

# Veriopt Theories

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## 1 Runtime Values and Arithmetic

**theory** *Values*

**imports**

*HOL-Library.Word*

*HOL-Library.Signed-Division*

*HOL-Library.Float*

*HOL-Library.LaTeXsugar*

**begin**

**lemma**  $-((x::\text{float})-y) = (y-x)$   
**by** *simp*

In order to properly implement the IR semantics we first introduce a type that represents runtime values. These runtime values represent the full range of primitive types currently allowed by our semantics, ranging from basic integer types to object references and arrays.

Note that Java supports 64, 32, 16, 8 signed ints, plus 1 bit (boolean) ints, and char is 16-bit unsigned. E.g. an 8-bit stamp has a default range of -128..+127. And a 1-bit stamp has a default range of -1..0, surprisingly.

During calculations the smaller sizes are sign-extended to 32 bits, but explicit widening nodes will do that, so most binary calculations should see equal input sizes.

An object reference is an option type where the *None* object reference points to the static fields. This is examined more closely in our definition of the heap.

**type-synonym** *int64* = 64 word — long  
**type-synonym** *int32* = 32 word — int  
**type-synonym** *int16* = 16 word — short  
**type-synonym** *int8* = 8 word — char  
**type-synonym** *int1* = 1 word — boolean

**abbreviation** *valid-int-widths* :: nat set **where**  
*valid-int-widths*  $\equiv \{1, 8, 16, 32, 64\}$

Option 2: explicit width stored with each integer value. However, this does not help us to distinguish between short (signed) and char (unsigned).

**typedef** *IntWidth* = { *w* :: nat . *w*=1  $\vee$  *w*=8  $\vee$  *w*=16  $\vee$  *w*=32  $\vee$  *w*=64 }  
**by** *blast*

**setup-lifting** *type-definition-IntWidth*

**lift-definition** *IntWidthBits* :: *IntWidth*  $\Rightarrow$  nat  
**is**  $\lambda w. w$  .

Option 3: explicit type stored with each integer value.

**datatype** *IntType* = *ILong* | *IInt* | *IShort* | *IChar* | *IByte* | *IBoolean*

**fun** *int-bits* :: *IntType*  $\Rightarrow$  nat **where**  
*int-bits* *ILong* = 64 |  
*int-bits* *IInt* = 32 |  
*int-bits* *IShort* = 16 |  
*int-bits* *IChar* = 16 |  
*int-bits* *IByte* = 8 |

*int-bits IBoolean = 1*

```
fun int-signed :: IntType  $\Rightarrow$  bool where  
  int-signed ILong = True |  
  int-signed IInt  = True |  
  int-signed IShort = True |  
  int-signed IChar  = False |  
  int-signed IByte  = True |  
  int-signed IBoolean = True
```

Option 4: int64 with the number of significant bits.

```
type-synonym iwidth = nat  
type-synonym objref = nat option
```

```
datatype (discs-sels) Value =  
  UndefVal |
```

*IntVal iwidth int64* |

*ObjRef objref* |  
*ObjStr string*

```
fun intval-bits :: Value  $\Rightarrow$  nat where  
  intval-bits (IntVal b v) = b
```

```
fun intval-word :: Value  $\Rightarrow$  int64 where  
  intval-word (IntVal b v) = v
```

```
fun bit-bounds :: nat  $\Rightarrow$  (int  $\times$  int) where  
  bit-bounds bits = ((( $2^{\text{bits}}$ ) div 2) * -1, (( $2^{\text{bits}}$ ) div 2) - 1)
```

```
definition logic-negate :: ('a::len) word  $\Rightarrow$  'a word where  
  logic-negate x = (if x = 0 then 1 else 0)
```

```
fun int-signed-value :: iwidth  $\Rightarrow$  int64  $\Rightarrow$  int where  
  int-signed-value b v = sint (signed-take-bit (b - 1) v)
```

```
fun int-unsigned-value :: iwidth  $\Rightarrow$  int64  $\Rightarrow$  int where  
  int-unsigned-value b v = uint v
```

Converts an integer word into a Java value.

```
fun new-int :: iwidth  $\Rightarrow$  int64  $\Rightarrow$  Value where  
  new-int b w = IntVal b (take-bit b w)
```

Converts an integer word into a Java value, iff the two types are equal.

```
fun new-int-bin :: iwidth  $\Rightarrow$  iwidth  $\Rightarrow$  int64  $\Rightarrow$  Value where
  new-int-bin b1 b2 w = (if b1=b2 then new-int b1 w else UndefVal)
```

```
fun wf-bool :: Value  $\Rightarrow$  bool where
  wf-bool (IntVal b w) = (b = 1) |
  wf-bool - = False
```

```
fun val-to-bool :: Value  $\Rightarrow$  bool where
  val-to-bool (IntVal b val) = (if val = 0 then False else True) |
  val-to-bool val = False
```

```
fun bool-to-val :: bool  $\Rightarrow$  Value where
  bool-to-val True = (IntVal 32 1) |
  bool-to-val False = (IntVal 32 0)
```

Converts an Isabelle bool into a Java value, iff the two types are equal.

```
fun bool-to-val-bin :: iwidth  $\Rightarrow$  iwidth  $\Rightarrow$  bool  $\Rightarrow$  Value where
  bool-to-val-bin t1 t2 b = (if t1 = t2 then bool-to-val b else UndefVal)
```

```
fun is-int-val :: Value  $\Rightarrow$  bool where
  is-int-val v = is-IntVal v
```

A convenience function for directly constructing -1 values of a given bit size.

```
fun neg-one :: iwidth  $\Rightarrow$  int64 where
  neg-one b = mask b
```

```
lemma neg-one-value[simp]: new-int b (neg-one b) = IntVal b (mask b)
by simp
```

```
lemma neg-one-signed[simp]:
  assumes 0 < b
  shows int-signed-value b (neg-one b) = -1
  by (smt (verit, best) assms diff-le-self diff-less int-signed-value.simps less-one
    mask-eq-take-bit-minus-one neg-one.simps nle-le signed-minus-1 signed-take-bit-of-minus-1
    signed-take-bit-take-bit verit-comp-simplify1(1))
```

## 1.1 Arithmetic Operators

We need to introduce arithmetic operations which agree with the JVM.

Within the JVM, bytecode arithmetic operations are performed on 32 or 64 bit integers, unboxing where appropriate.

The following collection of intval functions correspond to the JVM arithmetic operations. We merge the 32 and 64 bit operations into a single

function, even though the stamp of each IRNode tells us exactly what the bit widths will be. These merged functions make it easier to do the instantiation of Value as 'plus', etc. It might be worse for reasoning, because it could cause more case analysis, but this does not seem to be a problem in practice.

```
fun intval-add :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-add (IntVal b1 v1) (IntVal b2 v2) =
    (if b1 = b2 then IntVal b1 (take-bit b1 (v1+v2)) else UndefVal) |
  intval-add - - = UndefVal
```

```
fun intval-sub :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-sub (IntVal b1 v1) (IntVal b2 v2) = new-int-bin b1 b2 (v1-v2) |
  intval-sub - - = UndefVal
```

```
instantiation Value :: minus
begin
```

```
definition minus-Value :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  minus-Value = intval-sub
```

```
instance proof qed
end
```

```
fun intval-mul :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-mul (IntVal b1 v1) (IntVal b2 v2) = new-int-bin b1 b2 (v1*v2) |
  intval-mul - - = UndefVal
```

```
instantiation Value :: times
begin
```

```
definition times-Value :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  times-Value = intval-mul
```

```
instance proof qed
end
```

```
fun intval-div :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-div (IntVal b1 v1) (IntVal b2 v2) =
    new-int-bin b1 b2 (word-of-int
```

```

      ((int-signed-value b1 v1) sdiv (int-signed-value b2 v2))) |
intval-div - - = UndefVal

```

```

instantiation Value :: divide
begin

```

```

definition divide-Value :: Value ⇒ Value ⇒ Value where
  divide-Value = intval-div

```

```

instance proof qed
end

```

```

fun intval-mod :: Value ⇒ Value ⇒ Value where
  intval-mod (IntVal b1 v1) (IntVal b2 v2) =
    new-int-bin b1 b2 (word-of-int
      ((int-signed-value b1 v1) smod (int-signed-value b2 v2))) |
  intval-mod - - = UndefVal

```

```

instantiation Value :: modulo
begin

```

```

definition modulo-Value :: Value ⇒ Value ⇒ Value where
  modulo-Value = intval-mod

```

```

instance proof qed
end

```

## 1.2 Bitwise Operators and Comparisons

```

context
  includes bit-operations-syntax
begin

```

```

fun intval-and :: Value ⇒ Value ⇒ Value where
  intval-and (IntVal b1 v1) (IntVal b2 v2) = new-int-bin b1 b2 (v1 AND v2) |
  intval-and - - = UndefVal

```

```

fun intval-or :: Value ⇒ Value ⇒ Value where
  intval-or (IntVal b1 v1) (IntVal b2 v2) = new-int-bin b1 b2 (v1 OR v2) |
  intval-or - - = UndefVal

```

```

fun intval-xor :: Value ⇒ Value ⇒ Value where
  intval-xor (IntVal b1 v1) (IntVal b2 v2) = new-int-bin b1 b2 (v1 XOR v2) |
  intval-xor - - = UndefVal

```

```

fun intval-short-circuit-or :: Value ⇒ Value ⇒ Value where
  intval-short-circuit-or (IntVal b1 v1) (IntVal b2 v2) = bool-to-val-bin b1 b2 (((v1

```

```

≠ 0) ∨ (v2 ≠ 0))) |
  intval-short-circuit-or - - =.UndefVal

```

```

fun intval-equals :: Value ⇒ Value ⇒ Value where
  intval-equals (IntVal b1 v1) (IntVal b2 v2) = bool-to-val-bin b1 b2 (v1 = v2) |
  intval-equals - - =.UndefVal

```

```

fun intval-less-than :: Value ⇒ Value ⇒ Value where
  intval-less-than (IntVal b1 v1) (IntVal b2 v2) =
    bool-to-val-bin b1 b2 (int-signed-value b1 v1 < int-signed-value b2 v2) |
  intval-less-than - - =.UndefVal

```

```

fun intval-below :: Value ⇒ Value ⇒ Value where
  intval-below (IntVal b1 v1) (IntVal b2 v2) = bool-to-val-bin b1 b2 (v1 < v2) |
  intval-below - - =.UndefVal

```

```

fun intval-not :: Value ⇒ Value where
  intval-not (IntVal t v) = new-int t (NOT v) |
  intval-not - =.UndefVal

```

```

fun intval-negate :: Value ⇒ Value where
  intval-negate (IntVal t v) = new-int t (- v) |
  intval-negate - =.UndefVal

```

```

fun intval-abs :: Value ⇒ Value where
  intval-abs (IntVal t v) = new-int t (if int-signed-value t v < 0 then - v else v) |
  intval-abs - =.UndefVal

```

```

fun intval-conditional :: Value ⇒ Value ⇒ Value ⇒ Value where
  intval-conditional cond tv fv = (if (val-to-bool cond) then tv else fv)

```

TODO: clarify which widths this should work on: just 1-bit or all?

```

fun intval-logic-negation :: Value ⇒ Value where
  intval-logic-negation (IntVal b v) = new-int b (logic-negate v) |
  intval-logic-negation - =.UndefVal

```

### 1.3 Narrowing and Widening Operators

Note: we allow these operators to have inBits=outBits, because the Graal compiler also seems to allow that case, even though it should rarely / never arise in practice.

```

value sint(signed-take-bit 0 (1 :: int32))

```

```

fun intval-narrow :: nat ⇒ nat ⇒ Value ⇒ Value where
  intval-narrow inBits outBits (IntVal b v) =
    (if inBits = b ∧ 0 < outBits ∧ outBits ≤ inBits ∧ inBits ≤ 64
     then new-int outBits v
     else.UndefVal) |

```

```

    intval-narrow - - - = UndefVal

value intval(intval-narrow 16 8 (IntVal32 (512 - 2)))

value sint (signed-take-bit 7 ((256 + 128) :: int64))

fun intval-sign-extend :: nat ⇒ nat ⇒ Value ⇒ Value where
  intval-sign-extend inBits outBits (IntVal b v) =
    (if inBits = b ∧ 0 < inBits ∧ inBits ≤ outBits ∧ outBits ≤ 64
     then new-int outBits (signed-take-bit (inBits - 1) v)
     else UndefVal) |
  intval-sign-extend - - - = UndefVal

fun intval-zero-extend :: nat ⇒ nat ⇒ Value ⇒ Value where
  intval-zero-extend inBits outBits (IntVal b v) =
    (if inBits = b ∧ 0 < inBits ∧ inBits ≤ outBits ∧ outBits ≤ 64
     then new-int outBits (take-bit inBits v)
     else UndefVal) |
  intval-zero-extend - - - = UndefVal

```

Some well-formedness results to help reasoning about narrowing and widening operators

**lemma** *intval-narrow-ok*:

```

assumes intval-narrow inBits outBits val ≠ UndefVal
shows 0 < outBits ∧ outBits ≤ inBits ∧ inBits ≤ 64 ∧ outBits ≤ 64 ∧
  is-IntVal val ∧
  intval-bits val = inBits
using assms intval-narrow.simps neq0-conv intval-bits.simps
by (metis Value.disc(2) intval-narrow.elims le-trans)

```

**lemma** *intval-sign-extend-ok*:

```

assumes intval-sign-extend inBits outBits val ≠ UndefVal
shows 0 < inBits ∧
  inBits ≤ outBits ∧ outBits ≤ 64 ∧
  is-IntVal val ∧
  intval-bits val = inBits
using assms intval-sign-extend.simps neq0-conv
by (metis intval-bits.simps intval-sign-extend.elims is-IntVal-def)

```

**lemma** *intval-zero-extend-ok*:

```

assumes intval-zero-extend inBits outBits val ≠ UndefVal
shows 0 < inBits ∧
  inBits ≤ outBits ∧ outBits ≤ 64 ∧
  is-IntVal val ∧
  intval-bits val = inBits

```



```

using assms intval-sign-extend.simps neq0-conv
by (metis intval-bits.simps intval-zero-extend.elims is-IntVal-def)

```

## 1.4 Bit-Shifting Operators

```

definition shiftl (infix << 75) where
  shiftl w n = (push-bit n) w

```

```

lemma shiftl-power[simp]: (x::('a::len) word) * ( $2^j$ ) = x << j
unfolding shiftl-def apply (induction j)
apply simp unfolding funpow-Suc-right
by (metis (no-types, opaque-lifting) push-bit-eq-mult)

```

```

lemma (x::('a::len) word) * ( $(2^j + 1)$ ) = x << j + x
by (simp add: distrib-left)

```

```

lemma (x::('a::len) word) * ( $(2^j - 1)$ ) = x << j - x
by (simp add: right-diff-distrib)

```

```

lemma (x::('a::len) word) * ( $(2^j) + (2^k)$ ) = x << j + x << k
by (simp add: distrib-left)

```

```

lemma (x::('a::len) word) * ( $(2^j) - (2^k)$ ) = x << j - x << k
by (simp add: right-diff-distrib)

```

```

definition shiftr (infix >>> 75) where
  shiftr w n = (drop-bit n) w

```

```

value (255 :: 8 word) >>> (2 :: nat)

```

```

definition sshiftr :: 'a :: len word  $\Rightarrow$  nat  $\Rightarrow$  'a :: len word (infix >> 75) where
  sshiftr w n = word-of-int ((sint w) div ( $2^n$ ))

```

```

value (128 :: 8 word) >> 2

```

Note that Java shift operators use unary numeric promotion, unlike other binary operators, which use binary numeric promotion (see the Java language reference manual). This means that the left-hand input determines the output size, while the right-hand input can be any size.

```

fun shift-amount :: iwidth  $\Rightarrow$  int64  $\Rightarrow$  nat where
  shift-amount b val = unat (val AND (if b = 64 then 0x3F else 0x1f))

```

```

fun intval-left-shift :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-left-shift (IntVal b1 v1) (IntVal b2 v2) = new-int b1 (v1 << shift-amount b1 v2) |
  intval-left-shift - - = UndefVal

