-as-set-eq
    using BinaryNodeNew.hyps(7) BinaryNodeNew.hyps(8) fresh-ids by presburger
  then have kind-eq: ∀ n'. n' ∉ new ⟶ kind g3 n' = kind g' n'
    by (metis BinaryNodeNew.hyps(8) add-node-as-set equalityD1 local.new-def
not-excluded-keep-type not-in-g)
  from unchanged have stamp-eq: ∀ n' ∈ ids g . n' ∉ new ⟶ stamp g3 n' =
stamp g' n'
    using not-excluded-keep-type new-def new
    by (metis BinaryNodeNew.hyps(1) BinaryNodeNew.hyps(3) BinaryNode-
New.hyps(8) add-node-as-set unrep-preserves-contains)
  have max-g3: maximal-sharing g3
    using BinaryNodeNew.hyps(1) BinaryNodeNew.hyps(2) BinaryNodeNew.hyps(4)
BinaryNodeNew.premis(1) BinaryNodeNew.premis(2) unrep-preserves-closure by blast

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```

show ?case unfolding maximal-sharing apply (rule allI; rule allI; rule impI)
using max-g3 unfolding maximal-sharing apply auto
proof –
fix  $n_1\ n_2\ e$ 
assume  $1: \forall n_1\ n_2.$ 
 $n_1 \in \text{true-ids } g3 \wedge n_2 \in \text{true-ids } g3 \longrightarrow$ 
 $(\exists e. (g3 \vdash n_1 \simeq e) \wedge (g3 \vdash n_2 \simeq e) \wedge \text{stamp } g3\ n_1 = \text{stamp } g3\ n_2) \longrightarrow$ 
 $n_1 = n_2$ 
assume  $n_1 \in \text{true-ids } g'$ 
assume  $n_2 \in \text{true-ids } g'$ 
show  $g' \vdash n_1 \simeq e \implies g' \vdash n_2 \simeq e \implies \text{stamp } g'\ n_1 = \text{stamp } g'\ n_2 \implies n_1 =$ 
 $n_2$ 
proof (cases  $n_1 \in \text{true-ids } g3$ )
case n1: True
then show  $g' \vdash n_1 \simeq e \implies g' \vdash n_2 \simeq e \implies \text{stamp } g'\ n_1 = \text{stamp } g'\ n_2$ 
 $\implies n_1 = n_2$ 
proof (cases  $n_2 \in \text{true-ids } g3$ )
case n2: True
assume  $n1rep': g' \vdash n_1 \simeq e$ 
assume  $n2rep': g' \vdash n_2 \simeq e$ 
assume  $\text{stamp } g'\ n_1 = \text{stamp } g'\ n_2$ 
have  $n1rep: g3 \vdash n_1 \simeq e$ 
using n1rep' kind-eq stamp-eq new-def add-preserves-rep
by (metis (no-types, lifting) BinaryNodeNew.hyps(1) BinaryNode-
New.hyps(3) BinaryNodeNew.prem(1) Diff-iff IRGraph.true-ids-def n1 unchanged
unrep-preserves-closure)
have  $n2rep: g3 \vdash n_2 \simeq e$ 
using n2rep' kind-eq stamp-eq new-def add-preserves-rep
by (metis BinaryNodeNew.hyps(1) BinaryNodeNew.hyps(3) BinaryNode-
New.prem(1) DiffE n2 true-ids unchanged unrep-preserves-closure)
have  $\text{stamp } g3\ n_1 = \text{stamp } g3\ n_2$ 
by (metis BinaryNodeNew.hyps(7) BinaryNodeNew.hyps(8) ⟨stamp g' n1
 $= \text{stamp } g'\ n_2⟩$  fresh-ids fresh-node-subset n1rep n2rep subset-stamp)
then show ?thesis using 1
using n1 n2
using n1rep n2rep by blast
next
case n2: False
assume  $n1rep': g' \vdash n_1 \simeq e$ 
assume  $n2rep': g' \vdash n_2 \simeq e$ 
assume  $\text{stamp } g'\ n_1 = \text{stamp } g'\ n_2$ 
have  $\text{new-n2}: n_2 = n$ 
using  $\langle n_2 \in \text{true-ids } g' \rangle \text{ dom } n2$  by auto
then have  $ne: n_2 \notin \text{ids } g3$ 
using new n2
using BinaryNodeNew.hyps(7) fresh-ids by presburger
have unrep-bin:  $g3 \vdash n_1 \simeq \text{BinaryExpr } op\ xe\ ye$ 
using n1rep' kind-eq stamp-eq new-def add-preserves-rep
by (metis BinaryNodeNew.hyps(1) BinaryNodeNew.hyps(3) BinaryNode-

```



```

New.prem(1) DiffE ⟨new = true-ids g' - true-ids g3⟩ encodes-contains ids-some
n1 n2rep' new-n2 repDet repn unchanged unrep-preserves-closure)
  have rep: (g3 ⊢ x ≃ xe) ∧ (g3 ⊢ y ≃ ye)
  by (meson BinaryNodeNew.hyps(1) BinaryNodeNew.hyps(3) term-graph-reconstruction
unrep-contains unrep-unchanged)
  have not-ref: ¬(∃ n'. kind g3 n1 = RefNode n')
  using TreeToGraphThms.true-ids-def n1 by force
  then have kind g3 n1 = bin-node op x y
  using unrep-bin binary-rep-kind rep by simp
  then show ?thesis using find-none BinaryNodeNew.hyps(6)
  by (metis BinaryNodeNew.hyps(8) ⟨stamp g' n = s'⟩ ⟨stamp g' n1 =
stamp g' n2⟩ fresh-node-subset ne new-n2 no-encoding subset-stamp unrep-bin)
qed
next
case n1: False
  then show g' ⊢ n1 ≃ e ⇒ g' ⊢ n2 ≃ e ⇒ stamp g' n1 = stamp g' n2
⇒ n1 = n2
  proof (cases n2 ∈ true-ids g3)
  case n2: True
    assume n1rep': g' ⊢ n1 ≃ e
    assume n2rep': g' ⊢ n2 ≃ e
    assume stamp g' n1 = stamp g' n2
    have new-n1: n1 = n
    using ⟨n1 ∈ true-ids g'⟩ dom n1 by auto
    then have ne: n1 ∉ ids g3
    using new n1
    using BinaryNodeNew.hyps(7) fresh-ids by blast
    have unrep-bin: g3 ⊢ n2 ≃ BinaryExpr op xe ye
    using n1rep' kind-eq stamp-eq new-def add-preserves-rep
    by (metis (mono-tags, lifting) BinaryNodeNew.hyps(1) BinaryN-
odeNew.hyps(3) BinaryNodeNew.prem(1) Diff-iff IRGraph.true-ids-def n2 n2rep'
new-n1 repDet repn unchanged unrep-preserves-closure)
    have rep: (g3 ⊢ x ≃ xe) ∧ (g3 ⊢ y ≃ ye)
    using BinaryNodeNew.hyps(1) BinaryNodeNew.hyps(3) term-graph-reconstruction
unrep-contains unrep-unchanged by blast
    have not-ref: ¬(∃ n'. kind g3 n2 = RefNode n')
    using TreeToGraphThms.true-ids-def n2 by fastforce
    then have kind g3 n2 = bin-node op x y
    using unrep-bin binary-rep-kind rep by simp
    then show ?thesis using find-none BinaryNodeNew.hyps(6)
    by (metis BinaryNodeNew.hyps(8) ⟨stamp g' n = s'⟩ ⟨stamp g' n1 =
stamp g' n2⟩ fresh-node-subset ne new-n1 no-encoding subset-stamp unrep-bin)
  next
  case n2: False
    assume n1rep': g' ⊢ n1 ≃ e
    assume n2rep': g' ⊢ n2 ≃ e
    assume stamp g' n1 = stamp g' n2
    have n1 = n ∧ n2 = n
    using ⟨n1 ∈ true-ids g'⟩ dom n1

```

```

      using ⟨ $n_2 \in \text{true-ids } g'$ ⟩  $n_2$ 
      by simp
    then show ?thesis
      by simp
  qed
qed
qed
next
  case (AllLeafNodes  $g \ n \ s$ )
  then show ?case by blast
qed
end

```

8 Control-flow Semantics

```

theory IRStepObj
  imports
    TreeToGraph
begin

```

8.1 Object Heap

The heap model we introduce maps field references to object instances to runtime values. We use the $H[f][p]$ heap representation. See [\cite{heap-reps-2011}](#). We also introduce the `DynamicHeap` type which allocates new object references sequentially storing the next free object reference as 'Free'.

heapdef

```

type-synonym ('a, 'b) Heap = 'a  $\Rightarrow$  'b  $\Rightarrow$  Value
type-synonym Free = nat
type-synonym ('a, 'b) DynamicHeap = ('a, 'b) Heap  $\times$  Free

fun h-load-field :: 'a  $\Rightarrow$  'b  $\Rightarrow$  ('a, 'b) DynamicHeap  $\Rightarrow$  Value where
  h-load-field f r (h, n) = h f r

fun h-store-field :: 'a  $\Rightarrow$  'b  $\Rightarrow$  Value  $\Rightarrow$  ('a, 'b) DynamicHeap  $\Rightarrow$  ('a, 'b)
  DynamicHeap where
  h-store-field f r v (h, n) = (h(f := ((h f)(r := v))), n)

fun h-new-inst :: ('a, 'b) DynamicHeap  $\Rightarrow$  ('a, 'b) DynamicHeap  $\times$  Value
where
  h-new-inst (h, n) = ((h,n+1), (ObjRef (Some n)))

type-synonym FieldRefHeap = (string, objref) DynamicHeap

```

```

definition new-heap :: ('a, 'b) DynamicHeap where

```

$new_heap = ((\lambda f. \lambda p. \text{UndefVal}), 0)$

8.2 Intraprocedural Semantics

fun $find_index :: 'a \Rightarrow 'a \text{ list} \Rightarrow \text{nat}$ **where**
 $find_index - [] = 0 \mid$
 $find_index\ v\ (x \# xs) = (\text{if } (x=v) \text{ then } 0 \text{ else } find_index\ v\ xs + 1)$

fun $phi_list :: \text{IRGraph} \Rightarrow \text{ID} \Rightarrow \text{ID list}$ **where**
 $phi_list\ g\ n =$
 $(\text{filter } (\lambda x. (is_PhiNode\ (kind\ g\ x)))$
 $(\text{sorted-list-of-set } (usages\ g\ n)))$

fun $input_index :: \text{IRGraph} \Rightarrow \text{ID} \Rightarrow \text{ID} \Rightarrow \text{nat}$ **where**
 $input_index\ g\ n\ n' = find_index\ n' (inputs_of\ (kind\ g\ n))$

fun $phi_inputs :: \text{IRGraph} \Rightarrow \text{nat} \Rightarrow \text{ID list} \Rightarrow \text{ID list}$ **where**
 $phi_inputs\ g\ i\ nodes = (\text{map } (\lambda n. (inputs_of\ (kind\ g\ n))!(i + 1))\ nodes)$

fun $set_phis :: \text{ID list} \Rightarrow \text{Value list} \Rightarrow \text{MapState} \Rightarrow \text{MapState}$ **where**
 $set_phis\ []\ []\ m = m \mid$
 $set_phis\ (n \# xs)\ (v \# vs)\ m = (set_phis\ xs\ vs\ (m(n := v))) \mid$
 $set_phis\ []\ (v \# vs)\ m = m \mid$
 $set_phis\ (x \# xs)\ []\ m = m$

Intraprocedural semantics are given as a small-step semantics.

Within the context of a graph, the configuration triple, (ID, MethodState, Heap), is related to the subsequent configuration.

inductive $step :: \text{IRGraph} \Rightarrow \text{Params} \Rightarrow (\text{ID} \times \text{MapState} \times \text{FieldRefHeap}) \Rightarrow (\text{ID} \times \text{MapState} \times \text{FieldRefHeap}) \Rightarrow \text{bool}$
 $(-, - \vdash - \rightarrow - \ 55)$ **for** $g\ p$ **where**

SequentialNode:

$\llbracket is_sequential_node\ (kind\ g\ nid);$
 $nid' = (successors_of\ (kind\ g\ nid))!0 \rrbracket$
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m, h) \mid$

IfNode:

$\llbracket kind\ g\ nid = (IfNode\ cond\ tb\ fb);$
 $g \vdash cond \simeq condE;$
 $[m, p] \vdash condE \mapsto val;$
 $nid' = (\text{if } val_to_bool\ val \text{ then } tb \text{ else } fb) \rrbracket$
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m, h) \mid$

EndNodes:

$\llbracket is_AbstractEndNode\ (kind\ g\ nid);$
 $merge = any_usage\ g\ nid;$
 $is_AbstractMergeNode\ (kind\ g\ merge);$

$i = \text{find-index } nid \text{ (inputs-of (kind } g \text{ merge))};$
 $phis = (\text{phi-list } g \text{ merge});$
 $inps = (\text{phi-inputs } g \text{ } i \text{ } phis);$
 $g \vdash inps \simeq_L inpsE;$
 $[m, p] \vdash inpsE \mapsto_L vs;$

$m' = \text{set-phis } phis \text{ vs } m$
 $\implies g, p \vdash (nid, m, h) \rightarrow (\text{merge}, m', h) \mid$

NewInstanceNode:

$\llbracket \text{kind } g \text{ } nid = (\text{NewInstanceNode } nid \text{ } f \text{ } obj \text{ } nid') \rrbracket;$
 $(h', \text{ref}) = h\text{-new-inst } h;$
 $m' = m(nid := \text{ref})$
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m', h') \mid$

LoadFieldNode:

$\llbracket \text{kind } g \text{ } nid = (\text{LoadFieldNode } nid \text{ } f \text{ } (\text{Some } obj) \text{ } nid') \rrbracket;$
 $g \vdash obj \simeq objE;$
 $[m, p] \vdash objE \mapsto \text{ObjRef } \text{ref};$
 $h\text{-load-field } f \text{ } \text{ref } h = v;$
 $m' = m(nid := v)$
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m', h) \mid$

SignedDivNode:

$\llbracket \text{kind } g \text{ } nid = (\text{SignedDivNode } nid \text{ } x \text{ } y \text{ } zero \text{ } sb \text{ } nxt) \rrbracket;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye;$
 $[m, p] \vdash xe \mapsto v1;$
 $[m, p] \vdash ye \mapsto v2;$
 $v = (\text{intval-div } v1 \text{ } v2);$
 $m' = m(nid := v)$
 $\implies g, p \vdash (nid, m, h) \rightarrow (nxt, m', h) \mid$

SignedRemNode:

$\llbracket \text{kind } g \text{ } nid = (\text{SignedRemNode } nid \text{ } x \text{ } y \text{ } zero \text{ } sb \text{ } nxt) \rrbracket;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye;$
 $[m, p] \vdash xe \mapsto v1;$
 $[m, p] \vdash ye \mapsto v2;$
 $v = (\text{intval-mod } v1 \text{ } v2);$
 $m' = m(nid := v)$
 $\implies g, p \vdash (nid, m, h) \rightarrow (nxt, m', h) \mid$

StaticLoadFieldNode:

$\llbracket \text{kind } g \text{ } nid = (\text{LoadFieldNode } nid \text{ } f \text{ } \text{None} \text{ } nid') \rrbracket;$
 $h\text{-load-field } f \text{ } \text{None} \text{ } h = v;$
 $m' = m(nid := v)$
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m', h) \mid$

StoreFieldNode:

$$\begin{aligned} & \llbracket \text{kind } g \text{ nid} = (\text{StoreFieldNode } \text{nid } f \text{ newval} - (\text{Some } \text{obj}) \text{ nid}') \rrbracket; \\ & g \vdash \text{newval} \simeq \text{newval}E; \\ & g \vdash \text{obj} \simeq \text{obj}E; \\ & [m, p] \vdash \text{newval}E \mapsto \text{val}; \\ & [m, p] \vdash \text{obj}E \mapsto \text{ObjRef } \text{ref}; \\ & h' = h\text{-store-field } f \text{ ref val } h; \\ & m' = m(\text{nid} := \text{val}) \\ \implies & g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h') \mid \end{aligned}$$

StaticStoreFieldNode:

$$\begin{aligned} & \llbracket \text{kind } g \text{ nid} = (\text{StoreFieldNode } \text{nid } f \text{ newval} - \text{None } \text{nid}') \rrbracket; \\ & g \vdash \text{newval} \simeq \text{newval}E; \\ & [m, p] \vdash \text{newval}E \mapsto \text{val}; \\ & h' = h\text{-store-field } f \text{ None val } h; \\ & m' = m(\text{nid} := \text{val}) \\ \implies & g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h') \end{aligned}$$

code-pred (*modes*: $i \Rightarrow i \Rightarrow i * i * i \Rightarrow o * o * o \Rightarrow \text{bool}$) *step* .

8.3 Interprocedural Semantics

type-synonym *Signature* = *string*

type-synonym *Program* = *Signature* \rightarrow *IRGraph*

inductive *step-top* :: *Program* \Rightarrow (*IRGraph* \times *ID* \times *MapState* \times *Params*) *list* \times *FieldRefHeap* \Rightarrow (*IRGraph* \times *ID* \times *MapState* \times *Params*) *list* \times *FieldRefHeap* \Rightarrow *bool*

($\vdash - \longrightarrow -$ 55)

for *P* **where**

Lift:

$$\begin{aligned} & \llbracket g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h') \rrbracket \\ \implies & P \vdash ((g, \text{nid}, m, p) \# \text{stk}, h) \longrightarrow ((g, \text{nid}', m', p) \# \text{stk}, h') \mid \end{aligned}$$

InvokeNodeStep:

$\llbracket \text{is-Invoke } (\text{kind } g \text{ nid}) \rrbracket;$

$$\begin{aligned} & \text{callTarget} = \text{ir-callTarget } (\text{kind } g \text{ nid}); \\ & \text{kind } g \text{ callTarget} = (\text{MethodCallTargetNode } \text{targetMethod } \text{arguments}); \\ & \text{Some } \text{targetGraph} = P \text{ targetMethod}; \\ & m' = \text{new-map-state}; \\ & g \vdash \text{arguments} \simeq_L \text{args}E; \\ & [m, p] \vdash \text{args}E \mapsto_L p \\ \implies & P \vdash ((g, \text{nid}, m, p) \# \text{stk}, h) \longrightarrow ((\text{targetGraph}, 0, m', p') \# (g, \text{nid}, m, p) \# \text{stk}, h) \end{aligned}$$

|

ReturnNode:

$\llbracket \text{kind } g \text{ nid} = (\text{ReturnNode } (\text{Some } \text{expr}) \text{ -}) \rrbracket$;

$g \vdash \text{expr} \simeq e$;

$[m, p] \vdash e \mapsto v$;

$cm' = cm(\text{cnid} := v)$;

$\text{cnid}' = (\text{successors-of } (\text{kind } cg \text{ cnid}))!0$

$\implies P \vdash ((g, \text{nid}, m, p) \# (cg, \text{cnid}, cm, cp) \# \text{stk}, h) \longrightarrow ((cg, \text{cnid}', cm', cp) \# \text{stk}, h) \mid$

ReturnNodeVoid:

$\llbracket \text{kind } g \text{ nid} = (\text{ReturnNode } \text{None} \text{ -}) \rrbracket$;

$cm' = cm(\text{cnid} := (\text{ObjRef } (\text{Some } (2048))))$;

$\text{cnid}' = (\text{successors-of } (\text{kind } cg \text{ cnid}))!0$

$\implies P \vdash ((g, \text{nid}, m, p) \# (cg, \text{cnid}, cm, cp) \# \text{stk}, h) \longrightarrow ((cg, \text{cnid}', cm', cp) \# \text{stk}, h) \mid$

UnwindNode:

$\llbracket \text{kind } g \text{ nid} = (\text{UnwindNode } \text{exception}) \rrbracket$;

$g \vdash \text{exception} \simeq \text{exceptionE}$;

$[m, p] \vdash \text{exceptionE} \mapsto e$;

$\text{kind } cg \text{ cnid} = (\text{InvokeWithExceptionNode } \text{ - - - - - } \text{exEdge})$;

$cm' = cm(\text{cnid} := e)$

$\implies P \vdash ((g, \text{nid}, m, p) \# (cg, \text{cnid}, cm, cp) \# \text{stk}, h) \longrightarrow ((cg, \text{exEdge}, cm', cp) \# \text{stk}, h)$

code-pred (*modes*: $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$) *step-top* .

8.4 Big-step Execution

type-synonym *Trace* = (*IRGraph* \times *ID* \times *MapState* \times *Params*) *list*

fun *has-return* :: *MapState* \Rightarrow *bool* **where**

has-return *m* = (*m* 0 \neq *UndefVal*)

inductive *exec* :: *Program*

\Rightarrow (*IRGraph* \times *ID* \times *MapState* \times *Params*) *list* \times *FieldRefHeap*

\Rightarrow *Trace*

\Rightarrow (*IRGraph* \times *ID* \times *MapState* \times *Params*) *list* \times *FieldRefHeap*

\Rightarrow *Trace*

\Rightarrow *bool*

(- \vdash - \mid - \longrightarrow * - \mid -)

for *P*

where

$\llbracket P \vdash (((g, \text{nid}, m, p) \# xs), h) \longrightarrow (((g', \text{nid}', m', p') \# ys), h') \rrbracket$;

$\neg(\text{has-return } m')$;

$l' = (l @ [(g, \text{nid}, m, p)])$;

$$\begin{aligned} & \text{exec } P \ ((g', \text{nid}', m', p') \# \text{ys}), h') \ l' \ \text{next-state } l'' \\ & \implies \text{exec } P \ ((g, \text{nid}, m, p) \# \text{xs}), h) \ l \ \text{next-state } l'' \\ & | \\ & \llbracket P \vdash ((g, \text{nid}, m, p) \# \text{xs}), h) \longrightarrow ((g', \text{nid}', m', p') \# \text{ys}), h'); \\ & \quad \text{has-return } m'; \\ & l' = (l \ @ \ [(g, \text{nid}, m, p)]) \\ & \implies \text{exec } P \ ((g, \text{nid}, m, p) \# \text{xs}), h) \ l \ ((g', \text{nid}', m', p') \# \text{ys}), h') \ l' \\ \text{code-pred } (& \text{modes: } i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow o \Rightarrow \text{bool as } \text{Exec}) \ \text{exec} . \end{aligned}$$

inductive *exec-debug* :: *Program*

$$\begin{aligned} & \Rightarrow (\text{IRGraph} \times \text{ID} \times \text{MapState} \times \text{Params}) \ \text{list} \times \text{FieldRefHeap} \\ & \Rightarrow \text{nat} \\ & \Rightarrow (\text{IRGraph} \times \text{ID} \times \text{MapState} \times \text{Params}) \ \text{list} \times \text{FieldRefHeap} \\ & \Rightarrow \text{bool} \\ & (- \vdash \longrightarrow * - * -) \\ \text{where} \\ & \llbracket n > 0; \\ & \quad p \vdash s \longrightarrow s'; \\ & \quad \text{exec-debug } p \ s' \ (n - 1) \ s' \rrbracket \\ & \implies \text{exec-debug } p \ s \ n \ s'' \mid \\ & \llbracket n = 0 \rrbracket \\ & \implies \text{exec-debug } p \ s \ n \ s \\ \text{code-pred } (& \text{modes: } i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}) \ \text{exec-debug} . \end{aligned}$$

8.4.1 Heap Testing

definition *p3* :: *Params* **where**

$$p3 = [\text{IntVal } 32 \ 3]$$

values $\{(prod.fst(prod.snd \ (prod.snd \ (hd \ (prod.fst \ res)))) \ 0$
 $\mid res. (\lambda x. \text{Some } eg2\text{-sq}) \vdash ([(eg2\text{-sq}, 0, \text{new-map-state}, p3), (eg2\text{-sq}, 0, \text{new-map-state}, p3)],$
 $\text{new-heap}) \rightarrow * 2 * \ res\}$

definition *field-sq* :: *string* **where**

$$\text{field-sq} = \text{"sq"}$$

definition *eg3-sq* :: *IRGraph* **where**

$$\begin{aligned} eg3\text{-sq} = & \text{irgraph } [\\ & (0, \text{StartNode } \text{None } 4, \text{VoidStamp}), \\ & (1, \text{ParameterNode } 0, \text{default-stamp}), \\ & (3, \text{MulNode } 1 \ 1, \text{default-stamp}), \\ & (4, \text{StoreFieldNode } 4 \ \text{field-sq } 3 \ \text{None } \text{None } 5, \text{VoidStamp}), \\ & (5, \text{ReturnNode } (\text{Some } 3) \ \text{None}, \text{default-stamp}) \end{aligned}$$

