Veriopt Theories

December 9, 2022

Contents

1 Canonicalization Optimizations

theory *Common*
  imports
    OptimizationDSL.Canonicalization
    Semantics.IRTreeEvalThms
begin

lemma *size-pos*[*size-simps*]: *0* < *size y*
  apply (*induction y*; *auto?*)
 by (*smt* (*z3*) *add-2-eq-Suc′ add-is-0 not-gr0 size.elims size.simps*(*12*) *size.simps*(*13*) *size.simps*(*14*) *size.simps*(*15*) *zero-neq-numeral zero-neq-one*)

lemma *size-non-add*[*size-simps*]: *size* (*BinaryExpr op a b*) = *size a* + *size b* + *2* $\longleftrightarrow$ ¬(*is-ConstantExpr b*)
by (*induction b*; *induction op*; *auto simp*: *is-ConstantExpr-def*)

lemma *size-non-const*[*size-simps*]:
¬ *is-ConstantExpr y* $\Longrightarrow$ *1* < *size y*
using *size-pos* **apply** (*induction y*; *auto*)
by (*metis Suc-lessI add-is-1 is-ConstantExpr-def le-less linorder-not-le n-not-Suc-n numeral-2-eq-2 pos2 size.simps*(*2*) *size-non-add*)

lemma *size-binary-const*[*size-simps*]:
size (*BinaryExpr op a b*) = *size a* + *2* $\longleftrightarrow$ (*is-ConstantExpr b*)
by (*induction b*; *auto simp*: *is-ConstantExpr-def size-pos*)

lemma *size-flip-binary*[*size-simps*]:
¬(*is-ConstantExpr y*) $\longrightarrow$ *size* (*BinaryExpr op* (*ConstantExpr x*) *y*) > *size* (*BinaryExpr op y* (*ConstantExpr x*))
by (*metis add-Suc not-less-eq order-less-asym plus-1-eq-Suc size.simps*(*11*) *size.simps*(*2*) *size-non-add*)

lemma *size-binary-lhs-a*[*size-simps*]:
size (*BinaryExpr op* (*BinaryExpr op′ a b*) *c*) > *size a*
by (*metis add-lessD1 less-add-same-cancel1 pos2 size-binary-const size-non-add*)

lemma *size-binary-lhs-b*[*size-simps*]:
size (*BinaryExpr op* (*BinaryExpr op′ a b*) *c*) > *size b*
by (*metis IRExpr.disc*(*42*) *One-nat-def add.left-commute add.right-neutral is-ConstantExpr-def less-add-Suc2 numeral-2-eq-2 plus-1-eq-Suc size.simps*(*11*) *size-binary-const size-non-add size-non-const trans-less-add1*)

lemma *size-binary-lhs-c*[*size-simps*]:
size (*BinaryExpr op* (*BinaryExpr op′ a b*) *c*) > *size c*
by (*metis IRExpr.disc*(*42*) *add.left-commute add.right-neutral is-ConstantExpr-def less-Suc-eq numeral-2-eq-2 plus-1-eq-Suc size.simps*(*11*) *size-non-add size-non-const trans-less-add2*)

lemma *size-binary-rhs-a*[*size-simps*]:
size (*BinaryExpr op c* (*BinaryExpr op′ a b*)) > *size a*
by (*smt* (*verit*, *best*) *less-Suc-eq less-add-Suc2 less-add-same-cancel1 linorder-neqE-nat not-add-less1 order-less-trans pos2 size.simps*(*4*) *size-binary-const size-non-add*)

lemma *size-binary-rhs-b*[*size-simps*]:
size (*BinaryExpr op c* (*BinaryExpr op′ a b*)) > *size b*
by (*metis add.left-commute add.right-neutral is-ConstantExpr-def lessI numeral-2-eq-2 plus-1-eq-Suc size.simps*(*11*) *size.simps*(*4*) *size-non-add trans-less-add2*)

lemma *size-binary-rhs-c*[*size-simps*]:
size (*BinaryExpr op c* (*BinaryExpr op′ a b*)) > *size c*

by *simp*

lemma *size-binary-lhs*[*size-simps*]:
size (*BinaryExpr op x y*) > *size x*
by (*metis One-nat-def Suc-eq-plus1 add-Suc-right less-add-Suc1 numeral-2-eq-2 size-binary-const size-non-add*)

lemma *size-binary-rhs*[*size-simps*]:
size (*BinaryExpr op x y*) > *size y*
by (*metis IRExpr.disc*(*42*) *add-strict-increasing is-ConstantExpr-def linorder-not-le not-add-less1 size.simps*(*11*) *size-non-add size-non-const size-pos*)

lemmas *arith*[*size-simps*] = *Suc-leI add-strict-increasing order-less-trans trans-less-add2*

definition *well-formed-equal* :: *Value* ⇒ *Value* ⇒ *bool*
(**infix** ≈ *50*) **where**
well-formed-equal v_1 v_2 = (v_1 ≠ *UndefVal* ⟶ v_1 = v_2)

lemma *well-formed-equal-defn* [*simp*]:
well-formed-equal v_1 v_2 = (v_1 ≠ *UndefVal* ⟶ v_1 = v_2)
unfolding *well-formed-equal-def* **by** *simp*

end

1.1 AbsNode Phase

theory *AbsPhase*
imports
Common
begin

phase *AbsNode*
terminating *size*
begin

lemma *abs-pos*:
fixes *v* :: (*'a* :: *len word*)
assumes *0* ≤*s v*
shows (*if v* <*s 0 then* − *v else v*) = *v*
by (*simp add*: *assms signed.leD*)

lemma *abs-neg*:
fixes *v* :: (*'a* :: *len word*)
assumes *v* <*s 0*
assumes −(*2* ^ (*Nat.size v* − *1*)) <*s v*
shows (*if v* <*s 0 then* − *v else v*) = − *v* ∧ *0* <*s* −*v*

```
  by (smt (verit, ccfv-SIG) assms(1) assms(2) signed-take-bit-int-greater-eq-minus-exp

     signed-take-bit-int-greater-eq-self-iff sint-0 sint-word-ariths(4) word-sless-alt)

lemma abs-max-neg:
  fixes v :: ('a :: len word)
  assumes v <s 0
  assumes − (2 ^ (Nat.size v − 1)) = v
  shows −v = v
  using assms
  by (metis One-nat-def add.inverse-neutral double-eq-zero-iff mult-minus-right size-word.rep-eq)

lemma final-abs:
  fixes v :: ('a :: len word)
  assumes take-bit (Nat.size v) v = v
  assumes − (2 ^ (Nat.size v − 1)) ≠ v
  shows 0 ≤s (if v <s 0 then −v else v)

proof (cases v <s 0)
  case True
  then show ?thesis
  proof (cases v = − (2 ^ (Nat.size v − 1)))
    case True
    then show ?thesis using abs-max-neg
      using assms by presburger
  next
    case False
    then have − (2 ^ (Nat.size v − 1)) <s v
      unfolding word-sless-def using signed-take-bit-int-greater-self-iff
        by (smt (verit, best) One-nat-def diff-less double-eq-zero-iff len-gt-0 lessI
less-irrefl
          mult-minus-right neg-equal-0-iff-equal signed.rep-eq signed-of-int
          signed-take-bit-int-greater-eq-self-iff signed-word-eqI sint-0 sint-range-size
         sint-sbintrunc' sint-word-ariths(4) size-word.rep-eq unsigned-0 word-2p-lem

          word-sless.rep-eq word-sless-def)
    then show ?thesis
      using abs-neg abs-pos signed.nless-le by auto
  qed
next
  case False
  then show ?thesis using abs-pos by auto
qed



lemma wf-abs: is-IntVal x ⟹ intval-abs x ≠ UndefVal
  using intval-abs.simps unfolding new-int.simps
  using is-IntVal-def by force
```

fun *bin-abs* :: *'a* ::*len word* ⇒ *'a* ::*len word* **where**
bin-abs v = (*if* (*v* <*s 0*) *then* (− *v*) *else v*)

lemma *val-abs-zero*:
intval-abs (*new-int b 0*) = *new-int b 0*
by *simp*

lemma *less-eq-zero*:
assumes *val-to-bool* (*val*[(*IntVal b 0*) < (*IntVal b v*)])
shows *int-signed-value b v* > *0*
using *assms* **unfolding** *intval-less-than.simps*(*1*) **apply** *simp*
by (*metis bool-to-val.elims val-to-bool.simps*(*1*))

lemma *val-abs-pos*:
assumes *val-to-bool*(*val*[(*new-int b 0*) < (*new-int b v*)])
shows *intval-abs* (*new-int b v*) = (*new-int b v*)
using *assms* **using** *less-eq-zero* **unfolding** *intval-abs.simps new-int.simps*
by *force*

lemma *val-abs-neg*:
assumes *val-to-bool*(*val*[(*new-int b v*) < (*new-int b 0*)])
shows *intval-abs* (*new-int b v*) = *intval-negate* (*new-int b v*)
using *assms* **using** *less-eq-zero* **unfolding** *intval-abs.simps new-int.simps*
by *force*

lemma *val-bool-unwrap*:
val-to-bool (*bool-to-val v*) = *v*
by (*metis bool-to-val.elims one-neq-zero val-to-bool.simps*(*1*))

lemma *take-bit-unwrap*:
b = *64* ⟹ *take-bit b* (*v1*::*64 word*) = *v1*
by (*metis size64 size-word.rep-eq take-bit-length-eq*)

lemma *bit-less-eq-def*:
fixes *v1 v2* :: *64 word*
assumes *b* ≤ *64*
shows *sint* (*signed-take-bit* (*b* − *Suc* (*0*::*nat*)) (*take-bit b v1*))
< *sint* (*signed-take-bit* (*b* − *Suc* (*0*::*nat*)) (*take-bit b v2*)) ⟷
signed-take-bit (*63*::*nat*) (*Word.rep v1*) < *signed-take-bit* (*63*::*nat*) (*Word.rep*
v2)
using *assms* **sorry**

lemma *less-eq-def*:

shows *val-to-bool*(*val*[(*new-int b v1*) < (*new-int b v2*)]) ⟷ *v1* <*s v2*
unfolding *new-int.simps intval-less-than.simps bool-to-val-bin.simps bool-to-val.simps*

int-signed-value.simps
apply (*simp add*: *val-bool-unwrap*) **apply** *auto*
unfolding *word-sless-def* **apply** *auto*
unfolding *signed-def* **apply** *auto*
using *bit-less-eq-def* **apply** (*metis bot-nat-0.extremum take-bit-0*)
by (*metis bit-less-eq-def bot-nat-0.extremum take-bit-0*)

lemma *val-abs-always-pos*:
assumes *intval-abs* (*new-int b v*) = (*new-int b v'*)
shows $0 \leq s\ v'$
using *assms*
proof (*cases v = 0*)
case *True*
then have $v' = 0$
using *val-abs-zero assms*
by (*smt* (*verit*, *ccfv-threshold*) *Suc-diff-1 bit-less-eq-def bot-nat-0.extremum*
diff-is-0-eq
len-gt-0 len-of-numeral-defs(*2*) *order-le-less signed-eq-0-iff take-bit-0*
take-bit-signed-take-bit take-bit-unwrap)
then show *?thesis* **by** *simp*
next
case *neq0*: *False*
then show *?thesis*
proof (*cases val-to-bool*(*val*[(*new-int b 0*) < (*new-int b v*)]))
case *True*
then show *?thesis* **using** *less-eq-def*
using *assms val-abs-pos*
by (*smt* (*verit*, *ccfv-SIG*) *One-nat-def Suc-leI bit.compl-one bit-less-eq-def*
cancel-comm-monoid-add-class.diff-cancel diff-zero len-gt-0 len-of-numeral-defs(*2*)

mask-0 mask-1 one-le-numeral one-neq-zero signed-word-eqI take-bit-dist-subL

take-bit-minus-one-eq-mask take-bit-not-eq-mask-diff take-bit-signed-take-bit

zero-le-numeral)
next
case *False*
then have *val-to-bool*(*val*[(*new-int b v*) < (*new-int b 0*)])
using *neq0 less-eq-def*
by (*metis signed.neqE*)
then show *?thesis* **using** *val-abs-neg less-eq-def* **unfolding** *new-int.simps*
intval-negate.simps
by (*metis signed.nless-le take-bit-0*)
qed

qed

lemma *intval-abs-elims*:
  assumes *intval-abs* $x \neq$ *UndefVal*
  shows $\exists$ *t v* . x = *IntVal t v* $\wedge$ *intval-abs x* = *new-int t* (*if int-signed-value t v* < *0 then* $-$ *v else v*)
  using *assms*
  by (*meson intval-abs.elims*)

lemma *wf-abs-new-int*:
  assumes *intval-abs* (*IntVal t v*) $\neq$ *UndefVal*
  shows *intval-abs* (*IntVal t v*) = *new-int t v* $\vee$ *intval-abs* (*IntVal t v*) = *new-int t* ($-v$)
  using *assms*
  using *intval-abs.simps*(*1*) **by** *presburger*

lemma *mono-undef-abs*:
  assumes *intval-abs* (*intval-abs x*) $\neq$ *UndefVal*
  shows *intval-abs x* $\neq$ *UndefVal*
  using *assms*
  by *force*

lemma *val-abs-idem*:
  assumes *intval-abs*(*intval-abs*(*x*)) $\neq$ *UndefVal*
  shows *intval-abs*(*intval-abs*(*x*)) = *intval-abs x*
  using *assms*
proof $-$
  obtain *b v* **where** *in-def*: *intval-abs x* = *new-int b v*
    using *assms intval-abs-elims mono-undef-abs* **by** *blast*
  then show *?thesis*
  proof (*cases val-to-bool*(*val*[(*new-int b v*) < (*new-int b 0*)]))
    case *True*
    then have *nested*: (*intval-abs* (*intval-abs x*)) = *new-int b* ($-v$)
      using *val-abs-neg intval-negate.simps in-def*
      by *simp*
    then have x = *new-int b* ($-v$)
      using *in-def True* **unfolding** *new-int.simps*
     by (*smt* (*verit*, *best*) *intval-abs.simps*(*1*) *less-eq-def less-eq-zero less-numeral-extra*(*1*)

        mask-1 mask-eq-take-bit-minus-one neg-one.elims neg-one-signed new-int.simps

            one-le-numeral one-neq-zero signed.neqE signed.not-less take-bit-of-0 val-abs-always-pos)
    then show *?thesis* **using** *val-abs-always-pos*
      using *True in-def less-eq-def signed.leD*
      using *signed.nless-le* **by** *blast*
  next
    case *False*
    then show *?thesis*
      using *in-def* **by** *force*

```
  qed
qed

lemma val-abs-negate:
  assumes intval-abs (intval-negate x) ≠ UndefVal
  shows   intval-abs (intval-negate x) = intval-abs x
  using assms apply (cases x; auto)
  apply (metis less-eq-def new-int.simps signed.dual-order.strict-iff-not signed.less-linear

         take-bit-0)
  by (smt (verit, ccfv-threshold) add.inverse-neutral intval-abs.simps(1) less-eq-def
less-eq-zero
    less-numeral-extra(1) mask-1 mask-eq-take-bit-minus-one neg-one.elims neg-one-signed

    new-int.simps one-le-numeral one-neq-zero signed.order.order-iff-strict take-bit-of-0

      val-abs-always-pos)
```

Optimisations

```
optimization AbsIdempotence: abs(abs(x)) ⟼  abs(x)
   apply auto
  by (metis UnaryExpr unary-eval.simps(1) val-abs-idem)

optimization AbsNegate: (abs(−x)) ⟼  abs(x)
    apply auto using val-abs-negate
  by (metis unary-eval.simps(1) unfold-unary)

end

end
```