```

Signed shift is more complex, because we sometimes have to insert 1 bits at the correct point, which is at  $b1$  bits.

```
fun intval-right-shift :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-right-shift (IntVal  $b1$   $v1$ ) (IntVal  $b2$   $v2$ ) =
    (let shift = shift-amount  $b1$   $v2$  in
     let ones = mask  $b1$  AND (NOT (mask ( $b1 - shift$ ) :: int64)) in
     (if int-signed-value  $b1$   $v1$  < 0
      then new-int  $b1$  (ones OR ( $v1$  >>> shift))
      else new-int  $b1$  ( $v1$  >>> shift))) |
  intval-right-shift - - = UndefVal

fun intval-uright-shift :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-uright-shift (IntVal  $b1$   $v1$ ) (IntVal  $b2$   $v2$ ) = new-int  $b1$  ( $v1$  >>> shift-amount
 $b1$   $v2$ ) |
  intval-uright-shift - - = UndefVal

end
```

## 2 Examples of Narrowing / Widening Functions

**experiment begin**

**corollary** *intval-narrow* 32 8 (*IntVal* 32 (256 + 128)) = *IntVal* 8 128 **by** *simp*

**corollary** *intval-narrow* 32 8 (*IntVal* 32 (-2)) = *IntVal* 8 254 **by** *simp*

**corollary** *intval-narrow* 32 1 (*IntVal* 32 (-2)) = *IntVal* 1 0 **by** *simp*

**corollary** *intval-narrow* 32 1 (*IntVal* 32 (-3)) = *IntVal* 1 1 **by** *simp*

**corollary** *intval-narrow* 32 8 (*IntVal* 64 (-2)) = *UndefVal* **by** *simp*

**corollary** *intval-narrow* 64 8 (*IntVal* 32 (-2)) = *UndefVal* **by** *simp*

**corollary** *intval-narrow* 64 8 (*IntVal* 64 254) = *IntVal* 8 254 **by** *simp*

**corollary** *intval-narrow* 64 8 (*IntVal* 64 (256+127)) = *IntVal* 8 127 **by** *simp*

**corollary** *intval-narrow* 64 64 (*IntVal* 64 (-2)) = *IntVal* 64 (-2) **by** *simp*

**end**

**experiment begin**

**corollary** *intval-sign-extend* 8 32 (*IntVal* 8 (256 + 128)) = *IntVal* 32 ( $2^{32} - 128$ ) **by** *simp*

**corollary** *intval-sign-extend* 8 32 (*IntVal* 8 (-2)) = *IntVal* 32 ( $2^{32} - 2$ ) **by** *simp*

**corollary** *intval-sign-extend* 1 32 (*IntVal* 1 (-2)) = *IntVal* 32 0 **by** *simp*

**corollary** *intval-sign-extend* 1 32 (*IntVal* 1 (-3)) = *IntVal* 32 (*mask* 32) **by** *simp*

**corollary** *intval-sign-extend* 8 32 (*IntVal* 64 254) = *UndefVal* **by** *simp*

**corollary** *intval-sign-extend* 8 64 (*IntVal* 32 254) = *UndefVal* **by** *simp*

**corollary** *intval-sign-extend* 8 64 (*IntVal* 8 254) = *IntVal* 64 (-2) **by** *simp*

**corollary** *intval-sign-extend* 32 64 (*IntVal* 32 ( $2^{32} - 2$ )) = *IntVal* 64 (-2) **by** *simp*

**corollary** *intval-sign-extend* 64 64 (*IntVal* 64 (-2)) = *IntVal* 64 (-2) **by** *simp*  
**end**

**experiment begin**

**corollary** *intval-zero-extend* 8 32 (*IntVal* 8 (256 + 128)) = *IntVal* 32 128 **by** *simp*

**corollary** *intval-zero-extend* 8 32 (*IntVal* 8 (-2)) = *IntVal* 32 254 **by** *simp*

**corollary** *intval-zero-extend* 1 32 (*IntVal* 1 (-1)) = *IntVal* 32 1 **by** *simp*

**corollary** *intval-zero-extend* 1 32 (*IntVal* 1 (-2)) = *IntVal* 32 0 **by** *simp*

**corollary** *intval-zero-extend* 8 32 (*IntVal* 64 (-2)) = *UndefVal* **by** *simp*

**corollary** *intval-zero-extend* 8 64 (*IntVal* 64 (-2)) = *UndefVal* **by** *simp*

**corollary** *intval-zero-extend* 8 64 (*IntVal* 8 254) = *IntVal* 64 254 **by** *simp*

**corollary** *intval-zero-extend* 32 64 (*IntVal* 32 ( $2^{32} - 2$ )) = *IntVal* 64 ( $2^{32} - 2$ ) **by** *simp*

**corollary** *intval-zero-extend* 64 64 (*IntVal* 64 (-2)) = *IntVal* 64 (-2) **by** *simp*  
**end**

**experiment begin**

**corollary** *intval-right-shift* (*IntVal* 8 128) (*IntVal* 8 0) = *IntVal* 8 128 **by** *eval*

**corollary** *intval-right-shift* (*IntVal* 8 128) (*IntVal* 8 1) = *IntVal* 8 192 **by** *eval*

**corollary** *intval-right-shift* (*IntVal* 8 128) (*IntVal* 8 2) = *IntVal* 8 224 **by** *eval*

**corollary** *intval-right-shift* (*IntVal* 8 128) (*IntVal* 8 8) = *IntVal* 8 255 **by** *eval*

**corollary** *intval-right-shift* (*IntVal* 8 128) (*IntVal* 8 31) = *IntVal* 8 255 **by** *eval*  
**end**

**lemma** *intval-add-sym*:

**shows** *intval-add* a b = *intval-add* b a

**by** (*induction* a; *induction* b; *auto simp: add.commute*)

**code-deps** *intval-add*

**code-thms** *intval-add*

**lemma** *intval-add* (*IntVal* 32 ( $2^{31} - 1$ )) (*IntVal* 32 ( $2^{31} - 1$ )) = *IntVal* 32 ( $2^{32} - 2$ )

**by** *eval*

**lemma** *intval-add* (*IntVal* 64 ( $2^{31} - 1$ )) (*IntVal* 64 ( $2^{31} - 1$ )) = *IntVal* 64 4294967294

**by** *eval*

**end**

### 3 Extra Theorems for Fixed-Width Integer Words

```
theory ValueThms
  imports Values
begin
```

#### 3.0.1 Support Lemmas for Upper/Lower Bounds

```
lemma size32: size v = 32 for v :: 32 word
  using size-word.rep-eq
  using One-nat-def add.right-neutral add-Suc-right len-of-numeral-defs(2) len-of-numeral-defs(3)
  mult.right-neutral mult-Suc-right numeral-2-eq-2 numeral-Bit0
  by (smt (verit, del-ists) mult.commute)
```

```
lemma size64: size v = 64 for v :: 64 word
  using size-word.rep-eq
  using One-nat-def add.right-neutral add-Suc-right len-of-numeral-defs(2) len-of-numeral-defs(3)
  mult.right-neutral mult-Suc-right numeral-2-eq-2 numeral-Bit0
  by (smt (verit, del-ists) mult.commute)
```

```
lemma lower-bounds-equiv:
  assumes 0 < N
  shows  $\neg((2::int) \wedge (N-1)) = (2::int) \wedge N \text{ div } 2 * - 1$ 
  by (simp add: asms int-power-div-base)
```

```
lemma upper-bounds-equiv:
  assumes 0 < N
  shows  $(2::int) \wedge (N-1) = (2::int) \wedge N \text{ div } 2$ 
  by (simp add: asms int-power-div-base)
```

Some min/max bounds for 64-bit words

```
lemma bit-bounds-min64: ((fst (bit-bounds 64))) ≤ (sint (v::int64))
  unfolding bit-bounds.simps fst-def
  using sint-ge[of v] by simp
```

```
lemma bit-bounds-max64: ((snd (bit-bounds 64))) ≥ (sint (v::int64))
  unfolding bit-bounds.simps fst-def
  using sint-lt[of v] by simp
```

Extend these min/max bounds to extracting smaller signed words using *signed\_take\_bit*.

Note: we could use *signed* to convert between bit-widths, instead of *signed\_take\_bit*. But that would have to be done separately for each bit-width type.

```
value sint(signed-take-bit 7 (128 :: int8))
```

ML-val  $\langle @\{thm\ signed\ take\ bit\ decr\ length\ iff\}\rangle$   
 declare  $[[show\ types = true]]$   
 ML-val  $\langle @\{thm\ signed\ take\ bit\ int\ less\ exp\}\rangle$

**lemma** *signed-take-bit-int-less-exp-word*:  
 fixes  $ival :: 'a :: len\ word$   
 assumes  $n < LENGTH('a)$   
 shows  $sint(signed\ take\ bit\ n\ ival) < (2::int) ^ n$   
 apply *transfer*  
 by (*smt* (*verit*, *best*) *not-take-bit-negative signed-take-bit-eq-take-bit-shift*  
*signed-take-bit-int-less-exp take-bit-int-greater-self-iff*)

**lemma** *signed-take-bit-int-greater-eq-minus-exp-word*:  
 fixes  $ival :: 'a :: len\ word$   
 assumes  $n < LENGTH('a)$   
 shows  $-(2 ^ n) \leq sint(signed\ take\ bit\ n\ ival)$   
 apply *transfer*  
 by (*smt* (*verit*, *best*) *signed-take-bit-int-greater-eq-minus-exp*  
*signed-take-bit-int-greater-eq-self-iff signed-take-bit-int-less-exp*)

**lemma** *signed-take-bit-range*:  
 fixes  $ival :: 'a :: len\ word$   
 assumes  $n < LENGTH('a)$   
 assumes  $val = sint(signed\ take\ bit\ n\ ival)$   
 shows  $-(2 ^ n) \leq val \wedge val < 2 ^ n$   
 using *signed-take-bit-int-greater-eq-minus-exp-word signed-take-bit-int-less-exp-word*  
 using *assms* by *blast*

A *bit\_bounds* version of the above lemma.

**lemma** *signed-take-bit-bounds*:  
 fixes  $ival :: 'a :: len\ word$   
 assumes  $n \leq LENGTH('a)$   
 assumes  $0 < n$   
 assumes  $val = sint(signed\ take\ bit\ (n - 1)\ ival)$   
 shows  $fst\ (bit\ bounds\ n) \leq val \wedge val \leq snd\ (bit\ bounds\ n)$   
 using *assms signed-take-bit-range lower-bounds-equiv upper-bounds-equiv*  
 by (*metis bit-bounds.simps fst-conv less-imp-diff-less nat-less-le sint-ge sint-lt*  
*snd-conv zle-diff1-eq*)

**lemma** *signed-take-bit-bounds64*:  
 fixes  $ival :: int64$   
 assumes  $n \leq 64$   
 assumes  $0 < n$   
 assumes  $val = sint(signed\ take\ bit\ (n - 1)\ ival)$   
 shows  $fst\ (bit\ bounds\ n) \leq val \wedge val \leq snd\ (bit\ bounds\ n)$   
 using *assms signed-take-bit-bounds*  
 by (*metis size64 word-size*)

**lemma** *int-signed-value-bounds*:

**assumes**  $b1 \leq 64$

**assumes**  $0 < b1$

**shows**  $\text{fst } (\text{bit-bounds } b1) \leq \text{int-signed-value } b1 \ v2 \wedge$   
 $\text{int-signed-value } b1 \ v2 \leq \text{snd } (\text{bit-bounds } b1)$

**using** *assms int-signed-value.simps signed-take-bit-bounds64* **by** *blast*

**lemma** *int-signed-value-range*:

**fixes** *ival* :: *int64*

**assumes**  $\text{val} = \text{int-signed-value } n \ \text{ival}$

**shows**  $-(2^{(n-1)}) \leq \text{val} \wedge \text{val} < 2^{(n-1)}$

**using** *signed-take-bit-range assms*

**by** (*smt (verit, ccfv-SIG) One-nat-def diff-less int-signed-value.elims len-gt-0 len-num1 power-less-imp-less-exp power-strict-increasing sint-greater-eq sint-less*)

Some lemmas about unsigned words smaller than 64-bit, for zero-extend operators.

**lemma** *take-bit-smaller-range*:

**fixes** *ival* :: '*a* :: *len word*

**assumes**  $n < \text{LENGTH}('a)$

**assumes**  $\text{val} = \text{sint}(\text{take-bit } n \ \text{ival})$

**shows**  $0 \leq \text{val} \wedge \text{val} < (2::\text{int})^n$

**by** (*simp add: assms signed-take-bit-eq*)

**lemma** *take-bit-same-size-nochange*:

**fixes** *ival* :: '*a* :: *len word*

**assumes**  $n = \text{LENGTH}('a)$

**shows**  $\text{ival} = \text{take-bit } n \ \text{ival}$

**by** (*simp add: assms*)

A simplification lemma for *new\_int*, showing that upper bits can be ignored.

**lemma** *take-bit-redundant*[*simp*]:

**fixes** *ival* :: '*a* :: *len word*

**assumes**  $0 < n$

**assumes**  $n < \text{LENGTH}('a)$

**shows**  $\text{signed-take-bit } (n-1) (\text{take-bit } n \ \text{ival}) = \text{signed-take-bit } (n-1) \ \text{ival}$

**proof** –

**have**  $\neg (n \leq n-1)$  **using** *assms* **by** *arith*

**then have**  $\bigwedge i. \text{signed-take-bit } (n-1) (\text{take-bit } n \ i) = \text{signed-take-bit } (n-1) \ i$

**using** *signed-take-bit-take-bit* **by** (*metis (mono-tags)*)

**then show** *?thesis*

**by** *blast*

**qed**

**lemma** *take-bit-same-size-range*:

**fixes** *ival* :: '*a* :: *len word*

**assumes**  $n = \text{LENGTH}('a)$

**assumes**  $\text{ival2} = \text{take-bit } n \ \text{ival}$

**shows**  $-(2 \wedge n \text{ div } 2) \leq \text{sint } \text{ival2} \wedge \text{sint } \text{ival2} < 2 \wedge n \text{ div } 2$   
**using** *assms lower-bounds-equiv sint-ge sint-lt* **by** *auto*

**lemma** *take-bit-same-bounds*:  
**fixes** *ival* :: 'a :: len word  
**assumes**  $n = \text{LENGTH}('a)$   
**assumes**  $\text{ival2} = \text{take-bit } n \text{ ival}$   
**shows**  $\text{fst } (\text{bit-bounds } n) \leq \text{sint } \text{ival2} \wedge \text{sint } \text{ival2} \leq \text{snd } (\text{bit-bounds } n)$   
**unfolding** *bit-bounds.simps*  
**using** *assms take-bit-same-size-range*  
**by** *force*

Next we show that casting a word to a wider word preserves any upper/lower bounds. (These lemmas may not be needed any more, since we are not using *scast* now?)