```

]

values {h-load-field field-sq None (prod.snd res)
        | res. (λx. Some eg3-sq) ⊢ [(eg3-sq, 0, new-map-state, p3), (eg3-sq, 0,
new-map-state, p3)], new-heap) →*3* res}

definition eg4-sq :: IRGraph where
  eg4-sq = irgraph [
    (0, StartNode None 4, VoidStamp),
    (1, ParameterNode 0, default-stamp),
    (3, MulNode 1 1, default-stamp),
    (4, NewInstanceNode 4 "obj-class" None 5, ObjectStamp "obj-class" True True
True),
    (5, StoreFieldNode 5 field-sq 3 None (Some 4) 6, VoidStamp),
    (6, ReturnNode (Some 3) None, default-stamp)
  ]

values {h-load-field field-sq (Some 0) (prod.snd res) | res.
        (λx. Some eg4-sq) ⊢ [(eg4-sq, 0, new-map-state, p3), (eg4-sq, 0,
new-map-state, p3)], new-heap) →*3* res}

end

```

8.5 Control-flow Semantics Theorems

```

theory IRStepThms
imports
  IRStepObj
  TreeToGraphThms
begin

```

We prove that within the same graph, a configuration triple will always transition to the same subsequent configuration. Therefore, our step semantics is deterministic.

8.5.1 Control-flow Step is Deterministic

```

theorem stepDet:
  (g, p ⊢ (nid,m,h) → next) ⇒
  (∀ next'. ((g, p ⊢ (nid,m,h) → next') ⟶ next = next'))
proof (induction rule: step.induct)
case (SequentialNode nid next m h)
have notif: ¬(is-IfNode (kind g nid))
using SequentialNode.hyps(1) is-sequential-node.simps
by (metis is-IfNode-def)
have notend: ¬(is-AbstractEndNode (kind g nid))
using SequentialNode.hyps(1) is-sequential-node.simps

```



```

    by (metis is-AbstractEndNode.simps is-EndNode.elims(2) is-LoopEndNode-def)
  have notnew:  $\neg$ (is-NewInstanceNode (kind g nid))
    using SequentialNode.hyps(1) is-sequential-node.simps
    by (metis is-NewInstanceNode-def)
  have notload:  $\neg$ (is-LoadFieldNode (kind g nid))
    using SequentialNode.hyps(1) is-sequential-node.simps
    by (metis is-LoadFieldNode-def)
  have notstore:  $\neg$ (is-StoreFieldNode (kind g nid))
    using SequentialNode.hyps(1) is-sequential-node.simps
    by (metis is-StoreFieldNode-def)
  have notdivrem:  $\neg$ (is-IntegerDivRemNode (kind g nid))
    using SequentialNode.hyps(1) is-sequential-node.simps is-SignedDivNode-def
    is-SignedRemNode-def
    by (metis is-IntegerDivRemNode.simps)
  from notif notend notnew notload notstore notdivrem
  show ?case using SequentialNode.step.cases
    by (smt (z3) IRNode.disc(1028) IRNode.disc(2270) IRNode.discI(31) Pair-inject
    is-sequential-node.simps(18) is-sequential-node.simps(43) is-sequential-node.simps(44))
next
case (IfNode nid cond tb fb m val next h)
then have notseq:  $\neg$ (is-sequential-node (kind g nid))
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: IfNode.hyps(1))
have notend:  $\neg$ (is-AbstractEndNode (kind g nid))
  using is-AbstractEndNode.simps
  by (simp add: IfNode.hyps(1))
have notdivrem:  $\neg$ (is-IntegerDivRemNode (kind g nid))
  using is-AbstractEndNode.simps
  by (simp add: IfNode.hyps(1))
from notseq notend notdivrem show ?case using IfNode repDet evalDet IRN-
ode.distinct IRNode.inject(11) Pair-inject step.simps
  by (smt (z3) IRNode.distinct IRNode.inject(12) Pair-inject step.simps)
next
case (EndNodes nid merge i phis inputs m vs m' h)
have notseq:  $\neg$ (is-sequential-node (kind g nid))
  using EndNodes.hyps(1) is-AbstractEndNode.simps is-sequential-node.simps
  by (metis is-EndNode.elims(2) is-LoopEndNode-def)
have notif:  $\neg$ (is-IfNode (kind g nid))
  using EndNodes.hyps(1) is-IfNode-def is-AbstractEndNode.elims
  by (metis IRNode.distinct-disc(1058) is-EndNode.simps(12))
have notref:  $\neg$ (is-RefNode (kind g nid))
  using EndNodes.hyps(1) is-sequential-node.simps
  using IRNode.disc(1899) IRNode.distinct(1473) is-AbstractEndNode.simps
  is-EndNode.elims(2) is-LoopEndNode-def is-RefNode-def
  by metis
have notnew:  $\neg$ (is-NewInstanceNode (kind g nid))
  using EndNodes.hyps(1) is-AbstractEndNode.simps
  using IRNode.distinct-disc(1442) is-EndNode.simps(29) is-NewInstanceNode-def
  by (metis IRNode.distinct-disc(1901) is-EndNode.simps(32))

```

```

have notload: ¬(is-LoadFieldNode (kind g nid))
  using EndNodes.hyps(1) is-AbstractEndNode.simps
  using is-LoadFieldNode-def
  by (metis IRNode.distinct-disc(1706) is-EndNode.simps(21))
have notstore: ¬(is-StoreFieldNode (kind g nid))
  using EndNodes.hyps(1) is-AbstractEndNode.simps is-StoreFieldNode-def
  by (metis IRNode.distinct-disc(1926) is-EndNode.simps(44))
have notdivrem: ¬(is-IntegerDivRemNode (kind g nid))
  using EndNodes.hyps(1) is-AbstractEndNode.simps is-SignedDivNode-def is-SignedRemNode-def
  using IRNode.distinct-disc(1498) IRNode.distinct-disc(1500) is-IntegerDivRemNode.simps
is-EndNode.simps(36) is-EndNode.simps(37)
  by auto
from notseq notif notref notnew notload notstore notdivrem
show ?case using EndNodes.repAllDet evalAllDet
  by (smt (z3) is-IfNode-def is-LoadFieldNode-def is-NewInstanceNode-def is-RefNode-def
is-StoreFieldNode-def is-SignedDivNode-def is-SignedRemNode-def Pair-inject is-IntegerDivRemNode.elims(3)
step.cases)
next
case (NewInstanceNode nid f obj nxt h' ref h m' m)
then have notseq: ¬(is-sequential-node (kind g nid))
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notend: ¬(is-AbstractEndNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notif: ¬(is-IfNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notref: ¬(is-RefNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notload: ¬(is-LoadFieldNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notstore: ¬(is-StoreFieldNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notdivrem: ¬(is-IntegerDivRemNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
from notseq notend notif notref notload notstore notdivrem
show ?case using NewInstanceNode.step.cases
  by (smt (z3) IRNode.disc(1028) IRNode.disc(2270) IRNode.discI(11) IRN-
ode.distinct(2311) IRNode.distinct(2313) IRNode.inject(31) Pair-inject)
next
case (LoadFieldNode nid f obj nxt m ref h v m')
then have notseq: ¬(is-sequential-node (kind g nid))
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: LoadFieldNode.hyps(1))

```

```

have notend: ¬(is-AbstractEndNode (kind g nid))
  using is-AbstractEndNode.simps
  by (simp add: LoadFieldNode.hyps(1))
have notdivrem: ¬(is-IntegerDivRemNode (kind g nid))
  using is-AbstractEndNode.simps
  by (simp add: LoadFieldNode.hyps(1))
from notseq notend notdivrem
show ?case using LoadFieldNode step.cases repDet evalDet
  by (smt (z3) IRNode.distinct(1051) IRNode.distinct(1721) IRNode.distinct(1739)
IRNode.distinct(1741) IRNode.distinct(1745) IRNode.inject(20) Pair-inject Value.inject(2)
option.distinct(1) option.inject)
next
case (StaticLoadFieldNode nid f nrt h v m' m)
then have notseq: ¬(is-sequential-node (kind g nid))
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: StaticLoadFieldNode.hyps(1))
have notend: ¬(is-AbstractEndNode (kind g nid))
  using is-AbstractEndNode.simps
  by (simp add: StaticLoadFieldNode.hyps(1))
have notdivrem: ¬(is-IntegerDivRemNode (kind g nid))
  by (simp add: StaticLoadFieldNode.hyps(1))
from notseq notend notdivrem
show ?case using StaticLoadFieldNode step.cases
  by (smt (z3) IRNode.distinct(1051) IRNode.distinct(1721) IRNode.distinct(1739)
IRNode.distinct(1741) IRNode.distinct(1745) IRNode.inject(20) Pair-inject option.distinct(1))
next
case (StoreFieldNode nid f newval uu obj nrt m val ref h' h m')
then have notseq: ¬(is-sequential-node (kind g nid))
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: StoreFieldNode.hyps(1))
have notend: ¬(is-AbstractEndNode (kind g nid))
  using is-AbstractEndNode.simps
  by (simp add: StoreFieldNode.hyps(1))
have notdivrem: ¬(is-IntegerDivRemNode (kind g nid))
  by (simp add: StoreFieldNode.hyps(1))
from notseq notend notdivrem
show ?case using StoreFieldNode step.cases repDet evalDet
  by (smt (z3) IRNode.distinct(1097) IRNode.distinct(1745) IRNode.distinct(2317)
IRNode.distinct(2605) IRNode.distinct(2627) IRNode.inject(43) Pair-inject Value.inject(2)
option.distinct(1) option.inject)
next
case (StaticStoreFieldNode nid f newval uv nrt m val h' h m')
then have notseq: ¬(is-sequential-node (kind g nid))
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: StaticStoreFieldNode.hyps(1))
have notend: ¬(is-AbstractEndNode (kind g nid))
  using is-AbstractEndNode.simps
  by (simp add: StaticStoreFieldNode.hyps(1))
have notdivrem: ¬(is-IntegerDivRemNode (kind g nid))

```