1.2 AddNode Phase

```
theory AddPhase
  imports
    Common
begin

phase AddNode
  terminating size
begin

lemma binadd-commute:
  assumes bin-eval BinAdd x y ≠ UndefVal
  shows bin-eval BinAdd x y = bin-eval BinAdd y x
  using assms intval-add-sym by simp
```

```
optimization AddShiftConstantRight: ((const v) + y) ⟼ y + (const v) when
¬(is-ConstantExpr y)
  using size-non-const
 apply (metis add-2-eq-Suc' less-Suc-eq plus-1-eq-Suc size.simps(11) size-non-add)
  unfolding le-expr-def
  apply (rule impI)
  subgoal premises 1
    apply (rule allI impI)+

    subgoal premises 2 for m p va
      apply (rule BinaryExprE[OF 2])
      subgoal premises 3 for x ya
        apply (rule BinaryExpr)
        using 3 apply simp
        using 3 apply simp
        using 3 binadd-commute apply auto
        done
      done
    done
  done


optimization AddShiftConstantRight2: ((const v) + y) ⟼ y + (const v) when
¬(is-ConstantExpr y)
  unfolding le-expr-def
   apply (auto simp: intval-add-sym)

  using size-non-const
  by (metis add-2-eq-Suc' lessI plus-1-eq-Suc size.simps(11) size-non-add)




lemma is-neutral-0 [simp]:
  assumes 1: intval-add (IntVal b x) (IntVal b 0) ≠ UndefVal
  shows intval-add (IntVal b x) (IntVal b 0) = (new-int b x)
  using 1 by auto

optimization AddNeutral: (e + (const (IntVal 32 0))) ⟼ e
  unfolding le-expr-def apply auto
  using is-neutral-0 eval-unused-bits-zero
  by (smt (verit) add-cancel-left-right intval-add.elims val-to-bool.simps(1))


ML-val ‹@{term ‹x = y›}›

lemma NeutralLeftSubVal:
  assumes e1 = new-int b ival
```

shows *val*[(*e1* − *e2*) + *e2*] ≈ *e1*
apply *simp* **using** *assms* **by** (*cases e1*; *cases e2*; *auto*)

optimization *RedundantSubAdd*: $((e_1 - e_2) + e_2) \longmapsto e_1$
apply *auto* **using** *eval-unused-bits-zero NeutralLeftSubVal*
unfolding *well-formed-equal-defn*
by (*smt* (*verit*) *evalDet intval-sub.elims new-int.elims*)

lemma *allE2*: $(\forall\, x\ y.\ P\ x\ y) \Longrightarrow (P\ a\ b \Longrightarrow R) \Longrightarrow R$
by *simp*

lemma *just-goal2*:
assumes *1*: (∀ *a b*. (*intval-add* (*intval-sub a b*) *b* ≠ *UndefVal* ∧ *a* ≠ *UndefVal*
⟶
intval-add (*intval-sub a b*) *b* = *a*))
shows (*BinaryExpr BinAdd* (*BinaryExpr BinSub* e_1 e_2) e_2) ≥ e_1
unfolding *le-expr-def unfold-binary bin-eval.simps*
by (*metis 1 evalDet evaltree-not-undef*)

optimization *RedundantSubAdd2*: $e_2 + (e_1 - e_2) \longmapsto e_1$
apply (*metis add.commute add-less-cancel-right less-add-Suc2 plus-1-eq-Suc size-binary-const size-non-add trans-less-add2*)
by (*smt* (*verit*, *del-insts*) *BinaryExpr BinaryExprE RedundantSubAdd*(*1*) *bin-add-commute le-expr-def rewrite-preservation.simps*(*1*))

lemma *AddToSubHelperLowLevel*:
shows *intval-add* (*intval-negate e*) *y* = *intval-sub y e* (**is** *?x* = *?y*)
by (*induction y*; *induction e*; *auto*)

print-phases

lemma *val-redundant-add-sub*:
assumes *a* = *new-int bb ival*
assumes *val*[*b* + *a*] ≠ *UndefVal*
shows *val*[(*b* + *a*) − *b*] = *a*

using *assms* **apply** (*cases a*; *cases b*; *auto*)
by *presburger*

lemma *val-add-right-negate-to-sub*:
assumes $val[x + e] \neq UndefVal$
shows $val[x + (-e)] = val[x - e]$
using *assms* **by** (*cases x*; *cases e*; *auto*)

lemma *exp-add-left-negate-to-sub*:
$exp[-e + y] \geq exp[y - e]$
apply (*cases e*; *cases y*; *auto*)
using *AddToSubHelperLowLevel* **by** *auto+*

Optimisations

optimization *RedundantAddSub*: $(b + a) - b \longmapsto a$
apply *auto*
by (*smt* (*verit*) *evalDet intval-add.elims new-int.elims val-redundant-add-sub eval-unused-bits-zero*)

optimization *AddRightNegateToSub*: $x + -e \longmapsto x - e$
apply (*metis Nat.add-0-right add-2-eq-Suc′ add-less-mono1 add-mono-thms-linordered-field*(*2*)
less-SucI not-less-less-Suc-eq size-binary-const size-non-add size-pos)
using *AddToSubHelperLowLevel intval-add-sym* **by** *auto*

optimization *AddLeftNegateToSub*: $-e + y \longmapsto y - e$
apply (*smt* (*verit*, *best*) *One-nat-def add.commute add-Suc-right is-ConstantExpr-def less-add-Suc2*
numeral-2-eq-2 plus-1-eq-Suc size.simps(*1*) *size.simps*(*11*) *size-binary-const size-non-add*)
using *exp-add-left-negate-to-sub* **by** *blast*

end

end

1.3 AndNode Phase

theory *AndPhase*
imports
Common
Proofs.StampEvalThms
begin

```
context stamp-mask
begin

lemma AndRightFallthrough: (((and (not (↓ x)) (↑ y)) = 0)) ⟶ exp[x & y] ≥
exp[y]
  apply simp apply (rule impI; (rule allI)+)
  apply (rule impI)
  subgoal premises p for m p v
  proof −
    obtain xv where xv: [m, p] ⊢ x ↦ xv
      using p(2) by blast
    obtain yv where yv: [m, p] ⊢ y ↦ yv
      using p(2) by blast
    have v = val[xv & yv]
      using p(2) xv yv
      by (metis BinaryExprE bin-eval.simps(4) evalDet)
    then have v = yv
      using p(1) not-down-up-mask-and-zero-implies-zero
     by (smt (verit) eval-unused-bits-zero intval-and.elims new-int.elims new-int-bin.elims
p(2)
          unfold-binary xv yv)
    then show ?thesis using yv by simp
  qed
  done

lemma AndLeftFallthrough: (((and (not (↓ y)) (↑ x)) = 0)) ⟶ exp[x & y] ≥
exp[x]
  apply simp apply (rule impI; (rule allI)+)
  apply (rule impI)
  subgoal premises p for m p v
  proof −
    obtain xv where xv: [m, p] ⊢ x ↦ xv
      using p(2) by blast
    obtain yv where yv: [m, p] ⊢ y ↦ yv
      using p(2) by blast
    have v = val[xv & yv]
      using p(2) xv yv
      by (metis BinaryExprE bin-eval.simps(4) evalDet)
    then have v = xv
      using p(1) not-down-up-mask-and-zero-implies-zero
     by (smt (verit) and.commute eval-unused-bits-zero intval-and.elims new-int.simps

          new-int-bin.simps p(2) unfold-binary xv yv)
    then show ?thesis using xv by simp
  qed
  done
end
```

phase *AndNode*
  terminating *size*
begin

lemma *bin-and-nots*:
 $(\sim x \ \& \ \sim y) = (\sim(x \mid y))$
  by *simp*

lemma *bin-and-neutral*:
 $(x \ \& \ \sim False) = x$
  by *simp*

lemma *val-and-equal*:
  assumes $x = new\text{-}int\ b\ v$
  and $val[x \ \& \ x] \neq UndefVal$
  shows $val[x \ \& \ x] = x$
   using *assms* **by** (*cases x*; *auto*)

lemma *val-and-nots*:
  $val[\sim x \ \& \ \sim y] = val[\sim(x \mid y)]$
  apply (*cases x*; *cases y*; *auto*) **by** (*simp add*: *take-bit-not-take-bit*)

lemma *val-and-neutral*:
  assumes $x = new\text{-}int\ b\ v$
  and $val[x \ \& \ \sim(new\text{-}int\ b'\ 0)] \neq UndefVal$
  shows $val[x \ \& \ \sim(new\text{-}int\ b'\ 0)] = x$
   using *assms* **apply** (*cases x*; *auto*) **apply** (*simp add*: *take-bit-eq-mask*)
   by *presburger*

lemma *val-and-zero*:
  assumes $x = new\text{-}int\ b\ v$
  shows $val[x \ \& \ (IntVal\ b\ 0)] = IntVal\ b\ 0$
   using *assms* **by** (*cases x*; *auto*)

lemma *exp-and-equal*:
  $exp[x \ \& \ x] \geq exp[x]$
   apply *auto*
 by (*smt* (*verit*) *evalDet intval-and.elims new-int.elims val-and-equal eval-unused-bits-zero*)

lemma *exp-and-nots*:
  $exp[\sim x \ \& \ \sim y] \geq exp[\sim(x \mid y)]$
   apply (*cases x*; *cases y*; *auto*) **using** *val-and-nots*

by *fastforce+*

lemma *exp-sign-extend*:
assumes $e = (1 << In) - 1$
shows *BinaryExpr BinAnd* (*UnaryExpr* (*UnarySignExtend In Out*) *x*)
(*ConstantExpr* (*new-int b e*))
$\geq$ (*UnaryExpr* (*UnaryZeroExtend In Out*) *x*)
apply *auto*
subgoal premises *p* **for** *m p va*
proof −
obtain *va* **where** *va*: [*m*,*p*] ⊢ *x* ↦ *va*
using *p*(*2*) **by** *auto*
then have *va* ≠ *UndefVal*
by (*simp add*: *evaltree-not-undef*)
then have *1*: *intval-and* (*intval-sign-extend In Out va*) (*IntVal b* (*take-bit b*
e)) ≠ *UndefVal*
using *evalDet p*(*1*) *p*(*2*) *va* **by** *blast*
then have *2*: *intval-sign-extend In Out va* ≠ *UndefVal*
by *auto*
then have *21*: (*0*::*nat*) < *b*
using *eval-bits-1-64 p*(*4*) **by** *blast*
then have *3*: *b* ⊑ (*64*::*nat*)
using *eval-bits-1-64 p*(*4*) **by** *blast*
then have *4*: − ((*2*::*int*) ^ *b div* (*2*::*int*)) ⊑ *sint* (*signed-take-bit* (*b* − *Suc*
(*0*::*nat*)) (*take-bit b e*))
by (*simp add*: *21 int-power-div-base signed-take-bit-int-greater-eq-minus-exp-word*)
then have *5*: *sint* (*signed-take-bit* (*b* − *Suc* (*0*::*nat*)) (*take-bit b e*)) < (*2*::*int*)
^ *b div* (*2*::*int*)
by (*simp add*: *21 3 Suc-le-lessD int-power-div-base signed-take-bit-int-less-exp-word*)
then have *6*: [*m*,*p*] ⊢ *UnaryExpr* (*UnaryZeroExtend In Out*)
x ↦ *intval-and* (*intval-sign-extend In Out va*) (*IntVal b* (*take-bit b e*))
apply (*cases va*; *simp*)
apply (*simp add*: ‹(*va*::*Value*) ≠ *UndefVal*›) **defer**
subgoal premises *p* **for** *x3*
proof −
have *va* = *ObjRef x3*
using *p*(*1*) **by** *auto*
then have *sint* (*signed-take-bit* (*b* − *Suc* (*0*::*nat*)) (*take-bit b e*)) <
(*2*::*int*) ^ *b div* (*2*::*int*)
by (*simp add*: *5*)
then show *?thesis*
using *2 intval-sign-extend.simps*(*3*) *p*(*1*) **by** *blast*
qed

subgoal premises *p* **for** *x4*
proof −
have *sg1*: *va* = *ObjStr x4*
using *2 p*(*1*) **by** *auto*
then have *sint* (*signed-take-bit* (*b* − *Suc* (*0*::*nat*)) (*take-bit b e*)) <

```
(2::int) ^ b div (2::int)
             by (simp add: 5)
           then show ?thesis
             using 1 sg1 by auto
         qed

         subgoal premises p for x21 x22
           proof -
             have sgg1: va = IntVal x21 x22
               by (simp add: p(1))
            then have sgg2: sint (signed-take-bit (b − Suc (0::nat)) (take-bit b e))
< (2::int) ^ b div (2::int)
               by (simp add: 5)
             then show ?thesis
               sorry
             qed
           done
     then show ?thesis
       by (metis evalDet p(2) va)
   qed
 done
```

```
lemma val-and-commute[simp]:
   val[x & y] = val[y & x]
   apply (cases x; cases y; auto)
  by (simp add: word-bw-comms(1))
```

Optimisations

```
optimization AndEqual: x & x ⟼ x
  using exp-and-equal by blast

optimization AndShiftConstantRight: ((const x) & y) ⟼ y & (const x)
                                        when ¬(is-ConstantExpr y)
  using size-flip-binary by auto

optimization AndNots: (~x) & (~y) ⟼ ~(x | y)
   apply (metis add-2-eq-Suc' less-SucI less-add-Suc1 not-less-eq size-binary-const
size-non-add)
  using exp-and-nots by presburger


optimization AndSignExtend: BinaryExpr BinAnd (UnaryExpr (UnarySignExtend
In Out) (x))
```

$(const\ (new\text{-}int\ b\ e))$
$\longmapsto (UnaryExpr\ (UnaryZeroExtend\ In\ Out)\ (x))$
when $(e = (1 << In) - 1)$
using *exp-sign-extend* **by** *simp*

optimization *AndNeutral*: $(x\ \&\ {}^{\sim}(const\ (IntVal\ b\ 0))) \longmapsto x$
when $(wf\text{-}stamp\ x \wedge stamp\text{-}expr\ x = IntegerStamp\ b\ lo\ hi)$
apply *auto*
by (*smt* (*verit*) *Value.sel*(*1*) *eval-unused-bits-zero intval-and.elims intval-word.simps*

new-int.simps new-int-bin.simps take-bit-eq-mask)

optimization *AndRightFallThrough*: $(x\ \&\ y) \longmapsto y$
when $(((and\ (not\ (IRExpr\text{-}down\ x))\ (IRExpr\text{-}up\ y)) = 0))$
by (*simp add*: *IRExpr-down-def IRExpr-up-def*)

optimization *AndLeftFallThrough*: $(x\ \&\ y) \longmapsto x$
when $(((and\ (not\ (IRExpr\text{-}down\ y))\ (IRExpr\text{-}up\ x)) = 0))$
by (*simp add*: *IRExpr-down-def IRExpr-up-def*)

end

end

1.4 BinaryNode Phase

theory *BinaryNode*
imports
Common
begin

phase *BinaryNode*
terminating *size*
begin

optimization *BinaryFoldConstant*: *BinaryExpr op* (*const v1*) (*const v2*) $\longmapsto$ *ConstantExpr* (*bin-eval op v1 v2*)
unfolding *le-expr-def*
apply (*rule allI impI*)+
subgoal premises *bin* **for** *m p v*
print-facts
apply (*rule BinaryExprE*[*OF bin*])
subgoal premises *prems* **for** *x y*
print-facts

proof –

```
    have x: x = v1 using prems by auto
    have y: y = v2 using prems by auto
    have xy: v = bin-eval op x y using prems x y by simp
    have int: ∃ b vv . v = new-int b vv using bin-eval-new-int prems by fast
    show ?thesis
      unfolding prems x y xy
      apply (rule ConstantExpr)
      using prems x y xy int sorry
    qed
  done
done

print-facts

end

end
```

1.5 ConditionalNode Phase

```
theory ConditionalPhase
  imports
    Common
    Proofs.StampEvalThms
begin

phase ConditionalNode
  terminating size
begin

lemma negates: ∃ v b. e = IntVal b v ∧ b > 0 ⟹ val-to-bool (val[e]) ⟷
¬(val-to-bool (val[!e]))
  unfolding intval-logic-negation.simps
  by (metis (mono-tags, lifting) intval-logic-negation.simps(1) logic-negate-def new-int.simps
of-bool-eq(2) one-neq-zero take-bit-of-0 take-bit-of-1 val-to-bool.simps(1))

lemma negation-condition-intval:
  assumes e = IntVal b ie
  assumes 0 < b
  shows val[(!e) ? x : y] = val[e ? y : x]
  using assms by (cases e; auto simp: negates logic-negate-def)

lemma negation-preserve-eval:
  assumes [m, p] ⊢ exp[!e] ↦ v
  shows ∃ v'. ([m, p] ⊢ exp[e] ↦ v') ∧ v = val[!v']
  using assms by auto

lemma negation-preserve-eval-intval:
  assumes [m, p] ⊢ exp[!e] ↦ v
```

shows $\exists\, v'\ b\ vv.\ ([m,\ p] \vdash\ exp[e] \mapsto v') \wedge v' = IntVal\ b\ vv \wedge b > 0$
using *assms*
by (*metis eval-bits-1-64 intval-logic-negation.elims negation-preserve-eval unfold-unary*)

optimization *NegateConditionFlipBranches*: $((!e)\ ?\ x : y) \longmapsto (e\ ?\ y : x)$
apply *simp* **using** *negation-condition-intval negation-preserve-eval-intval*
by (*smt* (*verit*, *best*) *ConditionalExpr ConditionalExprE Value.distinct*(*1*) *evalDet negates negation-preserve-eval*)

optimization *DefaultTrueBranch*: $(true\ ?\ x : y) \longmapsto x$ **.**

optimization *DefaultFalseBranch*: $(false\ ?\ x : y) \longmapsto y$ **.**

optimization *ConditionalEqualBranches*: $(e\ ?\ x : x) \longmapsto x$ **.**

optimization *condition-bounds-x*: $((u < v)\ ?\ x : y) \longmapsto x$
when (*stamp-under* (*stamp-expr u*) (*stamp-expr v*) $\wedge$ *wf-stamp u* $\wedge$ *wf-stamp v*)
using *stamp-under-defn* **by** *fastforce*

optimization *condition-bounds-y*: $((u < v)\ ?\ x : y) \longmapsto y$
when (*stamp-under* (*stamp-expr v*) (*stamp-expr u*) $\wedge$ *wf-stamp u* $\wedge$ *wf-stamp v*)
using *stamp-under-defn-inverse* **by** *fastforce*

