

# Verifying term graph optimizations using Isabelle/HOL

Isabelle/HOL Theories

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## Abstract

Our objective is to formally verify the correctness of the hundreds of expression optimization rules used within the GraalVM compiler. When defining the semantics of a programming language, expressions naturally form abstract syntax trees, or, terms. However, in order to facilitate sharing of common subexpressions, modern compilers represent expressions as term graphs. Defining the semantics of term graphs is more complicated than defining the semantics of their equivalent term representations. More significantly, defining optimizations directly on term graphs and proving semantics preservation is considerably more complicated than on the equivalent term representations. On terms, optimizations can be expressed as conditional term rewriting rules, and proofs that the rewrites are semantics preserving are relatively straightforward. In this paper, we explore an approach to using term rewrites to verify term graph transformations of optimizations within the GraalVM compiler. This approach significantly reduces the overall verification effort and allows for simpler encoding of optimization rules.

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# 1 Operator Semantics

```
theory Values
imports
  HOL-Library.Word
  HOL-Library.Signed-Division
  HOL-Library.Float
  HOL-Library.LaTeXsugar
begin
```

In order to properly implement the IR semantics we first introduce a type that represents runtime values. These runtime values represent the full range of primitive types currently allowed by our semantics, ranging from basic integer types to object references and arrays.

Note that Java supports 64, 32, 16, 8 signed ints, plus 1 bit (boolean) ints, and char is 16-bit unsigned. E.g. an 8-bit stamp has a default range of -128..+127. And a 1-bit stamp has a default range of -1..0, surprisingly.

During calculations the smaller sizes are sign-extended to 32 bits, but explicit widening nodes will do that, so most binary calculations should see equal input sizes.

An object reference is an option type where the *None* object reference points to the static fields. This is examined more closely in our definition of the heap.

```
type-synonym int64 = 64 word — long
type-synonym int32 = 32 word — int
type-synonym int16 = 16 word — short
type-synonym int8 = 8 word — char
type-synonym int1 = 1 word — boolean
```

```
abbreviation valid-int-widths :: nat set where
  valid-int-widths  $\equiv \{1, 8, 16, 32, 64\}$ 
```

```
experiment begin
```

Option 2: explicit width stored with each integer value. However, this does not help us to distinguish between short (signed) and char (unsigned).

```
typedef IntWidth = { w :: nat . w=1  $\vee$  w=8  $\vee$  w=16  $\vee$  w=32  $\vee$  w=64 }
by blast
```

```
setup-lifting type-definition-IntWidth
```

```
lift-definition IntWidthBits :: IntWidth  $\Rightarrow$  nat
is  $\lambda w. w$  .
end
```

```
experiment begin
```

Option 3: explicit type stored with each integer value.

**datatype** *IntType* = *ILong* | *IInt* | *IShort* | *IChar* | *IByte* | *IBoolean*

**fun** *int-bits* :: *IntType*  $\Rightarrow$  *nat* **where**

*int-bits* *ILong* = 64 |  
*int-bits* *IInt* = 32 |  
*int-bits* *IShort* = 16 |  
*int-bits* *IChar* = 16 |  
*int-bits* *IByte* = 8 |  
*int-bits* *IBoolean* = 1

**fun** *int-signed* :: *IntType*  $\Rightarrow$  *bool* **where**

*int-signed* *ILong* = *True* |  
*int-signed* *IInt* = *True* |  
*int-signed* *IShort* = *True* |  
*int-signed* *IChar* = *False* |  
*int-signed* *IByte* = *True* |  
*int-signed* *IBoolean* = *True*

**end**

Option 4: int64 with the number of significant bits.

**type-synonym** *iwidth* = *nat*

**type-synonym** *objref* = *nat option*

**datatype** (*discs-sels*) *Value* =  
*UndefVal* |

*IntVal* *iwidth* *int64* |

*ObjRef* *objref* |  
*ObjStr* *string*

**fun** *intval-bits* :: *Value*  $\Rightarrow$  *nat* **where**

*intval-bits* (*IntVal* *b* *v*) = *b*

**fun** *intval-word* :: *Value*  $\Rightarrow$  *int64* **where**

*intval-word* (*IntVal* *b* *v*) = *v*

**fun** *bit-bounds* :: *nat*  $\Rightarrow$  (*int*  $\times$  *int*) **where**

*bit-bounds* *bits* = ((( $2^{\text{bits}}$ ) *div* 2) \* -1, (( $2^{\text{bits}}$ ) *div* 2) - 1)

**definition** *logic-negate* :: ('*a*::*len*) *word*  $\Rightarrow$  '*a* *word* **where**

*logic-negate* *x* = (if *x* = 0 then 1 else 0)

**fun** *int-signed-value* :: *iwidth*  $\Rightarrow$  *int64*  $\Rightarrow$  *int* **where**  
*int-signed-value* *b v* = *sint* (*signed-take-bit* (*b* - 1) *v*)

**fun** *int-unsigned-value* :: *iwidth*  $\Rightarrow$  *int64*  $\Rightarrow$  *int* **where**  
*int-unsigned-value* *b v* = *uint* *v*

Converts an integer word into a Java value.

**fun** *new-int* :: *iwidth*  $\Rightarrow$  *int64*  $\Rightarrow$  *Value* **where**  
*new-int* *b w* = *IntVal* *b* (*take-bit* *b w*)

Converts an integer word into a Java value, iff the two types are equal.

**fun** *new-int-bin* :: *iwidth*  $\Rightarrow$  *iwidth*  $\Rightarrow$  *int64*  $\Rightarrow$  *Value* **where**  
*new-int-bin* *b1 b2 w* = (if *b1=b2* then *new-int* *b1 w* else *UndefVal*)

**fun** *wf-bool* :: *Value*  $\Rightarrow$  *bool* **where**  
*wf-bool* (*IntVal* *b w*) = (*b* = 1) |  
*wf-bool* - = *False*

**fun** *val-to-bool* :: *Value*  $\Rightarrow$  *bool* **where**  
*val-to-bool* (*IntVal* *b val*) = (if *val* = 0 then *False* else *True*) |  
*val-to-bool* *val* = *False*

**fun** *bool-to-val* :: *bool*  $\Rightarrow$  *Value* **where**  
*bool-to-val* *True* = (*IntVal* 32 1) |  
*bool-to-val* *False* = (*IntVal* 32 0)

Converts an Isabelle bool into a Java value, iff the two types are equal.

**fun** *bool-to-val-bin* :: *iwidth*  $\Rightarrow$  *iwidth*  $\Rightarrow$  *bool*  $\Rightarrow$  *Value* **where**  
*bool-to-val-bin* *t1 t2 b* = (if *t1* = *t2* then *bool-to-val* *b* else *UndefVal*)

**fun** *is-int-val* :: *Value*  $\Rightarrow$  *bool* **where**  
*is-int-val* *v* = *is-IntVal* *v*

A convenience function for directly constructing -1 values of a given bit size.

**fun** *neg-one* :: *iwidth*  $\Rightarrow$  *int64* **where**  
*neg-one* *b* = *mask* *b*

**lemma** *neg-one-value[simp]*: *new-int* *b* (*neg-one* *b*) = *IntVal* *b* (*mask* *b*)  
**by** *simp*

**lemma** *neg-one-signed[simp]*:  
**assumes**  $0 < b$   
**shows** *int-signed-value* *b* (*neg-one* *b*) = -1

by (smt (verit, best) assms diff-le-self diff-less int-signed-value.simps less-one mask-eq-take-bit-minus-one neg-one.simps nle-le signed-minus-1 signed-take-bit-of-minus-1 signed-take-bit-take-bit verit-comp-simplify1(1))

## 1.1 Arithmetic Operators

We need to introduce arithmetic operations which agree with the JVM.

Within the JVM, bytecode arithmetic operations are performed on 32 or 64 bit integers, unboxing where appropriate.

The following collection of intval functions correspond to the JVM arithmetic operations. We merge the 32 and 64 bit operations into a single function, even though the stamp of each IRNode tells us exactly what the bit widths will be. These merged functions make it easier to do the instantiation of Value as 'plus', etc. It might be worse for reasoning, because it could cause more case analysis, but this does not seem to be a problem in practice.

```
fun intval-add :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-add (IntVal b1 v1) (IntVal b2 v2) =
    (if b1 = b2 then IntVal b1 (take-bit b1 (v1+v2)) else.UndefVal) |
  intval-add - - =.UndefVal
```

```
fun intval-sub :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-sub (IntVal b1 v1) (IntVal b2 v2) = new-int-bin b1 b2 (v1-v2) |
  intval-sub - - =.UndefVal
```

```
fun intval-mul :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-mul (IntVal b1 v1) (IntVal b2 v2) = new-int-bin b1 b2 (v1*v2) |
  intval-mul - - =.UndefVal
```

```
fun intval-div :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-div (IntVal b1 v1) (IntVal b2 v2) =
    new-int-bin b1 b2 (word-of-int
      ((int-signed-value b1 v1) sdiv (int-signed-value b2 v2))) |
  intval-div - - =.UndefVal
```

```
fun intval-mod :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-mod (IntVal b1 v1) (IntVal b2 v2) =
    new-int-bin b1 b2 (word-of-int
      ((int-signed-value b1 v1) smod (int-signed-value b2 v2))) |
  intval-mod - - =.UndefVal
```

```
fun intval-negate :: Value  $\Rightarrow$  Value where
```



```

intval-negate (IntVal t v) = new-int t (- v) |
intval-negate - = UndefVal

```

```

fun intval-abs :: Value ⇒ Value where
  intval-abs (IntVal t v) = new-int t (if int-signed-value t v < 0 then - v else v) |
  intval-abs - = UndefVal

```

TODO: clarify which widths this should work on: just 1-bit or all?

```

fun intval-logic-negation :: Value ⇒ Value where
  intval-logic-negation (IntVal b v) = new-int b (logic-negate v) |
  intval-logic-negation - = UndefVal

```

## 1.2 Bitwise Operators

```

fun intval-and :: Value ⇒ Value ⇒ Value where
  intval-and (IntVal b1 v1) (IntVal b2 v2) = new-int-bin b1 b2 (and v1 v2) |
  intval-and - - = UndefVal

```

```

fun intval-or :: Value ⇒ Value ⇒ Value where
  intval-or (IntVal b1 v1) (IntVal b2 v2) = new-int-bin b1 b2 (or v1 v2) |
  intval-or - - = UndefVal

```

```

fun intval-xor :: Value ⇒ Value ⇒ Value where
  intval-xor (IntVal b1 v1) (IntVal b2 v2) = new-int-bin b1 b2 (xor v1 v2) |
  intval-xor - - = UndefVal

```

```

fun intval-not :: Value ⇒ Value where
  intval-not (IntVal t v) = new-int t (not v) |
  intval-not - = UndefVal

```

## 1.3 Comparison Operators

```

fun intval-short-circuit-or :: Value ⇒ Value ⇒ Value where
  intval-short-circuit-or (IntVal b1 v1) (IntVal b2 v2) = bool-to-val-bin b1 b2 (((v1
  ≠ 0) ∨ (v2 ≠ 0))) |
  intval-short-circuit-or - - = UndefVal

```

```

fun intval-equals :: Value ⇒ Value ⇒ Value where
  intval-equals (IntVal b1 v1) (IntVal b2 v2) = bool-to-val-bin b1 b2 (v1 = v2) |
  intval-equals - - = UndefVal

```

```

fun intval-less-than :: Value ⇒ Value ⇒ Value where
  intval-less-than (IntVal b1 v1) (IntVal b2 v2) =
    bool-to-val-bin b1 b2 (int-signed-value b1 v1 < int-signed-value b2 v2) |
  intval-less-than - - = UndefVal

```

```

fun intval-below :: Value ⇒ Value ⇒ Value where
  intval-below (IntVal b1 v1) (IntVal b2 v2) = bool-to-val-bin b1 b2 (v1 < v2) |
  intval-below - - = UndefVal

```

```
fun intval-conditional :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-conditional cond tv fv = (if (val-to-bool cond) then tv else fv)
```

## 1.4 Narrowing and Widening Operators

Note: we allow these operators to have  $\text{inBits} = \text{outBits}$ , because the Graal compiler also seems to allow that case, even though it should rarely / never arise in practice.

```
value sint(signed-take-bit 0 (1 :: int32))
```

```
fun intval-narrow :: nat  $\Rightarrow$  nat  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-narrow inBits outBits (IntVal b v) =
    (if inBits = b  $\wedge$  0 < outBits  $\wedge$  outBits  $\leq$  inBits  $\wedge$  inBits  $\leq$  64
     then new-int outBits v
     else UndefVal) |
  intval-narrow - - - = UndefVal
```

```
value sint (signed-take-bit 7 ((256 + 128) :: int64))
```

```
fun intval-sign-extend :: nat  $\Rightarrow$  nat  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-sign-extend inBits outBits (IntVal b v) =
    (if inBits = b  $\wedge$  0 < inBits  $\wedge$  inBits  $\leq$  outBits  $\wedge$  outBits  $\leq$  64
     then new-int outBits (signed-take-bit (inBits - 1) v)
     else UndefVal) |
  intval-sign-extend - - - = UndefVal
```

```
fun intval-zero-extend :: nat  $\Rightarrow$  nat  $\Rightarrow$  Value  $\Rightarrow$  Value where
  intval-zero-extend inBits outBits (IntVal b v) =
    (if inBits = b  $\wedge$  0 < inBits  $\wedge$  inBits  $\leq$  outBits  $\wedge$  outBits  $\leq$  64
     then new-int outBits (take-bit inBits v)
     else UndefVal) |
  intval-zero-extend - - - = UndefVal
```

Some well-formedness results to help reasoning about narrowing and widening operators

**lemma** *intval-narrow-ok*:

```
assumes intval-narrow inBits outBits val  $\neq$  UndefVal
shows 0 < outBits  $\wedge$  outBits  $\leq$  inBits  $\wedge$  inBits  $\leq$  64  $\wedge$  outBits  $\leq$  64  $\wedge$ 
  is-IntVal val  $\wedge$ 
  intval-bits val = inBits
using assms intval-narrow.simps neq0-conv intval-bits.simps
by (metis Value.disc(2) intval-narrow.elims le-trans)
```

**lemma** *intval-sign-extend-ok*:

```
assumes intval-sign-extend inBits outBits val  $\neq$  UndefVal
shows 0 < inBits  $\wedge$ 
  inBits  $\leq$  outBits  $\wedge$  outBits  $\leq$  64  $\wedge$ 
```

```

    is-IntVal val ∧
    intval-bits val = inBits
  using assms intval-sign-extend.simps neq0-conv
  by (metis intval-bits.simps intval-sign-extend.elims is-IntVal-def)

```

```

lemma intval-zero-extend-ok:
  assumes intval-zero-extend inBits outBits val ≠ UndefVal
  shows 0 < inBits ∧
        inBits ≤ outBits ∧ outBits ≤ 64 ∧
        is-IntVal val ∧
        intval-bits val = inBits
  using assms intval-sign-extend.simps neq0-conv
  by (metis intval-bits.simps intval-zero-extend.elims is-IntVal-def)

```

## 1.5 Bit-Shifting Operators

```

definition shiftl (infix << 75) where
  shiftl w n = (push-bit n) w

```

```

lemma shiftl-power[simp]: (x::('a::len) word) * (2 ^ j) = x << j
  unfolding shiftl-def apply (induction j)
  apply simp unfolding funpow-Suc-right
  by (metis (no-types, opaque-lifting) push-bit-eq-mult)

```

```

lemma (x::('a::len) word) * ((2 ^ j) + 1) = x << j + x
  by (simp add: distrib-left)

```

```

lemma (x::('a::len) word) * ((2 ^ j) - 1) = x << j - x
  by (simp add: right-diff-distrib)

```

```

lemma (x::('a::len) word) * ((2 ^ j) + (2 ^ k)) = x << j + x << k
  by (simp add: distrib-left)

```

```

lemma (x::('a::len) word) * ((2 ^ j) - (2 ^ k)) = x << j - x << k
  by (simp add: right-diff-distrib)

```

```

definition shiftr (infix >>> 75) where
  shiftr w n = (drop-bit n) w

```

```

value (255 :: 8 word) >>> (2 :: nat)

```

```

definition sshiftr :: 'a :: len word ⇒ nat ⇒ 'a :: len word (infix >> 75) where
  sshiftr w n = word-of-int ((sint w) div (2 ^ n))

```

```

value (128 :: 8 word) >> 2

```

Note that Java shift operators use unary numeric promotion, unlike other binary operators, which use binary numeric promotion (see the Java lan-

guage reference manual). This means that the left-hand input determines the output size, while the right-hand input can be any size.

```
fun shift-amount :: iwidth ⇒ int64 ⇒ nat where
  shift-amount b val = unat (and val (if b = 64 then 0x3F else 0x1f))
```

```
fun intval-left-shift :: Value ⇒ Value ⇒ Value where
  intval-left-shift (IntVal b1 v1) (IntVal b2 v2) = new-int b1 (v1 << shift-amount
b1 v2) |
  intval-left-shift - - = UndefVal
```

Signed shift is more complex, because we sometimes have to insert 1 bits at the correct point, which is at b1 bits.

```
fun intval-right-shift :: Value ⇒ Value ⇒ Value where
  intval-right-shift (IntVal b1 v1) (IntVal b2 v2) =
    (let shift = shift-amount b1 v2 in
     let ones = and (mask b1) (not (mask (b1 - shift) :: int64)) in
     (if int-signed-value b1 v1 < 0
      then new-int b1 (or ones (v1 >>> shift))
      else new-int b1 (v1 >>> shift))) |
  intval-right-shift - - = UndefVal
```

```
fun intval-uright-shift :: Value ⇒ Value ⇒ Value where
  intval-uright-shift (IntVal b1 v1) (IntVal b2 v2) = new-int b1 (v1 >>> shift-amount
b1 v2) |
  intval-uright-shift - - = UndefVal
```

### 1.5.1 Examples of Narrowing / Widening Functions

**experiment begin**

**corollary** intval-narrow 32 8 (IntVal 32 (256 + 128)) = IntVal 8 128 **by simp**

**corollary** intval-narrow 32 8 (IntVal 32 (-2)) = IntVal 8 254 **by simp**

**corollary** intval-narrow 32 1 (IntVal 32 (-2)) = IntVal 1 0 **by simp**

**corollary** intval-narrow 32 1 (IntVal 32 (-3)) = IntVal 1 1 **by simp**

**corollary** intval-narrow 32 8 (IntVal 64 (-2)) = UndefVal **by simp**

**corollary** intval-narrow 64 8 (IntVal 32 (-2)) = UndefVal **by simp**

**corollary** intval-narrow 64 8 (IntVal 64 254) = IntVal 8 254 **by simp**

**corollary** intval-narrow 64 8 (IntVal 64 (256+127)) = IntVal 8 127 **by simp**

**corollary** intval-narrow 64 64 (IntVal 64 (-2)) = IntVal 64 (-2) **by simp**

**end**

**experiment begin**

**corollary** intval-sign-extend 8 32 (IntVal 8 (256 + 128)) = IntVal 32 (2<sup>32</sup> - 128) **by simp**

**corollary** intval-sign-extend 8 32 (IntVal 8 (-2)) = IntVal 32 (2<sup>32</sup> - 2) **by simp**

**corollary** intval-sign-extend 1 32 (IntVal 1 (-2)) = IntVal 32 0 **by simp**

**corollary** intval-sign-extend 1 32 (IntVal 1 (-3)) = IntVal 32 (mask 32) **by simp**

```

corollary intval-sign-extend 8 32 (IntVal 64 254) = UndefVal by simp
corollary intval-sign-extend 8 64 (IntVal 32 254) = UndefVal by simp
corollary intval-sign-extend 8 64 (IntVal 8 254) = IntVal 64 (-2) by simp
corollary intval-sign-extend 32 64 (IntVal 32 ( $2^{32} - 2$ )) = IntVal 64 (-2) by
simp
corollary intval-sign-extend 64 64 (IntVal 64 (-2)) = IntVal 64 (-2) by simp
end

```

**experiment begin**

```

corollary intval-zero-extend 8 32 (IntVal 8 (256 + 128)) = IntVal 32 128 by simp
corollary intval-zero-extend 8 32 (IntVal 8 (-2)) = IntVal 32 254 by simp
corollary intval-zero-extend 1 32 (IntVal 1 (-1)) = IntVal 32 1 by simp
corollary intval-zero-extend 1 32 (IntVal 1 (-2)) = IntVal 32 0 by simp

```

```

corollary intval-zero-extend 8 32 (IntVal 64 (-2)) = UndefVal by simp
corollary intval-zero-extend 8 64 (IntVal 64 (-2)) = UndefVal by simp
corollary intval-zero-extend 8 64 (IntVal 8 254) = IntVal 64 254 by simp
corollary intval-zero-extend 32 64 (IntVal 32 ( $2^{32} - 2$ )) = IntVal 64 ( $2^{32} - 2$ ) by simp
corollary intval-zero-extend 64 64 (IntVal 64 (-2)) = IntVal 64 (-2) by simp
end

```

**experiment begin**

```

corollary intval-right-shift (IntVal 8 128) (IntVal 8 0) = IntVal 8 128 by eval
corollary intval-right-shift (IntVal 8 128) (IntVal 8 1) = IntVal 8 192 by eval
corollary intval-right-shift (IntVal 8 128) (IntVal 8 2) = IntVal 8 224 by eval
corollary intval-right-shift (IntVal 8 128) (IntVal 8 8) = IntVal 8 255 by eval
corollary intval-right-shift (IntVal 8 128) (IntVal 8 31) = IntVal 8 255 by eval
end

```

**lemma** *intval-add-sym*:

```

  shows intval-add a b = intval-add b a
  by (induction a; induction b; auto simp: add.commute)

```

```

code-deps intval-add
code-thms intval-add

```

```

lemma intval-add (IntVal 32 ( $2^{31}-1$ )) (IntVal 32 ( $2^{31}-1$ )) = IntVal 32 ( $2^{32} - 2$ )
by eval

```

```

lemma intval-add (IntVal 64 ( $2^{31}-1$ )) (IntVal 64 ( $2^{31}-1$ )) = IntVal 64 4294967294
  by eval

```

```

end

```

## 1.6 Fixed-width Word Theories

```

theory ValueThms
  imports Values
begin

```

### 1.6.1 Support Lemmas for Upper/Lower Bounds

```

lemma size32: size v = 32 for v :: 32 word
  using size-word.rep-eq
  using One-nat-def add.right-neutral add-Suc-right len-of-numeral-defs(2) len-of-numeral-defs(3)
  mult.right-neutral mult-Suc-right numeral-2-eq-2 numeral-Bit0
  by (smt (verit, del-insts) mult.commute)

```

```

lemma size64: size v = 64 for v :: 64 word
  using size-word.rep-eq
  using One-nat-def add.right-neutral add-Suc-right len-of-numeral-defs(2) len-of-numeral-defs(3)
  mult.right-neutral mult-Suc-right numeral-2-eq-2 numeral-Bit0
  by (smt (verit, del-insts) mult.commute)

```

```

lemma lower-bounds-equiv:
  assumes  $0 < N$ 
  shows  $\neg((2::int) \wedge (N-1)) = (2::int) \wedge N \text{ div } 2 * - 1$ 
  by (simp add: assms int-power-div-base)

```

```

lemma upper-bounds-equiv:
  assumes  $0 < N$ 
  shows  $(2::int) \wedge (N-1) = (2::int) \wedge N \text{ div } 2$ 
  by (simp add: assms int-power-div-base)

```

Some min/max bounds for 64-bit words

```

lemma bit-bounds-min64: ((fst (bit-bounds 64))) ≤ (sint (v::int64))
  unfolding bit-bounds.simps fst-def
  using sint-ge[of v] by simp

```

```

lemma bit-bounds-max64: ((snd (bit-bounds 64))) ≥ (sint (v::int64))
  unfolding bit-bounds.simps fst-def
  using sint-lt[of v] by simp

```

Extend these min/max bounds to extracting smaller signed words using *signed\_take\_bit*.

Note: we could use `signed` to convert between bit-widths, instead of `signed_take_bit`. But that would have to be done separately for each bit-width type.

```
value sint(signed-take-bit 7 (128 :: int8))
```

```
ML-val <@{thm signed-take-bit-decr-length-iff}>
declare [[show-types=true]]
ML-val <@{thm signed-take-bit-int-less-exp}>
```

```
lemma signed-take-bit-int-less-exp-word:
  fixes ival :: 'a :: len word
  assumes n < LENGTH('a)
  shows sint(signed-take-bit n ival) < (2::int) ^ n
  apply transfer
  by (smt (verit, best) not-take-bit-negative signed-take-bit-eq-take-bit-shift
      signed-take-bit-int-less-exp take-bit-int-greater-self-iff)
```

```
lemma signed-take-bit-int-greater-eq-minus-exp-word:
  fixes ival :: 'a :: len word
  assumes n < LENGTH('a)
  shows - (2 ^ n) ≤ sint(signed-take-bit n ival)
  apply transfer
  by (smt (verit, best) signed-take-bit-int-greater-eq-minus-exp
      signed-take-bit-int-greater-eq-self-iff signed-take-bit-int-less-exp)
```

```
lemma signed-take-bit-range:
  fixes ival :: 'a :: len word
  assumes n < LENGTH('a)
  assumes val = sint(signed-take-bit n ival)
  shows - (2 ^ n) ≤ val ∧ val < 2 ^ n
  using signed-take-bit-int-greater-eq-minus-exp-word signed-take-bit-int-less-exp-word
  using assms by blast
```

A `bit_bounds` version of the above lemma.

```
lemma signed-take-bit-bounds:
  fixes ival :: 'a :: len word
  assumes n ≤ LENGTH('a)
  assumes 0 < n
  assumes val = sint(signed-take-bit (n - 1) ival)
  shows fst (bit-bounds n) ≤ val ∧ val ≤ snd (bit-bounds n)
  using assms signed-take-bit-range lower-bounds-equiv upper-bounds-equiv
  by (metis bit-bounds.simps fst-conv less-imp-diff-less nat-less-le sint-ge sint-lt
      snd-conv zle-diff1-eq)
```

```
lemma signed-take-bit-bounds64:
  fixes ival :: int64
  assumes n ≤ 64
```

```

assumes  $0 < n$ 
assumes  $val = \text{sint}(\text{signed-take-bit } (n - 1) \text{ ival})$ 
shows  $\text{fst } (\text{bit-bounds } n) \leq val \wedge val \leq \text{snd } (\text{bit-bounds } n)$ 
using assms signed-take-bit-bounds
by (metis size64 word-size)

```

```

lemma int-signed-value-bounds:
assumes  $b1 \leq 64$ 
assumes  $0 < b1$ 
shows  $\text{fst } (\text{bit-bounds } b1) \leq \text{int-signed-value } b1 \text{ v2} \wedge$ 
 $\text{int-signed-value } b1 \text{ v2} \leq \text{snd } (\text{bit-bounds } b1)$ 
using assms int-signed-value.simps signed-take-bit-bounds64 by blast

```

```

lemma int-signed-value-range:
fixes  $ival :: \text{int64}$ 
assumes  $val = \text{int-signed-value } n \text{ ival}$ 
shows  $-(2^{(n-1)}) \leq val \wedge val < 2^n$ 
using signed-take-bit-range assms
by (smt (verit, ccfv-SIG) One-nat-def diff-less int-signed-value.elims len-gt-0
 $\text{len-num1 power-less-imp-less-exp power-strict-increasing sint-greater-eq sint-less}$ )

```

Some lemmas about unsigned words smaller than 64-bit, for zero-extend operators.

```

lemma take-bit-smaller-range:
fixes  $ival :: 'a :: \text{len word}$ 
assumes  $n < \text{LENGTH}('a)$ 
assumes  $val = \text{sint}(\text{take-bit } n \text{ ival})$ 
shows  $0 \leq val \wedge val < (2::\text{int})^n$ 
by (simp add: assms signed-take-bit-eq)

```

```

lemma take-bit-same-size-nochange:
fixes  $ival :: 'a :: \text{len word}$ 
assumes  $n = \text{LENGTH}('a)$ 
shows  $ival = \text{take-bit } n \text{ ival}$ 
by (simp add: assms)

```

A simplification lemma for *new\_int*, showing that upper bits can be ignored.

```

lemma take-bit-redundant[simp]:
fixes  $ival :: 'a :: \text{len word}$ 
assumes  $0 < n$ 
assumes  $n < \text{LENGTH}('a)$ 
shows  $\text{signed-take-bit } (n - 1) (\text{take-bit } n \text{ ival}) = \text{signed-take-bit } (n - 1) \text{ ival}$ 
proof -
have  $\neg (n \leq n - 1)$  using assms by arith
then have  $\bigwedge i. \text{signed-take-bit } (n - 1) (\text{take-bit } n \text{ i}) = \text{signed-take-bit } (n - 1) \text{ i}$ 
using signed-take-bit-take-bit by (metis (mono-tags))
then show ?thesis
by blast
qed

```



```

lemma take-bit-same-size-range:
  fixes ival :: 'a :: len word
  assumes n = LENGTH('a)
  assumes ival2 = take-bit n ival
  shows  $-(2 \wedge n \text{ div } 2) \leq \text{sint } \text{ival2} \wedge \text{sint } \text{ival2} < 2 \wedge n \text{ div } 2$ 
  using assms lower-bounds-equiv sint-ge sint-lt by auto

lemma take-bit-same-bounds:
  fixes ival :: 'a :: len word
  assumes n = LENGTH('a)
  assumes ival2 = take-bit n ival
  shows fst (bit-bounds n)  $\leq \text{sint } \text{ival2} \wedge \text{sint } \text{ival2} \leq \text{snd } (\text{bit-bounds } n)$ 
  unfolding bit-bounds.simps
  using assms take-bit-same-size-range
  by force

```

Next we show that casting a word to a wider word preserves any upper/lower bounds. (These lemmas may not be needed any more, since we are not using *scast* now?)

```

lemma scast-max-bound:
  assumes sint (v :: 'a :: len word) < M
  assumes LENGTH('a) < LENGTH('b)
  shows sint ((scast v) :: 'b :: len word) < M
  unfolding Word.scast-eq Word.sint-sbintrunc'
  using Bit-Operations.signed-take-bit-int-eq-self-iff
  by (smt (verit, best) One-nat-def assms(1) assms(2) decr-length-less-iff linorder-not-le
power-strict-increasing-iff signed-take-bit-int-less-self-iff sint-greater-eq)

```

```

lemma scast-min-bound:
  assumes M  $\leq \text{sint } (v :: 'a :: len word)$ 
  assumes LENGTH('a) < LENGTH('b)
  shows M  $\leq \text{sint } ((\text{scast } v) :: 'b :: len word)$ 
  unfolding Word.scast-eq Word.sint-sbintrunc'
  using Bit-Operations.signed-take-bit-int-eq-self-iff
  by (smt (verit) One-nat-def Suc-pred assms(1) assms(2) len-gt-0 less-Suc-eq or-
der-less-le order-less-le-trans power-le-imp-le-exp signed-take-bit-int-greater-eq-self-iff
sint-lt)

```

```

lemma scast-bigger-max-bound:
  assumes (result :: 'b :: len word) = scast (v :: 'a :: len word)
  shows sint result <  $2 \wedge \text{LENGTH}('a) \text{ div } 2$ 
  using sint-lt upper-bounds-equiv scast-max-bound
  by (smt (verit, best) assms(1) len-gt-0 signed-scast-eq signed-take-bit-int-greater-self-iff
sint-ge sint-less upper-bounds-equiv)

```

```

lemma scast-bigger-min-bound:
  assumes (result :: 'b :: len word) = scast (v :: 'a :: len word)

```

**shows**  $-(2 \wedge \text{LENGTH}('a) \text{ div } 2) \leq \text{sint result}$   
**using** *sint-ge lower-bounds-equiv scast-min-bound*  
**by** (*smt (verit) assms len-gt-0 nat-less-le not-less scast-max-bound*)

**lemma** *scast-bigger-bit-bounds*:  
**assumes** (*result :: 'b :: len word*) = *scast (v :: 'a :: len word)*  
**shows** *fst (bit-bounds (LENGTH('a))) ≤ sint result ∧ sint result ≤ snd (bit-bounds (LENGTH('a)))*  
**using** *assms scast-bigger-min-bound scast-bigger-max-bound*  
**by** *auto*

Results about *new\_int*.

**lemma** *new-int-take-bits*:  
**assumes** *IntVal b val = new-int b ival*  
**shows** *take-bit b val = val*  
**using** *assms* **by** *force*

### 1.6.2 Support lemmas for take bit and signed take bit.

Lemmas for removing redundant take\_bit wrappers.

**lemma** *take-bit-dist-addL[simp]*:  
**fixes** *x :: 'a :: len word*  
**shows** *take-bit b (take-bit b x + y) = take-bit b (x + y)*  
**proof** (*induction b*)  
**case** *0*  
**then show** *?case*  
**by** *simp*  
**next**  
**case** (*Suc b*)  
**then show** *?case*  
**by** (*simp add: add.commute mask-egs(2) take-bit-eq-mask*)  
**qed**

**lemma** *take-bit-dist-addR[simp]*:  
**fixes** *x :: 'a :: len word*  
**shows** *take-bit b (x + take-bit b y) = take-bit b (x + y)*  
**using** *take-bit-dist-addL* **by** (*metis add.commute*)

**lemma** *take-bit-dist-subL[simp]*:  
**fixes** *x :: 'a :: len word*  
**shows** *take-bit b (take-bit b x - y) = take-bit b (x - y)*  
**by** (*metis take-bit-dist-addR uminus-add-conv-diff*)

**lemma** *take-bit-dist-subR[simp]*:  
**fixes** *x :: 'a :: len word*  
**shows** *take-bit b (x - take-bit b y) = take-bit b (x - y)*  
**using** *take-bit-dist-subL*  
**by** (*metis (no-types, opaque-lifting) diff-add-cancel diff-right-commute diff-self*)

```

lemma take-bit-dist-neg[simp]:
  fixes ix :: 'a :: len word
  shows take-bit b ( $-$  take-bit b (ix)) = take-bit b ( $-$  ix)
  by (metis diff-0 take-bit-dist-subR)

lemma signed-take-take-bit[simp]:
  fixes x :: 'a :: len word
  assumes  $0 < b$ 
  shows signed-take-bit ( $b - 1$ ) (take-bit b x) = signed-take-bit ( $b - 1$ ) x
  by (smt (verit, best) Suc-diff-1 assms lessI linorder-not-less signed-take-bit-take-bit)

lemma mod-larger-ignore:
  fixes a :: int
  fixes m n :: nat
  assumes  $n < m$ 
  shows (a mod  $2^m$ ) mod  $2^n$  = a mod  $2^n$ 
  by (smt (verit, del-insts) assms exp-mod-exp linorder-not-le mod-0-imp-dvd mod-mod-cancel
mod-self order-less-imp-le)

lemma mod-dist-over-add:
  fixes a b c :: int64
  fixes n :: nat
  assumes  $1: 0 < n$ 
  assumes  $2: n < 64$ 
  shows (a mod  $2^n$  + b) mod  $2^n$  = (a + b) mod  $2^n$ 
proof –
  have  $3: (0 :: \text{int64}) < 2^n$ 
  using assms by (simp add: size64 word-2p-lem)
  then show ?thesis
  unfolding word-mod-2p-is-mask[OF 3]
  apply transfer
  by (metis (no-types, opaque-lifting) and.right-idem take-bit-add take-bit-eq-mask)
qed

end

```

## 2 Stamp Typing

```

theory Stamp
  imports Values
begin

```

The GraalVM compiler uses the Stamp class to store range and type information for a given node in the IR graph. We model the Stamp class as a

datatype, Stamp, and provide a number of functions on the datatype which correspond to the class methods within the compiler.

Stamp information is used in a variety of ways in optimizations, and so, we additionally provide a number of lemmas which help to prove future optimizations.

```
datatype Stamp =
  VoidStamp
| IntegerStamp (stp-bits: nat) (stpi-lower: int) (stpi-upper: int)

| KlassPointerStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
| MethodCountersPointerStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
| MethodPointersStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
| ObjectStamp (stp-type: string) (stp-exactType: bool) (stp-nonNull: bool) (stp-alwaysNull:
bool)
| RawPointerStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
| IllegalStamp
```

```
fun is-stamp-empty :: Stamp  $\Rightarrow$  bool where
  is-stamp-empty (IntegerStamp b lower upper) = (upper < lower) |

  is-stamp-empty x = False
```

Just like the IntegerStamp class, we need to know that our lo/hi bounds fit into the given number of bits (either signed or unsigned). Our integer stamps have infinite lo/hi bounds, so if the lower bound is non-negative, we can assume that all values are positive, and the integer bits of a related value can be interpreted as unsigned. This is similar (but slightly more general) to what IntegerStamp.java does with its test: if (sameSignBounds()) in the unsignedUpperBound() method.

Note that this is a bit different and more accurate than what StampFactory.forUnsignedInteger does (it widens large unsigned ranges to the max signed range to allow all bit patterns) because its lo/hi values are only 64-bit.

```
fun valid-stamp :: Stamp  $\Rightarrow$  bool where
  valid-stamp (IntegerStamp bits lo hi) =
    (0 < bits  $\wedge$  bits  $\leq$  64  $\wedge$ 
     fst (bit-bounds bits)  $\leq$  lo  $\wedge$  lo  $\leq$  snd (bit-bounds bits)  $\wedge$ 
     fst (bit-bounds bits)  $\leq$  hi  $\wedge$  hi  $\leq$  snd (bit-bounds bits)) |
  valid-stamp s = True
```

**experiment begin**

```
corollary bit-bounds 1 = (-1, 0) by simp
end
```

— A stamp which includes the full range of the type

```
fun unrestricted-stamp :: Stamp ⇒ Stamp where
  unrestricted-stamp VoidStamp = VoidStamp |
  unrestricted-stamp (IntegerStamp bits lower upper) = (IntegerStamp bits (fst
(bit-bounds bits)) (snd (bit-bounds bits))) |

  unrestricted-stamp (KlassPointerStamp nonNull alwaysNull) = (KlassPointerStamp
False False) |
  unrestricted-stamp (MethodCountersPointerStamp nonNull alwaysNull) = (MethodCountersPointerStamp
False False) |
  unrestricted-stamp (MethodPointersStamp nonNull alwaysNull) = (MethodPointersStamp
False False) |
  unrestricted-stamp (ObjectStamp type exactType nonNull alwaysNull) = (ObjectStamp
"" False False False) |
  unrestricted-stamp - = IllegalStamp
```

```
fun is-stamp-unrestricted :: Stamp ⇒ bool where
  is-stamp-unrestricted s = (s = unrestricted-stamp s)
```

— A stamp which provides type information but has an empty range of values

```
fun empty-stamp :: Stamp ⇒ Stamp where
  empty-stamp VoidStamp = VoidStamp |
  empty-stamp (IntegerStamp bits lower upper) = (IntegerStamp bits (snd (bit-bounds
bits)) (fst (bit-bounds bits))) |

  empty-stamp (KlassPointerStamp nonNull alwaysNull) = (KlassPointerStamp
nonNull alwaysNull) |
  empty-stamp (MethodCountersPointerStamp nonNull alwaysNull) = (MethodCountersPointerStamp
nonNull alwaysNull) |
  empty-stamp (MethodPointersStamp nonNull alwaysNull) = (MethodPointersStamp
nonNull alwaysNull) |
  empty-stamp (ObjectStamp type exactType nonNull alwaysNull) = (ObjectStamp
"" True True False) |
  empty-stamp stamp = IllegalStamp
```

— Calculate the meet stamp of two stamps

```
fun meet :: Stamp ⇒ Stamp ⇒ Stamp where
  meet VoidStamp VoidStamp = VoidStamp |
  meet (IntegerStamp b1 l1 u1) (IntegerStamp b2 l2 u2) = (
    if b1 ≠ b2 then IllegalStamp else
    (IntegerStamp b1 (min l1 l2) (max u1 u2))
  ) |
```

```

meet (KlassPointerStamp nn1 an1) (KlassPointerStamp nn2 an2) = (
  KlassPointerStamp (nn1  $\wedge$  nn2) (an1  $\wedge$  an2)
) |
meet (MethodCountersPointerStamp nn1 an1) (MethodCountersPointerStamp
nn2 an2) = (
  MethodCountersPointerStamp (nn1  $\wedge$  nn2) (an1  $\wedge$  an2)
) |
meet (MethodPointersStamp nn1 an1) (MethodPointersStamp nn2 an2) = (
  MethodPointersStamp (nn1  $\wedge$  nn2) (an1  $\wedge$  an2)
) |
meet s1 s2 = IllegalStamp

```

— Calculate the join stamp of two stamps

```

fun join :: Stamp  $\Rightarrow$  Stamp  $\Rightarrow$  Stamp where
  join VoidStamp VoidStamp = VoidStamp |
  join (IntegerStamp b1 l1 u1) (IntegerStamp b2 l2 u2) = (
    if b1  $\neq$  b2 then IllegalStamp else
    (IntegerStamp b1 (max l1 l2) (min u1 u2))
  ) |
  join (KlassPointerStamp nn1 an1) (KlassPointerStamp nn2 an2) = (
    if ((nn1  $\vee$  nn2)  $\wedge$  (an1  $\vee$  an2))
    then (empty-stamp (KlassPointerStamp nn1 an1))
    else (KlassPointerStamp (nn1  $\vee$  nn2) (an1  $\vee$  an2))
  ) |
  join (MethodCountersPointerStamp nn1 an1) (MethodCountersPointerStamp nn2
an2) = (
    if ((nn1  $\vee$  nn2)  $\wedge$  (an1  $\vee$  an2))
    then (empty-stamp (MethodCountersPointerStamp nn1 an1))
    else (MethodCountersPointerStamp (nn1  $\vee$  nn2) (an1  $\vee$  an2))
  ) |
  join (MethodPointersStamp nn1 an1) (MethodPointersStamp nn2 an2) = (
    if ((nn1  $\vee$  nn2)  $\wedge$  (an1  $\vee$  an2))
    then (empty-stamp (MethodPointersStamp nn1 an1))
    else (MethodPointersStamp (nn1  $\vee$  nn2) (an1  $\vee$  an2))
  ) |
  join s1 s2 = IllegalStamp

```

— In certain circumstances a stamp provides enough information to evaluate a value as a stamp, the asConstant function converts the stamp to a value where one can be inferred.

```

fun asConstant :: Stamp  $\Rightarrow$  Value where
  asConstant (IntegerStamp b l h) = (if l = h then IntVal b (word-of-int l) else
UndefVal) |
  asConstant - = UndefVal

```

— Determine if two stamps never have value overlaps i.e. their join is empty

```

fun alwaysDistinct :: Stamp  $\Rightarrow$  Stamp  $\Rightarrow$  bool where

```

*alwaysDistinct stamp1 stamp2 = is-stamp-empty (join stamp1 stamp2)*

— Determine if two stamps must always be the same value i.e. two equal constants

**fun** *neverDistinct* :: *Stamp*  $\Rightarrow$  *Stamp*  $\Rightarrow$  *bool* **where**  
*neverDistinct stamp1 stamp2 = (asConstant stamp1 = asConstant stamp2  $\wedge$*   
*asConstant stamp1  $\neq$  UndefVal)*

**fun** *constantAsStamp* :: *Value*  $\Rightarrow$  *Stamp* **where**  
*constantAsStamp (IntVal b v) = (IntegerStamp b (int-signed-value b v) (int-signed-value*  
*b v)) |*

*constantAsStamp - = IllegalStamp*

— Define when a runtime value is valid for a stamp. The stamp bounds must be valid, and val must be zero-extended.

**fun** *valid-value* :: *Value*  $\Rightarrow$  *Stamp*  $\Rightarrow$  *bool* **where**  
*valid-value (IntVal b1 val) (IntegerStamp b l h) =*  
*(if b1 = b then*  
*valid-stamp (IntegerStamp b l h)  $\wedge$*   
*take-bit b val = val  $\wedge$*   
*l  $\leq$  int-signed-value b val  $\wedge$  int-signed-value b val  $\leq$  h*  
*else False) |*

*valid-value (ObjRef ref) (ObjectStamp klass exact nonNull alwaysNull) =*  
*((alwaysNull  $\longrightarrow$  ref = None)  $\wedge$  (ref=None  $\longrightarrow$   $\neg$  nonNull)) |*  
*valid-value stamp val = False*

**definition** *wf-value* :: *Value*  $\Rightarrow$  *bool* **where**  
*wf-value v = valid-value v (constantAsStamp v)*

**lemma** *unfold-wf-value[simp]*:  
*wf-value v  $\Longrightarrow$  valid-value v (constantAsStamp v)*  
**using** *wf-value-def* **by** *auto*

**fun** *compatible* :: *Stamp*  $\Rightarrow$  *Stamp*  $\Rightarrow$  *bool* **where**  
*compatible (IntegerStamp b1 lo1 hi1) (IntegerStamp b2 lo2 hi2) =*  
*(b1 = b2  $\wedge$  valid-stamp (IntegerStamp b1 lo1 hi1)  $\wedge$  valid-stamp (IntegerStamp*  
*b2 lo2 hi2)) |*  
*compatible (VoidStamp) (VoidStamp) = True |*  
*compatible - - = False*

**fun** *stamp-under* :: *Stamp*  $\Rightarrow$  *Stamp*  $\Rightarrow$  *bool* **where**  
*stamp-under (IntegerStamp b1 lo1 hi1) (IntegerStamp b2 lo2 hi2) = (b1 = b2  $\wedge$*   
*hi1 < lo2) |*  
*stamp-under - - = False*

— The most common type of stamp within the compiler (apart from the Void-Stamp) is a 32 bit integer stamp with an unrestricted range. We use *default-stamp* as it is a frequently used stamp.

**definition** *default-stamp* :: *Stamp* **where**  
*default-stamp* = (*unrestricted-stamp* (*IntegerStamp* 32 0 0))

**value** *valid-value* (*IntVal* 8 (255)) (*IntegerStamp* 8 (−128) 127)  
**end**

## 3 Graph Representation

### 3.1 IR Graph Nodes

**theory** *IRNodes*  
**imports**  
*Values*  
**begin**

The GraalVM IR is represented using a graph data structure. Here we define the nodes that are contained within the graph. Each node represents a Node subclass in the GraalVM compiler, the node classes have annotated fields to indicate input and successor edges.

We represent these classes with each *IRNode* constructor explicitly labelling a reference to the node IDs that it stores as inputs and successors.

The *inputs\_of* and *successors\_of* functions partition those labelled references into input edges and successor edges of a node.

To identify each Node, we use a simple natural number index. Zero is always the start node in a graph. For human readability, within nodes we write *INPUT* (or special case thereof) instead of *ID* for input edges, and *SUCC* instead of *ID* for control-flow successor edges. Optional edges are handled as "INPUT option" etc.