**lemma** *scast-max-bound*:  
**assumes**  $\text{sint } (v :: 'a :: \text{len word}) < M$   
**assumes**  $\text{LENGTH}('a) < \text{LENGTH}('b)$   
**shows**  $\text{sint } ((\text{scast } v) :: 'b :: \text{len word}) < M$   
**unfolding** *Word.scast-eq Word.sint-sbintrunc'*  
**using** *Bit-Operations.signed-take-bit-int-eq-self-iff*  
**by** (*smt (verit, best) One-nat-def assms(1) assms(2) decr-length-less-iff linorder-not-le power-strict-increasing-iff signed-take-bit-int-less-self-iff sint-greater-eq*)

**lemma** *scast-min-bound*:  
**assumes**  $M \leq \text{sint } (v :: 'a :: \text{len word})$   
**assumes**  $\text{LENGTH}('a) < \text{LENGTH}('b)$   
**shows**  $M \leq \text{sint } ((\text{scast } v) :: 'b :: \text{len word})$   
**unfolding** *Word.scast-eq Word.sint-sbintrunc'*  
**using** *Bit-Operations.signed-take-bit-int-eq-self-iff*  
**by** (*smt (verit) One-nat-def Suc-pred assms(1) assms(2) len-gt-0 less-Suc-eq order-less-le order-less-le-trans power-le-imp-le-exp signed-take-bit-int-greater-eq-self-iff sint-lt*)

**lemma** *scast-bigger-max-bound*:  
**assumes**  $(\text{result} :: 'b :: \text{len word}) = \text{scast } (v :: 'a :: \text{len word})$   
**shows**  $\text{sint } \text{result} < 2 \wedge \text{LENGTH}('a) \text{ div } 2$   
**using** *sint-lt upper-bounds-equiv scast-max-bound*  
**by** (*smt (verit, best) assms(1) len-gt-0 signed-scast-eq signed-take-bit-int-greater-self-iff sint-ge sint-less upper-bounds-equiv*)

**lemma** *scast-bigger-min-bound*:  
**assumes**  $(\text{result} :: 'b :: \text{len word}) = \text{scast } (v :: 'a :: \text{len word})$   
**shows**  $-(2 \wedge \text{LENGTH}('a) \text{ div } 2) \leq \text{sint } \text{result}$   
**using** *sint-ge lower-bounds-equiv scast-min-bound*  
**by** (*smt (verit) assms len-gt-0 nat-less-le not-less scast-max-bound*)

**lemma** *scast-bigger-bit-bounds*:

```

assumes (result :: 'b :: len word) = scast (v :: 'a :: len word)
shows fst (bit-bounds (LENGTH('a))) ≤ sint result ∧ sint result ≤ snd (bit-bounds
(LENGTH('a)))
using assms scast-bigger-min-bound scast-bigger-max-bound
by auto

```

Results about *new\_int*.

```

lemma new-int-take-bits:
  assumes IntVal b val = new-int b ival
  shows take-bit b val = val
  using assms by force

```

### 3.0.2 Support lemmas for take bit and signed take bit.

Lemmas for removing redundant take\_bit wrappers.

```

lemma take-bit-dist-addL[simp]:
  fixes x :: 'a :: len word
  shows take-bit b (take-bit b x + y) = take-bit b (x + y)
proof (induction b)
  case 0
  then show ?case
  by simp
next
  case (Suc b)
  then show ?case
  by (simp add: add.commute mask-eqs(2) take-bit-eq-mask)
qed

```

```

lemma take-bit-dist-addR[simp]:
  fixes x :: 'a :: len word
  shows take-bit b (x + take-bit b y) = take-bit b (x + y)
  using take-bit-dist-addL by (metis add.commute)

```

```

lemma take-bit-dist-subL[simp]:
  fixes x :: 'a :: len word
  shows take-bit b (take-bit b x - y) = take-bit b (x - y)
  by (metis take-bit-dist-addR uminus-add-conv-diff)

```

```

lemma take-bit-dist-subR[simp]:
  fixes x :: 'a :: len word
  shows take-bit b (x - take-bit b y) = take-bit b (x - y)
  using take-bit-dist-subL
  by (metis (no-types, opaque-lifting) diff-add-cancel diff-right-commute diff-self)

```

```

lemma take-bit-dist-neg[simp]:
  fixes ix :: 'a :: len word
  shows take-bit b (- take-bit b (ix)) = take-bit b (- ix)

```



```

by (metis diff-0 take-bit-dist-subR)

lemma signed-take-take-bit[simp]:
  fixes x :: 'a :: len word
  assumes 0 < b
  shows signed-take-bit (b - 1) (take-bit b x) = signed-take-bit (b - 1) x
by (smt (verit, best) Suc-diff-1 assms lessI linorder-not-less signed-take-bit-take-bit)

lemma mod-larger-ignore:
  fixes a :: int
  fixes m n :: nat
  assumes n < m
  shows (a mod 2 ^ m) mod 2 ^ n = a mod 2 ^ n
by (smt (verit, del-insts) assms exp-mod-exp linorder-not-le mod-0-imp-dvd mod-mod-cancel
mod-self order-less-imp-le)

lemma mod-dist-over-add:
  fixes a b c :: int64
  fixes n :: nat
  assumes 1: 0 < n
  assumes 2: n < 64
  shows (a mod 2 ^ n + b) mod 2 ^ n = (a + b) mod 2 ^ n
proof -
  have 3: (0 :: int64) < 2 ^ n
  using assms by (simp add: size64 word-2p-lem)
  then show ?thesis
  unfolding word-mod-2p-is-mask[OF 3]
  apply transfer
  by (metis (no-types, opaque-lifting) and.right-idem take-bit-add take-bit-eq-mask)
qed

end

```

## 4 Nodes

### 4.1 Types of Nodes

```

theory IRNodes
  imports
    Values
begin

```

The GraalVM IR is represented using a graph data structure. Here we define the nodes that are contained within the graph. Each node represents a Node subclass in the GraalVM compiler, the node classes have annotated fields to indicate input and successor edges.

We represent these classes with each `IRNode` constructor explicitly labelling a reference to the node IDs that it stores as inputs and successors.

The `inputs_of` and `successors_of` functions partition those labelled references into input edges and successor edges of a node.

To identify each Node, we use a simple natural number index. Zero is always the start node in a graph. For human readability, within nodes we write `INPUT` (or special case thereof) instead of `ID` for input edges, and `SUCC` instead of `ID` for control-flow successor edges. Optional edges are handled as "INPUT option" etc.

```
type-synonym ID = nat
type-synonym INPUT = ID
type-synonym INPUT-ASSOC = ID
type-synonym INPUT-STATE = ID
type-synonym INPUT-GUARD = ID
type-synonym INPUT-COND = ID
type-synonym INPUT-EXT = ID
type-synonym SUCC = ID
```

```
datatype (discs-sels) IRNode =
  AbsNode (ir-value: INPUT)
| AddNode (ir-x: INPUT) (ir-y: INPUT)
| AndNode (ir-x: INPUT) (ir-y: INPUT)
| BeginNode (ir-next: SUCC)
| BytecodeExceptionNode (ir-arguments: INPUT list) (ir-stateAfter-opt: INPUT-STATE
option) (ir-next: SUCC)
| ConditionalNode (ir-condition: INPUT-COND) (ir-trueValue: INPUT) (ir-falseValue:
INPUT)
| ConstantNode (ir-const: Value)
| DynamicNewArrayNode (ir-elementType: INPUT) (ir-length: INPUT) (ir-voidClass-opt:
INPUT option) (ir-stateBefore-opt: INPUT-STATE option) (ir-next: SUCC)
| EndNode
| ExceptionObjectNode (ir-stateAfter-opt: INPUT-STATE option) (ir-next: SUCC)

| FrameState (ir-monitorIds: INPUT-ASSOC list) (ir-outerFrameState-opt: IN-
PUT-STATE option) (ir-values-opt: INPUT list option) (ir-virtualObjectMappings-opt:
INPUT-STATE list option)
| IfNode (ir-condition: INPUT-COND) (ir-trueSuccessor: SUCC) (ir-falseSuccessor:
SUCC)
| IntegerBelowNode (ir-x: INPUT) (ir-y: INPUT)
| IntegerEqualsNode (ir-x: INPUT) (ir-y: INPUT)
| IntegerLessThanNode (ir-x: INPUT) (ir-y: INPUT)
| InvokeNode (ir-nid: ID) (ir-callTarget: INPUT-EXT) (ir-classInit-opt: IN-
PUT option) (ir-stateDuring-opt: INPUT-STATE option) (ir-stateAfter-opt: IN-
PUT-STATE option) (ir-next: SUCC)
| InvokeWithExceptionNode (ir-nid: ID) (ir-callTarget: INPUT-EXT) (ir-classInit-opt:
INPUT option) (ir-stateDuring-opt: INPUT-STATE option) (ir-stateAfter-opt: IN-
```

*PUT-STATE option*) (*ir-next: SUCC*) (*ir-exceptionEdge: SUCC*)  
 | *IsNullNode* (*ir-value: INPUT*)  
 | *KillingBeginNode* (*ir-next: SUCC*)  
 | *LeftShiftNode* (*ir-x: INPUT*) (*ir-y: INPUT*)  
 | *LoadFieldNode* (*ir-nid: ID*) (*ir-field: string*) (*ir-object-opt: INPUT option*)  
 (*ir-next: SUCC*)  
 | *LogicNegationNode* (*ir-value: INPUT-COND*)  
 | *LoopBeginNode* (*ir-ends: INPUT-ASSOC list*) (*ir-overflowGuard-opt: INPUT-GUARD option*) (*ir-stateAfter-opt: INPUT-STATE option*) (*ir-next: SUCC*)  
 | *LoopEndNode* (*ir-loopBegin: INPUT-ASSOC*)  
 | *LoopExitNode* (*ir-loopBegin: INPUT-ASSOC*) (*ir-stateAfter-opt: INPUT-STATE option*) (*ir-next: SUCC*)  
 | *MergeNode* (*ir-ends: INPUT-ASSOC list*) (*ir-stateAfter-opt: INPUT-STATE option*) (*ir-next: SUCC*)  
 | *MethodCallTargetNode* (*ir-targetMethod: string*) (*ir-arguments: INPUT list*)  
 | *MulNode* (*ir-x: INPUT*) (*ir-y: INPUT*)  
 | *NarrowNode* (*ir-inputBits: nat*) (*ir-resultBits: nat*) (*ir-value: INPUT*)  
 | *NegateNode* (*ir-value: INPUT*)  
 | *NewArrayNode* (*ir-length: INPUT*) (*ir-stateBefore-opt: INPUT-STATE option*)  
 (*ir-next: SUCC*)  
 | *NewInstanceNode* (*ir-nid: ID*) (*ir-instanceClass: string*) (*ir-stateBefore-opt: INPUT-STATE option*) (*ir-next: SUCC*)  
 | *NotNode* (*ir-value: INPUT*)  
 | *OrNode* (*ir-x: INPUT*) (*ir-y: INPUT*)  
 | *ParameterNode* (*ir-index: nat*)  
 | *PiNode* (*ir-object: INPUT*) (*ir-guard-opt: INPUT-GUARD option*)  
 | *ReturnNode* (*ir-result-opt: INPUT option*) (*ir-memoryMap-opt: INPUT-EXT option*)  
 | *RightShiftNode* (*ir-x: INPUT*) (*ir-y: INPUT*)  
 | *ShortCircuitOrNode* (*ir-x: INPUT-COND*) (*ir-y: INPUT-COND*)  
 | *SignExtendNode* (*ir-inputBits: nat*) (*ir-resultBits: nat*) (*ir-value: INPUT*)  
 | *SignedDivNode* (*ir-nid: ID*) (*ir-x: INPUT*) (*ir-y: INPUT*) (*ir-zeroCheck-opt: INPUT-GUARD option*) (*ir-stateBefore-opt: INPUT-STATE option*) (*ir-next: SUCC*)  
 | *SignedRemNode* (*ir-nid: ID*) (*ir-x: INPUT*) (*ir-y: INPUT*) (*ir-zeroCheck-opt: INPUT-GUARD option*) (*ir-stateBefore-opt: INPUT-STATE option*) (*ir-next: SUCC*)  
 | *StartNode* (*ir-stateAfter-opt: INPUT-STATE option*) (*ir-next: SUCC*)  
 | *StoreFieldNode* (*ir-nid: ID*) (*ir-field: string*) (*ir-value: INPUT*) (*ir-stateAfter-opt: INPUT-STATE option*) (*ir-object-opt: INPUT option*) (*ir-next: SUCC*)  
 | *SubNode* (*ir-x: INPUT*) (*ir-y: INPUT*)  
 | *UnsignedRightShiftNode* (*ir-x: INPUT*) (*ir-y: INPUT*)  
 | *UnwindNode* (*ir-exception: INPUT*)  
 | *ValuePhiNode* (*ir-nid: ID*) (*ir-values: INPUT list*) (*ir-merge: INPUT-ASSOC*)  
 | *ValueProxyNode* (*ir-value: INPUT*) (*ir-loopExit: INPUT-ASSOC*)  
 | *XorNode* (*ir-x: INPUT*) (*ir-y: INPUT*)  
 | *ZeroExtendNode* (*ir-inputBits: nat*) (*ir-resultBits: nat*) (*ir-value: INPUT*)  
 | *NoNode*

| *RefNode* (*ir-ref:ID*)

```
fun opt-to-list :: 'a option ⇒ 'a list where
  opt-to-list None = [] |
  opt-to-list (Some v) = [v]
```

```
fun opt-list-to-list :: 'a list option ⇒ 'a list where
  opt-list-to-list None = [] |
  opt-list-to-list (Some x) = x
```

The following functions, *inputs\_of* and *successors\_of*, are automatically generated from the GraalVM compiler. Their purpose is to partition the node edges into input or successor edges.

```
fun inputs-of :: IRNode ⇒ ID list where
  inputs-of-AbsNode:
  inputs-of (AbsNode value) = [value] |
  inputs-of-AddNode:
  inputs-of (AddNode x y) = [x, y] |
  inputs-of-AndNode:
  inputs-of (AndNode x y) = [x, y] |
  inputs-of-BEGINNode:
  inputs-of (BeginNode next) = [] |
  inputs-of-BytecodeExceptionNode:
  inputs-of (BytecodeExceptionNode arguments stateAfter next) = arguments @
  (opt-to-list stateAfter) |
  inputs-of-ConditionalNode:
  inputs-of (ConditionalNode condition trueValue falseValue) = [condition, true-
  Value, falseValue] |
  inputs-of-ConstantNode:
  inputs-of (ConstantNode const) = [] |
  inputs-of-DynamicNewArrayNode:
  inputs-of (DynamicNewArrayNode elementType length0 voidClass stateBefore
  next) = [elementType, length0] @ (opt-to-list voidClass) @ (opt-to-list stateBefore)
  |
  inputs-of-EndNode:
  inputs-of (EndNode) = [] |
  inputs-of-ExceptionObjectNode:
  inputs-of (ExceptionObjectNode stateAfter next) = (opt-to-list stateAfter) |
  inputs-of-FrameState:
  inputs-of (FrameState monitorIds outerFrameState values virtualObjectMappings)
  = monitorIds @ (opt-to-list outerFrameState) @ (opt-list-to-list values) @ (opt-list-to-list
  virtualObjectMappings) |
  inputs-of-IfNode:
  inputs-of (IfNode condition trueSuccessor falseSuccessor) = [condition] |
```

*inputs-of-IntegerBelowNode:*  
*inputs-of (IntegerBelowNode x y) = [x, y] |*  
*inputs-of-IntegerEqualsNode:*  
*inputs-of (IntegerEqualsNode x y) = [x, y] |*  
*inputs-of-IntegerLessThanNode:*  
*inputs-of (IntegerLessThanNode x y) = [x, y] |*  
*inputs-of-InvokeNode:*  
*inputs-of (InvokeNode nid0 callTarget classInit stateDuring stateAfter next) =*  
*callTarget # (opt-to-list classInit) @ (opt-to-list stateDuring) @ (opt-to-list stateAfter)*  
*|*  
*inputs-of-InvokeWithExceptionNode:*  
*inputs-of (InvokeWithExceptionNode nid0 callTarget classInit stateDuring stateAfter*  
*next exceptionEdge) = callTarget # (opt-to-list classInit) @ (opt-to-list stateDur-*  
*ing) @ (opt-to-list stateAfter) |*  
*inputs-of-IsNullNode:*  
*inputs-of (IsNullNode value) = [value] |*  
*inputs-of-KillingBeginNode:*  
*inputs-of (KillingBeginNode next) = [] |*  
*inputs-of-LeftShiftNode:*  
*inputs-of (LeftShiftNode x y) = [x, y] |*  
*inputs-of-LoadFieldNode:*  
*inputs-of (LoadFieldNode nid0 field object next) = (opt-to-list object) |*  
*inputs-of-LogicNegationNode:*  
*inputs-of (LogicNegationNode value) = [value] |*  
*inputs-of-LoopBeginNode:*  
*inputs-of (LoopBeginNode ends overflowGuard stateAfter next) = ends @ (opt-to-list*  
*overflowGuard) @ (opt-to-list stateAfter) |*  
*inputs-of-LoopEndNode:*  
*inputs-of (LoopEndNode loopBegin) = [loopBegin] |*  
*inputs-of-LoopExitNode:*  
*inputs-of (LoopExitNode loopBegin stateAfter next) = loopBegin # (opt-to-list*  
*stateAfter) |*  
*inputs-of-MergeNode:*  
*inputs-of (MergeNode ends stateAfter next) = ends @ (opt-to-list stateAfter) |*  
*inputs-of-MethodCallTargetNode:*  
*inputs-of (MethodCallTargetNode targetMethod arguments) = arguments |*  
*inputs-of-MulNode:*  
*inputs-of (MulNode x y) = [x, y] |*  
*inputs-of-NarrowNode:*  
*inputs-of (NarrowNode inputBits resultBits value) = [value] |*  
*inputs-of-NegateNode:*  
*inputs-of (NegateNode value) = [value] |*  
*inputs-of-NewArrayNode:*  
*inputs-of (NewArrayNode length0 stateBefore next) = length0 # (opt-to-list state-*  
*Before) |*  
*inputs-of-NewInstanceNode:*  
*inputs-of (NewInstanceNode nid0 instanceClass stateBefore next) = (opt-to-list*  
*stateBefore) |*  
*inputs-of-NotNode:*