lemma *val-optimise-integer-test*:
assumes $\exists\, v.\ x = IntVal\ 32\ v$
shows $val[((x\ \&\ (IntVal\ 32\ 1))\ eq\ (IntVal\ 32\ 0))\ ?\ (IntVal\ 32\ 0) : (IntVal\ 32\ 1)] =$
$val[x\ \&\ IntVal\ 32\ 1]$
using *assms* **apply** *auto*
apply (*metis* (*full-types*) *bool-to-val.simps*(*2*) *val-to-bool.simps*(*1*))
by (*metis* (*mono-tags*, *lifting*) *and-one-eq bool-to-val.simps*(*1*) *even-iff-mod-2-eq-zero odd-iff-mod-2-eq-one val-to-bool.simps*(*1*))

optimization *ConditionalEliminateKnownLess*: $((x < y)\ ?\ x : y) \longmapsto x$
when (*stamp-under* (*stamp-expr x*) (*stamp-expr y*)
$\wedge$ *wf-stamp x* $\wedge$ *wf-stamp y*)
using *stamp-under-defn* **by** *fastforce*

optimization *ConditionalEqualIsRHS*: $((x\ eq\ y)\ ?\ x : y) \longmapsto y$
apply *auto*
by (*smt* (*verit*) *Value.inject*(*1*) *bool-to-val.simps*(*2*) *bool-to-val-bin.simps evalDet*

intval-equals.elims val-to-bool.elims(*1*))

optimization *normalizeX*: ((*x eq const* (*IntVal 32 0*)) *?*
(*const* (*IntVal 32 0*)) : (*const* (*IntVal 32 1*))) $\longmapsto$ *x*
when (*IRExpr-up x* = *1*) $\wedge$ *stamp-expr x* = *IntegerStamp b 0 1*
apply *auto*
subgoal premises *p* **for** *m p v xa*
proof −
obtain *xa* **where** *xa*: [*m*,*p*] $\vdash$ *x* $\mapsto$ *xa*
using *p* **by** *blast*
have *3*: [*m*,*p*] $\vdash$ *if val-to-bool* (*intval-equals xa* (*IntVal* (*32*::*nat*) (*0*::*64 word*)))
then ConstantExpr (*IntVal* (*32*::*nat*) (*0*::*64 word*))
else ConstantExpr (*IntVal* (*32*::*nat*) (*1*::*64 word*)) $\mapsto$ *v*
using *evalDet p*(*3*) *p*(*5*) *xa*
using *p*(*4*) *p*(*6*) **by** *blast*
then have *4*: *xa* = *IntVal 32 0* | *xa* = *IntVal 32 1*
sorry
then have *6*: *v* = *xa*
sorry
then show *?thesis*
using *xa* **by** *auto*
qed
done

optimization *normalizeX2*: ((*x eq* (*const* (*IntVal 32 1*))) *?*
(*const* (*IntVal 32 1*)) : (*const* (*IntVal 32 0*))) $\longmapsto$ *x*
when (*x* = *ConstantExpr* (*IntVal 32 0*) | (*x* = *ConstantExpr* (*IntVal 32 1*))) **.**

optimization *flipX*: ((*x eq* (*const* (*IntVal 32 0*))) *?*
(*const* (*IntVal 32 1*)) : (*const* (*IntVal 32 0*))) $\longmapsto$
x $\oplus$ (*const* (*IntVal 32 1*))
when (*x* = *ConstantExpr* (*IntVal 32 0*) | (*x* = *ConstantExpr* (*IntVal 32 1*))) **.**

optimization *flipX2*: ((*x eq* (*const* (*IntVal 32 1*))) *?*
(*const* (*IntVal 32 0*)) : (*const* (*IntVal 32 1*))) $\longmapsto$
x $\oplus$ (*const* (*IntVal 32 1*))
when (*x* = *ConstantExpr* (*IntVal 32 0*) | (*x* = *ConstantExpr* (*IntVal 32 1*))) **.**

lemma *stamp-of-default*:
assumes *stamp-expr x* = *default-stamp*
assumes *wf-stamp x*
shows ([*m*, *p*] $\vdash$ *x* $\mapsto$ *v*) $\longrightarrow$ ($\exists$ *vv*. *v* = *IntVal 32 vv*)
using *assms*
by (*metis default-stamp valid-value-elims*(*3*) *wf-stamp-def*)

optimization *OptimiseIntegerTest*:
(((*x* & (*const* (*IntVal 32 1*))) *eq* (*const* (*IntVal 32 0*))) *?*
(*const* (*IntVal 32 0*)) : (*const* (*IntVal 32 1*))) $\longmapsto$
x & (*const* (*IntVal 32 1*))
when (*stamp-expr x* = *default-stamp* $\wedge$ *wf-stamp x*)
apply *simp* **apply** (*rule impI*; (*rule allI*)+; *rule impI*)
subgoal premises *eval* **for** *m p v*
proof –
obtain *xv* **where** *xv*: [*m*, *p*] $\vdash$ *x* $\mapsto$ *xv*
using *eval* **by** *fast*
then have *x32*: $\exists$ *v*. *xv* = *IntVal 32 v*
using *stamp-of-default eval* **by** *auto*
obtain *lhs* **where** *lhs*: [*m*, *p*] $\vdash$ *exp*[(((*x* & (*const* (*IntVal 32 1*))) *eq* (*const* (*IntVal 32 0*))) *?*
(*const* (*IntVal 32 0*)) : (*const* (*IntVal 32 1*)))] $\mapsto$ *lhs*
using *eval*(*2*) **by** *auto*
then have *lhsV*: *lhs* = *val*[((*xv* & (*IntVal 32 1*)) *eq* (*IntVal 32 0*)) *?* (*IntVal 32 0*) : (*IntVal 32 1*)]
using *xv evaltree.BinaryExpr evaltree.ConstantExpr evaltree.ConditionalExpr*
by (*smt* (*verit*) *ConditionalExprE ConstantExprE bin-eval.simps*(*11*) *bin-eval.simps*(*4*) *evalDet intval-conditional.simps unfold-binary*)
obtain *rhs* **where** *rhs*: [*m*, *p*] $\vdash$ *exp*[*x* & (*const* (*IntVal 32 1*))] $\mapsto$ *rhs*
using *eval*(*2*) **by** *blast*
then have *rhsV*: *rhs* = *val*[*xv* & *IntVal 32 1*]
by (*metis BinaryExprE ConstantExprE bin-eval.simps*(*4*) *evalDet xv*)
have *lhs* = *rhs* **using** *val-optimise-integer-test x32*
using *lhsV rhsV* **by** *presburger*
then show *?thesis*
by (*metis eval*(*2*) *evalDet lhs rhs*)
qed
done

optimization *opt-optimise-integer-test-2*:
(((*x* & (*const* (*IntVal 32 1*))) *eq* (*const* (*IntVal 32 0*))) *?*
(*const* (*IntVal 32 0*)) : (*const* (*IntVal 32 1*))) $\longmapsto$
x
when (*x* = *ConstantExpr* (*IntVal 32 0*) | (*x* = *ConstantExpr* (*IntVal 32 1*))) **.**

end

end

1.6 MulNode Phase

theory *MulPhase*
  imports
    Common
    Proofs.StampEvalThms
begin

fun *mul-size* :: *IRExpr* ⇒ *nat* **where**
  mul-size (*UnaryExpr op e*) = (*mul-size e*) + *2* |
  mul-size (*BinaryExpr BinMul x y*) = ((*mul-size x*) + (*mul-size y*) + *2*) ∗ *2* |
  mul-size (*BinaryExpr op x y*) = (*mul-size x*) + (*mul-size y*) + *2* |
  mul-size (*ConditionalExpr cond t f*) = (*mul-size cond*) + (*mul-size t*) + (*mul-size f*) + *2* |
  mul-size (*ConstantExpr c*) = *1* |
  mul-size (*ParameterExpr ind s*) = *2* |
  mul-size (*LeafExpr nid s*) = *2* |
  mul-size (*ConstantVar c*) = *2* |
  mul-size (*VariableExpr x s*) = *2*

phase *MulNode*
  terminating *mul-size*
begin

lemma *bin-eliminate-redundant-negative*:
  uminus (*x* :: ′*a*::*len word*) ∗ *uminus* (*y* :: ′*a*::*len word*) = *x* ∗ *y*
  by *simp*

lemma *bin-multiply-identity*:
 (*x* :: ′*a*::*len word*) ∗ *1* = *x*
  by *simp*

lemma *bin-multiply-eliminate*:
 (*x* :: ′*a*::*len word*) ∗ *0* = *0*
  by *simp*

lemma *bin-multiply-negative*:
 (*x* :: ′*a*::*len word*) ∗ *uminus 1* = *uminus x*
  by *simp*

lemma *bin-multiply-power-2*:
 (*x*:: ′*a*::*len word*) ∗ (*2*^*j*) = *x* << *j*
  by *simp*

lemma *take-bit64*[*simp*]:

fixes *w* :: *int64*
shows *take-bit 64 w* = *w*
proof −
have *Nat.size w* = *64*
by (*simp add*: *size64*)
then show *?thesis*
by (*metis lt2p-lem mask-eq-iff take-bit-eq-mask verit-comp-simplify1*(*2*) *wsst-TYs*(*3*))
qed

lemma *mergeTakeBit*:
fixes *a* :: *nat*
fixes *b c* :: *64 word*
shows *take-bit a* (*take-bit a* (*b*) ∗ *take-bit a* (*c*)) =
take-bit a (*b* ∗ *c*)
by (*smt* (*verit*, *ccfv-SIG*) *take-bit-mult take-bit-of-int unsigned-take-bit-eq word-mult-def*)

lemma *val-eliminate-redundant-negative*:
assumes *val*[−*x* ∗ −*y*] ≠ *UndefVal*
shows *val*[−*x* ∗ −*y*] = *val*[*x* ∗ *y*]
using *assms* **apply** (*cases x*; *cases y*; *auto*)
using *mergeTakeBit* **by** *auto*

lemma *val-multiply-neutral*:
assumes *x* = *new-int b v*
shows *val*[*x* ∗ (*IntVal b 1*)] = *val*[*x*]
using *assms* **by** *force*

lemma *val-multiply-zero*:
assumes *x* = *new-int b v*
shows *val*[*x* ∗ (*IntVal b 0*)] = *IntVal b 0*
using *assms* **by** *simp*

lemma *val-multiply-negative*:
assumes *x* = *new-int b v*
shows *val*[*x* ∗ *intval-negate* (*IntVal b 1*)] = *intval-negate x*
by (*smt* (*verit*) *Value.disc*(*1*) *Value.inject*(*1*) *add.inverse-neutral intval-negate.simps*(*1*)

is-IntVal-def mask-0 mask-eq-take-bit-minus-one new-int.elims of-bool-eq(*2*)
take-bit-dist-neg
take-bit-of-1 val-eliminate-redundant-negative val-multiply-neutral val-multiply-zero

verit-minus-simplify(*4*) *zero-neq-one assms*)

lemma *val-MulPower2*:

fixes *i* :: *64 word*
assumes $y = IntVal\ 64\ (2\ \hat{}\ unat(i))$
and $0 < i$
and $i < 64$
and $val[x * y] \neq UndefVal$
shows $val[x * y] = val[x << IntVal\ 64\ i]$
using *assms* **apply** (*cases x*; *cases y*; *auto*)
subgoal premises *p* **for** *x2*
proof −
have *63*: (*63* :: *int64*) = *mask 6*
by *eval*
then have (*2*::*int*) ^ *6* = *64*
by *eval*
then have *uint i* < (*2*::*int*) ^ *6*
by (*metis linorder-not-less lt2p-lem of-int-numeral p*(*4*) *size64 word-2p-lem word-of-int-2p*
wsst-TYs(*3*))
then have *and i* (*mask 6*) = *i*
using *mask-eq-iff* **by** *blast*
then show *x2* << *unat i* = *x2* << *unat* (*and i* (*63*::*64 word*))
unfolding *63*
by *force*
qed
by *presburger*

lemma *val-MulPower2Add1*:
fixes *i* :: *64 word*
assumes $y = IntVal\ 64\ ((2\ \hat{}\ unat(i)) + 1)$
and $0 < i$
and $i < 64$
and *val-to-bool*(*val*[*IntVal 64 0* < *x*])
and *val-to-bool*(*val*[*IntVal 64 0* < *y*])
shows $val[x * y] = val[(x << IntVal\ 64\ i) + x]$
using *assms* **apply** (*cases x*; *cases y*; *auto*)
subgoal premises *p* **for** *x2*
proof −
have *63*: (*63* :: *int64*) = *mask 6*
by *eval*
then have (*2*::*int*) ^ *6* = *64*
by *eval*
then have *and i* (*mask 6*) = *i*
using *mask-eq-iff* **by** (*simp add*: *less-mask-eq p*(*6*))
then have *x2* ∗ ((*2*::*64 word*) ^ *unat i* + (*1*::*64 word*)) = (*x2* ∗ ((*2*::*64 word*) ^ *unat i*)) + *x2*
by (*simp add*: *distrib-left*)
then show *x2* ∗ ((*2*::*64 word*) ^ *unat i* + (*1*::*64 word*)) = *x2* << *unat* (*and i* (*63*::*64 word*)) + *x2*
by (*simp add*: *63* ‹*and* (*i*::*64 word*) (*mask* (*6*::*nat*)) = *i*›)

qed
using *val-to-bool.simps*(*2*) **by** *presburger*

lemma *val-MulPower2Sub1*:
fixes *i* :: *64 word*
assumes $y = IntVal\ 64\ ((2\ \hat{}\ unat(i)) - 1)$
and $0 < i$
and $i < 64$
and *val-to-bool*(*val*[*IntVal 64 0* < *x*])
and *val-to-bool*(*val*[*IntVal 64 0* < *y*])
shows *val*[*x* ∗ *y*] = *val*[(*x* << *IntVal 64 i*) − *x*]
using *assms* **apply** (*cases x*; *cases y*; *auto*)
subgoal premises *p* **for** *x2*
proof −
have *63*: (*63* :: *int64*) = *mask 6*
by *eval*
then have (*2*::*int*) ^ *6* = *64*
by *eval*
then have *and i* (*mask 6*) = *i*
using *mask-eq-iff* **by** (*simp add*: *less-mask-eq p*(*6*))
then have *x2* ∗ ((*2*::*64 word*) ^ *unat i* − (*1*::*64 word*)) = (*x2* ∗ ((*2*::*64 word*) ^ *unat i*)) − *x2*
by (*simp add*: *right-diff-distrib′*)
then show *x2* ∗ ((*2*::*64 word*) ^ *unat i* − (*1*::*64 word*)) = *x2* << *unat* (*and i* (*63*::*64 word*)) − *x2*
by (*simp add*: *63* ‹*and* (*i*::*64 word*) (*mask* (*6*::*nat*)) = *i*›)
qed
using *val-to-bool.simps*(*2*) **by** *presburger*

lemma *val-distribute-multiplication*:
assumes *x* = *new-int 64 xx* ∧ *q* = *new-int 64 qq* ∧ *a* = *new-int 64 aa*
shows *val*[*x* ∗ (*q* + *a*)] = *val*[(*x* ∗ *q*) + (*x* ∗ *a*)]
apply (*cases x*; *cases q*; *cases a*; *auto*) **using** *distrib-left assms* **by** *auto*

lemma *val-MulPower2AddPower2*:
fixes *i j* :: *64 word*
assumes $y = IntVal\ 64\ ((2\ \hat{}\ unat(i)) + (2\ \hat{}\ unat(j)))$
and $0 < i$
and $0 < j$
and $i < 64$
and $j < 64$
and *x* = *new-int 64 xx*
shows *val*[*x* ∗ *y*] = *val*[(*x* << *IntVal 64 i*) + (*x* << *IntVal 64 j*)]
using *assms*
proof −

have *63*: (*63* :: *int64*) = *mask 6*
by *eval*
then have (*2*::*int*) ^ *6* = *64*
by *eval*
then have *n*: *IntVal 64* ((*2* ^ *unat*(*i*)) + (*2* ^ *unat*(*j*))) =
val[(*IntVal 64* (*2* ^ *unat*(*i*))) + (*IntVal 64* (*2* ^ *unat*(*j*)))]

using *assms* **by** (*cases i*; *cases j*; *auto*)
then have *1*: *val*[*x* ∗ ((*IntVal 64* (*2* ^ *unat*(*i*))) + (*IntVal 64* (*2* ^ *unat*(*j*))))]
=
val[(*x* ∗ *IntVal 64* (*2* ^ *unat*(*i*))) + (*x* ∗ *IntVal 64* (*2* ^ *unat*(*j*)))]

using *assms val-distribute-multiplication val-MulPower2* **by** *simp*
then have *2*: *val*[(*x* ∗ *IntVal 64* (*2* ^ *unat*(*i*)))] = *val*[*x* << *IntVal 64 i*]
by (*smt* (*verit*) *Value.distinct*(*1*) *intval-mul.simps*(*1*) *new-int.simps new-int-bin.simps*
assms
val-MulPower2)
then show *?thesis*
by (*smt* (*verit*, *del-insts*) *1 Value.distinct*(*1*) *assms*(*1*) *assms*(*3*) *assms*(*5*)
assms(*6*)
intval-mul.simps(*1*) *n new-int.simps new-int-bin.elims val-MulPower2*)
qed