**type-synonym** *ID* = *nat*  
**type-synonym** *INPUT* = *ID*  
**type-synonym** *INPUT-ASSOC* = *ID*  
**type-synonym** *INPUT-STATE* = *ID*  
**type-synonym** *INPUT-GUARD* = *ID*  
**type-synonym** *INPUT-COND* = *ID*  
**type-synonym** *INPUT-EXT* = *ID*  
**type-synonym** *SUCC* = *ID*

**datatype** (*discs-sels*) *IRNode* =  
*AbsNode* (*ir-value*: *INPUT*)  
| *AddNode* (*ir-x*: *INPUT*) (*ir-y*: *INPUT*)  
| *AndNode* (*ir-x*: *INPUT*) (*ir-y*: *INPUT*)  
| *BeginNode* (*ir-next*: *SUCC*)



- | *BytecodeExceptionNode* (*ir-arguments*: INPUT list) (*ir-stateAfter-opt*: INPUT-STATE option) (*ir-next*: SUCC)
- | *ConditionalNode* (*ir-condition*: INPUT-COND) (*ir-trueValue*: INPUT) (*ir-falseValue*: INPUT)
- | *ConstantNode* (*ir-const*: Value)
- | *DynamicNewArrayNode* (*ir-elementType*: INPUT) (*ir-length*: INPUT) (*ir-voidClass-opt*: INPUT option) (*ir-stateBefore-opt*: INPUT-STATE option) (*ir-next*: SUCC)
- | *EndNode*
- | *ExceptionObjectNode* (*ir-stateAfter-opt*: INPUT-STATE option) (*ir-next*: SUCC)
  
- | *FrameState* (*ir-monitorIds*: INPUT-ASSOC list) (*ir-outerFrameState-opt*: INPUT-STATE option) (*ir-values-opt*: INPUT list option) (*ir-virtualObjectMappings-opt*: INPUT-STATE list option)
- | *IfNode* (*ir-condition*: INPUT-COND) (*ir-trueSuccessor*: SUCC) (*ir-falseSuccessor*: SUCC)
- | *IntegerBelowNode* (*ir-x*: INPUT) (*ir-y*: INPUT)
- | *IntegerEqualsNode* (*ir-x*: INPUT) (*ir-y*: INPUT)
- | *IntegerLessThanNode* (*ir-x*: INPUT) (*ir-y*: INPUT)
- | *InvokeNode* (*ir-nid*: ID) (*ir-callTarget*: INPUT-EXT) (*ir-classInit-opt*: INPUT option) (*ir-stateDuring-opt*: INPUT-STATE option) (*ir-stateAfter-opt*: INPUT-STATE option) (*ir-next*: SUCC)
- | *InvokeWithExceptionNode* (*ir-nid*: ID) (*ir-callTarget*: INPUT-EXT) (*ir-classInit-opt*: INPUT option) (*ir-stateDuring-opt*: INPUT-STATE option) (*ir-stateAfter-opt*: INPUT-STATE option) (*ir-next*: SUCC) (*ir-exceptionEdge*: SUCC)
- | *IsNullNode* (*ir-value*: INPUT)
- | *KillingBeginNode* (*ir-next*: SUCC)
- | *LeftShiftNode* (*ir-x*: INPUT) (*ir-y*: INPUT)
- | *LoadFieldNode* (*ir-nid*: ID) (*ir-field*: string) (*ir-object-opt*: INPUT option) (*ir-next*: SUCC)
- | *LogicNegationNode* (*ir-value*: INPUT-COND)
- | *LoopBeginNode* (*ir-ends*: INPUT-ASSOC list) (*ir-overflowGuard-opt*: INPUT-GUARD option) (*ir-stateAfter-opt*: INPUT-STATE option) (*ir-next*: SUCC)
- | *LoopEndNode* (*ir-loopBegin*: INPUT-ASSOC)
- | *LoopExitNode* (*ir-loopBegin*: INPUT-ASSOC) (*ir-stateAfter-opt*: INPUT-STATE option) (*ir-next*: SUCC)
- | *MergeNode* (*ir-ends*: INPUT-ASSOC list) (*ir-stateAfter-opt*: INPUT-STATE option) (*ir-next*: SUCC)
- | *MethodCallTargetNode* (*ir-targetMethod*: string) (*ir-arguments*: INPUT list)
- | *MulNode* (*ir-x*: INPUT) (*ir-y*: INPUT)
- | *NarrowNode* (*ir-inputBits*: nat) (*ir-resultBits*: nat) (*ir-value*: INPUT)
- | *NegateNode* (*ir-value*: INPUT)
- | *NewArrayNode* (*ir-length*: INPUT) (*ir-stateBefore-opt*: INPUT-STATE option) (*ir-next*: SUCC)
- | *NewInstanceNode* (*ir-nid*: ID) (*ir-instanceClass*: string) (*ir-stateBefore-opt*: INPUT-STATE option) (*ir-next*: SUCC)
- | *NotNode* (*ir-value*: INPUT)
- | *OrNode* (*ir-x*: INPUT) (*ir-y*: INPUT)
- | *ParameterNode* (*ir-index*: nat)
- | *PiNode* (*ir-object*: INPUT) (*ir-guard-opt*: INPUT-GUARD option)

```

| ReturnNode (ir-result-opt: INPUT option) (ir-memoryMap-opt: INPUT-EXT
option)
| RightShiftNode (ir-x: INPUT) (ir-y: INPUT)
| ShortCircuitOrNode (ir-x: INPUT-COND) (ir-y: INPUT-COND)
| SignExtendNode (ir-inputBits: nat) (ir-resultBits: nat) (ir-value: INPUT)
| SignedDivNode (ir-nid: ID) (ir-x: INPUT) (ir-y: INPUT) (ir-zeroCheck-opt: IN-
PUT-GUARD option) (ir-stateBefore-opt: INPUT-STATE option) (ir-next: SUCC)

| SignedRemNode (ir-nid: ID) (ir-x: INPUT) (ir-y: INPUT) (ir-zeroCheck-opt:
INPUT-GUARD option) (ir-stateBefore-opt: INPUT-STATE option) (ir-next: SUCC)

| StartNode (ir-stateAfter-opt: INPUT-STATE option) (ir-next: SUCC)
| StoreFieldNode (ir-nid: ID) (ir-field: string) (ir-value: INPUT) (ir-stateAfter-opt:
INPUT-STATE option) (ir-object-opt: INPUT option) (ir-next: SUCC)
| SubNode (ir-x: INPUT) (ir-y: INPUT)
| UnsignedRightShiftNode (ir-x: INPUT) (ir-y: INPUT)
| UnwindNode (ir-exception: INPUT)
| ValuePhiNode (ir-nid: ID) (ir-values: INPUT list) (ir-merge: INPUT-ASSOC)
| ValueProxyNode (ir-value: INPUT) (ir-loopExit: INPUT-ASSOC)
| XORNode (ir-x: INPUT) (ir-y: INPUT)
| ZeroExtendNode (ir-inputBits: nat) (ir-resultBits: nat) (ir-value: INPUT)
| NoNode

| RefNode (ir-ref:ID)

```

```

fun opt-to-list :: 'a option ⇒ 'a list where
  opt-to-list None = [] |
  opt-to-list (Some v) = [v]

```

```

fun opt-list-to-list :: 'a list option ⇒ 'a list where
  opt-list-to-list None = [] |
  opt-list-to-list (Some x) = x

```

The following functions, `inputs_of` and `successors_of`, are automatically generated from the GraalVM compiler. Their purpose is to partition the node edges into input or successor edges.

```

fun inputs-of :: IRNode ⇒ ID list where
  inputs-of-AbsNode:
  inputs-of (AbsNode value) = [value] |
  inputs-of-AddNode:
  inputs-of (AddNode x y) = [x, y] |
  inputs-of-AndNode:
  inputs-of (AndNode x y) = [x, y] |
  inputs-of-BEGINNode:
  inputs-of (BeginNode next) = [] |

```

*inputs-of-BytecodeExceptionNode:*  
*inputs-of (BytecodeExceptionNode arguments stateAfter next) = arguments @*  
*(opt-to-list stateAfter) |*  
*inputs-of-ConditionalNode:*  
*inputs-of (ConditionalNode condition trueValue falseValue) = [condition, true-*  
*Value, falseValue] |*  
*inputs-of-ConstantNode:*  
*inputs-of (ConstantNode const) = [] |*  
*inputs-of-DynamicNewArrayNode:*  
*inputs-of (DynamicNewArrayNode elementType length0 voidClass stateBefore*  
*next) = [elementType, length0] @ (opt-to-list voidClass) @ (opt-to-list stateBefore)*  
*|*  
*inputs-of-EndNode:*  
*inputs-of (EndNode) = [] |*  
*inputs-of-ExceptionObjectNode:*  
*inputs-of (ExceptionObjectNode stateAfter next) = (opt-to-list stateAfter) |*  
*inputs-of-FrameState:*  
*inputs-of (FrameState monitorIds outerFrameState values virtualObjectMappings)*  
*= monitorIds @ (opt-to-list outerFrameState) @ (opt-list-to-list values) @ (opt-list-to-list*  
*virtualObjectMappings) |*  
*inputs-of-IfNode:*  
*inputs-of (IfNode condition trueSuccessor falseSuccessor) = [condition] |*  
*inputs-of-IntegerBelowNode:*  
*inputs-of (IntegerBelowNode x y) = [x, y] |*  
*inputs-of-IntegerEqualsNode:*  
*inputs-of (IntegerEqualsNode x y) = [x, y] |*  
*inputs-of-IntegerLessThanNode:*  
*inputs-of (IntegerLessThanNode x y) = [x, y] |*  
*inputs-of-InvokeNode:*  
*inputs-of (InvokeNode nid0 callTarget classInit stateDuring stateAfter next)*  
*= callTarget # (opt-to-list classInit) @ (opt-to-list stateDuring) @ (opt-to-list*  
*stateAfter) |*  
*inputs-of-InvokeWithExceptionNode:*  
*inputs-of (InvokeWithExceptionNode nid0 callTarget classInit stateDuring stateAfter*  
*next exceptionEdge) = callTarget # (opt-to-list classInit) @ (opt-to-list stateDur-*  
*ing) @ (opt-to-list stateAfter) |*  
*inputs-of-IsNullNode:*  
*inputs-of (IsNullNode value) = [value] |*  
*inputs-of-KillingBeginNode:*  
*inputs-of (KillingBeginNode next) = [] |*  
*inputs-of-LeftShiftNode:*  
*inputs-of (LeftShiftNode x y) = [x, y] |*  
*inputs-of-LoadFieldNode:*  
*inputs-of (LoadFieldNode nid0 field object next) = (opt-to-list object) |*  
*inputs-of-LogicNegationNode:*  
*inputs-of (LogicNegationNode value) = [value] |*  
*inputs-of-LoopBeginNode:*  
*inputs-of (LoopBeginNode ends overflowGuard stateAfter next) = ends @ (opt-to-list*  
*overflowGuard) @ (opt-to-list stateAfter) |*

*inputs-of-LoopEndNode:*  
*inputs-of (LoopEndNode loopBegin) = [loopBegin] |*  
*inputs-of-LoopExitNode:*  
*inputs-of (LoopExitNode loopBegin stateAfter next) = loopBegin # (opt-to-list stateAfter) |*  
*inputs-of-MergeNode:*  
*inputs-of (MergeNode ends stateAfter next) = ends @ (opt-to-list stateAfter) |*  
*inputs-of-MethodCallTargetNode:*  
*inputs-of (MethodCallTargetNode targetMethod arguments) = arguments |*  
*inputs-of-MulNode:*  
*inputs-of (MulNode x y) = [x, y] |*  
*inputs-of-NarrowNode:*  
*inputs-of (NarrowNode inputBits resultBits value) = [value] |*  
*inputs-of-NegateNode:*  
*inputs-of (NegateNode value) = [value] |*  
*inputs-of-NewArrayNode:*  
*inputs-of (NewArrayNode length0 stateBefore next) = length0 # (opt-to-list stateBefore) |*  
*inputs-of-NewInstanceNode:*  
*inputs-of (NewInstanceNode nid0 instanceClass stateBefore next) = (opt-to-list stateBefore) |*  
*inputs-of-NotNode:*  
*inputs-of (NotNode value) = [value] |*  
*inputs-of-OrNode:*  
*inputs-of (OrNode x y) = [x, y] |*  
*inputs-of-ParameterNode:*  
*inputs-of (ParameterNode index) = [] |*  
*inputs-of-PiNode:*  
*inputs-of (PiNode object guard) = object # (opt-to-list guard) |*  
*inputs-of-ReturnNode:*  
*inputs-of (ReturnNode result memoryMap) = (opt-to-list result) @ (opt-to-list memoryMap) |*  
*inputs-of-RightShiftNode:*  
*inputs-of (RightShiftNode x y) = [x, y] |*  
*inputs-of-ShortCircuitOrNode:*  
*inputs-of (ShortCircuitOrNode x y) = [x, y] |*  
*inputs-of-SignExtendNode:*  
*inputs-of (SignExtendNode inputBits resultBits value) = [value] |*  
*inputs-of-SignedDivNode:*  
*inputs-of (SignedDivNode nid0 x y zeroCheck stateBefore next) = [x, y] @ (opt-to-list zeroCheck) @ (opt-to-list stateBefore) |*  
*inputs-of-SignedRemNode:*  
*inputs-of (SignedRemNode nid0 x y zeroCheck stateBefore next) = [x, y] @ (opt-to-list zeroCheck) @ (opt-to-list stateBefore) |*  
*inputs-of-StartNode:*  
*inputs-of (StartNode stateAfter next) = (opt-to-list stateAfter) |*  
*inputs-of-StoreFieldNode:*  
*inputs-of (StoreFieldNode nid0 field value stateAfter object next) = value # (opt-to-list stateAfter) @ (opt-to-list object) |*

*inputs-of-SubNode:*  
*inputs-of (SubNode x y) = [x, y] |*  
*inputs-of-UnsignedRightShiftNode:*  
*inputs-of (UnsignedRightShiftNode x y) = [x, y] |*  
*inputs-of-UnwindNode:*  
*inputs-of (UnwindNode exception) = [exception] |*  
*inputs-of-ValuePhiNode:*  
*inputs-of (ValuePhiNode nid0 values merge) = merge # values |*  
*inputs-of-ValueProxyNode:*  
*inputs-of (ValueProxyNode value loopExit) = [value, loopExit] |*  
*inputs-of-XorNode:*  
*inputs-of (XorNode x y) = [x, y] |*  
*inputs-of-ZeroExtendNode:*  
*inputs-of (ZeroExtendNode inputBits resultBits value) = [value] |*  
*inputs-of-NoNode: inputs-of (NoNode) = [] |*

*inputs-of-RefNode: inputs-of (RefNode ref) = [ref]*

**fun** *successors-of* :: *IRNode*  $\Rightarrow$  *ID list* **where**

*successors-of-AbsNode:*  
*successors-of (AbsNode value) = [] |*  
*successors-of-AddNode:*  
*successors-of (AddNode x y) = [] |*  
*successors-of-AndNode:*  
*successors-of (AndNode x y) = [] |*  
*successors-of-BeginNode:*  
*successors-of (BeginNode next) = [next] |*  
*successors-of-BytecodeExceptionNode:*  
*successors-of (BytecodeExceptionNode arguments stateAfter next) = [next] |*  
*successors-of-ConditionalNode:*  
*successors-of (ConditionalNode condition trueValue falseValue) = [] |*  
*successors-of-ConstantNode:*  
*successors-of (ConstantNode const) = [] |*  
*successors-of-DynamicNewArrayNode:*  
*successors-of (DynamicNewArrayNode elementType length0 voidClass stateBefore*  
*next) = [next] |*  
*successors-of-EndNode:*  
*successors-of (EndNode) = [] |*  
*successors-of-ExceptionObjectNode:*  
*successors-of (ExceptionObjectNode stateAfter next) = [next] |*  
*successors-of-FrameState:*  
*successors-of (FrameState monitorIds outerFrameState values virtualObjectMap-*  
*pings) = [] |*  
*successors-of-IfNode:*  
*successors-of (IfNode condition trueSuccessor falseSuccessor) = [trueSuccessor,*  
*falseSuccessor] |*  
*successors-of-IntegerBelowNode:*

*successors-of* (*IntegerBelowNode* *x y*) = [] |  
*successors-of-IntegerEqualsNode*:  
*successors-of* (*IntegerEqualsNode* *x y*) = [] |  
*successors-of-IntegerLessThanNode*:  
*successors-of* (*IntegerLessThanNode* *x y*) = [] |  
*successors-of-InvokeNode*:  
*successors-of* (*InvokeNode* *nid0 callTarget classInit stateDuring stateAfter next*)  
= [*next*] |  
*successors-of-InvokeWithExceptionNode*:  
*successors-of* (*InvokeWithExceptionNode* *nid0 callTarget classInit stateDuring*  
*stateAfter next exceptionEdge*) = [*next*, *exceptionEdge*] |  
*successors-of-IsNullNode*:  
*successors-of* (*IsNullNode* *value*) = [] |  
*successors-of-KillingBeginNode*:  
*successors-of* (*KillingBeginNode* *next*) = [*next*] |  
*successors-of-LeftShiftNode*:  
*successors-of* (*LeftShiftNode* *x y*) = [] |  
*successors-of-LoadFieldNode*:  
*successors-of* (*LoadFieldNode* *nid0 field object next*) = [*next*] |  
*successors-of-LogicNegationNode*:  
*successors-of* (*LogicNegationNode* *value*) = [] |  
*successors-of-LoopBeginNode*:  
*successors-of* (*LoopBeginNode* *ends overflowGuard stateAfter next*) = [*next*] |  
*successors-of-LoopEndNode*:  
*successors-of* (*LoopEndNode* *loopBegin*) = [] |  
*successors-of-LoopExitNode*:  
*successors-of* (*LoopExitNode* *loopBegin stateAfter next*) = [*next*] |  
*successors-of-MergeNode*:  
*successors-of* (*MergeNode* *ends stateAfter next*) = [*next*] |  
*successors-of-MethodCallTargetNode*:  
*successors-of* (*MethodCallTargetNode* *targetMethod arguments*) = [] |  
*successors-of-MulNode*:  
*successors-of* (*MulNode* *x y*) = [] |  
*successors-of-NarrowNode*:  
*successors-of* (*NarrowNode* *inputBits resultBits value*) = [] |  
*successors-of-NegateNode*:  
*successors-of* (*NegateNode* *value*) = [] |  
*successors-of-NewArrayNode*:  
*successors-of* (*NewArrayNode* *length0 stateBefore next*) = [*next*] |  
*successors-of-NewInstanceNode*:  
*successors-of* (*NewInstanceNode* *nid0 instanceClass stateBefore next*) = [*next*] |  
*successors-of-NotNode*:  
*successors-of* (*NotNode* *value*) = [] |  
*successors-of-OrNode*:  
*successors-of* (*OrNode* *x y*) = [] |  
*successors-of-ParameterNode*:  
*successors-of* (*ParameterNode* *index*) = [] |  
*successors-of-PiNode*:  
*successors-of* (*PiNode* *object guard*) = [] |

*successors-of-ReturnNode:*  
*successors-of (ReturnNode result memoryMap) = [] |*  
*successors-of-RightShiftNode:*  
*successors-of (RightShiftNode x y) = [] |*  
*successors-of-ShortCircuitOrNode:*  
*successors-of (ShortCircuitOrNode x y) = [] |*  
*successors-of-SignExtendNode:*  
*successors-of (SignExtendNode inputBits resultBits value) = [] |*  
*successors-of-SignedDivNode:*  
*successors-of (SignedDivNode nid0 x y zeroCheck stateBefore next) = [next] |*  
*successors-of-SignedRemNode:*  
*successors-of (SignedRemNode nid0 x y zeroCheck stateBefore next) = [next] |*  
*successors-of-StartNode:*  
*successors-of (StartNode stateAfter next) = [next] |*  
*successors-of-StoreFieldNode:*  
*successors-of (StoreFieldNode nid0 field value stateAfter object next) = [next] |*  
*successors-of-SubNode:*  
*successors-of (SubNode x y) = [] |*  
*successors-of-UnsignedRightShiftNode:*  
*successors-of (UnsignedRightShiftNode x y) = [] |*  
*successors-of-UnwindNode:*  
*successors-of (UnwindNode exception) = [] |*  
*successors-of-ValuePhiNode:*  
*successors-of (ValuePhiNode nid0 values merge) = [] |*  
*successors-of-ValueProxyNode:*  
*successors-of (ValueProxyNode value loopExit) = [] |*  
*successors-of-XorNode:*  
*successors-of (XorNode x y) = [] |*  
*successors-of-ZeroExtendNode:*  
*successors-of (ZeroExtendNode inputBits resultBits value) = [] |*  
*successors-of-NoNode: successors-of (NoNode) = [] |*

*successors-of-RefNode: successors-of (RefNode ref) = [ref]*

**lemma** *inputs-of (FrameState x (Some y) (Some z) None) = x @ [y] @ z*

**unfolding** *inputs-of-FrameState by simp*

**lemma** *successors-of (FrameState x (Some y) (Some z) None) = []*

**unfolding** *inputs-of-FrameState by simp*

**lemma** *inputs-of (IfNode c t f) = [c]*

**unfolding** *inputs-of-IfNode by simp*

**lemma** *successors-of (IfNode c t f) = [t, f]*

**unfolding** *successors-of-IfNode by simp*

**lemma** *inputs-of (EndNode) = [] ∧ successors-of (EndNode) = []*

**unfolding** *inputs-of-EndNode successors-of-EndNode by simp*

end

### 3.2 IR Graph Node Hierarchy

```
theory IRNodeHierarchy
imports IRNodes
begin
```

It is helpful to introduce a node hierarchy into our formalization. Often the GraalVM compiler relies on explicit type checks to determine which operations to perform on a given node, we try to mimic the same functionality by using a suite of predicate functions over the *IRNode* class to determine inheritance.

As one would expect, the function *is<ClassName>Type* will be true if the node parameter is a subclass of the *ClassName* within the GraalVM compiler.

These functions have been automatically generated from the compiler.

```
fun is-EndNode :: IRNode  $\Rightarrow$  bool where
  is-EndNode EndNode = True |
  is-EndNode - = False
```

```
fun is-VirtualState :: IRNode  $\Rightarrow$  bool where
  is-VirtualState n = (is-FrameState n)
```

```
fun is-BinaryArithmeticNode :: IRNode  $\Rightarrow$  bool where
  is-BinaryArithmeticNode n = (is-AddNode n)  $\vee$  (is-AndNode n)  $\vee$  (is-MulNode n)
   $\vee$  (is-OrNode n)  $\vee$  (is-SubNode n)  $\vee$  (is-XorNode n)
```

```
fun is-ShiftNode :: IRNode  $\Rightarrow$  bool where
  is-ShiftNode n = (is-LeftShiftNode n)  $\vee$  (is-RightShiftNode n)  $\vee$  (is-UnsignedRightShiftNode n)
```

```
fun is-BinaryNode :: IRNode  $\Rightarrow$  bool where
  is-BinaryNode n = (is-BinaryArithmeticNode n)  $\vee$  (is-ShiftNode n)
```

```
fun is-AbstractLocalNode :: IRNode  $\Rightarrow$  bool where
  is-AbstractLocalNode n = (is-ParameterNode n)
```

```
fun is-IntegerConvertNode :: IRNode  $\Rightarrow$  bool where
  is-IntegerConvertNode n = (is-NarrowNode n)  $\vee$  (is-SignExtendNode n)  $\vee$ 
  (is-ZeroExtendNode n)
```

```
fun is-UnaryArithmeticNode :: IRNode  $\Rightarrow$  bool where
  is-UnaryArithmeticNode n = (is-AbsNode n)  $\vee$  (is-NegateNode n)  $\vee$  (is-NotNode n)
```



```

fun is-UnaryNode :: IRNode  $\Rightarrow$  bool where
  is-UnaryNode n = ((is-IntegerConvertNode n)  $\vee$  (is-UnaryArithmeticNode n))

fun is-PhiNode :: IRNode  $\Rightarrow$  bool where
  is-PhiNode n = ((is-ValuePhiNode n))

fun is-FloatingGuardedNode :: IRNode  $\Rightarrow$  bool where
  is-FloatingGuardedNode n = ((is-PiNode n))

fun is-UnaryOpLogicNode :: IRNode  $\Rightarrow$  bool where
  is-UnaryOpLogicNode n = ((is-IsNullNode n))

fun is-IntegerLowerThanNode :: IRNode  $\Rightarrow$  bool where
  is-IntegerLowerThanNode n = ((is-IntegerBelowNode n)  $\vee$  (is-IntegerLessThanNode n))

fun is-CompareNode :: IRNode  $\Rightarrow$  bool where
  is-CompareNode n = ((is-IntegerEqualsNode n)  $\vee$  (is-IntegerLowerThanNode n))

fun is-BinaryOpLogicNode :: IRNode  $\Rightarrow$  bool where
  is-BinaryOpLogicNode n = ((is-CompareNode n))

fun is-LogicNode :: IRNode  $\Rightarrow$  bool where
  is-LogicNode n = ((is-BinaryOpLogicNode n)  $\vee$  (is-LogicNegationNode n)  $\vee$ 
    (is-ShortCircuitOrNode n)  $\vee$  (is-UnaryOpLogicNode n))

fun is-ProxyNode :: IRNode  $\Rightarrow$  bool where
  is-ProxyNode n = ((is-ValueProxyNode n))

fun is-FloatingNode :: IRNode  $\Rightarrow$  bool where
  is-FloatingNode n = ((is-AbstractLocalNode n)  $\vee$  (is-BinaryNode n)  $\vee$  (is-ConditionalNode n)
     $\vee$  (is-ConstantNode n)  $\vee$  (is-FloatingGuardedNode n)  $\vee$  (is-LogicNode n)  $\vee$ 
    (is-PhiNode n)  $\vee$  (is-ProxyNode n)  $\vee$  (is-UnaryNode n))

fun is-AccessFieldNode :: IRNode  $\Rightarrow$  bool where
  is-AccessFieldNode n = ((is-LoadFieldNode n)  $\vee$  (is-StoreFieldNode n))

fun is-AbstractNewArrayNode :: IRNode  $\Rightarrow$  bool where
  is-AbstractNewArrayNode n = ((is-DynamicNewArrayNode n)  $\vee$  (is-NewArrayNode n))

fun is-AbstractNewObjectNode :: IRNode  $\Rightarrow$  bool where
  is-AbstractNewObjectNode n = ((is-AbstractNewArrayNode n)  $\vee$  (is-NewInstanceNode n))

fun is-IntegerDivRemNode :: IRNode  $\Rightarrow$  bool where
  is-IntegerDivRemNode n = ((is-SignedDivNode n)  $\vee$  (is-SignedRemNode n))

fun is-FixedBinaryNode :: IRNode  $\Rightarrow$  bool where

```

```

is-FixedBinaryNode n = ((is-IntegerDivRemNode n))

fun is-DeoptimizingFixedWithNextNode :: IRNode ⇒ bool where
  is-DeoptimizingFixedWithNextNode n = ((is-AbstractNewObjectNode n) ∨ (is-FixedBinaryNode
n))

fun is-AbstractMemoryCheckpoint :: IRNode ⇒ bool where
  is-AbstractMemoryCheckpoint n = ((is-BytecodeExceptionNode n) ∨ (is-InvokeNode
n))

fun is-AbstractStateSplit :: IRNode ⇒ bool where
  is-AbstractStateSplit n = ((is-AbstractMemoryCheckpoint n))

fun is-AbstractMergeNode :: IRNode ⇒ bool where
  is-AbstractMergeNode n = ((is-LoopBeginNode n) ∨ (is-MergeNode n))

fun is-BeginStateSplitNode :: IRNode ⇒ bool where
  is-BeginStateSplitNode n = ((is-AbstractMergeNode n) ∨ (is-ExceptionObjectNode
n) ∨ (is-LoopExitNode n) ∨ (is-StartNode n))

fun is-AbstractBeginNode :: IRNode ⇒ bool where
  is-AbstractBeginNode n = ((is-BeginNode n) ∨ (is-BeginStateSplitNode n) ∨
(is-KillingBeginNode n))

fun is-FixedWithNextNode :: IRNode ⇒ bool where
  is-FixedWithNextNode n = ((is-AbstractBeginNode n) ∨ (is-AbstractStateSplit n)
∨ (is-AccessFieldNode n) ∨ (is-DeoptimizingFixedWithNextNode n))

fun is-WithExceptionNode :: IRNode ⇒ bool where
  is-WithExceptionNode n = ((is-InvokeWithExceptionNode n))

fun is-ControlSplitNode :: IRNode ⇒ bool where
  is-ControlSplitNode n = ((is-IfNode n) ∨ (is-WithExceptionNode n))

fun is-ControlSinkNode :: IRNode ⇒ bool where
  is-ControlSinkNode n = ((is-ReturnNode n) ∨ (is-UnwindNode n))

fun is-AbstractEndNode :: IRNode ⇒ bool where
  is-AbstractEndNode n = ((is-EndNode n) ∨ (is-LoopEndNode n))

fun is-FixedNode :: IRNode ⇒ bool where
  is-FixedNode n = ((is-AbstractEndNode n) ∨ (is-ControlSinkNode n) ∨ (is-ControlSplitNode
n) ∨ (is-FixedWithNextNode n))

fun is-CallTargetNode :: IRNode ⇒ bool where
  is-CallTargetNode n = ((is-MethodCallTargetNode n))

fun is-ValueNode :: IRNode ⇒ bool where
  is-ValueNode n = ((is-CallTargetNode n) ∨ (is-FixedNode n) ∨ (is-FloatingNode

```

*n*))

**fun** *is-Node* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-Node* *n* = ((*is-ValueNode* *n*)  $\vee$  (*is-VirtualState* *n*))

**fun** *is-MemoryKill* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-MemoryKill* *n* = ((*is-AbstractMemoryCheckpoint* *n*))

**fun** *is-NarrowableArithmeticNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-NarrowableArithmeticNode* *n* = ((*is-AbsNode* *n*)  $\vee$  (*is-AddNode* *n*)  $\vee$  (*is-AndNode* *n*)  $\vee$  (*is-MulNode* *n*)  $\vee$  (*is-NegateNode* *n*)  $\vee$  (*is-NotNode* *n*)  $\vee$  (*is-OrNode* *n*)  $\vee$  (*is-ShiftNode* *n*)  $\vee$  (*is-SubNode* *n*)  $\vee$  (*is-XorNode* *n*))

**fun** *is-AnchoringNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-AnchoringNode* *n* = ((*is-AbstractBeginNode* *n*))

**fun** *is-DeoptBefore* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-DeoptBefore* *n* = ((*is-DeoptimizingFixedWithNextNode* *n*))

**fun** *is-IndirectCanonicalization* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-IndirectCanonicalization* *n* = ((*is-LogicNode* *n*))

**fun** *is-IterableNodeType* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-IterableNodeType* *n* = ((*is-AbstractBeginNode* *n*)  $\vee$  (*is-AbstractMergeNode* *n*)  $\vee$  (*is-FrameState* *n*)  $\vee$  (*is-IfNode* *n*)  $\vee$  (*is-IntegerDivRemNode* *n*)  $\vee$  (*is-InvokeWithExceptionNode* *n*)  $\vee$  (*is-LoopBeginNode* *n*)  $\vee$  (*is-LoopExitNode* *n*)  $\vee$  (*is-MethodCallTargetNode* *n*)  $\vee$  (*is-ParameterNode* *n*)  $\vee$  (*is-ReturnNode* *n*)  $\vee$  (*is-ShortCircuitOrNode* *n*))

**fun** *is-Invoke* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-Invoke* *n* = ((*is-InvokeNode* *n*)  $\vee$  (*is-InvokeWithExceptionNode* *n*))

**fun** *is-Proxy* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-Proxy* *n* = ((*is-ProxyNode* *n*))

**fun** *is-ValueProxy* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-ValueProxy* *n* = ((*is-PiNode* *n*)  $\vee$  (*is-ValueProxyNode* *n*))

**fun** *is-ValueNodeInterface* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-ValueNodeInterface* *n* = ((*is-ValueNode* *n*))

**fun** *is-ArrayLengthProvider* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-ArrayLengthProvider* *n* = ((*is-AbstractNewArrayNode* *n*)  $\vee$  (*is-ConstantNode* *n*))

**fun** *is-StampInverter* :: *IRNode*  $\Rightarrow$  *bool* **where**  
  *is-StampInverter* *n* = ((*is-IntegerConvertNode* *n*)  $\vee$  (*is-NegateNode* *n*)  $\vee$  (*is-NotNode* *n*))

**fun** *is-GuardingNode* :: *IRNode*  $\Rightarrow$  *bool* **where**

```

is-GuardingNode n = ((is-AbstractBeginNode n))

fun is-SingleMemoryKill :: IRNode ⇒ bool where
  is-SingleMemoryKill n = ((is-BytecodeExceptionNode n) ∨ (is-ExceptionObjectNode
n) ∨ (is-InvokeNode n) ∨ (is-InvokeWithExceptionNode n) ∨ (is-KillingBeginNode
n) ∨ (is-StartNode n))

fun is-LIRLowerable :: IRNode ⇒ bool where
  is-LIRLowerable n = ((is-AbstractBeginNode n) ∨ (is-AbstractEndNode n) ∨
(is-AbstractMergeNode n) ∨ (is-BinaryOpLogicNode n) ∨ (is-CallTargetNode n) ∨
(is-ConditionalNode n) ∨ (is-ConstantNode n) ∨ (is-IfNode n) ∨ (is-InvokeNode n)
∨ (is-InvokeWithExceptionNode n) ∨ (is-IsNullNode n) ∨ (is-LoopBeginNode n) ∨
(is-PiNode n) ∨ (is-ReturnNode n) ∨ (is-SignedDivNode n) ∨ (is-SignedRemNode
n) ∨ (is-UnaryOpLogicNode n) ∨ (is-UnwindNode n))

fun is-GuardedNode :: IRNode ⇒ bool where
  is-GuardedNode n = ((is-FloatingGuardedNode n))

fun is-ArithmeticLIRLowerable :: IRNode ⇒ bool where
  is-ArithmeticLIRLowerable n = ((is-AbsNode n) ∨ (is-BinaryArithmeticNode n) ∨
(is-IntegerConvertNode n) ∨ (is-NotNode n) ∨ (is-ShiftNode n) ∨ (is-UnaryArithmeticNode
n))

fun is-SwitchFoldable :: IRNode ⇒ bool where
  is-SwitchFoldable n = ((is-IfNode n))

fun is-VirtualizableAllocation :: IRNode ⇒ bool where
  is-VirtualizableAllocation n = ((is-NewArrayNode n) ∨ (is-NewInstanceNode n))

fun is-Unary :: IRNode ⇒ bool where
  is-Unary n = ((is-LoadFieldNode n) ∨ (is-LogicNegationNode n) ∨ (is-UnaryNode
n) ∨ (is-UnaryOpLogicNode n))

fun is-FixedNodeInterface :: IRNode ⇒ bool where
  is-FixedNodeInterface n = ((is-FixedNode n))

fun is-BinaryCommutative :: IRNode ⇒ bool where
  is-BinaryCommutative n = ((is-AddNode n) ∨ (is-AndNode n) ∨ (is-IntegerEqualsNode
n) ∨ (is-MulNode n) ∨ (is-OrNode n) ∨ (is-XorNode n))

fun is-Canonicalizable :: IRNode ⇒ bool where
  is-Canonicalizable n = ((is-BytecodeExceptionNode n) ∨ (is-ConditionalNode n) ∨
(is-DynamicNewArrayNode n) ∨ (is-PhiNode n) ∨ (is-PiNode n) ∨ (is-ProxyNode
n) ∨ (is-StoreFieldNode n) ∨ (is-ValueProxyNode n))

fun is-UncheckedInterfaceProvider :: IRNode ⇒ bool where
  is-UncheckedInterfaceProvider n = ((is-InvokeNode n) ∨ (is-InvokeWithExceptionNode
n) ∨ (is-LoadFieldNode n) ∨ (is-ParameterNode n))

```

```
fun is-Binary :: IRNode  $\Rightarrow$  bool where
  is-Binary n = ((is-BinaryArithmeticNode n)  $\vee$  (is-BinaryNode n)  $\vee$  (is-BinaryOpLogicNode
n)  $\vee$  (is-CompareNode n)  $\vee$  (is-FixedBinaryNode n)  $\vee$  (is-ShortCircuitOrNode n))
```

```
fun is-ArithmeticOperation :: IRNode  $\Rightarrow$  bool where
  is-ArithmeticOperation n = ((is-BinaryArithmeticNode n)  $\vee$  (is-IntegerConvertNode
n)  $\vee$  (is-ShiftNode n)  $\vee$  (is-UnaryArithmeticNode n))
```

```
fun is-ValueNumberable :: IRNode  $\Rightarrow$  bool where
  is-ValueNumberable n = ((is-FloatingNode n)  $\vee$  (is-ProxyNode n))
```

```
fun is-Lowerable :: IRNode  $\Rightarrow$  bool where
  is-Lowerable n = ((is-AbstractNewObjectNode n)  $\vee$  (is-AccessFieldNode n)  $\vee$ 
(is-BytecodeExceptionNode n)  $\vee$  (is-ExceptionObjectNode n)  $\vee$  (is-IntegerDivRemNode
n)  $\vee$  (is-UnwindNode n))
```

```
fun is-Virtualizable :: IRNode  $\Rightarrow$  bool where
  is-Virtualizable n = ((is-IsNullNode n)  $\vee$  (is-LoadFieldNode n)  $\vee$  (is-PiNode n)
 $\vee$  (is-StoreFieldNode n)  $\vee$  (is-ValueProxyNode n))
```

```
fun is-Simplifiable :: IRNode  $\Rightarrow$  bool where
  is-Simplifiable n = ((is-AbstractMergeNode n)  $\vee$  (is-BEGINNode n)  $\vee$  (is-IfNode
n)  $\vee$  (is-LoopExitNode n)  $\vee$  (is-MethodCallTargetNode n)  $\vee$  (is-NewArrayNode n))
```

```
fun is-StateSplit :: IRNode  $\Rightarrow$  bool where
  is-StateSplit n = ((is-AbstractStateSplit n)  $\vee$  (is-BEGINStateSplitNode n)  $\vee$  (is-StoreFieldNode
n))
```

```
fun is-ConvertNode :: IRNode  $\Rightarrow$  bool where
  is-ConvertNode n = ((is-IntegerConvertNode n))
```

```
fun is-sequential-node :: IRNode  $\Rightarrow$  bool where
  is-sequential-node (StartNode -) = True |
  is-sequential-node (BeginNode -) = True |
  is-sequential-node (KillingBeginNode -) = True |
  is-sequential-node (LoopBeginNode - - -) = True |
  is-sequential-node (LoopExitNode - - -) = True |
  is-sequential-node (MergeNode - - -) = True |
  is-sequential-node (RefNode -) = True |
  is-sequential-node - = False
```

The following convenience function is useful in determining if two IRNodes are of the same type irregardless of their edges. It will return true if both the node parameters are the same node class.

```
fun is-same-ir-node-type :: IRNode  $\Rightarrow$  IRNode  $\Rightarrow$  bool where
is-same-ir-node-type n1 n2 = (
  ((is-AbsNode n1)  $\wedge$  (is-AbsNode n2))  $\vee$ 
  ((is-AddNode n1)  $\wedge$  (is-AddNode n2))  $\vee$ 
```

```

((is-AndNode n1) ∧ (is-AndNode n2)) ∨
((is-BEGINNode n1) ∧ (is-BEGINNode n2)) ∨
((is-BytecodeExceptionNode n1) ∧ (is-BytecodeExceptionNode n2)) ∨
((is-ConditionalNode n1) ∧ (is-ConditionalNode n2)) ∨
((is-ConstantNode n1) ∧ (is-ConstantNode n2)) ∨
((is-DynamicNewArrayNode n1) ∧ (is-DynamicNewArrayNode n2)) ∨
((is-EndNode n1) ∧ (is-EndNode n2)) ∨
((is-ExceptionObjectNode n1) ∧ (is-ExceptionObjectNode n2)) ∨
((is-FrameState n1) ∧ (is-FrameState n2)) ∨
((is-IfNode n1) ∧ (is-IfNode n2)) ∨
((is-IntegerBelowNode n1) ∧ (is-IntegerBelowNode n2)) ∨
((is-IntegerEqualsNode n1) ∧ (is-IntegerEqualsNode n2)) ∨
((is-IntegerLessThanNode n1) ∧ (is-IntegerLessThanNode n2)) ∨
((is-InvokeNode n1) ∧ (is-InvokeNode n2)) ∨
((is-InvokeWithExceptionNode n1) ∧ (is-InvokeWithExceptionNode n2)) ∨
((is-IsNullNode n1) ∧ (is-IsNullNode n2)) ∨
((is-KillingBeginNode n1) ∧ (is-KillingBeginNode n2)) ∨
((is-LoadFieldNode n1) ∧ (is-LoadFieldNode n2)) ∨
((is-LogicNegationNode n1) ∧ (is-LogicNegationNode n2)) ∨
((is-LoopBeginNode n1) ∧ (is-LoopBeginNode n2)) ∨
((is-LoopEndNode n1) ∧ (is-LoopEndNode n2)) ∨
((is-LoopExitNode n1) ∧ (is-LoopExitNode n2)) ∨
((is-MergeNode n1) ∧ (is-MergeNode n2)) ∨
((is-MethodCallTargetNode n1) ∧ (is-MethodCallTargetNode n2)) ∨
((is-MulNode n1) ∧ (is-MulNode n2)) ∨
((is-NegateNode n1) ∧ (is-NegateNode n2)) ∨
((is-NewArrayNode n1) ∧ (is-NewArrayNode n2)) ∨
((is-NewInstanceNode n1) ∧ (is-NewInstanceNode n2)) ∨
((is-NotNode n1) ∧ (is-NotNode n2)) ∨
((is-OrNode n1) ∧ (is-OrNode n2)) ∨
((is-ParameterNode n1) ∧ (is-ParameterNode n2)) ∨
((is-PiNode n1) ∧ (is-PiNode n2)) ∨
((is-ReturnNode n1) ∧ (is-ReturnNode n2)) ∨
((is-ShortCircuitOrNode n1) ∧ (is-ShortCircuitOrNode n2)) ∨
((is-SignedDivNode n1) ∧ (is-SignedDivNode n2)) ∨
((is-StartNode n1) ∧ (is-StartNode n2)) ∨
((is-StoreFieldNode n1) ∧ (is-StoreFieldNode n2)) ∨
((is-SubNode n1) ∧ (is-SubNode n2)) ∨
((is-UnwindNode n1) ∧ (is-UnwindNode n2)) ∨
((is-ValuePhiNode n1) ∧ (is-ValuePhiNode n2)) ∨
((is-ValueProxyNode n1) ∧ (is-ValueProxyNode n2)) ∨
((is-XorNode n1) ∧ (is-XorNode n2))

```

end

### 3.3 IR Graph Type

```

theory IRGraph
  imports

```

```

    IRNodeHierarchy
    Stamp
    HOL-Library.FSet
    HOL.Relation
begin

```

This theory defines the main Graal data structure - an entire IR Graph.

IRGraph is defined as a partial map with a finite domain. The finite domain is required to be able to generate code and produce an interpreter.

```

typedef IRGraph = {g :: ID  $\rightharpoonup$  (IRNode  $\times$  Stamp) . finite (dom g)}
proof -
  have finite(dom(Map.empty))  $\wedge$  ran Map.empty = {} by auto
  then show ?thesis
    by fastforce
qed

```

**setup-lifting** type-definition-IRGraph

```

lift-definition ids :: IRGraph  $\Rightarrow$  ID set
is  $\lambda g. \{nid \in \text{dom } g . \nexists s. g \text{ nid} = (\text{Some } (\text{NoNode}, s))\}$  .

```

```

fun with-default :: 'c  $\Rightarrow$  ('b  $\Rightarrow$  'c)  $\Rightarrow$  (('a  $\rightharpoonup$  'b)  $\Rightarrow$  'a  $\Rightarrow$  'c) where
  with-default def conv = ( $\lambda m k.$ 
    (case m k of None  $\Rightarrow$  def | Some v  $\Rightarrow$  conv v))

```

```

lift-definition kind :: IRGraph  $\Rightarrow$  (ID  $\Rightarrow$  IRNode)
is with-default NoNode fst .

```

```

lift-definition stamp :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  Stamp
is with-default IllegalStamp snd .

```

```

lift-definition add-node :: ID  $\Rightarrow$  (IRNode  $\times$  Stamp)  $\Rightarrow$  IRGraph  $\Rightarrow$  IRGraph
is  $\lambda nid k g.$  if fst k = NoNode then g else g(nid  $\mapsto$  k) by simp

```

```

lift-definition remove-node :: ID  $\Rightarrow$  IRGraph  $\Rightarrow$  IRGraph
is  $\lambda nid g.$  g(nid := None) by simp

```

```

lift-definition replace-node :: ID  $\Rightarrow$  (IRNode  $\times$  Stamp)  $\Rightarrow$  IRGraph  $\Rightarrow$  IRGraph
is  $\lambda nid k g.$  if fst k = NoNode then g else g(nid  $\mapsto$  k) by simp

```

```

lift-definition as-list :: IRGraph  $\Rightarrow$  (ID  $\times$  IRNode  $\times$  Stamp) list
is  $\lambda g.$  map ( $\lambda k. (k, \text{the } (g k))$ ) (sorted-list-of-set (dom g)) .