```

  by (simp add: StaticStoreFieldNode.hyps(1))
from notseq notend notdivrem
show ?case using StoreFieldNode step.cases repDet evalDet
  by (smt (z3) IRNode.distinct(1097) IRNode.distinct(1745) IRNode.distinct(2317)
IRNode.distinct(2605) IRNode.distinct(2627) IRNode.inject(43) Pair-inject Static-
StoreFieldNode.hyps(1) StaticStoreFieldNode.hyps(2) StaticStoreFieldNode.hyps(3)
StaticStoreFieldNode.hyps(4) StaticStoreFieldNode.hyps(5) option.distinct(1))
next
case (SignedDivNode nid x y zero sb nxt m v1 v2 v m' h)
then have notseq:  $\neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$ 
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: SignedDivNode.hyps(1))
have notend:  $\neg(\text{is-AbstractEndNode } (\text{kind } g \text{ nid}))$ 
  using is-AbstractEndNode.simps
  by (simp add: SignedDivNode.hyps(1))
from notseq notend
show ?case using SignedDivNode step.cases repDet evalDet
  by (smt (z3) IRNode.distinct(1091) IRNode.distinct(1739) IRNode.distinct(2311)
IRNode.distinct(2601) IRNode.distinct(2605) IRNode.inject(40) Pair-inject)
next
case (SignedRemNode nid x y zero sb nxt m v1 v2 v m' h)
then have notseq:  $\neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$ 
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: SignedRemNode.hyps(1))
have notend:  $\neg(\text{is-AbstractEndNode } (\text{kind } g \text{ nid}))$ 
  using is-AbstractEndNode.simps
  by (simp add: SignedRemNode.hyps(1))
from notseq notend
show ?case using SignedRemNode step.cases repDet evalDet
  by (smt (z3) IRNode.distinct(1093) IRNode.distinct(1741) IRNode.distinct(2313)
IRNode.distinct(2601) IRNode.distinct(2627) IRNode.inject(41) Pair-inject)
qed

```

lemma *stepRefNode*:

```

 $\llbracket \text{kind } g \text{ nid} = \text{RefNode nid}' \rrbracket \implies g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m, h)$ 
  using SequentialNode
  by (metis IRNodes.successors-of-RefNode is-sequential-node.simps(7) nth-Cons-0)

```

lemma *IfNodeStepCases*:

```

assumes kind g nid = IfNode cond tb fb
assumes  $g \vdash \text{cond} \simeq \text{condE}$ 
assumes  $[m, p] \vdash \text{condE} \mapsto v$ 
assumes  $g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m, h)$ 
shows  $\text{nid}' \in \{\text{tb}, \text{fb}\}$ 
  using step.IfNode repDet stepDet assms
  by (metis insert-iff old.prod.inject)

```

lemma *IfNodeSeq*:

```

shows kind g nid = IfNode cond tb fb  $\longrightarrow \neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$ 

```

unfolding *is-sequential-node.simps*
using *is-sequential-node.simps(18)* **by** *presburger*

lemma *IfNodeCond*:

assumes *kind g nid = IfNode cond tb fb*
assumes *g, p ⊢ (nid, m, h) → (nid', m, h)*
shows $\exists \text{ condE } v. ((g \vdash \text{cond} \simeq \text{condE}) \wedge ([m, p] \vdash \text{condE} \mapsto v))$
using *assms(2,1)* **by** (*induct (nid,m,h) (nid',m,h)* *rule: step.induct; auto*)

lemma *step-in-ids*:

assumes *g, p ⊢ (nid, m, h) → (nid', m', h')*
shows *nid ∈ ids g*
using *assms* **apply** (*induct (nid, m, h) (nid', m', h')* *rule: step.induct*)
using *is-sequential-node.simps(45)* *not-in-g*
apply *simp*
apply (*metis is-sequential-node.simps(53)*)
using *ids-some*
using *IRNode.distinct(1113)* **apply** *presburger*
using *EndNodes(1)* *is-AbstractEndNode.simps is-EndNode.simps(45)* *ids-some*
apply (*metis IRNode.disc(1218) is-EndNode.simps(52)*)
by *simp+*

end

9 Proof Infrastructure

9.1 Bisimulation

theory *Bisimulation*

imports

Stuttering

begin

inductive *weak-bisimilar* :: *ID ⇒ IRGraph ⇒ IRGraph ⇒ bool*

(*- . - ~ -*) **for** *nid* **where**

$\llbracket \forall P'. (g \text{ m } p \text{ h} \vdash \text{nid} \rightsquigarrow P') \longrightarrow (\exists Q'. (g' \text{ m } p \text{ h} \vdash \text{nid} \rightsquigarrow Q') \wedge P' = Q');$
 $\forall Q'. (g' \text{ m } p \text{ h} \vdash \text{nid} \rightsquigarrow Q') \longrightarrow (\exists P'. (g \text{ m } p \text{ h} \vdash \text{nid} \rightsquigarrow P') \wedge P' = Q') \rrbracket$
 $\implies \text{nid} . g \sim g'$

A strong bisimilution between no-op transitions

inductive *strong-noop-bisimilar* :: *ID ⇒ IRGraph ⇒ IRGraph ⇒ bool*

(*- | - ~ -*) **for** *nid* **where**

$\llbracket \forall P'. (g, p \vdash (\text{nid}, m, h) \rightarrow P') \longrightarrow (\exists Q'. (g', p \vdash (\text{nid}, m, h) \rightarrow Q') \wedge P' = Q');$
 $\forall Q'. (g', p \vdash (\text{nid}, m, h) \rightarrow Q') \longrightarrow (\exists P'. (g, p \vdash (\text{nid}, m, h) \rightarrow P') \wedge P' = Q') \rrbracket$
 $\implies \text{nid} \mid g \sim g'$

lemma *lockstep-strong-bisimulation*:
assumes $g' = \text{replace-node } \textit{nid} \text{ node } g$
assumes $g, p \vdash (\textit{nid}, m, h) \rightarrow (\textit{nid}', m, h)$
assumes $g', p \vdash (\textit{nid}, m, h) \rightarrow (\textit{nid}', m, h)$
shows $\textit{nid} \mid g \sim g'$
using *assms(2) assms(3) stepDet strong-noop-bisimilar.simps* **by** *metis*

lemma *no-step-bisimulation*:
assumes $\forall m p h \textit{nid}' m' h'. \neg(g, p \vdash (\textit{nid}, m, h) \rightarrow (\textit{nid}', m', h'))$
assumes $\forall m p h \textit{nid}' m' h'. \neg(g', p \vdash (\textit{nid}, m, h) \rightarrow (\textit{nid}', m', h'))$
shows $\textit{nid} \mid g \sim g'$
using *assms*
by (*simp add: assms(1) assms(2) strong-noop-bisimilar.intros*)

end

9.2 Graph Rewriting

theory

Rewrites

imports

Stuttering

begin

fun *replace-usages* :: $ID \Rightarrow ID \Rightarrow IRGraph \Rightarrow IRGraph$ **where**
replace-usages $\textit{nid} \textit{nid}' g = \text{replace-node } \textit{nid} \text{ (RefNode } \textit{nid}', \text{ stamp } g \textit{nid}') } g$

lemma *replace-usages-effect*:
assumes $g' = \text{replace-usages } \textit{nid} \textit{nid}' g$
shows $\text{kind } g' \textit{nid} = \text{RefNode } \textit{nid}'$
using *assms replace-node-lookup replace-usages.simps*
by (*metis IRNode.distinct(2755)*)

lemma *replace-usages-changeonly*:
assumes $\textit{nid} \in \textit{ids } g$
assumes $g' = \text{replace-usages } \textit{nid} \textit{nid}' g$
shows *changeonly* $\{\textit{nid}\} g g'$
using *assms unfolding replace-usages.simps*
by (*metis add-changed add-node-def replace-node-def*)

lemma *replace-usages-unchanged*:
assumes $\textit{nid} \in \textit{ids } g$
assumes $g' = \text{replace-usages } \textit{nid} \textit{nid}' g$
shows *unchanged* $(\textit{ids } g - \{\textit{nid}\}) g g'$
using *assms unfolding replace-usages.simps*
using *assms(2) disjoint-change replace-usages-changeonly* **by** *presburger*

```

fun nextNid :: IRGraph ⇒ ID where
  nextNid g = (Max (ids g)) + 1

lemma max-plus-one:
  fixes c :: ID set
  shows  $\llbracket \text{finite } c; c \neq \{\} \rrbracket \implies (\text{Max } c) + 1 \notin c$ 
  by (meson Max-gr-iff less-add-one less-irrefl)

lemma ids-finite:
  finite (ids g)
  by simp

lemma nextNidNotIn:
  ids g ≠ {} ⟶ nextNid g ∉ ids g
  unfolding nextNid.simps
  using ids-finite max-plus-one by blast

fun constantCondition :: bool ⇒ ID ⇒ IRNode ⇒ IRGraph ⇒ IRGraph where
  constantCondition val nid (IfNode cond t f) g =
    replace-node nid (IfNode (nextNid g) t f, stamp g nid)
      (add-node (nextNid g) ((ConstantNode (bool-to-val val)), constantAsStamp
        (bool-to-val val)) g) |
    constantCondition cond nid - g = g

lemma constantConditionTrue:
  assumes kind g ifcond = IfNode cond t f
  assumes g' = constantCondition True ifcond (kind g ifcond) g
  shows g', p ⊢ (ifcond, m, h) → (t, m, h)
proof –
  have ifn:  $\bigwedge c\ t\ f. \text{IfNode } c\ t\ f \neq \text{NoNode}$ 
    by simp
  then have if': kind g' ifcond = IfNode (nextNid g) t f
    using assms(1) assms(2) constantCondition.simps(1) replace-node-lookup
    by presburger
  have truedef: bool-to-val True = (IntVal 32 1)
    by auto
  from ifn have ifcond ≠ (nextNid g)
    by (metis assms(1) emptyE ids-some nextNidNotIn)
  moreover have  $\bigwedge c. \text{ConstantNode } c \neq \text{NoNode}$  by simp
  ultimately have kind g' (nextNid g) = ConstantNode (bool-to-val True)
    using add-changed add-node-def assms(1) assms(2) constantCondition.simps(1)
    not-in-g other-node-unchanged replace-node-def replace-node-lookup singletonD
    by (smt (z3) DiffI add-node-lookup replace-node-unchanged)
  then have c': kind g' (nextNid g) = ConstantNode (IntVal 32 1)
    using truedef by simp
  have valid-value (IntVal 32 1) (constantAsStamp (IntVal 32 1))
    unfolding constantAsStamp.simps valid-value.simps
    using nat-numeral by force

```

then have $[g', m, p] \vdash \text{nextNid } g \mapsto \text{IntVal } 32 \ 1$
using *ConstantExpr ConstantNode Value.distinct(1) $\langle \text{kind } g' (\text{nextNid } g) = \text{ConstantNode } (\text{bool-to-val True}) \rangle \text{ encodeeval-def truedef}$*
by *metis*
from *if' c' show ?thesis using IfNode*
by *(metis (no-types, opaque-lifting) val-to-bool.simps(1) $\langle [g', m, p] \vdash \text{nextNid } g \mapsto \text{IntVal } 32 \ 1 \rangle \text{ encodeeval-def zero-neg-one}$)*
qed

lemma *constantConditionFalse:*

assumes *kind g ifcond = IfNode cond t f*
assumes *$g' = \text{constantCondition False ifcond } (\text{kind } g \text{ ifcond}) \ g$*
shows *$g', p \vdash (\text{ifcond}, m, h) \rightarrow (f, m, h)$*
proof –
have *$\bigwedge c \ t \ f. \text{IfNode } c \ t \ f \neq \text{NoNode}$*
by *simp*
then have *$\text{if}' : \text{kind } g' \text{ ifcond} = \text{IfNode } (\text{nextNid } g) \ t \ f$*
by *(metis assms(1) assms(2) constantCondition.simps(1) replace-node-lookup)*
have *$\text{falsedef} : \text{bool-to-val False} = (\text{IntVal } 32 \ 0)$*
by *auto*
from *ifn have ifcond \neq (nextNid g)*
by *(metis assms(1) equals0D ids-some nextNidNotIn)*
moreover have *$\bigwedge c. \text{ConstantNode } c \neq \text{NoNode}$* **by** *simp*
ultimately have *$\text{kind } g' (\text{nextNid } g) = \text{ConstantNode } (\text{bool-to-val False})$*
by *(smt (z3) add-changed add-node-def assms(1) assms(2) constantCondition.simps(1) not-in-g other-node-unchanged replace-node-def replace-node-lookup singletonD)*
then have *$c' : \text{kind } g' (\text{nextNid } g) = \text{ConstantNode } (\text{IntVal } 32 \ 0)$*
using *falsedef* **by** *simp*
have *$\text{valid-value } (\text{IntVal } 32 \ 0) (\text{constantAsStamp } (\text{IntVal } 32 \ 0))$*
unfolding *constantAsStamp.simps valid-value.simps*
using *nat-numeral* **by** *force*
then have $[g', m, p] \vdash \text{nextNid } g \mapsto \text{IntVal } 32 \ 0$
by *(metis ConstantExpr ConstantNode $\langle \text{kind } g' (\text{nextNid } g) = \text{ConstantNode } (\text{bool-to-val False}) \rangle \text{ encodeeval-def falsedef}$)*
from *if' c' show ?thesis using IfNode*
by *(metis (no-types, opaque-lifting) val-to-bool.simps(1) $\langle [g', m, p] \vdash \text{nextNid } g \mapsto \text{IntVal } 32 \ 0 \rangle \text{ encodeeval-def}$)*
qed

lemma *diff-forall:*

assumes *$\forall n \in \text{ids } g - \{nid\}. \text{cond } n$*
shows *$\forall n. n \in \text{ids } g \wedge n \notin \{nid\} \longrightarrow \text{cond } n$*
by *(meson Diff-iff assms)*

lemma *replace-node-changeonly:*

assumes *$g' = \text{replace-node nid node } g$*
shows *$\text{changeonly } \{nid\} \ g \ g'$*
using *assms replace-node-unchanged*


```

unfolding changeonly.simps using diff-forall
by (metis add-changed add-node-def changeonly.simps replace-node-def)

lemma add-node-changeonly:
  assumes  $g' = \text{add-node } \textit{nid} \textit{ node } g$ 
  shows changeonly  $\{\textit{nid}\} g g'$ 
  by (metis Rep-IRGraph-inverse add-node.rep-eq assms replace-node.rep-eq replace-node-changeonly)

lemma constantConditionNoEffect:
  assumes  $\neg(\textit{is-IfNode } (\textit{kind } g \textit{ nid}))$ 
  shows  $g = \text{constantCondition } b \textit{ nid } (\textit{kind } g \textit{ nid}) g$ 
  using assms apply (cases kind g nid)
  using constantCondition.simps
  apply presburger+
  apply (metis is-IfNode-def)
  using constantCondition.simps
  by presburger+

lemma constantConditionIfNode:
  assumes  $\textit{kind } g \textit{ nid} = \textit{IfNode cond t f}$ 
  shows  $\text{constantCondition } \textit{val } \textit{nid } (\textit{kind } g \textit{ nid}) g =$ 
     $\text{replace-node } \textit{nid } (\textit{IfNode } (\textit{nextNid } g) \textit{ t f, stamp } g \textit{ nid})$ 
     $(\text{add-node } (\textit{nextNid } g) ((\text{ConstantNode } (\textit{bool-to-val } \textit{val})), \text{constantAsStamp}$ 
     $(\textit{bool-to-val } \textit{val}))) g)$ 
  using constantCondition.simps
  by (simp add: assms)

lemma constantCondition-changeonly:
  assumes  $\textit{nid} \in \textit{ids } g$ 
  assumes  $g' = \text{constantCondition } b \textit{ nid } (\textit{kind } g \textit{ nid}) g$ 
  shows changeonly  $\{\textit{nid}\} g g'$ 
proof (cases is-IfNode (kind g nid))
  case True
  have  $\textit{nextNid } g \notin \textit{ids } g$ 
  using nextNidNotIn by (metis emptyE)
  then show ?thesis using assms
  using replace-node-changeonly add-node-changeonly unfolding changeonly.simps
  using True constantCondition.simps(1) is-IfNode-def
  by (metis (no-types, lifting) insert-iff)
next
  case False
  have  $g = g'$ 
  using constantConditionNoEffect
  using False assms(2) by blast
  then show ?thesis by simp
qed

```

```

lemma constantConditionNoIf:
  assumes  $\forall \text{ cond } t f. \text{ kind } g \text{ ifcond} \neq \text{ IfNode cond } t f$ 
  assumes  $g' = \text{constantCondition val ifcond (kind } g \text{ ifcond)} g$ 
  shows  $\exists \text{ nid}' . (g \text{ m } p \text{ h} \vdash \text{ifcond} \rightsquigarrow \text{nid}') \longleftrightarrow (g' \text{ m } p \text{ h} \vdash \text{ifcond} \rightsquigarrow \text{nid}')$ 
proof -
  have  $g' = g$ 
  using assms(2) assms(1)
  using constantConditionNoEffect
  by (metis IRNode.collapse(11))
  then show ?thesis by simp
qed

```