thm-oracles *val-MulPower2AddPower2*

lemma *exp-multiply-zero-64*:
exp[*x* ∗ (*const* (*IntVal 64 0*))] ≥ *ConstantExpr* (*IntVal 64 0*)
using *val-multiply-zero* **apply** *auto*
by (*smt* (*verit*) *Value.inject*(*1*) *constantAsStamp.simps*(*1*) *int-signed-value-bounds*
intval-mul.elims
mult-zero-right new-int.simps new-int-bin.simps nle-le numeral-eq-Suc take-bit-of-0

unfold-const valid-stamp.simps(*1*) *valid-value.simps*(*1*) *zero-less-Suc wf-value-def*)

lemma *exp-multiply-neutral*:
exp[*x* ∗ (*const* (*IntVal b 1*))] ≥ *x*
using *val-multiply-neutral* **apply** *auto*
by (*smt* (*verit*) *Value.inject*(*1*) *eval-unused-bits-zero intval-mul.elims mult.right-neutral*

new-int.elims new-int-bin.elims)

thm-oracles *exp-multiply-neutral*

lemma *exp-MulPower2*:
fixes *i* :: *64 word*
assumes *y* = *ConstantExpr* (*IntVal 64* (*2* ^ *unat*(*i*)))
and *0* < *i*
and *i* < *64*

and $exp[x > (const\ IntVal\ b\ 0)]$
and $exp[y > (const\ IntVal\ b\ 0)]$
shows $exp[x * y] \geq exp[x << ConstantExpr\ (IntVal\ 64\ i)]$
using *assms* **apply** *simp*
by (*metis ConstantExprE equiv-exprs-def unfold-binary*)

lemma *exp-MulPower2Add1*:
fixes $i :: 64\ word$
assumes $y = ConstantExpr\ (IntVal\ 64\ ((2\ \hat{}\ unat(i)) + 1))$
and $0 < i$
and $i < 64$
and $exp[x > (const\ IntVal\ b\ 0)]$
and $exp[y > (const\ IntVal\ b\ 0)]$
shows $exp[x * y] \geq exp[(x << ConstantExpr\ (IntVal\ 64\ i)) + x]$
using *assms* **apply** *simp*
by (*metis* (*no-types*, *lifting*) *ConstantExprE equiv-exprs-def unfold-binary*)

lemma *exp-MulPower2Sub1*:
fixes $i :: 64\ word$
assumes $y = ConstantExpr\ (IntVal\ 64\ ((2\ \hat{}\ unat(i)) - 1))$
and $0 < i$
and $i < 64$
and $exp[x > (const\ IntVal\ b\ 0)]$
and $exp[y > (const\ IntVal\ b\ 0)]$
shows $exp[x * y] \geq exp[(x << ConstantExpr\ (IntVal\ 64\ i)) - x]$
using *assms* **apply** *simp*
by (*metis* (*no-types*, *lifting*) *ConstantExprE equiv-exprs-def unfold-binary*)

lemma *exp-MulPower2AddPower2*:
fixes $i\ j :: 64\ word$
assumes $y = ConstantExpr\ (IntVal\ 64\ ((2\ \hat{}\ unat(i)) + (2\ \hat{}\ unat(j))))$
and $0 < i$
and $0 < j$
and $i < 64$
and $j < 64$
and $exp[x > (const\ IntVal\ b\ 0)]$
and $exp[y > (const\ IntVal\ b\ 0)]$
shows $exp[x * y] \geq exp[(x << ConstantExpr\ (IntVal\ 64\ i)) + (x << ConstantExpr\ (IntVal\ 64\ j))]$
using *assms* **apply** *simp*
by (*metis* (*no-types*, *lifting*) *ConstantExprE equiv-exprs-def unfold-binary*)

lemma *greaterConstant*:
fixes $a\ b :: 64\ word$
assumes $a > b$
and $y = ConstantExpr\ (IntVal\ 64\ a)$

and $x = ConstantExpr\ (IntVal\ 64\ b)$
shows $exp[y > x]$
apply *auto*
sorry

lemma *exp-distribute-multiplication*:
shows $exp[(x * q) + (x * a)] \geq exp[x * (q + a)]$
sorry

Optimisations

optimization *EliminateRedundantNegative*: $-x * -y \longmapsto x * y$
using *mul-size.simps* **apply** *auto*
by (*metis BinaryExpr val-eliminate-redundant-negative bin-eval.simps*(*2*))

optimization *MulNeutral*: $x * ConstantExpr\ (IntVal\ b\ 1) \longmapsto x$
using *exp-multiply-neutral* **by** *blast*

optimization *MulEliminator*: $x * ConstantExpr\ (IntVal\ b\ 0) \longmapsto const\ (IntVal\ b\ 0)$
apply *auto*
by (*smt* (*verit*) *Value.inject*(*1*) *constantAsStamp.simps*(*1*) *int-signed-value-bounds intval-mul.elims*
mult-zero-right new-int.simps new-int-bin.simps take-bit-of-0 unfold-const
valid-stamp.simps(*1*) *valid-value.simps*(*1*) *val-multiply-zero*)

optimization *MulNegate*: $x * -(const\ (IntVal\ b\ 1)) \longmapsto -x$
apply *auto*
by (*smt* (*verit*) *Value.distinct*(*1*) *Value.sel*(*1*) *add.inverse-inverse intval-mul.elims*

intval-negate.simps(*1*) *mask-eq-take-bit-minus-one new-int.simps new-int-bin.simps*

take-bit-dist-neg unary-eval.simps(*2*) *unfold-unary val-multiply-negative*
val-eliminate-redundant-negative val-multiply-negative wf-value-def)

fun *isNonZero* :: *Stamp* ⇒ *bool* **where**
isNonZero (*IntegerStamp b lo hi*) = ($lo > 0$) |
isNonZero - = *False*

lemma *isNonZero-defn*:
assumes *isNonZero* (*stamp-expr x*)
assumes *wf-stamp x*
shows $([m, p] \vdash x \mapsto v) \longrightarrow (\exists\, vv\ b.\ (v = IntVal\ b\ vv \wedge val\text{-}to\text{-}bool\ val[(IntVal\ b\ 0) < v]))$
apply (*rule impI*) **subgoal premises** *eval*
proof −
obtain *b lo hi* **where** *xstamp*: *stamp-expr x = IntegerStamp b lo hi*
by (*meson isNonZero.elims*(*2*) *assms*)
then obtain *vv* **where** *vdef*: *v = IntVal b vv*
by (*metis assms*(*2*) *eval valid-int wf-stamp-def*)

have *lo* > *0*
using *assms*(*1*) *xstamp* **by** *force*
then have *signed-above*: *int-signed-value b vv* > *0*
using *assms* **unfolding** *wf-stamp-def*
using *eval vdef xstamp* **by** *fastforce*
have *take-bit b vv* = *vv*
using *eval eval-unused-bits-zero vdef* **by** *auto*
then have *vv* > *0*
by (*metis bit-take-bit-iff int-signed-value.simps not-less-zero signed-eq-0-iff*
signed-take-bit-eq-if-positive take-bit-0 take-bit-of-0 verit-comp-simplify1(*1*)
word-gt-0
signed-above)
then show *?thesis*
using *vdef signed-above*
by *simp*
qed
done

optimization *MulPower2*: $x * y \longmapsto x << const\ (IntVal\ 64\ i)$
when $(i > 0 \wedge$
$64 > i \wedge$
$y = exp[const\ (IntVal\ 64\ (2\ \widehat{}\ unat(i)))])$
defer
apply *simp* **apply** (*rule impI*; (*rule allI*)+; *rule impI*)
subgoal premises *eval* **for** *m p v*
proof −
obtain *xv* **where** *xv*: $[m, p] \vdash x \mapsto xv$
using *eval*(*2*) **by** *blast*
then obtain *xvv* **where** *xvv*: *xv* = *IntVal 64 xvv*
by (*smt* (*verit*) *ConstantExprE bin-eval.simps*(*2*) *evalDet intval-bits.simps int-*
val-mul.elims
new-int-bin.simps unfold-binary eval)
obtain *yv* **where** *yv*: $[m, p] \vdash y \mapsto yv$
using *eval*(*1*) *eval*(*2*) **by** *blast*
then have *lhs*: $[m, p] \vdash exp[x * y] \mapsto val[xv * yv]$
by (*metis bin-eval.simps*(*2*) *eval*(*1*) *eval*(*2*) *evalDet unfold-binary xv*)
have $[m, p] \vdash exp[const\ (IntVal\ 64\ i)] \mapsto val[(IntVal\ 64\ i)]$
by (*smt* (*verit*, *ccfv-SIG*) *ConstantExpr constantAsStamp.simps*(*1*) *eval-bits-1-64*
take-bit64
validStampIntConst wf-value-def valid-value.simps(*1*) *xv xvv*)
then have *rhs*: $[m, p] \vdash exp[x << const\ (IntVal\ 64\ i)] \mapsto val[xv << (IntVal\ 64$
$i)]$
using *xv xvv* **using** *evaltree.BinaryExpr*
by (*metis Value.simps*(*5*) *bin-eval.simps*(*8*) *intval-left-shift.simps*(*1*) *new-int.simps*)
have $val[xv * yv] = val[xv << (IntVal\ 64\ i)]$
by (*metis ConstantExprE eval*(*1*) *evaltree-not-undef lhs yv val-MulPower2*)
then show *?thesis*
by (*metis eval*(*1*) *eval*(*2*) *evalDet lhs rhs*)
qed

```
  done

optimization MulPower2Add1: x * y ⟼ (x << const (IntVal 64 i)) + x
                              when (i > 0 ∧
                                    64 > i ∧
                                    y = ConstantExpr (IntVal 64 ((2 ^ unat(i)) + 1)) )
   defer
   apply simp apply (rule impI; (rule allI)+; rule impI)
  subgoal premises p for m p v
  proof -
    obtain xv where xv: [m, p] ⊢ x ↦ xv
      using p by fast
    then obtain xvv where xvv: xv = IntVal 64 xvv
      by (smt (verit) p ConstantExprE bin-eval.simps(2) evalDet intval-bits.simps
intval-mul.elims
          new-int-bin.simps unfold-binary)
    obtain yv where yv: [m, p] ⊢ y ↦ yv
      using p  by blast
    have ygezero: y > ConstantExpr (IntVal 64 0)
      using greaterConstant p wf-value-def by fastforce
    then have 1: 0 < i ∧
                 i < 64 ∧
                 y = ConstantExpr (IntVal 64 ((2 ^ unat(i)) + 1))
      using p by blast
    then have lhs: [m, p] ⊢ exp[x * y] ↦ val[xv * yv]
      by (metis bin-eval.simps(2) evalDet p(1) p(2) xv yv unfold-binary)
    then have [m, p] ⊢ exp[const (IntVal 64 i)] ↦ val[(IntVal 64 i)]
     by (metis wf-value-def verit-comp-simplify1(2) zero-less-numeral ConstantExpr

        constantAsStamp.simps(1) take-bit64 validStampIntConst valid-value.simps(1))
    then have rhs2: [m, p] ⊢ exp[x << const (IntVal 64 i)] ↦ val[xv << (IntVal
64 i)]
     by (metis Value.simps(5) bin-eval.simps(8) intval-left-shift.simps(1) new-int.simps
xv xvv
          evaltree.BinaryExpr)
     then have rhs: [m, p] ⊢ exp[(x << const (IntVal 64 i)) + x] ↦ val[(xv <<
(IntVal 64 i)) + xv]
          by (metis (no-types, lifting) intval-add.simps(1) rhs2 bin-eval.simps(1)
Value.simps(5)
           evaltree.BinaryExpr intval-left-shift.simps(1) new-int.simps xv xvv)
     then have simple: val[xv * (IntVal 64 (2 ^ unat(i)))] = val[xv << (IntVal 64
i)]
       using val-MulPower2  sorry
     then have val[xv * yv] = val[(xv << (IntVal 64 i)) + xv]
        sorry
     then show ?thesis
       by (metis 1 evalDet lhs p(2) rhs)
  qed
```

done

optimization *MulPower2Sub1*: $x * y \longmapsto (x << const\ (IntVal\ 64\ i)) - x$
$when\ (i > 0 \land$
$64 > i \land$
$y = ConstantExpr\ (IntVal\ 64\ ((2\ \hat{}\ unat(i)) - 1))\ )$
defer
apply *simp* **apply** (*rule impI*; (*rule allI*)+; *rule impI*)
subgoal premises *p* **for** *m p v*
proof −
obtain *xv* **where** *xv*: $[m,p] \vdash x \mapsto xv$
using *p* **by** *fast*
then obtain *xvv* **where** *xvv*: $xv = IntVal\ 64\ xvv$
by (*smt* (*verit*) *p ConstantExprE bin-eval.simps*(*2*) *evalDet intval-bits.simps intval-mul.elims*
new-int-bin.simps unfold-binary)
obtain *yv* **where** *yv*: $[m,p] \vdash y \mapsto yv$
using *p* **by** *blast*
have *ygezero*: $y > ConstantExpr\ (IntVal\ 64\ 0)$
by (*smt* (*verit*, *del-insts*) *eq-iff-diff-eq-0 mask-0 mask-eq-exp-minus-1 power-inject-exp*

uint-2p unat-eq-zero word-gt-0 zero-neq-one greaterConstant p)
then have *1*: $0 < i \land$
$i < 64 \land$
$y = ConstantExpr\ (IntVal\ 64\ ((2\ \hat{}\ unat(i)) - 1))$
using *p* **by** *blast*
then have *lhs*: $[m, p] \vdash exp[x * y] \mapsto val[xv * yv]$
by (*metis bin-eval.simps*(*2*) *evalDet p*(*1*) *p*(*2*) *xv yv unfold-binary*)
then have $[m, p] \vdash exp[const\ (IntVal\ 64\ i)] \mapsto val[(IntVal\ 64\ i)]$
by (*metis wf-value-def verit-comp-simplify1*(*2*) *zero-less-numeral ConstantExpr*

constantAsStamp.simps(*1*) *take-bit64 validStampIntConst valid-value.simps*(*1*))
then have *rhs2*: $[m, p] \vdash exp[x << const\ (IntVal\ 64\ i)] \mapsto val[xv << (IntVal\ 64\ i)]$
by (*metis Value.simps*(*5*) *bin-eval.simps*(*8*) *intval-left-shift.simps*(*1*) *new-int.simps xv xvv*
evaltree.BinaryExpr)
then have *rhs*: $[m, p] \vdash exp[(x << const\ (IntVal\ 64\ i)) - x] \mapsto val[(xv << (IntVal\ 64\ i)) - xv]$
by (*smt* (*verit*, *ccfv-threshold*) *bin-eval.simps*(*3*) *new-int-bin.simps intval-sub.simps*(*1*)

rhs2 bin-eval.simps(*1*) *Value.simps*(*5*) *evaltree.BinaryExpr intval-left-shift.simps*(*1*)

new-int.simps xv xvv)
then have $val[xv * yv] = val[(xv << (IntVal\ 64\ i)) - xv]$
using *1 exp-MulPower2Sub1 ygezero* **sorry**
then show *?thesis*
by (*metis evalDet lhs p*(*1*) *p*(*2*) *rhs*)

qed
done

end

end

1.7 Experimental AndNode Phase

```
theory NewAnd
  imports
    Common
    Graph.Long
begin

lemma bin-distribute-and-over-or:
  bin[z & (x | y)] = bin[(z & x) | (z & y)]
  by (smt (verit, best) bit-and-iff bit-eqI bit-or-iff)

lemma intval-distribute-and-over-or:
  val[z & (x | y)] = val[(z & x) | (z & y)]
  apply (cases x; cases y; cases z; auto)
  using bin-distribute-and-over-or by blast+

lemma exp-distribute-and-over-or:
  exp[z & (x | y)] ≥ exp[(z & x) | (z & y)]
  apply simp using intval-distribute-and-over-or
  using BinaryExpr bin-eval.simps(4,5)
 using intval-or.simps(1) unfolding new-int-bin.simps new-int.simps apply auto
 by (metis bin-eval.simps(4) bin-eval.simps(5) intval-or.simps(2) intval-or.simps(5))


lemma intval-and-commute:
  val[x & y] = val[y & x]
  by (cases x; cases y; auto simp: and.commute)

lemma intval-or-commute:
  val[x | y] = val[y | x]
  by (cases x; cases y; auto simp: or.commute)

lemma intval-xor-commute:
  val[x ⊕ y] = val[y ⊕ x]
  by (cases x; cases y; auto simp: xor.commute)

lemma exp-and-commute:
  exp[x & z] ≥ exp[z & x]
  apply simp using intval-and-commute by auto
```

lemma *exp-or-commute*:
  $exp[x \mid y] \geq exp[y \mid x]$
  apply *simp* **using** *intval-or-commute* **by** *auto*