```

```

fun no-node :: (ID  $\times$  (IRNode  $\times$  Stamp)) list  $\Rightarrow$  (ID  $\times$  (IRNode  $\times$  Stamp)) list
where
  no-node g = filter ( $\lambda n. \text{fst } (\text{snd } n) \neq \text{NoNode}$ ) g

```

```

lift-definition irgraph :: (ID  $\times$  (IRNode  $\times$  Stamp)) list  $\Rightarrow$  IRGraph

```

**is** *map-of*  $\circ$  *no-node*  
**by** (*simp add: finite-dom-map-of*)

**definition** *as-set* :: *IRGraph*  $\Rightarrow$  (*ID*  $\times$  (*IRNode*  $\times$  *Stamp*)) *set* **where**  
*as-set* *g* =  $\{(n, \text{kind } g \ n, \text{stamp } g \ n) \mid n . n \in \text{ids } g\}$

**definition** *true-ids* :: *IRGraph*  $\Rightarrow$  *ID* *set* **where**  
*true-ids* *g* = *ids* *g* -  $\{n \in \text{ids } g . \exists n' . \text{kind } g \ n = \text{RefNode } n'\}$

**definition** *domain-subtraction* :: '*a* *set*  $\Rightarrow$  ('*a*  $\times$  '*b*) *set*  $\Rightarrow$  ('*a*  $\times$  '*b*) *set*  
 (**infix**  $\trianglelefteq 30$ ) **where**  
*domain-subtraction* *s* *r* =  $\{(x, y) . (x, y) \in r \wedge x \notin s\}$

**notation** (*latex*)  
*domain-subtraction* ( $- \trianglelefteq -$ )

**code-datatype** *irgraph*

**fun** *filter-none* **where**  
*filter-none* *g* =  $\{nid \in \text{dom } g . \nexists s . g \ nid = (\text{Some } (\text{NoNode}, s))\}$

**lemma** *no-node-clears*:  
*res* = *no-node* *xs*  $\longrightarrow (\forall x \in \text{set } res . \text{fst } (\text{snd } x) \neq \text{NoNode})$   
**by** *simp*

**lemma** *dom-eq*:  
**assumes**  $\forall x \in \text{set } xs . \text{fst } (\text{snd } x) \neq \text{NoNode}$   
**shows** *filter-none* (*map-of* *xs*) = *dom* (*map-of* *xs*)  
**unfolding** *filter-none.simps* **using** *assms map-of-SomeD*  
**by** *fastforce*

**lemma** *fil-eq*:  
*filter-none* (*map-of* (*no-node* *xs*)) = *set* (*map* *fst* (*no-node* *xs*))  
**using** *no-node-clears*  
**by** (*metis dom-eq dom-map-of-conv-image-fst list.set-map*)

**lemma** *irgraph[code]: ids* (*irgraph* *m*) = *set* (*map* *fst* (*no-node* *m*))  
**unfolding** *irgraph-def ids-def* **using** *fil-eq*  
**by** (*smt Rep-IRGraph comp-apply eq-onp-same-args filter-none.simps ids.abs-eq*  
*ids-def irgraph.abs-eq irgraph.rep-eq irgraph-def mem-Collect-eq*)

**lemma** [*code*]: *Rep-IRGraph* (*irgraph* *m*) = *map-of* (*no-node* *m*)  
**using** *Abs-IRGraph-inverse*  
**by** (*simp add: irgraph.rep-eq*)

— Get the inputs set of a given node ID  
**fun** *inputs* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *ID* *set* **where**



```

    inputs g nid = set (inputs-of (kind g nid))
  — Get the successor set of a given node ID
fun succ :: IRGraph ⇒ ID ⇒ ID set where
    succ g nid = set (successors-of (kind g nid))
  — Gives a relation between node IDs - between a node and its input nodes
fun input-edges :: IRGraph ⇒ ID rel where
    input-edges g = (⋃ i ∈ ids g. {(i,j)|j. j ∈ (inputs g i)})
  — Find all the nodes in the graph that have nid as an input - the usages of nid
fun usages :: IRGraph ⇒ ID ⇒ ID set where
    usages g nid = {i. i ∈ ids g ∧ nid ∈ inputs g i}
fun successor-edges :: IRGraph ⇒ ID rel where
    successor-edges g = (⋃ i ∈ ids g. {(i,j)|j. j ∈ (succ g i)})
fun predecessors :: IRGraph ⇒ ID ⇒ ID set where
    predecessors g nid = {i. i ∈ ids g ∧ nid ∈ succ g i}
fun nodes-of :: IRGraph ⇒ (IRNode ⇒ bool) ⇒ ID set where
    nodes-of g sel = {nid ∈ ids g . sel (kind g nid)}
fun edge :: (IRNode ⇒ 'a) ⇒ ID ⇒ IRGraph ⇒ 'a where
    edge sel nid g = sel (kind g nid)

fun filtered-inputs :: IRGraph ⇒ ID ⇒ (IRNode ⇒ bool) ⇒ ID list where
    filtered-inputs g nid f = filter (f ∘ (kind g)) (inputs-of (kind g nid))
fun filtered-successors :: IRGraph ⇒ ID ⇒ (IRNode ⇒ bool) ⇒ ID list where
    filtered-successors g nid f = filter (f ∘ (kind g)) (successors-of (kind g nid))
fun filtered-usages :: IRGraph ⇒ ID ⇒ (IRNode ⇒ bool) ⇒ ID set where
    filtered-usages g nid f = {n ∈ (usages g nid). f (kind g n)}

fun is-empty :: IRGraph ⇒ bool where
    is-empty g = (ids g = {})

fun any-usage :: IRGraph ⇒ ID ⇒ ID where
    any-usage g nid = hd (sorted-list-of-set (usages g nid))

lemma ids-some[simp]: x ∈ ids g ⟷ kind g x ≠ NoNode
proof —
  have that: x ∈ ids g ⟶ kind g x ≠ NoNode
    using ids.rep-eq kind.rep-eq by force
  have kind g x ≠ NoNode ⟶ x ∈ ids g
    unfolding with-default.simps kind-def ids-def
    by (cases Rep-IRGraph g x = None; auto)
  from this that show ?thesis by auto
qed

lemma not-in-g:
  assumes nid ∉ ids g
  shows kind g nid = NoNode
  using asms ids-some by blast

lemma valid-creation[simp]:
  finite (dom g) ⟷ Rep-IRGraph (Abs-IRGraph g) = g

```

```

using Abs-IRGraph-inverse by (metis Rep-IRGraph mem-Collect-eq)

lemma [simp]: finite (ids g)
using Rep-IRGraph ids.rep-eq by simp

lemma [simp]: finite (ids (irgraph g))
by (simp add: finite-dom-map-of)

lemma [simp]: finite (dom g)  $\longrightarrow$  ids (Abs-IRGraph g) = {nid  $\in$  dom g .  $\nexists$  s. g
nid = Some (NoNode, s)}
using ids.rep-eq by simp

lemma [simp]: finite (dom g)  $\longrightarrow$  kind (Abs-IRGraph g) = ( $\lambda x$  . (case g x of None
 $\Rightarrow$  NoNode | Some n  $\Rightarrow$  fst n))
by (simp add: kind.rep-eq)

lemma [simp]: finite (dom g)  $\longrightarrow$  stamp (Abs-IRGraph g) = ( $\lambda x$  . (case g x of
None  $\Rightarrow$  IllegalStamp | Some n  $\Rightarrow$  snd n))
using stamp.abs-eq stamp.rep-eq by auto

lemma [simp]: ids (irgraph g) = set (map fst (no-node g))
using irgraph by auto

lemma [simp]: kind (irgraph g) = ( $\lambda$ nid. (case (map-of (no-node g)) nid of None
 $\Rightarrow$  NoNode | Some n  $\Rightarrow$  fst n))
using irgraph.rep-eq kind.transfer kind.rep-eq by auto

lemma [simp]: stamp (irgraph g) = ( $\lambda$ nid. (case (map-of (no-node g)) nid of None
 $\Rightarrow$  IllegalStamp | Some n  $\Rightarrow$  snd n))
using irgraph.rep-eq stamp.transfer stamp.rep-eq by auto

lemma map-of-upd: (map-of g)(k  $\mapsto$  v) = (map-of ((k, v)  $\#$  g))
by simp

lemma [code]: replace-node nid k (irgraph g) = (irgraph ( ((nid, k)  $\#$  g)))
proof (cases fst k = NoNode)
  case True
    then show ?thesis
    by (metis (mono-tags, lifting) Rep-IRGraph-inject filter.simps(2) irgraph.abs-eq
no-node.simps replace-node.rep-eq snd-conv)
  next
    case False
    then show ?thesis unfolding irgraph-def replace-node-def no-node.simps
    by (smt (verit, best) Rep-IRGraph comp-apply eq-onp-same-args filter.simps(2)
id-def irgraph.rep-eq map-fun-apply map-of-upd mem-Collect-eq no-node.elims re-
place-node.abs-eq replace-node-def snd-eqD)
qed

```

**lemma** [code]: *add-node* *nid* *k* (*irgraph* *g*) = (*irgraph* (((*nid*, *k*) # *g*)))  
 by (*smt* (*z3*) *Rep-IRGraph-inject add-node.rep-eq filter.simps(2) irgraph.rep-eq map-of-upd no-node.simps snd-conv*)

**lemma** *add-node-lookup*:  
*gup* = *add-node* *nid* (*k*, *s*) *g*  $\longrightarrow$   
 (if *k*  $\neq$  *NoNode* then *kind* *gup* *nid* = *k*  $\wedge$  *stamp* *gup* *nid* = *s* else *kind* *gup* *nid* = *kind* *g* *nid*)  
**proof** (*cases* *k* = *NoNode*)  
 case *True*  
 then show ?thesis  
 by (*simp* add: *add-node.rep-eq kind.rep-eq*)  
next  
 case *False*  
 then show ?thesis  
 by (*simp* add: *kind.rep-eq add-node.rep-eq stamp.rep-eq*)  
qed

**lemma** *remove-node-lookup*:  
*gup* = *remove-node* *nid* *g*  $\longrightarrow$  *kind* *gup* *nid* = *NoNode*  $\wedge$  *stamp* *gup* *nid* = *IllegalStamp*  
 by (*simp* add: *kind.rep-eq remove-node.rep-eq stamp.rep-eq*)

**lemma** *replace-node-lookup*[*simp*]:  
*gup* = *replace-node* *nid* (*k*, *s*) *g*  $\wedge$  *k*  $\neq$  *NoNode*  $\longrightarrow$  *kind* *gup* *nid* = *k*  $\wedge$  *stamp* *gup* *nid* = *s*  
 by (*simp* add: *replace-node.rep-eq kind.rep-eq stamp.rep-eq*)

**lemma** *replace-node-unchanged*:  
*gup* = *replace-node* *nid* (*k*, *s*) *g*  $\longrightarrow$  ( $\forall$  *n*  $\in$  (*ids* *g* - {*nid*}) . *n*  $\in$  *ids* *g*  $\wedge$  *n*  $\in$  *ids* *gup*  $\wedge$  *kind* *g* *n* = *kind* *gup* *n*)  
 by (*simp* add: *kind.rep-eq replace-node.rep-eq*)

### 3.3.1 Example Graphs

Example 1: empty graph (just a start and end node)

**definition** *start-end-graph*:: *IRGraph* **where**  
*start-end-graph* = *irgraph* [(0, *StartNode* *None* 1, *VoidStamp*), (1, *ReturnNode* *None* *None*, *VoidStamp*)]

Example 2: public static int sq(int x) return x \* x;  
 [1 P(0)] / [0 Start] [4 \*] | / V / [5 Return]

**definition** *eg2-sq* :: *IRGraph* **where**  
*eg2-sq* = *irgraph* [  
 (0, *StartNode* *None* 5, *VoidStamp*),  
 (1, *ParameterNode* 0, *default-stamp*),  
 (4, *MulNode* 1 1, *default-stamp*),  
 (5, *ReturnNode* (*Some* 4) *None*, *default-stamp*)

]

```

value input-edges eg2-sq
value usages eg2-sq 1

end

```

## 4 java.lang.Long

Utility functions from the Long class that Graal occasionally makes use of.

```

theory Long
  imports ValueThms
begin

```

```

lemma negative-all-set-32:
   $n < 32 \implies \text{bit } (-1::\text{int}32) \ n$ 
  apply transfer by auto

```

```

definition MaxOrNeg :: nat set  $\Rightarrow$  int where
  MaxOrNeg s = (if s = {} then -1 else Max s)

```

```

definition MinOrHighest :: nat set  $\Rightarrow$  nat  $\Rightarrow$  nat where
  MinOrHighest s m = (if s = {} then m else Min s)

```

```

definition highestOneBit :: ('a::len) word  $\Rightarrow$  int where
  highestOneBit v = MaxOrNeg {n . bit v n}

```

```

definition lowestOneBit :: ('a::len) word  $\Rightarrow$  nat where
  lowestOneBit v = MinOrHighest {n . bit v n} (size v)

```

```

lemma max-bit: bit (v::('a::len) word) n  $\implies n < \text{size } v$ 
  by (simp add: bit-imp-le-length size-word.rep-eq)

```

```

lemma max-set-bit: MaxOrNeg {n . bit (v::('a::len) word) n} < Nat.size v
  using max-bit unfolding MaxOrNeg-def
  by force

```

### 4.1 Long.numberOfLeadingZeros

```

definition numberOfLeadingZeros :: ('a::len) word  $\Rightarrow$  nat where
  numberOfLeadingZeros v = nat (Nat.size v - highestOneBit v - 1)

```

```

lemma MaxOrNeg-neg: MaxOrNeg {} = -1

```

**by** (*simp add: MaxOrNeg-def*)

**lemma** *MaxOrNeg-max*:  $s \neq \{\} \implies \text{MaxOrNeg } s = \text{Max } s$   
**by** (*simp add: MaxOrNeg-def*)

**lemma** *zero-no-bits*:  
 $\{n \mid \text{bit } 0 \ n\} = \{\}$   
**by** *simp*

**lemma** *highestOneBit* ( $0::64 \text{ word}$ ) =  $-1$   
**by** (*simp add: MaxOrNeg-neg highestOneBit-def*)

**lemma** *numberOfLeadingZeros* ( $0::64 \text{ word}$ ) =  $64$   
**unfolding** *numberOfLeadingZeros-def* **using** *MaxOrNeg-neg highestOneBit-def size64*  
**by** (*smt (verit) nat-int zero-no-bits*)

**lemma** *highestOneBit-top*:  $\text{Max } \{\text{highestOneBit } (v::64 \text{ word})\} < 64$   
**unfolding** *highestOneBit-def*  
**by** (*metis Max-singleton int-eq-iff-numeral max-set-bit size64*)

**lemma** *numberOfLeadingZeros-top*:  $\text{Max } \{\text{numberOfLeadingZeros } (v::64 \text{ word})\} \leq 64$   
**unfolding** *numberOfLeadingZeros-def*  
**using** *size64*  
**by** (*simp add: MaxOrNeg-def highestOneBit-def nat-le-iff*)

**lemma** *numberOfLeadingZeros-range*:  $0 \leq \text{numberOfLeadingZeros } a \wedge \text{numberOfLeadingZeros } a \leq \text{Nat.size } a$   
**unfolding** *numberOfLeadingZeros-def*  
**using** *MaxOrNeg-def highestOneBit-def nat-le-iff*  
**by** (*smt (verit) bot-nat-0.extremum int-eq-iff*)

**lemma** *leadingZerosAddHighestOne*:  $\text{numberOfLeadingZeros } v + \text{highestOneBit } v = \text{Nat.size } v - 1$   
**unfolding** *numberOfLeadingZeros-def highestOneBit-def*  
**using** *MaxOrNeg-def int-nat-eq int-ops(6) max-bit order-less-irrefl* **by** *fastforce*

## 4.2 Long.numberOfTrailingZeros

**definition** *numberOfTrailingZeros* ::  $('a::\text{len}) \text{ word} \Rightarrow \text{nat}$  **where**  
 $\text{numberOfTrailingZeros } v = \text{lowestOneBit } v$

**lemma** *lowestOneBit-bot*:  $\text{lowestOneBit } (0::64 \text{ word}) = 64$   
**unfolding** *lowestOneBit-def MinOrHighest-def*  
**by** (*simp add: size64*)

**lemma** *bit-zero-set-in-top*:  $\text{bit } (-1::'a::\text{len} \text{ word}) \ 0$   
**by** *auto*

**lemma** *nat-bot-set*:  $(0::nat) \in xs \longrightarrow (\forall x \in xs . 0 \leq x)$   
**by** *fastforce*

**lemma** *numberOfTrailingZeros*  $(0::64 \text{ word}) = 64$   
**unfolding** *numberOfTrailingZeros-def*  
**using** *lowestOneBit-bot* **by** *simp*

### 4.3 Long.bitCount

**definition** *bitCount* ::  $('a::len) \text{ word} \Rightarrow nat$  **where**  
*bitCount*  $v = \text{card } \{n . \text{bit } v \ n\}$

**lemma** *bitCount 0 = 0*  
**unfolding** *bitCount-def*  
**by**  $(metis \text{ card.empty zero-no-bits})$

### 4.4 Long.zeroCount

**definition** *zeroCount* ::  $('a::len) \text{ word} \Rightarrow nat$  **where**  
*zeroCount*  $v = \text{card } \{n . n < \text{Nat.size } v \wedge \neg(\text{bit } v \ n)\}$

**lemma** *zeroCount-finite*:  $\text{finite } \{n . n < \text{Nat.size } v \wedge \neg(\text{bit } v \ n)\}$   
**using** *finite-nat-set-iff-bounded* **by** *blast*

**lemma** *negone-set*:  
 $\text{bit } (-1::('a::len) \text{ word}) \ n \longleftrightarrow n < \text{LENGTH}('a)$   
**by** *simp*

**lemma** *negone-all-bits*:  
 $\{n . \text{bit } (-1::('a::len) \text{ word}) \ n\} = \{n . 0 \leq n \wedge n < \text{LENGTH}('a)\}$   
**using** *negone-set*  
**by** *auto*

**lemma** *bitCount-finite*:  
 $\text{finite } \{n . \text{bit } (v::('a::len) \text{ word}) \ n\}$   
**by** *simp*

**lemma** *card-of-range*:  
 $x = \text{card } \{n . 0 \leq n \wedge n < x\}$   
**by** *simp*

**lemma** *range-of-nat*:  
 $\{(n::nat) . 0 \leq n \wedge n < x\} = \{n . n < x\}$   
**by** *simp*

**lemma** *finite-range*:  
 $\text{finite } \{n::nat . n < x\}$   
**by** *simp*

```

lemma range-eq:
  fixes  $x\ y :: \text{nat}$ 
  shows  $\text{card } \{y..<x\} = \text{card } \{y<..x\}$ 
  using card-atLeastLessThan card-greaterThanAtMost by presburger

lemma card-of-range-bound:
  fixes  $x\ y :: \text{nat}$ 
  assumes  $x > y$ 
  shows  $x - y = \text{card } \{n . y < n \wedge n \leq x\}$ 
proof -
  have finite:  $\text{finite } \{n . y \leq n \wedge n < x\}$ 
    by auto
  have nonempty:  $\{n . y \leq n \wedge n < x\} \neq \{\}$ 
    using assms by blast
  have simp:  $\{n . y < n \wedge n \leq x\} = \{y<..x\}$ 
    by auto
  have  $x - y = \text{card } \{y<..x\}$ 
    by auto
  then show ?thesis
    unfolding simp by blast
qed

lemma bitCount  $(-1::('a::\text{len}) \text{ word}) = \text{LENGTH}('a)$ 
  unfolding bitCount-def using card-of-range
  by (metis (no-types, lifting) Collect-cong negone-all-bits)

lemma bitCount-range:
  fixes  $n :: ('a::\text{len}) \text{ word}$ 
  shows  $0 \leq \text{bitCount } n \wedge \text{bitCount } n \leq \text{Nat.size } n$ 
  unfolding bitCount-def
  by (metis atLeastLessThan-iff bot-nat-0.extremum max-bit mem-Collect-eq subsetI
subset-eq-atLeast0-lessThan-card)

lemma zerosAboveHighestOne:
   $n > \text{highestOneBit } a \implies \neg(\text{bit } a\ n)$ 
  unfolding highestOneBit-def MaxOrNeg-def
  by (metis (mono-tags, opaque-lifting) Collect-empty-eq Max-ge finite-bit-word
less-le-not-le mem-Collect-eq of-nat-le-iff)

lemma zerosBelowLowestOne:
  assumes  $n < \text{lowestOneBit } a$ 
  shows  $\neg(\text{bit } a\ n)$ 
proof (cases  $\{i. \text{bit } a\ i\} = \{\}$ )
  case True
  then show ?thesis by simp
next
  case False
  have  $n < \text{Min } (\text{Collect } (\text{bit } a)) \implies \neg \text{bit } a\ n$ 

```

```

    using False by auto
  then show ?thesis
    by (metis False MinOrHighest-def assms lowestOneBit-def)
qed

```

```

lemma union-bit-sets:
  fixes a :: ('a::len) word
  shows  $\{n . n < \text{Nat.size } a \wedge \text{bit } a \ n\} \cup \{n . n < \text{Nat.size } a \wedge \neg(\text{bit } a \ n)\} = \{n . n < \text{Nat.size } a\}$ 
  by fastforce

```

```

lemma disjoint-bit-sets:
  fixes a :: ('a::len) word
  shows  $\{n . n < \text{Nat.size } a \wedge \text{bit } a \ n\} \cap \{n . n < \text{Nat.size } a \wedge \neg(\text{bit } a \ n)\} = \{\}$ 
  by blast

```

```

lemma qualified-bitCount:
  bitCount v = card  $\{n . n < \text{Nat.size } v \wedge \text{bit } v \ n\}$ 
  by (metis (no-types, lifting) Collect-cong bitCount-def max-bit)

```

```

lemma card-eq:
  assumes finite x  $\wedge$  finite y  $\wedge$  finite z
  assumes  $x \cup y = z$ 
  assumes  $y \cap x = \{\}$ 
  shows  $\text{card } z - \text{card } y = \text{card } x$ 
  using assms add-diff-cancel-right' card-Un-disjoint
  by (metis inf.commute)

```

```

lemma card-add:
  assumes finite x  $\wedge$  finite y  $\wedge$  finite z
  assumes  $x \cup y = z$ 
  assumes  $y \cap x = \{\}$ 
  shows  $\text{card } x + \text{card } y = \text{card } z$ 
  using assms card-Un-disjoint
  by (metis inf.commute)

```

```

lemma card-add-inverses:
  assumes finite  $\{n. Q \ n \wedge \neg(P \ n)\} \wedge$  finite  $\{n. Q \ n \wedge P \ n\} \wedge$  finite  $\{n. Q \ n\}$ 
  shows  $\text{card } \{n. Q \ n \wedge P \ n\} + \text{card } \{n. Q \ n \wedge \neg(P \ n)\} = \text{card } \{n. Q \ n\}$ 
  apply (rule card-add)
  using assms apply simp
  apply auto[1]
  by auto

```

```

lemma ones-zero-sum-to-width:
  bitCount a + zeroCount a = Nat.size a
proof -
  have add-cards:  $\text{card } \{n. (\lambda n. n < \text{size } a) \ n \wedge (\text{bit } a \ n)\} + \text{card } \{n. (\lambda n. n <$ 

```



```

size a) n ∧ ¬(bit a n)} = card {n. (λn. n < size a) n}
  apply (rule card-add-inverses) by simp
  then have ... = Nat.size a
    by auto
  then show ?thesis
    unfolding bitCount-def zeroCount-def using max-bit
    by (metis (mono-tags, lifting) Collect-cong add-cards)
qed

lemma intersect-bitCount-helper:
  card {n . n < Nat.size a} - bitCount a = card {n . n < Nat.size a ∧ ¬(bit a n)}
proof -
  have size-def: Nat.size a = card {n . n < Nat.size a}
    using card-of-range by simp
  have bitCount-def: bitCount a = card {n . n < Nat.size a ∧ bit a n}
    using qualified-bitCount by auto
  have disjoint: {n . n < Nat.size a ∧ bit a n} ∩ {n . n < Nat.size a ∧ ¬(bit a n)} = {}
    using disjoint-bit-sets by auto
  have union: {n . n < Nat.size a ∧ bit a n} ∪ {n . n < Nat.size a ∧ ¬(bit a n)}
    = {n . n < Nat.size a}
    using union-bit-sets by auto
  show ?thesis
    unfolding bitCount-def
    apply (rule card-eq)
    using finite-range apply simp
    using union apply blast
    using disjoint by simp
qed

lemma intersect-bitCount:
  Nat.size a - bitCount a = card {n . n < Nat.size a ∧ ¬(bit a n)}
  using card-of-range intersect-bitCount-helper by auto

hide-fact intersect-bitCount-helper

end

```

## 5 Data-flow Semantics

```

theory IRTreeEval
  imports
    Graph.Stamp
begin

```

We define a tree representation of data-flow nodes, as an abstraction of the graph view.

Data-flow trees are evaluated in the context of a method state (currently

called MapState in the theories for historical reasons).

The method state consists of the values for each method parameter, references to method parameters use an index of the parameter within the parameter list, as such we store a list of parameter values which are looked up at parameter references.

The method state also stores a mapping of node ids to values. The contents of this mapping is calculated during the traversal of the control flow graph. As a concrete example, as the *SignedDivNode* can have side-effects (during division by zero), it is treated as part of the control-flow, since the data-flow phase is specified to be side-effect free. As a result, the control-flow semantics for *SignedDivNode* calculates the value of a node and maps the node identifier to the value within the method state. The data-flow semantics then just reads the value stored in the method state for the node.

**type-synonym** *ID* = *nat*

**type-synonym** *MapState* = *ID*  $\Rightarrow$  *Value*

**type-synonym** *Params* = *Value list*

**definition** *new-map-state* :: *MapState* **where**

*new-map-state* = ( $\lambda x.$  *UndefVal*)

## 5.1 Data-flow Tree Representation

**datatype** *IRUnaryOp* =

*UnaryAbs*  
| *UnaryNeg*  
| *UnaryNot*  
| *UnaryLogicNegation*  
| *UnaryNarrow* (*ir-inputBits*: *nat*) (*ir-resultBits*: *nat*)  
| *UnarySignExtend* (*ir-inputBits*: *nat*) (*ir-resultBits*: *nat*)  
| *UnaryZeroExtend* (*ir-inputBits*: *nat*) (*ir-resultBits*: *nat*)

**datatype** *IRBinaryOp* =

*BinAdd*  
| *BinMul*  
| *BinSub*  
| *BinAnd*  
| *BinOr*  
| *BinXor*  
| *BinShortCircuitOr*  
| *BinLeftShift*  
| *BinRightShift*  
| *BinURightShift*  
| *BinIntegerEquals*  
| *BinIntegerLessThan*  
| *BinIntegerBelow*

**datatype** (*discs-sels*) *IRExpr* =

```

    UnaryExpr (ir-uop: IRUnaryOp) (ir-value: IRExpr)
  | BinaryExpr (ir-op: IRBinaryOp) (ir-x: IRExpr) (ir-y: IRExpr)
  | ConditionalExpr (ir-condition: IRExpr) (ir-trueValue: IRExpr) (ir-falseValue:
IRExpr)

  | ParameterExpr (ir-index: nat) (ir-stamp: Stamp)

  | LeafExpr (ir-nid: ID) (ir-stamp: Stamp)

  | ConstantExpr (ir-const: Value)
  | ConstantVar (ir-name: string)
  | VariableExpr (ir-name: string) (ir-stamp: Stamp)

```

```

fun is-ground :: IRExpr ⇒ bool where
  is-ground (UnaryExpr op e) = is-ground e |
  is-ground (BinaryExpr op e1 e2) = (is-ground e1 ∧ is-ground e2) |
  is-ground (ConditionalExpr b e1 e2) = (is-ground b ∧ is-ground e1 ∧ is-ground
e2) |
  is-ground (ParameterExpr i s) = True |
  is-ground (LeafExpr n s) = True |
  is-ground (ConstantExpr v) = True |
  is-ground (ConstantVar name) = False |
  is-ground (VariableExpr name s) = False

```

```

typedef GroundExpr = { e :: IRExpr . is-ground e }
using is-ground.simps(6) by blast

```

## 5.2 Functions for re-calculating stamps

Note: in Java all integer calculations are done as 32 or 64 bit calculations. However, here we generalise the operators to allow any size calculations. Many operators have the same output bits as their inputs. However, the unary integer operators that are not *normal\_unary* are narrowing or widening operators, so the result bits is specified by the operator. The binary integer operators are divided into three groups: (1) *binary\_fixed\_32* operators always output 32 bits, (2) *binary\_shift\_ops* operators output size is determined by their left argument, and (3) other operators output the same number of bits as both their inputs.

**abbreviation** *binary-fixed-32-ops* :: IRBinaryOp set **where**  
*binary-fixed-32-ops* ≡ { BinShortCircuitOr, BinIntegerEquals, BinIntegerLessThan, BinIntegerBelow }

**abbreviation** *binary-shift-ops* :: IRBinaryOp set **where**  
*binary-shift-ops* ≡ { BinLeftShift, BinRightShift, BinURightShift }

**abbreviation** *normal-unary* :: IRUnaryOp set **where**  
*normal-unary* ≡ { UnaryAbs, UnaryNeg, UnaryNot, UnaryLogicNegation }

```

fun stamp-unary :: IRUnaryOp ⇒ Stamp ⇒ Stamp where

  stamp-unary op (IntegerStamp b lo hi) =
    unrestricted-stamp (IntegerStamp (if op ∈ normal-unary then b else (ir-resultBits
    op)) lo hi) |

  stamp-unary op - = IllegalStamp

fun stamp-binary :: IRBinaryOp ⇒ Stamp ⇒ Stamp ⇒ Stamp where
  stamp-binary op (IntegerStamp b1 lo1 hi1) (IntegerStamp b2 lo2 hi2) =
    (if op ∈ binary-shift-ops then unrestricted-stamp (IntegerStamp b1 lo1 hi1)
    else if b1 ≠ b2 then IllegalStamp else
    (if op ∈ binary-fixed-32-ops
    then unrestricted-stamp (IntegerStamp 32 lo1 hi1)
    else unrestricted-stamp (IntegerStamp b1 lo1 hi1))) |

  stamp-binary op - - = IllegalStamp

fun stamp-expr :: IRExpr ⇒ Stamp where
  stamp-expr (UnaryExpr op x) = stamp-unary op (stamp-expr x) |
  stamp-expr (BinaryExpr bop x y) = stamp-binary bop (stamp-expr x) (stamp-expr
  y) |
  stamp-expr (ConstantExpr val) = constantAsStamp val |
  stamp-expr (LeafExpr i s) = s |
  stamp-expr (ParameterExpr i s) = s |
  stamp-expr (ConditionalExpr c t f) = meet (stamp-expr t) (stamp-expr f)

export-code stamp-unary stamp-binary stamp-expr

```

### 5.3 Data-flow Tree Evaluation

```

fun unary-eval :: IRUnaryOp ⇒ Value ⇒ Value where
  unary-eval UnaryAbs v = intval-abs v |
  unary-eval UnaryNeg v = intval-negate v |
  unary-eval UnaryNot v = intval-not v |
  unary-eval UnaryLogicNegation v = intval-logic-negation v |
  unary-eval (UnaryNarrow inBits outBits) v = intval-narrow inBits outBits v |
  unary-eval (UnarySignExtend inBits outBits) v = intval-sign-extend inBits outBits
  v |
  unary-eval (UnaryZeroExtend inBits outBits) v = intval-zero-extend inBits outBits
  v

```

```

fun bin-eval :: IRBinaryOp ⇒ Value ⇒ Value ⇒ Value where
  bin-eval BinAdd v1 v2 = intval-add v1 v2 |
  bin-eval BinMul v1 v2 = intval-mul v1 v2 |
  bin-eval BinSub v1 v2 = intval-sub v1 v2 |
  bin-eval BinAnd v1 v2 = intval-and v1 v2 |

```

$\text{bin-eval BinOr } v1 \ v2 = \text{intval-or } v1 \ v2 \mid$   
 $\text{bin-eval BinXor } v1 \ v2 = \text{intval-xor } v1 \ v2 \mid$   
 $\text{bin-eval BinShortCircuitOr } v1 \ v2 = \text{intval-short-circuit-or } v1 \ v2 \mid$   
 $\text{bin-eval BinLeftShift } v1 \ v2 = \text{intval-left-shift } v1 \ v2 \mid$   
 $\text{bin-eval BinRightShift } v1 \ v2 = \text{intval-right-shift } v1 \ v2 \mid$   
 $\text{bin-eval BinURightShift } v1 \ v2 = \text{intval-uright-shift } v1 \ v2 \mid$   
 $\text{bin-eval BinIntegerEquals } v1 \ v2 = \text{intval-equals } v1 \ v2 \mid$   
 $\text{bin-eval BinIntegerLessThan } v1 \ v2 = \text{intval-less-than } v1 \ v2 \mid$   
 $\text{bin-eval BinIntegerBelow } v1 \ v2 = \text{intval-below } v1 \ v2$

**lemmas** *eval-thms* =

$\text{intval-abs.simps}$   $\text{intval-negate.simps}$   $\text{intval-not.simps}$   
 $\text{intval-logic-negation.simps}$   $\text{intval-narrow.simps}$   
 $\text{intval-sign-extend.simps}$   $\text{intval-zero-extend.simps}$   
 $\text{intval-add.simps}$   $\text{intval-mul.simps}$   $\text{intval-sub.simps}$   
 $\text{intval-and.simps}$   $\text{intval-or.simps}$   $\text{intval-xor.simps}$   
 $\text{intval-left-shift.simps}$   $\text{intval-right-shift.simps}$   
 $\text{intval-uright-shift.simps}$   $\text{intval-equals.simps}$   
 $\text{intval-less-than.simps}$   $\text{intval-below.simps}$

**inductive** *not-undef-or-fail* :: *Value*  $\Rightarrow$  *Value*  $\Rightarrow$  *bool* **where**

$\llbracket \text{value} \neq \text{UndefVal} \rrbracket \Longrightarrow \text{not-undef-or-fail value value}$

**notation** (*latex output*)

*not-undef-or-fail* (- = -)

**inductive**

*evaltree* :: *MapState*  $\Rightarrow$  *Params*  $\Rightarrow$  *IRExpr*  $\Rightarrow$  *Value*  $\Rightarrow$  *bool* ( $\llbracket -, - \rrbracket \vdash - \mapsto -$  55)

**for** *m p* **where**

*ConstantExpr*:

$\llbracket \text{wf-value } c \rrbracket$   
 $\Longrightarrow [m, p] \vdash (\text{ConstantExpr } c) \mapsto c \mid$

*ParameterExpr*:

$\llbracket i < \text{length } p; \text{valid-value } (p!i) \ s \rrbracket$   
 $\Longrightarrow [m, p] \vdash (\text{ParameterExpr } i \ s) \mapsto p!i \mid$

*ConditionalExpr*:

$\llbracket [m, p] \vdash ce \mapsto \text{cond};$   
 $\text{branch} = (\text{if val-to-bool cond then te else fe});$   
 $[m, p] \vdash \text{branch} \mapsto \text{result};$   
 $\text{result} \neq \text{UndefVal} \rrbracket$   
 $\Longrightarrow [m, p] \vdash (\text{ConditionalExpr } ce \ te \ fe) \mapsto \text{result} \mid$

*UnaryExpr*:

$\llbracket [m, p] \vdash xe \mapsto x; \rrbracket$

$result = (unary\text{-}eval\ op\ x);$   
 $result \neq UndefVal]$   
 $\implies [m,p] \vdash (UnaryExpr\ op\ xe) \mapsto result \mid$

*BinaryExpr:*  
 $[[m,p] \vdash xe \mapsto x;$   
 $[m,p] \vdash ye \mapsto y;$   
 $result = (bin\text{-}eval\ op\ x\ y);$   
 $result \neq UndefVal]$   
 $\implies [m,p] \vdash (BinaryExpr\ op\ xe\ ye) \mapsto result \mid$

*LeafExpr:*  
 $[[val = m\ n;$   
 $valid\text{-}value\ val\ s]]$   
 $\implies [m,p] \vdash LeafExpr\ n\ s \mapsto val$

**code-pred** (*modes:  $i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow bool$  as evalT*)  
 $[show\text{-}steps, show\text{-}mode\text{-}inference, show\text{-}intermediate\text{-}results]$   
*evaltree .*

**inductive**

*evaltrees :: MapState  $\Rightarrow$  Params  $\Rightarrow$  IRExpr list  $\Rightarrow$  Value list  $\Rightarrow$  bool* ( $[-, -] \vdash - \mapsto_L$   
- 55)

**for** *m p* **where**

*EvalNil:*  
 $[m,p] \vdash [] \mapsto_L [] \mid$

*EvalCons:*  
 $[[m,p] \vdash x \mapsto xval;$   
 $[m,p] \vdash yy \mapsto_L yyval]$   
 $\implies [m,p] \vdash (x\#yy) \mapsto_L (xval\#yyval)$

**code-pred** (*modes:  $i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow bool$  as evalTs*)  
*evaltrees .*

**definition** *sq-param0 :: IRExpr* **where**

*sq-param0 = BinaryExpr BinMul*  
 $(ParameterExpr\ 0\ (IntegerStamp\ 32\ (-\ 2147483648)\ 2147483647))$   
 $(ParameterExpr\ 0\ (IntegerStamp\ 32\ (-\ 2147483648)\ 2147483647))$

**values**  $\{v.\ evaltree\ new\text{-}map\text{-}state\ [IntVal\ 32\ 5]\ sq\text{-}param0\ v\}$

**declare** *evaltree.intros* [*intro*]  
**declare** *evaltrees.intros* [*intro*]

## 5.4 Data-flow Tree Refinement

We define the induced semantic equivalence relation between expressions. Note that syntactic equality implies semantic equivalence, but not vice versa.

**definition** *equiv-exprs* :: *IRExpr*  $\Rightarrow$  *IRExpr*  $\Rightarrow$  *bool* (*-*  $\doteq$  *-* 55) **where**  
 $(e1 \doteq e2) = (\forall m p v. ([m,p] \vdash e1 \mapsto v) \longleftrightarrow ([m,p] \vdash e2 \mapsto v))$

We also prove that this is a total equivalence relation (*equivp equiv-exprs*) (HOL.Equiv\_Relations), so that we can reuse standard results about equivalence relations.

**lemma** *equivp equiv-exprs*  
**apply** (*auto simp add: equivp-def equiv-exprs-def*)  
**by** (*metis equiv-exprs-def*)<sup>+</sup>

We define a refinement ordering over *IRExpr* and show that it is a preorder. Note that it is asymmetric because *e2* may refer to fewer variables than *e1*.

**instantiation** *IRExpr* :: *preorder* **begin**

**notation** *less-eq* (*infix*  $\sqsubseteq$  65)

**definition**

*le-expr-def* [*simp*]:  
 $(e_2 \leq e_1) \longleftrightarrow (\forall m p v. ([m,p] \vdash e_1 \mapsto v) \longrightarrow ([m,p] \vdash e_2 \mapsto v))$

**definition**

*lt-expr-def* [*simp*]:  
 $(e_1 < e_2) \longleftrightarrow (e_1 \leq e_2 \wedge \neg (e_1 \doteq e_2))$

**instance proof**

**fix** *x y z* :: *IRExpr*  
**show**  $x < y \longleftrightarrow x \leq y \wedge \neg (y \leq x)$  **by** (*simp add: equiv-exprs-def; auto*)  
**show**  $x \leq x$  **by** *simp*  
**show**  $x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$  **by** *simp*  
**qed**

**end**

**abbreviation** (*output*) *Refines* :: *IRExpr*  $\Rightarrow$  *IRExpr*  $\Rightarrow$  *bool* (*infix*  $\sqsupseteq$  64)  
**where**  $e_1 \sqsupseteq e_2 \equiv (e_2 \leq e_1)$

## 5.5 Stamp Masks

A stamp can contain additional range information in the form of masks. A stamp has an up mask and a down mask, corresponding to the bits that may be set and the bits that must be set.

Examples: A stamp where no range information is known will have; an up mask of -1 as all bits may be set, and a down mask of 0 as no bits must be set.

A stamp known to be one should have; an up mask of 1 as only the first bit may be set, no others, and a down mask of 1 as the first bit must be set and no others.

We currently don't carry mask information in stamps, and instead assume correct masks to prove optimizations.

```

locale stamp-mask =
  fixes up :: IRExp ⇒ int64 (↑)
  fixes down :: IRExp ⇒ int64 (↓)
  assumes up-spec: [m, p] ⊢ e ↦ IntVal b v ⇒ (and v (not ((ucast (↑e))))) = 0
  and down-spec: [m, p] ⊢ e ↦ IntVal b v ⇒ (and (not v) (ucast (↓e))) = 0
begin

```

```

lemma may-implies-either:
  [m, p] ⊢ e ↦ IntVal b v ⇒ bit (↑e) n ⇒ bit v n = False ∨ bit v n = True
by simp

```

```

lemma not-may-implies-false:
  [m, p] ⊢ e ↦ IntVal b v ⇒ ¬(bit (↑e) n) ⇒ bit v n = False
using up-spec
using bit-and-iff bit-eq-iff bit-not-iff bit-unsigned-iff down-spec
by (smt (verit, best) bit.double-compl)

```

```

lemma must-implies-true:
  [m, p] ⊢ e ↦ IntVal b v ⇒ bit (↓e) n ⇒ bit v n = True
using down-spec
by (metis bit.compl-one bit-and-iff bit-minus-1-iff bit-not-iff impossible-bit ucast-id)

```

```

lemma not-must-implies-either:
  [m, p] ⊢ e ↦ IntVal b v ⇒ ¬(bit (↓e) n) ⇒ bit v n = False ∨ bit v n = True
by simp

```

```

lemma must-implies-may:
  [m, p] ⊢ e ↦ IntVal b v ⇒ n < 32 ⇒ bit (↓e) n ⇒ bit (↑e) n
by (meson must-implies-true not-may-implies-false)

```

```

lemma up-mask-and-zero-implies-zero:
  assumes and (↑x) (↑y) = 0
  assumes [m, p] ⊢ x ↦ IntVal b xv
  assumes [m, p] ⊢ y ↦ IntVal b yv
  shows and xv yv = 0
  using assms
  by (smt (z3) and.commute and.right-neutral and-zero-eq bit.compl-zero bit.conj-cancel-right
    bit.conj-disj-distrib(1) ucast-id up-spec word-bw-assocs(1) word-not-dist(2))

```

```

lemma not-down-up-mask-and-zero-implies-zero:
  assumes and (not (↓x)) (↑y) = 0
  assumes [m, p] ⊢ x ↦ IntVal b xv

```



```

assumes  $[m, p] \vdash y \mapsto \text{IntVal } b \ yv$ 
shows  $\text{and } xv \ yv = yv$ 
using assms
by (smt (z3) and-zero-eq bit.conj-cancel-left bit.conj-disj-distrib(1) bit.conj-disj-distrib(2)
bit.de-Morgan-disj down-spec or-eq-not-not-and ucast-id up-spec word-ao-absorbs(2)
word-ao-absorbs(8) word-bw-lcs(1) word-not-dist(2))

```

**end**

```

definition IRExpr-up :: IRExpr  $\Rightarrow$  int64 where
  IRExpr-up e = not 0

```

```

definition IRExpr-down :: IRExpr  $\Rightarrow$  int64 where
  IRExpr-down e = 0

```

```

lemma ucast-zero: (ucast (0::int64)::int32) = 0
by simp

```

```

lemma ucast-minus-one: (ucast (-1::int64)::int32) = -1
apply transfer by auto

```

```

interpretation simple-mask: stamp-mask
  IRExpr-up :: IRExpr  $\Rightarrow$  int64
  IRExpr-down :: IRExpr  $\Rightarrow$  int64
unfolding IRExpr-up-def IRExpr-down-def
apply unfold-locales
by (simp add: ucast-minus-one)+

```

**end**

## 5.6 Data-flow Tree Theorems

```

theory IRTreeEvalThms
imports
  Graph.ValueThms
  IRTreeEval
begin

```

### 5.6.1 Deterministic Data-flow Evaluation

```

lemma evalDet:
   $[m, p] \vdash e \mapsto v_1 \Rightarrow$ 
   $[m, p] \vdash e \mapsto v_2 \Rightarrow$ 
   $v_1 = v_2$ 
apply (induction arbitrary: v2 rule: evaltree.induct)
by (elim EvalTreeE; auto)+

```

```

lemma evalAllDet:
   $[m, p] \vdash e \mapsto_L v1 \Rightarrow$ 
   $[m, p] \vdash e \mapsto_L v2 \Rightarrow$ 

```

```

v1 = v2
apply (induction arbitrary: v2 rule: evaltrees.induct)
apply (elim EvalTreeE; auto)
using evalDet by force

```

### 5.6.2 Typing Properties for Integer Evaluation Functions

We use three simple typing properties on integer values: *isIntVal32*, *isIntVal64* and the more general *isIntVal*.

```

lemma unary-eval-not-obj-ref:
  shows unary-eval op x  $\neq$  ObjRef v
  by (cases op; cases x; auto)

```

```

lemma unary-eval-not-obj-str:
  shows unary-eval op x  $\neq$  ObjStr v
  by (cases op; cases x; auto)

```

```

lemma unary-eval-int:
  assumes def: unary-eval op x  $\neq$  UndefVal
  shows is-IntVal (unary-eval op x)
  unfolding is-IntVal-def using def
  apply (cases unary-eval op x; auto)
  using unary-eval-not-obj-ref unary-eval-not-obj-str by simp+

```

```

lemma bin-eval-int:
  assumes def: bin-eval op x y  $\neq$  UndefVal
  shows is-IntVal (bin-eval op x y)
  apply (cases op; cases x; cases y)
  unfolding is-IntVal-def using def apply auto
    apply presburger+
    apply (meson bool-to-val.elims)
    apply (meson bool-to-val.elims)
    apply (smt (verit) new-int.simps)+
  by (meson bool-to-val.elims)+

```

```

lemma IntVal0:
  (IntVal 32 0) = (new-int 32 0)
  unfolding new-int.simps
  by auto

```

```

lemma IntVal1:
  (IntVal 32 1) = (new-int 32 1)
  unfolding new-int.simps

```

by *auto*

**lemma** *bin-eval-new-int*:  
**assumes** *def*: *bin-eval op x y*  $\neq$  *UndefVal*  
**shows**  $\exists b v. (bin\text{-}eval\ op\ x\ y) = new\text{-}int\ b\ v \wedge$   
 $b = (if\ op \in binary\text{-}fixed\text{-}32\text{-}ops\ then\ 32\ else\ intval\text{-}bits\ x)$   
**apply** (*cases op*; *cases x*; *cases y*)  
**unfolding** *is-IntVal-def* **using** *def* **apply** *auto*  
**apply** *presburger+*  
**apply** (*metis take-bit-and*)  
**apply** *presburger*  
**apply** (*metis take-bit-or*)  
**apply** *presburger*  
**apply** (*metis take-bit-xor*)  
**apply** *presburger*  
**using** *IntVal0 IntVal1*  
**apply** (*metis bool-to-val.elims new-int.simps*)  
**apply** *presburger*  
**apply** (*smt (verit) new-int.elims*)  
**apply** (*smt (verit, best) new-int.elims*)  
**apply** (*metis IntVal0 IntVal1 bool-to-val.elims new-int.simps*)  
**apply** *presburger*  
**apply** (*metis IntVal0 IntVal1 bool-to-val.elims new-int.simps*)  
**apply** *presburger*  
**apply** (*metis IntVal0 IntVal1 bool-to-val.elims new-int.simps*)  
**by** *meson*

**lemma** *int-stamp*:  
**assumes** *i*: *is-IntVal v*  
**shows** *is-IntegerStamp (constantAsStamp v)*  
**using** *i* **unfolding** *is-IntegerStamp-def is-IntVal-def* **by** *auto*

**lemma** *validStampIntConst*:  
**assumes** *v* = *IntVal b ival*  
**assumes**  $0 < b \wedge b \leq 64$   
**shows** *valid-stamp (constantAsStamp v)*  
**proof** –  
**have** *bnds*: *fst (bit-bounds b)*  $\leq int\text{-}signed\text{-}value\ b\ ival \wedge int\text{-}signed\text{-}value\ b\ ival$   
 $\leq snd\ (bit\text{-}bounds\ b)$   
**using** *assms int-signed-value-bounds*  
**by** *presburger*  
**have** *s*: *constantAsStamp v* = *IntegerStamp b (int-signed-value b ival) (int-signed-value b ival)*  
**using** *assms(1) constantAsStamp.simps(1)* **by** *blast*  
**then show** *?thesis*  
**unfolding** *s valid-stamp.simps*

```

    using assms(2) assms bnds by linarith
qed

lemma validDefIntConst:
  assumes v: v = IntVal b ival
  assumes 0 < b ∧ b ≤ 64
  assumes take-bit b ival = ival
  shows valid-value v (constantAsStamp v)
proof -
  have bnds: fst (bit-bounds b) ≤ int-signed-value b ival ∧ int-signed-value b ival
    ≤ snd (bit-bounds b)
    using assms int-signed-value-bounds
    by presburger
  have s: constantAsStamp v = IntegerStamp b (int-signed-value b ival) (int-signed-value
    b ival)
    using assms(1) constantAsStamp.simps(1) by blast
  then show ?thesis
    unfolding v unfolding v unfolding valid-value.simps
    using assms validStampIntConst
    by simp
qed

```

### 5.6.3 Evaluation Results are Valid

A valid value cannot be *UndefVal*.

```

lemma valid-not-undef:
  assumes a1: valid-value val s
  assumes a2: s ≠ VoidStamp
  shows val ≠ UndefVal
  apply (rule valid-value.elims(1)[of val s True])
  using a1 a2 by auto

```

```

lemma valid-VoidStamp[elim]:
  shows valid-value val VoidStamp ⇒
    val = UndefVal
  using valid-value.simps by metis

```

```

lemma valid-ObjStamp[elim]:
  shows valid-value val (ObjectStamp klass exact nonNull alwaysNull) ⇒
    (∃ v. val = ObjRef v)
  using valid-value.simps by (metis val-to-bool.cases)

```

```

lemma valid-int[elim]:
  shows valid-value val (IntegerStamp b lo hi) ⇒
    (∃ v. val = IntVal b v)
  using valid-value.elims(2) by fastforce

```

```

lemmas valid-value-elim =

```

*valid-VoidStamp*  
*valid-ObjStamp*  
*valid-int*

**lemma** *evaltree-not-undef*:  
**fixes** *m p e v*  
**shows**  $([m,p] \vdash e \mapsto v) \implies v \neq \text{UndefVal}$   
**apply** (*induction rule: evaltree.induct*)  
**using** *valid-not-undef wf-value-def* **by** *auto*

**lemma** *leafint*:  
**assumes** *ev*:  $[m,p] \vdash \text{LeafExpr } i \ (\text{IntegerStamp } b \text{ lo } hi) \mapsto val$   
**shows**  $\exists b \ v. \ val = (\text{IntVal } b \ v)$

**proof** –  
**have** *valid-value val* (*IntegerStamp b lo hi*)  
**using** *ev* **by** (*rule LeafExprE; simp*)  
**then show** *?thesis* **by** *auto*  
**qed**

**lemma** *default-stamp [simp]*: *default-stamp* = *IntegerStamp 32* (*-2147483648*)  
*2147483647*  
**using** *default-stamp-def* **by** *auto*

**lemma** *valid-value-signed-int-range [simp]*:  
**assumes** *valid-value val* (*IntegerStamp b lo hi*)  
**assumes** *lo < 0*  
**shows**  $\exists v. (val = \text{IntVal } b \ v \wedge$   
 $lo \leq \text{int-signed-value } b \ v \wedge$   
 $\text{int-signed-value } b \ v \leq hi)$   
**using** *assms valid-int*  
**by** (*metis valid-value.simps(1)*)

## 5.6.4 Example Data-flow Optimisations

## 5.6.5 Monotonicity of Expression Refinement

We prove that each subexpression position is monotonic. That is, optimizing a subexpression anywhere deep inside a top-level expression also optimizes that top-level expression.

Note that we might also be able to do this via reusing Isabelle’s *mono* operator (HOL.Orderings theory), proving instantiations like *mono(UnaryExprop)*, but it is not obvious how to do this for both arguments of the binary expressions.

**lemma** *mono-unary*:

**assumes**  $e \geq e'$   
**shows**  $(UnaryExpr\ op\ e) \geq (UnaryExpr\ op\ e')$   
**using** *UnaryExpr assms* **by** *auto*

**lemma** *mono-binary*:

**assumes**  $x \geq x'$   
**assumes**  $y \geq y'$   
**shows**  $(BinaryExpr\ op\ x\ y) \geq (BinaryExpr\ op\ x'\ y')$   
**using** *BinaryExpr assms* **by** *auto*

**lemma** *never-void*:

**assumes**  $[m, p] \vdash x \mapsto xv$   
**assumes** *valid-value xv (stamp-expr xe)*  
**shows** *stamp-expr xe*  $\neq$  *VoidStamp*  
**using** *valid-value.simps*  
**using** *assms(2)* **by** *force*

**lemma** *compatible-trans*:

*compatible x y*  $\wedge$  *compatible y z*  $\implies$  *compatible x z*  
**by** (*cases x*; *cases y*; *cases z*; *simp del: valid-stamp.simps*)

**lemma** *compatible-refl*:

*compatible x y*  $\implies$  *compatible y x*  
**using** *compatible.elims(2)* **by** *fastforce*

**lemma** *mono-conditional*:

**assumes**  $ce \geq ce'$   
**assumes**  $te \geq te'$   
**assumes**  $fe \geq fe'$   
**shows**  $(ConditionalExpr\ ce\ te\ fe) \geq (ConditionalExpr\ ce'\ te'\ fe')$   
**proof** (*simp only: le-expr-def*; (*rule allI*) $+$ ; *rule impI*)  
**fix**  $m\ p\ v$   
**assume**  $a: [m, p] \vdash ConditionalExpr\ ce\ te\ fe \mapsto v$   
**then obtain** *cond* **where**  $ce: [m, p] \vdash ce \mapsto cond$  **by** *auto*  
**then have**  $ce': [m, p] \vdash ce' \mapsto cond$  **using** *assms* **by** *auto*  
  
**define** *branch* **where**  $b: branch = (if\ val\text{-}to\text{-}bool\ cond\ then\ te\ else\ fe)$   
**define** *branch'* **where**  $b': branch' = (if\ val\text{-}to\text{-}bool\ cond\ then\ te'\ else\ fe')$   
**then have** *beval*:  $[m, p] \vdash branch \mapsto v$  **using**  $a\ b\ ce\ evalDet$  **by** *blast*  
  
**from** *beval* **have**  $[m, p] \vdash branch' \mapsto v$  **using** *assms b b'* **by** *auto*  
**then show**  $[m, p] \vdash ConditionalExpr\ ce'\ te'\ fe' \mapsto v$   
**using** *ConditionalExpr ce' b'*

using  $a$  by *blast*  
 qed

## 5.7 Unfolding rules for evaltree quadruples down to bin-eval level

These rewrite rules can be useful when proving optimizations. They support top-down rewriting of each level of the tree into the lower-level *bin\_eval* / *unary\_eval* level, simply by saying *unfoldingunfold\_evaltree*.