*inputs-of* (*NotNode value*) = [*value*] |  
*inputs-of-OrNode*:  
*inputs-of* (*OrNode x y*) = [*x*, *y*] |  
*inputs-of-ParameterNode*:  
*inputs-of* (*ParameterNode index*) = [] |  
*inputs-of-PiNode*:  
*inputs-of* (*PiNode object guard*) = *object* # (*opt-to-list guard*) |  
*inputs-of-ReturnNode*:  
*inputs-of* (*ReturnNode result memoryMap*) = (*opt-to-list result*) @ (*opt-to-list memoryMap*) |  
*inputs-of-RightShiftNode*:  
*inputs-of* (*RightShiftNode x y*) = [*x*, *y*] |  
*inputs-of-ShortCircuitOrNode*:  
*inputs-of* (*ShortCircuitOrNode x y*) = [*x*, *y*] |  
*inputs-of-SignExtendNode*:  
*inputs-of* (*SignExtendNode inputBits resultBits value*) = [*value*] |  
*inputs-of-SignedDivNode*:  
*inputs-of* (*SignedDivNode nid0 x y zeroCheck stateBefore next*) = [*x*, *y*] @ (*opt-to-list zeroCheck*) @ (*opt-to-list stateBefore*) |  
*inputs-of-SignedRemNode*:  
*inputs-of* (*SignedRemNode nid0 x y zeroCheck stateBefore next*) = [*x*, *y*] @ (*opt-to-list zeroCheck*) @ (*opt-to-list stateBefore*) |  
*inputs-of-StartNode*:  
*inputs-of* (*StartNode stateAfter next*) = (*opt-to-list stateAfter*) |  
*inputs-of-StoreFieldNode*:  
*inputs-of* (*StoreFieldNode nid0 field value stateAfter object next*) = *value* # (*opt-to-list stateAfter*) @ (*opt-to-list object*) |  
*inputs-of-SubNode*:  
*inputs-of* (*SubNode x y*) = [*x*, *y*] |  
*inputs-of-UnsignedRightShiftNode*:  
*inputs-of* (*UnsignedRightShiftNode x y*) = [*x*, *y*] |  
*inputs-of-UnwindNode*:  
*inputs-of* (*UnwindNode exception*) = [*exception*] |  
*inputs-of-ValuePhiNode*:  
*inputs-of* (*ValuePhiNode nid0 values merge*) = *merge* # *values* |  
*inputs-of-ValueProxyNode*:  
*inputs-of* (*ValueProxyNode value loopExit*) = [*value*, *loopExit*] |  
*inputs-of-XorNode*:  
*inputs-of* (*XorNode x y*) = [*x*, *y*] |  
*inputs-of-ZeroExtendNode*:  
*inputs-of* (*ZeroExtendNode inputBits resultBits value*) = [*value*] |  
*inputs-of-NoNode*: *inputs-of* (*NoNode*) = [] |

*inputs-of-RefNode*: *inputs-of* (*RefNode ref*) = [*ref*]

**fun** *successors-of* :: *IRNode* ⇒ *ID list* **where**  
*successors-of-AbsNode*:

*successors-of* (*AbsNode value*) = [] |  
*successors-of-AddNode*:  
*successors-of* (*AddNode x y*) = [] |  
*successors-of-AndNode*:  
*successors-of* (*AndNode x y*) = [] |  
*successors-of-BeginNode*:  
*successors-of* (*BeginNode next*) = [next] |  
*successors-of-BytecodeExceptionNode*:  
*successors-of* (*BytecodeExceptionNode arguments stateAfter next*) = [next] |  
*successors-of-ConditionalNode*:  
*successors-of* (*ConditionalNode condition trueValue falseValue*) = [] |  
*successors-of-ConstantNode*:  
*successors-of* (*ConstantNode const*) = [] |  
*successors-of-DynamicNewArrayNode*:  
*successors-of* (*DynamicNewArrayNode elementType length0 voidClass stateBefore next*) = [next] |  
*successors-of-EndNode*:  
*successors-of* (*EndNode*) = [] |  
*successors-of-ExceptionObjectNode*:  
*successors-of* (*ExceptionObjectNode stateAfter next*) = [next] |  
*successors-of-FrameState*:  
*successors-of* (*FrameState monitorIds outerFrameState values virtualObjectMappings*) = [] |  
*successors-of-IfNode*:  
*successors-of* (*IfNode condition trueSuccessor falseSuccessor*) = [trueSuccessor, falseSuccessor] |  
*successors-of-IntegerBelowNode*:  
*successors-of* (*IntegerBelowNode x y*) = [] |  
*successors-of-IntegerEqualsNode*:  
*successors-of* (*IntegerEqualsNode x y*) = [] |  
*successors-of-IntegerLessThanNode*:  
*successors-of* (*IntegerLessThanNode x y*) = [] |  
*successors-of-InvokeNode*:  
*successors-of* (*InvokeNode nid0 callTarget classInit stateDuring stateAfter next*) = [next] |  
*successors-of-InvokeWithExceptionNode*:  
*successors-of* (*InvokeWithExceptionNode nid0 callTarget classInit stateDuring stateAfter next exceptionEdge*) = [next, exceptionEdge] |  
*successors-of-IsNullNode*:  
*successors-of* (*IsNullNode value*) = [] |  
*successors-of-KillingBeginNode*:  
*successors-of* (*KillingBeginNode next*) = [next] |  
*successors-of-LeftShiftNode*:  
*successors-of* (*LeftShiftNode x y*) = [] |  
*successors-of-LoadFieldNode*:  
*successors-of* (*LoadFieldNode nid0 field object next*) = [next] |  
*successors-of-LogicNegationNode*:  
*successors-of* (*LogicNegationNode value*) = [] |  
*successors-of-LoopBeginNode*:

*successors-of* (*LoopBeginNode ends overflowGuard stateAfter next*) = [*next*] |  
*successors-of-LoopEndNode*:  
*successors-of* (*LoopEndNode loopBegin*) = [] |  
*successors-of-LoopExitNode*:  
*successors-of* (*LoopExitNode loopBegin stateAfter next*) = [*next*] |  
*successors-of-MergeNode*:  
*successors-of* (*MergeNode ends stateAfter next*) = [*next*] |  
*successors-of-MethodCallTargetNode*:  
*successors-of* (*MethodCallTargetNode targetMethod arguments*) = [] |  
*successors-of-MulNode*:  
*successors-of* (*MulNode x y*) = [] |  
*successors-of-NarrowNode*:  
*successors-of* (*NarrowNode inputBits resultBits value*) = [] |  
*successors-of-NegateNode*:  
*successors-of* (*NegateNode value*) = [] |  
*successors-of-NewArrayNode*:  
*successors-of* (*NewArrayNode length0 stateBefore next*) = [*next*] |  
*successors-of-NewInstanceNode*:  
*successors-of* (*NewInstanceNode nid0 instanceClass stateBefore next*) = [*next*] |  
*successors-of-NotNode*:  
*successors-of* (*NotNode value*) = [] |  
*successors-of-OrNode*:  
*successors-of* (*OrNode x y*) = [] |  
*successors-of-ParameterNode*:  
*successors-of* (*ParameterNode index*) = [] |  
*successors-of-PiNode*:  
*successors-of* (*PiNode object guard*) = [] |  
*successors-of-ReturnNode*:  
*successors-of* (*ReturnNode result memoryMap*) = [] |  
*successors-of-RightShiftNode*:  
*successors-of* (*RightShiftNode x y*) = [] |  
*successors-of-ShortCircuitOrNode*:  
*successors-of* (*ShortCircuitOrNode x y*) = [] |  
*successors-of-SignExtendNode*:  
*successors-of* (*SignExtendNode inputBits resultBits value*) = [] |  
*successors-of-SignedDivNode*:  
*successors-of* (*SignedDivNode nid0 x y zeroCheck stateBefore next*) = [*next*] |  
*successors-of-SignedRemNode*:  
*successors-of* (*SignedRemNode nid0 x y zeroCheck stateBefore next*) = [*next*] |  
*successors-of-StartNode*:  
*successors-of* (*StartNode stateAfter next*) = [*next*] |  
*successors-of-StoreFieldNode*:  
*successors-of* (*StoreFieldNode nid0 field value stateAfter object next*) = [*next*] |  
*successors-of-SubNode*:  
*successors-of* (*SubNode x y*) = [] |  
*successors-of-UnsignedRightShiftNode*:  
*successors-of* (*UnsignedRightShiftNode x y*) = [] |  
*successors-of-UnwindNode*:  
*successors-of* (*UnwindNode exception*) = [] |



```

successors-of-ValuePhiNode:
successors-of (ValuePhiNode nid0 values merge) = [] |
successors-of-ValueProxyNode:
successors-of (ValueProxyNode value loopExit) = [] |
successors-of-XorNode:
successors-of (XorNode x y) = [] |
successors-of-ZeroExtendNode:
successors-of (ZeroExtendNode inputBits resultBits value) = [] |
successors-of-NoNode: successors-of (NoNode) = [] |

```

```

successors-of-RefNode: successors-of (RefNode ref) = [ref]

```

```

lemma inputs-of (FrameState x (Some y) (Some z) None) = x @ [y] @ z
unfolding inputs-of-FrameState by simp
lemma successors-of (FrameState x (Some y) (Some z) None) = []
unfolding inputs-of-FrameState by simp

```

```

lemma inputs-of (IfNode c t f) = [c]
unfolding inputs-of-IfNode by simp
lemma successors-of (IfNode c t f) = [t, f]
unfolding successors-of-IfNode by simp

```

```

lemma inputs-of (EndNode) = [] ∧ successors-of (EndNode) = []
unfolding inputs-of-EndNode successors-of-EndNode by simp

```

**end**

## 4.2 Hierarchy of Nodes

```

theory IRNodeHierarchy
imports IRNodes
begin

```

It is helpful to introduce a node hierarchy into our formalization. Often the GraalVM compiler relies on explicit type checks to determine which operations to perform on a given node, we try to mimic the same functionality by using a suite of predicate functions over the `IRNode` class to determine inheritance.

As one would expect, the function `is<ClassName>Type` will be true if the node parameter is a subclass of the `ClassName` within the GraalVM compiler.

These functions have been automatically generated from the compiler.

```

fun is-EndNode :: IRNode ⇒ bool where
  is-EndNode EndNode = True |
  is-EndNode - = False

```

```

fun is-VirtualState :: IRNode  $\Rightarrow$  bool where
  is-VirtualState n = ((is-FrameState n))

fun is-BinaryArithmeticNode :: IRNode  $\Rightarrow$  bool where
  is-BinaryArithmeticNode n = ((is-AddNode n)  $\vee$  (is-AndNode n)  $\vee$  (is-MulNode
n)  $\vee$  (is-OrNode n)  $\vee$  (is-SubNode n)  $\vee$  (is-XorNode n))

fun is-ShiftNode :: IRNode  $\Rightarrow$  bool where
  is-ShiftNode n = ((is-LeftShiftNode n)  $\vee$  (is-RightShiftNode n)  $\vee$  (is-UnsignedRightShiftNode
n))

fun is-BinaryNode :: IRNode  $\Rightarrow$  bool where
  is-BinaryNode n = ((is-BinaryArithmeticNode n)  $\vee$  (is-ShiftNode n))

fun is-AbstractLocalNode :: IRNode  $\Rightarrow$  bool where
  is-AbstractLocalNode n = ((is-ParameterNode n))

fun is-IntegerConvertNode :: IRNode  $\Rightarrow$  bool where
  is-IntegerConvertNode n = ((is-NarrowNode n)  $\vee$  (is-SignExtendNode n)  $\vee$  (is-ZeroExtendNode
n))

fun is-UnaryArithmeticNode :: IRNode  $\Rightarrow$  bool where
  is-UnaryArithmeticNode n = ((is-AbsNode n)  $\vee$  (is-NegateNode n)  $\vee$  (is-NotNode
n))

fun is-UnaryNode :: IRNode  $\Rightarrow$  bool where
  is-UnaryNode n = ((is-IntegerConvertNode n)  $\vee$  (is-UnaryArithmeticNode n))

fun is-PhiNode :: IRNode  $\Rightarrow$  bool where
  is-PhiNode n = ((is-ValuePhiNode n))

fun is-FloatingGuardedNode :: IRNode  $\Rightarrow$  bool where
  is-FloatingGuardedNode n = ((is-PiNode n))

fun is-UnaryOpLogicNode :: IRNode  $\Rightarrow$  bool where
  is-UnaryOpLogicNode n = ((is-IsNullNode n))

fun is-IntegerLowerThanNode :: IRNode  $\Rightarrow$  bool where
  is-IntegerLowerThanNode n = ((is-IntegerBelowNode n)  $\vee$  (is-IntegerLessThanNode
n))

fun is-CompareNode :: IRNode  $\Rightarrow$  bool where
  is-CompareNode n = ((is-IntegerEqualsNode n)  $\vee$  (is-IntegerLowerThanNode n))

fun is-BinaryOpLogicNode :: IRNode  $\Rightarrow$  bool where
  is-BinaryOpLogicNode n = ((is-CompareNode n))

```

```

fun is-LogicNode :: IRNode  $\Rightarrow$  bool where
  is-LogicNode n = ((is-BinaryOpLogicNode n)  $\vee$  (is-LogicNegationNode n)  $\vee$ 
    (is-ShortCircuitOrNode n)  $\vee$  (is-UnaryOpLogicNode n))

fun is-ProxyNode :: IRNode  $\Rightarrow$  bool where
  is-ProxyNode n = ((is-ValueProxyNode n))

fun is-FloatingNode :: IRNode  $\Rightarrow$  bool where
  is-FloatingNode n = ((is-AbstractLocalNode n)  $\vee$  (is-BinaryNode n)  $\vee$  (is-ConditionalNode
    n)  $\vee$  (is-ConstantNode n)  $\vee$  (is-FloatingGuardedNode n)  $\vee$  (is-LogicNode n)  $\vee$ 
    (is-PhiNode n)  $\vee$  (is-ProxyNode n)  $\vee$  (is-UnaryNode n))

fun is-AccessFieldNode :: IRNode  $\Rightarrow$  bool where
  is-AccessFieldNode n = ((is-LoadFieldNode n)  $\vee$  (is-StoreFieldNode n))

fun is-AbstractNewArrayNode :: IRNode  $\Rightarrow$  bool where
  is-AbstractNewArrayNode n = ((is-DynamicNewArrayNode n)  $\vee$  (is-NewArrayNode
    n))

fun is-AbstractNewObjectNode :: IRNode  $\Rightarrow$  bool where
  is-AbstractNewObjectNode n = ((is-AbstractNewArrayNode n)  $\vee$  (is-NewInstanceNode
    n))

fun is-IntegerDivRemNode :: IRNode  $\Rightarrow$  bool where
  is-IntegerDivRemNode n = ((is-SignedDivNode n)  $\vee$  (is-SignedRemNode n))

fun is-FixedBinaryNode :: IRNode  $\Rightarrow$  bool where
  is-FixedBinaryNode n = ((is-IntegerDivRemNode n))

fun is-DeoptimizingFixedWithNextNode :: IRNode  $\Rightarrow$  bool where
  is-DeoptimizingFixedWithNextNode n = ((is-AbstractNewObjectNode n)  $\vee$  (is-FixedBinaryNode
    n))

fun is-AbstractMemoryCheckpoint :: IRNode  $\Rightarrow$  bool where
  is-AbstractMemoryCheckpoint n = ((is-BytecodeExceptionNode n)  $\vee$  (is-InvokeNode
    n))

fun is-AbstractStateSplit :: IRNode  $\Rightarrow$  bool where
  is-AbstractStateSplit n = ((is-AbstractMemoryCheckpoint n))

fun is-AbstractMergeNode :: IRNode  $\Rightarrow$  bool where
  is-AbstractMergeNode n = ((is-LoopBeginNode n)  $\vee$  (is-MergeNode n))

fun is-BeginStateSplitNode :: IRNode  $\Rightarrow$  bool where
  is-BeginStateSplitNode n = ((is-AbstractMergeNode n)  $\vee$  (is-ExceptionObjectNode
    n)  $\vee$  (is-LoopExitNode n)  $\vee$  (is-StartNode n))

fun is-AbstractBeginNode :: IRNode  $\Rightarrow$  bool where
  is-AbstractBeginNode n = ((is-BeginNode n)  $\vee$  (is-BeginStateSplitNode n)  $\vee$ 