```

lemma constantConditionValid:
  assumes  $\text{kind } g \text{ ifcond} = \text{IfNode cond } t f$ 
  assumes  $[g, m, p] \vdash \text{cond} \mapsto v$ 
  assumes  $\text{const} = \text{val-to-bool } v$ 
  assumes  $g' = \text{constantCondition const ifcond (kind } g \text{ ifcond)} g$ 
  shows  $\exists \text{ nid}' . (g \text{ m } p \text{ h} \vdash \text{ifcond} \rightsquigarrow \text{nid}') \longleftrightarrow (g' \text{ m } p \text{ h} \vdash \text{ifcond} \rightsquigarrow \text{nid}')$ 
proof (cases const)
  case True
  have ifstep:  $g, p \vdash (\text{ifcond}, m, h) \rightarrow (t, m, h)$ 
  by (meson IfNode True assms(1) assms(2) assms(3) encodeeval-def)
  have ifstep':  $g', p \vdash (\text{ifcond}, m, h) \rightarrow (t, m, h)$ 
  using constantConditionTrue
  using True assms(1) assms(4) by presburger
  from ifstep ifstep' show ?thesis
  using StutterStep by blast
next
  case False
  have ifstep:  $g, p \vdash (\text{ifcond}, m, h) \rightarrow (f, m, h)$ 
  by (meson IfNode False assms(1) assms(2) assms(3) encodeeval-def)
  have ifstep':  $g', p \vdash (\text{ifcond}, m, h) \rightarrow (f, m, h)$ 
  using constantConditionFalse
  using False assms(1) assms(4) by presburger
  from ifstep ifstep' show ?thesis
  using StutterStep by blast
qed

```

end

9.3 Stuttering

```

theory Stuttering
  imports
    Semantics.IRStepThms
begin

```

```

inductive stutter:: IRGraph  $\Rightarrow$  MapState  $\Rightarrow$  Params  $\Rightarrow$  FieldRefHeap  $\Rightarrow$  ID  $\Rightarrow$ 
ID  $\Rightarrow$  bool (- - - - -  $\rightsquigarrow$  - 55)

```

for $g \ m \ p \ h$ **where**

StutterStep:

$\llbracket g, p \vdash (nid, m, h) \rightarrow (nid', m, h) \rrbracket$
 $\implies g \ m \ p \ h \vdash nid \rightsquigarrow nid' \mid$

Transitive:

$\llbracket g, p \vdash (nid, m, h) \rightarrow (nid'', m, h);$
 $g \ m \ p \ h \vdash nid'' \rightsquigarrow nid' \rrbracket$
 $\implies g \ m \ p \ h \vdash nid \rightsquigarrow nid'$

lemma *stuttering-successor*:

assumes $(g, p \vdash (nid, m, h) \rightarrow (nid', m, h))$

shows $\{P'. (g \ m \ p \ h \vdash nid \rightsquigarrow P')\} = \{nid'\} \cup \{nid''. (g \ m \ p \ h \vdash nid' \rightsquigarrow nid'')\}$

proof –

have *nextin*: $nid' \in \{P'. (g \ m \ p \ h \vdash nid \rightsquigarrow P')\}$

using *assms StutterStep* **by** *blast*

have *nextsubset*: $\{nid''. (g \ m \ p \ h \vdash nid' \rightsquigarrow nid'')\} \subseteq \{P'. (g \ m \ p \ h \vdash nid \rightsquigarrow P')\}$

by (*metis Collect-mono assms stutter.Transitive*)

have $\forall n \in \{P'. (g \ m \ p \ h \vdash nid \rightsquigarrow P')\} . n = nid' \vee n \in \{nid''. (g \ m \ p \ h \vdash nid' \rightsquigarrow nid'')\}$

using *stepDet*

by (*metis (no-types, lifting) Pair-inject assms mem-Collect-eq stutter.simps*)

then show *?thesis*

using *insert-absorb mk-disjoint-insert nextin nextsubset* **by** *auto*

qed

end

9.4 Evaluation Stamp Theorems

theory *StampEvalThms*

imports *Graph.ValueThms*

Semantics.IRTreeEvalThms

begin

lemma *unrestricted-new-int-always-valid* [*simp*]:

assumes $0 < b \wedge b \leq 64$

shows *valid-value* (*new-int* $b \ v$) (*unrestricted-stamp* (*IntegerStamp* $b \ lo \ hi$))

unfolding *unrestricted-stamp.simps new-int.simps valid-value.simps*

by (*simp; metis One-nat-def assms int-power-div-base int-signed-value.simps int-signed-value-range linorder-not-le not-exp-less-eq-0-int zero-less-numeral*)

lemma *unary-undef*: $val = \text{UndefVal} \implies \text{unary-eval } op \ val = \text{UndefVal}$

by (*cases op; auto*)

lemma *unary-obj*: $val = \text{ObjRef } x \implies \text{unary-eval } op \ val = \text{UndefVal}$

by (*cases op; auto*)

```

lemma unrestricted-stamp-valid:
  assumes  $s = \text{unrestricted-stamp } (\text{IntegerStamp } b \text{ lo hi})$ 
  assumes  $0 < b \wedge b \leq 64$ 
  shows valid-stamp  $s$ 
  using assms
  by (smt (z3) Stamp.inject(1) bit-bounds.simps not-exp-less-eq-0-int prod.sel(1)
prod.sel(2) unrestricted-stamp.simps(2) upper-bounds-equiv valid-stamp.elims(1))

lemma unrestricted-stamp-valid-value [simp]:
  assumes  $1: \text{result} = \text{IntVal } b \text{ ival}$ 
  assumes take-bit  $b \text{ ival} = \text{ival}$ 
  assumes  $0 < b \wedge b \leq 64$ 
  shows valid-value result (unrestricted-stamp (IntegerStamp  $b \text{ lo hi}$ ))
proof –
  have valid-stamp (unrestricted-stamp (IntegerStamp  $b \text{ lo hi}$ ))
    using assms unrestricted-stamp-valid by blast
  then show ?thesis
    unfolding  $1$  unrestricted-stamp.simps valid-value.simps
    using assms int-signed-value-bounds by presburger
qed

```

9.4.1 Support Lemmas for Integer Stamps and Associated IntVal values

Valid int implies some useful facts.

```

lemma valid-int-gives:
  assumes valid-value (IntVal  $b \text{ val}$ ) stamp
  obtains  $lo \ hi$  where  $\text{stamp} = \text{IntegerStamp } b \text{ lo hi} \wedge$ 
     $\text{valid-stamp } (\text{IntegerStamp } b \text{ lo hi}) \wedge$ 
     $\text{take-bit } b \text{ val} = \text{val} \wedge$ 
     $lo \leq \text{int-signed-value } b \text{ val} \wedge \text{int-signed-value } b \text{ val} \leq hi$ 
  using assms
  by (smt (z3) Value.distinct(7) Value.inject(1) valid-value.elims(1))

```

And the corresponding lemma where we know the stamp rather than the value.

```

lemma valid-int-stamp-gives:
  assumes valid-value val (IntegerStamp  $b \text{ lo hi}$ )
  obtains ival where  $\text{val} = \text{IntVal } b \text{ ival} \wedge$ 
     $\text{valid-stamp } (\text{IntegerStamp } b \text{ lo hi}) \wedge$ 
     $\text{take-bit } b \text{ ival} = \text{ival} \wedge$ 
     $lo \leq \text{int-signed-value } b \text{ ival} \wedge \text{int-signed-value } b \text{ ival} \leq hi$ 
  by (metis assms valid-int valid-value.simps(1))

```

A valid int must have the expected number of bits.

```

lemma valid-int-same-bits:

```

assumes *valid-value* (*IntVal b val*) (*IntegerStamp bits lo hi*)
shows $b = bits$
by (*meson assms valid-value.simps(1)*)

A valid value means a valid stamp.

lemma *valid-int-valid-stamp*:
assumes *valid-value* (*IntVal b val*) (*IntegerStamp bits lo hi*)
shows *valid-stamp* (*IntegerStamp bits lo hi*)
by (*metis assms valid-value.simps(1)*)

A valid int means a valid non-empty stamp.

lemma *valid-int-not-empty*:
assumes *valid-value* (*IntVal b val*) (*IntegerStamp bits lo hi*)
shows $lo \leq hi$
by (*metis assms order.trans valid-value.simps(1)*)

A valid int fits into the given number of bits (and other bits are zero).

lemma *valid-int-fits*:
assumes *valid-value* (*IntVal b val*) (*IntegerStamp bits lo hi*)
shows *take-bit bits val = val*
by (*metis assms valid-value.simps(1)*)

lemma *valid-int-is-zero-masked*:
assumes *valid-value* (*IntVal b val*) (*IntegerStamp bits lo hi*)
shows *and val (not (mask bits)) = 0*
by (*metis (no-types, lifting) assms bit.conj-cancel-right take-bit-eq-mask valid-int-fits*)

word-bw-assocs(1) word-log-esimps(1)

Unsigned ints have bounds 0 up to 2^{bits} .

lemma *valid-int-unsigned-bounds*:
assumes *valid-value* (*IntVal b val*) (*IntegerStamp bits lo hi*)

shows $uint\ val < 2^{bits}$
by (*metis assms(1) mask-eq-iff take-bit-eq-mask valid-value.simps(1)*)

Signed ints have the usual two-complement bounds.

lemma *valid-int-signed-upper-bound*:
assumes *valid-value* (*IntVal b val*) (*IntegerStamp bits lo hi*)
shows *int-signed-value bits val* $< 2^{(bits - 1)}$
by (*metis (mono-tags, opaque-lifting) diff-le-mono int-signed-value.simps less-imp-diff-less*)

linorder-not-le one-le-numeral order-less-le-trans power-increasing signed-take-bit-int-less-exp-word sint-lt

lemma *valid-int-signed-lower-bound*:
assumes *valid-value* (*IntVal b val*) (*IntegerStamp bits lo hi*)
shows $-(2^{(bits - 1)}) \leq int-signed-value\ bits\ val$

by (*smt* (*verit*) *diff-le-self int-signed-value.simps linorder-not-less power-increasing-iff signed-take-bit-int-greater-eq-minus-exp-word sint-greater-eq*)

and *bit_bounds* versions of the above bounds.

lemma *valid-int-signed-upper-bit-bound*:
assumes *valid-value* (*IntVal* *b val*) (*IntegerStamp* *bits lo hi*)
shows *int-signed-value* *bits val* \leq *snd* (*bit-bounds* *bits*)
proof –
have *b = bits* **using** *assms valid-int-same-bits* **by** *blast*
then show *?thesis*
using *assms* **by** *force*
qed

lemma *valid-int-signed-lower-bit-bound*:
assumes *valid-value* (*IntVal* *b val*) (*IntegerStamp* *bits lo hi*)
shows *fst* (*bit-bounds* *bits*) \leq *int-signed-value* *bits val*
proof –
have *b = bits* **using** *assms valid-int-same-bits* **by** *blast*
then show *?thesis*
using *assms* **by** *force*
qed

Valid values satisfy their stamp bounds.

lemma *valid-int-signed-range*:
assumes *valid-value* (*IntVal* *b val*) (*IntegerStamp* *bits lo hi*)
shows *lo* \leq *int-signed-value* *bits val* \wedge *int-signed-value* *bits val* \leq *hi*
by (*metis* *assms valid-value.simps(1)*)

9.4.2 Validity of all Unary Operators

We split the validity proof for unary operators into two lemmas, one for normal unary operators whose output bits equals their input bits, and the other case for the widen and narrow operators.

lemma *eval-normal-unary-implies-valid-value*:
assumes $[m, p] \vdash \text{expr} \mapsto \text{val}$
assumes *result* = *unary-eval* *op val*
assumes *op*: *op* \in *normal-unary*
assumes *result* \neq *UndefVal*
assumes *valid-value* *val* (*stamp-expr* *expr*)
shows *valid-value* *result* (*stamp-expr* (*UnaryExpr* *op expr*))
proof –
obtain *b1 v1* **where** *v1*: *val* = *IntVal* *b1 v1*
by (*metis* *Value.exhaust* *assms(1)* *assms(2)* *assms(4)* *assms(5)* *evaltree-not-undef unary-obj valid-value.simps(11)*)
then obtain *b2 v2* **where** *v2*: *result* = *IntVal* *b2 v2*
using *assms(2)* *assms(4)* *is-IntVal-def unary-eval-int* **by** *presburger*
then have *result* = *unary-eval* *op* (*IntVal* *b1 v1*)
using *assms(2)* *v1* **by** *blast*

```

then obtain vtmp where vtmp: result = new-int b2 vtmp
using assms(3) v2 by auto
obtain b' lo' hi' where stamp-expr expr = IntegerStamp b' lo' hi'
by (metis assms(5) v1 valid-int-gives)
then have stamp-unary op (stamp-expr expr) =
  unrestricted-stamp
  (IntegerStamp (if op ∈ normal-unary then b' else ir-resultBits op) lo' hi')
using stamp-unary.simps(1) by presburger
then obtain lo2 hi2 where s: (stamp-expr (UnaryExpr op expr)) = unrestricted-stamp (IntegerStamp b2 lo2 hi2)
unfolding stamp-expr.simps
using vtmp op
by (smt (verit, best) Value.inject(1) <(result::Value) = unary-eval (op::IRUnaryOp)
(IntVal (b1::nat) (v1::64 word))> <stamp-expr (expr::IRExpr) = IntegerStamp (b'::nat)
(lo'::int) (hi'::int)> assms(5) insertE intval-abs.simps(1) intval-logic-negation.simps(1)
intval-negate.simps(1) intval-not.simps(1) new-int.elims singleton-iff unary-eval.simps(1)
unary-eval.simps(2) unary-eval.simps(3) unary-eval.simps(4) v1 valid-int-same-bits)
then have  $0 < b1 \wedge b1 \leq 64$ 
using valid-int-gives
by (metis assms(5) v1 valid-stamp.simps(1))
then have fst (bit-bounds b2) ≤ int-signed-value b2 v2 ∧
  int-signed-value b2 v2 ≤ snd (bit-bounds b2)
by (smt (verit, del-insts) Stamp.inject(1) assms(3) assms(5) int-signed-value-bounds
s stamp-expr.simps(1) stamp-unary.simps(1) unrestricted-stamp.simps(2) v1 valid-int-gives)
then show ?thesis
unfolding s v2 unrestricted-stamp.simps valid-value.simps
by (smt (z3) assms(3) assms(5) is-stamp-empty.simps(1) new-int-take-bits s
stamp-expr.simps(1) stamp-unary.simps(1) unrestricted-stamp.simps(2) v1 v2 valid-int-gives
valid-stamp.simps(1) vtmp)
qed

```

```

lemma narrow-widen-output-bits:
assumes unary-eval op val ≠ UndefVal
assumes op ∉ normal-unary
shows  $0 < (\text{ir-resultBits } op) \wedge (\text{ir-resultBits } op) \leq 64$ 
proof –
consider ib ob where op = UnaryNarrow ib ob
  | ib ob where op = UnarySignExtend ib ob
  | ib ob where op = UnaryZeroExtend ib ob
using IRUnaryOp.exhaust-sel assms(2) by blast
then show ?thesis
proof (cases)
  case 1
    then show ?thesis using assms intval-narrow-ok by force
  next
    case 2
      then show ?thesis using assms intval-sign-extend-ok by force
  next
    case 3

```

```

    then show ?thesis using assms intval-zero-extend-ok by force
qed
qed

```

lemma *eval-widen-narrow-unary-implies-valid-value:*

```

assumes [m,p] ⊢ expr ↦ val
assumes result = unary-eval op val
assumes op: op ∉ normal-unary
assumes result ≠ UndefVal
assumes valid-value val (stamp-expr expr)
shows valid-value result (stamp-expr (UnaryExpr op expr))
proof -
  obtain b1 v1 where v1: val = IntVal b1 v1
  by (metis Value.exhaust assms(1) assms(2) assms(4) assms(5) evaltree-not-undef
unary-obj valid-value.simps(11))
  then have result = unary-eval op (IntVal b1 v1)
  using assms(2) v1 by blast
  then obtain v2 where v2: result = new-int (ir-resultBits op) v2
  using assms by (cases op; simp; (meson new-int.simps)+)
  then obtain v3 where v3: result = IntVal (ir-resultBits op) v3
  using assms by (cases op; simp; (meson new-int.simps)+)
  then obtain lo2 hi2 where s: (stamp-expr (UnaryExpr op expr)) = unre-
stricted-stamp (IntegerStamp (ir-resultBits op) lo2 hi2)
  unfolding stamp-expr.simps stamp-unary.simps
  using assms(3) assms(5) v1 valid-int-gives by fastforce
  then have outBits: 0 < (ir-resultBits op) ∧ (ir-resultBits op) ≤ 64
  using assms narrow-widen-output-bits
  by blast
  then have fst (bit-bounds (ir-resultBits op)) ≤ int-signed-value (ir-resultBits op)
v3 ∧
    int-signed-value (ir-resultBits op) v3 ≤ snd (bit-bounds (ir-resultBits op))
  using int-signed-value-bounds
  by (smt (verit, del-ists) Stamp.inject(1) assms(3) assms(5) int-signed-value-bounds
s stamp-expr.simps(1) stamp-unary.simps(1) unrestricted-stamp.simps(2) v1 valid-int-gives)
  then show ?thesis
  unfolding s v3 unrestricted-stamp.simps valid-value.simps
  using outBits v2 v3 by auto
qed

```

lemma *eval-unary-implies-valid-value:*

```

assumes [m,p] ⊢ expr ↦ val
assumes result = unary-eval op val
assumes result ≠ UndefVal
assumes valid-value val (stamp-expr expr)
shows valid-value result (stamp-expr (UnaryExpr op expr))
proof (cases op ∈ normal-unary)
  case True
  then show ?thesis by (metis assms eval-normal-unary-implies-valid-value)

```



```

next
  case False
  then show ?thesis by (metis assms eval-widen-narrow-unary-implies-valid-value)
qed

```

9.4.3 Support Lemmas for Binary Operators

```

lemma binary-undef:  $v1 = \text{UndefVal} \vee v2 = \text{UndefVal} \implies \text{bin-eval } op \ v1 \ v2 = \text{UndefVal}$ 
  by (cases op; auto)

```

```

lemma binary-obj:  $v1 = \text{ObjRef } x \vee v2 = \text{ObjRef } y \implies \text{bin-eval } op \ v1 \ v2 = \text{UndefVal}$ 
  by (cases op; auto)

```

Some lemmas about the three different output sizes for binary operators.