**lemma** *unfold-const*:

shows  $([m,p] \vdash \text{ConstantExpr } c \mapsto v) = (\text{wf-value } v \wedge v = c)$   
 by *blast*

**lemma** *unfold-binary*:

shows  $([m,p] \vdash \text{BinaryExpr } op \ x \ y \mapsto val) = (\exists \ x \ y. \\
 ([m,p] \vdash x \mapsto x) \wedge \\
 ([m,p] \vdash y \mapsto y) \wedge \\
 (val = \text{bin-eval } op \ x \ y) \wedge \\
 (val \neq \text{UndefVal}) \\
 )) \text{ (is ?L = ?R)}$

**proof** (*intro iffI*)

assume  $\mathcal{I}$ : ?L

show ?R by (rule *evaltree.cases*[OF  $\mathcal{I}$ ]; *blast+*)

**next**

assume ?R

then obtain  $x \ y$  where  $[m,p] \vdash x \mapsto x$

and  $[m,p] \vdash y \mapsto y$

and  $val = \text{bin-eval } op \ x \ y$

and  $val \neq \text{UndefVal}$

by *auto*

then show ?L

by (rule *BinaryExpr*)

qed

**lemma** *unfold-unary*:

shows  $([m,p] \vdash \text{UnaryExpr } op \ x \mapsto val) \\
 = (\exists \ x. \\
 ([m,p] \vdash x \mapsto x) \wedge \\
 (val = \text{unary-eval } op \ x) \wedge \\
 (val \neq \text{UndefVal}) \\
 )) \text{ (is ?L = ?R)}$

by *auto*

**lemmas** *unfold-evaltree* =

*unfold-binary*  
*unfold-unary*

## 5.8 Lemmas about *new\_\_int* and integer eval results.

**lemma** *unary-eval-new-int*:

**assumes** *def: unary-eval op x ≠ UndefVal*

**shows**  $\exists b v. \text{unary-eval } op \ x = \text{new-int } b \ v \wedge$

$b = (\text{if } op \in \text{normal-unary} \text{ then } \text{intval-bits } x \text{ else } \text{ir-resultBits } op)$

**proof** (*cases op ∈ normal-unary*)

**case** *True*

**then show** *?thesis*

**by** (*metis def-empty-iff insert-iff intval-abs.elims intval-bits.simps intval-logic-negation.elims intval-negate.elims intval-not.elims unary-eval.simps(1) unary-eval.simps(2) unary-eval.simps(3) unary-eval.simps(4)*)

**next**

**case** *False*

**consider** *ib ob where op = UnaryNarrow ib ob |*

*ib ob where op = UnaryZeroExtend ib ob |*

*ib ob where op = UnarySignExtend ib ob*

**by** (*metis False IRUnaryOp.exhaust insert-iff*)

**then show** *?thesis*

**proof** (*cases*)

**case** *1*

**then show** *?thesis*

**by** (*metis False IRUnaryOp.sel(4) def intval-narrow.elims unary-eval.simps(5)*)

**next**

**case** *2*

**then show** *?thesis*

**by** (*metis False IRUnaryOp.sel(6) def intval-zero-extend.elims unary-eval.simps(7)*)

**next**

**case** *3*

**then show** *?thesis*

**by** (*metis False IRUnaryOp.sel(5) def intval-sign-extend.elims unary-eval.simps(6)*)

**qed**

**qed**

**lemma** *new-int-unused-bits-zero*:

**assumes** *IntVal b ival = new-int b ival0*

**shows** *take-bit b ival = ival*

**using** *assms(1) new-int-take-bits by blast*

**lemma** *unary-eval-unused-bits-zero*:

**assumes** *unary-eval op x = IntVal b ival*

**shows** *take-bit b ival = ival*

**using** *assms unary-eval-new-int*

**by** (*metis Value.inject(1) Value.simps(5) new-int.elims new-int-unused-bits-zero*)

**lemma** *bin-eval-unused-bits-zero*:



```

assumes bin-eval op x y = (IntVal b ival)
shows take-bit b ival = ival
using assms bin-eval-new-int
by (metis Value.distinct(1) Value.inject(1) new-int.elims new-int-take-bits)

lemma eval-unused-bits-zero:
   $[m,p] \vdash xe \mapsto (IntVal\ b\ ix) \implies take-bit\ b\ ix = ix$ 
proof (induction xe)
  case (UnaryExpr x1 xe)
  then show ?case
    using unary-eval-unused-bits-zero by force
next
  case (BinaryExpr x1 xe1 xe2)
  then show ?case
    using bin-eval-unused-bits-zero by force
next
  case (ConditionalExpr xe1 xe2 xe3)
  then show ?case
    by (metis (full-types) EvalTreeE(3))
next
  case (ParameterExpr i s)
  then have valid-value (p!i) s
    by fastforce
  then show ?case
    by (metis ParameterExprE Value.distinct(7) intval-bits.simps intval-word.simps
local.ParameterExpr valid-value.elims(2))
next
  case (LeafExpr x1 x2)
  then show ?case
    by (smt (z3) EvalTreeE(6) Value.simps(11) valid-value.elims(1) valid-value.simps(1))

next
  case (ConstantExpr x)
  then show ?case using wf-value-def
    by (metis EvalTreeE(1) constantAsStamp.simps(1) valid-value.simps(1))
next
  case (ConstantVar x)
  then show ?case
    by fastforce
next
  case (VariableExpr x1 x2)
  then show ?case
    by fastforce
qed

```

```

lemma unary-normal-bitsize:
  assumes unary-eval op x = IntVal b ival
  assumes op ∈ normal-unary

```

```

shows  $\exists ix. x = \text{IntVal } b \text{ } ix$ 
apply (cases op)
  prefer 7 using assms apply blast
  prefer 6 using assms apply blast
  prefer 5 using assms apply blast
using Value.distinct(1) Value.sel(1) assms(1) new-int.simps unary-eval.simps
  intval-abs.elims intval-negate.elims intval-not.elims intval-logic-negation.elims
  applymetis+
done

lemma unary-not-normal-bitsize:
  assumes unary-eval op x = IntVal b ival
  assumes op  $\notin$  normal-unary
  shows  $b = \text{ir-resultBits } op \wedge 0 < b \wedge b \leq 64$ 
  apply (cases op)
  using assms apply blast+
  apply (metis IRUnaryOp.sel(4) Value.distinct(1) Value.sel(1) assms(1) intval-narrow.elims
    intval-narrow-ok new-int.simps unary-eval.simps(5))
  apply (smt (verit) IRUnaryOp.sel(5) Value.distinct(1) Value.sel(1) assms(1)
    intval-sign-extend.elims new-int.simps order-less-le-trans unary-eval.simps(6))
  apply (metis IRUnaryOp.sel(6) Value.distinct(1) assms(1) intval-bits.simps int-
    val-zero-extend.elims linorder-not-less neq0-conv new-int.simps unary-eval.simps(7))
  done

lemma unary-eval-bitsize:
  assumes unary-eval op x = IntVal b ival
  assumes 2:  $x = \text{IntVal } bx \text{ } ix$ 
  assumes  $0 < bx \wedge bx \leq 64$ 
  shows  $0 < b \wedge b \leq 64$ 
proof (cases op  $\in$  normal-unary)
  case True
  then obtain tmp where unary-eval op x = new-int bx tmp
  by (cases op; simp; auto simp: 2)
  then show ?thesis
  using assms by simp
next
  case False
  then obtain tmp where unary-eval op x = new-int b tmp  $\wedge 0 < b \wedge b \leq 64$ 
  apply (cases op; simp; auto simp: 2)
  apply (metis 2 Value.inject(1) Value.simps(5) assms(1) intval-narrow.simps(1)
    intval-narrow-ok new-int.simps unary-eval.simps(5))
  apply (metis 2 Value.distinct(1) Value.inject(1) assms(1) bot-nat-0.not-eq-extremum
    diff-is-0-eq intval-sign-extend.elims new-int.simps unary-eval.simps(6) zero-less-diff)
  by (smt (verit, del-insts) 2 Value.simps(5) assms(1) intval-bits.simps int-
    val-zero-extend.simps(1) new-int.simps order-less-le-trans unary-eval.simps(7))
  then show ?thesis
  by blast
qed

```

```

lemma bin-eval-inputs-are-ints:
  assumes bin-eval op x y = IntVal b ix
  obtains xb yb xi yi where x = IntVal xb xi ∧ y = IntVal yb yi
proof –
  have bin-eval op x y ≠ UndefVal
    by (simp add: assms)
  then show ?thesis
    using assms apply (cases op; cases x; cases y; simp)
    using that by blast+
qed

lemma eval-bits-1-64:
   $[m,p] \vdash xe \mapsto (IntVal\ b\ ix) \implies 0 < b \wedge b \leq 64$ 
proof (induction xe arbitrary: b ix)
  case (UnaryExpr op x2)
  then obtain xv where
    xv: ([m,p] ⊢ x2 ↦ xv) ∧
    IntVal b ix = unary-eval op xv
    using unfold-binary by auto
  then have b = (if op ∈ normal-unary then intval-bits xv else ir-resultBits op)
    using unary-eval-new-int
    by (metis Value.disc(1) Value.discI(1) Value.sel(1) new-int.simps)
  then show ?case
    by (metis xv UnaryExpr.IH unary-normal-bitsize unary-not-normal-bitsize)
next
  case (BinaryExpr op x y)
  then obtain xv yv where
    xy: ([m,p] ⊢ x ↦ xv) ∧
    ([m,p] ⊢ y ↦ yv) ∧
    IntVal b ix = bin-eval op xv yv
    using unfold-binary by auto
  then have def: bin-eval op xv yv ≠ UndefVal and xv: xv ≠ UndefVal and yv ≠
UndefVal
    using evaltree-not-undef xy by (force, blast, blast)
  then have b = (if op ∈ binary-fixed-32-ops then 32 else intval-bits xv)
    by (metis xy intval-bits.simps new-int.simps bin-eval-new-int)
  then show ?case
    by (metis BinaryExpr.IH(1) Value.distinct(7) Value.distinct(9) xv bin-eval-inputs-are-ints
intval-bits.elims le-add-same-cancel1 less-or-eq-imp-le numeral-Bit0 xy zero-less-numeral)
next
  case (ConditionalExpr xe1 xe2 xe3)
  then show ?case
    by (metis (full-types) EvalTreeE(3))
next
  case (ParameterExpr x1 x2)

```

```

    then show ?case
      using ParameterExprE intval-bits.simps valid-stamp.simps(1) valid-value.elims(2)
      valid-value.simps(17)
      by (metis (no-types, lifting))
  next
    case (LeafExpr x1 x2)
    then show ?case
      by (smt (z3) EvalTreeE(6) Value.distinct(7) Value.inject(1) valid-stamp.simps(1)
      valid-value.elims(1))
  next
    case (ConstantExpr x)
    then show ?case using wf-value-def
      by (metis EvalTreeE(1) constantAsStamp.simps(1) valid-stamp.simps(1) valid-value.simps(1))
  next
    case (ConstantVar x)
    then show ?case
      by blast
  next
    case (VariableExpr x1 x2)
    then show ?case
      by blast
qed

```

**lemma** *unfold-binary-width:*

```

  assumes op ∉ binary-fixed-32-ops ∧ op ∉ binary-shift-ops
  shows ([m,p] ⊢ BinaryExpr op xe ye ↦ IntVal b val) = (∃ x y.
    ([m,p] ⊢ xe ↦ IntVal b x) ∧
    ([m,p] ⊢ ye ↦ IntVal b y) ∧
    (IntVal b val = bin-eval op (IntVal b x) (IntVal b y)) ∧
    (IntVal b val ≠ UndefVal)
  )) (is ?L = ?R)
proof (intro iffI)
  assume 3: ?L
  show ?R apply (rule evaltree.cases[OF 3])
    apply force+ apply auto[1]
  using assms apply (cases op; auto)
    apply (smt (verit) intval-add.elims Value.inject(1))
  using intval-mul.elims Value.inject(1)
    apply (smt (verit) new-int.simps new-int-bin.simps)
  using intval-sub.elims Value.inject(1)
    apply (smt (verit) new-int.simps new-int-bin.simps)
  using intval-and.elims Value.inject(1)
    apply (smt (verit) new-int.simps new-int-bin.simps take-bit-and)
  using intval-or.elims Value.inject(1)
    apply (smt (verit) new-int.simps new-int-bin.simps take-bit-or)
  using intval-xor.elims Value.inject(1)
    apply (smt (verit) new-int.simps new-int-bin.simps take-bit-xor)
  by blast

```

```

next
  assume  $R: ?R$ 
  then obtain  $x\ y$  where  $[m,p] \vdash xe \mapsto \text{IntVal } b\ x$ 
    and  $[m,p] \vdash ye \mapsto \text{IntVal } b\ y$ 
    and  $\text{new-int } b\ \text{val} = \text{bin-eval } op\ (\text{IntVal } b\ x)\ (\text{IntVal } b\ y)$ 
    and  $\text{new-int } b\ \text{val} \neq \text{UndefVal}$ 
  using  $\text{bin-eval-unused-bits-zero}$  by force
  then show  $?L$ 
  using  $R$  by blast
qed

end

```

## 6 Tree to Graph

```

theory TreeToGraph
  imports
    Semantics.IRTreeEval
    Graph.IRGraph
begin

```

### 6.1 Subgraph to Data-flow Tree

```

fun find-node-and-stamp :: IRGraph  $\Rightarrow$  (IRNode  $\times$  Stamp)  $\Rightarrow$  ID option where
  find-node-and-stamp  $g\ (n,s) =$ 
    find  $(\lambda i. \text{kind } g\ i = n \wedge \text{stamp } g\ i = s)\ (\text{sorted-list-of-set}(\text{ids } g))$ 

export-code find-node-and-stamp

```

```

fun is-preevaluated :: IRNode  $\Rightarrow$  bool where
  is-preevaluated (InvokeNode  $n\ -\ -\ -\ -$ ) = True |
  is-preevaluated (InvokeWithExceptionNode  $n\ -\ -\ -\ -$ ) = True |
  is-preevaluated (NewInstanceNode  $n\ -\ -$ ) = True |
  is-preevaluated (LoadFieldNode  $n\ -\ -$ ) = True |
  is-preevaluated (SignedDivNode  $n\ -\ -\ -\ -$ ) = True |
  is-preevaluated (SignedRemNode  $n\ -\ -\ -\ -$ ) = True |
  is-preevaluated (ValuePhiNode  $n\ -$ ) = True |
  is-preevaluated - = False

```

```

inductive
  rep :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  IRExpr  $\Rightarrow$  bool  $(-\vdash - \simeq -\ 55)$ 
  for  $g$  where

  ConstantNode:
     $\llbracket \text{kind } g\ n = \text{ConstantNode } c \rrbracket$ 
     $\implies g \vdash n \simeq (\text{ConstantExpr } c) \mid$ 

```

*ParameterNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{ParameterNode } i; \\ & \quad \text{stamp } g \ n = s \rrbracket \\ & \implies g \vdash n \simeq (\text{ParameterExpr } i \ s) \mid \end{aligned}$$

*ConditionalNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{ConditionalNode } c \ t \ f; \\ & \quad g \vdash c \simeq ce; \\ & \quad g \vdash t \simeq te; \\ & \quad g \vdash f \simeq fe \rrbracket \\ & \implies g \vdash n \simeq (\text{ConditionalExpr } ce \ te \ fe) \mid \end{aligned}$$

*AbsNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{AbsNode } x; \\ & \quad g \vdash x \simeq xe \rrbracket \\ & \implies g \vdash n \simeq (\text{UnaryExpr } \text{UnaryAbs } xe) \mid \end{aligned}$$

*NotNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{NotNode } x; \\ & \quad g \vdash x \simeq xe \rrbracket \\ & \implies g \vdash n \simeq (\text{UnaryExpr } \text{UnaryNot } xe) \mid \end{aligned}$$

*NegateNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{NegateNode } x; \\ & \quad g \vdash x \simeq xe \rrbracket \\ & \implies g \vdash n \simeq (\text{UnaryExpr } \text{UnaryNeg } xe) \mid \end{aligned}$$

*LogicNegationNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{LogicNegationNode } x; \\ & \quad g \vdash x \simeq xe \rrbracket \\ & \implies g \vdash n \simeq (\text{UnaryExpr } \text{UnaryLogicNegation } xe) \mid \end{aligned}$$

*AddNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{AddNode } x \ y; \\ & \quad g \vdash x \simeq xe; \\ & \quad g \vdash y \simeq ye \rrbracket \\ & \implies g \vdash n \simeq (\text{BinaryExpr } \text{BinAdd } xe \ ye) \mid \end{aligned}$$

*MulNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{MulNode } x \ y; \\ & \quad g \vdash x \simeq xe; \\ & \quad g \vdash y \simeq ye \rrbracket \\ & \implies g \vdash n \simeq (\text{BinaryExpr } \text{BinMul } xe \ ye) \mid \end{aligned}$$

*SubNode:*

$$\llbracket \text{kind } g \ n = \text{SubNode } x \ y;$$

$$\begin{aligned}
&g \vdash x \simeq xe; \\
&g \vdash y \simeq ye \\
\implies &g \vdash n \simeq (\text{BinaryExpr BinSub } xe \ ye) \mid
\end{aligned}$$

*AndNode:*

$$\begin{aligned}
&\llbracket \text{kind } g \ n = \text{AndNode } x \ y; \\
&g \vdash x \simeq xe; \\
&g \vdash y \simeq ye \rrbracket \\
\implies &g \vdash n \simeq (\text{BinaryExpr BinAnd } xe \ ye) \mid
\end{aligned}$$

*OrNode:*

$$\begin{aligned}
&\llbracket \text{kind } g \ n = \text{OrNode } x \ y; \\
&g \vdash x \simeq xe; \\
&g \vdash y \simeq ye \rrbracket \\
\implies &g \vdash n \simeq (\text{BinaryExpr BinOr } xe \ ye) \mid
\end{aligned}$$

*XorNode:*

$$\begin{aligned}
&\llbracket \text{kind } g \ n = \text{XorNode } x \ y; \\
&g \vdash x \simeq xe; \\
&g \vdash y \simeq ye \rrbracket \\
\implies &g \vdash n \simeq (\text{BinaryExpr BinXor } xe \ ye) \mid
\end{aligned}$$

*ShortCircuitOrNode:*

$$\begin{aligned}
&\llbracket \text{kind } g \ n = \text{ShortCircuitOrNode } x \ y; \\
&g \vdash x \simeq xe; \\
&g \vdash y \simeq ye \rrbracket \\
\implies &g \vdash n \simeq (\text{BinaryExpr BinShortCircuitOr } xe \ ye) \mid
\end{aligned}$$

*LeftShiftNode:*

$$\begin{aligned}
&\llbracket \text{kind } g \ n = \text{LeftShiftNode } x \ y; \\
&g \vdash x \simeq xe; \\
&g \vdash y \simeq ye \rrbracket \\
\implies &g \vdash n \simeq (\text{BinaryExpr BinLeftShift } xe \ ye) \mid
\end{aligned}$$

*RightShiftNode:*

$$\begin{aligned}
&\llbracket \text{kind } g \ n = \text{RightShiftNode } x \ y; \\
&g \vdash x \simeq xe; \\
&g \vdash y \simeq ye \rrbracket \\
\implies &g \vdash n \simeq (\text{BinaryExpr BinRightShift } xe \ ye) \mid
\end{aligned}$$

*UnsignedRightShiftNode:*

$$\begin{aligned}
&\llbracket \text{kind } g \ n = \text{UnsignedRightShiftNode } x \ y; \\
&g \vdash x \simeq xe; \\
&g \vdash y \simeq ye \rrbracket \\
\implies &g \vdash n \simeq (\text{BinaryExpr BinURightShift } xe \ ye) \mid
\end{aligned}$$

*IntegerBelowNode:*

$$\begin{aligned}
&\llbracket \text{kind } g \ n = \text{IntegerBelowNode } x \ y; \\
&g \vdash x \simeq xe;
\end{aligned}$$

$g \vdash y \simeq ye$   
 $\implies g \vdash n \simeq (\text{BinaryExpr BinIntegerBelow } xe \ ye) \mid$

*IntegerEqualsNode:*

$\llbracket \text{kind } g \ n = \text{IntegerEqualsNode } x \ y;$   
 $g \vdash x \simeq xe;$   
 $g \vdash y \simeq ye \rrbracket$   
 $\implies g \vdash n \simeq (\text{BinaryExpr BinIntegerEquals } xe \ ye) \mid$

*IntegerLessThanNode:*

$\llbracket \text{kind } g \ n = \text{IntegerLessThanNode } x \ y;$   
 $g \vdash x \simeq xe;$   
 $g \vdash y \simeq ye \rrbracket$   
 $\implies g \vdash n \simeq (\text{BinaryExpr BinIntegerLessThan } xe \ ye) \mid$

*NarrowNode:*

$\llbracket \text{kind } g \ n = \text{NarrowNode inputBits resultBits } x;$   
 $g \vdash x \simeq xe \rrbracket$   
 $\implies g \vdash n \simeq (\text{UnaryExpr (UnaryNarrow inputBits resultBits) } xe) \mid$

*SignExtendNode:*

$\llbracket \text{kind } g \ n = \text{SignExtendNode inputBits resultBits } x;$   
 $g \vdash x \simeq xe \rrbracket$   
 $\implies g \vdash n \simeq (\text{UnaryExpr (UnarySignExtend inputBits resultBits) } xe) \mid$

*ZeroExtendNode:*

$\llbracket \text{kind } g \ n = \text{ZeroExtendNode inputBits resultBits } x;$   
 $g \vdash x \simeq xe \rrbracket$   
 $\implies g \vdash n \simeq (\text{UnaryExpr (UnaryZeroExtend inputBits resultBits) } xe) \mid$

*LeafNode:*

$\llbracket \text{is-preevaluated (kind } g \ n);$   
 $\text{stamp } g \ n = s \rrbracket$   
 $\implies g \vdash n \simeq (\text{LeafExpr } n \ s) \mid$

*RefNode:*

$\llbracket \text{kind } g \ n = \text{RefNode } n';$   
 $g \vdash n' \simeq e \rrbracket$   
 $\implies g \vdash n \simeq e$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool as exprE}$ ) *rep* .

**inductive**

$\text{replist} :: \text{IRGraph} \Rightarrow \text{ID list} \Rightarrow \text{IRExpr list} \Rightarrow \text{bool} \ (- \vdash - \simeq_L - \ 55)$   
**for** *g* **where**



*RepNil:*  
 $g \vdash [] \simeq_L []$

*RepCons:*  
 $\llbracket g \vdash x \simeq xe; \\ g \vdash xs \simeq_L xse \rrbracket \\ \implies g \vdash x\#xs \simeq_L xe\#xse$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$  as *exprListE*) *replist* .

**definition** *wf-term-graph* :: *MapState*  $\Rightarrow$  *Params*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *bool* **where**  
*wf-term-graph* *m p g n* = ( $\exists e. (g \vdash n \simeq e) \wedge (\exists v. ([m, p] \vdash e \mapsto v))$ )

**values** {*t. eg2-sq*  $\vdash 4 \simeq t$ }

## 6.2 Data-flow Tree to Subgraph

**fun** *unary-node* :: *IRUnaryOp*  $\Rightarrow$  *ID*  $\Rightarrow$  *IRNode* **where**  
*unary-node* *UnaryAbs* *v* = *AbsNode* *v* |  
*unary-node* *UnaryNot* *v* = *NotNode* *v* |  
*unary-node* *UnaryNeg* *v* = *NegateNode* *v* |  
*unary-node* *UnaryLogicNegation* *v* = *LogicNegationNode* *v* |  
*unary-node* (*UnaryNarrow* *ib rb*) *v* = *NarrowNode* *ib rb v* |  
*unary-node* (*UnarySignExtend* *ib rb*) *v* = *SignExtendNode* *ib rb v* |  
*unary-node* (*UnaryZeroExtend* *ib rb*) *v* = *ZeroExtendNode* *ib rb v*

**fun** *bin-node* :: *IRBinaryOp*  $\Rightarrow$  *ID*  $\Rightarrow$  *ID*  $\Rightarrow$  *IRNode* **where**  
*bin-node* *BinAdd* *x y* = *AddNode* *x y* |  
*bin-node* *BinMul* *x y* = *MulNode* *x y* |  
*bin-node* *BinSub* *x y* = *SubNode* *x y* |  
*bin-node* *BinAnd* *x y* = *AndNode* *x y* |  
*bin-node* *BinOr* *x y* = *OrNode* *x y* |  
*bin-node* *BinXor* *x y* = *XorNode* *x y* |  
*bin-node* *BinShortCircuitOr* *x y* = *ShortCircuitOrNode* *x y* |  
*bin-node* *BinLeftShift* *x y* = *LeftShiftNode* *x y* |  
*bin-node* *BinRightShift* *x y* = *RightShiftNode* *x y* |  
*bin-node* *BinURightShift* *x y* = *UnsignedRightShiftNode* *x y* |  
*bin-node* *BinIntegerEquals* *x y* = *IntegerEqualsNode* *x y* |  
*bin-node* *BinIntegerLessThan* *x y* = *IntegerLessThanNode* *x y* |  
*bin-node* *BinIntegerBelow* *x y* = *IntegerBelowNode* *x y*

**inductive** *fresh-id* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *bool* **where**  
 $n \notin \text{ids } g \implies \text{fresh-id } g \ n$

**code-pred** *fresh-id* .

**fun** *get-fresh-id* :: *IRGraph*  $\Rightarrow$  *ID* **where**

*get-fresh-id* *g* = *last*(*sorted-list-of-set*(*ids* *g*)) + 1

**export-code** *get-fresh-id*

**value** *get-fresh-id* *eg2-sq*

**value** *get-fresh-id* (*add-node* 6 (*ParameterNode* 2, *default-stamp*) *eg2-sq*)

**inductive**

*unrep* :: *IRGraph*  $\Rightarrow$  *IRExpr*  $\Rightarrow$  (*IRGraph*  $\times$  *ID*)  $\Rightarrow$  *bool* (-  $\oplus$  -  $\rightsquigarrow$  - 55)  
**where**

*ConstantNodeSame*:

$\llbracket \text{find-node-and-stamp } g \text{ (ConstantNode } c, \text{ constantAsStamp } c) = \text{Some } n \rrbracket$   
 $\implies g \oplus (\text{ConstantExpr } c) \rightsquigarrow (g, n) \mid$

*ConstantNodeNew*:

$\llbracket \text{find-node-and-stamp } g \text{ (ConstantNode } c, \text{ constantAsStamp } c) = \text{None};$   
*n* = *get-fresh-id* *g*;  
*g'* = *add-node* *n* (*ConstantNode* *c*, *constantAsStamp* *c*) *g*  $\rrbracket$   
 $\implies g \oplus (\text{ConstantExpr } c) \rightsquigarrow (g', n) \mid$

*ParameterNodeSame*:

$\llbracket \text{find-node-and-stamp } g \text{ (ParameterNode } i, s) = \text{Some } n \rrbracket$   
 $\implies g \oplus (\text{ParameterExpr } i \text{ } s) \rightsquigarrow (g, n) \mid$

*ParameterNodeNew*:

$\llbracket \text{find-node-and-stamp } g \text{ (ParameterNode } i, s) = \text{None};$   
*n* = *get-fresh-id* *g*;  
*g'* = *add-node* *n* (*ParameterNode* *i*, *s*) *g*  $\rrbracket$   
 $\implies g \oplus (\text{ParameterExpr } i \text{ } s) \rightsquigarrow (g', n) \mid$

*ConditionalNodeSame*:

$\llbracket g \oplus ce \rightsquigarrow (g2, c);$   
*g2*  $\oplus$  *te*  $\rightsquigarrow$  (*g3*, *t*);  
*g3*  $\oplus$  *fe*  $\rightsquigarrow$  (*g4*, *f*);  
*s'* = *meet* (*stamp* *g4* *t*) (*stamp* *g4* *f*);  
*find-node-and-stamp* *g4* (*ConditionalNode* *c* *t* *f*, *s'*) = *Some* *n*  $\rrbracket$   
 $\implies g \oplus (\text{ConditionalExpr } ce \text{ } te \text{ } fe) \rightsquigarrow (g4, n) \mid$

*ConditionalNodeNew*:

$\llbracket g \oplus ce \rightsquigarrow (g2, c);$   
*g2*  $\oplus$  *te*  $\rightsquigarrow$  (*g3*, *t*);

$g3 \oplus fe \rightsquigarrow (g4, f);$   
 $s' = \text{meet } (\text{stamp } g4 \ t) (\text{stamp } g4 \ f);$   
 $\text{find-node-and-stamp } g4 \ (\text{ConditionalNode } c \ t \ f, s') = \text{None};$   
 $n = \text{get-fresh-id } g4;$   
 $g' = \text{add-node } n \ (\text{ConditionalNode } c \ t \ f, s') \ g4]$   
 $\implies g \oplus (\text{ConditionalExpr } ce \ te \ fe) \rightsquigarrow (g', n) \mid$

*UnaryNodeSame:*

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$   
 $s' = \text{stamp-unary } op \ (\text{stamp } g2 \ x);$   
 $\text{find-node-and-stamp } g2 \ (\text{unary-node } op \ x, s') = \text{Some } n \rrbracket$   
 $\implies g \oplus (\text{UnaryExpr } op \ xe) \rightsquigarrow (g2, n) \mid$

*UnaryNodeNew:*

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$   
 $s' = \text{stamp-unary } op \ (\text{stamp } g2 \ x);$   
 $\text{find-node-and-stamp } g2 \ (\text{unary-node } op \ x, s') = \text{None};$   
 $n = \text{get-fresh-id } g2;$   
 $g' = \text{add-node } n \ (\text{unary-node } op \ x, s') \ g2]$   
 $\implies g \oplus (\text{UnaryExpr } op \ xe) \rightsquigarrow (g', n) \mid$

*BinaryNodeSame:*

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$   
 $g2 \oplus ye \rightsquigarrow (g3, y);$   
 $s' = \text{stamp-binary } op \ (\text{stamp } g3 \ x) (\text{stamp } g3 \ y);$   
 $\text{find-node-and-stamp } g3 \ (\text{bin-node } op \ x \ y, s') = \text{Some } n \rrbracket$   
 $\implies g \oplus (\text{BinaryExpr } op \ xe \ ye) \rightsquigarrow (g3, n) \mid$

*BinaryNodeNew:*

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$   
 $g2 \oplus ye \rightsquigarrow (g3, y);$   
 $s' = \text{stamp-binary } op \ (\text{stamp } g3 \ x) (\text{stamp } g3 \ y);$   
 $\text{find-node-and-stamp } g3 \ (\text{bin-node } op \ x \ y, s') = \text{None};$   
 $n = \text{get-fresh-id } g3;$   
 $g' = \text{add-node } n \ (\text{bin-node } op \ x \ y, s') \ g3]$   
 $\implies g \oplus (\text{BinaryExpr } op \ xe \ ye) \rightsquigarrow (g', n) \mid$

*AllLeafNodes:*

$\llbracket \text{stamp } g \ n = s;$   
 $\text{is-preevaluated } (\text{kind } g \ n) \rrbracket$   
 $\implies g \oplus (\text{LeafExpr } n \ s) \rightsquigarrow (g, n)$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$  as *unrepE*)  
*unrep* .

$$\frac{\text{find-node-and-stamp } g \text{ (ConstantNode } c, \text{ constantAsStamp } c) = \text{Some } n}{g \oplus \text{ConstantExpr } c \rightsquigarrow (g, n)}$$

$$\frac{\begin{array}{l} \text{find-node-and-stamp } g \text{ (ConstantNode } c, \text{ constantAsStamp } c) = \text{None} \\ n = \text{get-fresh-id } g \\ g' = \text{add-node } n \text{ (ConstantNode } c, \text{ constantAsStamp } c) \end{array} \quad g}{g \oplus \text{ConstantExpr } c \rightsquigarrow (g', n)}$$

$$\frac{\text{find-node-and-stamp } g \text{ (ParameterNode } i, s) = \text{Some } n}{g \oplus \text{ParameterExpr } i \text{ } s \rightsquigarrow (g, n)}$$

$$\frac{\begin{array}{l} \text{find-node-and-stamp } g \text{ (ParameterNode } i, s) = \text{None} \\ n = \text{get-fresh-id } g \quad g' = \text{add-node } n \text{ (ParameterNode } i, s) \end{array} \quad g}{g \oplus \text{ParameterExpr } i \text{ } s \rightsquigarrow (g', n)}$$

$$\frac{\begin{array}{l} g \oplus ce \rightsquigarrow (g2, c) \quad g2 \oplus te \rightsquigarrow (g3, t) \\ g3 \oplus fe \rightsquigarrow (g4, f) \quad s' = \text{meet (stamp } g4 \text{ } t) \text{ (stamp } g4 \text{ } f) \\ \text{find-node-and-stamp } g4 \text{ (ConditionalNode } c \text{ } t \text{ } f, s') = \text{Some } n \end{array}}{g \oplus \text{ConditionalExpr } ce \text{ } te \text{ } fe \rightsquigarrow (g4, n)}$$

$$\frac{\begin{array}{l} g \oplus ce \rightsquigarrow (g2, c) \quad g2 \oplus te \rightsquigarrow (g3, t) \\ g3 \oplus fe \rightsquigarrow (g4, f) \quad s' = \text{meet (stamp } g4 \text{ } t) \text{ (stamp } g4 \text{ } f) \\ \text{find-node-and-stamp } g4 \text{ (ConditionalNode } c \text{ } t \text{ } f, s') = \text{None} \\ n = \text{get-fresh-id } g4 \quad g' = \text{add-node } n \text{ (ConditionalNode } c \text{ } t \text{ } f, s') \end{array} \quad g4}{g \oplus \text{ConditionalExpr } ce \text{ } te \text{ } fe \rightsquigarrow (g', n)}$$

$$\frac{\begin{array}{l} g \oplus xe \rightsquigarrow (g2, x) \\ g2 \oplus ye \rightsquigarrow (g3, y) \quad s' = \text{stamp-binary op (stamp } g3 \text{ } x) \text{ (stamp } g3 \text{ } y) \\ \text{find-node-and-stamp } g3 \text{ (bin-node op } x \text{ } y, s') = \text{Some } n \end{array}}{g \oplus \text{BinaryExpr op } xe \text{ } ye \rightsquigarrow (g3, n)}$$

$$\frac{\begin{array}{l} g \oplus xe \rightsquigarrow (g2, x) \\ g2 \oplus ye \rightsquigarrow (g3, y) \quad s' = \text{stamp-binary op (stamp } g3 \text{ } x) \text{ (stamp } g3 \text{ } y) \\ \text{find-node-and-stamp } g3 \text{ (bin-node op } x \text{ } y, s') = \text{None} \\ n = \text{get-fresh-id } g3 \quad g' = \text{add-node } n \text{ (bin-node op } x \text{ } y, s') \end{array} \quad g3}{g \oplus \text{BinaryExpr op } xe \text{ } ye \rightsquigarrow (g', n)}$$

$$\frac{\begin{array}{l} g \oplus xe \rightsquigarrow (g2, x) \quad s' = \text{stamp-unary op (stamp } g2 \text{ } x) \\ \text{find-node-and-stamp } g2 \text{ (unary-node op } x, s') = \text{Some } n \end{array}}{g \oplus \text{UnaryExpr op } xe \rightsquigarrow (g2, n)}$$

$$\frac{\begin{array}{l} g \oplus xe \rightsquigarrow (g2, x) \quad s' = \text{stamp-unary op (stamp } g2 \text{ } x) \\ \text{find-node-and-stamp } g2 \text{ (unary-node op } x, s') = \text{None} \\ n = \text{get-fresh-id } g2 \quad g' = \text{add-node } n \text{ (unary-node op } x, s') \end{array} \quad g2}{g \oplus \text{UnaryExpr op } xe \rightsquigarrow (g', n)}$$

$$\frac{\text{stamp } g \text{ } n = s \quad \text{is-preevaluated (kind } g \text{ } n)}{g \oplus \text{LeafExpr } n \text{ } s \rightsquigarrow (g, n)}$$

*values*  $\{(n, g) . (eg2\text{-}sq \oplus sq\text{-}param0 \rightsquigarrow (g, n))\}$

### 6.3 Lift Data-flow Tree Semantics

**definition** *encodeeval* :: *IRGraph*  $\Rightarrow$  *MapState*  $\Rightarrow$  *Params*  $\Rightarrow$  *ID*  $\Rightarrow$  *Value*  $\Rightarrow$  *bool*  
 $([\cdot, \cdot, \cdot] \vdash \cdot \mapsto \cdot \ 50)$   
**where**  
*encodeeval* *g m p n v* =  $(\exists e. (g \vdash n \simeq e) \wedge ([m, p] \vdash e \mapsto v))$

### 6.4 Graph Refinement

**definition** *graph-represents-expression* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *IRExpr*  $\Rightarrow$  *bool*  
 $(\vdash \cdot \leq \cdot \ 50)$   
**where**  
 $(g \vdash n \leq e) = (\exists e'. (g \vdash n \simeq e') \wedge (e' \leq e))$

**definition** *graph-refinement* :: *IRGraph*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *bool* **where**  
*graph-refinement* *g1 g2* =  
 $((ids\ g_1 \subseteq ids\ g_2) \wedge$   
 $(\forall n. n \in ids\ g_1 \longrightarrow (\forall e. (g_1 \vdash n \simeq e) \longrightarrow (g_2 \vdash n \leq e))))$

**lemma** *graph-refinement*:

*graph-refinement* *g1 g2*  $\implies (\forall n\ m\ p\ v. n \in ids\ g1 \longrightarrow ([g1, m, p] \vdash n \mapsto v) \longrightarrow$   
 $([g2, m, p] \vdash n \mapsto v))$   
**by** (*meson encodeeval-def graph-refinement-def graph-represents-expression-def le-expr-def*)

### 6.5 Maximal Sharing

**definition** *maximal-sharing*:

*maximal-sharing* *g* =  $(\forall n_1\ n_2. n_1 \in true\text{-}ids\ g \wedge n_2 \in true\text{-}ids\ g \longrightarrow$   
 $(\forall e. (g \vdash n_1 \simeq e) \wedge (g \vdash n_2 \simeq e) \wedge (stamp\ g\ n_1 = stamp\ g\ n_2) \longrightarrow n_1 =$   
 $n_2))$

**end**

### 6.6 Formedness Properties

**theory** *Form*

**imports**

*Semantics.TreeToGraph*

**begin**

**definition** *wf-start* **where**

*wf-start* *g* =  $(0 \in ids\ g \wedge$   
 $is\text{-}StartNode\ (kind\ g\ 0))$

**definition** *wf-closed* **where**

*wf-closed* *g* =  
 $(\forall n \in ids\ g .$

$$\begin{aligned} &inputs\ g\ n \subseteq ids\ g \wedge \\ &succ\ g\ n \subseteq ids\ g \wedge \\ &kind\ g\ n \neq NoNode) \end{aligned}$$

**definition** *wf-phs* **where**

$$\begin{aligned} wf-phs\ g = & \\ &(\forall\ n \in ids\ g. \\ &\quad is-PhiNode\ (kind\ g\ n) \longrightarrow \\ &\quad length\ (ir-values\ (kind\ g\ n)) \\ &= length\ (ir-ends \\ &\quad (kind\ g\ (ir-merge\ (kind\ g\ n)))))) \end{aligned}$$

**definition** *wf-ends* **where**

$$\begin{aligned} wf-ends\ g = & \\ &(\forall\ n \in ids\ g . \\ &\quad is-AbstractEndNode\ (kind\ g\ n) \longrightarrow \\ &\quad card\ (usages\ g\ n) > 0) \end{aligned}$$

**fun** *wf-graph* :: *IRGraph*  $\Rightarrow$  *bool* **where**

$$wf-graph\ g = (wf-start\ g \wedge wf-closed\ g \wedge wf-phs\ g \wedge wf-ends\ g)$$

**lemmas** *wf-folds* =

$$\begin{aligned} &wf-graph.simps \\ &wf-start-def \\ &wf-closed-def \\ &wf-phs-def \\ &wf-ends-def \end{aligned}$$

**fun** *wf-stamps* :: *IRGraph*  $\Rightarrow$  *bool* **where**

$$\begin{aligned} wf-stamps\ g = &(\forall\ n \in ids\ g . \\ &(\forall\ v\ m\ p\ e . (g \vdash n \simeq e) \wedge ([m, p] \vdash e \mapsto v) \longrightarrow valid-value\ v\ (stamp-expr\ e))) \end{aligned}$$

**fun** *wf-stamp* :: *IRGraph*  $\Rightarrow$  (*ID*  $\Rightarrow$  *Stamp*)  $\Rightarrow$  *bool* **where**

$$\begin{aligned} wf-stamp\ g\ s = &(\forall\ n \in ids\ g . \\ &(\forall\ v\ m\ p\ e . (g \vdash n \simeq e) \wedge ([m, p] \vdash e \mapsto v) \longrightarrow valid-value\ v\ (s\ n))) \end{aligned}$$

**lemma** *wf-empty*: *wf-graph start-end-graph*

**unfolding** *start-end-graph-def wf-folds by simp*

**lemma** *wf-eg2-sq*: *wf-graph eg2-sq*

**unfolding** *eg2-sq-def wf-folds by simp*

**fun** *wf-logic-node-inputs* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *bool* **where**

$$\begin{aligned} wf-logic-node-inputs\ g\ n = & \\ &(\forall\ inp \in set\ (inputs-of\ (kind\ g\ n)) . (\forall\ v\ m\ p . ([g, m, p] \vdash inp \mapsto v) \longrightarrow wf-bool \\ &v)) \end{aligned}$$

**fun** *wf-values* :: *IRGraph*  $\Rightarrow$  *bool* **where**

$$wf-values\ g = (\forall\ n \in ids\ g .$$

$$\begin{aligned}
& (\forall v \ m \ p . ([g, m, p] \vdash n \mapsto v) \longrightarrow \\
& \quad (is-LogicNode (kind \ g \ n) \longrightarrow \\
& \quad \quad wf-bool \ v \wedge wf-logic-node-inputs \ g \ n)))
\end{aligned}$$

**end**

## 6.7 Dynamic Frames

This theory defines two operators, 'unchanged' and 'changeonly', that are useful for specifying which nodes in an IRGraph can change. The dynamic framing idea originates from 'Dynamic Frames' in software verification, started by Ioannis T. Kassios in "Dynamic frames: Support for framing, dependencies and sharing without restrictions", In FM 2006.