lemma *exp-xor-commute*:
  $exp[x \oplus y] \geq exp[y \oplus x]$
  apply *simp* **using** *intval-xor-commute* **by** *auto*

lemma *bin-eliminate-y*:
  assumes $bin[y \ \& \ z] = 0$
  shows $bin[(x \mid y) \ \& \ z] = bin[x \ \& \ z]$
  using *assms*
  by (*simp add*: *and.commute bin-distribute-and-over-or*)

lemma *intval-eliminate-y*:
  assumes $val[y \ \& \ z] = IntVal\ b\ 0$
  shows $val[(x \mid y) \ \& \ z] = val[x \ \& \ z]$
  using *assms bin-eliminate-y* **by** (*cases x*; *cases y*; *cases z*; *auto*)

lemma *intval-and-associative*:
  $val[(x \ \& \ y) \ \& \ z] = val[x \ \& \ (y \ \& \ z)]$
  apply (*cases x*; *cases y*; *cases z*; *auto*)
  by (*simp add*: *and.assoc*)+

lemma *intval-or-associative*:
  $val[(x \mid y) \mid z] = val[x \mid (y \mid z)]$
  apply (*cases x*; *cases y*; *cases z*; *auto*)
  by (*simp add*: *or.assoc*)+

lemma *intval-xor-associative*:
  $val[(x \oplus y) \oplus z] = val[x \oplus (y \oplus z)]$
  apply (*cases x*; *cases y*; *cases z*; *auto*)
  by (*simp add*: *xor.assoc*)+

lemma *exp-and-associative*:
  $exp[(x \ \& \ y) \ \& \ z] \geq exp[x \ \& \ (y \ \& \ z)]$
  apply *simp* **using** *intval-and-associative* **by** *fastforce*

lemma *exp-or-associative*:
  $exp[(x \mid y) \mid z] \geq exp[x \mid (y \mid z)]$
  apply *simp* **using** *intval-or-associative* **by** *fastforce*

lemma *exp-xor-associative*:
  $exp[(x \oplus y) \oplus z] \geq exp[x \oplus (y \oplus z)]$
  apply *simp* **using** *intval-xor-associative* **by** *fastforce*

lemma *intval-and-absorb-or*:

assumes $\exists\, b\ v\ .\ x = \textit{new-int}\ b\ v$
assumes $\textit{val}[x\ \&\ (x \mid y)] \neq \textit{UndefVal}$
shows $\textit{val}[x\ \&\ (x \mid y)] = \textit{val}[x]$
using *assms* **apply** (*cases x*; *cases y*; *auto*)
by (*metis* (*mono-tags*, *lifting*) *intval-and.simps*(*5*))

lemma *intval-or-absorb-and*:
assumes $\exists\, b\ v\ .\ x = \textit{new-int}\ b\ v$
assumes $\textit{val}[x \mid (x\ \&\ y)] \neq \textit{UndefVal}$
shows $\textit{val}[x \mid (x\ \&\ y)] = \textit{val}[x]$
using *assms* **apply** (*cases x*; *cases y*; *auto*)
by (*metis* (*mono-tags*, *lifting*) *intval-or.simps*(*5*))

lemma *exp-and-absorb-or*:
$\textit{exp}[x\ \&\ (x \mid y)] \geq \textit{exp}[x]$
apply *auto* **using** *intval-and-absorb-or eval-unused-bits-zero*
by (*smt* (*verit*) *evalDet intval-or.elims new-int.elims*)

lemma *exp-or-absorb-and*:
$\textit{exp}[x \mid (x\ \&\ y)] \geq \textit{exp}[x]$
apply *auto* **using** *intval-or-absorb-and eval-unused-bits-zero*
by (*smt* (*verit*) *evalDet intval-or.elims new-int.elims*)

lemma
assumes $y = \mathit{0}$
shows $x + y = \textit{or}\ x\ y$
using *assms*
by *simp*

lemma *no-overlap-or*:
assumes $\textit{and}\ x\ y = \mathit{0}$
shows $x + y = \textit{or}\ x\ y$
using *assms*
by (*metis bit-and-iff bit-xor-iff disjunctive-add xor-self-eq*)

context *stamp-mask*
begin

lemma *intval-up-and-zero-implies-zero*:
assumes $\textit{and}\ (\uparrow x)\ (\uparrow y) = \mathit{0}$

assumes $[m, p] \vdash x \mapsto xv$
assumes $[m, p] \vdash y \mapsto yv$
assumes $val[xv \ \& \ yv] \neq UndefVal$
shows $\exists \ b \ . \ val[xv \ \& \ yv] = new\text{-}int \ b \ 0$
using *assms* **apply** (*cases xv*; *cases yv*; *auto*)
using *up-mask-and-zero-implies-zero*
apply (*smt* (*verit*, *best*) *take-bit-and take-bit-of-0*)
by *presburger*

lemma *exp-eliminate-y*:
and $(\uparrow y)$ $(\uparrow z) = 0 \longrightarrow$ *BinaryExpr BinAnd* (*BinaryExpr BinOr x y*) $z \geq$ *BinaryExpr BinAnd x z*
apply *simp* **apply** (*rule impI*; *rule allI*; *rule allI*; *rule allI*)
subgoal premises *p* **for** *m p v* **apply** (*rule impI*) **subgoal premises** *e*
proof −
obtain *xv* **where** *xv*: $[m,p] \vdash x \mapsto xv$
using *e* **by** *auto*
obtain *yv* **where** *yv*: $[m,p] \vdash y \mapsto yv$
using *e* **by** *auto*
obtain *zv* **where** *zv*: $[m,p] \vdash z \mapsto zv$
using *e* **by** *auto*
have *lhs*: $v = val[(xv \mid yv) \ \& \ zv]$
using *xv yv zv*
by (*smt* (*verit*, *best*) *BinaryExprE bin-eval.simps*(*4*) *bin-eval.simps*(*5*) *e evalDet*)
then have $v = val[(xv \ \& \ zv) \mid (yv \ \& \ zv)]$
by (*simp add*: *intval-and-commute intval-distribute-and-over-or*)
also have $\exists b. \ val[yv \ \& \ zv] = new\text{-}int \ b \ 0$
using *intval-up-and-zero-implies-zero*
by (*metis calculation e intval-or.simps*(*5*) *p unfold-binary yv zv*)
ultimately have *rhs*: $v = val[xv \ \& \ zv]$
using *intval-eliminate-y lhs* **by** *force*
from *lhs rhs* **show** *?thesis*
by (*metis BinaryExpr BinaryExprE bin-eval.simps*(*4*) *e xv zv*)
qed
done
done

lemma *leadingZeroBounds*:
fixes *x* :: *'a::len word*
assumes $n = numberOfLeadingZeros \ x$
shows $0 \leq n \wedge n \leq Nat.size \ x$
using *assms* **unfolding** *numberOfLeadingZeros-def*
by (*simp add*: *MaxOrNeg-def highestOneBit-def nat-le-iff*)

lemma *above-nth-not-set*:
fixes *x* :: *int64*
assumes $n = 64 - numberOfLeadingZeros \ x$
shows $j > n \longrightarrow \neg(bit \ x \ j)$

using *assms* **unfolding** *numberOfLeadingZeros-def*
by (*smt* (*verit*, *ccfv-SIG*) *highestOneBit-def int-nat-eq int-ops*(*6*) *less-imp-of-nat-less max-set-bit size64 zerosAboveHighestOne*)

no-notation *LogicNegationNotation* (!-)

lemma *zero-horner*:
horner-sum of-bool 2 (*map* (*λx. False*) *xs*) = *0*
apply (*induction xs*) **apply** *simp*
by *force*

lemma *zero-map*:
assumes $j \leq n$
assumes $\forall\, i.\ j \leq i \longrightarrow \neg(f\ i)$
shows *map f* [*0*..<*n*] = *map f* [*0*..<*j*] @ *map* (*λx. False*) [*j*..<*n*]
apply (*insert assms*)
by (*smt* (*verit*, *del-insts*) *add-diff-inverse-nat atLeastLessThan-iff bot-nat-0.extremum leD map-append map-eq-conv set-upt upt-add-eq-append*)

lemma *map-join-horner*:
assumes *map f* [*0*..<*n*] = *map f* [*0*..<*j*] @ *map* (*λx. False*) [*j*..<*n*]
shows *horner-sum of-bool* (*2*::'*a*::*len word*) (*map f* [*0*..<*n*]) = *horner-sum of-bool 2* (*map f* [*0*..<*j*])
proof −
have *horner-sum of-bool* (*2*::'*a*::*len word*) (*map f* [*0*..<*n*]) = *horner-sum of-bool 2* (*map f* [*0*..<*j*]) + *2* ^ *length* [*0*..<*j*] ∗ *horner-sum of-bool 2* (*map f* [*j*..<*n*])
using *horner-sum-append*
by (*smt* (*verit*) *assms diff-le-self diff-zero le-add-same-cancel2 length-append length-map length-upt map-append upt-add-eq-append*)
also have ... = *horner-sum of-bool 2* (*map f* [*0*..<*j*]) + *2* ^ *length* [*0*..<*j*] ∗ *horner-sum of-bool 2* (*map* (*λx. False*) [*j*..<*n*])
using *assms*
by (*metis calculation horner-sum-append length-map*)
also have ... = *horner-sum of-bool 2* (*map f* [*0*..<*j*])
using *zero-horner*
using *mult-not-zero* **by** *auto*
finally show *?thesis* **by** *simp*
qed

lemma *split-horner*:
assumes $j \leq n$
assumes $\forall\, i.\ j \leq i \longrightarrow \neg(f\ i)$
shows *horner-sum of-bool* (*2*::'*a*::*len word*) (*map f* [*0*..<*n*]) = *horner-sum of-bool 2* (*map f* [*0*..<*j*])
apply (*rule map-join-horner*)
apply (*rule zero-map*)
using *assms* **by** *auto*

lemma *transfer-map*:

assumes $\forall\, i.\ i < n \longrightarrow f\ i = f'\ i$
shows $(map\ f\ [0..{<}n]) = (map\ f'\ [0..{<}n])$
using *assms* **by** *simp*

lemma *transfer-horner*:
assumes $\forall\, i.\ i < n \longrightarrow f\ i = f'\ i$
shows $horner\text{-}sum\ of\text{-}bool\ (2::'a::len\ word)\ (map\ f\ [0..{<}n]) = horner\text{-}sum\ of\text{-}bool\ 2\ (map\ f'\ [0..{<}n])$
using *assms* **using** *transfer-map*
by (*smt* (*verit*, *best*))

lemma *L1*:
assumes $n = 64 - numberOfLeadingZeros\ (\uparrow z)$
assumes $[m, p] \vdash z \mapsto IntVal\ b\ zv$
shows $and\ v\ zv = and\ (v\ mod\ 2\hat{}n)\ zv$
proof –
have *nle*: $n \leq 64$
using *assms*
using *diff-le-self* **by** *blast*
also have $and\ v\ zv = horner\text{-}sum\ of\text{-}bool\ 2\ (map\ (bit\ (and\ v\ zv))\ [0..{<}64])$
using *horner-sum-bit-eq-take-bit size64*
by (*metis size-word.rep-eq take-bit-length-eq*)
also have ... $= horner\text{-}sum\ of\text{-}bool\ 2\ (map\ (\lambda i.\ bit\ (and\ v\ zv)\ i)\ [0..{<}64])$
by *blast*
also have ... $= horner\text{-}sum\ of\text{-}bool\ 2\ (map\ (\lambda i.\ ((bit\ v\ i) \wedge (bit\ zv\ i)))\ [0..{<}64])$
using *bit-and-iff* **by** *metis*
also have ... $= horner\text{-}sum\ of\text{-}bool\ 2\ (map\ (\lambda i.\ ((bit\ v\ i) \wedge (bit\ zv\ i)))\ [0..{<}n])$
proof –
have $\forall\, i.\ i \geq n \longrightarrow \neg(bit\ zv\ i)$
using *above-nth-not-set assms*(*1*)
using *assms*(*2*) *not-may-implies-false*
by (*smt* (*verit*, *ccfv-SIG*) *One-nat-def diff-less int-ops*(*6*) *leadingZerosAddHighestOne linorder-not-le nat-int-comparison*(*2*) *not-numeral-le-zero size64 zero-less-Suc zerosAboveHighestOne*)
then have $\forall\, i.\ i \geq n \longrightarrow \neg((bit\ v\ i) \wedge (bit\ zv\ i))$
by *auto*
then show *?thesis* **using** *nle split-horner*
by (*metis* (*no-types*, *lifting*))
qed
also have ... $= horner\text{-}sum\ of\text{-}bool\ 2\ (map\ (\lambda i.\ ((bit\ (v\ mod\ 2\hat{}n)\ i) \wedge (bit\ zv\ i)))\ [0..{<}n])$
proof –
have $\forall\, i.\ i < n \longrightarrow bit\ (v\ mod\ 2\hat{}n)\ i = bit\ v\ i$
by (*metis bit-take-bit-iff take-bit-eq-mod*)
then have $\forall\, i.\ i < n \longrightarrow ((bit\ v\ i) \wedge (bit\ zv\ i)) = ((bit\ (v\ mod\ 2\hat{}n)\ i) \wedge (bit\ zv\ i))$
by *force*
then show *?thesis*
by (*rule transfer-horner*)

qed
also have ... = *horner-sum of-bool 2* (*map* (λ*i.* ((*bit* (*v mod 2^n*) *i*) $\wedge$ (*bit zv i*))) [*0..<64*])
proof −
have $\forall$ *i. i* $\geq$ *n* $\longrightarrow$ $\neg$(*bit zv i*)
using *above-nth-not-set assms*(*1*)
using *assms*(*2*) *not-may-implies-false*
by (*smt* (*verit*, *ccfv-SIG*) *One-nat-def diff-less int-ops*(*6*) *leadingZerosAddHighestOne linorder-not-le nat-int-comparison*(*2*) *not-numeral-le-zero size64 zero-less-Suc zerosAboveHighestOne*)
then show *?thesis*
by (*metis* (*no-types*, *lifting*) *assms*(*1*) *diff-le-self split-horner*)
qed
also have ... = *horner-sum of-bool 2* (*map* (*bit* (*and* (*v mod 2^n*) *zv*)) [*0..<64*])
by (*meson bit-and-iff*)
also have ... = *and* (*v mod 2^n*) *zv*
using *horner-sum-bit-eq-take-bit size64*
by (*metis size-word.rep-eq take-bit-length-eq*)
finally show *?thesis*
using ‹*and* (*v*::*64 word*) (*zv*::*64 word*) = *horner-sum of-bool* (*2*::*64 word*) (*map* (*bit* (*and v zv*)) [*0*::*nat..<64*::*nat*])› ‹*horner-sum of-bool* (*2*::*64 word*) (*map* (λ*i*::*nat. bit* ((*v*::*64 word*) *mod* (*2*::*64 word*) ^ (*n*::*nat*)) *i* $\wedge$ *bit* (*zv*::*64 word*) *i*) [*0*::*nat..<64*::*nat*]) = *horner-sum of-bool* (*2*::*64 word*) (*map* (*bit* (*and* (*v mod* (*2*::*64 word*) ^ *n*) *zv*)) [*0*::*nat..<64*::*nat*])› ‹*horner-sum of-bool* (*2*::*64 word*) (*map* (λ*i*::*nat. bit* ((*v*::*64 word*) *mod* (*2*::*64 word*) ^ (*n*::*nat*)) *i* $\wedge$ *bit* (*zv*::*64 word*) *i*) [*0*::*nat..<n*]) = *horner-sum of-bool* (*2*::*64 word*) (*map* (λ*i*::*nat. bit* (*v mod* (*2*::*64 word*) ^ *n*) *i* $\wedge$ *bit zv i*) [*0*::*nat..<64*::*nat*])› ‹*horner-sum of-bool* (*2*::*64 word*) (*map* (λ*i*::*nat. bit* (*v*::*64 word*) *i* $\wedge$ *bit* (*zv*::*64 word*) *i*) [*0*::*nat..<64*::*nat*]) = *horner-sum of-bool* (*2*::*64 word*) (*map* (λ*i*::*nat. bit v i* $\wedge$ *bit zv i*) [*0*::*nat..<n*::*nat*])› ‹*horner-sum of-bool* (*2*::*64 word*) (*map* (λ*i*::*nat. bit* (*v*::*64 word*) *i* $\wedge$ *bit* (*zv*::*64 word*) *i*) [*0*::*nat..<n*::*nat*]) = *horner-sum of-bool* (*2*::*64 word*) (*map* (λ*i*::*nat. bit* (*v mod* (*2*::*64 word*) ^ *n*) *i* $\wedge$ *bit zv i*) [*0*::*nat..<n*])› ‹*horner-sum of-bool* (*2*::*64 word*) (*map* (*bit* (*and* ((*v*::*64 word*) *mod* (*2*::*64 word*) ^ (*n*::*nat*)) (*zv*::*64 word*))) [*0*::*nat..<64*::*nat*]) = *and* (*v mod* (*2*::*64 word*) ^ *n*) *zv*› ‹*horner-sum of-bool* (*2*::*64 word*) (*map* (*bit* (*and* (*v*::*64 word*) (*zv*::*64 word*))) [*0*::*nat..<64*::*nat*]) = *horner-sum of-bool* (*2*::*64 word*) (*map* (λ*i*::*nat. bit v i* $\wedge$ *bit zv i*) [*0*::*nat..<64*::*nat*])› **by** *presburger*
qed