```

lemma bin-eval-bits-binary-shift-ops:
  assumes  $result = \text{bin-eval } op \ (\text{IntVal } b1 \ v1) \ (\text{IntVal } b2 \ v2)$ 
  assumes  $result \neq \text{UndefVal}$ 
  assumes  $op \in \text{binary-shift-ops}$ 
  shows  $\exists v. result = \text{new-int } b1 \ v$ 
  using assms
  by (cases op; simp; smt (verit, best) new-int.simps) +

```

```

lemma bin-eval-bits-fixed-32-ops:
  assumes  $result = \text{bin-eval } op \ (\text{IntVal } b1 \ v1) \ (\text{IntVal } b2 \ v2)$ 
  assumes  $result \neq \text{UndefVal}$ 
  assumes  $op \in \text{binary-fixed-32-ops}$ 
  shows  $\exists v. result = \text{new-int } 32 \ v$ 
  using assms
  apply (cases op; simp)
  using assms bool-to-val.simps bin-eval-new-int new-int.simps bin-eval-unused-bits-zero
  by metis+

```

```

lemma bin-eval-bits-normal-ops:
  assumes  $result = \text{bin-eval } op \ (\text{IntVal } b1 \ v1) \ (\text{IntVal } b2 \ v2)$ 
  assumes  $result \neq \text{UndefVal}$ 
  assumes  $op \notin \text{binary-shift-ops}$ 
  assumes  $op \notin \text{binary-fixed-32-ops}$ 
  shows  $\exists v. result = \text{new-int } b1 \ v$ 
  using assms apply (cases op; simp)
  using assms apply (metis (mono-tags)) +
  using take-bit-and apply metis
  using take-bit-or apply metis
  using take-bit-xor by metis

```

```

lemma bin-eval-input-bits-equal:
  assumes  $result = \text{bin-eval } op \ (\text{IntVal } b1 \ v1) \ (\text{IntVal } b2 \ v2)$ 
  assumes  $result \neq \text{UndefVal}$ 

```

```

assumes  $op \notin \text{binary-shift-ops}$ 
shows  $b1 = b2$ 
using assms apply (cases op; simp)
by presburger+

```

lemma *bin-eval-implies-valid-value*:

```

assumes  $[m,p] \vdash \text{expr1} \mapsto \text{val1}$ 
assumes  $[m,p] \vdash \text{expr2} \mapsto \text{val2}$ 
assumes  $\text{result} = \text{bin-eval } op \text{ val1 val2}$ 
assumes  $\text{result} \neq \text{UndefVal}$ 
assumes valid-value val1 (stamp-expr expr1)
assumes valid-value val2 (stamp-expr expr2)
shows valid-value result (stamp-expr (BinaryExpr op expr1 expr2))

```

proof –

```

obtain  $b1 \ v1$  where  $v1: \text{val1} = \text{IntVal } b1 \ v1$ 
by (metis Value.collapse(1) assms(3) assms(4) bin-eval-inputs-are-ints bin-eval-int)
obtain  $b2 \ v2$  where  $v2: \text{val2} = \text{IntVal } b2 \ v2$ 
by (metis Value.collapse(1) assms(3) assms(4) bin-eval-inputs-are-ints bin-eval-int)
then obtain  $lo1 \ hi1$  where  $s1: \text{stamp-expr expr1} = \text{IntegerStamp } b1 \ lo1 \ hi1$ 
by (metis assms(5) v1 valid-int-gives)
then obtain  $lo2 \ hi2$  where  $s2: \text{stamp-expr expr2} = \text{IntegerStamp } b2 \ lo2 \ hi2$ 
by (metis assms(6) v2 valid-int-gives)
then have  $r: \text{result} = \text{bin-eval } op \ (\text{IntVal } b1 \ v1) \ (\text{IntVal } b2 \ v2)$ 
using assms(3) v1 v2 by blast
then obtain  $bres \ vtmp$  where  $vtmp: \text{result} = \text{new-int } bres \ vtmp$ 
using assms bin-eval-bits-binary-shift-ops
by (meson bin-eval-new-int)
then obtain  $vres$  where  $vres: \text{result} = \text{IntVal } bres \ vres$ 
by force

```

```

then have  $sres: \text{stamp-expr (BinaryExpr op expr1 expr2)} =$ 
   $\text{unrestricted-stamp (IntegerStamp } bres \ lo1 \ hi1)$ 
   $\wedge 0 < bres \wedge bres \leq 64$ 

```

proof (*cases op ∈ binary-shift-ops*)

case *True*

then show *?thesis*

unfolding $s1 \ s2 \ \text{stamp-binary.simps} \ \text{stamp-expr.simps}$

using *assms bin-eval-bits-binary-shift-ops*

by (*metis Value.inject(1) eval-bits-1-64 new-int.simps r v1 vres*)

next

case *False*

then have $op \notin \text{binary-shift-ops}$

by *simp*

then have *beq: b1 = b2*

using $v1 \ v2 \ \text{assms bin-eval-input-bits-equal}$ **by** *simp*

then show *?thesis*

proof (*cases op ∈ binary-fixed-32-ops*)

case *True*

```

    then show ?thesis
    unfolding s1 s2 stamp-binary.simps stamp-expr.simps
    using assms bin-eval-bits-fixed-32-ops
    by (metis False Value.inject(1) beq bin-eval-new-int le-add-same-cancel1
new-int.simps numeral-Bit0 vres zero-le-numeral zero-less-numeral)
  next
    case False
    then show ?thesis
    unfolding s1 s2 stamp-binary.simps stamp-expr.simps
    using assms
    by (metis beq bin-eval-new-int eval-bits-1-64 intval-bits.simps unrestricted-new-int-always-valid
unrestricted-stamp.simps(2) v1 valid-int-same-bits vres)
  qed
qed
then show ?thesis
  unfolding vres
  using unrestricted-new-int-always-valid vres vtmp by presburger
qed

```

9.4.4 Validity of Stamp Meet and Join Operators

```

lemma stamp-meet-integer-is-valid-stamp:
  assumes valid-stamp stamp1
  assumes valid-stamp stamp2
  assumes is-IntegerStamp stamp1
  assumes is-IntegerStamp stamp2
  shows valid-stamp (meet stamp1 stamp2)
  using assms unfolding is-IntegerStamp-def valid-stamp.simps meet.simps
  by (smt (verit, del-Insts) meet.simps(2) valid-stamp.simps(1) valid-stamp.simps(8))

```

```

lemma stamp-meet-is-valid-stamp:
  assumes 1: valid-stamp stamp1
  assumes 2: valid-stamp stamp2
  shows valid-stamp (meet stamp1 stamp2)
  by (cases stamp1; cases stamp2; insert stamp-meet-integer-is-valid-stamp[OF 1
2]; auto)

```

```

lemma stamp-meet-commutes: meet stamp1 stamp2 = meet stamp2 stamp1
  by (cases stamp1; cases stamp2; auto)

```

```

lemma stamp-meet-is-valid-value1:
  assumes valid-value val stamp1
  assumes valid-stamp stamp2
  assumes stamp1 = IntegerStamp b1 lo1 hi1
  assumes stamp2 = IntegerStamp b2 lo2 hi2
  assumes meet stamp1 stamp2 ≠ IllegalStamp
  shows valid-value val (meet stamp1 stamp2)
proof –

```

```

have m: meet stamp1 stamp2 = IntegerStamp b1 (min lo1 lo2) (max hi1 hi2)
  using assms by (metis meet.simps(2))
obtain ival where val: val = IntVal b1 ival
  using assms valid-int by blast
then have v: valid-stamp (IntegerStamp b1 lo1 hi1) ∧
  take-bit b1 ival = ival ∧
  lo1 ≤ int-signed-value b1 ival ∧ int-signed-value b1 ival ≤ hi1
  using assms by (metis valid-value.simps(1))
then have mm: min lo1 lo2 ≤ int-signed-value b1 ival ∧ int-signed-value b1 ival
≤ max hi1 hi2
  by linarith
then have valid-stamp (IntegerStamp b1 (min lo1 lo2) (max hi1 hi2))
  using assms v stamp-meet-is-valid-stamp
  by (metis meet.simps(2))
then show ?thesis
  unfolding m val valid-value.simps
  using mm v by presburger
qed

```

and the symmetric lemma follows by the commutativity of meet.

```

lemma stamp-meet-is-valid-value:
  assumes valid-value val stamp2
  assumes valid-stamp stamp1
  assumes stamp1 = IntegerStamp b1 lo1 hi1
  assumes stamp2 = IntegerStamp b2 lo2 hi2
  assumes meet stamp1 stamp2 ≠ IllegalStamp
  shows valid-value val (meet stamp1 stamp2)
  using assms stamp-meet-commutes stamp-meet-is-valid-value1
  by metis

```

9.4.5 Validity of conditional expressions

```

lemma conditional-eval-implies-valid-value:
  assumes [m,p] ⊢ cond ⇨ condv
  assumes expr = (if val-to-bool condv then expr1 else expr2)
  assumes [m,p] ⊢ expr ⇨ val
  assumes val ≠ UndefVal
  assumes valid-value condv (stamp-expr cond)
  assumes valid-value val (stamp-expr expr)
  assumes compatible (stamp-expr expr1) (stamp-expr expr2)
  shows valid-value val (stamp-expr (ConditionalExpr cond expr1 expr2))
proof –
  have def: meet (stamp-expr expr1) (stamp-expr expr2) ≠ IllegalStamp
    using assms
  by (metis Stamp.distinct(13) Stamp.distinct(25) compatible.elims(2) meet.simps(1)
meet.simps(2))
  then have valid-stamp (meet (stamp-expr expr1) (stamp-expr expr2))
    using assms
  by (smt (verit, best) compatible.elims(2) stamp-meet-is-valid-stamp valid-stamp.simps(2))

```

```

then show ?thesis using stamp-meet-is-valid-value
  using assms def
  by (smt (verit, best) compatible.elims(2) never-void stamp-expr.simps(6) stamp-meet-commutes)

qed

```

9.4.6 Validity of Whole Expression Tree Evaluation

TODO: find a way to encode that conditional expressions must have compatible (and valid) stamps? One approach would be for all the stamp_expr operators to require that all input stamps are valid.

experiment begin

```

lemma stamp-implies-valid-value:
  assumes [m,p] ⊢ expr ↦ val
  shows valid-value val (stamp-expr expr)
  using assms proof (induction expr val)
  case (UnaryExpr expr val result op)
  then show ?case using eval-unary-implies-valid-value by simp
  next
    case (BinaryExpr expr1 val1 expr2 val2 result op)
    then show ?case using bin-eval-implies-valid-value by simp
  next
    case (ConditionalExpr cond condv expr expr1 expr2 val)
    have compatible (stamp-expr expr1) (stamp-expr expr2)
      using assms sorry
    then show ?case
      using assms conditional-eval-implies-valid-value
      using ConditionalExpr.IH(1) ConditionalExpr.IH(2) ConditionalExpr.hyps(1)
      ConditionalExpr.hyps(2) ConditionalExpr.hyps(3) ConditionalExpr.hyps(4) by blast
  next
    case (ParameterExpr x1 x2)
    then show ?case by auto
  next
    case (LeafExpr x1 x2)
    then show ?case by auto
  next
    case (ConstantExpr x)
    then show ?case by auto
qed

```

```

lemma value-range:
  assumes [m, p] ⊢ e ↦ v
  shows v ∈ {val . valid-value val (stamp-expr e)}
  using assms sorry
end

```

lemma stamp-under-semantics:

```

assumes stamp-under (stamp-expr x) (stamp-expr y)
assumes  $[m, p] \vdash (BinaryExpr\ BinIntegerLessThan\ x\ y) \mapsto v$ 
assumes xvalid:  $(\forall m\ p\ v. ([m, p] \vdash x \mapsto v) \longrightarrow valid\_value\ v\ (stamp\_expr\ x))$ 
assumes yvalid:  $(\forall m\ p\ v. ([m, p] \vdash y \mapsto v) \longrightarrow valid\_value\ v\ (stamp\_expr\ y))$ 
shows val-to-bool v
sorry

lemma stamp-under-semantics-inversed:
assumes stamp-under (stamp-expr y) (stamp-expr x)
assumes  $[m, p] \vdash (BinaryExpr\ BinIntegerLessThan\ x\ y) \mapsto v$ 
assumes xvalid:  $(\forall m\ p\ v. ([m, p] \vdash x \mapsto v) \longrightarrow valid\_value\ v\ (stamp\_expr\ x))$ 
assumes yvalid:  $(\forall m\ p\ v. ([m, p] \vdash y \mapsto v) \longrightarrow valid\_value\ v\ (stamp\_expr\ y))$ 
shows  $\neg(val\_to\_bool\ v)$ 
sorry

```

end

10 Optization DSLs

```

theory Markup
  imports Semantics.IRTreeEval Snippets.Snipping
begin

```

```

datatype 'a Rewrite =
  Transform 'a 'a (-  $\mapsto$  - 10) |
  Conditional 'a 'a bool (-  $\mapsto$  - when - 70) |
  Sequential 'a Rewrite 'a Rewrite |
  Transitive 'a Rewrite

```

```

datatype 'a ExtraNotation =
  ConditionalNotation 'a 'a 'a (- ? - : -) |
  EqualsNotation 'a 'a (- eq -) |
  ConstantNotation 'a (const - 120) |
  TrueNotation (true) |
  FalseNotation (false) |
  ExclusiveOr 'a 'a (-  $\oplus$  -) |
  LogicNegationNotation 'a (!-) |
  ShortCircuitOr 'a 'a (-  $\parallel$  -)

```

```

definition word :: ('a::len) word  $\Rightarrow$  'a word where
  word x = x

```

ML-file $\langle markup.ML \rangle$

```

ML  $\langle$ 
  structure IRExprTranslator : DSL-TRANSLATION =
  struct
    fun markup DSL-Tokens.Add = @{term BinaryExpr} $ @{term BinAdd}
      | markup DSL-Tokens.Sub = @{term BinaryExpr} $ @{term BinSub}

```

```

| markup DSL-Tokens.Mul = @{term BinaryExpr} $ @{term BinMul}
| markup DSL-Tokens.And = @{term BinaryExpr} $ @{term BinAnd}
| markup DSL-Tokens.Or = @{term BinaryExpr} $ @{term BinOr}
| markup DSL-Tokens.Xor = @{term BinaryExpr} $ @{term BinXor}
| markup DSL-Tokens.ShortCircuitOr = @{term BinaryExpr} $ @{term Bin-
ShortCircuitOr}
| markup DSL-Tokens.Abs = @{term UnaryExpr} $ @{term UnaryAbs}
| markup DSL-Tokens.Less = @{term BinaryExpr} $ @{term BinIntegerLessThan}
| markup DSL-Tokens.Equals = @{term BinaryExpr} $ @{term BinIntegerEquals}
| markup DSL-Tokens.Not = @{term UnaryExpr} $ @{term UnaryNot}
| markup DSL-Tokens.Negate = @{term UnaryExpr} $ @{term UnaryNeg}
| markup DSL-Tokens.LogicNegate = @{term UnaryExpr} $ @{term UnaryLog-
icNegation}
| markup DSL-Tokens.LeftShift = @{term BinaryExpr} $ @{term BinLeftShift}
| markup DSL-Tokens.RightShift = @{term BinaryExpr} $ @{term BinRightShift}
| markup DSL-Tokens.UnsignedRightShift = @{term BinaryExpr} $ @{term Bin-
URightShift}
| markup DSL-Tokens.Conditional = @{term ConditionalExpr}
| markup DSL-Tokens.Constant = @{term ConstantExpr}
| markup DSL-Tokens.TrueConstant = @{term ConstantExpr (IntVal 32 1)}
| markup DSL-Tokens.FalseConstant = @{term ConstantExpr (IntVal 32 0)}
end

```

```

structure IntValTranslator : DSL-TRANSLATION =
struct
fun markup DSL-Tokens.Add = @{term intval-add}
| markup DSL-Tokens.Sub = @{term intval-sub}
| markup DSL-Tokens.Mul = @{term intval-mul}
| markup DSL-Tokens.And = @{term intval-and}
| markup DSL-Tokens.Or = @{term intval-or}
| markup DSL-Tokens.ShortCircuitOr = @{term intval-short-circuit-or}
| markup DSL-Tokens.Xor = @{term intval-xor}
| markup DSL-Tokens.Abs = @{term intval-abs}
| markup DSL-Tokens.Less = @{term intval-less-than}
| markup DSL-Tokens.Equals = @{term intval-equals}
| markup DSL-Tokens.Not = @{term intval-not}
| markup DSL-Tokens.Negate = @{term intval-negate}
| markup DSL-Tokens.LogicNegate = @{term intval-logic-negation}
| markup DSL-Tokens.LeftShift = @{term intval-left-shift}
| markup DSL-Tokens.RightShift = @{term intval-right-shift}
| markup DSL-Tokens.UnsignedRightShift = @{term intval-uright-shift}
| markup DSL-Tokens.Conditional = @{term intval-conditional}
| markup DSL-Tokens.Constant = @{term IntVal 32}
| markup DSL-Tokens.TrueConstant = @{term IntVal 32 1}
| markup DSL-Tokens.FalseConstant = @{term IntVal 32 0}
end

```