**theory** *IRGraphFrames*

**imports**

*Form*

**begin**

**fun** *unchanged* :: *ID set*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *bool* **where**

*unchanged ns g1 g2* =  $(\forall n . n \in ns \longrightarrow$   
 $(n \in ids \ g1 \wedge n \in ids \ g2 \wedge kind \ g1 \ n = kind \ g2 \ n \wedge stamp \ g1 \ n = stamp \ g2 \ n))$

**fun** *changeonly* :: *ID set*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *bool* **where**

*changeonly ns g1 g2* =  $(\forall n . n \in ids \ g1 \wedge n \notin ns \longrightarrow$   
 $(n \in ids \ g1 \wedge n \in ids \ g2 \wedge kind \ g1 \ n = kind \ g2 \ n \wedge stamp \ g1 \ n = stamp \ g2 \ n))$

**lemma** *node-unchanged*:

**assumes** *unchanged ns g1 g2*

**assumes** *nid*  $\in$  *ns*

**shows** *kind g1 nid* = *kind g2 nid*

**using** *assms* **by** *auto*

**lemma** *other-node-unchanged*:

**assumes** *changeonly ns g1 g2*

**assumes** *nid*  $\in$  *ids g1*

**assumes** *nid*  $\notin$  *ns*

**shows** *kind g1 nid* = *kind g2 nid*

**using** *assms*

**using** *changeonly.simps* **by** *blast*

Some notation for input nodes used

**inductive** *eval-uses*:: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *ID*  $\Rightarrow$  *bool*

**for** *g* **where**

*use0*: *nid*  $\in$  *ids g*

$\implies eval-uses \ g \ nid \ nid \mid$

```

use-inp:  $nid' \in inputs\ g\ n$ 
 $\implies eval\text{-}uses\ g\ nid\ nid' \mid$ 

use-trans:  $\llbracket eval\text{-}uses\ g\ nid\ nid';$ 
 $eval\text{-}uses\ g\ nid'\ nid'' \rrbracket$ 
 $\implies eval\text{-}uses\ g\ nid\ nid''$ 

fun eval-usages :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID set where
  eval-usages g nid = {n  $\in$  ids g . eval-uses g nid n}

lemma eval-usages-self:
  assumes nid  $\in$  ids g
  shows nid  $\in$  eval-usages g nid
  using assms eval-usages.simps eval-uses.intros(1)
  by (simp add: ids.rep-eq)

lemma not-in-g-inputs:
  assumes nid  $\notin$  ids g
  shows inputs g nid = {}
proof –
  have k: kind g nid = NoNode using assms not-in-g by blast
  then show ?thesis by (simp add: k)
qed

lemma child-member:
  assumes n = kind g nid
  assumes n  $\neq$  NoNode
  assumes List.member (inputs-of n) child
  shows child  $\in$  inputs g nid
  unfolding inputs.simps using assms
  by (metis in-set-member)

lemma child-member-in:
  assumes nid  $\in$  ids g
  assumes List.member (inputs-of (kind g nid)) child
  shows child  $\in$  inputs g nid
  unfolding inputs.simps using assms
  by (metis child-member ids-some inputs.elims)

lemma inp-in-g:
  assumes n  $\in$  inputs g nid
  shows nid  $\in$  ids g
proof –
  have inputs g nid  $\neq$  {}
  using assms
  by (metis empty-iff empty-set)

```



```

then have kind g nid  $\neq$  NoNode
  using not-in-g-inputs
  using ids-some by blast
then show ?thesis
  using not-in-g
  by metis
qed

```

```

lemma inp-in-g-wf:
  assumes wf-graph g
  assumes  $n \in \text{inputs } g \text{ nid}$ 
  shows  $n \in \text{ids } g$ 
  using assms unfolding wf-folds
  using inp-in-g by blast

```

```

lemma kind-unchanged:
  assumes nid  $\in \text{ids } g1$ 
  assumes unchanged (eval-usages g1 nid) g1 g2
  shows kind g1 nid = kind g2 nid
proof -
  show ?thesis
    using assms eval-usages-self
    using unchanged.simps by blast
qed

```

```

lemma stamp-unchanged:
  assumes nid  $\in \text{ids } g1$ 
  assumes unchanged (eval-usages g1 nid) g1 g2
  shows stamp g1 nid = stamp g2 nid
  by (meson assms(1) assms(2) eval-usages-self unchanged.elims(2))

```

```

lemma child-unchanged:
  assumes child  $\in \text{inputs } g1 \text{ nid}$ 
  assumes unchanged (eval-usages g1 nid) g1 g2
  shows unchanged (eval-usages g1 child) g1 g2
  by (smt assms(1) assms(2) eval-usages.simps mem-Collect-eq
      unchanged.simps use-inp use-trans)

```

```

lemma eval-usages:
  assumes us = eval-usages g nid
  assumes nid'  $\in \text{ids } g$ 
  shows eval-uses g nid nid'  $\longleftrightarrow$  nid'  $\in \text{us}$  (is ?P  $\longleftrightarrow$  ?Q)
  using assms eval-usages.simps
  by (simp add: ids.rep-eq)

```

```

lemma inputs-are-uses:
  assumes nid'  $\in \text{inputs } g \text{ nid}$ 

```

**shows** *eval-uses* *g nid nid'*  
**by** (*metis assms use-inp*)

**lemma** *inputs-are-usages*:  
**assumes** *nid' ∈ inputs g nid*  
**assumes** *nid' ∈ ids g*  
**shows** *nid' ∈ eval-usages g nid*  
**using** *assms(1) assms(2) eval-usages inputs-are-uses* **by** *blast*

**lemma** *inputs-of-are-usages*:  
**assumes** *List.member (inputs-of (kind g nid)) nid'*  
**assumes** *nid' ∈ ids g*  
**shows** *nid' ∈ eval-usages g nid*  
**by** (*metis assms(1) assms(2) in-set-member inputs.elims inputs-are-usages*)

**lemma** *usage-includes-inputs*:  
**assumes** *us = eval-usages g nid*  
**assumes** *ls = inputs g nid*  
**assumes** *ls ⊆ ids g*  
**shows** *ls ⊆ us*  
**using** *inputs-are-usages eval-usages*  
**using** *assms(1) assms(2) assms(3)* **by** *blast*

**lemma** *elim-inp-set*:  
**assumes** *k = kind g nid*  
**assumes** *k ≠ NoNode*  
**assumes** *child ∈ set (inputs-of k)*  
**shows** *child ∈ inputs g nid*  
**using** *assms* **by** *auto*

**lemma** *encode-in-ids*:  
**assumes** *g ⊢ nid ≃ e*  
**shows** *nid ∈ ids g*  
**using** *assms*  
**apply** (*induction rule: rep.induct*)  
**apply** *simp+*  
**by** *fastforce+*

**lemma** *eval-in-ids*:  
**assumes** *[g, m, p] ⊢ nid ↦ v*  
**shows** *nid ∈ ids g*  
**using** *assms* **using** *encodeeval-def encode-in-ids*  
**by** *auto*

**lemma** *transitive-kind-same*:  
**assumes** *unchanged (eval-usages g1 nid) g1 g2*  
**shows**  $\forall \text{nid}' \in (\text{eval-usages } g1 \text{ nid}) . \text{kind } g1 \text{ nid}' = \text{kind } g2 \text{ nid}'$   
**using** *assms*  
**by** (*meson unchanged.elims(1)*)

```

theorem stay-same-encoding:
  assumes nc: unchanged (eval-usages g1 nid) g1 g2
  assumes g1: g1  $\vdash$  nid  $\simeq$  e
  assumes wf: wf-graph g1
  shows g2  $\vdash$  nid  $\simeq$  e
proof –
  have dom: nid  $\in$  ids g1
  using g1 encode-in-ids by simp
  show ?thesis
using g1 nc wf dom proof (induction e rule: rep.induct)
  case (ConstantNode n c)
  then have kind g2 n = ConstantNode c
  using dom nc kind-unchanged
  by metis
  then show ?case using rep.ConstantNode
  by presburger
next
  case (ParameterNode n i s)
  then have kind g2 n = ParameterNode i
  by (metis kind-unchanged)
  then show ?case
  by (metis ParameterNode.hyps(2) ParameterNode.premis(1) ParameterNode.premis(3)
  rep.ParameterNode stamp-unchanged)
next
  case (ConditionalNode n c t f ce te fe)
  then have kind g2 n = ConditionalNode c t f
  by (metis kind-unchanged)
  have c  $\in$  eval-usages g1 n  $\wedge$  t  $\in$  eval-usages g1 n  $\wedge$  f  $\in$  eval-usages g1 n
  using inputs-of-ConditionalNode
  by (metis ConditionalNode.hyps(1) ConditionalNode.hyps(2) ConditionalNode.hyps(3)
  ConditionalNode.hyps(4) encode-in-ids inputs.simps inputs-are-usages
  list.set-intros(1) set-subset-Cons subset-code(1))
  then show ?case using transitive-kind-same
  by (metis ConditionalNode.hyps(1) ConditionalNode.premis(1) IRNodes.inputs-of-ConditionalNode
   $\langle$ kind g2 n = ConditionalNode c t f $\rangle$  child-unchanged inputs.simps list.set-intros(1)
  local.ConditionalNode(5) local.ConditionalNode(6) local.ConditionalNode(7) local.ConditionalNode(9)
  rep.ConditionalNode set-subset-Cons subset-code(1) unchanged.elims(2))
next
  case (AbsNode n x xe)
  then have kind g2 n = AbsNode x
  using kind-unchanged
  by metis
  then have x  $\in$  eval-usages g1 n
  using inputs-of-AbsNode
  by (metis AbsNode.hyps(1) AbsNode.hyps(2) encode-in-ids inputs.simps inputs-are-usages
  list.set-intros(1))
  then show ?case
  by (metis AbsNode.IH AbsNode.hyps(1) AbsNode.premis(1) AbsNode.premis(3))

```

```

IRNodes.inputs-of-AbsNode ⟨kind g2 n = AbsNode x⟩ child-member-in child-unchanged
local.wf member-rec(1) rep.AbsNode unchanged.simps)
next
  case (NotNode n x xe)
  then have kind g2 n = NotNode x
    using kind-unchanged
    by metis
  then have x ∈ eval-usages g1 n
    using inputs-of-NotNode
    by (metis NotNode.hyps(1) NotNode.hyps(2) encode-in-ids inputs.simps in-
      puts-are-usages list.set-intros(1))
  then show ?case
    by (metis NotNode.IH NotNode.hyps(1) NotNode.prem(1) NotNode.prem(3)
      IRNodes.inputs-of-NotNode ⟨kind g2 n = NotNode x⟩ child-member-in child-unchanged
      local.wf member-rec(1) rep.NotNode unchanged.simps)
next
  case (NegateNode n x xe)
  then have kind g2 n = NegateNode x
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n
    using inputs-of-NegateNode
    by (metis NegateNode.hyps(1) NegateNode.hyps(2) encode-in-ids inputs.simps
      inputs-are-usages list.set-intros(1))
  then show ?case
    by (metis IRNodes.inputs-of-NegateNode NegateNode.IH NegateNode.hyps(1)
      NegateNode.prem(1) NegateNode.prem(3) ⟨kind g2 n = NegateNode x⟩ child-member-in
      child-unchanged local.wf member-rec(1) rep.NegateNode unchanged.elims(1))
next
  case (LogicNegationNode n x xe)
  then have kind g2 n = LogicNegationNode x
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
    by (metis LogicNegationNode.hyps(1) LogicNegationNode.hyps(2) encode-in-ids
      member-rec(1))
  then show ?case
    by (metis IRNodes.inputs-of-LogicNegationNode LogicNegationNode.IH Logic-
      NegationNode.hyps(1) LogicNegationNode.hyps(2) LogicNegationNode.prem(1) ⟨kind
      g2 n = LogicNegationNode x⟩ child-unchanged encode-in-ids inputs.simps list.set-intros(1)
      local.wf rep.LogicNegationNode)
next
  case (AddNode n x y xe ye)
  then have kind g2 n = AddNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
    by (metis AddNode.hyps(1) AddNode.hyps(2) AddNode.hyps(3) IRNodes.inputs-of-AddNode
      encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case

```

```

    by (metis AddNode.IH(1) AddNode.IH(2) AddNode.hyps(1) AddNode.hyps(2)
AddNode.hyps(3) AddNode.premis(1) IRNodes.inputs-of-AddNode ⟨kind g2 n = AddNode
x y⟩ child-unchanged encode-in-ids in-set-member inputs.simps local.wf member-rec(1)
rep.AddNode)
next
  case (MulNode n x y xe ye)
  then have kind g2 n = MulNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
  by (metis MulNode.hyps(1) MulNode.hyps(2) MulNode.hyps(3) IRNodes.inputs-of-MulNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using MulNode inputs-of-MulNode
  by (metis ⟨kind g2 n = MulNode x y⟩ child-unchanged inputs.simps list.set-intros(1)
rep.MulNode set-subset-Cons subset-iff unchanged.elims(2))
next
  case (SubNode n x y xe ye)
  then have kind g2 n = SubNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
  by (metis SubNode.hyps(1) SubNode.hyps(2) SubNode.hyps(3) IRNodes.inputs-of-SubNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using SubNode inputs-of-SubNode
  by (metis ⟨kind g2 n = SubNode x y⟩ child-member child-unchanged encode-in-ids
ids-some member-rec(1) rep.SubNode)
next
  case (AndNode n x y xe ye)
  then have kind g2 n = AndNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
  by (metis AndNode.hyps(1) AndNode.hyps(2) AndNode.hyps(3) IRNodes.inputs-of-AndNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using AndNode inputs-of-AndNode
  by (metis ⟨kind g2 n = AndNode x y⟩ child-unchanged inputs.simps list.set-intros(1)
rep.AndNode set-subset-Cons subset-iff unchanged.elims(2))
next
  case (OrNode n x y xe ye)
  then have kind g2 n = OrNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-OrNode inputs-of-are-usages
  by (metis OrNode.hyps(1) OrNode.hyps(2) OrNode.hyps(3) IRNodes.inputs-of-OrNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using OrNode inputs-of-OrNode
  by (metis ⟨kind g2 n = OrNode x y⟩ child-member child-unchanged encode-in-ids
ids-some member-rec(1) rep.OrNode)
next

```

```

case (XorNode n x y xe ye)
then have kind g2 n = XorNode x y
  using kind-unchanged by metis
then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
  using inputs-of-XorNode inputs-of-are-usages
  by (metis XorNode.hyps(1) XorNode.hyps(2) XorNode.hyps(3) IRNodes.inputs-of-XorNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using XorNode inputs-of-XorNode
  by (metis ⟨kind g2 n = XorNode x y⟩ child-member child-unchanged encode-in-ids
ids-some member-rec(1) rep.XorNode)
next
case (ShortCircuitOrNode n x y xe ye)
then have kind g2 n = ShortCircuitOrNode x y
  using kind-unchanged by metis
then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
  using inputs-of-XorNode inputs-of-are-usages
  by (metis ShortCircuitOrNode.hyps(1) ShortCircuitOrNode.hyps(2) ShortCircuitOrNode.hyps(3) IRNodes.inputs-of-ShortCircuitOrNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using ShortCircuitOrNode inputs-of-ShortCircuitOrNode
  by (metis ⟨kind g2 n = ShortCircuitOrNode x y⟩ child-member child-unchanged
encode-in-ids ids-some member-rec(1) rep.ShortCircuitOrNode)
next
case (LeftShiftNode n x y xe ye)
then have kind g2 n = LeftShiftNode x y
  using kind-unchanged by metis
then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
  using inputs-of-XorNode inputs-of-are-usages
  by (metis LeftShiftNode.hyps(1) LeftShiftNode.hyps(2) LeftShiftNode.hyps(3) IRNodes.inputs-of-LeftShiftNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using LeftShiftNode inputs-of-LeftShiftNode
  by (metis ⟨kind g2 n = LeftShiftNode x y⟩ child-member child-unchanged
encode-in-ids ids-some member-rec(1) rep.LeftShiftNode)
next
case (RightShiftNode n x y xe ye)
then have kind g2 n = RightShiftNode x y
  using kind-unchanged by metis
then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
  using inputs-of-RightShiftNode inputs-of-are-usages
  by (metis RightShiftNode.hyps(1) RightShiftNode.hyps(2) RightShiftNode.hyps(3) IRNodes.inputs-of-RightShiftNode
encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using RightShiftNode inputs-of-RightShiftNode
  by (metis ⟨kind g2 n = RightShiftNode x y⟩ child-member child-unchanged
encode-in-ids ids-some member-rec(1) rep.RightShiftNode)
next
case (UnsignedRightShiftNode n x y xe ye)
then have kind g2 n = UnsignedRightShiftNode x y

```

```

    using kind-unchanged by metis
  then have  $x \in \text{eval-usages } g1 \ n \wedge y \in \text{eval-usages } g1 \ n$ 
    using inputs-of-UnsignedRightShiftNode inputs-of-are-usages
    by (metis UnsignedRightShiftNode.hyps(1) UnsignedRightShiftNode.hyps(2) UnsignedRightShiftNode.hyps(3) IRNodes.inputs-of-UnsignedRightShiftNode encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using UnsignedRightShiftNode inputs-of-UnsignedRightShiftNode
    by (metis  $\langle \text{kind } g2 \ n = \text{UnsignedRightShiftNode } x \ y \rangle$  child-member child-unchanged encode-in-ids ids-some member-rec(1) rep.UnsignedRightShiftNode)
next
  case (IntegerBelowNode  $n \ x \ y \ xe \ ye$ )
  then have  $\text{kind } g2 \ n = \text{IntegerBelowNode } x \ y$ 
    using kind-unchanged by metis
  then have  $x \in \text{eval-usages } g1 \ n \wedge y \in \text{eval-usages } g1 \ n$ 
    using inputs-of-IntegerBelowNode inputs-of-are-usages
    by (metis IntegerBelowNode.hyps(1) IntegerBelowNode.hyps(2) IntegerBelowNode.hyps(3) IRNodes.inputs-of-IntegerBelowNode encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using IntegerBelowNode inputs-of-IntegerBelowNode
    by (metis  $\langle \text{kind } g2 \ n = \text{IntegerBelowNode } x \ y \rangle$  child-member child-unchanged encode-in-ids ids-some member-rec(1) rep.IntegerBelowNode)
next
  case (IntegerEqualsNode  $n \ x \ y \ xe \ ye$ )
  then have  $\text{kind } g2 \ n = \text{IntegerEqualsNode } x \ y$ 
    using kind-unchanged by metis
  then have  $x \in \text{eval-usages } g1 \ n \wedge y \in \text{eval-usages } g1 \ n$ 
    using inputs-of-IntegerEqualsNode inputs-of-are-usages
    by (metis IntegerEqualsNode.hyps(1) IntegerEqualsNode.hyps(2) IntegerEqualsNode.hyps(3) IRNodes.inputs-of-IntegerEqualsNode encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using IntegerEqualsNode inputs-of-IntegerEqualsNode
    by (metis  $\langle \text{kind } g2 \ n = \text{IntegerEqualsNode } x \ y \rangle$  child-member child-unchanged encode-in-ids ids-some member-rec(1) rep.IntegerEqualsNode)
next
  case (IntegerLessThanNode  $n \ x \ y \ xe \ ye$ )
  then have  $\text{kind } g2 \ n = \text{IntegerLessThanNode } x \ y$ 
    using kind-unchanged by metis
  then have  $x \in \text{eval-usages } g1 \ n \wedge y \in \text{eval-usages } g1 \ n$ 
    using inputs-of-IntegerLessThanNode inputs-of-are-usages
    by (metis IntegerLessThanNode.hyps(1) IntegerLessThanNode.hyps(2) IntegerLessThanNode.hyps(3) IRNodes.inputs-of-IntegerLessThanNode encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using IntegerLessThanNode inputs-of-IntegerLessThanNode
    by (metis  $\langle \text{kind } g2 \ n = \text{IntegerLessThanNode } x \ y \rangle$  child-member child-unchanged encode-in-ids ids-some member-rec(1) rep.IntegerLessThanNode)
next
  case (NarrowNode  $n \ ib \ rb \ x \ xe$ )
  then have  $\text{kind } g2 \ n = \text{NarrowNode } ib \ rb \ x$ 
    using kind-unchanged by metis

```

```

then have  $x \in \text{eval-usages } g1 \ n$ 
  using inputs-of-NarrowNode inputs-of-are-usages
  by (metis NarrowNode.hyps(1) NarrowNode.hyps(2) IRNodes.inputs-of-NarrowNode
encode-in-ids inputs.simps inputs-are-usages list.set-intros(1))
  then show ?case using NarrowNode inputs-of-NarrowNode
    by (metis  $\langle \text{kind } g2 \ n = \text{NarrowNode } ib \ rb \ x \rangle$  child-unchanged inputs.elims
list.set-intros(1) rep.NarrowNode unchanged.simps)
next
  case (SignExtendNode n ib rb x xe)
  then have  $\text{kind } g2 \ n = \text{SignExtendNode } ib \ rb \ x$ 
    using kind-unchanged by metis
  then have  $x \in \text{eval-usages } g1 \ n$ 
    using inputs-of-SignExtendNode inputs-of-are-usages
    by (metis SignExtendNode.hyps(1) SignExtendNode.hyps(2) encode-in-ids in-
puts.simps inputs-are-usages list.set-intros(1))
    then show ?case using SignExtendNode inputs-of-SignExtendNode
      by (metis  $\langle \text{kind } g2 \ n = \text{SignExtendNode } ib \ rb \ x \rangle$  child-member-in child-unchanged
in-set-member list.set-intros(1) rep.SignExtendNode unchanged.elims(2))
  next
  case (ZeroExtendNode n ib rb x xe)
  then have  $\text{kind } g2 \ n = \text{ZeroExtendNode } ib \ rb \ x$ 
    using kind-unchanged by metis
  then have  $x \in \text{eval-usages } g1 \ n$ 
    using inputs-of-ZeroExtendNode inputs-of-are-usages
    by (metis ZeroExtendNode.hyps(1) ZeroExtendNode.hyps(2) IRNodes.inputs-of-ZeroExtendNode
encode-in-ids inputs.simps inputs-are-usages list.set-intros(1))
    then show ?case using ZeroExtendNode inputs-of-ZeroExtendNode
      by (metis  $\langle \text{kind } g2 \ n = \text{ZeroExtendNode } ib \ rb \ x \rangle$  child-member-in child-unchanged
member-rec(1) rep.ZeroExtendNode unchanged.simps)
  next
  case (LeafNode n s)
  then show ?case
    by (metis kind-unchanged rep.LeafNode stamp-unchanged)
  next
  case (RefNode n n')
  then have  $\text{kind } g2 \ n = \text{RefNode } n'$ 
    using kind-unchanged by metis
  then have  $n' \in \text{eval-usages } g1 \ n$ 
    by (metis IRNodes.inputs-of-RefNode RefNode.hyps(1) RefNode.hyps(2) en-
code-in-ids inputs.elims inputs-are-usages list.set-intros(1))
    then show ?case
      by (metis IRNodes.inputs-of-RefNode RefNode.IH RefNode.hyps(1) RefNode.hyps(2)
RefNode.prem(1)  $\langle \text{kind } g2 \ n = \text{RefNode } n' \rangle$  child-unchanged encode-in-ids in-
puts.elims list.set-intros(1) local.wf rep.RefNode)
qed
qed

```



```

theorem stay-same:
  assumes nc: unchanged (eval-usages g1 nid) g1 g2
  assumes g1:  $[g1, m, p] \vdash \text{id} \mapsto v1$ 
  assumes wf: wf-graph g1
  shows  $[g2, m, p] \vdash \text{id} \mapsto v1$ 
proof –
  have nid:  $\text{id} \in \text{ids } g1$ 
    using g1 eval-in-ids by simp
  then have  $\text{id} \in \text{eval-usages } g1 \text{ nid}$ 
    using eval-usages-self by blast
  then have kind-same:  $\text{kind } g1 \text{ nid} = \text{kind } g2 \text{ nid}$ 
    using nc node-unchanged by blast
  obtain e where  $e: (g1 \vdash \text{id} \simeq e) \wedge ([m,p] \vdash e \mapsto v1)$ 
    using encodeeval-def g1
    by auto
  then have val:  $[m,p] \vdash e \mapsto v1$ 
    using g1 encodeeval-def
    by simp
  then show ?thesis using e nid nc
    unfolding encodeeval-def
proof (induct e v1 arbitrary: nid rule: evaltree.induct)
  case (ConstantExpr c)
    then show ?case
      by (meson local.wf stay-same-encoding)
  next
    case (ParameterExpr i s)
    have  $g2 \vdash \text{id} \simeq \text{ParameterExpr } i \text{ s}$ 
      using stay-same-encoding ParameterExpr
      by (meson local.wf)
    then show ?case using evaltree.ParameterExpr
      by (meson ParameterExpr.hyps)
  next
    case (ConditionalExpr ce cond branch te fe v)
    then have  $g2 \vdash \text{id} \simeq \text{ConditionalExpr } ce \text{ te } fe$ 
      using ConditionalExpr.premis(1) ConditionalExpr.premis(3) local.wf stay-same-encoding
      by presburger
    then show ?case
      by (meson ConditionalExpr.premis(1) ConditionalExpr.premis(3) local.wf
stay-same-encoding)
  next
    case (UnaryExpr xe v op)
    then show ?case
      using local.wf stay-same-encoding by blast
  next
    case (BinaryExpr xe x ye y op)
    then show ?case
      using local.wf stay-same-encoding by blast
  next
    case (LeafExpr val nid s)

```

```

    then show ?case
    by (metis local.wf stay-same-encoding)
qed
qed

```

```

lemma add-changed:
  assumes gup = add-node new k g
  shows changeonly {new} g gup
  using assms unfolding add-node-def changeonly.simps
  using add-node.rep-eq add-node-def kind.rep-eq stamp.rep-eq by simp

```

```

lemma disjoint-change:
  assumes changeonly change g gup
  assumes nochange = ids g - change
  shows unchanged nochange g gup
  using assms unfolding changeonly.simps unchanged.simps
  by blast

```

```

lemma add-node-unchanged:
  assumes new  $\notin$  ids g
  assumes nid  $\in$  ids g
  assumes gup = add-node new k g
  assumes wf-graph g
  shows unchanged (eval-usages g nid) g gup
proof -
  have new  $\notin$  (eval-usages g nid) using assms
  using eval-usages.simps by blast
  then have changeonly {new} g gup
  using assms add-changed by blast
  then show ?thesis using assms add-node-def disjoint-change
  using Diff-insert-absorb by auto
qed

```

```

lemma eval-uses-imp:
  ((nid'  $\in$  ids g  $\wedge$  nid = nid')
   $\vee$  nid'  $\in$  inputs g nid
   $\vee$  ( $\exists$  nid'' . eval-uses g nid nid''  $\wedge$  eval-uses g nid'' nid'))
 $\longleftrightarrow$  eval-uses g nid nid'
  using use0 use-inp use-trans
  by (meson eval-uses.simps)

```

```

lemma wf-use-ids:
  assumes wf-graph g
  assumes nid  $\in$  ids g
  assumes eval-uses g nid nid'
  shows nid'  $\in$  ids g
  using assms(3)
proof (induction rule: eval-uses.induct)

```

```

    case use0
    then show ?case by simp
next
    case use-inp
    then show ?case
        using assms(1) inp-in-g-wf by blast
next
    case use-trans
    then show ?case by blast
qed

lemma no-external-use:
  assumes wf-graph g
  assumes nid'  $\notin$  ids g
  assumes nid  $\in$  ids g
  shows  $\neg$ (eval-uses g nid nid')
proof -
  have 0: nid  $\neq$  nid'
    using assms by blast
  have inp: nid'  $\notin$  inputs g nid
    using assms
    using inp-in-g-wf by blast
  have rec-0:  $\nexists n . n \in$  ids g  $\wedge$  n = nid'
    using assms by blast
  have rec-inp:  $\nexists n . n \in$  ids g  $\wedge$  n  $\in$  inputs g nid'
    using assms(2) inp-in-g by blast
  have rec:  $\nexists$  nid'' . eval-uses g nid nid''  $\wedge$  eval-uses g nid'' nid'
    using wf-use-ids assms(1) assms(2) assms(3) by blast
  from inp 0 rec show ?thesis
    using eval-uses-imp by blast
qed

end

```

## 6.8 Tree to Graph Theorems

```

theory TreeToGraphThms
imports
  IRTreeEvalThms
  IRGraphFrames
  HOL-Eisbach.Eisbach
  HOL-Eisbach.Eisbach-Tools
begin

```

### 6.8.1 Extraction and Evaluation of Expression Trees is Deterministic.

First, we prove some extra rules that relate each type of IRNode to the corresponding IRExpr type that 'rep' will produce. These are very helpful

for proving that 'rep' is deterministic.

**named-theorems** *rep*

**lemma** *rep-constant* [*rep*]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = ConstantNode\ c \implies$   
 $e = ConstantExpr\ c$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-parameter* [*rep*]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = ParameterNode\ i \implies$   
 $(\exists\ s.\ e = ParameterExpr\ i\ s)$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-conditional* [*rep*]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = ConditionalNode\ c\ t\ f \implies$   
 $(\exists\ ce\ te\ fe.\ e = ConditionalExpr\ ce\ te\ fe)$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-abs* [*rep*]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = AbsNode\ x \implies$   
 $(\exists\ xe.\ e = UnaryExpr\ UnaryAbs\ xe)$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-not* [*rep*]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = NotNode\ x \implies$   
 $(\exists\ xe.\ e = UnaryExpr\ UnaryNot\ xe)$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-negate* [*rep*]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = NegateNode\ x \implies$   
 $(\exists\ xe.\ e = UnaryExpr\ UnaryNeg\ xe)$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-logicnegation* [*rep*]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = LogicNegationNode\ x \implies$   
 $(\exists\ xe.\ e = UnaryExpr\ UnaryLogicNegation\ xe)$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-add* [*rep*]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = AddNode\ x\ y \implies$   
 $(\exists\ xe\ ye.\ e = BinaryExpr\ BinAdd\ xe\ ye)$

**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-sub* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = SubNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinSub\ xe\ ye)$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-mul* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = MulNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinMul\ xe\ ye)$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-and* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = AndNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinAnd\ xe\ ye)$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-or* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = OrNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinOr\ xe\ ye)$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-xor* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = XorNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinXor\ xe\ ye)$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-short-circuit-or* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = ShortCircuitOrNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinShortCircuitOr\ xe\ ye)$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-left-shift* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = LeftShiftNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinLeftShift\ xe\ ye)$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-right-shift* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = RightShiftNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinRightShift\ xe\ ye)$   
**by** (*induction rule: rep.induct; auto*)

**lemma** *rep-unsigned-right-shift* [rep]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = UnsignedRightShiftNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinURightShift\ xe\ ye)$   
**by** (induction rule: *rep.induct*; auto)

**lemma** *rep-integer-below* [rep]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = IntegerBelowNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinIntegerBelow\ xe\ ye)$   
**by** (induction rule: *rep.induct*; auto)

**lemma** *rep-integer-equals* [rep]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = IntegerEqualsNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinIntegerEquals\ xe\ ye)$   
**by** (induction rule: *rep.induct*; auto)

**lemma** *rep-integer-less-than* [rep]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = IntegerLessThanNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinIntegerLessThan\ xe\ ye)$   
**by** (induction rule: *rep.induct*; auto)

**lemma** *rep-narrow* [rep]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = NarrowNode\ ib\ rb\ x \implies$   
 $(\exists\ x. e = UnaryExpr\ (UnaryNarrow\ ib\ rb)\ x)$   
**by** (induction rule: *rep.induct*; auto)

**lemma** *rep-sign-extend* [rep]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = SignExtendNode\ ib\ rb\ x \implies$   
 $(\exists\ x. e = UnaryExpr\ (UnarySignExtend\ ib\ rb)\ x)$   
**by** (induction rule: *rep.induct*; auto)

**lemma** *rep-zero-extend* [rep]:

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = ZeroExtendNode\ ib\ rb\ x \implies$   
 $(\exists\ x. e = UnaryExpr\ (UnaryZeroExtend\ ib\ rb)\ x)$   
**by** (induction rule: *rep.induct*; auto)

**lemma** *rep-load-field* [rep]:

$g \vdash n \simeq e \implies$   
 $is\ preevaluated\ (kind\ g\ n) \implies$   
 $(\exists\ s. e = LeafExpr\ n\ s)$   
**by** (induction rule: *rep.induct*; auto)

```

lemma rep-ref [rep]:
  g ⊢ n ≃ e ⇒
    kind g n = RefNode n' ⇒
      g ⊢ n' ≃ e
by (induction rule: rep.induct; auto)

```

```

method solve-det uses node =
  (match node in kind - - = node - for node ⇒
    ⟨match rep in r: - ⇒ - = node - ⇒ - ⇒
      ⟨match IRNode.inject in i: (node - = node -) = - ⇒
        ⟨match RepE in e: - ⇒ (∧x. - = node x ⇒ -) ⇒ - ⇒
          ⟨match IRNode.distinct in d: node - ≠ RefNode - ⇒
            ⟨metis i e r d⟩⟩⟩⟩ |
    match node in kind - - = node - - for node ⇒
      ⟨match rep in r: - ⇒ - = node - - ⇒ - ⇒
        ⟨match IRNode.inject in i: (node - - = node - -) = - ⇒
          ⟨match RepE in e: - ⇒ (∧x y. - = node x y ⇒ -) ⇒ - ⇒
            ⟨match IRNode.distinct in d: node - - ≠ RefNode - ⇒
              ⟨metis i e r d⟩⟩⟩⟩ |
    match node in kind - - = node - - - for node ⇒
      ⟨match rep in r: - ⇒ - = node - - - ⇒ - ⇒
        ⟨match IRNode.inject in i: (node - - - = node - - -) = - ⇒
          ⟨match RepE in e: - ⇒ (∧x y z. - = node x y z ⇒ -) ⇒ - ⇒
            ⟨match IRNode.distinct in d: node - - - ≠ RefNode - ⇒
              ⟨metis i e r d⟩⟩⟩⟩ |
    match node in kind - - = node - - - for node ⇒
      ⟨match rep in r: - ⇒ - = node - - - ⇒ - ⇒
        ⟨match IRNode.inject in i: (node - - - = node - - -) = - ⇒
          ⟨match RepE in e: - ⇒ (∧x. - = node - - x ⇒ -) ⇒ - ⇒
            ⟨match IRNode.distinct in d: node - - - ≠ RefNode - ⇒
              ⟨metis i e r d⟩⟩⟩⟩)

```

Now we can prove that 'rep' and 'eval', and their list versions, are deterministic.

```

lemma repDet:
  shows (g ⊢ n ≃ e1) ⇒ (g ⊢ n ≃ e2) ⇒ e1 = e2
proof (induction arbitrary: e2 rule: rep.induct)
  case (ConstantNode n c)
  then show ?case using rep-constant by auto
next
  case (ParameterNode n i s)
  then show ?case
    by (metis IRNode.disc(2685) ParameterNodeE is-RefNode-def rep-parameter)
next
  case (ConditionalNode n c t f ce te fe)
  then show ?case
    using IRNode.distinct(593)
    using IRNode.inject(6) ConditionalNodeE rep-conditional

```

```

    by metis
next
  case (AbsNode n x xe)
  then show ?case
    by (solve-det node: AbsNode)
next
  case (NotNode n x xe)
  then show ?case
    by (solve-det node: NotNode)
next
  case (NegateNode n x xe)
  then show ?case
    by (solve-det node: NegateNode)
next
  case (LogicNegationNode n x xe)
  then show ?case
    by (solve-det node: LogicNegationNode)
next
  case (AddNode n x y xe ye)
  then show ?case
    by (solve-det node: AddNode)
next
  case (MulNode n x y xe ye)
  then show ?case
    by (solve-det node: MulNode)
next
  case (SubNode n x y xe ye)
  then show ?case
    by (solve-det node: SubNode)
next
  case (AndNode n x y xe ye)
  then show ?case
    by (solve-det node: AndNode)
next
  case (OrNode n x y xe ye)
  then show ?case
    by (solve-det node: OrNode)
next
  case (XorNode n x y xe ye)
  then show ?case
    by (solve-det node: XorNode)
next
  case (ShortCircuitOrNode n x y xe ye)
  then show ?case
    by (solve-det node: ShortCircuitOrNode)
next
  case (LeftShiftNode n x y xe ye)
  then show ?case
    by (solve-det node: LeftShiftNode)

```



```

next
  case (RightShiftNode n x y xe ye)
  then show ?case
    by (solve-det node: RightShiftNode)
next
  case (UnsignedRightShiftNode n x y xe ye)
  then show ?case
    by (solve-det node: UnsignedRightShiftNode)
next
  case (IntegerBelowNode n x y xe ye)
  then show ?case
    by (solve-det node: IntegerBelowNode)
next
  case (IntegerEqualsNode n x y xe ye)
  then show ?case
    by (solve-det node: IntegerEqualsNode)
next
  case (IntegerLessThanNode n x y xe ye)
  then show ?case
    by (solve-det node: IntegerLessThanNode)
next
  case (NarrowNode n x xe)
  then show ?case
    by (metis IRNode.distinct(2203) IRNode.inject(28) NarrowNodeE rep-narrow)
next
  case (SignExtendNode n x xe)
  then show ?case
    by (metis IRNode.distinct(2599) IRNode.inject(39) SignExtendNodeE rep-sign-extend)
next
  case (ZeroExtendNode n x xe)
  then show ?case
    by (metis IRNode.distinct(2753) IRNode.inject(50) ZeroExtendNodeE rep-zero-extend)
next
  case (LeafNode n s)
  then show ?case using rep-load-field LeafNodeE
    by (metis is-preevaluated.simps(53))
next
  case (RefNode n')
  then show ?case
    using rep-ref by blast
qed

lemma repAllDet:
   $g \vdash xs \simeq_L e1 \implies$ 
   $g \vdash xs \simeq_L e2 \implies$ 
   $e1 = e2$ 
proof (induction arbitrary: e2 rule: replist.induct)
  case RepNil
  then show ?case

```

```

    using replist.cases by auto
next
  case (RepCons x xe xs xse)
  then show ?case
    by (metis list.distinct(1) list.sel(1) list.sel(3) repDet replist.cases)
qed

```

```

lemma encodeEvalDet:
  [g,m,p] ⊢ e ↦ v1 ⟹
  [g,m,p] ⊢ e ↦ v2 ⟹
  v1 = v2
by (metis encodeeval-def evalDet repDet)

```

```

lemma graphDet: ([g,m,p] ⊢ n ↦ v1) ∧ ([g,m,p] ⊢ n ↦ v2) ⟹ v1 = v2
using encodeEvalDet by blast

```

### 6.8.2 Monotonicity of Graph Refinement

Lift refinement monotonicity to graph level. Hopefully these shouldn't really be required.

```

lemma mono-abs:
  assumes kind g1 n = AbsNode x ∧ kind g2 n = AbsNode x
  assumes (g1 ⊢ x ≃ xe1) ∧ (g2 ⊢ x ≃ xe2)
  assumes xe1 ≥ xe2
  assumes (g1 ⊢ n ≃ e1) ∧ (g2 ⊢ n ≃ e2)
  shows e1 ≥ e2
by (metis AbsNode assms(1) assms(2) assms(3) assms(4) mono-unary repDet)

```

```

lemma mono-not:
  assumes kind g1 n = NotNode x ∧ kind g2 n = NotNode x
  assumes (g1 ⊢ x ≃ xe1) ∧ (g2 ⊢ x ≃ xe2)
  assumes xe1 ≥ xe2
  assumes (g1 ⊢ n ≃ e1) ∧ (g2 ⊢ n ≃ e2)
  shows e1 ≥ e2
by (metis NotNode assms(1) assms(2) assms(3) assms(4) mono-unary repDet)

```

```

lemma mono-negate:
  assumes kind g1 n = NegateNode x ∧ kind g2 n = NegateNode x
  assumes (g1 ⊢ x ≃ xe1) ∧ (g2 ⊢ x ≃ xe2)
  assumes xe1 ≥ xe2
  assumes (g1 ⊢ n ≃ e1) ∧ (g2 ⊢ n ≃ e2)
  shows e1 ≥ e2
by (metis NegateNode assms(1) assms(2) assms(3) assms(4) mono-unary repDet)

```

```

lemma mono-logic-negation:
  assumes kind g1 n = LogicNegationNode x ∧ kind g2 n = LogicNegationNode x
  assumes (g1 ⊢ x ≃ xe1) ∧ (g2 ⊢ x ≃ xe2)
  assumes xe1 ≥ xe2
  assumes (g1 ⊢ n ≃ e1) ∧ (g2 ⊢ n ≃ e2)

```

**shows**  $e1 \geq e2$   
**by** (*metis* *LogicNegationNode* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *mono-unary* *repDet*)

**lemma** *mono-narrow*:

**assumes**  $\text{kind } g1 \ n = \text{NarrowNode } ib \ rb \ x \wedge \text{kind } g2 \ n = \text{NarrowNode } ib \ rb \ x$   
**assumes**  $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$   
**assumes**  $xe1 \geq xe2$   
**assumes**  $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$   
**shows**  $e1 \geq e2$   
**using** *assms* *mono-unary* *repDet* *NarrowNode*  
**by** *metis*

**lemma** *mono-sign-extend*:

**assumes**  $\text{kind } g1 \ n = \text{SignExtendNode } ib \ rb \ x \wedge \text{kind } g2 \ n = \text{SignExtendNode } ib \ rb \ x$   
**assumes**  $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$   
**assumes**  $xe1 \geq xe2$   
**assumes**  $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$   
**shows**  $e1 \geq e2$   
**by** (*metis* *SignExtendNode* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *mono-unary* *repDet*)

**lemma** *mono-zero-extend*:

**assumes**  $\text{kind } g1 \ n = \text{ZeroExtendNode } ib \ rb \ x \wedge \text{kind } g2 \ n = \text{ZeroExtendNode } ib \ rb \ x$   
**assumes**  $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$   
**assumes**  $xe1 \geq xe2$   
**assumes**  $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$   
**shows**  $e1 \geq e2$   
**using** *assms* *mono-unary* *repDet* *ZeroExtendNode*  
**by** *metis*

**lemma** *mono-conditional-graph*:

**assumes**  $\text{kind } g1 \ n = \text{ConditionalNode } c \ t \ f \wedge \text{kind } g2 \ n = \text{ConditionalNode } c \ t \ f$   
**assumes**  $(g1 \vdash c \simeq ce1) \wedge (g2 \vdash c \simeq ce2)$   
**assumes**  $(g1 \vdash t \simeq te1) \wedge (g2 \vdash t \simeq te2)$   
**assumes**  $(g1 \vdash f \simeq fe1) \wedge (g2 \vdash f \simeq fe2)$   
**assumes**  $ce1 \geq ce2 \wedge te1 \geq te2 \wedge fe1 \geq fe2$   
**assumes**  $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$   
**shows**  $e1 \geq e2$   
**using** *ConditionalNodeE* *IRNode.inject*(6) *assms*(1) *assms*(2) *assms*(3) *assms*(4) *assms*(5) *assms*(6) *mono-conditional* *repDet* *rep-conditional*  
**by** (*smt* (*verit*, *best*) *ConditionalNode*)

**lemma** *mono-add*:

**assumes**  $\text{kind } g1 \ n = \text{AddNode } x \ y \wedge \text{kind } g2 \ n = \text{AddNode } x \ y$   
**assumes**  $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$   
**assumes**  $(g1 \vdash y \simeq ye1) \wedge (g2 \vdash y \simeq ye2)$

```

assumes  $xe1 \geq xe2 \wedge ye1 \geq ye2$ 
assumes  $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$ 
shows  $e1 \geq e2$ 
using mono-binary assms AddNodeE IRNode.inject(2) repDet rep-add
by (metis IRNode.distinct(205))

```

**lemma** *mono-mul*:

```

assumes  $kind\ g1\ n = MulNode\ x\ y \wedge kind\ g2\ n = MulNode\ x\ y$ 
assumes  $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$ 
assumes  $(g1 \vdash y \simeq ye1) \wedge (g2 \vdash y \simeq ye2)$ 
assumes  $xe1 \geq xe2 \wedge ye1 \geq ye2$ 
assumes  $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$ 
shows  $e1 \geq e2$ 
using mono-binary assms IRNode.inject(27) MulNodeE repDet rep-mul
by (smt (verit, best) MulNode)

```

**lemma** *term-graph-evaluation*:

```

 $(g \vdash n \sqsubseteq e) \implies (\forall\ m\ p\ v.\ ([m,p] \vdash e \mapsto v) \longrightarrow ([g,m,p] \vdash n \mapsto v))$ 
unfolding graph-represents-expression-def apply auto
by (meson encodeeval-def)

```

**lemma** *encodes-contains*:

```

 $g \vdash n \simeq e \implies$ 
 $kind\ g\ n \neq NoNode$ 
apply (induction rule: rep.induct)
apply (match IRNode.distinct in e: ?n ≠ NoNode ⇒
 $\langle presburger\ add: e \rangle +$ 
apply force
by fastforce)

```

**lemma** *no-encoding*:

```

assumes  $n \notin ids\ g$ 
shows  $\neg(g \vdash n \simeq e)$ 
using assms apply simp apply (rule notI) by (induction e; simp add: en-
codes-contains)

```

**lemma** *not-excluded-keep-type*:

```

assumes  $n \in ids\ g1$ 
assumes  $n \notin excluded$ 
assumes  $(excluded \sqsubseteq as-set\ g1) \subseteq as-set\ g2$ 
shows  $kind\ g1\ n = kind\ g2\ n \wedge stamp\ g1\ n = stamp\ g2\ n$ 
using assms unfolding as-set-def domain-subtraction-def by blast

```

**method** *metis-node-eq-unary* **for**  $node :: 'a \Rightarrow IRNode =$

```

 $(match\ IRNode.inject\ in\ i: (node\ - = node\ -) = - \Rightarrow$ 
 $\langle metis\ i \rangle)$ 

```

**method** *metis-node-eq-binary* **for**  $node :: 'a \Rightarrow 'a \Rightarrow IRNode =$

```

 $(match\ IRNode.inject\ in\ i: (node\ -\ - = node\ -\ -) = - \Rightarrow$ 

```

```

    ⟨metis i⟩
method metis-node-eq-ternary for node :: 'a ⇒ 'a ⇒ 'a ⇒ IRNode =
  (match IRNode.inject in i: (node - - - = node - - -) = - ⇒
    ⟨metis i⟩)

```

### 6.8.3 Lift Data-flow Tree Refinement to Graph Refinement

```

theorem graph-semantic-preservation:
  assumes a:  $e1' \geq e2'$ 
  assumes b:  $(\{n'\} \trianglelefteq \text{as-set } g1) \subseteq \text{as-set } g2$ 
  assumes c:  $g1 \vdash n' \simeq e1'$ 
  assumes d:  $g2 \vdash n' \simeq e2'$ 
  shows graph-refinement g1 g2
  unfolding graph-refinement-def apply rule
  apply (metis b d ids-some no-encoding not-excluded-keep-type singleton-iff sub-
    setI)
  apply (rule allI) apply (rule impI) apply (rule allI) apply (rule impI)
  unfolding graph-represents-expression-def
proof –
  fix n e1
  assume e:  $n \in \text{ids } g1$ 
  assume f:  $(g1 \vdash n \simeq e1)$ 

  show  $\exists e2. (g2 \vdash n \simeq e2) \wedge e1 \geq e2$ 
proof (cases  $n = n'$ )
  case True
  have g:  $e1 = e1'$  using c f True repDet by simp
  have h:  $(g2 \vdash n \simeq e2') \wedge e1' \geq e2'$ 
    using True a d by blast
  then show ?thesis
    using g by blast
next
  case False
  have n  $\notin \{n'\}$ 
    using False by simp
  then have i:  $\text{kind } g1 \ n = \text{kind } g2 \ n \wedge \text{stamp } g1 \ n = \text{stamp } g2 \ n$ 
    using not-excluded-keep-type
    using b e by presburger
  show ?thesis using f i
proof (induction e1)
  case (ConstantNode n c)
  then show ?case
    by (metis eq-refl rep.ConstantNode)
next
  case (ParameterNode n i s)
  then show ?case
    by (metis eq-refl rep.ParameterNode)
next
  case (ConditionalNode n c t f ce1 te1 fe1)

```

```

have k: g1 ⊢ n ≃ ConditionalExpr ce1 te1 fe1 using f ConditionalNode
  by (simp add: ConditionalNode.hyps(2) rep.ConditionalNode)
obtain cn tn fn where l: kind g1 n = ConditionalNode cn tn fn
  using ConditionalNode.hyps(1) by blast
then have mc: g1 ⊢ cn ≃ ce1
  using ConditionalNode.hyps(1) ConditionalNode.hyps(2) by fastforce
from l have mt: g1 ⊢ tn ≃ te1
  using ConditionalNode.hyps(1) ConditionalNode.hyps(3) by fastforce
from l have mf: g1 ⊢ fn ≃ fe1
  using ConditionalNode.hyps(1) ConditionalNode.hyps(4) by fastforce
then show ?case
proof -
  have g1 ⊢ cn ≃ ce1 using mc by simp
  have g1 ⊢ tn ≃ te1 using mt by simp
  have g1 ⊢ fn ≃ fe1 using mf by simp
  have cer: ∃ ce2. (g2 ⊢ cn ≃ ce2) ∧ ce1 ≥ ce2
    using ConditionalNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-ternary ConditionalNode)
  have ter: ∃ te2. (g2 ⊢ tn ≃ te2) ∧ te1 ≥ te2
    using ConditionalNode a b c d l no-encoding not-excluded-keep-type repDet
    singletonD
    by (metis-node-eq-ternary ConditionalNode)
  have ∃ fe2. (g2 ⊢ fn ≃ fe2) ∧ fe1 ≥ fe2
    using ConditionalNode a b c d l no-encoding not-excluded-keep-type repDet
    singletonD
    by (metis-node-eq-ternary ConditionalNode)
  then have ∃ ce2 te2 fe2. (g2 ⊢ n ≃ ConditionalExpr ce2 te2 fe2) ∧
    ConditionalExpr ce1 te1 fe1 ≥ ConditionalExpr ce2 te2 fe2
    using ConditionalNode.premis l rep.ConditionalNode cer ter
    by (smt (verit) mono-conditional)
  then show ?thesis
    by meson
qed
next
case (AbsNode n x xe1)
have k: g1 ⊢ n ≃ UnaryExpr UnaryAbs xe1 using f AbsNode
  by (simp add: AbsNode.hyps(2) rep.AbsNode)
obtain xn where l: kind g1 n = AbsNode xn
  using AbsNode.hyps(1) by blast
then have m: g1 ⊢ xn ≃ xe1
  using AbsNode.hyps(1) AbsNode.hyps(2) by fastforce
then show ?case
proof (cases xn = n')
case True
  then have n: xe1 = e1' using c m repDet by simp
  then have ev: g2 ⊢ n ≃ UnaryExpr UnaryAbs e2' using AbsNode.hyps(1)
    l m n
    using AbsNode.premis True d rep.AbsNode by simp

```

```

    then have r: UnaryExpr UnaryAbs e1' ≥ UnaryExpr UnaryAbs e2'
      by (meson a mono-unary)
    then show ?thesis using ev r
      by (metis n)
  next
    case False
    have g1 ⊢ xn ≃ xe1 using m by simp
    have ∃ xe2. (g2 ⊢ xn ≃ xe2) ∧ xe1 ≥ xe2
      using AbsNode
    using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
      by (metis-node-eq-unary AbsNode)
    then have ∃ xe2. (g2 ⊢ n ≃ UnaryExpr UnaryAbs xe2) ∧ UnaryExpr
UnaryAbs xe1 ≥ UnaryExpr UnaryAbs xe2
      by (metis AbsNode.premis l mono-unary rep.AbsNode)
    then show ?thesis
      by meson
  qed
next
  case (NotNode n x xe1)
  have k: g1 ⊢ n ≃ UnaryExpr UnaryNot xe1 using f NotNode
    by (simp add: NotNode.hyps(2) rep.NotNode)
  obtain xn where l: kind g1 n = NotNode xn
    using NotNode.hyps(1) by blast
  then have m: g1 ⊢ xn ≃ xe1
    using NotNode.hyps(1) NotNode.hyps(2) by fastforce
  then show ?case
  proof (cases xn = n')
    case True
    then have n: xe1 = e1' using c m repDet by simp
    then have ev: g2 ⊢ n ≃ UnaryExpr UnaryNot e2' using NotNode.hyps(1)
l m n
      using NotNode.premis True d rep.NotNode by simp
    then have r: UnaryExpr UnaryNot e1' ≥ UnaryExpr UnaryNot e2'
      by (meson a mono-unary)
    then show ?thesis using ev r
      by (metis n)
  next
    case False
    have g1 ⊢ xn ≃ xe1 using m by simp
    have ∃ xe2. (g2 ⊢ xn ≃ xe2) ∧ xe1 ≥ xe2
      using NotNode
    using False i b l not-excluded-keep-type singletonD no-encoding
      by (metis-node-eq-unary NotNode)
    then have ∃ xe2. (g2 ⊢ n ≃ UnaryExpr UnaryNot xe2) ∧ UnaryExpr
UnaryNot xe1 ≥ UnaryExpr UnaryNot xe2
      by (metis NotNode.premis l mono-unary rep.NotNode)
    then show ?thesis
      by meson
  qed