```

*(is-KillingBeginNode n)*)

**fun** *is-FixedWithNextNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-FixedWithNextNode* *n* = ((*is-AbstractBeginNode* *n*)  $\vee$  (*is-AbstractStateSplit* *n*)  
   $\vee$  (*is-AccessFieldNode* *n*)  $\vee$  (*is-DeoptimizingFixedWithNextNode* *n*))

**fun** *is-WithExceptionNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-WithExceptionNode* *n* = ((*is-InvokeWithExceptionNode* *n*))

**fun** *is-ControlSplitNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-ControlSplitNode* *n* = ((*is-IfNode* *n*)  $\vee$  (*is-WithExceptionNode* *n*))

**fun** *is-ControlSinkNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-ControlSinkNode* *n* = ((*is-ReturnNode* *n*)  $\vee$  (*is-UnwindNode* *n*))

**fun** *is-AbstractEndNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-AbstractEndNode* *n* = ((*is-EndNode* *n*)  $\vee$  (*is-LoopEndNode* *n*))

**fun** *is-FixedNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-FixedNode* *n* = ((*is-AbstractEndNode* *n*)  $\vee$  (*is-ControlSinkNode* *n*)  $\vee$  (*is-ControlSplitNode*  
  *n*)  $\vee$  (*is-FixedWithNextNode* *n*))

**fun** *is-CallTargetNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-CallTargetNode* *n* = ((*is-MethodCallTargetNode* *n*))

**fun** *is-ValueNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-ValueNode* *n* = ((*is-CallTargetNode* *n*)  $\vee$  (*is-FixedNode* *n*)  $\vee$  (*is-FloatingNode*  
  *n*))

**fun** *is-Node* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-Node* *n* = ((*is-ValueNode* *n*)  $\vee$  (*is-VirtualState* *n*))

**fun** *is-MemoryKill* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-MemoryKill* *n* = ((*is-AbstractMemoryCheckpoint* *n*))

**fun** *is-NarrowableArithmeticNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-NarrowableArithmeticNode* *n* = ((*is-AbsNode* *n*)  $\vee$  (*is-AddNode* *n*)  $\vee$  (*is-AndNode*  
  *n*)  $\vee$  (*is-MulNode* *n*)  $\vee$  (*is-NegateNode* *n*)  $\vee$  (*is-NotNode* *n*)  $\vee$  (*is-OrNode* *n*)  $\vee$   
  (*is-ShiftNode* *n*)  $\vee$  (*is-SubNode* *n*)  $\vee$  (*is-XorNode* *n*))

**fun** *is-AnchoringNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-AnchoringNode* *n* = ((*is-AbstractBeginNode* *n*))

**fun** *is-DeoptBefore* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-DeoptBefore* *n* = ((*is-DeoptimizingFixedWithNextNode* *n*))

**fun** *is-IndirectCanonicalization* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-IndirectCanonicalization* *n* = ((*is-LogicNode* *n*))

```

fun is-IterableNodeType :: IRNode  $\Rightarrow$  bool where
  is-IterableNodeType n = ((is-AbstractBeginNode n)  $\vee$  (is-AbstractMergeNode n)  $\vee$ 
    (is-FrameState n)  $\vee$  (is-IfNode n)  $\vee$  (is-IntegerDivRemNode n)  $\vee$  (is-InvokeWithExceptionNode
    n)  $\vee$  (is-LoopBeginNode n)  $\vee$  (is-LoopExitNode n)  $\vee$  (is-MethodCallTargetNode n)
     $\vee$  (is-ParameterNode n)  $\vee$  (is-ReturnNode n)  $\vee$  (is-ShortCircuitOrNode n))

fun is-Invoke :: IRNode  $\Rightarrow$  bool where
  is-Invoke n = ((is-InvokeNode n)  $\vee$  (is-InvokeWithExceptionNode n))

fun is-Proxy :: IRNode  $\Rightarrow$  bool where
  is-Proxy n = ((is-ProxyNode n))

fun is-ValueProxy :: IRNode  $\Rightarrow$  bool where
  is-ValueProxy n = ((is-PiNode n)  $\vee$  (is-ValueProxyNode n))

fun is-ValueNodeInterface :: IRNode  $\Rightarrow$  bool where
  is-ValueNodeInterface n = ((is-ValueNode n))

fun is-ArrayLengthProvider :: IRNode  $\Rightarrow$  bool where
  is-ArrayLengthProvider n = ((is-AbstractNewArrayNode n)  $\vee$  (is-ConstantNode
    n))

fun is-StampInverter :: IRNode  $\Rightarrow$  bool where
  is-StampInverter n = ((is-IntegerConvertNode n)  $\vee$  (is-NegateNode n)  $\vee$  (is-NotNode
    n))

fun is-GuardingNode :: IRNode  $\Rightarrow$  bool where
  is-GuardingNode n = ((is-AbstractBeginNode n))

fun is-SingleMemoryKill :: IRNode  $\Rightarrow$  bool where
  is-SingleMemoryKill n = ((is-BytecodeExceptionNode n)  $\vee$  (is-ExceptionObjectNode
    n)  $\vee$  (is-InvokeNode n)  $\vee$  (is-InvokeWithExceptionNode n)  $\vee$  (is-KillingBeginNode
    n)  $\vee$  (is-StartNode n))

fun is-LIRLowerable :: IRNode  $\Rightarrow$  bool where
  is-LIRLowerable n = ((is-AbstractBeginNode n)  $\vee$  (is-AbstractEndNode n)  $\vee$ 
    (is-AbstractMergeNode n)  $\vee$  (is-BinaryOpLogicNode n)  $\vee$  (is-CallTargetNode n)
     $\vee$  (is-ConditionalNode n)  $\vee$  (is-ConstantNode n)  $\vee$  (is-IfNode n)  $\vee$  (is-InvokeNode
    n)  $\vee$  (is-InvokeWithExceptionNode n)  $\vee$  (is-IsNullNode n)  $\vee$  (is-LoopBeginNode n)
     $\vee$  (is-PiNode n)  $\vee$  (is-ReturnNode n)  $\vee$  (is-SignedDivNode n)  $\vee$  (is-SignedRemNode
    n)  $\vee$  (is-UnaryOpLogicNode n)  $\vee$  (is-UnwindNode n))

fun is-GuardedNode :: IRNode  $\Rightarrow$  bool where
  is-GuardedNode n = ((is-FloatingGuardedNode n))

fun is-ArithmeticLIRLowerable :: IRNode  $\Rightarrow$  bool where
  is-ArithmeticLIRLowerable n = ((is-AbsNode n)  $\vee$  (is-BinaryArithmeticNode n)  $\vee$ 
    (is-IntegerConvertNode n)  $\vee$  (is-NotNode n)  $\vee$  (is-ShiftNode n)  $\vee$  (is-UnaryArithmeticNode
    n))

```

```

fun is-SwitchFoldable :: IRNode  $\Rightarrow$  bool where
  is-SwitchFoldable n = ((is-IfNode n))

fun is-VirtualizableAllocation :: IRNode  $\Rightarrow$  bool where
  is-VirtualizableAllocation n = ((is-NewArrayNode n)  $\vee$  (is-NewInstanceNode n))

fun is-Unary :: IRNode  $\Rightarrow$  bool where
  is-Unary n = ((is-LoadFieldNode n)  $\vee$  (is-LogicNegationNode n)  $\vee$  (is-UnaryNode n)  $\vee$  (is-UnaryOpLogicNode n))

fun is-FixedNodeInterface :: IRNode  $\Rightarrow$  bool where
  is-FixedNodeInterface n = ((is-FixedNode n))

fun is-BinaryCommutative :: IRNode  $\Rightarrow$  bool where
  is-BinaryCommutative n = ((is-AddNode n)  $\vee$  (is-AndNode n)  $\vee$  (is-IntegerEqualsNode n)  $\vee$  (is-MulNode n)  $\vee$  (is-OrNode n)  $\vee$  (is-XorNode n))

fun is-Canonicalizable :: IRNode  $\Rightarrow$  bool where
  is-Canonicalizable n = ((is-BytecodeExceptionNode n)  $\vee$  (is-ConditionalNode n)  $\vee$  (is-DynamicNewArrayNode n)  $\vee$  (is-PhiNode n)  $\vee$  (is-PiNode n)  $\vee$  (is-ProxyNode n)  $\vee$  (is-StoreFieldNode n)  $\vee$  (is-ValueProxyNode n))

fun is-UncheckedInterfaceProvider :: IRNode  $\Rightarrow$  bool where
  is-UncheckedInterfaceProvider n = ((is-InvokeNode n)  $\vee$  (is-InvokeWithExceptionNode n)  $\vee$  (is-LoadFieldNode n)  $\vee$  (is-ParameterNode n))

fun is-Binary :: IRNode  $\Rightarrow$  bool where
  is-Binary n = ((is-BinaryArithmeticNode n)  $\vee$  (is-BinaryNode n)  $\vee$  (is-BinaryOpLogicNode n)  $\vee$  (is-CompareNode n)  $\vee$  (is-FixedBinaryNode n)  $\vee$  (is-ShortCircuitOrNode n))

fun is-ArithmeticOperation :: IRNode  $\Rightarrow$  bool where
  is-ArithmeticOperation n = ((is-BinaryArithmeticNode n)  $\vee$  (is-IntegerConvertNode n)  $\vee$  (is-ShiftNode n)  $\vee$  (is-UnaryArithmeticNode n))

fun is-ValueNumberable :: IRNode  $\Rightarrow$  bool where
  is-ValueNumberable n = ((is-FloatingNode n)  $\vee$  (is-ProxyNode n))

fun is-Lowerable :: IRNode  $\Rightarrow$  bool where
  is-Lowerable n = ((is-AbstractNewObjectNode n)  $\vee$  (is-AccessFieldNode n)  $\vee$  (is-BytecodeExceptionNode n)  $\vee$  (is-ExceptionObjectNode n)  $\vee$  (is-IntegerDivRemNode n)  $\vee$  (is-UnwindNode n))

fun is-Virtualizable :: IRNode  $\Rightarrow$  bool where
  is-Virtualizable n = ((is-IsNullNode n)  $\vee$  (is-LoadFieldNode n)  $\vee$  (is-PiNode n)  $\vee$  (is-StoreFieldNode n)  $\vee$  (is-ValueProxyNode n))

fun is-Simplifiable :: IRNode  $\Rightarrow$  bool where
  is-Simplifiable n = ((is-AbstractMergeNode n)  $\vee$  (is-BeginNode n)  $\vee$  (is-IfNode

```

$n) \vee (is-LoopExitNode\ n) \vee (is-MethodCallTargetNode\ n) \vee (is-NewArrayNode\ n))$

**fun** *is-StateSplit* :: *IRNode*  $\Rightarrow$  *bool* **where**  
*is-StateSplit* *n* = ((*is-AbstractStateSplit* *n*)  $\vee$  (*is-BeginStateSplitNode* *n*)  $\vee$  (*is-StoreFieldNode* *n*))

**fun** *is-ConvertNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
*is-ConvertNode* *n* = ((*is-IntegerConvertNode* *n*))

**fun** *is-sequential-node* :: *IRNode*  $\Rightarrow$  *bool* **where**  
*is-sequential-node* (*StartNode* -) = *True* |  
*is-sequential-node* (*BeginNode* -) = *True* |  
*is-sequential-node* (*KillingBeginNode* -) = *True* |  
*is-sequential-node* (*LoopBeginNode* - - -) = *True* |  
*is-sequential-node* (*LoopExitNode* - - -) = *True* |  
*is-sequential-node* (*MergeNode* - - -) = *True* |  
*is-sequential-node* (*RefNode* -) = *True* |  
*is-sequential-node* - = *False*

The following convenience function is useful in determining if two IRNodes are of the same type irregardless of their edges. It will return true if both the node parameters are the same node class.

**fun** *is-same-ir-node-type* :: *IRNode*  $\Rightarrow$  *IRNode*  $\Rightarrow$  *bool* **where**  
*is-same-ir-node-type* *n1* *n2* = (  
 ((*is-AbsNode* *n1*)  $\wedge$  (*is-AbsNode* *n2*))  $\vee$   
 ((*is-AddNode* *n1*)  $\wedge$  (*is-AddNode* *n2*))  $\vee$   
 ((*is-AndNode* *n1*)  $\wedge$  (*is-AndNode* *n2*))  $\vee$   
 ((*is-BeginNode* *n1*)  $\wedge$  (*is-BeginNode* *n2*))  $\vee$   
 ((*is-BytecodeExceptionNode* *n1*)  $\wedge$  (*is-BytecodeExceptionNode* *n2*))  $\vee$   
 ((*is-ConditionalNode* *n1*)  $\wedge$  (*is-ConditionalNode* *n2*))  $\vee$   
 ((*is-ConstantNode* *n1*)  $\wedge$  (*is-ConstantNode* *n2*))  $\vee$   
 ((*is-DynamicNewArrayNode* *n1*)  $\wedge$  (*is-DynamicNewArrayNode* *n2*))  $\vee$   
 ((*is-EndNode* *n1*)  $\wedge$  (*is-EndNode* *n2*))  $\vee$   
 ((*is-ExceptionObjectNode* *n1*)  $\wedge$  (*is-ExceptionObjectNode* *n2*))  $\vee$   
 ((*is-FrameState* *n1*)  $\wedge$  (*is-FrameState* *n2*))  $\vee$   
 ((*is-IfNode* *n1*)  $\wedge$  (*is-IfNode* *n2*))  $\vee$   
 ((*is-IntegerBelowNode* *n1*)  $\wedge$  (*is-IntegerBelowNode* *n2*))  $\vee$   
 ((*is-IntegerEqualsNode* *n1*)  $\wedge$  (*is-IntegerEqualsNode* *n2*))  $\vee$   
 ((*is-IntegerLessThanNode* *n1*)  $\wedge$  (*is-IntegerLessThanNode* *n2*))  $\vee$   
 ((*is-InvokeNode* *n1*)  $\wedge$  (*is-InvokeNode* *n2*))  $\vee$   
 ((*is-InvokeWithExceptionNode* *n1*)  $\wedge$  (*is-InvokeWithExceptionNode* *n2*))  $\vee$   
 ((*is-IsNullNode* *n1*)  $\wedge$  (*is-IsNullNode* *n2*))  $\vee$   
 ((*is-KillingBeginNode* *n1*)  $\wedge$  (*is-KillingBeginNode* *n2*))  $\vee$   
 ((*is-LoadFieldNode* *n1*)  $\wedge$  (*is-LoadFieldNode* *n2*))  $\vee$   
 ((*is-LogicNegationNode* *n1*)  $\wedge$  (*is-LogicNegationNode* *n2*))  $\vee$   
 ((*is-LoopBeginNode* *n1*)  $\wedge$  (*is-LoopBeginNode* *n2*))  $\vee$   
 ((*is-LoopEndNode* *n1*)  $\wedge$  (*is-LoopEndNode* *n2*))  $\vee$   
 ((*is-LoopExitNode* *n1*)  $\wedge$  (*is-LoopExitNode* *n2*))  $\vee$

```

((is-MergeNode n1) ∧ (is-MergeNode n2)) ∨
((is-MethodCallTargetNode n1) ∧ (is-MethodCallTargetNode n2)) ∨
((is-MulNode n1) ∧ (is-MulNode n2)) ∨
((is-NegateNode n1) ∧ (is-NegateNode n2)) ∨
((is-NewArrayNode n1) ∧ (is-NewArrayNode n2)) ∨
((is-NewInstanceNode n1) ∧ (is-NewInstanceNode n2)) ∨
((is-NotNode n1) ∧ (is-NotNode n2)) ∨
((is-OrNode n1) ∧ (is-OrNode n2)) ∨
((is-ParameterNode n1) ∧ (is-ParameterNode n2)) ∨
((is-PiNode n1) ∧ (is-PiNode n2)) ∨
((is-ReturnNode n1) ∧ (is-ReturnNode n2)) ∨
((is-ShortCircuitOrNode n1) ∧ (is-ShortCircuitOrNode n2)) ∨
((is-SignedDivNode n1) ∧ (is-SignedDivNode n2)) ∨
((is-StartNode n1) ∧ (is-StartNode n2)) ∨
((is-StoreFieldNode n1) ∧ (is-StoreFieldNode n2)) ∨
((is-SubNode n1) ∧ (is-SubNode n2)) ∨
((is-UnwindNode n1) ∧ (is-UnwindNode n2)) ∨
((is-ValuePhiNode n1) ∧ (is-ValuePhiNode n2)) ∨
((is-ValueProxyNode n1) ∧ (is-ValueProxyNode n2)) ∨
((is-XorNode n1) ∧ (is-XorNode n2)))

```

end

## 5 Stamp Typing

```

theory Stamp
  imports Values
begin

```

The GraalVM compiler uses the Stamp class to store range and type information for a given node in the IR graph. We model the Stamp class as a datatype, Stamp, and provide a number of functions on the datatype which correspond to the class methods within the compiler.