```

structure WordTranslator : DSL-TRANSLATION =
struct

```

```

fun markup DSL-Tokens.Add = @{term plus}
| markup DSL-Tokens.Sub = @{term minus}
| markup DSL-Tokens.Mul = @{term times}
| markup DSL-Tokens.And = @{term Bit-Operations.semiring-bit-operations-class.and}
| markup DSL-Tokens.Or = @{term or}
| markup DSL-Tokens.Xor = @{term xor}
| markup DSL-Tokens.Abs = @{term abs}
| markup DSL-Tokens.Less = @{term less}
| markup DSL-Tokens.Equals = @{term HOL.eq}
| markup DSL-Tokens.Not = @{term not}
| markup DSL-Tokens.Negate = @{term uminus}
| markup DSL-Tokens.LogicNegate = @{term logic-negate}
| markup DSL-Tokens.LeftShift = @{term shiftl}
| markup DSL-Tokens.RightShift = @{term signed-shiftr}
| markup DSL-Tokens.UnsignedRightShift = @{term shiftr}
| markup DSL-Tokens.Constant = @{term word}
| markup DSL-Tokens.TrueConstant = @{term 1}
| markup DSL-Tokens.FalseConstant = @{term 0}
end

```

```

structure IRExprMarkup = DSL-Markup(IRExprTranslator);
structure IntValMarkup = DSL-Markup(IntValTranslator);
structure WordMarkup = DSL-Markup(WordTranslator);
>

```

ir expression translation

```

syntax -expandExpr :: term ⇒ term (exp[-])
parse-translation < [( @{syntax-const -expandExpr} , IRExprMarkup.markup-expr []) >

```

value expression translation

```

syntax -expandIntVal :: term ⇒ term (val[-])
parse-translation < [( @{syntax-const -expandIntVal} , IntValMarkup.markup-expr []) >

```

word expression translation

```

syntax -expandWord :: term ⇒ term (bin[-])
parse-translation < [( @{syntax-const -expandWord} , WordMarkup.markup-expr []) >

```

ir expression example

```

value exp[(e1 < e2) ? e1 : e2]

ConditionalExpr (BinaryExpr BinIntegerLessThan e1 e2) e1 e2

```


value expression example

```
value val[( $e_1 < e_2$ ) ?  $e_1$  :  $e_2$ ]
```

```
intval-conditional (intval-less-than  $e_1$   $e_2$ )  $e_1$   $e_2$ 
```

```
value exp[( $(e_1 - e_2) + (\text{const } (\text{IntVal } 32\ 0)) + e_2$ )  $\mapsto e_1$  when True]
```

word expression example

```
value bin[ $x \ \& \ y \mid z$ ]
```

```
intval-conditional (intval-less-than  $e_1$   $e_2$ )  $e_1$   $e_2$ 
```

```
value bin[ $\neg x$ ]  
value val[ $\neg x$ ]  
value exp[ $\neg x$ ]
```

```
value bin[ $!x$ ]  
value val[ $!x$ ]  
value exp[ $!x$ ]
```

```
value bin[ $\neg x$ ]  
value val[ $\neg x$ ]  
value exp[ $\neg x$ ]
```

```
value bin[ $\sim x$ ]  
value val[ $\sim x$ ]  
value exp[ $\sim x$ ]
```

```
value  $\sim x$ 
```

```
end  
theory Phase  
  imports Main  
begin
```

```
ML-file map.ML  
ML-file phase.ML
```

```
end
```

10.1 Canonicalization DSL

```
theory Canonicalization  
  imports  
    Markup  
    Phase  
    HOL-Eisbach.Eisbach
```

```

keywords
  phase :: thy-decl and
  terminating :: quasi-command and
  print-phases :: diag and
  export-phases :: thy-decl and
  optimization :: thy-goal-defn
begin

print-methods

ML <
datatype 'a Rewrite =
  Transform of 'a * 'a |
  Conditional of 'a * 'a * term |
  Sequential of 'a Rewrite * 'a Rewrite |
  Transitive of 'a Rewrite

type rewrite = {
  name: binding,
  rewrite: term Rewrite,
  proofs: thm list,
  code: thm list,
  source: term
}

structure RewriteRule : Rule =
struct
type T = rewrite;

(*
fun pretty-rewrite ctxt (Transform (from, to)) =
  Pretty.block [
    Syntax.pretty-term ctxt from,
    Pretty.str  $\mapsto$  ,
    Syntax.pretty-term ctxt to
  ]
| pretty-rewrite ctxt (Conditional (from, to, cond)) =
  Pretty.block [
    Syntax.pretty-term ctxt from,
    Pretty.str  $\mapsto$  ,
    Syntax.pretty-term ctxt to,
    Pretty.str when ,
    Syntax.pretty-term ctxt cond
  ]
| pretty-rewrite - - = Pretty.str not implemented*)

fun pretty-thm ctxt thm =
  (Proof-Context.pretty-fact ctxt (, [thm]))

```

```

fun pretty ctxt obligations t =
  let
    val is-skipped = Thm-Deps.has-skip-proof (#proofs t);

    val warning = (if is-skipped
      then [Pretty.str (proof skipped), Pretty.brk 0]
      else []);

    val obligations = (if obligations
      then [Pretty.big-list
        obligations:
          (map (pretty-thm ctxt) (#proofs t)),
          Pretty.brk 0]
      else []);

    fun pretty-bind binding =
      Pretty.markup
        (Position.markup (Binding.pos-of binding) Markup.position)
        [Pretty.str (Binding.name-of binding)];

  in
    Pretty.block ([
      pretty-bind (#name t), Pretty.str : ,
      Syntax.pretty-term ctxt (#source t), Pretty.fbrk
    ] @ obligations @ warning)
  end
end

structure RewritePhase = DSL-Phase(RewriteRule);

val - =
  Outer-Syntax.command command-keyword⟨phase⟩ enter an optimization phase
  (Parse.binding --| Parse.$$$ terminating -- Parse.const --| Parse.begin
  >> (Toplevel.begin-main-target true o RewritePhase.setup));

fun print-phases print-obligations ctxt =
  let
    val thy = Proof-Context.theory-of ctxt;
    fun print phase = RewritePhase.pretty print-obligations phase ctxt
  in
    map print (RewritePhase.phases thy)
  end

fun print-optimizations print-obligations thy =
  print-phases print-obligations thy |> Pretty.writeln-chunks

val - =
  Outer-Syntax.command command-keyword⟨print-phases⟩
  print debug information for optimizations

```

```

(Parse.opt-bang >>
  (fn b => Toplevel.keep ((print-optimizations b) o Toplevel.context-of)));

fun export-phases thy name =
  let
    val state = Toplevel.theory-tolevel thy;
    val ctxt = Toplevel.context-of state;
    val content = Pretty.string-of (Pretty.chunks (print-phases false ctxt));
    val cleaned = YXML.content-of content;

    val filename = Path.explode (name ^ ".rules");
    val directory = Path.explode optimizations;
    val path = Path.binding (
      Path.append directory filename,
      Position.none);
    val thy' = thy |> Generated-Files.add-files (path, content);

    val - = Export.export thy' path [YXML.parse cleaned];

    val - = writeln (Export.message thy' (Path.basic optimizations));
  in
    thy'
  end

val - =
  Outer-Syntax.command command-keyword ⟨export-phases⟩
  export information about encoded optimizations
  (Parse.text >>
    (fn name => Toplevel.theory (fn state => export-phases state name)))
,

```

ML-file *rewrites.ML*

```

phase Opt
  terminating size
begin

```

```

end

```

```

print-phases
export-phases ⟨MyPhases⟩

```

```

fun rewrite-preservation :: IRExp Rewrite ⇒ bool where
  rewrite-preservation (Transform x y) = (y ≤ x) |
  rewrite-preservation (Conditional x y cond) = (cond ⟶ (y ≤ x)) |
  rewrite-preservation (Sequential x y) = (rewrite-preservation x ∧ rewrite-preservation
y) |
  rewrite-preservation (Transitive x) = rewrite-preservation x

```

```

fun rewrite-termination :: IRExp Rewrite  $\Rightarrow$  (IRExp  $\Rightarrow$  nat)  $\Rightarrow$  bool where
  rewrite-termination (Transform x y) trm = (trm x > trm y) |
  rewrite-termination (Conditional x y cond) trm = (cond  $\longrightarrow$  (trm x > trm y)) |
  rewrite-termination (Sequential x y) trm = (rewrite-termination x trm  $\wedge$  rewrite-termination
y trm) |
  rewrite-termination (Transitive x) trm = rewrite-termination x trm

```

```

fun intval :: Value Rewrite  $\Rightarrow$  bool where
  intval (Transform x y) = (x  $\neq$  UndefVal  $\wedge$  y  $\neq$  UndefVal  $\longrightarrow$  x = y) |
  intval (Conditional x y cond) = (cond  $\longrightarrow$  (x = y)) |
  intval (Sequential x y) = (intval x  $\wedge$  intval y) |
  intval (Transitive x) = intval x

```

```

fun size :: IRExp  $\Rightarrow$  nat where
  size (UnaryExpr op e) = (size e) + 1 |
  size (BinaryExpr BinAdd x y) = (size x) + ((size y) * 2) |
  size (BinaryExpr op x y) = (size x) + (size y) |
  size (ConditionalExpr cond t f) = (size cond) + (size t) + (size f) + 2 |
  size (ConstantExpr c) = 1 |
  size (ParameterExpr ind s) = 2 |
  size (LeafExpr nid s) = 2 |
  size (ConstantVar c) = 2 |
  size (VariableExpr x s) = 2

```

```

method unfold-optimization =
  (unfold rewrite-preservation.simps, unfold rewrite-termination.simps,
   unfold intval.simps,
   rule conjE, simp, simp del: le-expr-def, force?)
| (unfold rewrite-preservation.simps, unfold rewrite-termination.simps,
   rule conjE, simp, simp del: le-expr-def, force?)

```

```

method unfold-size =
  (unfold size.simps, simp del: le-expr-def)?
| (unfold size.simps)?

```

print-methods

```

ML <
  structure System : RewriteSystem =
  struct
    val preservation = @{const rewrite-preservation};
    val termination = @{const rewrite-termination};
    val intval = @{const intval};
  end

```

```

  structure DSL = DSL-Rewrites(System);

```

```

  val - =

```

```

    Outer-Syntax.local-theory-to-proof command-keyword <optimization>
      define an optimization and open proof obligation
      (Parse-Spec.thm-name : -- Parse.term
        >> DSL.rewrite-cmd);
  >
end

```

11 Canonicalization Phase

```

theory Common
  imports
    OptimizationDSL.Canonicalization
    Semantics.IRTreeEvalThms
begin

lemma size-pos[simp]:  $0 < \text{size } y$ 
  apply (induction y; auto?)
  subgoal premises prems for op a b
    using prems by (induction op; auto)
  done

lemma size-non-add:  $op \neq \text{BinAdd} \implies \text{size } (\text{BinaryExpr } op \ a \ b) = \text{size } a + \text{size } b$ 
  by (induction op; auto)

lemma size-non-const:
   $\neg \text{is-ConstantExpr } y \implies 1 < \text{size } y$ 
  using size-pos apply (induction y; auto)
  subgoal premises prems for op a b
    apply (cases op = BinAdd)
    using size-non-add size-pos apply auto
    by (simp add: Suc-lessI one-is-add)+
  done

definition well-formed-equal :: Value  $\Rightarrow$  Value  $\Rightarrow$  bool
  (infix  $\approx$  50) where
    well-formed-equal v1 v2 = (v1  $\neq$  UndefVal  $\longrightarrow$  v1 = v2)

lemma well-formed-equal-defn [simp]:
  well-formed-equal v1 v2 = (v1  $\neq$  UndefVal  $\longrightarrow$  v1 = v2)
  unfolding well-formed-equal-def by simp

end

```

11.1 Conditional Expression

theory *ConditionalPhase*

imports

Common

begin

phase *ConditionalNode*

terminating *size*

begin

lemma *negates: is-IntVal e \implies val-to-bool (val[e]) $\equiv \neg$ (val-to-bool (val[!e]))*
using *intval-logic-negation.simps* **unfolding** *logic-negate-def*
sorry

lemma *negation-condition-intval:*

assumes *e = IntVal b ie*

assumes *0 < b*

shows *val[(!e) ? x : y] = val[e ? y : x]*

using *assms* **by** (*cases e; auto simp: negates logic-negate-def*)

optimization *NegateConditionFlipBranches: ((!e) ? x : y) \mapsto (e ? y : x)*

apply *simp using negation-condition-intval*

by (*smt (verit, ccfv-SIG) ConditionalExpr ConditionalExprE Value.collapse Value.exhaust-disc
evaltree-not-undef intval-logic-negation.simps(4) intval-logic-negation.simps negates
unary-eval.simps(4) unfold-unary*)

optimization *DefaultTrueBranch: (true ? x : y) \mapsto x .*

optimization *DefaultFalseBranch: (false ? x : y) \mapsto y .*

optimization *ConditionalEqualBranches: (e ? x : x) \mapsto x .*

definition *wff-stamps :: bool where*

wff-stamps = ($\forall m p \text{ expr val} . ([m, p] \vdash \text{expr} \mapsto \text{val}) \longrightarrow \text{valid-value val (stamp-expr expr)}$)

definition *wf-stamp :: IRExpr \Rightarrow bool where*

wf-stamp e = ($\forall m p v . ([m, p] \vdash e \mapsto v) \longrightarrow \text{valid-value v (stamp-expr e)}$)

lemma *val-optimise-integer-test:*

```

assumes is-IntVal32 x
shows intval-conditional (intval-equals val[(x & (IntVal32 1))] (IntVal32 0))
      (IntVal32 0) (IntVal32 1) =
      val[x & IntVal32 1]
apply simp-all
apply auto
using bool-to-val.elims intval-equals.elims val-to-bool.simps(1) val-to-bool.simps(3)
sorry

optimization ConditionalEliminateKnownLess: ((x < y) ? x : y)  $\mapsto$  x
      when (stamp-under (stamp-expr x) (stamp-expr y)
       $\wedge$  wf-stamp x  $\wedge$  wf-stamp y)

      apply auto
using stamp-under.simps wf-stamp-def val-to-bool.simps
sorry

optimization ConditionalEqualIsRHS: ((x eq y) ? x : y)  $\mapsto$  y
      apply simp-all apply auto using Canonicalization.intval.simps(1) evalDet
      intval-conditional.simps evaltree-not-undef
by (metis (no-types, opaque-lifting) Value.discI(2) Value.distinct(1) intval-and.simps(3)
intval-equals.simps(2) val-optimize-integer-test val-to-bool.simps(2))

optimization normalizeX: ((x eq const (IntVal 32 0)) ?
      (const (IntVal 32 0)) : (const (IntVal 32 1)))  $\mapsto$  x
      when (x = ConstantExpr (IntVal 32 0) | (x = ConstantExpr
(IntVal 32 1)))
      done

optimization normalizeX2: ((x eq (const (IntVal 32 1))) ?
      (const (IntVal 32 1)) : (const (IntVal 32 0)))  $\mapsto$  x
      when (x = ConstantExpr (IntVal 32 0) | (x = ConstantExpr
(IntVal 32 1)))
      done

optimization flipX: ((x eq (const (IntVal 32 0))) ?
      (const (IntVal 32 1)) : (const (IntVal 32 0)))  $\mapsto$ 
      x  $\oplus$  (const (IntVal 32 1))
      when (x = ConstantExpr (IntVal 32 0) | (x = ConstantExpr
(IntVal 32 1)))
      done

optimization flipX2: ((x eq (const (IntVal 32 1))) ?
      (const (IntVal 32 0)) : (const (IntVal 32 1)))  $\mapsto$ 
      x  $\oplus$  (const (IntVal 32 1))

```



```

when (x = ConstantExpr (IntVal 32 0) | (x = ConstantExpr
(IntVal 32 1)))
done

```

```

optimization OptimiseIntegerTest:
  (((x & (const (IntVal 32 1))) eq (const (IntVal 32 0))) ?
  (const (IntVal 32 0)) : (const (IntVal 32 1)))  $\mapsto$ 
  x & (const (IntVal 32 1))
  when (stamp-expr x = default-stamp)
apply simp-all
apply auto
using val-optimise-integer-test sorry

```

```

optimization opt-optimise-integer-test-2:
  (((x & (const (IntVal 32 1))) eq (const (IntVal 32 0))) ?
  (const (IntVal 32 0)) : (const (IntVal 32 1)))  $\mapsto$ 
  x
  when (x = ConstantExpr (IntVal 32 0) | (x = ConstantExpr (IntVal
32 1)))
done

```