```

```

next
  case (NegateNode n x xe1)
  have k:  $g1 \vdash n \simeq \text{UnaryExpr UnaryNeg xe1}$  using f NegateNode
    by (simp add: NegateNode.hyps(2) rep.NegateNode)
  obtain xn where l: kind g1 n = NegateNode xn
    using NegateNode.hyps(1) by blast
  then have m:  $g1 \vdash xn \simeq xe1$ 
    using NegateNode.hyps(1) NegateNode.hyps(2) by fastforce
  then show ?case
  proof (cases xn = n')
    case True
    then have n:  $xe1 = e1'$  using c m repDet by simp
    then have ev:  $g2 \vdash n \simeq \text{UnaryExpr UnaryNeg e2'}$  using NegateNode.hyps(1)
  l m n
    using NegateNode.premis True d rep.NegateNode by simp
  then have r:  $\text{UnaryExpr UnaryNeg e1'} \geq \text{UnaryExpr UnaryNeg e2'}$ 
    by (meson a mono-unary)
  then show ?thesis using ev r
    by (metis n)
  next
  case False
  have  $g1 \vdash xn \simeq xe1$  using m by simp
  have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using NegateNode
    using False i b l not-excluded-keep-type singletonD no-encoding
    by (metis-node-eq-unary NegateNode)
  then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr UnaryNeg xe2}) \wedge \text{UnaryExpr}$ 
  UnaryNeg xe1  $\geq \text{UnaryExpr UnaryNeg xe2}$ 
    by (metis NegateNode.premis l mono-unary rep.NegateNode)
  then show ?thesis
    by meson
  qed
  next
  case (LogicNegationNode n x xe1)
  have k:  $g1 \vdash n \simeq \text{UnaryExpr UnaryLogicNegation xe1}$  using f LogicNegationNode
  by (simp add: LogicNegationNode.hyps(2) rep.LogicNegationNode)
  obtain xn where l: kind g1 n = LogicNegationNode xn
    using LogicNegationNode.hyps(1) by blast
  then have m:  $g1 \vdash xn \simeq xe1$ 
    using LogicNegationNode.hyps(1) LogicNegationNode.hyps(2) by fastforce
  then show ?case
  proof (cases xn = n')
    case True
    then have n:  $xe1 = e1'$  using c m repDet by simp
    then have ev:  $g2 \vdash n \simeq \text{UnaryExpr UnaryLogicNegation e2'}$  using
  LogicNegationNode.hyps(1) l m n
    using LogicNegationNode.premis True d rep.LogicNegationNode by simp
  then have r:  $\text{UnaryExpr UnaryLogicNegation e1'} \geq \text{UnaryExpr UnaryLog-}$ 

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icNegation e2'
  by (meson a mono-unary)
  then show ?thesis using ev r
  by (metis n)
next
case False
have g1 ⊢ xn ≃ xe1 using m by simp
have ∃ xe2. (g2 ⊢ xn ≃ xe2) ∧ xe1 ≥ xe2
  using LogicNegationNode
  using False i b l not-excluded-keep-type singletonD no-encoding
  by (metis-node-eq-unary LogicNegationNode)
  then have ∃ xe2. (g2 ⊢ n ≃ UnaryExpr UnaryLogicNegation xe2) ∧
UnaryExpr UnaryLogicNegation xe1 ≥ UnaryExpr UnaryLogicNegation xe2
  by (metis LogicNegationNode.prem1 mono-unary rep.LogicNegationNode)
  then show ?thesis
  by meson
qed
next
case (AddNode n x y xe1 ye1)
have k: g1 ⊢ n ≃ BinaryExpr BinAdd xe1 ye1 using f AddNode
  by (simp add: AddNode.hyps(2) rep.AddNode)
obtain xn yn where l: kind g1 n = AddNode xn yn
  using AddNode.hyps(1) by blast
then have mx: g1 ⊢ xn ≃ xe1
  using AddNode.hyps(1) AddNode.hyps(2) by fastforce
from l have my: g1 ⊢ yn ≃ ye1
  using AddNode.hyps(1) AddNode.hyps(3) by fastforce
then show ?case
proof -
  have g1 ⊢ xn ≃ xe1 using mx by simp
  have g1 ⊢ yn ≃ ye1 using my by simp
  have xer: ∃ xe2. (g2 ⊢ xn ≃ xe2) ∧ xe1 ≥ xe2
    using AddNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary AddNode)
  have ∃ ye2. (g2 ⊢ yn ≃ ye2) ∧ ye1 ≥ ye2
    using AddNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary AddNode)
  then have ∃ xe2 ye2. (g2 ⊢ n ≃ BinaryExpr BinAdd xe2 ye2) ∧ BinaryExpr
BinAdd xe1 ye1 ≥ BinaryExpr BinAdd xe2 ye2
    by (metis AddNode.prem1 mono-binary rep.AddNode xer)
  then show ?thesis
  by meson
qed
next
case (MulNode n x y xe1 ye1)
have k: g1 ⊢ n ≃ BinaryExpr BinMul xe1 ye1 using f MulNode
  by (simp add: MulNode.hyps(2) rep.MulNode)

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obtain  $xn\ yn$  where  $l$ :  $kind\ g1\ n = MulNode\ xn\ yn$ 
  using  $MulNode.hyps(1)$  by blast
then have  $mx$ :  $g1 \vdash xn \simeq xe1$ 
  using  $MulNode.hyps(1)\ MulNode.hyps(2)$  by fastforce
from  $l$  have  $my$ :  $g1 \vdash yn \simeq ye1$ 
  using  $MulNode.hyps(1)\ MulNode.hyps(3)$  by fastforce
then show ?case
proof –
  have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
  have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
  have  $xer$ :  $\exists\ xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using  $MulNode$ 
    using  $a\ b\ c\ d\ l\ no-encoding\ not-excluded-keep-type\ repDet\ singletonD$ 
    by (metis-node-eq-binary  $MulNode$ )
  have  $\exists\ ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
    using  $MulNode$ 
    using  $a\ b\ c\ d\ l\ no-encoding\ not-excluded-keep-type\ repDet\ singletonD$ 
    by (metis-node-eq-binary  $MulNode$ )
  then have  $\exists\ xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinMul\ xe2\ ye2) \wedge BinaryExpr$ 
 $BinMul\ xe1\ ye1 \geq BinaryExpr\ BinMul\ xe2\ ye2$ 
    by (metis  $MulNode.premis\ l\ mono-binary\ rep.MulNode\ xer$ )
  then show ?thesis
    by meson
qed
next
case ( $SubNode\ n\ x\ y\ xe1\ ye1$ )
have  $k$ :  $g1 \vdash n \simeq BinaryExpr\ BinSub\ xe1\ ye1$  using  $f\ SubNode$ 
  by (simp  $add$ :  $SubNode.hyps(2)\ rep.SubNode$ )
obtain  $xn\ yn$  where  $l$ :  $kind\ g1\ n = SubNode\ xn\ yn$ 
  using  $SubNode.hyps(1)$  by blast
then have  $mx$ :  $g1 \vdash xn \simeq xe1$ 
  using  $SubNode.hyps(1)\ SubNode.hyps(2)$  by fastforce
from  $l$  have  $my$ :  $g1 \vdash yn \simeq ye1$ 
  using  $SubNode.hyps(1)\ SubNode.hyps(3)$  by fastforce
then show ?case
proof –
  have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
  have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
  have  $xer$ :  $\exists\ xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using  $SubNode$ 
    using  $a\ b\ c\ d\ l\ no-encoding\ not-excluded-keep-type\ repDet\ singletonD$ 
    by (metis-node-eq-binary  $SubNode$ )
  have  $\exists\ ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
using  $SubNode\ a\ b\ c\ d\ l\ no-encoding\ not-excluded-keep-type\ repDet\ singletonD$ 
  by (metis-node-eq-binary  $SubNode$ )
  then have  $\exists\ xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinSub\ xe2\ ye2) \wedge BinaryExpr$ 
 $BinSub\ xe1\ ye1 \geq BinaryExpr\ BinSub\ xe2\ ye2$ 
    by (metis  $SubNode.premis\ l\ mono-binary\ rep.SubNode\ xer$ )
  then show ?thesis

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      by meson
    qed
  next
    case (AndNode n x y xe1 ye1)
    have k:  $g1 \vdash n \simeq \text{BinaryExpr BinAnd } xe1 \text{ } ye1$  using f AndNode
      by (simp add: AndNode.hyps(2) rep.AndNode)
    obtain xn yn where l: kind g1 n = AndNode xn yn
      using AndNode.hyps(1) by blast
    then have mx:  $g1 \vdash xn \simeq xe1$ 
      using AndNode.hyps(1) AndNode.hyps(2) by fastforce
    from l have my:  $g1 \vdash yn \simeq ye1$ 
      using AndNode.hyps(1) AndNode.hyps(3) by fastforce
    then show ?case
    proof -
      have  $g1 \vdash xn \simeq xe1$  using mx by simp
      have  $g1 \vdash yn \simeq ye1$  using my by simp
      have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
        using AndNode
        using a b c d l no-encoding not-excluded-keep-type repDet singletonD
        by (metis-node-eq-binary AndNode)
      have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
        using AndNode a b c d l no-encoding not-excluded-keep-type repDet
        singletonD
        by (metis-node-eq-binary AndNode)
      then have  $\exists xe2 ye2. (g2 \vdash n \simeq \text{BinaryExpr BinAnd } xe2 \text{ } ye2) \wedge \text{BinaryExpr BinAnd } xe1 \text{ } ye1 \geq \text{BinaryExpr BinAnd } xe2 \text{ } ye2$ 
        by (metis AndNode.premis l mono-binary rep.AndNode xer)
      then show ?thesis
        by meson
    qed
  next
    case (OrNode n x y xe1 ye1)
    have k:  $g1 \vdash n \simeq \text{BinaryExpr BinOr } xe1 \text{ } ye1$  using f OrNode
      by (simp add: OrNode.hyps(2) rep.OrNode)
    obtain xn yn where l: kind g1 n = OrNode xn yn
      using OrNode.hyps(1) by blast
    then have mx:  $g1 \vdash xn \simeq xe1$ 
      using OrNode.hyps(1) OrNode.hyps(2) by fastforce
    from l have my:  $g1 \vdash yn \simeq ye1$ 
      using OrNode.hyps(1) OrNode.hyps(3) by fastforce
    then show ?case
    proof -
      have  $g1 \vdash xn \simeq xe1$  using mx by simp
      have  $g1 \vdash yn \simeq ye1$  using my by simp
      have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
        using OrNode
        using a b c d l no-encoding not-excluded-keep-type repDet singletonD
        by (metis-node-eq-binary OrNode)
      have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 

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    using OrNode a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary OrNode)
    then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr BinOr xe2 ye2) \wedge BinaryExpr$ 
    BinOr xe1 ye1  $\geq BinaryExpr BinOr xe2 ye2$ 
    by (metis OrNode.premis l mono-binary rep.OrNode xer)
    then show ?thesis
    by meson
  qed
next
case (XorNode n x y xe1 ye1)
have k:  $g1 \vdash n \simeq BinaryExpr BinXor xe1 ye1$  using f XorNode
by (simp add: XorNode.hyps(2) rep.XorNode)
obtain xn yn where l: kind g1 n = XorNode xn yn
using XorNode.hyps(1) by blast
then have mx:  $g1 \vdash xn \simeq xe1$ 
using XorNode.hyps(1) XorNode.hyps(2) by fastforce
from l have my:  $g1 \vdash yn \simeq ye1$ 
using XorNode.hyps(1) XorNode.hyps(3) by fastforce
then show ?case
proof -
  have  $g1 \vdash xn \simeq xe1$  using mx by simp
  have  $g1 \vdash yn \simeq ye1$  using my by simp
  have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
  using XorNode
  using a b c d l no-encoding not-excluded-keep-type repDet singletonD
  by (metis-node-eq-binary XorNode)
  have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
  using XorNode a b c d l no-encoding not-excluded-keep-type repDet
  singletonD
  by (metis-node-eq-binary XorNode)
  then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr BinXor xe2 ye2) \wedge BinaryExpr$ 
  BinXor xe1 ye1  $\geq BinaryExpr BinXor xe2 ye2$ 
  by (metis XorNode.premis l mono-binary rep.XorNode xer)
  then show ?thesis
  by meson
qed
next
case (ShortCircuitOrNode n x y xe1 ye1)
have k:  $g1 \vdash n \simeq BinaryExpr BinShortCircuitOr xe1 ye1$  using f ShortCir-
cuitOrNode
by (simp add: ShortCircuitOrNode.hyps(2) rep.ShortCircuitOrNode)
obtain xn yn where l: kind g1 n = ShortCircuitOrNode xn yn
using ShortCircuitOrNode.hyps(1) by blast
then have mx:  $g1 \vdash xn \simeq xe1$ 
using ShortCircuitOrNode.hyps(1) ShortCircuitOrNode.hyps(2) by fastforce
from l have my:  $g1 \vdash yn \simeq ye1$ 
using ShortCircuitOrNode.hyps(1) ShortCircuitOrNode.hyps(3) by fastforce
then show ?case
proof -

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    have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
    have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
    have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using ShortCircuitOrNode
      using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary ShortCircuitOrNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using ShortCircuitOrNode a b c d l no-encoding not-excluded-keep-type
repDet singletonD
      by (metis-node-eq-binary ShortCircuitOrNode)
    then have  $\exists xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinShortCircuitOr\ xe2\ ye2) \wedge$ 
BinaryExpr BinShortCircuitOr xe1 ye1  $\geq BinaryExpr BinShortCircuitOr xe2 ye2$ 
      by (metis ShortCircuitOrNode.premis l mono-binary rep.ShortCircuitOrNode
xer)
    then show ?thesis
      by meson
  qed
next
case (LeftShiftNode n x y xe1 ye1)
have  $k: g1 \vdash n \simeq BinaryExpr\ BinLeftShift\ xe1\ ye1$  using  $f$  LeftShiftNode
  by (simp add: LeftShiftNode.hyps(2) rep.LeftShiftNode)
obtain  $xn\ yn$  where  $l: kind\ g1\ n = LeftShiftNode\ xn\ yn$ 
  using LeftShiftNode.hyps(1) by blast
then have  $mx: g1 \vdash xn \simeq xe1$ 
  using LeftShiftNode.hyps(1) LeftShiftNode.hyps(2) by fastforce
from  $l$  have  $my: g1 \vdash yn \simeq ye1$ 
  using LeftShiftNode.hyps(1) LeftShiftNode.hyps(3) by fastforce
then show ?case
proof -
  have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
  have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
  have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using LeftShiftNode
    using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary LeftShiftNode)
  have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
    using LeftShiftNode a b c d l no-encoding not-excluded-keep-type repDet
singletonD
    by (metis-node-eq-binary LeftShiftNode)
  then have  $\exists xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinLeftShift\ xe2\ ye2) \wedge$ 
BinaryExpr BinLeftShift xe1 ye1  $\geq BinaryExpr BinLeftShift xe2 ye2$ 
    by (metis LeftShiftNode.premis l mono-binary rep.LeftShiftNode xer)
  then show ?thesis
    by meson
  qed
next
case (RightShiftNode n x y xe1 ye1)
have  $k: g1 \vdash n \simeq BinaryExpr\ BinRightShift\ xe1\ ye1$  using  $f$  RightShiftNode
  by (simp add: RightShiftNode.hyps(2) rep.RightShiftNode)

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obtain  $xn\ yn$  where  $l$ :  $\text{kind } g1\ n = \text{RightShiftNode } xn\ yn$ 
  using  $\text{RightShiftNode.hyps}(1)$  by blast
then have  $m\!x$ :  $g1 \vdash xn \simeq xe1$ 
  using  $\text{RightShiftNode.hyps}(1)\ \text{RightShiftNode.hyps}(2)$  by fastforce
from  $l$  have  $m\!y$ :  $g1 \vdash yn \simeq ye1$ 
  using  $\text{RightShiftNode.hyps}(1)\ \text{RightShiftNode.hyps}(3)$  by fastforce
then show ?case
proof –
  have  $g1 \vdash xn \simeq xe1$  using  $m\!x$  by simp
  have  $g1 \vdash yn \simeq ye1$  using  $m\!y$  by simp
  have  $x\!e\!r$ :  $\exists\ xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using  $\text{RightShiftNode}$ 
    using  $a\ b\ c\ d\ l\ \text{no-encoding not-excluded-keep-type repDet singletonD}$ 
    by  $(\text{metis-node-eq-binary } \text{RightShiftNode})$ 
  have  $\exists\ ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
    using  $\text{RightShiftNode } a\ b\ c\ d\ l\ \text{no-encoding not-excluded-keep-type repDet}$ 
    singletonD
    by  $(\text{metis-node-eq-binary } \text{RightShiftNode})$ 
  then have  $\exists\ xe2\ ye2. (g2 \vdash n \simeq \text{BinaryExpr BinRightShift } xe2\ ye2) \wedge$ 
BinaryExpr BinRightShift  $xe1\ ye1 \geq \text{BinaryExpr BinRightShift } xe2\ ye2$ 
    by  $(\text{metis } \text{RightShiftNode.premis } l\ \text{mono-binary rep.RightShiftNode } x\!e\!r)$ 
  then show ?thesis
    by meson
  qed
next
  case  $(\text{UnsignedRightShiftNode } n\ x\ y\ xe1\ ye1)$ 
  have  $k$ :  $g1 \vdash n \simeq \text{BinaryExpr BinURightShift } xe1\ ye1$  using  $f\ \text{UnsignedRightShiftNode}$ 
    by  $(\text{simp add: UnsignedRightShiftNode.hyps}(2)\ \text{rep.UnsignedRightShiftNode})$ 
  obtain  $xn\ yn$  where  $l$ :  $\text{kind } g1\ n = \text{UnsignedRightShiftNode } xn\ yn$ 
    using  $\text{UnsignedRightShiftNode.hyps}(1)$  by blast
  then have  $m\!x$ :  $g1 \vdash xn \simeq xe1$ 
    using  $\text{UnsignedRightShiftNode.hyps}(1)\ \text{UnsignedRightShiftNode.hyps}(2)$  by
fastforce
  from  $l$  have  $m\!y$ :  $g1 \vdash yn \simeq ye1$ 
    using  $\text{UnsignedRightShiftNode.hyps}(1)\ \text{UnsignedRightShiftNode.hyps}(3)$  by
fastforce
  then show ?case
  proof –
    have  $g1 \vdash xn \simeq xe1$  using  $m\!x$  by simp
    have  $g1 \vdash yn \simeq ye1$  using  $m\!y$  by simp
    have  $x\!e\!r$ :  $\exists\ xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using  $\text{UnsignedRightShiftNode}$ 
      using  $a\ b\ c\ d\ l\ \text{no-encoding not-excluded-keep-type repDet singletonD}$ 
      by  $(\text{metis-node-eq-binary } \text{UnsignedRightShiftNode})$ 
    have  $\exists\ ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using  $\text{UnsignedRightShiftNode } a\ b\ c\ d\ l\ \text{no-encoding not-excluded-keep-type}$ 
      repDet singletonD
      by  $(\text{metis-node-eq-binary } \text{UnsignedRightShiftNode})$ 

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      then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr\ BinURightShift\ xe2\ ye2) \wedge$ 
BinaryExpr BinURightShift xe1 ye1  $\geq BinaryExpr BinURightShift xe2 ye2$ 
      by (metis UnsignedRightShiftNode.premis l mono-binary rep.UnsignedRightShiftNode
xer)
      then show ?thesis
      by meson
    qed
  next
  case (IntegerBelowNode n x y xe1 ye1)
  have k:  $g1 \vdash n \simeq BinaryExpr\ BinIntegerBelow\ xe1\ ye1$  using f IntegerBe-
lowNode
  by (simp add: IntegerBelowNode.hyps(2) rep.IntegerBelowNode)
  obtain xn yn where l: kind g1 n = IntegerBelowNode xn yn
  using IntegerBelowNode.hyps(1) by blast
  then have mx:  $g1 \vdash xn \simeq xe1$ 
  using IntegerBelowNode.hyps(1) IntegerBelowNode.hyps(2) by fastforce
  from l have my:  $g1 \vdash yn \simeq ye1$ 
  using IntegerBelowNode.hyps(1) IntegerBelowNode.hyps(3) by fastforce
  then show ?case
  proof -
    have  $g1 \vdash xn \simeq xe1$  using mx by simp
    have  $g1 \vdash yn \simeq ye1$  using my by simp
    have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using IntegerBelowNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary IntegerBelowNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
    using IntegerBelowNode a b c d l no-encoding not-excluded-keep-type repDet
singletonD
    by (metis-node-eq-binary IntegerBelowNode)
    then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr\ BinIntegerBelow\ xe2\ ye2) \wedge$ 
BinaryExpr BinIntegerBelow xe1 ye1  $\geq BinaryExpr BinIntegerBelow xe2 ye2$ 
    by (metis IntegerBelowNode.premis l mono-binary rep.IntegerBelowNode
xer)
    then show ?thesis
    by meson
  qed
next
case (IntegerEqualsNode n x y xe1 ye1)
have k:  $g1 \vdash n \simeq BinaryExpr\ BinIntegerEquals\ xe1\ ye1$  using f IntegerEqual-
sNode
by (simp add: IntegerEqualsNode.hyps(2) rep.IntegerEqualsNode)
obtain xn yn where l: kind g1 n = IntegerEqualsNode xn yn
using IntegerEqualsNode.hyps(1) by blast
then have mx:  $g1 \vdash xn \simeq xe1$ 
using IntegerEqualsNode.hyps(1) IntegerEqualsNode.hyps(2) by fastforce
from l have my:  $g1 \vdash yn \simeq ye1$ 
using IntegerEqualsNode.hyps(1) IntegerEqualsNode.hyps(3) by fastforce
then show ?case

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proof –
  have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
  have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
  have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using IntegerEqualsNode
    using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary IntegerEqualsNode)
  have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
    using IntegerEqualsNode  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type
repDet singletonD
    by (metis-node-eq-binary IntegerEqualsNode)
  then have  $\exists xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinIntegerEquals\ xe2\ ye2) \wedge$ 
BinaryExpr BinIntegerEquals  $xe1\ ye1 \geq BinaryExpr\ BinIntegerEquals\ xe2\ ye2$ 
    by (metis IntegerEqualsNode.premis l mono-binary rep.IntegerEqualsNode
xer)
    then show ?thesis
      by meson
  qed
next
  case (IntegerLessThanNode  $n\ x\ y\ xe1\ ye1$ )
    have  $k: g1 \vdash n \simeq BinaryExpr\ BinIntegerLessThan\ xe1\ ye1$  using  $f$  IntegerLessThanNode
    by (simp add: IntegerLessThanNode.hyps(2) rep.IntegerLessThanNode)
    obtain  $xn\ yn$  where  $l: kind\ g1\ n = IntegerLessThanNode\ xn\ yn$ 
    using IntegerLessThanNode.hyps(1) by blast
    then have  $mx: g1 \vdash xn \simeq xe1$ 
    using IntegerLessThanNode.hyps(1) IntegerLessThanNode.hyps(2) by fast-force
    from  $l$  have  $my: g1 \vdash yn \simeq ye1$ 
    using IntegerLessThanNode.hyps(1) IntegerLessThanNode.hyps(3) by fast-force
    then show ?case
      proof –
        have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
        have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
        have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
          using IntegerLessThanNode
          using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
          by (metis-node-eq-binary IntegerLessThanNode)
        have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
          using IntegerLessThanNode  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type
repDet singletonD
          by (metis-node-eq-binary IntegerLessThanNode)
        then have  $\exists xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinIntegerLessThan\ xe2\ ye2)$ 
BinaryExpr BinIntegerLessThan  $xe1\ ye1 \geq BinaryExpr\ BinIntegerLessThan\ xe2\ ye2$ 
          by (metis IntegerLessThanNode.premis l mono-binary rep.IntegerLessThanNode
xer)
        then show ?thesis

```



```

      by meson
    qed
  next
    case (NarrowNode n inputBits resultBits x xe1)
    have k:  $g1 \vdash n \simeq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe1$  using
    f NarrowNode
      by (simp add: NarrowNode.hyps(2) rep.NarrowNode)
    obtain xn where l: kind g1 n = NarrowNode inputBits resultBits xn
      using NarrowNode.hyps(1) by blast
    then have m:  $g1 \vdash xn \simeq xe1$ 
      using NarrowNode.hyps(1) NarrowNode.hyps(2)
      by auto
    then show ?case
    proof (cases xn = n')
      case True
        then have n:  $xe1 = e1'$  using c m repDet by simp
        then have ev:  $g2 \vdash n \simeq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) e2'$ 
        using NarrowNode.hyps(1) l m n
          using NarrowNode.premis True d rep.NarrowNode by simp
        then have r:  $\text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) e1' \geq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) e2'$ 
          by (meson a mono-unary)
        then show ?thesis using ev r
          by (metis n)
      next
        case False
        have  $g1 \vdash xn \simeq xe1$  using m by simp
        have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
          using NarrowNode
          using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
          by (metis node-eq-ternary NarrowNode)
        then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe2) \wedge \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe1 \geq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe2$ 
          by (metis NarrowNode.premis l mono-unary rep.NarrowNode)
        then show ?thesis
          by meson
    qed
  next
    case (SignExtendNode n inputBits resultBits x xe1)
    have k:  $g1 \vdash n \simeq \text{UnaryExpr } (\text{UnarySignExtend inputBits resultBits}) xe1$ 
    using f SignExtendNode
      by (simp add: SignExtendNode.hyps(2) rep.SignExtendNode)
    obtain xn where l: kind g1 n = SignExtendNode inputBits resultBits xn
      using SignExtendNode.hyps(1) by blast
    then have m:  $g1 \vdash xn \simeq xe1$ 
      using SignExtendNode.hyps(1) SignExtendNode.hyps(2)
      by auto
    then show ?case

```

```

proof (cases  $xn = n'$ )
  case True
    then have  $n: xe1 = e1'$  using  $c\ m\ repDet$  by simp
    then have  $ev: g2 \vdash n \simeq UnaryExpr\ (UnarySignExtend\ inputBits\ resultBits)$ 
     $e2'$  using  $SignExtendNode.hyps(1)\ l\ m\ n$ 
      using  $SignExtendNode.premis\ True\ d\ rep.SignExtendNode$  by simp
      then have  $r: UnaryExpr\ (UnarySignExtend\ inputBits\ resultBits)\ e1' \geq$ 
 $UnaryExpr\ (UnarySignExtend\ inputBits\ resultBits)\ e2'$ 
        by (meson a mono-unary)
      then show ?thesis using  $ev\ r$ 
        by (metis n)
    next
      case False
        have  $g1 \vdash xn \simeq xe1$  using  $m$  by simp
        have  $\exists\ xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
          using  $SignExtendNode$ 
          using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
          by (metis-node-eq-ternary SignExtendNode)
        then have  $\exists\ xe2. (g2 \vdash n \simeq UnaryExpr\ (UnarySignExtend\ inputBits\ result-$ 
 $Bits)\ xe2) \wedge UnaryExpr\ (UnarySignExtend\ inputBits\ resultBits)\ xe1 \geq UnaryExpr$ 
 $(UnarySignExtend\ inputBits\ resultBits)\ xe2$ 
          by (metis SignExtendNode.premis l mono-unary rep.SignExtendNode)
        then show ?thesis
          by meson
      qed
    next
      case ( $ZeroExtendNode\ n\ inputBits\ resultBits\ x\ xe1$ )
        have  $k: g1 \vdash n \simeq UnaryExpr\ (UnaryZeroExtend\ inputBits\ resultBits)\ xe1$ 
using  $f\ ZeroExtendNode$ 
          by (simp add: ZeroExtendNode.hyps(2) rep.ZeroExtendNode)
        obtain  $xn$  where  $l: kind\ g1\ n = ZeroExtendNode\ inputBits\ resultBits\ xn$ 
          using  $ZeroExtendNode.hyps(1)$  by blast
        then have  $m: g1 \vdash xn \simeq xe1$ 
          using  $ZeroExtendNode.hyps(1)\ ZeroExtendNode.hyps(2)$ 
          by auto
        then show ?case
          proof (cases  $xn = n'$ )
            case True
              then have  $n: xe1 = e1'$  using  $c\ m\ repDet$  by simp
              then have  $ev: g2 \vdash n \simeq UnaryExpr\ (UnaryZeroExtend\ inputBits\ resultBits)$ 
               $e2'$  using  $ZeroExtendNode.hyps(1)\ l\ m\ n$ 
                using  $ZeroExtendNode.premis\ True\ d\ rep.ZeroExtendNode$  by simp
                then have  $r: UnaryExpr\ (UnaryZeroExtend\ inputBits\ resultBits)\ e1' \geq$ 
 $UnaryExpr\ (UnaryZeroExtend\ inputBits\ resultBits)\ e2'$ 
                  by (meson a mono-unary)
                then show ?thesis using  $ev\ r$ 
                  by (metis n)
            next
              case False

```

```

    have  $g1 \vdash xn \simeq xe1$  using  $m$  by simp
    have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using ZeroExtendNode
      using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
      by (metis-node-eq-ternary ZeroExtendNode)
    then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) xe2) \wedge \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) xe1 \geq \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) xe2$ 
      by (metis ZeroExtendNode.premis l mono-unary rep.ZeroExtendNode)
    then show ?thesis
      by meson
  qed
next
  case (LeafNode n s)
  then show ?case
    by (metis eq-refl rep.LeafNode)
next
  case (RefNode n')
  then show ?case
    by (metis a b c d no-encoding not-excluded-keep-type rep.RefNode repDet singletonD)
  qed
qed
qed

```

**lemma** *graph-antics-preservation-subscript*:

```

  assumes  $a: e_1' \geq e_2'$ 
  assumes  $b: (\{n\} \trianglelefteq \text{as-set } g_1) \subseteq \text{as-set } g_2$ 
  assumes  $c: g_1 \vdash n \simeq e_1'$ 
  assumes  $d: g_2 \vdash n \simeq e_2'$ 
  shows graph-refinement  $g_1 g_2$ 
  using graph-antics-preservation assms by simp

```

**lemma** *tree-to-graph-rewriting*:

```

   $e_1 \geq e_2$ 
   $\wedge (g_1 \vdash n \simeq e_1) \wedge \text{maximal-sharing } g_1$ 
   $\wedge (\{n\} \trianglelefteq \text{as-set } g_1) \subseteq \text{as-set } g_2$ 
   $\wedge (g_2 \vdash n \simeq e_2) \wedge \text{maximal-sharing } g_2$ 
 $\implies \text{graph-refinement } g_1 g_2$ 
  using graph-antics-preservation
  by auto

```

**declare**  $[[\text{simp-trace}]]$

**lemma** *equal-refines*:

```

  fixes  $e1 e2 :: \text{IRExpr}$ 
  assumes  $e1 = e2$ 
  shows  $e1 \geq e2$ 
  using assms

```

```

  by simp
declare [[simp-trace=false]]

```

```

lemma eval-contains-id[simp]:  $g1 \vdash n \simeq e \implies n \in \text{ids } g1$ 
  using no-encoding by blast

```

```

lemma subset-kind[simp]:  $\text{as-set } g1 \subseteq \text{as-set } g2 \implies g1 \vdash n \simeq e \implies \text{kind } g1 \ n = \text{kind } g2 \ n$ 
  using eval-contains-id unfolding as-set-def
  by blast

```

```

lemma subset-stamp[simp]:  $\text{as-set } g1 \subseteq \text{as-set } g2 \implies g1 \vdash n \simeq e \implies \text{stamp } g1 \ n = \text{stamp } g2 \ n$ 
  using eval-contains-id unfolding as-set-def
  by blast

```

```

method solve-subset-eval uses as-set eval =
  (metis eval as-set subset-kind subset-stamp |
   metis eval as-set subset-kind)

```

```

lemma subset-implies-evals:
  assumes  $\text{as-set } g1 \subseteq \text{as-set } g2$ 
  assumes  $(g1 \vdash n \simeq e)$ 
  shows  $(g2 \vdash n \simeq e)$ 
  using assms(2)
  apply (induction e)
    apply (solve-subset-eval as-set: assms(1) eval: ConstantNode)
    apply (solve-subset-eval as-set: assms(1) eval: ParameterNode)
    apply (solve-subset-eval as-set: assms(1) eval: ConditionalNode)
    apply (solve-subset-eval as-set: assms(1) eval: AbsNode)
    apply (solve-subset-eval as-set: assms(1) eval: NotNode)
    apply (solve-subset-eval as-set: assms(1) eval: NegateNode)
    apply (solve-subset-eval as-set: assms(1) eval: LogicNegationNode)
    apply (solve-subset-eval as-set: assms(1) eval: AddNode)
    apply (solve-subset-eval as-set: assms(1) eval: MulNode)
    apply (solve-subset-eval as-set: assms(1) eval: SubNode)
    apply (solve-subset-eval as-set: assms(1) eval: AndNode)
    apply (solve-subset-eval as-set: assms(1) eval: OrNode)
    apply (solve-subset-eval as-set: assms(1) eval: XorNode)
    apply (solve-subset-eval as-set: assms(1) eval: ShortCircuitOrNode)
    apply (solve-subset-eval as-set: assms(1) eval: LeftShiftNode)
    apply (solve-subset-eval as-set: assms(1) eval: RightShiftNode)
    apply (solve-subset-eval as-set: assms(1) eval: UnsignedRightShiftNode)
    apply (solve-subset-eval as-set: assms(1) eval: IntegerBelowNode)
    apply (solve-subset-eval as-set: assms(1) eval: IntegerEqualsNode)
    apply (solve-subset-eval as-set: assms(1) eval: IntegerLessThanNode)

```

```

    apply (solve-subset-eval as-set: assms(1) eval: NarrowNode)
    apply (solve-subset-eval as-set: assms(1) eval: SignExtendNode)
    apply (solve-subset-eval as-set: assms(1) eval: ZeroExtendNode)
    apply (solve-subset-eval as-set: assms(1) eval: LeafNode)
    by (solve-subset-eval as-set: assms(1) eval: RefNode)

lemma subset-refines:
  assumes as-set g1  $\subseteq$  as-set g2
  shows graph-refinement g1 g2
proof -
  have ids g1  $\subseteq$  ids g2 using assms unfolding as-set-def
  by blast
  then show ?thesis unfolding graph-refinement-def apply rule
  apply (rule allI) apply (rule impI) apply (rule allI) apply (rule impI)
  unfolding graph-represents-expression-def
  proof -
    fix n e1
    assume 1:n  $\in$  ids g1
    assume 2:g1  $\vdash$  n  $\simeq$  e1

    show  $\exists e2. (g2 \vdash n \simeq e2) \wedge e1 \geq e2$ 
    using assms 1 2 using subset-implies-evals
    by (meson equal-refines)
  qed
qed

```

```

lemma graph-construction:
  e1  $\geq$  e2
   $\wedge$  as-set g1  $\subseteq$  as-set g2
   $\wedge$  (g2  $\vdash$  n  $\simeq$  e2)
   $\implies$  (g2  $\vdash$  n  $\trianglelefteq$  e1)  $\wedge$  graph-refinement g1 g2
  using subset-refines
  by (meson encodeeval-def graph-represents-expression-def le-expr-def)

```

## 6.8.4 Term Graph Reconstruction

```

lemma find-exists-kind:
  assumes find-node-and-stamp g (node, s) = Some nid
  shows kind g nid = node
  using assms unfolding find-node-and-stamp.simps
  by (metis (mono-tags, lifting) find-Some-iff)

```

```

lemma find-exists-stamp:
  assumes find-node-and-stamp g (node, s) = Some nid
  shows stamp g nid = s
  using assms unfolding find-node-and-stamp.simps
  by (metis (mono-tags, lifting) find-Some-iff)

```

```

lemma find-new-kind:

```

```

assumes  $g' = \text{add-node } \text{nid} \ (\text{node}, s) \ g$ 
assumes  $\text{node} \neq \text{NoNode}$ 
shows  $\text{kind } g' \ \text{nid} = \text{node}$ 
using assms
using add-node-lookup by presburger

lemma find-new-stamp:
assumes  $g' = \text{add-node } \text{nid} \ (\text{node}, s) \ g$ 
assumes  $\text{node} \neq \text{NoNode}$ 
shows  $\text{stamp } g' \ \text{nid} = s$ 
using assms
using add-node-lookup by presburger

lemma sorted-bottom:
assumes finite xs
assumes  $x \in xs$ 
shows  $x \leq \text{last}(\text{sorted-list-of-set}(xs::\text{nat set}))$ 
using assms
using sorted2-simps(2) sorted-list-of-set(2)
by (smt (verit, del-insts) Diff-iff Max-ge Max-in empty-iff list.set(1) snoc-eq-iff-butlast
sorted-insort-is-snoc sorted-list-of-set(1) sorted-list-of-set.fold-insort-key.infinite sorted-list-of-set.fold-insort-ke)

lemma fresh:  $\text{finite } xs \implies \text{last}(\text{sorted-list-of-set}(xs::\text{nat set})) + 1 \notin xs$ 
using sorted-bottom
using not-le by auto

lemma fresh-ids:
assumes  $n = \text{get-fresh-id } g$ 
shows  $n \notin \text{ids } g$ 
proof –
  have finite (ids g) using Rep-IRGraph by auto
  then show ?thesis
    using assms fresh unfolding get-fresh-id.simps
    by blast
qed

lemma graph-unchanged-rep-unchanged:
assumes  $\forall n \in \text{ids } g. \text{kind } g \ n = \text{kind } g' \ n$ 
assumes  $\forall n \in \text{ids } g. \text{stamp } g \ n = \text{stamp } g' \ n$ 
shows  $(g \vdash n \simeq e) \longrightarrow (g' \vdash n \simeq e)$ 
apply (rule impI) subgoal premises e using e assms
  apply (induction  $n \ e$ )
    apply (metis no-encoding rep.ConstantNode)
    apply (metis no-encoding rep.ParameterNode)
    apply (metis no-encoding rep.ConditionalNode)
    apply (metis no-encoding rep.AbsNode)
    apply (metis no-encoding rep.NotNode)
    apply (metis no-encoding rep.NegateNode)
    apply (metis no-encoding rep.LogicNegationNode)

```

```

    apply (metis no-encoding rep.AddNode)
    apply (metis no-encoding rep.MulNode)
    apply (metis no-encoding rep.SubNode)
    apply (metis no-encoding rep.AndNode)
    apply (metis no-encoding rep.OrNode)
    apply (metis no-encoding rep.XorNode)
    apply (metis no-encoding rep.ShortCircuitOrNode)
    apply (metis no-encoding rep.LeftShiftNode)
    apply (metis no-encoding rep.RightShiftNode)
    apply (metis no-encoding rep.UnsignedRightShiftNode)
    apply (metis no-encoding rep.IntegerBelowNode)
    apply (metis no-encoding rep.IntegerEqualsNode)
    apply (metis no-encoding rep.IntegerLessThanNode)
    apply (metis no-encoding rep.NarrowNode)
    apply (metis no-encoding rep.SignExtendNode)
    apply (metis no-encoding rep.ZeroExtendNode)
    apply (metis no-encoding rep.LeafNode)
  by (metis no-encoding rep.RefNode)
done

```

**lemma** *fresh-node-subset*:

```

  assumes  $n \notin \text{ids } g$ 
  assumes  $g' = \text{add-node } n (k, s) g$ 
  shows  $\text{as-set } g \subseteq \text{as-set } g'$ 
  using assms
  by (smt (verit, del-Insts) Collect-mono-iff Diff-idemp Diff-insert-absorb add-changed
    as-set-def disjoint-change unchanged.simps)

```

**lemma** *unrep-subset*:

```

  assumes  $(g \oplus e \rightsquigarrow (g', n))$ 
  shows  $\text{as-set } g \subseteq \text{as-set } g'$ 
  using assms proof (induction  $g e (g', n)$  arbitrary:  $g' n$ )
  case (ConstantNodeSame  $g c n$ )
  then show ?case by blast
next
  case (ConstantNodeNew  $g c n g'$ )
  then show ?case using fresh-ids fresh-node-subset
    by presburger
next
  case (ParameterNodeSame  $g i s n$ )
  then show ?case by blast
next
  case (ParameterNodeNew  $g i s n g'$ )
  then show ?case using fresh-ids fresh-node-subset
    by presburger
next
  case (ConditionalNodeSame  $g ce g2 c te g3 t fe g4 f s' n$ )
  then show ?case by blast
next

```

```

    case (ConditionalNodeNew g ce g2 c te g3 t fe g4 f s' n g')
    then show ?case using fresh-ids fresh-node-subset
      by (meson subset-trans)
next
    case (UnaryNodeSame g xe g2 x s' op n)
    then show ?case by blast
next
    case (UnaryNodeNew g xe g2 x s' op n g')
    then show ?case using fresh-ids fresh-node-subset
      by (meson subset-trans)
next
    case (BinaryNodeSame g xe g2 x ye g3 y s' op n)
    then show ?case by blast
next
    case (BinaryNodeNew g xe g2 x ye g3 y s' op n g')
    then show ?case using fresh-ids fresh-node-subset
      by (meson subset-trans)
next
    case (AllLeafNodes g n s)
    then show ?case by blast
qed

```

**lemma** *fresh-node-preserves-other-nodes*:

```

  assumes n' = get-fresh-id g
  assumes g' = add-node n' (k, s) g
  shows  $\forall n \in \text{ids } g. (g \vdash n \simeq e) \longrightarrow (g' \vdash n \simeq e)$ 
  using assms
  by (smt (verit, ccfv-SIG) Diff-idemp Diff-insert-absorb add-changed disjoint-change
    fresh-ids graph-unchanged-rep-unchanged unchanged.elims(2))

```

**lemma** *found-node-preserves-other-nodes*:

```

  assumes find-node-and-stamp g (k, s) = Some n
  shows  $\forall n \in \text{ids } g. (g \vdash n \simeq e) \longleftrightarrow (g \vdash n \simeq e)$ 
  using assms
  by blast

```

**lemma** *unrep-ids-subset[simp]*:

```

  assumes  $g \oplus e \rightsquigarrow (g', n)$ 
  shows  $\text{ids } g \subseteq \text{ids } g'$ 
  using assms unrep-subset
  by (meson graph-refinement-def subset-refines)

```

**lemma** *unrep-unchanged*:

```

  assumes  $g \oplus e \rightsquigarrow (g', n)$ 
  shows  $\forall n \in \text{ids } g. \forall e. (g \vdash n \simeq e) \longrightarrow (g' \vdash n \simeq e)$ 
  using assms unrep-subset fresh-node-preserves-other-nodes
  by (meson subset-implies-evals)

```

**theorem** *term-graph-reconstruction*:



$g \oplus e \rightsquigarrow (g', n) \implies (g' \vdash n \simeq e) \wedge \text{as-set } g \subseteq \text{as-set } g'$   
**subgoal premises**  $e$  **apply** (rule *conjI*) **defer**  
   **using**  $e$  *unrep-subset* **apply** *blast* **using**  $e$   
**proof** (induction  $g$   $e$   $(g', n)$  arbitrary:  $g' n$ )  
   **case** (*ConstantNodeSame*  $g' c n$ )  
     **then have**  $\text{kind } g' n = \text{ConstantNode } c$   
       **using** *find-exists-kind* *local.ConstantNodeSame* **by** *blast*  
     **then show** ?*case* **using** *ConstantNode* **by** *blast*  
**next**  
   **case** (*ConstantNodeNew*  $g c$ )  
     **then show** ?*case*  
       **using** *ConstantNode* *IRNode.distinct(683)* *add-node-lookup* **by** *presburger*  
**next**  
   **case** (*ParameterNodeSame*  $i s$ )  
     **then show** ?*case*  
       **by** (*metis* *ParameterNode* *find-exists-kind* *find-exists-stamp*)  
**next**  
   **case** (*ParameterNodeNew*  $g i s$ )  
     **then show** ?*case*  
       **by** (*metis* *IRNode.distinct(2447)* *ParameterNode* *add-node-lookup*)  
**next**  
   **case** (*ConditionalNodeSame*  $g ce g2 c te g3 t fe g4 f s' n$ )  
     **then have**  $k$ :  $\text{kind } g4 n = \text{ConditionalNode } c t f$   
       **using** *find-exists-kind* **by** *blast*  
     **have**  $c$ :  $g4 \vdash c \simeq ce$  **using** *local.ConditionalNodeSame* *unrep-unchanged*  
       **using** *no-encoding* **by** *blast*  
     **have**  $t$ :  $g4 \vdash t \simeq te$  **using** *local.ConditionalNodeSame* *unrep-unchanged*  
       **using** *no-encoding* **by** *blast*  
     **have**  $f$ :  $g4 \vdash f \simeq fe$  **using** *local.ConditionalNodeSame* *unrep-unchanged*  
       **using** *no-encoding* **by** *blast*  
     **then show** ?*case* **using**  $c t f$   
       **using** *ConditionalNode*  $k$  **by** *blast*  
**next**  
   **case** (*ConditionalNodeNew*  $g ce g2 c te g3 t fe g4 f s' n g'$ )  
     **moreover have** *ConditionalNode*  $c t f \neq \text{NoNode}$   
       **using** *unary-node.elims* **by** *blast*  
     **ultimately have**  $k$ :  $\text{kind } g' n = \text{ConditionalNode } c t f$   
       **using** *find-new-kind* *local.ConditionalNodeNew*  
       **by** *presburger*  
     **then have**  $c$ :  $g' \vdash c \simeq ce$  **using** *local.ConditionalNodeNew* *unrep-unchanged*  
       **using** *no-encoding*  
       **by** (*metis* *ConditionalNodeNew.hyps(9)* *fresh-node-preserves-other-nodes*)  
     **then have**  $t$ :  $g' \vdash t \simeq te$  **using** *local.ConditionalNodeNew* *unrep-unchanged*  
       **using** *no-encoding* *fresh-node-preserves-other-nodes*  
       **by** *metis*  
     **then have**  $f$ :  $g' \vdash f \simeq fe$  **using** *local.ConditionalNodeNew* *unrep-unchanged*  
       **using** *no-encoding* *fresh-node-preserves-other-nodes*  
       **by** *metis*  
     **then show** ?*case* **using**  $c t f$

```

    using ConditionalNode k by blast
next
case (UnaryNodeSame g xe g' x s' op n)
then have k: kind g' n = unary-node op x
    using find-exists-kind local.UnaryNodeSame by blast
then have g' ⊢ x ≃ xe using local.UnaryNodeSame by blast
then show ?case using k
    apply (cases op)
    using AbsNode unary-node.simps(1) apply presburger
    using NegateNode unary-node.simps(3) apply presburger
    using NotNode unary-node.simps(2) apply presburger
    using LogicNegationNode unary-node.simps(4) apply presburger
    using NarrowNode unary-node.simps(5) apply presburger
    using SignExtendNode unary-node.simps(6) apply presburger
    using ZeroExtendNode unary-node.simps(7) by presburger
next
case (UnaryNodeNew g xe g2 x s' op n g')
moreover have unary-node op x ≠ NoNode
    using unary-node.elims by blast
ultimately have k: kind g' n = unary-node op x
    using find-new-kind local.UnaryNodeNew
    by presburger
have x ∈ ids g2 using local.UnaryNodeNew
    using eval-contains-id by blast
then have x ≠ n using local.UnaryNodeNew(5) fresh-ids by blast
have g' ⊢ x ≃ xe using local.UnaryNodeNew fresh-node-preserved-other-nodes
    using ⟨x ∈ ids g2⟩ by blast
then show ?case using k
    apply (cases op)
    using AbsNode unary-node.simps(1) apply presburger
    using NegateNode unary-node.simps(3) apply presburger
    using NotNode unary-node.simps(2) apply presburger
    using LogicNegationNode unary-node.simps(4) apply presburger
    using NarrowNode unary-node.simps(5) apply presburger
    using SignExtendNode unary-node.simps(6) apply presburger
    using ZeroExtendNode unary-node.simps(7) by presburger
next
case (BinaryNodeSame g xe g2 x ye g3 y s' op n)
then have k: kind g3 n = bin-node op x y
    using find-exists-kind by blast
have x: g3 ⊢ x ≃ xe using local.BinaryNodeSame unrep-unchanged
    using no-encoding by blast
have y: g3 ⊢ y ≃ ye using local.BinaryNodeSame unrep-unchanged
    using no-encoding by blast
then show ?case using x y k apply (cases op)
    using AddNode bin-node.simps(1) apply presburger
    using MulNode bin-node.simps(2) apply presburger
    using SubNode bin-node.simps(3) apply presburger
    using AndNode bin-node.simps(4) apply presburger

```

```

    using OrNode bin-node.simps(5) apply presburger
    using XorNode bin-node.simps(6) apply presburger
    using ShortCircuitOrNode bin-node.simps(7) apply presburger
    using LeftShiftNode bin-node.simps(8) apply presburger
    using RightShiftNode bin-node.simps(9) apply presburger
    using UnsignedRightShiftNode bin-node.simps(10) apply presburger
    using IntegerEqualsNode bin-node.simps(11) apply presburger
    using IntegerLessThanNode bin-node.simps(12) apply presburger
    using IntegerBelowNode bin-node.simps(13) by presburger
next
case (BinaryNodeNew g xe g2 x ye g3 y s' op n g')
moreover have bin-node op x y  $\neq$  NoNode
  using bin-node.elims by blast
ultimately have k: kind g' n = bin-node op x y
  using find-new-kind local.BinaryNodeNew
  by presburger
then have k: kind g' n = bin-node op x y
  using find-exists-kind by blast
have x: g'  $\vdash$  x  $\simeq$  xe using local.BinaryNodeNew unrep-unchanged
  using no-encoding
  by (meson fresh-node-preserves-other-nodes)
have y: g'  $\vdash$  y  $\simeq$  ye using local.BinaryNodeNew unrep-unchanged
  using no-encoding
  by (meson fresh-node-preserves-other-nodes)
then show ?case using x y k apply (cases op)
  using AddNode bin-node.simps(1) apply presburger
  using MulNode bin-node.simps(2) apply presburger
  using SubNode bin-node.simps(3) apply presburger
  using AndNode bin-node.simps(4) apply presburger
  using OrNode bin-node.simps(5) apply presburger
  using XorNode bin-node.simps(6) apply presburger
  using ShortCircuitOrNode bin-node.simps(7) apply presburger
  using LeftShiftNode bin-node.simps(8) apply presburger
  using RightShiftNode bin-node.simps(9) apply presburger
  using UnsignedRightShiftNode bin-node.simps(10) apply presburger
  using IntegerEqualsNode bin-node.simps(11) apply presburger
  using IntegerLessThanNode bin-node.simps(12) apply presburger
  using IntegerBelowNode bin-node.simps(13) by presburger
next
case (AllLeafNodes g n s)
then show ?case using rep.LeafNode by blast
qed
done