Stamp information is used in a variety of ways in optimizations, and so, we additionally provide a number of lemmas which help to prove future optimizations.

```

datatype Stamp =
  VoidStamp
  | IntegerStamp (stp-bits: nat) (stp-lower: int) (stp-upper: int)

  | KlassPointerStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
  | MethodCountersPointerStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
  | MethodPointersStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
  | ObjectStamp (stp-type: string) (stp-exactType: bool) (stp-nonNull: bool) (stp-alwaysNull:
bool)
  | RawPointerStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
  | IllegalStamp

```



```

fun is-stamp-empty :: Stamp  $\Rightarrow$  bool where
  is-stamp-empty (IntegerStamp b lower upper) = (upper < lower) |

  is-stamp-empty x = False

```

Just like the *IntegerStamp* class, we need to know that our lo/hi bounds fit into the given number of bits (either signed or unsigned). Our integer stamps have infinite lo/hi bounds, so if the lower bound is non-negative, we can assume that all values are positive, and the integer bits of a related value can be interpreted as unsigned. This is similar (but slightly more general) to what *IntegerStamp.java* does with its test: `if (sameSignBounds())` in the `unsignedUpperBound()` method.

Note that this is a bit different and more accurate than what *StampFactory.forUnsignedInteger* does (it widens large unsigned ranges to the max signed range to allow all bit patterns) because its lo/hi values are only 64-bit.

```

fun valid-stamp :: Stamp  $\Rightarrow$  bool where
  valid-stamp (IntegerStamp bits lo hi) =
    (0 < bits  $\wedge$  bits  $\leq$  64  $\wedge$ 
     fst (bit-bounds bits)  $\leq$  lo  $\wedge$  lo  $\leq$  snd (bit-bounds bits)  $\wedge$ 
     fst (bit-bounds bits)  $\leq$  hi  $\wedge$  hi  $\leq$  snd (bit-bounds bits)) |
  valid-stamp s = True

```

```

experiment begin
corollary bit-bounds 1 = (-1, 0) by simp
end

```

— A stamp which includes the full range of the type

```

fun unrestricted-stamp :: Stamp  $\Rightarrow$  Stamp where
  unrestricted-stamp VoidStamp = VoidStamp |
  unrestricted-stamp (IntegerStamp bits lower upper) = (IntegerStamp bits (fst
(bit-bounds bits)) (snd (bit-bounds bits))) |

  unrestricted-stamp (KlassPointerStamp nonNull alwaysNull) = (KlassPointerStamp
False False) |
  unrestricted-stamp (MethodCountersPointerStamp nonNull alwaysNull) = (MethodCountersPointerStamp
False False) |
  unrestricted-stamp (MethodPointersStamp nonNull alwaysNull) = (MethodPointersStamp
False False) |
  unrestricted-stamp (ObjectStamp type exactType nonNull alwaysNull) = (ObjectStamp

```

```

""" False False False) |
  unrestricted-stamp - = IllegalStamp

```

```

fun is-stamp-unrestricted :: Stamp ⇒ bool where
  is-stamp-unrestricted s = (s = unrestricted-stamp s)

```

— A stamp which provides type information but has an empty range of values

```

fun empty-stamp :: Stamp ⇒ Stamp where
  empty-stamp VoidStamp = VoidStamp |
  empty-stamp (IntegerStamp bits lower upper) = (IntegerStamp bits (snd (bit-bounds
bits)) (fst (bit-bounds bits))) |

  empty-stamp (KlassPointerStamp nonNull alwaysNull) = (KlassPointerStamp
nonNull alwaysNull) |
  empty-stamp (MethodCountersPointerStamp nonNull alwaysNull) = (MethodCountersPointerStamp
nonNull alwaysNull) |
  empty-stamp (MethodPointersStamp nonNull alwaysNull) = (MethodPointersStamp
nonNull alwaysNull) |
  empty-stamp (ObjectStamp type exactType nonNull alwaysNull) = (ObjectStamp
""" True True False) |
  empty-stamp stamp = IllegalStamp

```

— Calculate the meet stamp of two stamps

```

fun meet :: Stamp ⇒ Stamp ⇒ Stamp where
  meet VoidStamp VoidStamp = VoidStamp |
  meet (IntegerStamp b1 l1 u1) (IntegerStamp b2 l2 u2) = (
    if b1 ≠ b2 then IllegalStamp else
    (IntegerStamp b1 (min l1 l2) (max u1 u2))
  ) |

  meet (KlassPointerStamp nn1 an1) (KlassPointerStamp nn2 an2) = (
    KlassPointerStamp (nn1 ∧ nn2) (an1 ∧ an2)
  ) |
  meet (MethodCountersPointerStamp nn1 an1) (MethodCountersPointerStamp
nn2 an2) = (
    MethodCountersPointerStamp (nn1 ∧ nn2) (an1 ∧ an2)
  ) |
  meet (MethodPointersStamp nn1 an1) (MethodPointersStamp nn2 an2) = (
    MethodPointersStamp (nn1 ∧ nn2) (an1 ∧ an2)
  ) |
  meet s1 s2 = IllegalStamp

```

— Calculate the join stamp of two stamps

```

fun join :: Stamp ⇒ Stamp ⇒ Stamp where
  join VoidStamp VoidStamp = VoidStamp |
  join (IntegerStamp b1 l1 u1) (IntegerStamp b2 l2 u2) = (
    if b1 ≠ b2 then IllegalStamp else
    (IntegerStamp b1 (max l1 l2) (min u1 u2))
  )

```

```

) |

join (KlassPointerStamp nn1 an1) (KlassPointerStamp nn2 an2) = (
  if ((nn1 ∨ nn2) ∧ (an1 ∨ an2))
  then (empty-stamp (KlassPointerStamp nn1 an1))
  else (KlassPointerStamp (nn1 ∨ nn2) (an1 ∨ an2))
) |
join (MethodCountersPointerStamp nn1 an1) (MethodCountersPointerStamp nn2
an2) = (
  if ((nn1 ∨ nn2) ∧ (an1 ∨ an2))
  then (empty-stamp (MethodCountersPointerStamp nn1 an1))
  else (MethodCountersPointerStamp (nn1 ∨ nn2) (an1 ∨ an2))
) |
join (MethodPointersStamp nn1 an1) (MethodPointersStamp nn2 an2) = (
  if ((nn1 ∨ nn2) ∧ (an1 ∨ an2))
  then (empty-stamp (MethodPointersStamp nn1 an1))
  else (MethodPointersStamp (nn1 ∨ nn2) (an1 ∨ an2))
) |
join s1 s2 = IllegalStamp

```

— In certain circumstances a stamp provides enough information to evaluate a value as a stamp, the `asConstant` function converts the stamp to a value where one can be inferred.

```

fun asConstant :: Stamp ⇒ Value where
  asConstant (IntegerStamp b l h) = (if l = h then IntVal b (word-of-int l) else
  UndefVal) |
  asConstant - = UndefVal

```

— Determine if two stamps never have value overlaps i.e. their join is empty

```

fun alwaysDistinct :: Stamp ⇒ Stamp ⇒ bool where
  alwaysDistinct stamp1 stamp2 = is-stamp-empty (join stamp1 stamp2)

```

— Determine if two stamps must always be the same value i.e. two equal constants

```

fun neverDistinct :: Stamp ⇒ Stamp ⇒ bool where
  neverDistinct stamp1 stamp2 = (asConstant stamp1 = asConstant stamp2 ∧
  asConstant stamp1 ≠ UndefVal)

```

```

fun constantAsStamp :: Value ⇒ Stamp where
  constantAsStamp (IntVal b v) = (IntegerStamp b (int-signed-value b v) (int-signed-value
  b v)) |

```

```

  constantAsStamp - = IllegalStamp

```

— Define when a runtime value is valid for a stamp. The stamp bounds must be valid, and val must be zero-extended.

```

fun valid-value :: Value ⇒ Stamp ⇒ bool where
  valid-value (IntVal b1 val) (IntegerStamp b l h) =
    (if b1 = b then

```

```

    valid-stamp (IntegerStamp b l h) ∧
    take-bit b val = val ∧
    l ≤ int-signed-value b val ∧ int-signed-value b val ≤ h
    else False) |

```

```

valid-value (ObjRef ref) (ObjectStamp klass exact nonNull alwaysNull) =
  ((alwaysNull → ref = None) ∧ (ref=None → ¬ nonNull)) |
valid-value stamp val = False

```

```

fun compatible :: Stamp ⇒ Stamp ⇒ bool where
  compatible (IntegerStamp b1 lo1 hi1) (IntegerStamp b2 lo2 hi2) =
    (b1 = b2 ∧ valid-stamp (IntegerStamp b1 lo1 hi1) ∧ valid-stamp (IntegerStamp
b2 lo2 hi2)) |
  compatible (VoidStamp) (VoidStamp) = True |
  compatible - - = False

```

```

fun stamp-under :: Stamp ⇒ Stamp ⇒ bool where
  stamp-under x y = ((stpi-upper x) < (stpi-lower y))

```

— The most common type of stamp within the compiler (apart from the Void-Stamp) is a 32 bit integer stamp with an unrestricted range. We use *default-stamp* as it is a frequently used stamp.

```

definition default-stamp :: Stamp where
  default-stamp = (unrestricted-stamp (IntegerStamp 32 0 0))

```

```

value valid-value (IntVal 8 (255)) (IntegerStamp 8 (-128) 127)
end

```

## 6 Graph Representation

```

theory IRGraph
  imports
    IRNodeHierarchy
    Stamp
    HOL-Library.FSet
    HOL.Relation
  begin

```

This theory defines the main Graal data structure - an entire IR Graph.

IRGraph is defined as a partial map with a finite domain. The finite domain is required to be able to generate code and produce an interpreter.

```

typedef IRGraph = {g :: ID → (IRNode × Stamp) . finite (dom g)}
proof -
  have finite(dom(Map.empty)) ∧ ran Map.empty = {} by auto

```

then show *?thesis*  
 by *fastforce*  
 qed

**setup-lifting** *type-definition-IRGraph*

**lift-definition** *ids* :: *IRGraph*  $\Rightarrow$  *ID set*  
 is  $\lambda g. \{nid \in dom\ g \mid \nexists s. g\ nid = (Some\ (NoNode, s))\}$  .

**fun** *with-default* :: '*c*  $\Rightarrow$  ('*b*  $\Rightarrow$  '*c*)  $\Rightarrow$  (('a  $\rightarrow$  '*b*)  $\Rightarrow$  '*a*  $\Rightarrow$  '*c*) **where**  
*with-default* *def conv* = ( $\lambda m\ k.$   
 (case *m k* of *None*  $\Rightarrow$  *def* | *Some v*  $\Rightarrow$  *conv v*))

**lift-definition** *kind* :: *IRGraph*  $\Rightarrow$  (*ID*  $\Rightarrow$  *IRNode*)  
 is *with-default NoNode fst* .

**lift-definition** *stamp* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *Stamp*  
 is *with-default IllegalStamp snd* .

**lift-definition** *add-node* :: *ID*  $\Rightarrow$  (*IRNode*  $\times$  *Stamp*)  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *IRGraph*  
 is  $\lambda nid\ k\ g.$  if *fst k* = *NoNode* then *g* else *g*(*nid*  $\mapsto$  *k*) **by** *simp*

**lift-definition** *remove-node* :: *ID*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *IRGraph*  
 is  $\lambda nid\ g.$  *g*(*nid* := *None*) **by** *simp*

**lift-definition** *replace-node* :: *ID*  $\Rightarrow$  (*IRNode*  $\times$  *Stamp*)  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *IRGraph*  
 is  $\lambda nid\ k\ g.$  if *fst k* = *NoNode* then *g* else *g*(*nid*  $\mapsto$  *k*) **by** *simp*

**lift-definition** *as-list* :: *IRGraph*  $\Rightarrow$  (*ID*  $\times$  *IRNode*  $\times$  *Stamp*) *list*  
 is  $\lambda g.$  *map* ( $\lambda k. (k, the\ (g\ k))$ ) (*sorted-list-of-set* (*dom g*)) .

**fun** *no-node* :: (*ID*  $\times$  (*IRNode*  $\times$  *Stamp*)) *list*  $\Rightarrow$  (*ID*  $\times$  (*IRNode*  $\times$  *Stamp*)) *list*  
**where**  
*no-node g* = *filter* ( $\lambda n. fst\ (snd\ n) \neq NoNode$ ) *g*

**lift-definition** *irgraph* :: (*ID*  $\times$  (*IRNode*  $\times$  *Stamp*)) *list*  $\Rightarrow$  *IRGraph*  
 is *map-of*  $\circ$  *no-node*  
**by** (*simp add: finite-dom-map-of*)

**definition** *as-set* :: *IRGraph*  $\Rightarrow$  (*ID*  $\times$  (*IRNode*  $\times$  *Stamp*)) *set* **where**  
*as-set g* =  $\{(n, kind\ g\ n, stamp\ g\ n) \mid n. n \in ids\ g\}$

**definition** *true-ids* :: *IRGraph*  $\Rightarrow$  *ID set* **where**  
*true-ids g* = *ids g* -  $\{n \in ids\ g. \exists n'. kind\ g\ n = RefNode\ n'\}$

**definition** *domain-subtraction* :: '*a set*  $\Rightarrow$  ('*a*  $\times$  '*b*) *set*  $\Rightarrow$  ('*a*  $\times$  '*b*) *set*  
 (**infix**  $\leq 30$ ) **where**  
*domain-subtraction s r* =  $\{(x, y) \mid (x, y) \in r \wedge x \notin s\}$

**notation** (*latex*)

*domain-subtraction* ( $- \triangleleft -$ )

**code-datatype** *irgraph*

**fun** *filter-none* **where**

*filter-none*  $g = \{nid \in \text{dom } g \mid \nexists s. g \text{ nid} = (\text{Some } (\text{NoNode}, s))\}$

**lemma** *no-node-clears*:

*res* = *no-node*  $xs \longrightarrow (\forall x \in \text{set } res. \text{fst } (\text{snd } x) \neq \text{NoNode})$

**by** *simp*

**lemma** *dom-eq*:

**assumes**  $\forall x \in \text{set } xs. \text{fst } (\text{snd } x) \neq \text{NoNode}$

**shows** *filter-none* (*map-of*  $xs$ ) = *dom* (*map-of*  $xs$ )

**unfolding** *filter-none.simps* **using** *assms map-of-SomeD*

**by** *fastforce*

**lemma** *fil-eq*:

*filter-none* (*map-of* (*no-node*  $xs$ )) = *set* (*map fst* (*no-node*  $xs$ ))

**using** *no-node-clears*

**by** (*metis dom-eq dom-map-of-conv-image-fst list.set-map*)

**lemma** *irgraph[code]*: *ids* (*irgraph*  $m$ ) = *set* (*map fst* (*no-node*  $m$ ))

**unfolding** *irgraph-def ids-def* **using** *fil-eq*

**by** (*smt Rep-IRGraph comp-apply eq-onp-same-args filter-none.simps ids.abs-eq*  
*ids-def irgraph.abs-eq irgraph.rep-eq irgraph-def mem-Collect-eq*)