lemma *up-mask-upper-bound*:
assumes [*m*, *p*] $\vdash$ *x* $\mapsto$ *IntVal b xv*
shows *xv* $\leq$ ($\uparrow$*x*)
using *assms*
by (*metis* (*no-types*, *lifting*) *and.idem and.right-neutral bit.conj-cancel-left bit.conj-disj-distribs*(*1*) *bit.double-compl ucast-id up-spec word-and-le1 word-not-dist*(*2*))

lemma *L2*:
assumes *numberOfLeadingZeros* ($\uparrow$*z*) + *numberOfTrailingZeros* ($\uparrow$*y*) $\geq$ *64*
assumes *n* = *64* − *numberOfLeadingZeros* ($\uparrow$*z*)

```
  assumes [m, p] ⊢ z ↦ IntVal b zv
  assumes [m, p] ⊢ y ↦ IntVal b yv
  shows yv mod 2^n = 0
proof -
  have yv mod 2^n = horner-sum of-bool 2 (map (bit yv) [0..<n])
    by (simp add: horner-sum-bit-eq-take-bit take-bit-eq-mod)
  also have ... ≤ horner-sum of-bool 2 (map (bit (↑y)) [0..<n])
    using up-mask-upper-bound assms(4)
   by (metis (no-types, opaque-lifting) and.right-neutral bit.conj-cancel-right bit.conj-disj-distribs(1)
bit.double-compl horner-sum-bit-eq-take-bit take-bit-and ucast-id up-spec word-and-le1
word-not-dist(2))
  also have horner-sum of-bool 2 (map (bit (↑y)) [0..<n]) = horner-sum of-bool 2
(map (λx. False) [0..<n])
  proof -
    have ∀ i < n. ¬(bit (↑y) i)
      using assms(1,2) zerosBelowLowestOne
      by (metis add.commute add-diff-inverse-nat add-lessD1 leD le-diff-conv num-
berOfTrailingZeros-def)
    then show ?thesis
      by (metis (full-types) transfer-map)
  qed
  also have horner-sum of-bool 2 (map (λx. False) [0..<n]) = 0
    using zero-horner
    by blast
  finally show ?thesis
    by auto
qed

thm-oracles L1 L2

lemma unfold-binary-width-add:
  shows ([m,p] ⊢ BinaryExpr BinAdd xe ye ↦ IntVal b val) = (∃ x y.
          (([m,p] ⊢ xe ↦ IntVal b x) ∧
           ([m,p] ⊢ ye ↦ IntVal b y) ∧
           (IntVal b val = bin-eval BinAdd (IntVal b x) (IntVal b y)) ∧
           (IntVal b val ≠ UndefVal)
        )) (is ?L = ?R)
proof (intro iffI)
  assume 3: ?L
  show ?R apply (rule evaltree.cases[OF 3])
    apply force+ apply auto[1]
    apply (smt (verit) intval-add.elims intval-bits.simps)
    by blast
next
  assume R: ?R
  then obtain x y where [m,p] ⊢ xe ↦ IntVal b x
        and [m,p] ⊢ ye ↦ IntVal b y
        and new-int b val = bin-eval BinAdd (IntVal b x) (IntVal b y)
        and new-int b val ≠ UndefVal
```

```
    by auto
  then show ?L
    using R by blast
 qed

lemma unfold-binary-width-and:
  shows ([m,p] ⊢ BinaryExpr BinAnd xe ye ↦ IntVal b val) = (∃ x y.
          (([m,p] ⊢ xe ↦ IntVal b x) ∧
           ([m,p] ⊢ ye ↦ IntVal b y) ∧
           (IntVal b val = bin-eval BinAnd (IntVal b x) (IntVal b y)) ∧
           (IntVal b val ≠ UndefVal)
        )) (is ?L = ?R)
proof (intro iffI)
  assume 3: ?L
  show ?R apply (rule evaltree.cases[OF 3])
    apply force+ apply auto[1] using intval-and.elims intval-bits.simps
    apply (smt (verit) new-int.simps new-int-bin.simps take-bit-and)
    by blast
next
  assume R: ?R
  then obtain x y where [m,p] ⊢ xe ↦ IntVal b x
        and [m,p] ⊢ ye ↦ IntVal b y
        and new-int b val = bin-eval BinAnd (IntVal b x) (IntVal b y)
        and new-int b val ≠ UndefVal
    by auto
  then show ?L
    using R by blast
qed

lemma mod-dist-over-add-right:
  fixes a b c :: int64
  fixes n :: nat
  assumes 1: 0 < n
  assumes 2: n < 64
  shows (a + b mod 2^n) mod 2^n = (a + b) mod 2^n
  using mod-dist-over-add
  by (simp add: 1 2 add.commute)

lemma numberOfLeadingZeros-range:
  0 ≤ numberOfLeadingZeros n ∧ numberOfLeadingZeros n ≤ Nat.size n
  unfolding numberOfLeadingZeros-def highestOneBit-def using max-set-bit
  by (simp add: highestOneBit-def leadingZeroBounds numberOfLeadingZeros-def)

lemma improved-opt:
  assumes numberOfLeadingZeros (↑z) + numberOfTrailingZeros (↑y) ≥ 64
  shows exp[(x + y) & z] ≥ exp[x & z]
  apply simp apply ((rule allI)+; rule impI)
  subgoal premises eval for m p v
proof -
```

obtain *n* **where** *n*: $n = 64 - numberOfLeadingZeros\ (\uparrow z)$
by *simp*
obtain *b val* **where** *val*: $[m, p] \vdash exp[(x + y)\ \&\ z] \mapsto IntVal\ b\ val$
by (*metis BinaryExprE bin-eval-new-int eval new-int.simps*)
then obtain *xv yv* **where** *addv*: $[m, p] \vdash exp[x + y] \mapsto IntVal\ b\ (xv + yv)$
apply (*subst* (*asm*) *unfold-binary-width-and*) **by** (*metis add.right-neutral*)
then obtain *yv* **where** *yv*: $[m, p] \vdash y \mapsto IntVal\ b\ yv$
apply (*subst* (*asm*) *unfold-binary-width-add*) **by** *blast*
from *addv* **obtain** *xv* **where** *xv*: $[m, p] \vdash x \mapsto IntVal\ b\ xv$
apply (*subst* (*asm*) *unfold-binary-width-add*) **by** *blast*
from *val* **obtain** *zv* **where** *zv*: $[m, p] \vdash z \mapsto IntVal\ b\ zv$
apply (*subst* (*asm*) *unfold-binary-width-and*) **by** *blast*
have *addv*: $[m, p] \vdash exp[x + y] \mapsto new\text{-}int\ b\ (xv + yv)$
apply (*rule evaltree.BinaryExpr*)
using *xv* **apply** *simp*
using *yv* **apply** *simp*
by *simp+*
have *lhs*: $[m, p] \vdash exp[(x + y)\ \&\ z] \mapsto new\text{-}int\ b\ (and\ (xv + yv)\ zv)$
apply (*rule evaltree.BinaryExpr*)
using *addv* **apply** *simp*
using *zv* **apply** *simp*
using *addv* **apply** *auto*[*1*]
by *simp*
have *rhs*: $[m, p] \vdash exp[x\ \&\ z] \mapsto new\text{-}int\ b\ (and\ xv\ zv)$
apply (*rule evaltree.BinaryExpr*)
using *xv* **apply** *simp*
using *zv* **apply** *simp*
apply *force*
by *simp*
then show *?thesis*
proof (*cases* $numberOfLeadingZeros\ (\uparrow z) > 0$)
case *True*
have *n-bounds*: $0 \le n \land n < 64$
using *diff-le-self n numberOfLeadingZeros-range*
by (*simp add*: *True*)
have $and\ (xv + yv)\ zv = and\ ((xv + yv)\ mod\ 2\hat{}n)\ zv$
using *L1 n zv* **by** *blast*
also have ... $= and\ ((xv + (yv\ mod\ 2\hat{}n))\ mod\ 2\hat{}n)\ zv$
using *mod-dist-over-add-right n-bounds*
by (*metis take-bit-0 take-bit-eq-mod zero-less-iff-neq-zero*)
also have ... $= and\ (((xv\ mod\ 2\hat{}n) + (yv\ mod\ 2\hat{}n))\ mod\ 2\hat{}n)\ zv$
by (*metis bits-mod-by-1 mod-dist-over-add n-bounds order-le-imp-less-or-eq power-0*)
also have ... $= and\ ((xv\ mod\ 2\hat{}n)\ mod\ 2\hat{}n)\ zv$
using *L2 n zv yv*
using *assms* **by** *auto*
also have ... $= and\ (xv\ mod\ 2\hat{}n)\ zv$
using *mod-mod-trivial*
by (*smt* (*verit*, *best*) *and.idem take-bit-eq-mask take-bit-eq-mod word-bw-assocs*(*1*))

```
    also have ... = and xv zv
      using L1 n zv by metis
    finally show ?thesis
      using eval lhs rhs
      by (metis evalDet)
  next
    case False
    then have numberOfLeadingZeros (↑z) = 0
      by simp
    then have numberOfTrailingZeros (↑y) ≥ 64
      using assms(1)
      by fastforce
    then have yv = 0
      using yv
        by (metis (no-types, lifting) L1 L2 add-diff-cancel-left' and.comm-neutral
and.idem bit.compl-zero bit.conj-cancel-right bit.conj-disj-distribs(1) bit.double-compl
less-imp-diff-less linorder-not-le word-not-dist(2))
    then show ?thesis
      by (metis add.right-neutral eval evalDet lhs rhs)
  qed
qed
done

thm-oracles improved-opt

end

phase NewAnd
  terminating size
begin

optimization redundant-lhs-y-or: ((x | y) & z) ⟼ x & z
                              when (((and (IRExpr-up y) (IRExpr-up z)) = 0))
  apply (simp add: IRExpr-up-def)
  using simple-mask.exp-eliminate-y by blast

optimization redundant-lhs-x-or: ((x | y) & z) ⟼ y & z
                              when (((and (IRExpr-up x) (IRExpr-up z)) = 0))
  apply (simp add: IRExpr-up-def)
  using simple-mask.exp-eliminate-y
  by (meson exp-or-commute mono-binary order-refl order-trans)

optimization redundant-rhs-y-or: (z & (x | y)) ⟼ z & x
                              when (((and (IRExpr-up y) (IRExpr-up z)) = 0))
```

apply (*simp add*: *IRExpr-up-def*)
using *simple-mask.exp-eliminate-y*
by (*meson exp-and-commute order.trans*)

optimization *redundant-rhs-x-or*: $(z \ \& \ (x \mid y)) \longmapsto z \ \& \ y$
when (((*and* (*IRExpr-up x*) (*IRExpr-up z*)) = *0*))
apply (*simp add*: *IRExpr-up-def*)
using *simple-mask.exp-eliminate-y*
by (*meson dual-order.trans exp-and-commute exp-or-commute mono-binary order-refl*)

end

end

1.8 NotNode Phase

theory *NotPhase*
imports
Common
begin

phase *NotNode*
terminating *size*
begin

lemma *bin-not-cancel*:
bin[¬(¬(*e*))] = *bin*[*e*]
by *auto*

lemma *val-not-cancel*:
assumes *val*[~(*new-int b v*)] ≠ *UndefVal*
shows *val*[~(~(*new-int b v*))] = (*new-int b v*)
by (*simp add*: *take-bit-not-take-bit*)

lemma *exp-not-cancel*:
exp[~(~*a*)] ≥ *exp*[*a*]
using *val-not-cancel* **apply** *auto*
by (*metis eval-unused-bits-zero intval-logic-negation.cases new-int.simps intval-not.simps*(*1*)
intval-not.simps(*2*) *intval-not.simps*(*3*) *intval-not.simps*(*4*))

Optimisations

optimization *NotCancel*: *exp*[~(~*a*)] ⟼ *a*

by (*metis exp-not-cancel*)

end

end

1.9 OrNode Phase

theory *OrPhase*
  imports
    Common
begin

context *stamp-mask*
begin

Taking advantage of the truth table of or operations.

#	x	y	$x|y$
1	0	0	0
2	0	1	1
3	1	0	1
4	1	1	1

If row 2 never applies, that is, canBeZero x & canBeOne y = 0, then $(x|y) = x$.

Likewise, if row 3 never applies, canBeZero y & canBeOne x = 0, then $(x|y) = y$.

lemma *OrLeftFallthrough*:
  assumes (*and* (*not* (↓*x*)) (↑*y*)) = *0*
  shows *exp*[*x* | *y*] ≥ *exp*[*x*]
  using *assms*
  apply *simp* **apply** ((*rule allI*)+; *rule impI*)
  subgoal premises *eval* **for** *m p v*
  proof −
    obtain *b vv* **where** *e*: [*m*, *p*] ⊢ *exp*[*x* | *y*] ↦ *IntVal b vv*
      by (*metis BinaryExprE bin-eval-new-int new-int.simps eval*)
    from *e* **obtain** *xv* **where** *xv*: [*m*, *p*] ⊢ *x* ↦ *IntVal b xv*
      apply (*subst* (*asm*) *unfold-binary-width*)
      by *force*+
    from *e* **obtain** *yv* **where** *yv*: [*m*, *p*] ⊢ *y* ↦ *IntVal b yv*
      apply (*subst* (*asm*) *unfold-binary-width*)
      by *force*+
    have *vdef*: *v* = *intval-or* (*IntVal b xv*) (*IntVal b yv*)
      by (*metis bin-eval.simps*(*5*) *eval*(*2*) *evalDet unfold-binary xv yv*)
    have ∀ *i*. (*bit xv i*) | (*bit yv i*) = (*bit xv i*)
      by (*metis assms bit-and-iff not-down-up-mask-and-zero-implies-zero xv yv*)
    then have *IntVal b xv* = *intval-or* (*IntVal b xv*) (*IntVal b yv*)

```
    by (smt (verit, ccfv-threshold) and.idem assms bit.conj-disj-distrib eval-unused-bits-zero
        intval-or.simps(1) new-int.simps new-int-bin.simps not-down-up-mask-and-zero-implies-zero
          word-ao-absorbs(3) xv yv)
    then show ?thesis
      using xv vdef by presburger
  qed
  done

lemma OrRightFallthrough:
  assumes (and (not (↓y)) (↑x)) = 0
  shows exp[x | y] ≥ exp[y]
  using assms
  apply simp apply ((rule allI)+; rule impI)
  subgoal premises eval for m p v
  proof −
    obtain b vv where e: [m, p] ⊢ exp[x | y] ↦ IntVal b vv
      by (metis BinaryExprE bin-eval-new-int new-int.simps eval)
    from e obtain xv where xv: [m, p] ⊢ x ↦ IntVal b xv
      apply (subst (asm) unfold-binary-width)
      by force+
    from e obtain yv where yv: [m, p] ⊢ y ↦ IntVal b yv
      apply (subst (asm) unfold-binary-width)
      by force+
    have vdef: v = intval-or (IntVal b xv) (IntVal b yv)
      by (metis bin-eval.simps(5) eval(2) evalDet unfold-binary xv yv)
    have ∀ i. (bit xv i) | (bit yv i) = (bit yv i)
      by (metis assms bit-and-iff not-down-up-mask-and-zero-implies-zero xv yv)
    then have IntVal b yv = intval-or (IntVal b xv) (IntVal b yv)
       by (metis (no-types, lifting) assms eval-unused-bits-zero intval-or.simps(1)
new-int.elims
             new-int-bin.elims stamp-mask.not-down-up-mask-and-zero-implies-zero
stamp-mask-axioms
          word-ao-absorbs(8) xv yv)
    then show ?thesis
      using vdef yv by presburger
  qed
  done

end

phase OrNode
  terminating size
begin


lemma bin-or-equal:
  bin[x | x] = bin[x]
```

by *simp*

lemma *bin-shift-const-right-helper*:
$x \mid y = y \mid x$
by *simp*

lemma *bin-or-not-operands*:
$(\sim x \mid \sim y) = (\sim(x \ \& \ y))$
by *simp*

lemma *val-or-equal*:
assumes $x = \textit{new-int } b\ v$
and $(\textit{val}[x \mid x] \neq \textit{UndefVal})$
shows $\textit{val}[x \mid x] = \textit{val}[x]$
apply (*cases x*; *auto*) **using** *bin-or-equal assms*
by *auto*+

lemma *val-elim-redundant-false*:
assumes $x = \textit{new-int } b\ v$
and $\textit{val}[x \mid \textit{false}] \neq \textit{UndefVal}$
shows $\textit{val}[x \mid \textit{false}] = \textit{val}[x]$
using *assms* **apply** (*cases x*; *auto*) **by** *presburger*

lemma *val-shift-const-right-helper*:
$\textit{val}[x \mid y] = \textit{val}[y \mid x]$
apply (*cases x*; *cases y*; *auto*)
by (*simp add*: *or.commute*)+

lemma *val-or-not-operands*:
$\textit{val}[\sim x \mid \sim y] = \textit{val}[\sim(x \ \& \ y)]$
apply (*cases x*; *cases y*; *auto*)
by (*simp add*: *take-bit-not-take-bit*)

lemma *exp-or-equal*:
$\textit{exp}[x \mid x] \geq \textit{exp}[x]$
using *val-or-equal* **apply** *auto*
by (*smt* (*verit*, *ccfv-SIG*) *evalDet eval-unused-bits-zero intval-negate.elims int-val-or.simps*(*2*)
intval-or.simps(*6*) *intval-or.simps*(*7*) *new-int.simps val-or-equal*)

lemma *exp-elim-redundant-false*:
$\textit{exp}[x \mid \textit{false}] \geq \textit{exp}[x]$
using *val-elim-redundant-false* **apply** *auto*
by (*smt* (*verit*) *Value.sel*(*1*) *eval-unused-bits-zero intval-or.elims new-int.simps*

new-int-bin.simps val-elim-redundant-false)