```

lemma ref-refinement:  
 assumes g  $\vdash$  n  $\simeq$  e<sub>1</sub>  
 assumes kind g n' = RefNode n  
 shows g  $\vdash$  n'  $\trianglelefteq$  e<sub>1</sub>  
 using assms RefNode

```

by (meson equal-refines graph-represents-expression-def)

lemma unrep-refines:
  assumes  $g \oplus e \rightsquigarrow (g', n)$ 
  shows graph-refinement  $g$   $g'$ 
  using assms
  using graph-refinement-def subset-refines unrep-subset by blast

lemma add-new-node-refines:
  assumes  $n \notin \text{ids } g$ 
  assumes  $g' = \text{add-node } n \ (k, s) \ g$ 
  shows graph-refinement  $g$   $g'$ 
  using assms unfolding graph-refinement
  using fresh-node-subset subset-refines by presburger

lemma add-node-as-set:
  assumes  $g' = \text{add-node } n \ (k, s) \ g$ 
  shows  $(\{n\} \sqsubseteq \text{as-set } g) \subseteq \text{as-set } g'$ 
  using assms unfolding as-set-def domain-subtraction-def
  using add-changed
  by (smt (z3) case-prodE changeonly.simps mem-Collect-eq prod.sel(1) subsetI)

theorem refined-insert:
  assumes  $e_1 \geq e_2$ 
  assumes  $g_1 \oplus e_2 \rightsquigarrow (g_2, n')$ 
  shows  $(g_2 \vdash n' \sqsubseteq e_1) \wedge \text{graph-refinement } g_1 \ g_2$ 
  using assms
  using graph-construction term-graph-reconstruction by blast

lemma ids-finite: finite (ids  $g$ )
  using Rep-IRGraph ids.rep-eq by simp

lemma unwrap-sorted: set (sorted-list-of-set (ids  $g$ )) = ids  $g$ 
  using Rep-IRGraph set-sorted-list-of-set ids-finite
  by blast

lemma find-none:
  assumes find-node-and-stamp  $g \ (k, s) = \text{None}$ 
  shows  $\forall n \in \text{ids } g. \text{kind } g \ n \neq k \vee \text{stamp } g \ n \neq s$ 
proof -
  have  $(\nexists n. n \in \text{ids } g \wedge (\text{kind } g \ n = k \wedge \text{stamp } g \ n = s))$ 
    using assms unfolding find-node-and-stamp.simps using find-None-iff un-
  wrap-sorted
    by (metis (mono-tags, lifting))
  then show ?thesis
    by blast
qed

```

```

method ref-represents uses node =
  (metis IRNode.distinct(2755) RefNode dual-order.refl find-new-kind fresh-node-subset
node subset-implies-evals)

```

### 6.8.5 Data-flow Tree to Subgraph Preserves Maximal Sharing

```

lemma same-kind-stamp-encodes-equal:
  assumes kind g n = kind g n'
  assumes stamp g n = stamp g n'
  assumes  $\neg(\text{is-preevaluated } (\text{kind } g \ n))$ 
  shows  $\forall \ e. (g \vdash n \simeq e) \longrightarrow (g \vdash n' \simeq e)$ 
  apply (rule allI)
  subgoal for e
    apply (rule impI)
    subgoal premises eval using eval assms
      apply (induction e)
    using ConstantNode apply presburger
    using ParameterNode apply presburger
      apply (metis ConditionalNode)
      apply (metis AbsNode)
      apply (metis NotNode)
      apply (metis NegateNode)
      apply (metis LogicNegationNode)
      apply (metis AddNode)
      apply (metis MulNode)
      apply (metis SubNode)
      apply (metis AndNode)
      apply (metis OrNode)
      apply (metis XorNode)
      apply (metis ShortCircuitOrNode)
      apply (metis LeftShiftNode)
      apply (metis RightShiftNode)
      apply (metis UnsignedRightShiftNode)
      apply (metis IntegerBelowNode)
      apply (metis IntegerEqualsNode)
      apply (metis IntegerLessThanNode)
      apply (metis NarrowNode)

```

```

    apply (metis SignExtendNode)
    apply (metis ZeroExtendNode)
  defer
    apply (metis RefNode)
  by blast
done
done

```

**lemma** *new-node-not-present*:

```

assumes find-node-and-stamp  $g$  (node, s) = None
assumes  $n = \text{get-fresh-id } g$ 
assumes  $g' = \text{add-node } n \text{ (node, s) } g$ 
shows  $\forall n' \in \text{true-ids } g. (\forall e. ((g \vdash n \simeq e) \wedge (g \vdash n' \simeq e)) \longrightarrow n = n')$ 
using assms
using encode-in-ids fresh-ids by blast

```

**lemma** *true-ids-def*:

```

true-ids  $g = \{n \in \text{ids } g. \neg(\text{is-RefNode } (\text{kind } g \ n)) \wedge ((\text{kind } g \ n) \neq \text{NoNode})\}$ 
unfolding true-ids-def ids-def
using ids-def is-RefNode-def by fastforce

```

**lemma** *add-node-some-node-def*:

```

assumes  $k \neq \text{NoNode}$ 
assumes  $g' = \text{add-node } \text{nid} \ (k, s) \ g$ 
shows  $g' = \text{Abs-IRGraph } ((\text{Rep-IRGraph } g)(\text{nid} \mapsto (k, s)))$ 
using assms
by (metis Rep-IRGraph-inverse add-node.rep-eq fst-conv)

```

**lemma** *ids-add-update-v1*:

```

assumes  $g' = \text{add-node } \text{nid} \ (k, s) \ g$ 
assumes  $k \neq \text{NoNode}$ 
shows  $\text{dom } (\text{Rep-IRGraph } g') = \text{dom } (\text{Rep-IRGraph } g) \cup \{\text{nid}\}$ 
using assms ids.rep-eq add-node-some-node-def
by (simp add: add-node.rep-eq)

```

**lemma** *ids-add-update-v2*:

```

assumes  $g' = \text{add-node } \text{nid} \ (k, s) \ g$ 
assumes  $k \neq \text{NoNode}$ 
shows  $\text{nid} \in \text{ids } g'$ 
using assms
using find-new-kind ids-some by presburger

```

**lemma** *add-node-ids-subset*:

```

assumes  $n \in \text{ids } g$ 
assumes  $g' = \text{add-node } n \ \text{node } g$ 
shows  $\text{ids } g' = \text{ids } g \cup \{n\}$ 
using assms unfolding add-node-def
apply (cases fst node = NoNode)
using ids.rep-eq replace-node.rep-eq replace-node-def apply auto[1]

```

**unfolding** *ids-def*  
**by** (*smt* (*verit*, *best*) *Collect-cong Un-insert-right dom-fun-upd fst-conv fun-upd-apply*  
*ids.rep-eq ids-def insert-absorb mem-Collect-eq option.inject option.simps(3) re-*  
*place-node.rep-eq replace-node-def sup-bot.right-neutral*)

**lemma** *convert-maximal:*

**assumes**  $\forall n\ n'.\ n \in \text{true-ids } g \wedge n' \in \text{true-ids } g \longrightarrow (\forall e\ e'.\ (g \vdash n \simeq e) \wedge (g \vdash n' \simeq e') \longrightarrow e \neq e')$   
**shows** *maximal-sharing* *g*  
**using** *assms*  
**using** *maximal-sharing* **by** *blast*

**lemma** *add-node-set-eq:*

**assumes**  $k \neq \text{NoNode}$   
**assumes**  $n \notin \text{ids } g$   
**shows**  $\text{as-set } (\text{add-node } n\ (k, s)\ g) = \text{as-set } g \cup \{(n, (k, s))\}$   
**using** *assms* **unfolding** *as-set-def add-node-def* **apply** *transfer* **apply** *simp*  
**by** *blast*

**lemma** *add-node-as-set-eq:*

**assumes**  $g' = \text{add-node } n\ (k, s)\ g$   
**assumes**  $n \notin \text{ids } g$   
**shows**  $\{n\} \sqsubseteq \text{as-set } g' = \text{as-set } g$   
**using** *assms* **unfolding** *domain-subtraction-def*  
**using** *add-node-set-eq*  
**by** (*smt* (*z3*) *Collect-cong Rep-IRGraph-inverse UnCI UnE add-node.rep-eq as-set-def*  
*case-prodE2 case-prodI2 le-boolE le-boolI' mem-Collect-eq prod.sel(1) singletonD*  
*singletonI*)

**lemma** *true-ids:*

$\text{true-ids } g = \text{ids } g - \{n \in \text{ids } g.\ \text{is-RefNode } (\text{kind } g\ n)\}$   
**unfolding** *true-ids-def*  
**by** *fastforce*

**lemma** *as-set-ids:*

**assumes**  $\text{as-set } g = \text{as-set } g'$   
**shows**  $\text{ids } g = \text{ids } g'$   
**using** *assms*  
**by** (*metis antisym equalityD1 graph-refinement-def subset-refines*)

**lemma** *ids-add-update:*

**assumes**  $k \neq \text{NoNode}$   
**assumes**  $n \notin \text{ids } g$   
**assumes**  $g' = \text{add-node } n\ (k, s)\ g$   
**shows**  $\text{ids } g' = \text{ids } g \cup \{n\}$   
**using** *assms* **apply** (*subst* *assms(3)*) **using** *add-node-set-eq as-set-ids*  
**by** (*smt* (*verit*, *del-insts*) *Collect-cong Diff-idemp Diff-insert-absorb Un-commute*  
*add-node.rep-eq add-node-def ids.rep-eq ids-add-update-v1 ids-add-update-v2 insertE*  
*insert-Collect insert-is-Un map-upd-Some-unfold mem-Collect-eq replace-node-def*)

*replace-node-unchanged*)

**lemma** *true-ids-add-update*:

**assumes**  $k \neq \text{NoNode}$   
**assumes**  $n \notin \text{ids } g$   
**assumes**  $g' = \text{add-node } n \ (k, s) \ g$   
**assumes**  $\neg(\text{is-RefNode } k)$   
**shows**  $\text{true-ids } g' = \text{true-ids } g \cup \{n\}$   
**using** *assms* **using** *true-ids ids-add-update*  
**by** (*smt (z3) Collect-cong Diff-iff Diff-insert-absorb Un-commute add-node-def*  
*find-new-kind insert-Diff-if insert-is-Un mem-Collect-eq replace-node-def replace-node-unchanged*)

**lemma** *new-def*:

**assumes**  $(\text{new} \sqsubseteq \text{as-set } g') = \text{as-set } g$   
**shows**  $n \in \text{ids } g \longrightarrow n \notin \text{new}$   
**using** *assms*  
**by** (*smt (z3) as-set-def case-prodD domain-subtraction-def mem-Collect-eq*)

**lemma** *add-preserves-rep*:

**assumes** *unchanged*:  $(\text{new} \sqsubseteq \text{as-set } g') = \text{as-set } g$   
**assumes** *closed*: *wf-closed*  $g$   
**assumes** *existed*:  $n \in \text{ids } g$   
**assumes**  $g' \vdash n \simeq e$   
**shows**  $g \vdash n \simeq e$   
**proof** (*cases*  $n \in \text{new}$ )  
**case** *True*  
**have**  $n \notin \text{ids } g$   
**using** *unchanged True unfolding as-set-def domain-subtraction-def*  
**by** *blast*  
**then show** *?thesis* **using** *existed by simp*  
**next**  
**case** *False*  
**then have** *kind-eq*:  $\forall n'. n' \notin \text{new} \longrightarrow \text{kind } g \ n' = \text{kind } g' \ n'$   
— can be more general than *stamp\_eq* because *NoNode* default is equal  
**using** *unchanged not-excluded-keep-type*  
**by** (*smt (z3) case-prodE domain-subtraction-def ids-some mem-Collect-eq subsetI*)  
**from** *False* **have** *stamp-eq*:  $\forall n' \in \text{ids } g'. n' \notin \text{new} \longrightarrow \text{stamp } g \ n' = \text{stamp } g' \ n'$   
**using** *unchanged not-excluded-keep-type*  
**by** (*metis equalityE*)  
**show** *?thesis* **using** *assms(4) kind-eq stamp-eq False*  
**proof** (*induction*  $n \ e$  *rule: rep.induct*)  
**case** (*ConstantNode*  $n \ c$ )  
**then show** *?case*  
**using** *rep.ConstantNode kind-eq by presburger*  
**next**



```

    case (ParameterNode n i s)
    then show ?case
      using rep.ParameterNode
      by (metis no-encoding)
  next
    case (ConditionalNode n c t f ce te fe)
    have kind: kind g n = ConditionalNode c t f
      using ConditionalNode.hyps(1) ConditionalNode.prem(3) kind-eq by pres-
    burger
    then have isin: n ∈ ids g
      by simp
    have inputs: {c, t, f} = inputs g n
      using kind unfolding inputs.simps using inputs-of-ConditionalNode by simp
    have c ∈ ids g ∧ t ∈ ids g ∧ f ∈ ids g
      using closed unfolding wf-closed-def
      using isin inputs by blast
    then have c ∉ new ∧ t ∉ new ∧ f ∉ new
      using new-def unchanged by blast
    then show ?case using ConditionalNode apply simp
      using rep.ConditionalNode by presburger
  next
    case (AbsNode n x xe)
    then have kind: kind g n = AbsNode x
      by simp
    then have isin: n ∈ ids g
      by simp
    have inputs: {x} = inputs g n
      using kind unfolding inputs.simps by simp
    have x ∈ ids g
      using closed unfolding wf-closed-def
      using isin inputs by blast
    then have x ∉ new
      using new-def unchanged by blast
    then show ?case
      using AbsNode
      using rep.AbsNode by presburger
  next
    case (NotNode n x xe)
    then have kind: kind g n = NotNode x
      by simp
    then have isin: n ∈ ids g
      by simp
    have inputs: {x} = inputs g n
      using kind unfolding inputs.simps by simp
    have x ∈ ids g
      using closed unfolding wf-closed-def
      using isin inputs by blast
    then have x ∉ new
      using new-def unchanged by blast

```

```

then show ?case using NotNode
  using rep.NotNode by presburger
next
case (NegateNode n x xe)
then have kind: kind g n = NegateNode x
  by simp
then have isin:  $n \in \text{ids } g$ 
  by simp
have inputs:  $\{x\} = \text{inputs } g \ n$ 
  using kind unfolding inputs.simps by simp
have  $x \in \text{ids } g$ 
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have  $x \notin \text{new}$ 
  using new-def unchanged by blast
then show ?case using NegateNode
  using rep.NegateNode by presburger
next
case (LogicNegationNode n x xe)
then have kind: kind g n = LogicNegationNode x
  by simp
then have isin:  $n \in \text{ids } g$ 
  by simp
have inputs:  $\{x\} = \text{inputs } g \ n$ 
  using kind unfolding inputs.simps by simp
have  $x \in \text{ids } g$ 
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have  $x \notin \text{new}$ 
  using new-def unchanged by blast
then show ?case using LogicNegationNode
  using rep.LogicNegationNode by presburger
next
case (AddNode n x y xe ye)
then have kind: kind g n = AddNode x y
  by simp
then have isin:  $n \in \text{ids } g$ 
  by simp
have inputs:  $\{x, y\} = \text{inputs } g \ n$ 
  using kind unfolding inputs.simps by simp
have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
  using new-def unchanged by blast
then show ?case using AddNode
  using rep.AddNode by presburger
next
case (MulNode n x y xe ye)

```

```

then have kind: kind g n = MulNode x y
  by simp
then have isin: n ∈ ids g
  by simp
have inputs: {x, y} = inputs g n
  using kind unfolding inputs.simps by simp
have x ∈ ids g ∧ y ∈ ids g
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have x ∉ new ∧ y ∉ new
  using new-def unchanged by blast
then show ?case using MulNode
  using rep.MulNode by presburger
next
case (SubNode n x y xe ye)
then have kind: kind g n = SubNode x y
  by simp
then have isin: n ∈ ids g
  by simp
have inputs: {x, y} = inputs g n
  using kind unfolding inputs.simps by simp
have x ∈ ids g ∧ y ∈ ids g
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have x ∉ new ∧ y ∉ new
  using new-def unchanged by blast
then show ?case using SubNode
  using rep.SubNode by presburger
next
case (AndNode n x y xe ye)
then have kind: kind g n = AndNode x y
  by simp
then have isin: n ∈ ids g
  by simp
have inputs: {x, y} = inputs g n
  using kind unfolding inputs.simps by simp
have x ∈ ids g ∧ y ∈ ids g
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have x ∉ new ∧ y ∉ new
  using new-def unchanged by blast
then show ?case using AndNode
  using rep.AndNode by presburger
next
case (OrNode n x y xe ye)
then have kind: kind g n = OrNode x y
  by simp
then have isin: n ∈ ids g
  by simp

```

```

have inputs: {x, y} = inputs g n
  using kind unfolding inputs.simps by simp
have x ∈ ids g ∧ y ∈ ids g
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have x ∉ new ∧ y ∉ new
  using new-def unchanged by blast
then show ?case using OrNode
  using rep.OrNode by presburger
next
case (XorNode n x y xe ye)
then have kind: kind g n = XorNode x y
  by simp
then have isin: n ∈ ids g
  by simp
have inputs: {x, y} = inputs g n
  using kind unfolding inputs.simps by simp
have x ∈ ids g ∧ y ∈ ids g
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have x ∉ new ∧ y ∉ new
  using new-def unchanged by blast
then show ?case using XorNode
  using rep.XorNode by presburger
next
case (ShortCircuitOrNode n x y xe ye)
then have kind: kind g n = ShortCircuitOrNode x y
  by simp
then have isin: n ∈ ids g
  by simp
have inputs: {x, y} = inputs g n
  using kind unfolding inputs.simps by simp
have x ∈ ids g ∧ y ∈ ids g
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have x ∉ new ∧ y ∉ new
  using new-def unchanged by blast
then show ?case using ShortCircuitOrNode
  using rep.ShortCircuitOrNode by presburger
next
case (LeftShiftNode n x y xe ye)
then have kind: kind g n = LeftShiftNode x y
  by simp
then have isin: n ∈ ids g
  by simp
have inputs: {x, y} = inputs g n
  using kind unfolding inputs.simps by simp
have x ∈ ids g ∧ y ∈ ids g
  using closed unfolding wf-closed-def

```

```

    using isin inputs by blast
  then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using LeftShiftNode
    using rep.LeftShiftNode by presburger
next
case (RightShiftNode  $n\ x\ y\ xe\ ye$ )
then have kind:  $\text{kind } g\ n = \text{RightShiftNode } x\ y$ 
  by simp
then have isin:  $n \in \text{ids } g$ 
  by simp
have inputs:  $\{x, y\} = \text{inputs } g\ n$ 
  using kind unfolding inputs.simps by simp
have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
  using new-def unchanged by blast
then show ?case using RightShiftNode
  using rep.RightShiftNode by presburger
next
case (UnsignedRightShiftNode  $n\ x\ y\ xe\ ye$ )
then have kind:  $\text{kind } g\ n = \text{UnsignedRightShiftNode } x\ y$ 
  by simp
then have isin:  $n \in \text{ids } g$ 
  by simp
have inputs:  $\{x, y\} = \text{inputs } g\ n$ 
  using kind unfolding inputs.simps by simp
have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
  using new-def unchanged by blast
then show ?case using UnsignedRightShiftNode
  using rep.UnsignedRightShiftNode by presburger
next
case (IntegerBelowNode  $n\ x\ y\ xe\ ye$ )
then have kind:  $\text{kind } g\ n = \text{IntegerBelowNode } x\ y$ 
  by simp
then have isin:  $n \in \text{ids } g$ 
  by simp
have inputs:  $\{x, y\} = \text{inputs } g\ n$ 
  using kind unfolding inputs.simps by simp
have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
  using closed unfolding wf-closed-def
  using isin inputs by blast
then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
  using new-def unchanged by blast
then show ?case using IntegerBelowNode

```

```

    using rep.IntegerBelowNode by presburger
next
  case (IntegerEqualsNode n x y xe ye)
  then have kind: kind g n = IntegerEqualsNode x y
    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{x, y\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using IntegerEqualsNode
    using rep.IntegerEqualsNode by presburger
next
  case (IntegerLessThanNode n x y xe ye)
  then have kind: kind g n = IntegerLessThanNode x y
    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{x, y\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $x \in \text{ids } g \wedge y \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $x \notin \text{new} \wedge y \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using IntegerLessThanNode
    using rep.IntegerLessThanNode by presburger
next
  case (NarrowNode n inputBits resultBits x xe)
  then have kind: kind g n = NarrowNode inputBits resultBits x
    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{x\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $x \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $x \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using NarrowNode
    using rep.NarrowNode by presburger
next
  case (SignExtendNode n inputBits resultBits x xe)
  then have kind: kind g n = SignExtendNode inputBits resultBits x

```

```

    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{x\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $x \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $x \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using SignExtendNode
    using rep.SignExtendNode by presburger
next
case (ZeroExtendNode n inputBits resultBits x xe)
  then have kind:  $\text{kind } g \ n = \text{ZeroExtendNode inputBits resultBits } x$ 
    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{x\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $x \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $x \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case using ZeroExtendNode
    using rep.ZeroExtendNode by presburger
next
case (LeafNode n s)
  then show ?case
    by (metis no-encoding rep.LeanNode)
next
case (RefNode n n' e)
  then have kind:  $\text{kind } g \ n = \text{RefNode } n'$ 
    by simp
  then have isin:  $n \in \text{ids } g$ 
    by simp
  have inputs:  $\{n'\} = \text{inputs } g \ n$ 
    using kind unfolding inputs.simps by simp
  have  $n' \in \text{ids } g$ 
    using closed unfolding wf-closed-def
    using isin inputs by blast
  then have  $n' \notin \text{new}$ 
    using new-def unchanged by blast
  then show ?case
    using RefNode
    using rep.RefNode by presburger
qed
qed

```

```

lemma not-in-no-rep:
   $n \notin \text{ids } g \implies \forall e. \neg(g \vdash n \simeq e)$ 
  using eval-contains-id by blast

lemma unary-inputs:
  assumes  $\text{kind } g \ n = \text{unary-node } op \ x$ 
  shows  $\text{inputs } g \ n = \{x\}$ 
  using assms by (cases op; auto)

lemma unary-succ:
  assumes  $\text{kind } g \ n = \text{unary-node } op \ x$ 
  shows  $\text{succ } g \ n = \{\}$ 
  using assms by (cases op; auto)

lemma binary-inputs:
  assumes  $\text{kind } g \ n = \text{bin-node } op \ x \ y$ 
  shows  $\text{inputs } g \ n = \{x, y\}$ 
  using assms by (cases op; auto)

lemma binary-succ:
  assumes  $\text{kind } g \ n = \text{bin-node } op \ x \ y$ 
  shows  $\text{succ } g \ n = \{\}$ 
  using assms by (cases op; auto)

lemma unrep-contains:
  assumes  $g \oplus e \rightsquigarrow (g', n)$ 
  shows  $n \in \text{ids } g'$ 
  using assms
  using not-in-no-rep term-graph-reconstruction by blast

lemma unrep-preserves-contains:
  assumes  $n \in \text{ids } g$ 
  assumes  $g \oplus e \rightsquigarrow (g', n')$ 
  shows  $n \in \text{ids } g'$ 
  using assms
  by (meson subsetD unrep-ids-subset)

lemma unrep-preserves-closure:
  assumes wf-closed  $g$ 
  assumes  $g \oplus e \rightsquigarrow (g', n)$ 
  shows wf-closed  $g'$ 
  using assms(2,1) unfolding wf-closed-def
  proof (induction  $g \ e \ (g', n)$  arbitrary:  $g' \ n$ )
    case (ConstantNodeSame  $g \ c \ n$ )
    then show ?case
      by blast

```



```

next
  case (ConstantNodeNew g c n g')
  then have dom:  $ids\ g' = ids\ g \cup \{n\}$ 
    by (meson IRNode.distinct(683) add-node-ids-subset ids-add-update)
  have k:  $kind\ g'\ n = ConstantNode\ c$ 
    using ConstantNodeNew add-node-lookup by simp
  then have inp:  $\{\} = inputs\ g'\ n$ 
    unfolding inputs.simps by simp
  from k have suc:  $\{\} = succ\ g'\ n$ 
    unfolding succ.simps by simp
  have  $inputs\ g'\ n \subseteq ids\ g' \wedge succ\ g'\ n \subseteq ids\ g' \wedge kind\ g'\ n \neq NoNode$ 
    using inp suc k by simp
  then show ?case
    by (smt (verit) ConstantNodeNew.hyps(3) ConstantNodeNew.prem Un-insert-right
    add-changed changeonly.elims(2) dom inputs.simps insert-iff singleton-iff subset-insertI
    subset-trans succ.simps sup-bot-right)
  next
    case (ParameterNodeSame g i s n)
    then show ?case by blast
  next
    case (ParameterNodeNew g i s n g')
    then have dom:  $ids\ g' = ids\ g \cup \{n\}$ 
      using IRNode.distinct(2447) fresh-ids ids-add-update by presburger
    have k:  $kind\ g'\ n = ParameterNode\ i$ 
      using ParameterNodeNew add-node-lookup by simp
    then have inp:  $\{\} = inputs\ g'\ n$ 
      unfolding inputs.simps by simp
    from k have suc:  $\{\} = succ\ g'\ n$ 
      unfolding succ.simps by simp
    have  $inputs\ g'\ n \subseteq ids\ g' \wedge succ\ g'\ n \subseteq ids\ g' \wedge kind\ g'\ n \neq NoNode$ 
      using k inp suc by simp
    then show ?case
      by (smt (verit) ParameterNodeNew.hyps(3) ParameterNodeNew.prem Un-insert-right
      add-node-as-set dom inputs.elims insertE not-excluded-keep-type order-trans single-
      tonD subset-insertI succ.elims sup-bot-right)
    next
      case (ConditionalNodeSame g ce g2 c te g3 t fe g4 f s' n)
      then show ?case by blast
    next
      case (ConditionalNodeNew g ce g2 c te g3 t fe g4 f s' n g')
      then have dom:  $ids\ g' = ids\ g4 \cup \{n\}$ 
        by (meson IRNode.distinct(591) add-node-ids-subset ids-add-update)
      have k:  $kind\ g'\ n = ConditionalNode\ c\ t\ f$ 
        using ConditionalNodeNew add-node-lookup by simp
      then have inp:  $\{c, t, f\} = inputs\ g'\ n$ 
        unfolding inputs.simps by simp
      from k have suc:  $\{\} = succ\ g'\ n$ 
        unfolding succ.simps by simp
      have  $inputs\ g'\ n \subseteq ids\ g' \wedge succ\ g'\ n \subseteq ids\ g' \wedge kind\ g'\ n \neq NoNode$ 

```

```

    using k inp suc unrep-contains unrep-preserves-contains
    using ConditionalNodeNew(1,3,5,10)
    by (smt (verit) IRNode.simps(643) Un-insert-right bot.extremum dom in-
sert-absorb insert-subset subset-insertI sup-bot-right)
    then show ?case using dom
    by (smt (z3) ConditionalNodeNew.hyps(10) ConditionalNodeNew.hyps(2) Con-
ditionalNodeNew.hyps(4) ConditionalNodeNew.hyps(6) ConditionalNodeNew.prem
Diff-eq-empty-iff Diff-iff Un-insert-right Un-upper1 add-node-def inputs.simp
insertE replace-node-def replace-node-unchanged subset-trans succ.simps sup-
bot-right)
  next
    case (UnaryNodeSame g xe g2 x s' op n)
    then show ?case by blast
  next
    case (UnaryNodeNew g xe g2 x s' op n g')
    then have dom:  $ids\ g' = ids\ g2 \cup \{n\}$ 
    by (metis add-node-ids-subset add-node-lookup ids-add-update ids-some un-
rep.UnaryNodeNew unrep-contains)
    have k:  $kind\ g'\ n = unary-node\ op\ x$ 
    using UnaryNodeNew add-node-lookup
    by (metis fresh-ids ids-some)
    then have inp:  $\{x\} = inputs\ g'\ n$ 
    using unary-inputs by simp
    from k have suc:  $\{\} = succ\ g'\ n$ 
    using unary-succ by simp
    have  $inputs\ g'\ n \subseteq ids\ g' \wedge succ\ g'\ n \subseteq ids\ g' \wedge kind\ g'\ n \neq NoNode$ 
    using k inp suc unrep-contains unrep-preserves-contains
    using UnaryNodeNew(1,6)
    by (metis Un-upper1 dom empty-subsetI ids-some insert-not-empty in-
sert-subsetI not-in-g-inputs subset-iff)
    then show ?case
    by (smt (verit) Un-insert-right UnaryNodeNew.hyps(2) UnaryNodeNew.hyps(6)
UnaryNodeNew.prem
add-changed changeonly.elims(2) dom inputs.simps insert-iff
singleton-iff subset-insertI subset-trans succ.simps sup-bot-right)
  next
    case (BinaryNodeSame g xe g2 x ye g3 y s' op n)
    then show ?case by blast
  next
    case (BinaryNodeNew g xe g2 x ye g3 y s' op n g')
    then have dom:  $ids\ g' = ids\ g3 \cup \{n\}$ 
    by (metis binary-inputs fresh-ids ids-add-update ids-some insert-not-empty
not-in-g-inputs)
    have k:  $kind\ g'\ n = bin-node\ op\ x\ y$ 
    using BinaryNodeNew add-node-lookup
    by (metis fresh-ids ids-some)
    then have inp:  $\{x, y\} = inputs\ g'\ n$ 
    using binary-inputs by simp
    from k have suc:  $\{\} = succ\ g'\ n$ 
    using binary-succ by simp
    have  $inputs\ g'\ n \subseteq ids\ g' \wedge succ\ g'\ n \subseteq ids\ g' \wedge kind\ g'\ n \neq NoNode$ 

```

```

    using k inp suc unrep-contains unrep-preserves-contains
    using BinaryNodeNew(1,3,6)
    by (metis Un-upper1 dom empty-subsetI ids-some insert-not-empty in-
sert-subsetI not-in-g-inputs subset-iff)
    then show ?case using dom BinaryNodeNew
    by (smt (verit, del-insts) Diff-eq-empty-iff Diff-iff Un-insert-right Un-upper1
add-node-def inputs.simps insertE replace-node-def replace-node-unchanged subset-trans
succ.simps sup-bot-right)
  next
  case (AllLeafNodes g n s)
  then show ?case
  by blast
qed

```

**inductive-cases** *ConstUnrepE*:  $g \oplus (\text{ConstantExpr } x) \rightsquigarrow (g', n)$

**definition** *constant-value* **where**

*constant-value* = (*IntVal* 32 0)

**definition** *bad-graph* **where**

*bad-graph* = *irgraph* [  
 (0, *AbsNode* 1, *constantAsStamp* *constant-value*),  
 (1, *RefNode* 2, *constantAsStamp* *constant-value*),  
 (2, *ConstantNode* *constant-value*, *constantAsStamp* *constant-value*)  
 ]

**end**

## 7 Control-flow Semantics

**theory** *IRStepObj*

**imports**

*TreeToGraph*

**begin**

### 7.1 Object Heap

The heap model we introduce maps field references to object instances to runtime values. We use the  $H[f][p]$  heap representation. See \cite{heap-reps-2011}. We also introduce the *DynamicHeap* type which allocates new object references sequentially storing the next free object reference as 'Free'.

*heapdef*

```

type-synonym ('a, 'b) Heap = 'a  $\Rightarrow$  'b  $\Rightarrow$  Value
type-synonym Free = nat
type-synonym ('a, 'b) DynamicHeap = ('a, 'b) Heap  $\times$  Free

fun h-load-field :: 'a  $\Rightarrow$  'b  $\Rightarrow$  ('a, 'b) DynamicHeap  $\Rightarrow$  Value where
  h-load-field f r (h, n) = h f r

fun h-store-field :: 'a  $\Rightarrow$  'b  $\Rightarrow$  Value  $\Rightarrow$  ('a, 'b) DynamicHeap  $\Rightarrow$  ('a, 'b)
  DynamicHeap where
  h-store-field f r v (h, n) = (h(f := ((h f)(r := v))), n)

fun h-new-inst :: ('a, 'b) DynamicHeap  $\Rightarrow$  ('a, 'b) DynamicHeap  $\times$  Value
where
  h-new-inst (h, n) = ((h, n+1), (ObjRef (Some n)))

type-synonym FieldRefHeap = (string, objref) DynamicHeap

```

```

definition new-heap :: ('a, 'b) DynamicHeap where
  new-heap = (( $\lambda$ f.  $\lambda$ p. UndefVal), 0)

```

## 7.2 Intraprocedural Semantics

```

fun find-index :: 'a  $\Rightarrow$  'a list  $\Rightarrow$  nat where
  find-index - [] = 0 |
  find-index v (x # xs) = (if (x=v) then 0 else find-index v xs + 1)

fun phi-list :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID list where
  phi-list g n =
    (filter ( $\lambda$ x. (is-PhiNode (kind g x)))
     (sorted-list-of-set (usages g n)))

fun input-index :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID  $\Rightarrow$  nat where
  input-index g n n' = find-index n' (inputs-of (kind g n))

fun phi-inputs :: IRGraph  $\Rightarrow$  nat  $\Rightarrow$  ID list  $\Rightarrow$  ID list where
  phi-inputs g i nodes = (map ( $\lambda$ n. (inputs-of (kind g n))!(i + 1)) nodes)

fun set-phis :: ID list  $\Rightarrow$  Value list  $\Rightarrow$  MapState  $\Rightarrow$  MapState where
  set-phis [] [] m = m |
  set-phis (n # xs) (v # vs) m = (set-phis xs vs (m(n := v))) |
  set-phis [] (v # vs) m = m |
  set-phis (x # xs) [] m = m

```

Intraprocedural semantics are given as a small-step semantics.

Within the context of a graph, the configuration triple, (ID, MethodState, Heap), is related to the subsequent configuration.

```

inductive step :: IRGraph  $\Rightarrow$  Params  $\Rightarrow$  (ID  $\times$  MapState  $\times$  FieldRefHeap)  $\Rightarrow$  (ID

```

$\times \text{MapState} \times \text{FieldRefHeap}) \Rightarrow \text{bool}$   
 $(-, - \vdash - \rightarrow - \text{ 55})$  **for**  $g \ p$  **where**

*SequentialNode:*

$\llbracket \text{is-sequential-node } (kind \ g \ nid);$   
 $\quad nid' = (\text{successors-of } (kind \ g \ nid))!0 \rrbracket$   
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m, h) \mid$

*IfNode:*

$\llbracket kind \ g \ nid = (\text{IfNode } cond \ tb \ fb);$   
 $\quad g \vdash cond \simeq condE;$   
 $\quad [m, p] \vdash condE \mapsto val;$   
 $\quad nid' = (\text{if } val\text{-to-bool } val \text{ then } tb \text{ else } fb) \rrbracket$   
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m, h) \mid$

*EndNodes:*

$\llbracket \text{is-AbstractEndNode } (kind \ g \ nid);$   
 $\quad merge = \text{any-usage } g \ nid;$   
 $\quad \text{is-AbstractMergeNode } (kind \ g \ merge);$   
  
 $\quad i = \text{find-index } nid \ (\text{inputs-of } (kind \ g \ merge));$   
 $\quad phis = (\text{phi-list } g \ merge);$   
 $\quad inps = (\text{phi-inputs } g \ i \ phis);$   
 $\quad g \vdash inps \simeq_L inpsE;$   
 $\quad [m, p] \vdash inpsE \mapsto_L vs;$   
  
 $\quad m' = \text{set-phis } phis \ vs \ m \rrbracket$   
 $\implies g, p \vdash (nid, m, h) \rightarrow (merge, m', h) \mid$

*NewInstanceNode:*

$\llbracket kind \ g \ nid = (\text{NewInstanceNode } nid \ f \ obj \ nid');$   
 $\quad (h', ref) = h\text{-new-inst } h;$   
 $\quad m' = m(nid := ref) \rrbracket$   
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m', h') \mid$

*LoadFieldNode:*

$\llbracket kind \ g \ nid = (\text{LoadFieldNode } nid \ f \ (\text{Some } obj) \ nid');$   
 $\quad g \vdash obj \simeq objE;$   
 $\quad [m, p] \vdash objE \mapsto \text{ObjRef } ref;$   
 $\quad h\text{-load-field } f \ ref \ h = v;$   
 $\quad m' = m(nid := v) \rrbracket$   
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m', h) \mid$

*SignedDivNode:*

$\llbracket kind \ g \ nid = (\text{SignedDivNode } nid \ x \ y \ zero \ sb \ nxt);$   
 $\quad g \vdash x \simeq xe;$   
 $\quad g \vdash y \simeq ye;$   
 $\quad [m, p] \vdash xe \mapsto v1;$

$$\begin{aligned}
& [m, p] \vdash ye \mapsto v2; \\
& v = (\text{intval-div } v1 \ v2); \\
& m' = m(\text{nid} := v) \\
\implies & g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nxt}, m', h) \mid
\end{aligned}$$

*SignedRemNode:*

$$\begin{aligned}
& \llbracket \text{kind } g \text{ nid} = (\text{SignedRemNode } \text{nid } x \ y \ \text{zero } \text{sb } \text{nxt}); \\
& g \vdash x \simeq xe; \\
& g \vdash y \simeq ye; \\
& [m, p] \vdash xe \mapsto v1; \\
& [m, p] \vdash ye \mapsto v2; \\
& v = (\text{intval-mod } v1 \ v2); \\
& m' = m(\text{nid} := v) \\
\implies & g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nxt}, m', h) \mid
\end{aligned}$$

*StaticLoadFieldNode:*

$$\begin{aligned}
& \llbracket \text{kind } g \text{ nid} = (\text{LoadFieldNode } \text{nid } f \ \text{None } \text{nid}'); \\
& h\text{-load-field } f \ \text{None } h = v; \\
& m' = m(\text{nid} := v) \\
\implies & g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h) \mid
\end{aligned}$$

*StoreFieldNode:*

$$\begin{aligned}
& \llbracket \text{kind } g \text{ nid} = (\text{StoreFieldNode } \text{nid } f \ \text{newval} - (\text{Some } \text{obj}) \ \text{nid}'); \\
& g \vdash \text{newval} \simeq \text{newvalE}; \\
& g \vdash \text{obj} \simeq \text{objE}; \\
& [m, p] \vdash \text{newvalE} \mapsto \text{val}; \\
& [m, p] \vdash \text{objE} \mapsto \text{ObjRef } \text{ref}; \\
& h' = h\text{-store-field } f \ \text{ref } \text{val } h; \\
& m' = m(\text{nid} := \text{val}) \\
\implies & g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h') \mid
\end{aligned}$$

*StaticStoreFieldNode:*

$$\begin{aligned}
& \llbracket \text{kind } g \text{ nid} = (\text{StoreFieldNode } \text{nid } f \ \text{newval} - \text{None } \text{nid}'); \\
& g \vdash \text{newval} \simeq \text{newvalE}; \\
& [m, p] \vdash \text{newvalE} \mapsto \text{val}; \\
& h' = h\text{-store-field } f \ \text{None } \text{val } h; \\
& m' = m(\text{nid} := \text{val}) \\
\implies & g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h')
\end{aligned}$$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow i * i * i \Rightarrow o * o * o \Rightarrow \text{bool}$ ) *step* .

### 7.3 Interprocedural Semantics

**type-synonym** *Signature* = *string*

**type-synonym** *Program* = *Signature*  $\rightarrow$  *IRGraph*

**inductive** *step-top* :: *Program*  $\Rightarrow$  (*IRGraph*  $\times$  *ID*  $\times$  *MapState*  $\times$  *Params*) *list*  $\times$  *FieldRefHeap*  $\Rightarrow$  (*IRGraph*  $\times$  *ID*  $\times$  *MapState*  $\times$  *Params*) *list*  $\times$  *FieldRefHeap*  $\Rightarrow$

*bool*

$(- \vdash - \longrightarrow - \ 55)$

**for  $P$  where**

*Lift:*

$\llbracket g, p \vdash (nid, m, h) \rightarrow (nid', m', h') \rrbracket$   
 $\implies P \vdash ((g, nid, m, p) \# stk, h) \longrightarrow ((g, nid', m', p) \# stk, h') \mid$

*InvokeNodeStep:*

$\llbracket is-Invoke \ (kind \ g \ nid);$

$callTarget = ir-callTarget \ (kind \ g \ nid);$

$kind \ g \ callTarget = (MethodCallTargetNode \ targetMethod \ arguments);$

$Some \ targetGraph = P \ targetMethod;$

$m' = new-map-state;$

$g \vdash arguments \simeq_L argsE;$

$[m, p] \vdash argsE \mapsto_L p'$

$\implies P \vdash ((g, nid, m, p) \# stk, h) \longrightarrow ((targetGraph, 0, m', p') \# (g, nid, m, p) \# stk, h)$

$\mid$

*ReturnNode:*

$\llbracket kind \ g \ nid = (ReturnNode \ (Some \ expr) \ -);$

$g \vdash expr \simeq e;$

$[m, p] \vdash e \mapsto v;$

$cm' = cm(cnid := v);$

$cnid' = (successors-of \ (kind \ cg \ cnid))!0$

$\implies P \vdash ((g, nid, m, p) \# (cg, cnid, cm, cp) \# stk, h) \longrightarrow ((cg, cnid', cm', cp) \# stk, h) \mid$

*ReturnNodeVoid:*

$\llbracket kind \ g \ nid = (ReturnNode \ None \ -);$

$cm' = cm(cnid := (ObjRef \ (Some \ (2048))));$

$cnid' = (successors-of \ (kind \ cg \ cnid))!0$

$\implies P \vdash ((g, nid, m, p) \# (cg, cnid, cm, cp) \# stk, h) \longrightarrow ((cg, cnid', cm', cp) \# stk, h) \mid$

*UnwindNode:*

$\llbracket kind \ g \ nid = (UnwindNode \ exception);$

$g \vdash exception \simeq exceptionE;$

$[m, p] \vdash exceptionE \mapsto e;$

$kind \ cg \ cnid = (InvokeWithExceptionNode \ - \ - \ - \ - \ - \ exEdge);$

$cm' = cm(cnid := e)$

$\implies P \vdash ((g, nid, m, p) \# (cg, cnid, cm, cp) \# stk, h) \longrightarrow ((cg, exEdge, cm', cp) \# stk, h)$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow o \Rightarrow bool$ ) *step-top* .

## 7.4 Big-step Execution

**type-synonym**  $Trace = (IRGraph \times ID \times MapState \times Params) \text{ list}$

**fun**  $has\text{-}return :: MapState \Rightarrow bool$  **where**  
 $has\text{-}return\ m = (m\ 0 \neq Undefined)$

**inductive**  $exec :: Program$   
 $\Rightarrow (IRGraph \times ID \times MapState \times Params) \text{ list} \times FieldRefHeap$   
 $\Rightarrow Trace$   
 $\Rightarrow (IRGraph \times ID \times MapState \times Params) \text{ list} \times FieldRefHeap$   
 $\Rightarrow Trace$   
 $\Rightarrow bool$   
 $(- \vdash - \mid - \longrightarrow * - \mid -)$   
**for**  $P$   
**where**  
 $\llbracket P \vdash (((g, nid, m, p) \# xs), h) \longrightarrow (((g', nid', m', p') \# ys), h') ;$   
 $\neg(has\text{-}return\ m') ;$   
 $l' = (l \ @ \ [(g, nid, m, p)]) ;$   
 $exec\ P\ (((g', nid', m', p') \# ys), h')\ l'\ next\text{-}state\ l'' \rrbracket$   
 $\implies exec\ P\ (((g, nid, m, p) \# xs), h)\ l\ next\text{-}state\ l''$   
 $\mid$   
 $\llbracket P \vdash (((g, nid, m, p) \# xs), h) \longrightarrow (((g', nid', m', p') \# ys), h') ;$   
 $has\text{-}return\ m' ;$   
 $l' = (l \ @ \ [(g, nid, m, p)]) \rrbracket$   
 $\implies exec\ P\ (((g, nid, m, p) \# xs), h)\ l\ (((g', nid', m', p') \# ys), h')\ l'$   
**code-pred**  $(modes: i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow o \Rightarrow bool \text{ as } Exec)\ exec .$

**inductive**  $exec\text{-}debug :: Program$   
 $\Rightarrow (IRGraph \times ID \times MapState \times Params) \text{ list} \times FieldRefHeap$   
 $\Rightarrow nat$   
 $\Rightarrow (IRGraph \times ID \times MapState \times Params) \text{ list} \times FieldRefHeap$   
 $\Rightarrow bool$   
 $(\vdash \longrightarrow * - \mid -)$   
**where**  
 $\llbracket n > 0 ;$   
 $p \vdash s \longrightarrow s' ;$   
 $exec\text{-}debug\ p\ s'\ (n - 1)\ s' \rrbracket$   
 $\implies exec\text{-}debug\ p\ s\ n\ s'' \mid$   
 $\llbracket n = 0 \rrbracket$   
 $\implies exec\text{-}debug\ p\ s\ n\ s$   
**code-pred**  $(modes: i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow bool)\ exec\text{-}debug .$



### 7.4.1 Heap Testing

**definition**  $p3 :: Params$  **where**

$p3 = [IntVal\ 32\ 3]$

**values**  $\{(prod.fst(prod.snd\ (prod.snd\ (hd\ (prod.fst\ res))))\ 0$   
 $\mid res. (\lambda x. Some\ eg2\text{-}sq) \vdash [(eg2\text{-}sq, 0, new\text{-}map\text{-}state, p3), (eg2\text{-}sq, 0, new\text{-}map\text{-}state, p3)],$   
 $new\text{-}heap) \rightarrow^* 2^* res\}$

**definition**  $field\text{-}sq :: string$  **where**

$field\text{-}sq = "sq"$

**definition**  $eg3\text{-}sq :: IRGraph$  **where**

$eg3\text{-}sq = irgraph\ [$   
 $(0, StartNode\ None\ 4, VoidStamp),$   
 $(1, ParameterNode\ 0, default\text{-}stamp),$   
 $(3, MulNode\ 1\ 1, default\text{-}stamp),$   
 $(4, StoreFieldNode\ 4\ field\text{-}sq\ 3\ None\ None\ 5, VoidStamp),$   
 $(5, ReturnNode\ (Some\ 3)\ None, default\text{-}stamp)$   
 $]$

**values**  $\{h\text{-}load\text{-}field\ field\text{-}sq\ None\ (prod.snd\ res)$   
 $\mid res. (\lambda x. Some\ eg3\text{-}sq) \vdash [(eg3\text{-}sq, 0, new\text{-}map\text{-}state, p3), (eg3\text{-}sq, 0,$   
 $new\text{-}map\text{-}state, p3)], new\text{-}heap) \rightarrow^* 3^* res\}$

**definition**  $eg4\text{-}sq :: IRGraph$  **where**

$eg4\text{-}sq = irgraph\ [$   
 $(0, StartNode\ None\ 4, VoidStamp),$   
 $(1, ParameterNode\ 0, default\text{-}stamp),$   
 $(3, MulNode\ 1\ 1, default\text{-}stamp),$   
 $(4, NewInstanceNode\ 4\ "obj\text{-}class"\ None\ 5, ObjectStamp\ "obj\text{-}class"\ True\ True$   
 $True),$   
 $(5, StoreFieldNode\ 5\ field\text{-}sq\ 3\ None\ (Some\ 4)\ 6, VoidStamp),$   
 $(6, ReturnNode\ (Some\ 3)\ None, default\text{-}stamp)$   
 $]$

**values**  $\{h\text{-}load\text{-}field\ field\text{-}sq\ (Some\ 0)\ (prod.snd\ res) \mid res.$   
 $(\lambda x. Some\ eg4\text{-}sq) \vdash [(eg4\text{-}sq, 0, new\text{-}map\text{-}state, p3), (eg4\text{-}sq, 0,$   
 $new\text{-}map\text{-}state, p3)], new\text{-}heap) \rightarrow^* 3^* res\}$

**end**

## 7.5 Control-flow Semantics Theorems

**theory**  $IRStepThms$

**imports**

$IRStepObj$

**begin**

We prove that within the same graph, a configuration triple will always transition to the same subsequent configuration. Therefore, our step semantics is deterministic.