**lemma** [*code*]: *Rep-IRGraph* (*irgraph*  $m$ ) = *map-of* (*no-node*  $m$ )

**using** *Abs-IRGraph-inverse*

**by** (*simp add: irgraph.rep-eq*)

— Get the inputs set of a given node ID

**fun** *inputs* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *ID set* **where**

*inputs*  $g \text{ nid} = \text{set } (\text{inputs-of } (\text{kind } g \text{ nid}))$

— Get the successor set of a given node ID

**fun** *succ* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *ID set* **where**

*succ*  $g \text{ nid} = \text{set } (\text{successors-of } (\text{kind } g \text{ nid}))$

— Gives a relation between node IDs - between a node and its input nodes

**fun** *input-edges* :: *IRGraph*  $\Rightarrow$  *ID rel* **where**

*input-edges*  $g = (\bigcup i \in \text{ids } g. \{(i,j) \mid j \in (\text{inputs } g \ i)\})$

— Find all the nodes in the graph that have nid as an input - the usages of nid

**fun** *usages* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *ID set* **where**

*usages*  $g \text{ nid} = \{i. i \in \text{ids } g \wedge \text{nid} \in \text{inputs } g \ i\}$

**fun** *successor-edges* :: *IRGraph*  $\Rightarrow$  *ID rel* **where**

*successor-edges*  $g = (\bigcup i \in \text{ids } g. \{(i,j) \mid j \in (\text{succ } g \ i)\})$

**fun** *predecessors* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *ID set* **where**

```

    predecessors g nid = {i. i ∈ ids g ∧ nid ∈ succ g i}
fun nodes-of :: IRGraph ⇒ (IRNode ⇒ bool) ⇒ ID set where
    nodes-of g sel = {nid ∈ ids g . sel (kind g nid)}
fun edge :: (IRNode ⇒ 'a) ⇒ ID ⇒ IRGraph ⇒ 'a where
    edge sel nid g = sel (kind g nid)

fun filtered-inputs :: IRGraph ⇒ ID ⇒ (IRNode ⇒ bool) ⇒ ID list where
    filtered-inputs g nid f = filter (f ∘ (kind g)) (inputs-of (kind g nid))
fun filtered-successors :: IRGraph ⇒ ID ⇒ (IRNode ⇒ bool) ⇒ ID list where
    filtered-successors g nid f = filter (f ∘ (kind g)) (successors-of (kind g nid))
fun filtered-usages :: IRGraph ⇒ ID ⇒ (IRNode ⇒ bool) ⇒ ID set where
    filtered-usages g nid f = {n ∈ (usages g nid). f (kind g n)}

fun is-empty :: IRGraph ⇒ bool where
    is-empty g = (ids g = {})

fun any-usage :: IRGraph ⇒ ID ⇒ ID where
    any-usage g nid = hd (sorted-list-of-set (usages g nid))

lemma ids-some[simp]: x ∈ ids g ⟷ kind g x ≠ NoNode
proof -
    have that: x ∈ ids g ⟶ kind g x ≠ NoNode
    using ids.rep-eq kind.rep-eq by force
    have kind g x ≠ NoNode ⟶ x ∈ ids g
    unfolding with-default.simps kind-def ids-def
    by (cases Rep-IRGraph g x = None; auto)
    from this that show ?thesis by auto
qed

lemma not-in-g:
    assumes nid ∉ ids g
    shows kind g nid = NoNode
    using assms ids-some by blast

lemma valid-creation[simp]:
    finite (dom g) ⟷ Rep-IRGraph (Abs-IRGraph g) = g
    using Abs-IRGraph-inverse by (metis Rep-IRGraph mem-Collect-eq)

lemma [simp]: finite (ids g)
    using Rep-IRGraph ids.rep-eq by simp

lemma [simp]: finite (ids (irgraph g))
    by (simp add: finite-dom-map-of)

lemma [simp]: finite (dom g) ⟶ ids (Abs-IRGraph g) = {nid ∈ dom g . ∃ s. g
nid = Some (NoNode, s)}
    using ids.rep-eq by simp

lemma [simp]: finite (dom g) ⟶ kind (Abs-IRGraph g) = (λx . (case g x of None

```

$\Rightarrow \text{NoNode} \mid \text{Some } n \Rightarrow \text{fst } n))$   
**by** (*simp* *add*: *kind.rep-eq*)

**lemma** [*simp*]: *finite* (*dom g*)  $\longrightarrow$  *stamp* (*Abs-IRGraph g*) = ( $\lambda x . (\text{case } g \ x \text{ of}$   
*None*  $\Rightarrow$  *IllegalStamp*  $\mid$  *Some n*  $\Rightarrow$  *snd n*))  
**using** *stamp.abs-eq stamp.rep-eq* **by** *auto*

**lemma** [*simp*]: *ids* (*irgraph g*) = *set* (*map fst* (*no-node g*))  
**using** *irgraph* **by** *auto*

**lemma** [*simp*]: *kind* (*irgraph g*) = ( $\lambda \text{nid} . (\text{case } (\text{map-of } (\text{no-node } g)) \text{nid} \text{ of } \text{None}$   
 $\Rightarrow \text{NoNode} \mid \text{Some } n \Rightarrow \text{fst } n))$ )  
**using** *irgraph.rep-eq kind.transfer kind.rep-eq* **by** *auto*

**lemma** [*simp*]: *stamp* (*irgraph g*) = ( $\lambda \text{nid} . (\text{case } (\text{map-of } (\text{no-node } g)) \text{nid} \text{ of } \text{None}$   
 $\Rightarrow \text{IllegalStamp} \mid \text{Some } n \Rightarrow \text{snd } n))$ )  
**using** *irgraph.rep-eq stamp.transfer stamp.rep-eq* **by** *auto*

**lemma** *map-of-upd*: (*map-of g*)(*k*  $\mapsto$  *v*) = (*map-of* ((*k*, *v*)  $\#$  *g*))  
**by** *simp*

**lemma** [*code*]: *replace-node nid k* (*irgraph g*) = (*irgraph* ( ((*nid*, *k*)  $\#$  *g*)))  
**proof** (*cases fst k = NoNode*)  
  *case True*  
  **then show** ?*thesis*  
  **by** (*metis* (*mono-tags*, *lifting*) *Rep-IRGraph-inject filter.simps(2) irgraph.abs-eq*  
*no-node.simps replace-node.rep-eq snd-conv*)  
  **next**  
  *case False*  
  **then show** ?*thesis* **unfolding** *irgraph-def replace-node-def no-node.simps*  
  **by** (*smt* (*verit*, *best*) *Rep-IRGraph comp-apply eq-onp-same-args filter.simps(2)*  
*id-def irgraph.rep-eq map-fun-apply map-of-upd mem-Collect-eq no-node.elims re-*  
*place-node.abs-eq replace-node-def snd-eqD*)  
**qed**

**lemma** [*code*]: *add-node nid k* (*irgraph g*) = (*irgraph* (((*nid*, *k*)  $\#$  *g*)))  
**by** (*smt* (*z3*) *Rep-IRGraph-inject add-node.rep-eq filter.simps(2) irgraph.rep-eq*  
*map-of-upd no-node.simps snd-conv*)

**lemma** *add-node-lookup*:  
  *gup* = *add-node nid* (*k*, *s*) *g*  $\longrightarrow$   
  (*if k*  $\neq$  *NoNode* *then kind gup nid* = *k*  $\wedge$  *stamp gup nid* = *s* *else kind gup nid*  
  = *kind g nid*)  
**proof** (*cases k = NoNode*)  
  *case True*  
  **then show** ?*thesis*  
  **by** (*simp add*: *add-node.rep-eq kind.rep-eq*)  
**next**



```

    case False
  then show ?thesis
    by (simp add: kind.rep-eq add-node.rep-eq stamp.rep-eq)
qed

```

**lemma** *remove-node-lookup*:

```

  gup = remove-node nid g  $\longrightarrow$  kind gup nid = NoNode  $\wedge$  stamp gup nid = IllegalStamp
  by (simp add: kind.rep-eq remove-node.rep-eq stamp.rep-eq)

```

**lemma** *replace-node-lookup*[simp]:

```

  gup = replace-node nid (k, s) g  $\wedge$  k  $\neq$  NoNode  $\longrightarrow$  kind gup nid = k  $\wedge$  stamp gup nid = s
  by (simp add: replace-node.rep-eq kind.rep-eq stamp.rep-eq)

```

**lemma** *replace-node-unchanged*:

```

  gup = replace-node nid (k, s) g  $\longrightarrow$  ( $\forall$  n  $\in$  (ids g - {nid}) . n  $\in$  ids g  $\wedge$  n  $\in$ 
  ids gup  $\wedge$  kind g n = kind gup n)
  by (simp add: kind.rep-eq replace-node.rep-eq)

```

### 6.0.1 Example Graphs

Example 1: empty graph (just a start and end node)

**definition** *start-end-graph*:: *IRGraph* **where**

```

  start-end-graph = irgraph [(0, StartNode None 1, VoidStamp), (1, ReturnNode
  None None, VoidStamp)]

```

Example 2: public static int sq(int x) return x \* x;

```

[1 P(0)] / [0 Start] [4 *] | / V / [5 Return]

```

**definition** *eg2-sq*:: *IRGraph* **where**

```

  eg2-sq = irgraph [
    (0, StartNode None 5, VoidStamp),
    (1, ParameterNode 0, default-stamp),
    (4, MulNode 1 1, default-stamp),
    (5, ReturnNode (Some 4) None, default-stamp)
  ]

```

**value** *input-edges* *eg2-sq*

**value** *usages* *eg2-sq* 1

**end**

## 6.1 Control-flow Graph Traversal

**theory**

```

Traversal
imports
  IRGraph
begin

```

```

type-synonym Seen = ID set

```

`nextEdge` helps determine which node to traverse next by returning the first successor edge that isn't in the set of already visited nodes. If there is not an appropriate successor, `None` is returned instead.

```

fun nextEdge :: Seen  $\Rightarrow$  ID  $\Rightarrow$  IRGraph  $\Rightarrow$  ID option where
  nextEdge seen nid g =
    (let nids = (filter ( $\lambda$ nid'. nid'  $\notin$  seen) (successors-of (kind g nid))) in
      (if length nids > 0 then Some (hd nids) else None))

```

`pred` determines which node, if any, acts as the predecessor of another.

Merge nodes represent a special case where-in the predecessor exists as an input edge of the merge node, to simplify the traversal we treat only the first input end node as the predecessor, ignoring that multiple nodes may act as a successor.

For all other nodes, the predecessor is the first element of the predecessors set. Note that in a well-formed graph there should only be one element in the predecessor set.

```

fun pred :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID option where
  pred g nid = (case kind g nid of
    (MergeNode ends -)  $\Rightarrow$  Some (hd ends) |
    -  $\Rightarrow$ 
      (if IRGraph.predecessors g nid = {}
        then None else
        Some (hd (sorted-list-of-set (IRGraph.predecessors g nid))))
  )

```

Here we try to implement a generic fork of the control-flow traversal algorithm that was initially implemented for the ConditionalElimination phase

```

type-synonym 'a TraversalState = (ID  $\times$  Seen  $\times$  'a)

```

```

inductive Step

```

```

  :: ('a TraversalState  $\Rightarrow$  'a)  $\Rightarrow$  IRGraph  $\Rightarrow$  'a TraversalState  $\Rightarrow$  'a TraversalState
  option  $\Rightarrow$  bool

```

```

  for sa g where

```

— Hit a `BeginNode` with an `IfNode` predecessor which represents the start of a basic block for the `IfNode`. 1. `nid'` will be the successor of the begin node. 2. Find the first and only predecessor. 3. Extract condition from the preceding `IfNode`. 4. Negate condition if the begin node is second branch (we've taken the else branch of the condition) 5. Add the condition or the negated condition to stack 6. Perform any stamp updates based on the condition using the `registerNewCondition` function and place them on the top of the stack of stamp information

$\llbracket \text{kind } g \text{ nid} = \text{BeginNode } \text{nid}' ;$

$\text{nid} \notin \text{seen};$   
 $\text{seen}' = \{\text{nid}\} \cup \text{seen};$

$\text{Some } \text{ifcond} = \text{pred } g \text{ nid};$   
 $\text{kind } g \text{ ifcond} = \text{IfNode } \text{cond } t \text{ } f;$

$\text{analysis}' = \text{sa } (\text{nid}, \text{seen}, \text{analysis}) \rrbracket$   
 $\implies \text{Step } \text{sa } g (\text{nid}, \text{seen}, \text{analysis}) (\text{Some } (\text{nid}', \text{seen}', \text{analysis}')) \mid$

— Hit an EndNode 1. nid' will be the usage of EndNode 2. pop the conditions and stamp stack

$\llbracket \text{kind } g \text{ nid} = \text{EndNode};$

$\text{nid} \notin \text{seen};$   
 $\text{seen}' = \{\text{nid}\} \cup \text{seen};$

$\text{nid}' = \text{any-usage } g \text{ nid};$

$\text{analysis}' = \text{sa } (\text{nid}, \text{seen}, \text{analysis}) \rrbracket$   
 $\implies \text{Step } \text{sa } g (\text{nid}, \text{seen}, \text{analysis}) (\text{Some } (\text{nid}', \text{seen}', \text{analysis}')) \mid$

— We can find a successor edge that is not in seen, go there

$\llbracket \neg(\text{is-EndNode } (\text{kind } g \text{ nid}));$   
 $\neg(\text{is-BeginNode } (\text{kind } g \text{ nid}));$

$\text{nid} \notin \text{seen};$   
 $\text{seen}' = \{\text{nid}\} \cup \text{seen};$

$\text{Some } \text{nid}' = \text{nextEdge } \text{seen}' \text{ nid } g;$

$\text{analysis}' = \text{sa } (\text{nid}, \text{seen}, \text{analysis}) \rrbracket$   
 $\implies \text{Step } \text{sa } g (\text{nid}, \text{seen}, \text{analysis}) (\text{Some } (\text{nid}', \text{seen}', \text{analysis}')) \mid$

— We can cannot find a successor edge that is not in seen, give back None

$\llbracket \neg(\text{is-EndNode } (\text{kind } g \text{ nid}));$   
 $\neg(\text{is-BeginNode } (\text{kind } g \text{ nid}));$

$\text{nid} \notin \text{seen};$   
 $\text{seen}' = \{\text{nid}\} \cup \text{seen};$

$\text{None} = \text{nextEdge } \text{seen}' \text{ nid } g \rrbracket$   
 $\implies \text{Step } \text{sa } g (\text{nid}, \text{seen}, \text{analysis}) \text{ None} \mid$

— We've already seen this node, give back None

$\llbracket \text{nid} \in \text{seen} \rrbracket \implies \text{Step } \text{sa } g (\text{nid}, \text{seen}, \text{analysis}) \text{ None}$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$ ) *Step* .

end

## 6.2 Structural Graph Comparison

**theory**

*Comparison*

**imports**

*IRGraph*

**begin**

We introduce a form of structural graph comparison that is able to assert structural equivalence of graphs which differ in zero or more reference node chains for any given nodes.