```

optimization opt-conditional-eliminate-known-less: ((x < y) ? x : y)  $\mapsto$  x
  when (((stamp-under (stamp-expr x) (stamp-expr y)) |
  ((stpi-upper (stamp-expr x)) = (stpi-lower (stamp-expr
y))))
   $\wedge$  wf-stamp x  $\wedge$  wf-stamp y)
  unfolding le-expr-def apply auto
using stamp-under.simps wf-stamp-def
sorry

```

end

end

12 Conditional Elimination Phase

```

theory ConditionalElimination
imports
  Proofs.Rewrites
  Proofs.Bisimulation
begin

```

12.1 Individual Elimination Rules

We introduce a `TriState` as in the Graal compiler to represent when static analysis can tell us information about the value of a boolean expression. `Unknown` = No information can be inferred `KnownTrue`/`KnownFalse` = We can infer the expression will always be true or false.

datatype `TriState` = `Unknown` | `KnownTrue` | `KnownFalse`

The `implies` relation corresponds to the `LogicNode.implies` method from the compiler which attempts to infer when one logic nodes value can be inferred from a known logic node.

inductive `implies` :: `IRGraph` \Rightarrow `IRNode` \Rightarrow `IRNode` \Rightarrow `TriState` \Rightarrow `bool`

(- \vdash - & - \hookrightarrow -) **for** `g` **where**

eq-imp-less:

$g \vdash (\text{IntegerEqualsNode } x \ y) \ \& \ (\text{IntegerLessThanNode } x \ y) \hookrightarrow \text{KnownFalse} \mid$

eq-imp-less-rev:

$g \vdash (\text{IntegerEqualsNode } x \ y) \ \& \ (\text{IntegerLessThanNode } y \ x) \hookrightarrow \text{KnownFalse} \mid$

less-imp-rev-less:

$g \vdash (\text{IntegerLessThanNode } x \ y) \ \& \ (\text{IntegerLessThanNode } y \ x) \hookrightarrow \text{KnownFalse} \mid$

less-imp-not-eq:

$g \vdash (\text{IntegerLessThanNode } x \ y) \ \& \ (\text{IntegerEqualsNode } x \ y) \hookrightarrow \text{KnownFalse} \mid$

less-imp-not-eq-rev:

$g \vdash (\text{IntegerLessThanNode } x \ y) \ \& \ (\text{IntegerEqualsNode } y \ x) \hookrightarrow \text{KnownFalse} \mid$

x-imp-x:

$g \vdash x \ \& \ x \hookrightarrow \text{KnownTrue} \mid$

negate-false:

$\llbracket g \vdash x \ \& \ (\text{kind } g \ y) \hookrightarrow \text{KnownTrue} \rrbracket \implies g \vdash x \ \& \ (\text{LogicNegationNode } y) \hookrightarrow \text{KnownFalse} \mid$

negate-true:

$\llbracket g \vdash x \ \& \ (\text{kind } g \ y) \hookrightarrow \text{KnownFalse} \rrbracket \implies g \vdash x \ \& \ (\text{LogicNegationNode } y) \hookrightarrow \text{KnownTrue}$

Total relation over partial `implies` relation

inductive `condition-implies` :: `IRGraph` \Rightarrow `IRNode` \Rightarrow `IRNode` \Rightarrow `TriState` \Rightarrow `bool`

(- \vdash - & - \rightarrow -) **for** `g` **where**

$\llbracket \neg(g \vdash a \ \& \ b \hookrightarrow \text{imp}) \rrbracket \implies (g \vdash a \ \& \ b \rightarrow \text{Unknown}) \mid$

$\llbracket (g \vdash a \ \& \ b \hookrightarrow \text{imp}) \rrbracket \implies (g \vdash a \ \& \ b \rightarrow \text{imp})$

inductive `implies-tree` :: `IRExpr` \Rightarrow `IRExpr` \Rightarrow `bool` \Rightarrow `bool`

(- & - \hookrightarrow -) **where**

eq-imp-less:

$(\text{BinaryExpr } \text{BinIntegerEquals } x \ y) \ \& \ (\text{BinaryExpr } \text{BinIntegerLessThan } x \ y) \hookrightarrow \text{False} \mid$

eq-imp-less-rev:

```

  (BinaryExpr BinIntegerEquals x y) & (BinaryExpr BinIntegerLessThan y x)  $\hookrightarrow$ 
  False |
  less-imp-rev-less:
  (BinaryExpr BinIntegerLessThan x y) & (BinaryExpr BinIntegerLessThan y x)
 $\hookrightarrow$  False |
  less-imp-not-eq:
  (BinaryExpr BinIntegerLessThan x y) & (BinaryExpr BinIntegerEquals x y)  $\hookrightarrow$ 
  False |
  less-imp-not-eq-rev:
  (BinaryExpr BinIntegerLessThan x y) & (BinaryExpr BinIntegerEquals y x)  $\hookrightarrow$ 
  False |

```

```

x-imp-x:
x & x  $\hookrightarrow$  True |

```

```

negate-false:
[[x & y  $\hookrightarrow$  True]]  $\implies$  x & (UnaryExpr UnaryLogicNegation y)  $\hookrightarrow$  False |
negate-true:
[[x & y  $\hookrightarrow$  False]]  $\implies$  x & (UnaryExpr UnaryLogicNegation y)  $\hookrightarrow$  True

```

Proofs that the implies relation is correct with respect to the existing evaluation semantics.

experiment begin

lemma *logic-negate-type*:

```

  assumes [m, p]  $\vdash$  UnaryExpr UnaryLogicNegation x  $\mapsto$  v
  assumes v  $\neq$  UndefinedVal
  shows  $\exists v2. [m, p] \vdash x \mapsto \text{IntVal32 } v2$ 
proof –
  obtain ve where ve: [m, p]  $\vdash$  x  $\mapsto$  ve
  using assms(1) by blast
  then have [m, p]  $\vdash$  UnaryExpr UnaryLogicNegation x  $\mapsto$  unary-eval UnaryLogicNegation ve
  by (metis UnaryExprE assms(1) evalDet)
  then show ?thesis using assms unary-eval.elims evalDet ve IRUnaryOp.distinct
  sorry
qed

```

lemma *logic-negation-relation-tree*:

```

  assumes [m, p]  $\vdash$  y  $\mapsto$  val
  assumes [m, p]  $\vdash$  UnaryExpr UnaryLogicNegation y  $\mapsto$  invval
  assumes invval  $\neq$  UndefinedVal
  shows val-to-bool val  $\longleftrightarrow$   $\neg$ (val-to-bool invval)
proof –
  obtain v where invval = unary-eval UnaryLogicNegation v
  using assms(2) by blast
  then have [m, p]  $\vdash$  y  $\mapsto$  v using UnaryExprE assms(1,2) sorry
  then show ?thesis sorry
qed

```

```

lemma logic-negation-relation:
  assumes  $[g, m, p] \vdash y \mapsto val$ 
  assumes  $kind\ g\ neg = LogicNegationNode\ y$ 
  assumes  $[g, m, p] \vdash neg \mapsto invval$ 
  assumes  $invval \neq UndefinedVal$ 
  shows  $val\text{-}to\text{-}bool\ val \longleftrightarrow \neg(val\text{-}to\text{-}bool\ invval)$ 
proof -
  obtain yencode where 5:  $g \vdash y \simeq yencode$ 
  using assms(1) encodeeval-def by auto
  then have 6:  $g \vdash neg \simeq UnaryExpr\ UnaryLogicNegation\ yencode$ 
  using rep.intros(7) assms(2) by simp
  then have 7:  $[m, p] \vdash UnaryExpr\ UnaryLogicNegation\ yencode \mapsto invval$ 
  using assms(3) encodeeval-def
  by (metis repDet)
  obtain v1 where v1:  $[g, m, p] \vdash y \mapsto IntVal\ 32\ v1$ 
  using assms(1,2,3,4) using logic-negate-type sorry
  have  $invval = bool\text{-}to\text{-}val\ (\neg(val\text{-}to\text{-}bool\ val))$ 
  using assms(1,2,3) evalDet unary-eval.simps(4)
  sorry
  have  $val\text{-}to\text{-}bool\ invval \longleftrightarrow \neg(val\text{-}to\text{-}bool\ val)$ 
  using  $\langle invval = bool\text{-}to\text{-}val\ (\neg\ val\text{-}to\text{-}bool\ val) \rangle$  by force
  then show ?thesis
  by simp
qed
end

```

```

lemma implies-valid:
  assumes  $x \ \&\ y \hookrightarrow imp$ 
  assumes  $[m, p] \vdash x \mapsto v1$ 
  assumes  $[m, p] \vdash y \mapsto v2$ 
  assumes  $v1 \neq UndefinedVal \wedge v2 \neq UndefinedVal$ 
  shows  $(imp \longrightarrow (val\text{-}to\text{-}bool\ v1 \longrightarrow val\text{-}to\text{-}bool\ v2)) \wedge$ 
     $(\neg imp \longrightarrow (val\text{-}to\text{-}bool\ v1 \longrightarrow \neg(val\text{-}to\text{-}bool\ v2)))$ 
     $(is\ (?TP \longrightarrow ?TC) \wedge (?FP \longrightarrow ?FC))$ 
  apply (intro conjI; rule impI)
proof -
  assume KnownTrue: ?TP
  show ?TC
  using assms(1) KnownTrue assms(2-) proof (induct x y imp rule: implies-tree.induct)
    case (eq-imp-less x y)
    then show ?case by simp
  next
    case (eq-imp-less-rev x y)
    then show ?case by simp
  next
    case (less-imp-rev-less x y)
    then show ?case by simp
  next

```

```

    case (less-imp-not-eq x y)
    then show ?case by simp
next
    case (less-imp-not-eq-rev x y)
    then show ?case by simp
next
    case (x-imp-x)
    then show ?case
    by (metis evalDet)
next
    case (negate-false x1)
    then show ?case using evalDet
    using assms(2,3) by blast
next
    case (negate-true y)
    then show ?case
    sorry
qed
next
    assume KnownFalse: ?FP
    show ?FC using assms KnownFalse proof (induct x y imp rule: implies-tree.induct)
    case (eq-imp-less x y)
    obtain xval where xval: [m, p] ⊢ x ↦ xval
    using eq-imp-less(1) eq-imp-less.prem(3)
    by blast
    then obtain yval where yval: [m, p] ⊢ y ↦ yval
    using eq-imp-less.prem(3)
    using eq-imp-less.prem(2) by blast
    have egeval: [m, p] ⊢ (BinaryExpr BinIntegerEquals x y) ↦ intval-equals xval
    yval
    using xval yval evaltree.BinaryExpr
    by (metis BinaryExprE bin-eval.simps(11) eq-imp-less.prem(1) evalDet)
    have lesseval: [m, p] ⊢ (BinaryExpr BinIntegerLessThan x y) ↦ intval-less-than
    xval yval
    using xval yval evaltree.BinaryExpr
    by (metis BinaryExprE bin-eval.simps(12) eq-imp-less.prem(2) evalDet)
    have val-to-bool (intval-equals xval yval) ⟶ ¬(val-to-bool (intval-less-than xval
    yval))
    using assms(4) apply (cases xval; cases yval; auto) sorry

    then show ?case
    using egeval lesseval
    by (metis eq-imp-less.prem(1) eq-imp-less.prem(2) evalDet)
next
    case (eq-imp-less-rev x y)
    obtain xval where xval: [m, p] ⊢ x ↦ xval
    using eq-imp-less-rev.prem(3)
    using eq-imp-less-rev.prem(2) by blast
    obtain yval where yval: [m, p] ⊢ y ↦ yval

```

```

    using eq-imp-less-rev.premis(3)
    using eq-imp-less-rev.premis(2) by blast
    have egeval:  $[m, p] \vdash (\text{BinaryExpr BinIntegerEquals } x \ y) \mapsto \text{intval-equals } xval$ 
  yval
    using xval yval evaltree.BinaryExpr
    by (metis BinaryExprE bin-eval.simps(11) eq-imp-less-rev.premis(1) evalDet)
    have lesseval:  $[m, p] \vdash (\text{BinaryExpr BinIntegerLessThan } y \ x) \mapsto \text{intval-less-than}$ 
  yval xval
    using xval yval evaltree.BinaryExpr
    by (metis BinaryExprE bin-eval.simps(12) eq-imp-less-rev.premis(2) evalDet)
    have val-to-bool (intval-equals xval yval)  $\longrightarrow \neg(\text{val-to-bool } (\text{intval-less-than } yval$ 
  xval))
    using assms(4) apply (cases xval; cases yval; auto) sorry

  then show ?case
    using egeval lesseval
    by (metis eq-imp-less-rev.premis(1) eq-imp-less-rev.premis(2) evalDet)
next
  case (less-imp-rev-less x y)
  obtain xval where xval:  $[m, p] \vdash x \mapsto xval$ 
    using less-imp-rev-less.premis(3)
    using less-imp-rev-less.premis(2) by blast
  obtain yval where yval:  $[m, p] \vdash y \mapsto yval$ 
    using less-imp-rev-less.premis(3)
    using less-imp-rev-less.premis(2) by blast
  have lesseval:  $[m, p] \vdash (\text{BinaryExpr BinIntegerLessThan } x \ y) \mapsto \text{intval-less-than}$ 
  xval yval
    using xval yval evaltree.BinaryExpr
    by (metis BinaryExprE bin-eval.simps(12) evalDet less-imp-rev-less.premis(1))
    have revlesseval:  $[m, p] \vdash (\text{BinaryExpr BinIntegerLessThan } y \ x) \mapsto \text{int-}$ 
  val-less-than yval xval
    using xval yval evaltree.BinaryExpr
    by (metis BinaryExprE bin-eval.simps(12) evalDet less-imp-rev-less.premis(2))
    have val-to-bool (intval-less-than xval yval)  $\longrightarrow \neg(\text{val-to-bool } (\text{intval-less-than}$ 
  yval xval))
    using assms(4) apply (cases xval; cases yval; auto) sorry

  then show ?case
    by (metis evalDet less-imp-rev-less.premis(1) less-imp-rev-less.premis(2) lesseval
  revlesseval)
next
  case (less-imp-not-eq x y)
  obtain xval where xval:  $[m, p] \vdash x \mapsto xval$ 
    using less-imp-not-eq.premis(3)
    using less-imp-not-eq.premis(1) by blast
  obtain yval where yval:  $[m, p] \vdash y \mapsto yval$ 
    using less-imp-not-eq.premis(3)
    using less-imp-not-eq.premis(1) by blast
  have egeval:  $[m, p] \vdash (\text{BinaryExpr BinIntegerEquals } x \ y) \mapsto \text{intval-equals } xval$ 

```

```

yval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(11) evalDet less-imp-not-eq.prem(2))
  have lesseval:  $[m, p] \vdash (\text{BinaryExpr BinIntegerLessThan } x \ y) \mapsto \text{intval-less-than}$ 
xval yval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(12) evalDet less-imp-not-eq.prem(1))
  have val-to-bool (intval-less-than xval yval)  $\longrightarrow \neg(\text{val-to-bool (intval-equals xval yval)})$ 
  using assms(4) apply (cases xval; cases yval; auto) sorry

then show ?case
  by (metis egeval evalDet less-imp-not-eq.prem(1) less-imp-not-eq.prem(2) lesseval)
next
case (less-imp-not-eq-rev x y)
obtain xval where xval:  $[m, p] \vdash x \mapsto \text{xval}$ 
  using less-imp-not-eq-rev.prem(3)
  using less-imp-not-eq-rev.prem(1) by blast
obtain yval where yval:  $[m, p] \vdash y \mapsto \text{yval}$ 
  using less-imp-not-eq-rev.prem(3)
  using less-imp-not-eq-rev.prem(1) by blast
have egeval:  $[m, p] \vdash (\text{BinaryExpr BinIntegerEquals } y \ x) \mapsto \text{intval-equals yval xval}$ 
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(11) evalDet less-imp-not-eq-rev.prem(2))
  have lesseval:  $[m, p] \vdash (\text{BinaryExpr BinIntegerLessThan } x \ y) \mapsto \text{intval-less-than}$ 
xval yval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(12) evalDet less-imp-not-eq-rev.prem(1))
  have val-to-bool (intval-less-than xval yval)  $\longrightarrow \neg(\text{val-to-bool (intval-equals yval xval)})$ 
  using assms(4) apply (cases xval; cases yval; auto) sorry

then show ?case
  by (metis egeval evalDet less-imp-not-eq-rev.prem(1) less-imp-not-eq-rev.prem(2) lesseval)
next
case (x-imp-x x1)
then show ?case by simp
next
case (negate-false x y)
then show ?case sorry
next
case (negate-true x1)
then show ?case by simp
qed
qed

```

```

lemma implies-true-valid:
  assumes  $x \ \& \ y \hookrightarrow \text{imp}$ 
  assumes  $\text{imp}$ 
  assumes  $[m, p] \vdash x \mapsto v1$ 
  assumes  $[m, p] \vdash y \mapsto v2$ 
  assumes  $v1 \neq \text{UndefVal} \wedge v2 \neq \text{UndefVal}$ 
  shows  $\text{val-to-bool } v1 \longrightarrow \text{val-to-bool } v2$ 
  using assms implies-valid
  by blast

```

```

lemma implies-false-valid:
  assumes  $x \ \& \ y \hookrightarrow \text{imp}$ 
  assumes  $\neg \text{imp}$ 
  assumes  $[m, p] \vdash x \mapsto v1$ 
  assumes  $[m, p] \vdash y \mapsto v2$ 
  assumes  $v1 \neq \text{UndefVal} \wedge v2 \neq \text{UndefVal}$ 
  shows  $\text{val-to-bool } v1 \longrightarrow \neg(\text{val-to-bool } v2)$ 
  using assms implies-valid by blast

```

The following relation corresponds to the `UnaryOpLogicNode.tryFold` and `BinaryOpLogicNode.tryFold` methods and their associated concrete implementations.

The relation determines if a logic operation can be shown true or false through the stamp typing information.