Optimisations

optimization *OrEqual*: $x \mid x \longmapsto x$
  by (*meson exp-or-equal*)

optimization *OrShiftConstantRight*: $((const\ x) \mid y) \longmapsto y \mid (const\ x)$ *when* $\neg$(*is-ConstantExpr* *y*)
  using *size-flip-binary* **apply** *force*
  apply *auto*
  by (*simp add*: *BinaryExpr unfold-const val-shift-const-right-helper*)

optimization *EliminateRedundantFalse*: $x \mid false \longmapsto x$
  by (*meson exp-elim-redundant-false*)

optimization *OrNotOperands*: $(\sim x \mid \sim y) \longmapsto \sim(x\ \&\ y)$
   apply (*metis add-2-eq-Suc′ less-SucI not-add-less1 not-less-eq size-binary-const size-non-add*)
   apply *auto*
 by (*metis BinaryExpr UnaryExpr bin-eval.simps*(*4*) *intval-not.simps*(*2*) *unary-eval.simps*(*3*)

   val-or-not-operands)

optimization *OrLeftFallthrough*:
  $x \mid y \longmapsto x$ *when* ((*and* (*not* (*IRExpr-down x*)) (*IRExpr-up y*)) = *0*)
  using *simple-mask.OrLeftFallthrough* **by** *blast*

optimization *OrRightFallthrough*:
  $x \mid y \longmapsto y$ *when* ((*and* (*not* (*IRExpr-down y*)) (*IRExpr-up x*)) = *0*)
  using *simple-mask.OrRightFallthrough* **by** *blast*

end

end

1.10 ShiftNode Phase

theory *ShiftPhase*
  imports
    Common
begin

phase *ShiftNode*
  terminating *size*
begin

fun *intval-log2* :: *Value* ⇒ *Value* **where**
  intval-log2 (*IntVal b v*) = *IntVal b* (*word-of-int* (*SOME e. v=2^e*)) |
  intval-log2 - = *UndefVal*

fun *in-bounds* :: *Value* ⇒ *int* ⇒ *int* ⇒ *bool* **where**

in-bounds (*IntVal b v*) *l h* = (*l* < *sint v* ∧ *sint v* < *h*) |
in-bounds - *l h* = *False*

lemma
assumes *in-bounds* (*intval-log2 val-c*) *0 32*
shows *intval-left-shift x* (*intval-log2 val-c*) = *intval-mul x val-c*
apply (*cases val-c*; *auto*) **using** *intval-left-shift.simps*(*1*) *intval-mul.simps*(*1*) *intval-log2.simps*(*1*)
sorry

lemma *e-intval*:
n = *intval-log2 val-c* ∧ *in-bounds n 0 32* ⟶
intval-left-shift x (*intval-log2 val-c*) =
intval-mul x val-c
proof (*rule impI*)
assume *n* = *intval-log2 val-c* ∧ *in-bounds n 0 32*
show *intval-left-shift x* (*intval-log2 val-c*) =
intval-mul x val-c
proof (*cases* ∃ *v* . *val-c* = *IntVal 32 v*)
case *True*
obtain *vc* **where** *val-c* = *IntVal 32 vc*
using *True* **by** *blast*
then have *n* = *IntVal 32* (*word-of-int* (*SOME e. vc=2^e*))
using ‹*n* = *intval-log2 val-c* ∧ *in-bounds n 0 32*› *intval-log2.simps*(*1*) **by** *presburger*
then show *?thesis* **sorry**
next
case *False*
then have ∃ *v* . *val-c* = *IntVal 64 v*
sorry
then obtain *vc* **where** *val-c* = *IntVal 64 vc*
by *auto*
then have *n* = *IntVal 64* (*word-of-int* (*SOME e. vc=2^e*))
using ‹*n* = *intval-log2 val-c* ∧ *in-bounds n 0 32*› *intval-log2.simps*(*1*) **by** *presburger*
then show *?thesis* **sorry**
qed
qed

optimization *e*:
x ∗ (*const c*) ⟼ *x* << (*const n*) *when* (*n* = *intval-log2 c* ∧ *in-bounds n 0 32*)
using *e-intval*
using *BinaryExprE ConstantExprE bin-eval.simps*(*2*,*7*) **sorry**

end

end

1.11 SignedDivNode Phase

theory *SignedDivPhase*
  imports
    Common
begin

phase *SignedDivNode*
  terminating *size*
begin

lemma *val-division-by-one-is-self-32*:
  assumes *x* = *new-int 32 v*
  shows *intval-div x* (*IntVal 32 1*) = *x*
  using *assms* **apply** (*cases x*; *auto*)
  by (*simp add*: *take-bit-signed-take-bit*)

end

end

1.12 SignedRemNode Phase

theory *SignedRemPhase*
  imports
    Common
begin

phase *SignedRemNode*
  terminating *size*
begin

lemma *val-remainder-one*:
  assumes *intval-mod x* (*IntVal 32 1*) ≠ *UndefVal*
  shows *intval-mod x* (*IntVal 32 1*) = *IntVal 32 0*
  using *assms* **apply** (*cases x*; *auto*) **sorry**

value *word-of-int* (*sint* (*x2*::*32 word*) *smod 1*)

end

end

1.13 SubNode Phase

theory *SubPhase*
  imports
    Common
    Proofs.StampEvalThms
begin

phase *SubNode*
  terminating *size*
begin

lemma *bin-sub-after-right-add*:
  shows $((x::('a::len)\ word) + (y::('a::len)\ word)) - y = x$
  by *simp*

lemma *sub-self-is-zero*:
  shows $(x::('a::len)\ word) - x = 0$
  by *simp*

lemma *bin-sub-then-left-add*:
  shows $(x::('a::len)\ word) - (x + (y::('a::len)\ word)) = -y$
  by *simp*

lemma *bin-sub-then-left-sub*:
  shows $(x::('a::len)\ word) - (x - (y::('a::len)\ word)) = y$
  by *simp*

lemma *bin-subtract-zero*:
  shows $(x :: 'a::len\ word) - (0 :: 'a::len\ word) = x$
  by *simp*

lemma *bin-sub-negative-value*:
 $(x :: ('a::len)\ word) - (-(y :: ('a::len)\ word)) = x + y$
  by *simp*

lemma *bin-sub-self-is-zero*:
 $(x :: ('a::len)\ word) - x = 0$
  by *simp*

lemma *bin-sub-negative-const*:
$(x :: 'a::len\ word) - (-(y :: 'a::len\ word)) = x + y$
  by *simp*

lemma *val-sub-after-right-add-2*:
  assumes $x = new\text{-}int\ b\ v$
  assumes $val[(x + y) - y] \neq UndefVal$
  shows $val[(x + y) - y] = val[x]$

using *bin-sub-after-right-add*
using *assms* **apply** (*cases x*; *cases y*; *auto*)
by (*metis* (*full-types*) *intval-sub.simps*(*2*))

lemma *val-sub-after-left-sub*:
assumes *val*[$(x - y) - x$] $\neq$ *UndefVal*
shows *val*[$(x - y) - x$] = *val*[$-y$]
using *assms* **apply** (*cases x*; *cases y*; *auto*)
using *intval-sub.elims* **by** *fastforce*

lemma *val-sub-then-left-sub*:
assumes y = *new-int b v*
assumes *val*[$x - (x - y)$] $\neq$ *UndefVal*
shows *val*[$x - (x - y)$] = *val*[y]
using *assms* **apply** (*cases x*; *cases y*; *auto*)
by (*metis* (*mono-tags*) *intval-sub.simps*(*5*))

lemma *val-subtract-zero*:
assumes x = *new-int b v*
assumes *intval-sub x* (*IntVal b 0*) $\neq$ *UndefVal*
shows *intval-sub x* (*IntVal b 0*) = *val*[x]
using *assms* **by** (*induction x*; *simp*)

lemma *val-zero-subtract-value*:
assumes x = *new-int b v*
assumes *intval-sub* (*IntVal b 0*) $x \neq$ *UndefVal*
shows *intval-sub* (*IntVal b 0*) x = *val*[$-x$]
using *assms* **by** (*induction x*; *simp*)

lemma *val-sub-then-left-add*:
assumes *val*[$x - (x + y)$] $\neq$ *UndefVal*
shows *val*[$x - (x + y)$] = *val*[$-y$]
using *assms* **apply** (*cases x*; *cases y*; *auto*)
by (*metis* (*mono-tags*, *lifting*) *intval-sub.simps*(*5*))

lemma *val-sub-negative-value*:
assumes *val*[$x - (-y)$] $\neq$ *UndefVal*
shows *val*[$x - (-y)$] = *val*[$x + y$]
using *assms* **by** (*cases x*; *cases y*; *auto*)

lemma *val-sub-self-is-zero*:
assumes x = *new-int b v* $\wedge$ *val*[$x - x$] $\neq$ *UndefVal*
shows *val*[$x - x$] = *new-int b 0*
using *assms* **by** (*cases x*; *auto*)

lemma *val-sub-negative-const*:
assumes y = *new-int b v* $\wedge$ *val*[$x - (-y)$] $\neq$ *UndefVal*
shows *val*[$x - (-y)$] = *val*[$x + y$]
using *assms* **by** (*cases x*; *cases y*; *auto*)

```
lemma exp-sub-after-right-add:
  shows exp[(x + y) − y] ≥ exp[x]
   apply auto
  by (smt (verit) evalDet eval-unused-bits-zero intval-add.elims new-int.simps
      val-sub-after-right-add-2)

lemma exp-sub-after-right-add2:
  shows exp[(x + y) − x] ≥ exp[y]
  using exp-sub-after-right-add apply auto
 by (smt (z3) Value.inject(1) diff-eq-eq evalDet eval-unused-bits-zero intval-add.elims

    intval-sub.elims new-int.simps new-int-bin.simps take-bit-dist-subL bin-eval.simps(1)

      bin-eval.simps(3) intval-add-sym unfold-binary)

lemma exp-sub-negative-value:
 exp[x − (−y)] ≥ exp[x + y]
  apply simp
 by (smt (verit) bin-eval.simps(1) bin-eval.simps(3) evaltree-not-undef unary-eval.simps(2)

      unfold-binary unfold-unary val-sub-negative-value)

lemma exp-sub-then-left-sub:
  exp[x − (x − y)] ≥ exp[y]
  using val-sub-then-left-sub apply auto
  subgoal premises p for m p xa xaa ya
    proof−
      obtain xa where xa: [m, p] ⊢ x ↦ xa
        using p(2) by blast
      obtain ya where ya: [m, p] ⊢ y ↦ ya
        using p(5) by auto
      obtain xaa where xaa: [m, p] ⊢ x ↦ xaa
        using p(2) by blast
      have 1: val[xa − (xaa − ya)] ≠ UndefVal
        by (metis evalDet p(2) p(3) p(4) p(5) xa xaa ya)
      then have val[xaa − ya] ≠ UndefVal
        by auto
      then have [m, p] ⊢ y ↦ val[xa − (xaa − ya)]
        by (metis 1 Value.exhaust evalDet eval-unused-bits-zero evaltree-not-undef
         intval-sub.simps(6) intval-sub.simps(7) new-int.simps p(5) val-sub-then-left-sub
xa xaa
           ya)
      then show ?thesis
        by (metis evalDet p(2) p(4) p(5) xa xaa ya)
    qed
  done
```

thm-oracles *exp-sub-then-left-sub*

Optimisations

optimization *SubAfterAddRight*: $((x + y) - y) \longmapsto x$
using *exp-sub-after-right-add* **by** *blast*

optimization *SubAfterAddLeft*: $((x + y) - x) \longmapsto y$
using *exp-sub-after-right-add2* **by** *blast*

optimization *SubAfterSubLeft*: $((x - y) - x) \longmapsto -y$
apply (*metis Suc-lessI add-2-eq-Suc′ add-less-cancel-right less-trans-Suc not-add-less1*

size-binary-const size-binary-lhs size-binary-rhs size-non-add)
apply *auto*
by (*metis evalDet unary-eval.simps*(*2*) *unfold-unary val-sub-after-left-sub*)

optimization *SubThenAddLeft*: $(x - (x + y)) \longmapsto -y$
apply *auto*
by (*metis evalDet unary-eval.simps*(*2*) *unfold-unary val-sub-then-left-add*)

optimization *SubThenAddRight*: $(y - (x + y)) \longmapsto -x$
apply *auto*
by (*metis evalDet intval-add-sym unary-eval.simps*(*2*) *unfold-unary val-sub-then-left-add*)

optimization *SubThenSubLeft*: $(x - (x - y)) \longmapsto y$
using *size-simps* **apply** *simp*
using *exp-sub-then-left-sub* **by** *blast*

optimization *SubtractZero*: $(x - (const\ IntVal\ b\ 0)) \longmapsto x$
apply *auto*
by (*smt* (*verit*) *add.right-neutral diff-add-cancel eval-unused-bits-zero intval-sub.elims*

intval-word.simps new-int.simps new-int-bin.simps)

thm-oracles *SubtractZero*

optimization *SubNegativeValue*: $(x - (-y)) \longmapsto x + y$
apply (*metis add-2-eq-Suc′ less-SucI less-add-Suc1 not-less-eq size-binary-const size-non-add*)
using *exp-sub-negative-value* **by** *simp*

thm-oracles *SubNegativeValue*

lemma *negate-idempotent*:
assumes $x = IntVal\ b\ v \wedge take\text{-}bit\ b\ v = v$
shows $x = val[-(-x)]$

using *assms*
using *is-IntVal-def* **by** *force*

optimization *ZeroSubtractValue*: $((const\ IntVal\ b\ 0) - x) \longmapsto (-x)$
when $(wf\text{-}stamp\ x \wedge stamp\text{-}expr\ x = IntegerStamp\ b\ lo\ hi \wedge \neg(is\text{-}ConstantExpr\ x))$
defer
apply *auto* **unfolding** *wf-stamp-def*
apply (*smt* (*verit*) *diff-0 intval-negate.simps*(*1*) *intval-sub.elims intval-word.simps*

new-int-bin.simps unary-eval.simps(*2*) *unfold-unary*)
using *add-2-eq-Suc′ size.simps*(*2*) *size-flip-binary* **by** *presburger*

optimization *SubSelfIsZero*: $(x - x) \longmapsto const\ IntVal\ b\ 0$ *when*
$(wf\text{-}stamp\ x \wedge stamp\text{-}expr\ x = IntegerStamp\ b\ lo\ hi)$
apply *simp-all*
apply *auto*
using *IRExpr.disc*(*42*) *One-nat-def size-non-const* **apply** *presburger*
by (*smt* (*verit*, *best*) *wf-value-def ConstantExpr evalDet eval-bits-1-64 eval-unused-bits-zero*

new-int.simps take-bit-of-0 val-sub-self-is-zero validDefIntConst valid-int wf-stamp-def)

end

end

1.14 XorNode Phase

theory *XorPhase*
imports
Common
Proofs.StampEvalThms
begin

phase *XorNode*
terminating *size*
begin

lemma *bin-xor-self-is-false*:
$bin[x \oplus x] = 0$
by *simp*

lemma *bin-xor-commute*:
bin[$x \oplus y$] = *bin*[$y \oplus x$]
by (*simp add*: *xor.commute*)

lemma *bin-eliminate-redundant-false*:
bin[$x \oplus 0$] = *bin*[x]
by *simp*

lemma *val-xor-self-is-false*:
assumes *val*[$x \oplus x$] $\neq$ *UndefVal*
shows *val-to-bool* (*val*[$x \oplus x$]) = *False*
using *assms* **by** (*cases x*; *auto*)

lemma *val-xor-self-is-false-2*:
assumes (*val*[$x \oplus x$]) $\neq$ *UndefVal*
and x = *IntVal 32 v*
shows *val*[$x \oplus x$] = *bool-to-val False*
using *assms* **by** (*cases x*; *auto*)

lemma *val-xor-self-is-false-3*:
assumes *val*[$x \oplus x$] $\neq$ *UndefVal* $\wedge$ x = *IntVal 64 v*
shows *val*[$x \oplus x$] = *IntVal 64 0*
using *assms* **by** (*cases x*; *auto*)

lemma *val-xor-commute*:
val[$x \oplus y$] = *val*[$y \oplus x$]
apply (*cases x*; *cases y*; *auto*)
by (*simp add*: *xor.commute*)+

lemma *val-eliminate-redundant-false*:
assumes x = *new-int b v*
assumes *val*[$x \oplus$ (*bool-to-val False*)] $\neq$ *UndefVal*
shows *val*[$x \oplus$ (*bool-to-val False*)] = x
using *assms* **apply** (*cases x*; *auto*)
by *meson*

lemma *exp-xor-self-is-false*:
assumes *wf-stamp x* $\wedge$ *stamp-expr x* = *default-stamp*
shows *exp*[$x \oplus x$] $\geq$ *exp*[*false*]
using *assms* **apply** *auto* **unfolding** *wf-stamp-def*
by (*smt* (*z3*) *validDefIntConst IntVal0 Value.inject*(*1*) *bool-to-val.simps*(*2*)
constantAsStamp.simps(*1*) *evalDet int-signed-value-bounds new-int.simps unfold-const*
val-xor-self-is-false-2 valid-int valid-stamp.simps(*1*) *valid-value.simps*(*1*) *wf-value-def*)

lemma *exp-eliminate-redundant-false*:

shows $exp[x \oplus false] \geq exp[x]$
using *val-eliminate-redundant-false* **apply** *auto*
subgoal premises *p* **for** *m p xa*
proof −
obtain *xa* **where** *xa*: $[m, p] \vdash x \mapsto xa$
using *p(2)* **by** *blast*
then have $val[xa \oplus (IntVal\ 32\ 0)] \neq UndefVal$
using *evalDet p(2) p(3)* **by** *blast*
then have $[m, p] \vdash x \mapsto val[xa \oplus (IntVal\ 32\ 0)]$
apply (*cases xa*; *auto*) **using** *eval-unused-bits-zero xa* **by** *auto*
then show *?thesis*
using *evalDet p(2) xa* **by** *blast*
qed
done