**fun** *find-ref-nodes* :: *IRGraph*  $\Rightarrow$  (*ID*  $\rightarrow$  *ID*) **where**  
*find-ref-nodes* *g* = *map-of*  
 (*map* ( $\lambda n. (n, \text{ir-ref } (\text{kind } g \ n)))$ ) (*filter* ( $\lambda id. \text{is-RefNode } (\text{kind } g \ id)$ ) (*sorted-list-of-set* (*ids* *g*))))

**fun** *replace-ref-nodes* :: *IRGraph*  $\Rightarrow$  (*ID*  $\rightarrow$  *ID*)  $\Rightarrow$  *ID list*  $\Rightarrow$  *ID list* **where**  
*replace-ref-nodes* *g m xs* = *map* ( $\lambda id. (\text{case } (m \ id) \text{ of } \text{Some } other \Rightarrow other \mid \text{None} \Rightarrow id)$ ) *xs*

**fun** *find-next* :: *ID list*  $\Rightarrow$  *ID set*  $\Rightarrow$  *ID option* **where**  
*find-next* *to-see seen* = (*let* *l* = (*filter* ( $\lambda nid. nid \notin seen$ ) *to-see*)  
 in (*case* *l* of []  $\Rightarrow$  *None* | *xs*  $\Rightarrow$  *Some* (*hd* *xs*)))

**inductive** *reachables* :: *IRGraph*  $\Rightarrow$  *ID list*  $\Rightarrow$  *ID set*  $\Rightarrow$  *ID set*  $\Rightarrow$  *bool* **where**  
*reachables* *g* [] {} {} |  
 [|*None* = *find-next* *to-see* *seen*]  $\Longrightarrow$  *reachables* *g* *to-see* *seen* *seen* |  
 [|*Some* *n* = *find-next* *to-see* *seen*;  
   *node* = *kind* *g* *n*;  
   *new* = (*inputs-of* *node*) @ (*successors-of* *node*);  
   *reachables* *g* (*to-see* @ *new*) ({*n*}  $\cup$  *seen*) *seen'*]  $\Longrightarrow$  *reachables* *g* *to-see* *seen* *seen'*

**code-pred** (*modes*: *i*  $\Rightarrow$  *i*  $\Rightarrow$  *i*  $\Rightarrow$  *o*  $\Rightarrow$  *bool*) [*show-steps*,*show-mode-inference*,*show-intermediate-results*]

*reachables* .

**inductive** *nodeEq* :: (*ID*  $\rightarrow$  *ID*)  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *bool*  
**where**  
 [|*kind* *g1* *n1* = *RefNode* *ref*; *nodeEq* *m* *g1* *ref* *g2* *n2*]  $\Longrightarrow$  *nodeEq* *m* *g1* *n1* *g2* *n2* |  
 |  
 [|*x* = *kind* *g1* *n1*;  
   *y* = *kind* *g2* *n2*;  
   *is-same-ir-node-type* *x* *y*;  
   *replace-ref-nodes* *g1* *m* (*successors-of* *x*) = *successors-of* *y*;  
   *replace-ref-nodes* *g1* *m* (*inputs-of* *x*) = *inputs-of* *y*] |

$\implies \text{nodeEq } m \ g1 \ n1 \ g2 \ n2$

**code-pred** [show-modes] *nodeEq* .

**fun** *diffNodesGraph* :: *IRGraph*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *ID* *set* **where**  
*diffNodesGraph* *g1* *g2* = (let *refNodes* = *find-ref-nodes* *g1* in  
 { *n* . *n*  $\in$  *Predicate.the* (*reachables-i-i-i-o* *g1* [0] {})  $\wedge$  (*case* *refNodes* *n* of

*Some* -  $\Rightarrow$  *False* | -  $\Rightarrow$  *True*)  $\wedge$   $\neg(\text{nodeEq } \text{refNodes } g1 \ n \ g2 \ n)$  })

**fun** *diffNodesInfo* :: *IRGraph*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  (*ID*  $\times$  *IRNode*  $\times$  *IRNode*) *set* **where**  
*diffNodesInfo* *g1* *g2* = {(*nid*, *kind* *g1* *nid*, *kind* *g2* *nid*) | *nid* . *nid*  $\in$  *diffNodesGraph* *g1* *g2*}

**fun** *eqGraph* :: *IRGraph*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *bool* **where**  
*eqGraph* *isabelle-graph* *graal-graph* = ((*diffNodesGraph* *isabelle-graph* *graal-graph*)  
 = {})

**end**

## 7 java.lang.Long

Utility functions from the Long class that Graal occasionally makes use of.

**theory** *Long*  
**imports** *ValueThms*  
**begin**

**lemma** *negative-all-set-32*:  
 $n < 32 \implies \text{bit } (-1::\text{int}32) \ n$   
**apply** *transfer* **by** *auto*

**definition** *MaxOrNeg* :: *nat* *set*  $\Rightarrow$  *int* **where**  
*MaxOrNeg* *s* = (if *s* = {} then -1 else *Max* *s*)

**definition** *MinOrHighest* :: *nat* *set*  $\Rightarrow$  *nat*  $\Rightarrow$  *nat* **where**  
*MinOrHighest* *s* *m* = (if *s* = {} then *m* else *Min* *s*)

**definition** *highestOneBit* :: ('*a*::*len*) *word*  $\Rightarrow$  *int* **where**  
*highestOneBit* *v* = *MaxOrNeg* {*n* . *bit* *v* *n*}

**definition** *lowestOneBit* :: ('*a*::*len*) *word*  $\Rightarrow$  *nat* **where**  
*lowestOneBit* *v* = *MinOrHighest* {*n* . *bit* *v* *n*} (*size* *v*)

**lemma** *max-bit*: *bit* (*v*::('*a*::*len*) *word*) *n*  $\implies n < \text{size } v$   
**by** (*simp* *add*: *bit-imp-le-length* *size-word.rep-eq*)

**lemma** *max-set-bit*:  $\text{MaxOrNeg } \{n . \text{bit } (v::('a::\text{len}) \text{ word}) \ n\} < \text{Nat.size } v$   
**using** *max-bit* **unfolding** *MaxOrNeg-def*  
**by** *force*

**definition** *numberOfLeadingZeros* ::  $('a::\text{len}) \text{ word} \Rightarrow \text{nat}$  **where**  
*numberOfLeadingZeros*  $v = \text{nat } (\text{Nat.size } v - \text{highestOneBit } v - 1)$

**lemma** *MaxOrNeg-neg*:  $\text{MaxOrNeg } \{\} = -1$   
**by** (*simp add: MaxOrNeg-def*)

**lemma** *MaxOrNeg-max*:  $s \neq \{\} \implies \text{MaxOrNeg } s = \text{Max } s$   
**by** (*simp add: MaxOrNeg-def*)

**lemma** *zero-no-bits*:  
 $\{n . \text{bit } 0 \ n\} = \{\}$   
**by** *simp*

**lemma** *highestOneBit*  $(0::64 \text{ word}) = -1$   
**by** (*simp add: MaxOrNeg-neg highestOneBit-def*)

**lemma** *numberOfLeadingZeros*  $(0::64 \text{ word}) = 64$   
**unfolding** *numberOfLeadingZeros-def* **using** *MaxOrNeg-neg highestOneBit-def*  
*size64*  
**by** (*smt (verit) nat-int zero-no-bits*)

**lemma** *highestOneBit-top*:  $\text{Max } \{\text{highestOneBit } (v::64 \text{ word})\} < 64$   
**unfolding** *highestOneBit-def*  
**by** (*metis Max-singleton int-eq-iff-numeral max-set-bit size64*)

**lemma** *numberOfLeadingZeros-top*:  $\text{Max } \{\text{numberOfLeadingZeros } (v::64 \text{ word})\} \leq 64$   
**unfolding** *numberOfLeadingZeros-def*  
**using** *size64*  
**by** (*simp add: MaxOrNeg-def highestOneBit-def nat-le-iff*)

**lemma** *numberOfLeadingZeros-range*:  $0 \leq \text{numberOfLeadingZeros } a \wedge \text{numberOfLeadingZeros } a \leq \text{Nat.size } a$   
**unfolding** *numberOfLeadingZeros-def*  
**using** *MaxOrNeg-def highestOneBit-def nat-le-iff*  
**by** (*smt (verit) bot-nat-0.extremum int-eq-iff*)

**lemma** *leadingZerosAddHighestOne*:  $\text{numberOfLeadingZeros } v + \text{highestOneBit } v = \text{Nat.size } v - 1$   
**unfolding** *numberOfLeadingZeros-def highestOneBit-def*  
**using** *MaxOrNeg-def int-nat-eq int-ops(6) max-bit order-less-irrefl* **by** *fastforce*

**definition** *numberOfTrailingZeros* ::  $('a::\text{len}) \text{ word} \Rightarrow \text{nat}$  **where**  
*numberOfTrailingZeros*  $v = \text{lowestOneBit } v$

**lemma** *lowestOneBit-bot*:  $\text{lowestOneBit } (0::64 \text{ word}) = 64$   
**unfolding** *lowestOneBit-def MinOrHighest-def*  
**by** (*simp add: size64*)

**lemma** *bit-zero-set-in-top*:  $\text{bit } (-1::'a::\text{len word}) \ 0$   
**by** *auto*

**lemma** *nat-bot-set*:  $(0::\text{nat}) \in xs \longrightarrow (\forall x \in xs . 0 \leq x)$   
**by** *fastforce*

**lemma** *numberOfTrailingZeros*  $(0::64 \text{ word}) = 64$   
**unfolding** *numberOfTrailingZeros-def*  
**using** *lowestOneBit-bot* **by** *simp*

**definition** *bitCount* ::  $('a::\text{len}) \text{ word} \Rightarrow \text{nat}$  **where**  
*bitCount*  $v = \text{card } \{n . \text{bit } v \ n\}$

**lemma** *bitCount 0 = 0*  
**unfolding** *bitCount-def*  
**by** (*metis card.empty zero-no-bits*)

**definition** *zeroCount* ::  $('a::\text{len}) \text{ word} \Rightarrow \text{nat}$  **where**  
*zeroCount*  $v = \text{card } \{n . n < \text{Nat.size } v \wedge \neg(\text{bit } v \ n)\}$

**lemma** *zeroCount-finite*:  $\text{finite } \{n . n < \text{Nat.size } v \wedge \neg(\text{bit } v \ n)\}$   
**using** *finite-nat-set-iff-bounded* **by** *blast*

**lemma** *negone-set*:  
 $\text{bit } (-1::('a::\text{len}) \text{ word}) \ n \longleftrightarrow n < \text{LENGTH}('a)$   
**by** *simp*

**lemma** *negone-all-bits*:  
 $\{n . \text{bit } (-1::('a::\text{len}) \text{ word}) \ n\} = \{n . 0 \leq n \wedge n < \text{LENGTH}('a)\}$   
**using** *negone-set*  
**by** *auto*

**lemma** *bitCount-finite*:  
 $\text{finite } \{n . \text{bit } (v::('a::\text{len}) \text{ word}) \ n\}$   
**by** *simp*

**lemma** *card-of-range*:  
 $x = \text{card } \{n . 0 \leq n \wedge n < x\}$   
**by** *simp*

**lemma** *range-of-nat*:  
 $\{(n::\text{nat}) . 0 \leq n \wedge n < x\} = \{n . n < x\}$   
**by** *simp*

**lemma** *finite-range*:

```

finite {n::nat . n < x}
by simp

lemma range-eq:
  fixes x y :: nat
  shows card {y.. $x$ } = card {y<.. $x$ }
  using card-atLeastLessThan card-greaterThanAtMost by presburger

lemma card-of-range-bound:
  fixes x y :: nat
  assumes x > y
  shows x - y = card {n . y < n ∧ n ≤ x}
proof -
  have finite: finite {n . y ≤ n ∧ n < x}
  by auto
  have nonempty: {n . y ≤ n ∧ n < x} ≠ {}
  using assms by blast
  have simprep: {n . y < n ∧ n ≤ x} = {y<.. $x$ }
  by auto
  have x - y = card {y<.. $x$ }
  by auto
  then show ?thesis
  unfolding simprep by blast
qed

lemma bitCount (-1::('a::len) word) = LENGTH('a)
  unfolding bitCount-def using card-of-range
  by (metis (no-types, lifting) Collect-cong negone-all-bits)

lemma bitCount-range:
  fixes n :: ('a::len) word
  shows 0 ≤ bitCount n ∧ bitCount n ≤ Nat.size n
  unfolding bitCount-def
  by (metis atLeastLessThan-iff bot-nat-0.extremum max-bit mem-Collect-eq subsetI
  subset-eq-atLeast0-lessThan-card)

lemma zerosAboveHighestOne:
  n > highestOneBit a ⇒ ¬(bit a n)
  unfolding highestOneBit-def MaxOrNeg-def
  by (metis (mono-tags, opaque-lifting) Collect-empty-eq Max-ge finite-bit-word
  less-le-not-le mem-Collect-eq of-nat-le-iff)

lemma zerosBelowLowestOne:
  assumes n < lowestOneBit a
  shows ¬(bit a n)
proof (cases {i. bit a i} = {})
case True
  then show ?thesis by simp

```



```

next
  case False
  have  $n < \text{Min } (\text{Collect } (\text{bit } a)) \implies \neg \text{bit } a \ n$ 
  using False by auto
  then show ?thesis
  by (metis False MinOrHighest-def assms lowestOneBit-def)
qed

lemma union-bit-sets:
  fixes  $a :: ('a::\text{len}) \text{ word}$ 
  shows  $\{n . n < \text{Nat.size } a \wedge \text{bit } a \ n\} \cup \{n . n < \text{Nat.size } a \wedge \neg(\text{bit } a \ n)\} = \{n . n < \text{Nat.size } a\}$ 
  by fastforce

lemma disjoint-bit-sets:
  fixes  $a :: ('a::\text{len}) \text{ word}$ 
  shows  $\{n . n < \text{Nat.size } a \wedge \text{bit } a \ n\} \cap \{n . n < \text{Nat.size } a \wedge \neg(\text{bit } a \ n)\} = \{\}$ 
  by blast

lemma qualified-bitCount:
   $\text{bitCount } v = \text{card } \{n . n < \text{Nat.size } v \wedge \text{bit } v \ n\}$ 
  by (metis (no-types, lifting) Collect-cong bitCount-def max-bit)

lemma card-eq:
  assumes  $\text{finite } x \wedge \text{finite } y \wedge \text{finite } z$ 
  assumes  $x \cup y = z$ 
  assumes  $y \cap x = \{\}$ 
  shows  $\text{card } z - \text{card } y = \text{card } x$ 
  using assms add-diff-cancel-right' card-Un-disjoint
  by (metis inf.commute)

lemma card-add:
  assumes  $\text{finite } x \wedge \text{finite } y \wedge \text{finite } z$ 
  assumes  $x \cup y = z$ 
  assumes  $y \cap x = \{\}$ 
  shows  $\text{card } x + \text{card } y = \text{card } z$ 
  using assms card-Un-disjoint
  by (metis inf.commute)

lemma card-add-inverses:
  assumes  $\text{finite } \{n . Q \ n \wedge \neg(P \ n)\} \wedge \text{finite } \{n . Q \ n \wedge P \ n\} \wedge \text{finite } \{n . Q \ n\}$ 
  shows  $\text{card } \{n . Q \ n \wedge P \ n\} + \text{card } \{n . Q \ n \wedge \neg(P \ n)\} = \text{card } \{n . Q \ n\}$ 
  apply (rule card-add)
  using assms apply simp
  apply auto[1]
  by auto

lemma ones-zero-sum-to-width:

```

```

    bitCount a + zeroCount a = Nat.size a
  proof -
    have add-cards: card {n. (λn. n < size a) n ∧ (bit a n)} + card {n. (λn. n <
size a) n ∧ ¬(bit a n)} = card {n. (λn. n < size a) n}
    apply (rule card-add-inverses) by simp
    then have ... = Nat.size a
    by auto
  then show ?thesis
    unfolding bitCount-def zeroCount-def using max-bit
    by (metis (mono-tags, lifting) Collect-cong add-cards)
qed

```

```

lemma intersect-bitCount-helper:
  card {n . n < Nat.size a} - bitCount a = card {n . n < Nat.size a ∧ ¬(bit a n)}
proof -
  have size-def: Nat.size a = card {n . n < Nat.size a}
  using card-of-range by simp
  have bitCount-def: bitCount a = card {n . n < Nat.size a ∧ bit a n}
  using qualified-bitCount by auto
  have disjoint: {n . n < Nat.size a ∧ bit a n} ∩ {n . n < Nat.size a ∧ ¬(bit a
n)} = {}
  using disjoint-bit-sets by auto
  have union: {n . n < Nat.size a ∧ bit a n} ∪ {n . n < Nat.size a ∧ ¬(bit a n)}
= {n . n < Nat.size a}
  using union-bit-sets by auto
  show ?thesis
  unfolding bitCount-def
  apply (rule card-eq)
  using finite-range apply simp
  using union apply blast
  using disjoint by simp
qed

```

```

lemma intersect-bitCount:
  Nat.size a - bitCount a = card {n . n < Nat.size a ∧ ¬(bit a n)}
  using card-of-range intersect-bitCount-helper by auto

```

```

hide-fact intersect-bitCount-helper

```

```

end

```