```

inductive tryFold :: IRNode  $\Rightarrow$  (ID  $\Rightarrow$  Stamp)  $\Rightarrow$  bool  $\Rightarrow$  bool
  where
     $\llbracket \text{alwaysDistinct } (\text{stamps } x) \ (\text{stamps } y) \rrbracket$ 
       $\Longrightarrow \text{tryFold } (\text{IntegerEqualsNode } x \ y) \ \text{stamps } \text{False} \mid$ 
     $\llbracket \text{neverDistinct } (\text{stamps } x) \ (\text{stamps } y) \rrbracket$ 
       $\Longrightarrow \text{tryFold } (\text{IntegerEqualsNode } x \ y) \ \text{stamps } \text{True} \mid$ 
     $\llbracket \text{is-IntegerStamp } (\text{stamps } x);$ 
       $\text{is-IntegerStamp } (\text{stamps } y);$ 
       $\text{stpi-upper } (\text{stamps } x) < \text{stpi-lower } (\text{stamps } y) \rrbracket$ 
       $\Longrightarrow \text{tryFold } (\text{IntegerLessThanNode } x \ y) \ \text{stamps } \text{True} \mid$ 
     $\llbracket \text{is-IntegerStamp } (\text{stamps } x);$ 
       $\text{is-IntegerStamp } (\text{stamps } y);$ 
       $\text{stpi-lower } (\text{stamps } x) \geq \text{stpi-upper } (\text{stamps } y) \rrbracket$ 
       $\Longrightarrow \text{tryFold } (\text{IntegerLessThanNode } x \ y) \ \text{stamps } \text{False}$ 

```

Proofs that show that when the stamp lookup function is well-formed, the `tryFold` relation correctly predicts the output value with respect to our evaluation semantics.

```

lemma
  assumes  $\text{kind } g \ \text{nid} = \text{IntegerEqualsNode } x \ y$ 
  assumes  $[g, m, p] \vdash \text{nid} \mapsto v$ 
  assumes  $v \neq \text{UndefVal}$ 
  assumes  $([g, m, p] \vdash x \mapsto xval) \wedge ([g, m, p] \vdash y \mapsto yval)$ 
  shows  $\text{val-to-bool } (\text{intval-equals } xval \ yval) \longleftrightarrow v = \text{IntVal32 } 1$ 

```



```

proof –
  have  $v = \text{intval-equals } xval \ yval$ 
    using  $\text{assms}(1, 2, 3, 4) \ \text{BinaryExprE IntegerEqualsNode bin-eval.simps}(\gamma)$ 
    by  $(\text{smt } (\text{verit}) \ \text{bin-eval.simps}(11) \ \text{encodeeval-def evalDet repDet})$ 
  then show  $?thesis$  using  $\text{intval-equals.simps val-to-bool.simps}$  sorry
qed

lemma  $\text{tryFoldIntegerEqualsAlwaysDistinct}$ :
  assumes  $\text{wf-stamp } g \ \text{stamps}$ 
  assumes  $\text{kind } g \ \text{nid} = (\text{IntegerEqualsNode } x \ y)$ 
  assumes  $[g, m, p] \vdash \text{nid} \mapsto v$ 
  assumes  $\text{alwaysDistinct } (\text{stamps } x) \ (\text{stamps } y)$ 
  shows  $v = \text{IntVal32 } 0$ 
proof –
  have  $\forall \text{ val. } \neg(\text{valid-value val } (\text{join } (\text{stamps } x) \ (\text{stamps } y)))$ 
    using  $\text{assms}(1,4) \ \text{unfolding alwaysDistinct.simps}$ 
    by  $(\text{smt } (\text{verit}, \text{best}) \ \text{is-stamp-empty.elims}(2) \ \text{valid-int valid-value.simps}(1))$ 
  have  $\neg(\exists \text{ val} . ([g, m, p] \vdash x \mapsto \text{val}) \wedge ([g, m, p] \vdash y \mapsto \text{val}))$ 
    using  $\text{assms}(1,4) \ \text{unfolding alwaysDistinct.simps wf-stamp.simps encodeeval-def}$  sorry
  then show  $?thesis$  sorry
qed

lemma  $\text{tryFoldIntegerEqualsNeverDistinct}$ :
  assumes  $\text{wf-stamp } g \ \text{stamps}$ 
  assumes  $\text{kind } g \ \text{nid} = (\text{IntegerEqualsNode } x \ y)$ 
  assumes  $[g, m, p] \vdash \text{nid} \mapsto v$ 
  assumes  $\text{neverDistinct } (\text{stamps } x) \ (\text{stamps } y)$ 
  shows  $v = \text{IntVal32 } 1$ 
  using  $\text{assms IntegerEqualsNodeE}$  sorry

lemma  $\text{tryFoldIntegerLessThanTrue}$ :
  assumes  $\text{wf-stamp } g \ \text{stamps}$ 
  assumes  $\text{kind } g \ \text{nid} = (\text{IntegerLessThanNode } x \ y)$ 
  assumes  $[g, m, p] \vdash \text{nid} \mapsto v$ 
  assumes  $\text{stpi-upper } (\text{stamps } x) < \text{stpi-lower } (\text{stamps } y)$ 
  shows  $v = \text{IntVal32 } 1$ 
proof –
  have  $\text{stamp-type: is-IntegerStamp } (\text{stamps } x)$ 
    using  $\text{assms}$ 
    sorry
  obtain  $xval$  where  $xval: [g, m, p] \vdash x \mapsto xval$ 
    using  $\text{assms}(2,3)$  sorry
  obtain  $yval$  where  $yval: [g, m, p] \vdash y \mapsto yval$ 
    using  $\text{assms}(2,3)$  sorry
  have  $\text{is-IntegerStamp } (\text{stamps } x) \wedge \text{is-IntegerStamp } (\text{stamps } y)$ 
    using  $\text{assms}(4)$ 
    sorry
  then have  $\text{val-to-bool } (\text{intval-less-than } xval \ yval)$ 

```

```

    sorry
  then show ?thesis
    sorry
qed

```

```

lemma tryFoldIntegerLessThanFalse:
  assumes wf-stamp g stamps
  assumes kind g nid = (IntegerLessThanNode x y)
  assumes [g, m, p] ⊢ nid ↦ v
  assumes stpi-lower (stamps x) ≥ stpi-upper (stamps y)
  shows v = IntVal32 0
  proof -
    have stamp-type: is-IntegerStamp (stamps x)
      using assms
    sorry
    obtain xval where xval: [g, m, p] ⊢ x ↦ xval
      using assms(2,3) sorry
    obtain yval where yval: [g, m, p] ⊢ y ↦ yval
      using assms(2,3) sorry
    have is-IntegerStamp (stamps x) ∧ is-IntegerStamp (stamps y)
      using assms(4)
    sorry
    then have ¬(val-to-bool (intval-less-than xval yval))
      sorry
    then show ?thesis
      sorry
  qed

```

```

theorem tryFoldProofTrue:
  assumes wf-stamp g stamps
  assumes tryFold (kind g nid) stamps True
  assumes [g, m, p] ⊢ nid ↦ v
  shows val-to-bool v
  using assms(2) proof (induction kind g nid stamps True rule: tryFold.induct)
  case (1 stamps x y)
    then show ?case using tryFoldIntegerEqualsAlwaysDistinct assms sorry
  next
    case (2 stamps x y)
    then show ?case using tryFoldIntegerEqualsAlwaysDistinct assms sorry
  next
    case (3 stamps x y)
    then show ?case using tryFoldIntegerLessThanTrue assms sorry
  next
    case (4 stamps x y)
    then show ?case using tryFoldIntegerLessThanFalse assms sorry
  qed

```

```

theorem tryFoldProofFalse:
  assumes wf-stamp g stamps

```

```

assumes tryFold (kind g nid) stamps False
assumes  $[g, m, p] \vdash \text{nid} \mapsto v$ 
shows  $\neg(\text{val-to-bool } v)$ 
using assms(2) proof (induction kind g nid stamps False rule: tryFold.induct)
case (1 stamps x y)
  then show ?case using tryFoldIntegerEqualsAlwaysDistinct assms sorry
next
case (2 stamps x y)
  then show ?case using tryFoldIntegerEqualsNeverDistinct assms sorry
next
  case (3 stamps x y)
  then show ?case using tryFoldIntegerLessThanTrue assms sorry
next
  case (4 stamps x y)
  then show ?case using tryFoldIntegerLessThanFalse assms sorry

qed

```

inductive-cases *StepE*:

$g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h)$

Perform conditional elimination rewrites on the graph for a particular node. In order to determine conditional eliminations appropriately the rule needs two data structures produced by static analysis. The first parameter is the set of IRNodes that we know result in a true value when evaluated. The second parameter is a mapping from node identifiers to the flow-sensitive stamp.

The relation transforms the third parameter to the fifth parameter for a node identifier which represents the fourth parameter.

inductive *ConditionalEliminationStep* ::

$\text{IRExpr set} \Rightarrow (\text{ID} \Rightarrow \text{Stamp}) \Rightarrow \text{IRGraph} \Rightarrow \text{ID} \Rightarrow \text{IRGraph} \Rightarrow \text{bool}$ **where**
impliesTrue:

$\llbracket \text{kind } g \text{ ifcond} = (\text{IfNode } \text{cid } t \text{ } f);$
 $g \vdash \text{cid} \simeq \text{cond};$
 $\exists ce \in \text{conds} . (ce \ \& \ \text{cond} \hookrightarrow \text{True});$
 $g' = \text{constantCondition True ifcond (kind } g \text{ ifcond) } g$
 $\rrbracket \implies \text{ConditionalEliminationStep conds stamps } g \text{ ifcond } g' \mid$

impliesFalse:

$\llbracket \text{kind } g \text{ ifcond} = (\text{IfNode } \text{cid } t \text{ } f);$
 $g \vdash \text{cid} \simeq \text{cond};$
 $\exists ce \in \text{conds} . (ce \ \& \ \text{cond} \hookrightarrow \text{False});$
 $g' = \text{constantCondition False ifcond (kind } g \text{ ifcond) } g$
 $\rrbracket \implies \text{ConditionalEliminationStep conds stamps } g \text{ ifcond } g' \mid$

tryFoldTrue:

```

[[kind g ifcond = (IfNode cid t f);
  cond = kind g cid;
  tryFold (kind g cid) stamps True;
  g' = constantCondition True ifcond (kind g ifcond) g
]] ==> ConditionalEliminationStep conds stamps g ifcond g' |

```

```

tryFoldFalse:
[[kind g ifcond = (IfNode cid t f);
  cond = kind g cid;
  tryFold (kind g cid) stamps False;
  g' = constantCondition False ifcond (kind g ifcond) g
]] ==> ConditionalEliminationStep conds stamps g ifcond g'

```

code-pred (modes: $i \Rightarrow i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$) *ConditionalEliminationStep* .

thm *ConditionalEliminationStep.equation*

12.2 Control-flow Graph Traversal

```

type-synonym Seen = ID set
type-synonym Condition = IRNode
type-synonym Conditions = Condition list
type-synonym StampFlow = (ID  $\Rightarrow$  Stamp) list

```

`nextEdge` helps determine which node to traverse next by returning the first successor edge that isn't in the set of already visited nodes. If there is not an appropriate successor, `None` is returned instead.

```

fun nextEdge :: Seen  $\Rightarrow$  ID  $\Rightarrow$  IRGraph  $\Rightarrow$  ID option where
  nextEdge seen nid g =
    (let nids = (filter ( $\lambda \text{nid}'$ . nid'  $\notin$  seen) (successors-of (kind g nid))) in
     (if length nids > 0 then Some (hd nids) else None))

```

`pred` determines which node, if any, acts as the predecessor of another.

Merge nodes represent a special case where-in the predecessor exists as an input edge of the merge node, to simplify the traversal we treat only the first input end node as the predecessor, ignoring that multiple nodes may act as a successor.

For all other nodes, the predecessor is the first element of the predecessors set. Note that in a well-formed graph there should only be one element in the predecessor set.

```

fun pred :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID option where
  pred g nid = (case kind g nid of
    (MergeNode ends -)  $\Rightarrow$  Some (hd ends) |
    -  $\Rightarrow$ 
      (if IRGraph.predecessors g nid = {}
       then None else
       Some (hd (sorted-list-of-set (IRGraph.predecessors g nid))))

```

)
)

When the basic block of an if statement is entered, we know that the condition of the preceding if statement must be true. As in the GraalVM compiler, we introduce the `registerNewCondition` function which roughly corresponds to the `ConditionalEliminationPhase.registerNewCondition`. This method updates the flow-sensitive stamp information based on the condition which we know must be true.

```
fun clip-upper :: Stamp ⇒ int ⇒ Stamp where
  clip-upper (IntegerStamp b l h) c = (IntegerStamp b l c) |
  clip-upper s c = s
```

```
fun clip-lower :: Stamp ⇒ int ⇒ Stamp where
  clip-lower (IntegerStamp b l h) c = (IntegerStamp b c h) |
  clip-lower s c = s
```

```
fun registerNewCondition :: IRGraph ⇒ Condition ⇒ (ID ⇒ Stamp) ⇒ (ID ⇒ Stamp) where
```

```
  registerNewCondition g (IntegerEqualsNode x y) stamps =
    (stamps(x := join (stamps x) (stamps y)))(y := join (stamps x) (stamps y)) |
```

```
  registerNewCondition g (IntegerLessThanNode x y) stamps =
    (stamps
      (x := clip-upper (stamps x) (stpi-lower (stamps y))))
    (y := clip-lower (stamps y) (stpi-upper (stamps x))) |
  registerNewCondition g - stamps = stamps
```

```
fun hdOr :: 'a list ⇒ 'a ⇒ 'a where
  hdOr (x # xs) de = x |
  hdOr [] de = de
```

The Step relation is a small-step traversal of the graph which handles transitions between individual nodes of the graph.

It relates a pairs of tuple of the current node, the set of seen nodes, the always true stack of IfNode conditions, and the flow-sensitive stamp information.

inductive Step

```
:: IRGraph ⇒ (ID × Seen × Conditions × StampFlow) ⇒ (ID × Seen × Conditions × StampFlow) option ⇒ bool
```

for g **where**

— Hit a BeginNode with an IfNode predecessor which represents the start of a basic block for the IfNode. 1. nid' will be the successor of the begin node. 2. Find the first and only predecessor. 3. Extract condition from the preceding IfNode. 4. Negate condition if the begin node is second branch (we've taken the else branch of the condition) 5. Add the condition or the negated condition to stack 6. Perform any stamp updates based on the condition using the `registerNewCondition` function and place them on the top of the stack of stamp information

```
[[kind g nid = BeginNode nid';
```

$nid \notin seen;$
 $seen' = \{nid\} \cup seen;$

$Some\ ifcond = pred\ g\ nid;$
 $kind\ g\ ifcond = IfNode\ cond\ t\ f;$

$i = find-index\ nid\ (successors-of\ (kind\ g\ ifcond));$
 $c = (if\ i = 0\ then\ kind\ g\ cond\ else\ LogicNegationNode\ cond);$
 $conds' = c \# conds;$

$flow' = registerNewCondition\ g\ c\ (hdOr\ flow\ (stamp\ g))$
 $\implies Step\ g\ (nid, seen, conds, flow)\ (Some\ (nid', seen', conds', flow' \# flow)) \mid$

— Hit an EndNode 1. nid' will be the usage of EndNode 2. pop the conditions and stamp stack

$\llbracket kind\ g\ nid = EndNode;$

$nid \notin seen;$
 $seen' = \{nid\} \cup seen;$

$nid' = any-usage\ g\ nid;$

$conds' = tl\ conds;$
 $flow' = tl\ flow$

$\implies Step\ g\ (nid, seen, conds, flow)\ (Some\ (nid', seen', conds', flow')) \mid$

— We can find a successor edge that is not in seen, go there

$\llbracket \neg(is-EndNode\ (kind\ g\ nid));$
 $\neg(is-BeginNode\ (kind\ g\ nid));$

$nid \notin seen;$
 $seen' = \{nid\} \cup seen;$

$Some\ nid' = nextEdge\ seen'\ nid\ g$

$\implies Step\ g\ (nid, seen, conds, flow)\ (Some\ (nid', seen', conds, flow)) \mid$

— We can cannot find a successor edge that is not in seen, give back None

$\llbracket \neg(is-EndNode\ (kind\ g\ nid));$
 $\neg(is-BeginNode\ (kind\ g\ nid));$

$nid \notin seen;$
 $seen' = \{nid\} \cup seen;$

$None = nextEdge\ seen'\ nid\ g$

$\implies Step\ g\ (nid, seen, conds, flow)\ None \mid$

— We've already seen this node, give back None

$\llbracket nid \in seen \rrbracket \implies Step\ g\ (nid, seen, conds, flow)\ None$

code-pred (*modes: i \Rightarrow i \Rightarrow o \Rightarrow bool*) *Step* .

The ConditionalEliminationPhase relation is responsible for combining the individual traversal steps from the Step relation and the optimizations from the ConditionalEliminationStep relation to perform a transformation of the whole graph.

end