Optimisations

optimization *XorSelfIsFalse*: $(x \oplus x) \longmapsto false$ *when*
($wf\text{-}stamp\ x \wedge stamp\text{-}expr\ x = default\text{-}stamp$)
using *size-non-const* **apply** *force*
using *exp-xor-self-is-false* **by** *auto*

optimization *XorShiftConstantRight*: $((const\ x) \oplus y) \longmapsto y \oplus (const\ x)$ *when* $\neg(is\text{-}ConstantExpr\ y)$
using *size-flip-binary* **apply** *force*
unfolding *le-expr-def* **using** *val-xor-commute*
by *auto*

optimization *EliminateRedundantFalse*: $(x \oplus false) \longmapsto x$
using *exp-eliminate-redundant-false* **by** *blast*

end

end

1.15 NegateNode Phase

theory *NegatePhase*
imports
Common
begin

phase *NegateNode*
terminating *size*
begin

lemma *bin-negative-cancel*:
$-1 * (-1 * ((x::('a::len)\ word))) = x$
by *auto*

lemma *val-negative-cancel*:
assumes *intval-negate* (*new-int b v*) ≠ *UndefVal*
shows *val*[−(−(*new-int b v*))] = *val*[*new-int b v*]
using *assms* **by** *simp*

lemma *val-distribute-sub*:
assumes *x* ≠ *UndefVal* ∧ *y* ≠ *UndefVal*
shows *val*[−(*x* − *y*)] = *val*[*y* − *x*]
using *assms* **by** (*cases x*; *cases y*; *auto*)

lemma *exp-distribute-sub*:
shows *exp*[−(*x* − *y*)] ≥ *exp*[*y* − *x*]
using *val-distribute-sub* **apply** *auto*
using *evaltree-not-undef* **by** *auto*

thm-oracles *exp-distribute-sub*

lemma *exp-negative-cancel*:
shows *exp*[−(−*x*)] ≥ *exp*[*x*]
using *val-negative-cancel* **apply** *auto*
by (*metis* (*no-types*, *opaque-lifting*) *eval-unused-bits-zero intval-negate.elims intval-negate.simps*(*1*) *minus-equation-iff new-int.simps take-bit-dist-neg*)

lemma *exp-negative-shift*:
assumes *stamp-expr x* = *IntegerStamp b′ lo hi*
and *unat y* = (*b′* − *1*)
shows *exp*[−(*x* >> (*const* (*new-int b y*)))] ≥ *exp*[*x* >>> (*const* (*new-int b y*))]
apply *auto*
subgoal premises *p* **for** *m p xa*
proof −
obtain *xa* **where** *xa*: [*m*,*p*] ⊢ *x* ↦ *xa*
using *p*(*2*) **by** *auto*
then have *1*: *intval-negate* (*intval-right-shift xa* (*IntVal b* (*take-bit b y*))) ≠ *UndefVal*
using *evalDet p*(*1*) *p*(*2*) **by** *blast*
then have *2*: *intval-right-shift xa* (*IntVal b* (*take-bit b y*)) ≠ *UndefVal*
by *auto*
then have *3*: − ((*2*::*int*) ^ *b div* (*2*::*int*)) ⊑ *sint* (*signed-take-bit* (*b* − *Suc* (*0*::*nat*)) (*take-bit b y*))
by (*smt* (*verit*, *del-insts*) *One-nat-def diff-le-self gr0I half-nonnegative-int-iff linorder-not-le lower-bounds-equiv power-increasing-iff signed-0 signed-take-bit-int-greater-eq-minus-exp-word signed-take-bit-of-0 sint-greater-eq take-bit-0*)

then have *4*: *sint* (*signed-take-bit* (*b* − *Suc* (*0*::*nat*)) (*take-bit b y*)) < (*2*::*int*)
^ *b div* (*2*::*int*)
by (*metis Suc-le-lessD Suc-pred eval-bits-1-64 int-power-div-base p*(*4*) *signed-take-bit-int-less-exp-word size64 unfold-const wsst-TYs*(*3*) *zero-less-numeral*)
then have *5*: (*0*::*nat*) < *b*
using *eval-bits-1-64 p*(*4*) **by** *blast*
then have *6*: *b* ⊑ (*64*::*nat*)
using *eval-bits-1-64 p*(*4*) **by** *blast*
then have *7*: [*m*,*p*] ⊢ *BinaryExpr BinURightShift x*
(*ConstantExpr* (*IntVal b* (*take-bit b y*))) ↦
intval-negate (*intval-right-shift xa* (*IntVal b* (*take-bit b y*)))
apply (*cases y*; *auto*)

subgoal premises *p* **for** *n*
proof −
have *sg1*: *y* = *word-of-nat n*
by (*simp add*: *p*(*1*))
then have *sg2*: *n* < (*18446744073709551616*::*nat*)
by (*simp add*: *p*(*2*))
then have *sg3*: *b* ⊑ (*64*::*nat*)
by (*simp add*: *6*)
then have *sg4*: [*m*,*p*] ⊢ *BinaryExpr BinURightShift x*
(*ConstantExpr* (*IntVal b* (*take-bit b* (*word-of-nat n*)))) ↦
intval-negate (*intval-right-shift xa* (*IntVal b* (*take-bit b* (*word-of-nat n*))))
sorry
then show *?thesis*
by *simp*
qed
done
then show *?thesis*
by (*metis evalDet p*(*2*) *xa*)
qed
done

Optimisations

optimization *NegateCancel*: −(−(*x*)) ⟼ *x*
using *exp-negative-cancel* **by** *blast*

optimization *DistributeSubtraction*: −(*x* − *y*) ⟼ (*y* − *x*)
apply (*smt* (*z3*) *add.left-commute add-2-eq-Suc′ add-diff-cancel-left′ is-ConstantExpr-def*

less-Suc-eq-0-disj plus-1-eq-Suc size.simps(*11*) *size-binary-const size-non-add*

zero-less-diff)
using *exp-distribute-sub* **by** *simp*

optimization *NegativeShift*: $-(x >> (const\ (new\text{-}int\ b\ y))) \longmapsto x >>> (const\ (new\text{-}int\ b\ y))$
$\quad when\ (stamp\text{-}expr\ x = IntegerStamp\ b'\ lo\ hi \wedge unat\ y = (b' - 1))$
using *exp-negative-shift* **by** *simp*

end

end
theory *TacticSolving*
imports *Common*
begin

fun *size* :: *IRExpr* ⇒ *nat* **where**
size (*UnaryExpr op e*) = (*size e*) ∗ *2* |
size (*BinaryExpr BinAdd x y*) = (*size x*) + ((*size y*) ∗ *2*) |
size (*BinaryExpr op x y*) = (*size x*) + (*size y*) |
size (*ConditionalExpr cond t f*) = (*size cond*) + (*size t*) + (*size f*) + *2* |
size (*ConstantExpr c*) = *1* |
size (*ParameterExpr ind s*) = *2* |
size (*LeafExpr nid s*) = *2* |
size (*ConstantVar c*) = *2* |
size (*VariableExpr x s*) = *2*

lemma *size-pos*[*simp*]: *0* < *size y*
apply (*induction y*; *auto?*)
subgoal premises *prems* **for** *op a b*
using *prems* **by** (*induction op*; *auto*)
done

phase *TacticSolving*
terminating *size*
begin

1.16 AddNode

lemma *value-approx-implies-refinement*:
assumes *lhs* ≈ *rhs*
assumes ∀ *m p v*. ([*m*, *p*] ⊢ *elhs* ↦ *v*) ⟶ *v* = *lhs*
assumes ∀ *m p v*. ([*m*, *p*] ⊢ *erhs* ↦ *v*) ⟶ *v* = *rhs*
assumes ∀ *m p v1 v2*. ([*m*, *p*] ⊢ *elhs* ↦ *v1*) ⟶ ([*m*, *p*] ⊢ *erhs* ↦ *v2*)
shows *elhs* ≥ *erhs*
using *assms* **unfolding** *le-expr-def well-formed-equal-def*
using *evalDet evaltree-not-undef*
by *metis*

method *explore-cases* **for** *x y* :: *Value* =
(*cases x*; *cases y*; *auto*)

method *explore-cases-bin* **for** x :: *IRExpr* =
(*cases* x; *auto*)

method *obtain-approx-eq* **for** *lhs rhs* x y :: *Value* =
(*rule meta-mp*[**where** P=*lhs* $\approx$ *rhs*], *defer-tac*, *explore-cases* x y)

method *obtain-eval* **for** *exp*::*IRExpr* **and** *val*::*Value* =
(*rule meta-mp*[**where** $P=\bigwedge m\ p\ v.\ ([m,\ p] \vdash exp \mapsto v) \Longrightarrow v = val$], *defer-tac*)

method *solve* **for** *lhs rhs* x y :: *Value* =
(*match* **conclusion in** *size* - $<$ *size* - $\Rightarrow$ ‹*simp*›)?,
(*match* **conclusion in** (*elhs*::*IRExpr*) $\geq$ (*erhs*::*IRExpr*) **for** *elhs erhs* $\Rightarrow$ ‹
(*obtain-approx-eq lhs rhs* x y)?›)

print-methods

thm *BinaryExprE*
optimization *opt-add-left-negate-to-sub*:
$-x + y \longmapsto y - x$

apply (*solve val*[$-x1 + y1$] *val*[$y1 - x1$] *x1 y1*)
apply *simp* **apply** *auto* **using** *evaltree-not-undef* **sorry**

1.17 NegateNode

lemma *val-distribute-sub*:
val[$-(x-y)$] $\approx$ *val*[$y-x$]
by (*cases* x; *cases* y; *auto*)

optimization *distribute-sub*: $-(x-y) \longmapsto (y-x)$
apply *simp*
using *val-distribute-sub* **apply** *simp*
using *unfold-binary unfold-unary* **by** *auto*

lemma *val-xor-self-is-false*:
assumes x = *IntVal 32 v*
shows *val*[$x \oplus x$] $\approx$ *val*[*false*]
apply *simp* **using** *assms* **by** (*cases* x; *auto*)

definition *wf-stamp* :: *IRExpr* $\Rightarrow$ *bool* **where**
wf-stamp e = ($\forall m\ p\ v.\ ([m,\ p] \vdash e \mapsto v) \longrightarrow$ *valid-value* v (*stamp-expr* e))

lemma *exp-xor-self-is-false*:
assumes *stamp-expr* x = *IntegerStamp 32 l h*
assumes *wf-stamp* x
shows *exp*[$x \oplus x$] >= *exp*[*false*]

unfolding *le-expr-def* **using** *assms* **unfolding** *wf-stamp-def*
using *val-xor-self-is-false evaltree-not-undef*
by (*smt* (*z3*) *wf-value-def bin-eval.simps*(*6*) *bin-eval-new-int constantAsStamp.simps*(*1*) *evalDet*
int-signed-value-bounds new-int.simps new-int-take-bits unfold-binary unfold-const valid-int
valid-stamp.simps(*1*) *valid-value.simps*(*1*) *well-formed-equal-defn*)

lemma *val-or-commute*[*simp*]:
val[x | y] = *val*[y | x]
apply (*cases x*; *cases y*; *auto*)
by (*simp add*: *or.commute*)+

lemma *val-xor-commute*[*simp*]:
val[$x \oplus y$] = *val*[$y \oplus x$]
apply (*cases x*; *cases y*; *auto*)
by (*simp add*: *word-bw-comms*(*3*))

lemma *exp-or-commutative*:
exp[x | y] $\geq$ *exp*[y | x]
by *auto*

lemma *exp-xor-commutative*:
exp[$x \oplus y$] $\geq$ *exp*[$y \oplus x$]
by *auto*

lemma *OrInverseVal*:
assumes n = *IntVal 32 v*
shows *val*[n | $\sim n$] $\approx$ *new-int 32* (-1)
apply *simp* **using** *assms* **using** *word-or-not* **apply** (*cases n*; *auto*) **using** *take-bit-or*
by (*metis bit.disj-cancel-right mask-eq-take-bit-minus-one*)

optimization *OrInverse*: *exp*[n | $\sim n$] $\longmapsto$ (*const* (*new-int 32* (*not 0*)))
when (*stamp-expr n* = *IntegerStamp 32 l h* $\wedge$ *wf-stamp n*)
unfolding *size.simps* **apply** (*simp add*: *Suc-lessI*)
apply *auto* **using** *OrInverseVal* **unfolding** *wf-stamp-def*
by (*smt* (*z3*) *wf-value-def constantAsStamp.simps*(*1*) *evalDet int-signed-value-bounds*
mask-eq-take-bit-minus-one new-int.elims new-int-take-bits unfold-const valid-int
valid-stamp.simps(*1*) *valid-value.simps*(*1*) *well-formed-equal-defn*)

optimization *OrInverse2*: *exp*[$\sim n$ | n] $\longmapsto$ (*const* (*new-int 32* (*not 0*)))
when (*stamp-expr n* = *IntegerStamp 32 l h* $\wedge$ *wf-stamp n*)
using *OrInverse exp-or-commutative* **by** *auto*

lemma *XorInverseVal*:
  assumes $n = IntVal\ 32\ v$
  shows $val[n \oplus \,^{\sim}n] \approx new\text{-}int\ 32\ (-1)$
  apply *simp* **using** *assms* **using** *word-or-not* **apply** (*cases n*; *auto*)
  by (*metis* (*no-types*, *opaque-lifting*) *bit.compl-zero bit.xor-compl-right bit.xor-self*

mask-eq-take-bit-minus-one take-bit-xor)

optimization *XorInverse*: $exp[n \oplus \,^{\sim}n] \longmapsto (const\ (new\text{-}int\ 32\ (not\ 0)))$
  $when\ (stamp\text{-}expr\ n = IntegerStamp\ 32\ l\ h \wedge wf\text{-}stamp\ n)$
  unfolding *size.simps* **apply** (*simp add*: *Suc-lessI*)
  apply *auto* **using** *XorInverseVal*
  by (*smt* (*verit*) *wf-value-def constantAsStamp.simps*(*1*) *evalDet int-signed-value-bounds*

intval-xor.elims mask-eq-take-bit-minus-one new-int.elims new-int-take-bits
unfold-const
valid-stamp.simps(*1*) *valid-value.simps*(*1*) *well-formed-equal-defn wf-stamp-def*)

optimization *XorInverse2*: $exp[(^{\sim}n) \oplus n] \longmapsto (const\ (new\text{-}int\ 32\ (not\ 0)))$
  $when\ (stamp\text{-}expr\ n = IntegerStamp\ 32\ l\ h \wedge wf\text{-}stamp\ n)$
  using *XorInverse exp-xor-commutative* **by** *auto*

end

end
theory *ProofStatus*
  imports
    AbsPhase
    AddPhase
    AndPhase
    ConditionalPhase
    MulPhase

    NegatePhase
    NewAnd
    NotPhase
    OrPhase
    ShiftPhase
    SignedDivPhase
    SignedRemPhase
    SubPhase
    TacticSolving
    XorPhase
begin

declare [[*show-types=false*]]
print-phases
print-phases!

print-methods

print-theorems

thm *opt-add-left-negate-to-sub*
thm-oracles *AbsNegate*

export-phases ‹*Full*›

end