### 7.5.1 Control-flow Step is Deterministic

**theorem** *stepDet*:

$(g, p \vdash (nid, m, h) \rightarrow next) \implies$   
 $(\forall next'. ((g, p \vdash (nid, m, h) \rightarrow next') \longrightarrow next = next'))$

**proof** (*induction rule: step.induct*)

**case** (*SequentialNode nid next m h*)

**have** *notif*:  $\neg(is\_IfNode\ (kind\ g\ nid))$

**using** *SequentialNode.hyps(1) is-sequential-node.simps*

**by** (*metis is-IfNode-def*)

**have** *notend*:  $\neg(is\_AbstractEndNode\ (kind\ g\ nid))$

**using** *SequentialNode.hyps(1) is-sequential-node.simps*

**by** (*metis is-AbstractEndNode.simps is-EndNode.elims(2) is-LoopEndNode-def*)

**have** *notnew*:  $\neg(is\_NewInstanceNode\ (kind\ g\ nid))$

**using** *SequentialNode.hyps(1) is-sequential-node.simps*

**by** (*metis is-NewInstanceNode-def*)

**have** *notload*:  $\neg(is\_LoadFieldNode\ (kind\ g\ nid))$

**using** *SequentialNode.hyps(1) is-sequential-node.simps*

**by** (*metis is-LoadFieldNode-def*)

**have** *notstore*:  $\neg(is\_StoreFieldNode\ (kind\ g\ nid))$

**using** *SequentialNode.hyps(1) is-sequential-node.simps*

**by** (*metis is-StoreFieldNode-def*)

**have** *notdivrem*:  $\neg(is\_IntegerDivRemNode\ (kind\ g\ nid))$

**using** *SequentialNode.hyps(1) is-sequential-node.simps is-SignedDivNode-def*

*is-SignedRemNode-def*

**by** (*metis is-IntegerDivRemNode.simps*)

**from** *notif notend notnew notload notstore notdivrem*

**show** *?case using SequentialNode step.cases*

**by** (*smt (z3) IRNode.disc(1028) IRNode.disc(2270) IRNode.discI(31) Pair-inject*

*is-sequential-node.simps(18) is-sequential-node.simps(43) is-sequential-node.simps(44)*)

**next**

**case** (*IfNode nid cond tb fb m val next h*)

**then have** *notseq*:  $\neg(is\_sequential-node\ (kind\ g\ nid))$

**using** *is-sequential-node.simps is-AbstractMergeNode.simps*

**by** (*simp add: IfNode.hyps(1)*)

**have** *notend*:  $\neg(is\_AbstractEndNode\ (kind\ g\ nid))$

**using** *is-AbstractEndNode.simps*

**by** (*simp add: IfNode.hyps(1)*)

**have** *notdivrem*:  $\neg(is\_IntegerDivRemNode\ (kind\ g\ nid))$

**using** *is-AbstractEndNode.simps*

**by** (*simp add: IfNode.hyps(1)*)

**from** *notseq notend notdivrem show ?case using IfNode repDet evalDet IRN-*

```

ode.distinct IRNode.inject(11) Pair-inject step.simps
  by (smt (z3) IRNode.distinct IRNode.inject(12) Pair-inject step.simps)
next
case (EndNodes nid merge i phis inputs m vs m' h)
have notseq: ¬(is-sequential-node (kind g nid))
  using EndNodes.hyps(1) is-AbstractEndNode.simps is-sequential-node.simps
  by (metis is-EndNode.elims(2) is-LoopEndNode-def)
have notif: ¬(is-IfNode (kind g nid))
  using EndNodes.hyps(1) is-IfNode-def is-AbstractEndNode.elims
  by (metis IRNode.distinct-disc(1058) is-EndNode.simps(12))
have notref: ¬(is-RefNode (kind g nid))
  using EndNodes.hyps(1) is-sequential-node.simps
  using IRNode.disc(1899) IRNode.distinct(1473) is-AbstractEndNode.simps
is-EndNode.elims(2) is-LoopEndNode-def is-RefNode-def
  by metis
have notnew: ¬(is-NewInstanceNode (kind g nid))
  using EndNodes.hyps(1) is-AbstractEndNode.simps
  using IRNode.distinct-disc(1442) is-EndNode.simps(29) is-NewInstanceNode-def
  by (metis IRNode.distinct-disc(1901) is-EndNode.simps(32))
have notload: ¬(is-LoadFieldNode (kind g nid))
  using EndNodes.hyps(1) is-AbstractEndNode.simps
  using is-LoadFieldNode-def
  by (metis IRNode.distinct-disc(1706) is-EndNode.simps(21))
have notstore: ¬(is-StoreFieldNode (kind g nid))
  using EndNodes.hyps(1) is-AbstractEndNode.simps is-StoreFieldNode-def
  by (metis IRNode.distinct-disc(1926) is-EndNode.simps(44))
have notdivrem: ¬(is-IntegerDivRemNode (kind g nid))
  using EndNodes.hyps(1) is-AbstractEndNode.simps is-SignedDivNode-def is-SignedRemNode-def
  using IRNode.distinct-disc(1498) IRNode.distinct-disc(1500) is-IntegerDivRemNode.simps
is-EndNode.simps(36) is-EndNode.simps(37)
  by auto
from notseq notif notref notnew notload notstore notdivrem
show ?case using EndNodes repAllDet evalAllDet
  by (smt (z3) is-IfNode-def is-LoadFieldNode-def is-NewInstanceNode-def is-RefNode-def
is-StoreFieldNode-def is-SignedDivNode-def is-SignedRemNode-def Pair-inject is-IntegerDivRemNode.elims(3)
step.cases)
next
case (NewInstanceNode nid f obj nxt h' ref h m' m)
then have notseq: ¬(is-sequential-node (kind g nid))
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notend: ¬(is-AbstractEndNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notif: ¬(is-IfNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notref: ¬(is-RefNode (kind g nid))
  using is-AbstractMergeNode.simps

```

```

    by (simp add: NewInstanceNode.hyps(1))
  have notload:  $\neg(\text{is-LoadFieldNode } (\text{kind } g \text{ nid}))$ 
    using is-AbstractMergeNode.simps
    by (simp add: NewInstanceNode.hyps(1))
  have notstore:  $\neg(\text{is-StoreFieldNode } (\text{kind } g \text{ nid}))$ 
    using is-AbstractMergeNode.simps
    by (simp add: NewInstanceNode.hyps(1))
  have notdivrem:  $\neg(\text{is-IntegerDivRemNode } (\text{kind } g \text{ nid}))$ 
    using is-AbstractMergeNode.simps
    by (simp add: NewInstanceNode.hyps(1))
  from notseq notend notif notref notload notstore notdivrem
  show ?case using NewInstanceNode.step.cases
    by (smt (z3) IRNode.disc(1028) IRNode.disc(2270) IRNode.discI(11) IRNode.distinct(2311) IRNode.distinct(2313) IRNode.inject(31) Pair-inject)
next
  case (LoadFieldNode nid f obj nrt m ref h v m')
  then have notseq:  $\neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$ 
    using is-sequential-node.simps is-AbstractMergeNode.simps
    by (simp add: LoadFieldNode.hyps(1))
  have notend:  $\neg(\text{is-AbstractEndNode } (\text{kind } g \text{ nid}))$ 
    using is-AbstractEndNode.simps
    by (simp add: LoadFieldNode.hyps(1))
  have notdivrem:  $\neg(\text{is-IntegerDivRemNode } (\text{kind } g \text{ nid}))$ 
    using is-AbstractEndNode.simps
    by (simp add: LoadFieldNode.hyps(1))
  from notseq notend notdivrem
  show ?case using LoadFieldNode.step.cases repDet evalDet
    by (smt (z3) IRNode.distinct(1051) IRNode.distinct(1721) IRNode.distinct(1739) IRNode.distinct(1741) IRNode.distinct(1745) IRNode.inject(20) Pair-inject Value.inject(2) option.distinct(1) option.inject)
next
  case (StaticLoadFieldNode nid f nrt h v m' m)
  then have notseq:  $\neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$ 
    using is-sequential-node.simps is-AbstractMergeNode.simps
    by (simp add: StaticLoadFieldNode.hyps(1))
  have notend:  $\neg(\text{is-AbstractEndNode } (\text{kind } g \text{ nid}))$ 
    using is-AbstractEndNode.simps
    by (simp add: StaticLoadFieldNode.hyps(1))
  have notdivrem:  $\neg(\text{is-IntegerDivRemNode } (\text{kind } g \text{ nid}))$ 
    by (simp add: StaticLoadFieldNode.hyps(1))
  from notseq notend notdivrem
  show ?case using StaticLoadFieldNode.step.cases
    by (smt (z3) IRNode.distinct(1051) IRNode.distinct(1721) IRNode.distinct(1739) IRNode.distinct(1741) IRNode.distinct(1745) IRNode.inject(20) Pair-inject option.distinct(1))
next
  case (StoreFieldNode nid f newval uu obj nrt m val ref h' h m')
  then have notseq:  $\neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$ 
    using is-sequential-node.simps is-AbstractMergeNode.simps
    by (simp add: StoreFieldNode.hyps(1))

```

```

have notend: ¬(is-AbstractEndNode (kind g nid))
  using is-AbstractEndNode.simps
  by (simp add: StoreFieldNode.hyps(1))
have notdivrem: ¬(is-IntegerDivRemNode (kind g nid))
  by (simp add: StoreFieldNode.hyps(1))
from notseq notend notdivrem
show ?case using StoreFieldNode step.cases repDet evalDet
  by (smt (z3) IRNode.distinct(1097) IRNode.distinct(1745) IRNode.distinct(2317)
IRNode.distinct(2605) IRNode.distinct(2627) IRNode.inject(43) Pair-inject Value.inject(2)
option.distinct(1) option.inject)
next
case (StaticStoreFieldNode nid f newval uv nxt m val h' h m')
then have notseq: ¬(is-sequential-node (kind g nid))
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: StaticStoreFieldNode.hyps(1))
have notend: ¬(is-AbstractEndNode (kind g nid))
  using is-AbstractEndNode.simps
  by (simp add: StaticStoreFieldNode.hyps(1))
have notdivrem: ¬(is-IntegerDivRemNode (kind g nid))
  by (simp add: StaticStoreFieldNode.hyps(1))
from notseq notend notdivrem
show ?case using StoreFieldNode step.cases repDet evalDet
  by (smt (z3) IRNode.distinct(1097) IRNode.distinct(1745) IRNode.distinct(2317)
IRNode.distinct(2605) IRNode.distinct(2627) IRNode.inject(43) Pair-inject Static-
StoreFieldNode.hyps(1) StaticStoreFieldNode.hyps(2) StaticStoreFieldNode.hyps(3)
StaticStoreFieldNode.hyps(4) StaticStoreFieldNode.hyps(5) option.distinct(1))
next
case (SignedDivNode nid x y zero sb nxt m v1 v2 v m' h)
then have notseq: ¬(is-sequential-node (kind g nid))
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: SignedDivNode.hyps(1))
have notend: ¬(is-AbstractEndNode (kind g nid))
  using is-AbstractEndNode.simps
  by (simp add: SignedDivNode.hyps(1))
from notseq notend
show ?case using SignedDivNode step.cases repDet evalDet
  by (smt (z3) IRNode.distinct(1091) IRNode.distinct(1739) IRNode.distinct(2311)
IRNode.distinct(2601) IRNode.distinct(2605) IRNode.inject(40) Pair-inject)
next
case (SignedRemNode nid x y zero sb nxt m v1 v2 v m' h)
then have notseq: ¬(is-sequential-node (kind g nid))
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: SignedRemNode.hyps(1))
have notend: ¬(is-AbstractEndNode (kind g nid))
  using is-AbstractEndNode.simps
  by (simp add: SignedRemNode.hyps(1))
from notseq notend
show ?case using SignedRemNode step.cases repDet evalDet
  by (smt (z3) IRNode.distinct(1093) IRNode.distinct(1741) IRNode.distinct(2313)

```

*IRNode.distinct(2601) IRNode.distinct(2627) IRNode.inject(41) Pair-inject*  
**qed**

**lemma** *stepRefNode*:

$\llbracket \text{kind } g \text{ nid} = \text{RefNode nid}' \rrbracket \implies g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m, h)$

**using** *SequentialNode*

**by** (*metis IRNodes.successors-of-RefNode is-sequential-node.simps(7) nth-Cons-0*)

**lemma** *IfNodeStepCases*:

**assumes** *kind g nid = IfNode cond tb fb*

**assumes**  $g \vdash \text{cond} \simeq \text{condE}$

**assumes**  $[m, p] \vdash \text{condE} \mapsto v$

**assumes**  $g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m, h)$

**shows**  $\text{nid}' \in \{tb, fb\}$

**using** *step.IfNode repDet stepDet assms*

**by** (*metis insert-iff old.prod.inject*)

**lemma** *IfNodeSeq*:

**shows** *kind g nid = IfNode cond tb fb*  $\longrightarrow \neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$

**unfolding** *is-sequential-node.simps*

**using** *is-sequential-node.simps(18)* **by** *presburger*

**lemma** *IfNodeCond*:

**assumes** *kind g nid = IfNode cond tb fb*

**assumes**  $g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m, h)$

**shows**  $\exists \text{ condE } v. ((g \vdash \text{cond} \simeq \text{condE}) \wedge ([m, p] \vdash \text{condE} \mapsto v))$

**using** *assms(2,1)* **by** (*induct (nid,m,h) (nid',m,h) rule: step.induct; auto*)

**lemma** *step-in-ids*:

**assumes**  $g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h')$

**shows**  $\text{nid} \in \text{ids } g$

**using** *assms* **apply** (*induct (nid, m, h) (nid', m', h') rule: step.induct*)

**using** *is-sequential-node.simps(45) not-in-g*

**apply** *simp*

**apply** (*metis is-sequential-node.simps(53)*)

**using** *ids-some*

**using** *IRNode.distinct(1113)* **apply** *presburger*

**using** *EndNodes(1) is-AbstractEndNode.simps is-EndNode.simps(45) ids-some*

**apply** (*metis IRNode.disc(1218) is-EndNode.simps(52)*)

**by** *simp+*

**end**

## 7.6 Evaluation Stamp Theorems

**theory** *StampEvalThms*

**imports** *Graph.ValueThms*

*Semantics.IRTreeEvalThms*

**begin**

```

lemma
  assumes take-bit b v = v
  shows signed-take-bit b v = v
  using assms
  by (metis(full-types) eq-imp-le signed-take-bit-take-bit)

lemma unwrap-signed-take-bit:
  fixes v :: int64
  assumes 0 < b ∧ b ≤ 64
  assumes signed-take-bit (b - 1) v = v
  shows signed-take-bit 63 (Word.rep (signed-take-bit (b - Suc 0) v)) = sint v
  using assms using size64 unfolding signed-def by auto

lemma unrestricted-new-int-always-valid [simp]:
  assumes 0 < b ∧ b ≤ 64
  shows valid-value (new-int b v) (unrestricted-stamp (IntegerStamp b lo hi))
  unfolding unrestricted-stamp.simps new-int.simps valid-value.simps
  by (simp; metis One-nat-def assms int-power-div-base int-signed-value.simps
    int-signed-value-range linorder-not-le not-exp-less-eq-0-int zero-less-numeral)

lemma unary-undef: val = UndefVal ⇒ unary-eval op val = UndefVal
  by (cases op; auto)

lemma unary-obj: val = ObjRef x ⇒ unary-eval op val = UndefVal
  by (cases op; auto)

lemma unrestricted-stamp-valid:
  assumes s = unrestricted-stamp (IntegerStamp b lo hi)
  assumes 0 < b ∧ b ≤ 64
  shows valid-stamp s
  using assms
  by (smt (z3) Stamp.inject(1) bit-bounds.simps not-exp-less-eq-0-int prod.sel(1)
    prod.sel(2) unrestricted-stamp.simps(2) upper-bounds-equiv valid-stamp.elims(1))

lemma unrestricted-stamp-valid-value [simp]:
  assumes 1: result = IntVal b ival
  assumes take-bit b ival = ival
  assumes 0 < b ∧ b ≤ 64
  shows valid-value result (unrestricted-stamp (IntegerStamp b lo hi))
proof -
  have valid-stamp (unrestricted-stamp (IntegerStamp b lo hi))
    using assms unrestricted-stamp-valid by blast
  then show ?thesis
    unfolding 1 unrestricted-stamp.simps valid-value.simps
    using assms int-signed-value-bounds by presburger
qed

```

### 7.6.1 Support Lemmas for Integer Stamps and Associated IntVal values

Valid int implies some useful facts.

**lemma** *valid-int-gives:*

**assumes** *valid-value* (*IntVal b val*) *stamp*  
**obtains** *lo hi* **where** *stamp* = *IntegerStamp b lo hi*  $\wedge$   
*valid-stamp* (*IntegerStamp b lo hi*)  $\wedge$   
*take-bit b val* = *val*  $\wedge$   
 $lo \leq \text{int-signed-value } b \text{ val} \wedge \text{int-signed-value } b \text{ val} \leq hi$   
**using** *assms*  
**by** (*smt* (*z3*) *Value.distinct*(7) *Value.inject*(1) *valid-value.elims*(1))

And the corresponding lemma where we know the stamp rather than the value.

**lemma** *valid-int-stamp-gives:*

**assumes** *valid-value val* (*IntegerStamp b lo hi*)  
**obtains** *ival* **where** *val* = *IntVal b ival*  $\wedge$   
*valid-stamp* (*IntegerStamp b lo hi*)  $\wedge$   
*take-bit b ival* = *ival*  $\wedge$   
 $lo \leq \text{int-signed-value } b \text{ ival} \wedge \text{int-signed-value } b \text{ ival} \leq hi$   
**by** (*metis assms valid-int valid-value.simps*(1))

A valid int must have the expected number of bits.

**lemma** *valid-int-same-bits:*

**assumes** *valid-value* (*IntVal b val*) (*IntegerStamp bits lo hi*)  
**shows** *b* = *bits*  
**by** (*meson assms valid-value.simps*(1))

A valid value means a valid stamp.

**lemma** *valid-int-valid-stamp:*

**assumes** *valid-value* (*IntVal b val*) (*IntegerStamp bits lo hi*)  
**shows** *valid-stamp* (*IntegerStamp bits lo hi*)  
**by** (*metis assms valid-value.simps*(1))

A valid int means a valid non-empty stamp.

**lemma** *valid-int-not-empty:*

**assumes** *valid-value* (*IntVal b val*) (*IntegerStamp bits lo hi*)  
**shows**  $lo \leq hi$   
**by** (*metis assms order.trans valid-value.simps*(1))

A valid int fits into the given number of bits (and other bits are zero).

**lemma** *valid-int-fits:*

**assumes** *valid-value* (*IntVal b val*) (*IntegerStamp bits lo hi*)  
**shows** *take-bit bits val* = *val*  
**by** (*metis assms valid-value.simps*(1))



**lemma** *valid-int-is-zero-masked*:

**assumes** *valid-value* (*IntVal* *b val*) (*IntegerStamp* *bits lo hi*)

**shows** *and val (not (mask bits)) = 0*

**by** (*metis* (*no-types*, *lifting*) *assms* *bit.conj-cancel-right take-bit-eq-mask valid-int-fits*)

*word-bw-assocs(1) word-log-esimps(1)*

Unsigned ints have bounds 0 up to  $2^{\text{bits}}$ .

**lemma** *valid-int-unsigned-bounds*:

**assumes** *valid-value* (*IntVal* *b val*) (*IntegerStamp* *bits lo hi*)

**shows** *uint val < 2<sup>bits</sup>*

**by** (*metis* *assms(1) mask-eq-iff take-bit-eq-mask valid-value.simps(1)*)

Signed ints have the usual two-complement bounds.

**lemma** *valid-int-signed-upper-bound*:

**assumes** *valid-value* (*IntVal* *b val*) (*IntegerStamp* *bits lo hi*)

**shows** *int-signed-value bits val < 2<sup>(bits - 1)</sup>*

**by** (*metis* (*mono-tags*, *opaque-lifting*) *diff-le-mono int-signed-value.simps less-imp-diff-less*)

*linorder-not-le one-le-numeral order-less-le-trans power-increasing signed-take-bit-int-less-exp-word sint-lt*)

**lemma** *valid-int-signed-lower-bound*:

**assumes** *valid-value* (*IntVal* *b val*) (*IntegerStamp* *bits lo hi*)

**shows**  $-(2^{(\text{bits} - 1)}) \leq \text{int-signed-value bits val}$

**by** (*smt* (*verit*) *diff-le-self int-signed-value.simps linorder-not-less power-increasing-iff signed-take-bit-int-greater-eq-minus-exp-word sint-greater-eq*)

and *bit\_bounds* versions of the above bounds.

**lemma** *valid-int-signed-upper-bit-bound*:

**assumes** *valid-value* (*IntVal* *b val*) (*IntegerStamp* *bits lo hi*)

**shows** *int-signed-value bits val ≤ snd (bit-bounds bits)*

**proof** –

**have** *b = bits* **using** *assms valid-int-same-bits* **by** *blast*

**then show** *?thesis*

**using** *assms* **by** *force*

**qed**

**lemma** *valid-int-signed-lower-bit-bound*:

**assumes** *valid-value* (*IntVal* *b val*) (*IntegerStamp* *bits lo hi*)

**shows** *fst (bit-bounds bits) ≤ int-signed-value bits val*

**proof** –

**have** *b = bits* **using** *assms valid-int-same-bits* **by** *blast*

**then show** *?thesis*

**using** *assms* **by** *force*

**qed**

Valid values satisfy their stamp bounds.

**lemma** *valid-int-signed-range*:  
**assumes** *valid-value* (*IntVal* *b val*) (*IntegerStamp* *bits lo hi*)  
**shows**  $lo \leq \text{int-signed-value bits val} \wedge \text{int-signed-value bits val} \leq hi$   
**by** (*metis* *assms* *valid-value.simps*(1))

## 7.6.2 Validity of all Unary Operators

We split the validity proof for unary operators into two lemmas, one for normal unary operators whose output bits equals their input bits, and the other case for the widen and narrow operators.

**lemma** *eval-normal-unary-implies-valid-value*:  
**assumes**  $[m, p] \vdash \text{expr} \mapsto \text{val}$   
**assumes**  $\text{result} = \text{unary-eval op val}$   
**assumes**  $\text{op}: \text{op} \in \text{normal-unary}$   
**assumes**  $\text{result} \neq \text{UndefVal}$   
**assumes** *valid-value* *val* (*stamp-expr* *expr*)  
**shows** *valid-value* *result* (*stamp-expr* (*UnaryExpr* *op expr*))

**proof** –  
**obtain** *b1 v1* **where**  $v1: \text{val} = \text{IntVal } b1 \ v1$   
**by** (*metis* *Value.exhaust* *assms*(1) *assms*(2) *assms*(4) *assms*(5) *evaltree-not-undef* *unary-obj* *valid-value.simps*(11))  
**then obtain** *b2 v2* **where**  $v2: \text{result} = \text{IntVal } b2 \ v2$   
**using** *assms*(2) *assms*(4) *is-IntVal-def* *unary-eval-int* **by** *presburger*  
**then have**  $\text{result} = \text{unary-eval op (IntVal } b1 \ v1)$   
**using** *assms*(2) *v1* **by** *blast*  
**then obtain** *vtmp* **where**  $\text{result} = \text{new-int } b2 \ \text{vtmp}$   
**using** *assms*(3) *v2* **by** *auto*  
**obtain** *b' lo' hi'* **where**  $\text{stamp-expr expr} = \text{IntegerStamp } b' \ lo' \ hi'$   
**by** (*metis* *assms*(5) *v1* *valid-int-gives*)  
**then have**  $\text{stamp-unary op (stamp-expr expr)} =$   
 $\text{unrestricted-stamp}$   
 $(\text{IntegerStamp (if op} \in \text{normal-unary then } b' \text{ else ir-resultBits op) } lo' \ hi')$   
**using** *stamp-unary.simps*(1) **by** *presburger*  
**then obtain** *lo2 hi2* **where**  $s: (\text{stamp-expr (UnaryExpr op expr)}) = \text{unrestricted-stamp (IntegerStamp } b2 \ lo2 \ hi2)$   
**unfolding** *stamp-expr.simps*  
**using** *vtmp op*  
**by** (*smt* (*verit*, *best*) *Value.inject*(1)  $\langle (\text{result}::\text{Value}) = \text{unary-eval (op::IRUnaryOp)}$   
 $(\text{IntVal } (b1::\text{nat}) \ (v1::64 \ \text{word})) \rangle \langle \text{stamp-expr (expr::IRExpr)} = \text{IntegerStamp } (b'::\text{nat})$   
 $(lo'::\text{int}) \ (hi'::\text{int}) \rangle \text{assms}(5) \text{insertE intval-abs.simps}(1) \text{intval-logic-negation.simps}(1)$   
 $\text{intval-negate.simps}(1) \text{intval-not.simps}(1) \text{new-int.elims singleton-iff unary-eval.simps}(1)$   
 $\text{unary-eval.simps}(2) \text{unary-eval.simps}(3) \text{unary-eval.simps}(4) \ v1 \text{valid-int-same-bits}$ )  
**then have**  $0 < b1 \wedge b1 \leq 64$   
**using** *valid-int-gives*  
**by** (*metis* *assms*(5) *v1* *valid-stamp.simps*(1))  
**then have**  $\text{fst (bit-bounds } b2) \leq \text{int-signed-value } b2 \ v2 \wedge$   
 $\text{int-signed-value } b2 \ v2 \leq \text{snd (bit-bounds } b2)$   
**by** (*smt* (*verit*, *del-insts*) *Stamp.inject*(1) *assms*(3) *assms*(5) *int-signed-value-bounds* *s* *stamp-expr.simps*(1) *stamp-unary.simps*(1) *unrestricted-stamp.simps*(2) *v1* *valid-int-gives*)

```

then show ?thesis
  unfolding s v2 unrestricted-stamp.simps valid-value.simps
  by (smt (z3) assms(3) assms(5) is-stamp-empty.simps(1) new-int-take-bits s
stamp-expr.simps(1) stamp-unary.simps(1) unrestricted-stamp.simps(2) v1 v2 valid-int-gives
valid-stamp.simps(1) vtmp)
qed

```

```

lemma narrow-widen-output-bits:
  assumes unary-eval op val  $\neq$  UndefVal
  assumes op  $\notin$  normal-unary
  shows  $0 < (\text{ir-resultBits } op) \wedge (\text{ir-resultBits } op) \leq 64$ 
proof -
  consider ib ob where op = UnaryNarrow ib ob
    | ib ob where op = UnarySignExtend ib ob
    | ib ob where op = UnaryZeroExtend ib ob
  using IRUnaryOp.exhaust-sel assms(2) by blast
  then show ?thesis
  proof (cases)
    case 1
    then show ?thesis using assms intval-narrow-ok by force
  next
    case 2
    then show ?thesis using assms intval-sign-extend-ok by force
  next
    case 3
    then show ?thesis using assms intval-zero-extend-ok by force
  qed
qed

```

```

lemma eval-widen-narrow-unary-implies-valid-value:
  assumes  $[m, p] \vdash \text{expr} \mapsto \text{val}$ 
  assumes result = unary-eval op val
  assumes op: op  $\notin$  normal-unary
  assumes result  $\neq$  UndefVal
  assumes valid-value val (stamp-expr expr)
  shows valid-value result (stamp-expr (UnaryExpr op expr))
proof -
  obtain b1 v1 where v1: val = IntVal b1 v1
  by (metis Value.exhaust assms(1) assms(2) assms(4) assms(5) evaltree-not-undef
unary-obj valid-value.simps(11))
  then have result = unary-eval op (IntVal b1 v1)
  using assms(2) v1 by blast
  then obtain v2 where v2: result = new-int (ir-resultBits op) v2
  using assms by (cases op; simp; (meson new-int.simps)+)
  then obtain v3 where v3: result = IntVal (ir-resultBits op) v3
  using assms by (cases op; simp; (meson new-int.simps)+)
  then obtain lo2 hi2 where s: (stamp-expr (UnaryExpr op expr)) = unre-
stricted-stamp (IntegerStamp (ir-resultBits op) lo2 hi2)

```

```

    unfolding stamp-expr.simps stamp-unary.simps
    using assms(3) assms(5) v1 valid-int-gives by fastforce
  then have outBits:  $0 < (ir-resultBits\ op) \wedge (ir-resultBits\ op) \leq 64$ 
    using assms narrow-widen-output-bits
    by blast
  then have fst (bit-bounds (ir-resultBits op))  $\leq$  int-signed-value (ir-resultBits op)
v3  $\wedge$ 
    int-signed-value (ir-resultBits op) v3  $\leq$  snd (bit-bounds (ir-resultBits op))
    using int-signed-value-bounds
  by (smt (verit, del-insts) Stamp.inject(1) assms(3) assms(5) int-signed-value-bounds
s stamp-expr.simps(1) stamp-unary.simps(1) unrestricted-stamp.simps(2) v1 valid-int-gives)
  then show ?thesis
    unfolding s v3 unrestricted-stamp.simps valid-value.simps
    using outBits v2 v3 by auto
qed

```

```

lemma eval-unary-implies-valid-value:
  assumes  $[m, p] \vdash expr \mapsto val$ 
  assumes  $result = unary\text{-}eval\ op\ val$ 
  assumes  $result \neq UndefinedVal$ 
  assumes  $valid\text{-}value\ val\ (stamp\text{-}expr\ expr)$ 
  shows  $valid\text{-}value\ result\ (stamp\text{-}expr\ (UnaryExpr\ op\ expr))$ 
  proof (cases  $op \in normal\text{-}unary$ )
    case True
    then show ?thesis by (metis assms eval-normal-unary-implies-valid-value)
  next
    case False
    then show ?thesis by (metis assms eval-widen-narrow-unary-implies-valid-value)
  qed

```

### 7.6.3 Support Lemmas for Binary Operators

```

lemma binary-undef:  $v1 = UndefinedVal \vee v2 = UndefinedVal \implies bin\text{-}eval\ op\ v1\ v2 = UndefinedVal$ 
  by (cases op; auto)

```

```

lemma binary-obj:  $v1 = ObjRef\ x \vee v2 = ObjRef\ y \implies bin\text{-}eval\ op\ v1\ v2 = UndefinedVal$ 
  by (cases op; auto)

```

Some lemmas about the three different output sizes for binary operators.

```

lemma bin-eval-bits-binary-shift-ops:
  assumes  $result = bin\text{-}eval\ op\ (IntVal\ b1\ v1)\ (IntVal\ b2\ v2)$ 
  assumes  $result \neq UndefinedVal$ 
  assumes  $op \in binary\text{-}shift\text{-}ops$ 
  shows  $\exists v. result = new\text{-}int\ b1\ v$ 
  using assms
  by (cases op; simp; smt (verit, best) new-int.simps)+

```

```

lemma bin-eval-bits-fixed-32-ops:
  assumes result = bin-eval op (IntVal b1 v1) (IntVal b2 v2)
  assumes result ≠ UndefVal
  assumes op ∈ binary-fixed-32-ops
  shows ∃ v. result = new-int 32 v
  using assms
  apply (cases op; simp)
  using assms bool-to-val.simps bin-eval-new-int new-int.simps bin-eval-unused-bits-zero
  by metis+

```

```

lemma bin-eval-bits-normal-ops:
  assumes result = bin-eval op (IntVal b1 v1) (IntVal b2 v2)
  assumes result ≠ UndefVal
  assumes op ∉ binary-shift-ops
  assumes op ∉ binary-fixed-32-ops
  shows ∃ v. result = new-int b1 v
  using assms apply (cases op; simp)
  using assms apply (metis (mono-tags))+
  using take-bit-and apply metis
  using take-bit-or apply metis
  using take-bit-xor by metis

```

```

lemma bin-eval-input-bits-equal:
  assumes result = bin-eval op (IntVal b1 v1) (IntVal b2 v2)
  assumes result ≠ UndefVal
  assumes op ∉ binary-shift-ops
  shows b1 = b2
  using assms apply (cases op; simp)
  by presburger+

```

```

lemma bin-eval-implies-valid-value:
  assumes [m,p] ⊢ expr1 ↦ val1
  assumes [m,p] ⊢ expr2 ↦ val2
  assumes result = bin-eval op val1 val2
  assumes result ≠ UndefVal
  assumes valid-value val1 (stamp-expr expr1)
  assumes valid-value val2 (stamp-expr expr2)
  shows valid-value result (stamp-expr (BinaryExpr op expr1 expr2))
proof –
  obtain b1 v1 where v1: val1 = IntVal b1 v1
  by (metis Value.collapse(1) assms(3) assms(4) bin-eval-inputs-are-ints bin-eval-int)
  obtain b2 v2 where v2: val2 = IntVal b2 v2
  by (metis Value.collapse(1) assms(3) assms(4) bin-eval-inputs-are-ints bin-eval-int)
  then obtain lo1 hi1 where s1: stamp-expr expr1 = IntegerStamp b1 lo1 hi1
  by (metis assms(5) v1 valid-int-gives)
  then obtain lo2 hi2 where s2: stamp-expr expr2 = IntegerStamp b2 lo2 hi2
  by (metis assms(6) v2 valid-int-gives)
  then have r: result = bin-eval op (IntVal b1 v1) (IntVal b2 v2)

```

```

    using assms(3) v1 v2 by blast
  then obtain bres vtmp where vtmp: result = new-int bres vtmp
    using assms bin-eval-bits-binary-shift-ops
    by (meson bin-eval-new-int)
  then obtain vres where vres: result = IntVal bres vres
    by force

  then have sres: stamp-expr (BinaryExpr op expr1 expr2) =
    unrestricted-stamp (IntegerStamp bres lo1 hi1)
    ∧ 0 < bres ∧ bres ≤ 64
  proof (cases op ∈ binary-shift-ops)
    case True
    then show ?thesis
      unfolding s1 s2 stamp-binary.simps stamp-expr.simps
      using assms bin-eval-bits-binary-shift-ops
      by (metis Value.inject(1) eval-bits-1-64 new-int.simps r v1 vres)
  next
    case False
    then have op ∉ binary-shift-ops
      by simp
    then have beq: b1 = b2
      using v1 v2 assms bin-eval-input-bits-equal by simp
    then show ?thesis
      proof (cases op ∈ binary-fixed-32-ops)
        case True
        then show ?thesis
          unfolding s1 s2 stamp-binary.simps stamp-expr.simps
          using assms bin-eval-bits-fixed-32-ops
          by (metis False Value.inject(1) beq bin-eval-new-int le-add-same-cancel1
            new-int.simps numeral-Bit0 vres zero-le-numeral zero-less-numeral)
        case False
        then show ?thesis
          unfolding s1 s2 stamp-binary.simps stamp-expr.simps
          using assms
          by (metis beq bin-eval-new-int eval-bits-1-64 intval-bits.simps unrestricted-new-int-always-valid
            unrestricted-stamp.simps(2) v1 valid-int-same-bits vres)
      qed
    qed
    then show ?thesis
      unfolding vres
      using unrestricted-new-int-always-valid vres vtmp by presburger
  qed

```

#### 7.6.4 Validity of Stamp Meet and Join Operators

**lemma** *stamp-meet-integer-is-valid-stamp*:  
 assumes *valid-stamp stamp1*  
 assumes *valid-stamp stamp2*

```

assumes is-IntegerStamp stamp1
assumes is-IntegerStamp stamp2
shows valid-stamp (meet stamp1 stamp2)
using assms unfolding is-IntegerStamp-def valid-stamp.simps meet.simps
by (smt (verit, del-insts) meet.simps(2) valid-stamp.simps(1) valid-stamp.simps(8))

```

```

lemma stamp-meet-is-valid-stamp:
  assumes 1: valid-stamp stamp1
  assumes 2: valid-stamp stamp2
  shows valid-stamp (meet stamp1 stamp2)
  by (cases stamp1; cases stamp2; insert stamp-meet-integer-is-valid-stamp[OF 1
2]; auto)

```

```

lemma stamp-meet-commutes: meet stamp1 stamp2 = meet stamp2 stamp1
by (cases stamp1; cases stamp2; auto)

```

```

lemma stamp-meet-is-valid-value1:
  assumes valid-value val stamp1
  assumes valid-stamp stamp2
  assumes stamp1 = IntegerStamp b1 lo1 hi1
  assumes stamp2 = IntegerStamp b2 lo2 hi2
  assumes meet stamp1 stamp2 ≠ IllegalStamp
  shows valid-value val (meet stamp1 stamp2)
proof –
  have m: meet stamp1 stamp2 = IntegerStamp b1 (min lo1 lo2) (max hi1 hi2)
    using assms by (metis meet.simps(2))
  obtain ival where val: val = IntVal b1 ival
    using assms valid-int by blast
  then have v: valid-stamp (IntegerStamp b1 lo1 hi1) ∧
    take-bit b1 ival = ival ∧
    lo1 ≤ int-signed-value b1 ival ∧ int-signed-value b1 ival ≤ hi1
    using assms by (metis valid-value.simps(1))
  then have mm: min lo1 lo2 ≤ int-signed-value b1 ival ∧ int-signed-value b1 ival
≤ max hi1 hi2
    by linarith
  then have valid-stamp (IntegerStamp b1 (min lo1 lo2) (max hi1 hi2))
    using assms v stamp-meet-is-valid-stamp
    by (metis meet.simps(2))
  then show ?thesis
    unfolding m val valid-value.simps
    using mm v by presburger
qed

```

and the symmetric lemma follows by the commutativity of meet.

```

lemma stamp-meet-is-valid-value:
  assumes valid-value val stamp2
  assumes valid-stamp stamp1
  assumes stamp1 = IntegerStamp b1 lo1 hi1

```

```

assumes stamp2 = IntegerStamp b2 lo2 hi2
assumes meet stamp1 stamp2 ≠ IllegalStamp
shows valid-value val (meet stamp1 stamp2)
using assms stamp-meet-commutes stamp-meet-is-valid-value1
by metis

```

### 7.6.5 Validity of conditional expressions

**lemma** *conditional-eval-implies-valid-value*:

```

assumes [m,p] ⊢ cond ↦ condv
assumes expr = (if val-to-bool condv then expr1 else expr2)
assumes [m,p] ⊢ expr ↦ val
assumes val ≠ UndefinedVal
assumes valid-value condv (stamp-expr cond)
assumes valid-value val (stamp-expr expr)
assumes compatible (stamp-expr expr1) (stamp-expr expr2)
shows valid-value val (stamp-expr (ConditionalExpr cond expr1 expr2))
proof –
  have def: meet (stamp-expr expr1) (stamp-expr expr2) ≠ IllegalStamp
    using assms
  by (metis Stamp.distinct(13) Stamp.distinct(25) compatible.elims(2) meet.simps(1)
    meet.simps(2))
  then have valid-stamp (meet (stamp-expr expr1) (stamp-expr expr2))
    using assms
  by (smt (verit, best) compatible.elims(2) stamp-meet-is-valid-stamp valid-stamp.simps(2))

  then show ?thesis using stamp-meet-is-valid-value
    using assms def
  by (smt (verit, best) compatible.elims(2) never-void stamp-expr.simps(6) stamp-meet-commutes)

```

qed

### 7.6.6 Validity of Whole Expression Tree Evaluation

TODO: find a way to encode that conditional expressions must have compatible (and valid) stamps? One approach would be for all the stamp\_expr operators to require that all input stamps are valid.

**definition** *wf-stamp* :: *IRExpr* ⇒ *bool* **where**

*wf-stamp* e = (∀ m p v. ([m, p] ⊢ e ↦ v) ⟶ valid-value v (stamp-expr e))

**lemma** *stamp-under-defn*:

```

assumes stamp-under (stamp-expr x) (stamp-expr y)
assumes wf-stamp x ∧ wf-stamp y
assumes ([m, p] ⊢ x ↦ xv) ∧ ([m, p] ⊢ y ↦ yv)
shows val-to-bool (bin-eval BinIntegerLessThan xv yv)
proof –
  have yval: valid-value yv (stamp-expr y)
    using assms wf-stamp-def by blast
  obtain b lx hi where xstamp: stamp-expr x = IntegerStamp b lx hi

```



```

    using assms(1)
  by (metis stamp-under.elims(2))
then obtain lo hy where ystamp: stamp-expr y = IntegerStamp b lo hy
  using assms(1)
  by (metis Stamp.sel(1) stamp-under.elims(2))
obtain xv where xv: xv = IntVal b xv
  by (metis assms(2) assms(3) valid-int wf-stamp-def xstamp)
then have xval: valid-value (IntVal b xv) (stamp-expr x)
  using assms(2) assms(3) wf-stamp-def by blast
obtain yv where yv: yv = IntVal b yv
  by (metis valid-int ystamp yval)
then have yval: valid-value (IntVal b yv) (stamp-expr y)
  using yval by auto
have xunder: int-signed-value b xv ≤ hi
  using xv xval valid-value.simps
  by (metis assms(2) assms(3) wf-stamp-def xstamp)
have yunder: lo ≤ int-signed-value b yv
  using yv yval valid-value.simps
  by (metis ystamp)
have unwrap: ∀ cond. bool-to-val-bin b b cond = bool-to-val cond
  by simp
from xunder yunder have int-signed-value b xv < int-signed-value b yv
  using assms(1) xstamp ystamp by auto
then have (intval-less-than xv yv) = IntVal 32 1
  using xv yv
  using intval-less-than.simps(1) unwrap
  using bool-to-val.simps(1) by presburger
then show ?thesis
  by simp
qed

```

**lemma** stamp-under-defn-inverse:

```

  assumes stamp-under (stamp-expr y) (stamp-expr x)
  assumes wf-stamp x ∧ wf-stamp y
  assumes ([m, p] ⊢ x ↦ xv) ∧ ([m, p] ⊢ y ↦ yv)
  shows ¬(val-to-bool (bin-eval BinIntegerLessThan xv yv))
proof -
  have yval: valid-value yv (stamp-expr y)
    using assms wf-stamp-def by blast
  obtain b lo hx where xstamp: stamp-expr x = IntegerStamp b lo hx
    using assms(1)
    by (metis stamp-under.elims(2))
  then obtain ly hi where ystamp: stamp-expr y = IntegerStamp b ly hi
    using assms(1)
    by (metis Stamp.sel(1) stamp-under.elims(2))
  obtain xv where xv: xv = IntVal b xv
    by (metis assms(2) assms(3) valid-int wf-stamp-def xstamp)
  then have xval: valid-value (IntVal b xv) (stamp-expr x)
    using assms(2) assms(3) wf-stamp-def by blast

```

```

obtain yvv where yv = IntVal b yvv
  by (metis valid-int ystamp yval)
then have xval: valid-value (IntVal b yvv) (stamp-expr y)
  using yval by auto
have yunder: int-signed-value b yvv  $\leq$  hi
  using yvv yval valid-value.simps
  by (metis ystamp)
have xover: lo  $\leq$  int-signed-value b xvv
  using xvv xval valid-value.simps
  by (metis assms(2) assms(3) wf-stamp-def xstamp)
have unwrap:  $\forall$  cond. bool-to-val-bin b b cond = bool-to-val cond
  by simp
from xover yunder have int-signed-value b yvv  $<$  int-signed-value b xvv
  using assms(1) xstamp ystamp by auto
then have (intval-less-than xv yv) = IntVal 32 0
  using xvv yvv
  using intval-less-than.simps(1) unwrap
  by force
then show ?thesis
  by simp
qed

end

```

## 8 Optimization DSL

### 8.1 Markup

```

theory Markup
  imports Semantics.IRTreeEval Snippets.Snipping
begin

```

```

datatype 'a Rewrite =
  Transform 'a 'a (-  $\mapsto$  - 10) |
  Conditional 'a 'a bool (-  $\mapsto$  - when - 11) |
  Sequential 'a Rewrite 'a Rewrite |
  Transitive 'a Rewrite

```

```

datatype 'a ExtraNotation =
  ConditionalNotation 'a 'a 'a (- ? - : - 50) |
  EqualsNotation 'a 'a (- eq -) |
  ConstantNotation 'a (const - 120) |
  TrueNotation (true) |
  FalseNotation (false) |
  ExclusiveOr 'a 'a (-  $\oplus$  -) |
  LogicNegationNotation 'a (!-) |
  ShortCircuitOr 'a 'a (- || -)

```

```

definition word :: ('a::len) word  $\Rightarrow$  'a word where

```

word  $x = x$

**ML-file**  $\langle markup.ML \rangle$

### 8.1.1 Expression Markup

```

ML <
structure IRExprTranslator : DSL-TRANSLATION =
struct
fun markup DSL-Tokens.Add = @{term BinaryExpr} $ @{term BinAdd}
  | markup DSL-Tokens.Sub = @{term BinaryExpr} $ @{term BinSub}
  | markup DSL-Tokens.Mul = @{term BinaryExpr} $ @{term BinMul}
  | markup DSL-Tokens.And = @{term BinaryExpr} $ @{term BinAnd}
  | markup DSL-Tokens.Or = @{term BinaryExpr} $ @{term BinOr}
  | markup DSL-Tokens.Xor = @{term BinaryExpr} $ @{term BinXor}
  | markup DSL-Tokens.ShortCircuitOr = @{term BinaryExpr} $ @{term Bin-
ShortCircuitOr}
  | markup DSL-Tokens.Abs = @{term UnaryExpr} $ @{term UnaryAbs}
  | markup DSL-Tokens.Less = @{term BinaryExpr} $ @{term BinIntegerLessThan}
  | markup DSL-Tokens.Equals = @{term BinaryExpr} $ @{term BinIntegerEquals}
  | markup DSL-Tokens.Not = @{term UnaryExpr} $ @{term UnaryNot}
  | markup DSL-Tokens.Negate = @{term UnaryExpr} $ @{term UnaryNeg}
  | markup DSL-Tokens.LogicNegate = @{term UnaryExpr} $ @{term UnaryLog-
icNegation}
  | markup DSL-Tokens.LeftShift = @{term BinaryExpr} $ @{term BinLeftShift}
  | markup DSL-Tokens.RightShift = @{term BinaryExpr} $ @{term BinRightShift}
  | markup DSL-Tokens.UnsignedRightShift = @{term BinaryExpr} $ @{term Bin-
URightShift}
  | markup DSL-Tokens.Conditional = @{term ConditionalExpr}
  | markup DSL-Tokens.Constant = @{term ConstantExpr}
  | markup DSL-Tokens.TrueConstant = @{term ConstantExpr (IntVal 32 1)}
  | markup DSL-Tokens.FalseConstant = @{term ConstantExpr (IntVal 32 0)}
end
structure IRExprMarkup = DSL-Markup(IRExprTranslator);
>

```

*ir expression translation*

```

syntax -expandExpr :: term  $\Rightarrow$  term (exp[-])
parse-translation < [( @{syntax-const -expandExpr} , IREx-
prMarkup.markup-expr []) >

```

*ir expression example*

```

value exp[( $e_1 < e_2$ ) ?  $e_1 : e_2$ ]

ConditionalExpr (BinaryExpr BinIntegerLessThan  $e_1 e_2$ )  $e_1 e_2$ 

```

### 8.1.2 Value Markup

```

ML <
structure IntValTranslator : DSL-TRANSLATION =
struct
fun markup DSL-Tokens.Add = @{term intval-add}
  | markup DSL-Tokens.Sub = @{term intval-sub}
  | markup DSL-Tokens.Mul = @{term intval-mul}
  | markup DSL-Tokens.And = @{term intval-and}
  | markup DSL-Tokens.Or = @{term intval-or}
  | markup DSL-Tokens.ShortCircuitOr = @{term intval-short-circuit-or}
  | markup DSL-Tokens.Xor = @{term intval-xor}
  | markup DSL-Tokens.Abs = @{term intval-abs}
  | markup DSL-Tokens.Less = @{term intval-less-than}
  | markup DSL-Tokens.Equals = @{term intval-equals}
  | markup DSL-Tokens.Not = @{term intval-not}
  | markup DSL-Tokens.Negate = @{term intval-negate}
  | markup DSL-Tokens.LogicNegate = @{term intval-logic-negation}
  | markup DSL-Tokens.LeftShift = @{term intval-left-shift}
  | markup DSL-Tokens.RightShift = @{term intval-right-shift}
  | markup DSL-Tokens.UnsignedRightShift = @{term intval-uright-shift}
  | markup DSL-Tokens.Conditional = @{term intval-conditional}
  | markup DSL-Tokens.Constant = @{term IntVal 32}
  | markup DSL-Tokens.TrueConstant = @{term IntVal 32 1}
  | markup DSL-Tokens.FalseConstant = @{term IntVal 32 0}
end
structure IntValMarkup = DSL-Markup(IntValTranslator);
>

```

*value expression translation*

**syntax**  $-expandIntVal :: term \Rightarrow term (val[-])$   
**parse-translation** < [( @{syntax-const  $-expandIntVal$ } , IntVal-Markup.markup-expr []) ] >

*value expression example*

**value**  $val[(e_1 < e_2) ? e_1 : e_2]$   
*intval-conditional (intval-less-than  $e_1$   $e_2$ )  $e_1$   $e_2$*

### 8.1.3 Word Markup

```

ML <
structure WordTranslator : DSL-TRANSLATION =
struct
fun markup DSL-Tokens.Add = @{term plus}
  | markup DSL-Tokens.Sub = @{term minus}
  | markup DSL-Tokens.Mul = @{term times}

```

```

| markup DSL-Tokens.And = @{term Bit-Operations.semiring-bit-operations-class.and}
| markup DSL-Tokens.Or = @{term or}
| markup DSL-Tokens.Xor = @{term xor}
| markup DSL-Tokens.Abs = @{term abs}
| markup DSL-Tokens.Less = @{term less}
| markup DSL-Tokens.Equals = @{term HOL.eq}
| markup DSL-Tokens.Not = @{term not}
| markup DSL-Tokens.Negate = @{term uminus}
| markup DSL-Tokens.LogicNegate = @{term logic-negate}
| markup DSL-Tokens.LeftShift = @{term shiftl}
| markup DSL-Tokens.RightShift = @{term signed-shiftr}
| markup DSL-Tokens.UnsignedRightShift = @{term shiftr}
| markup DSL-Tokens.Constant = @{term word}
| markup DSL-Tokens.TrueConstant = @{term 1}
| markup DSL-Tokens.FalseConstant = @{term 0}
end
structure WordMarkup = DSL-Markup(WordTranslator);
>

```

#### word expression translation

```

syntax -expandWord :: term ⇒ term (bin[-])
parse-translation < [( @{syntax-const -expandWord} , Word-
Markup.markup-expr []) >

```

#### word expression example

```

value bin[x & y | z]

intval-conditional (intval-less-than e1 e2) e1 e2

```

```

value bin[¬x]
value val[¬x]
value exp[¬x]

```

```

value bin[!x]
value val[!x]
value exp[!x]

```

```

value bin[¬x]
value val[¬x]
value exp[¬x]

```

```

value bin[~x]
value val[~x]
value exp[~x]

```

```

value ~x

```

**end**

## 8.2 Optimization Phases

```
theory Phase
  imports Main
begin
```

```
ML-file map.ML
ML-file phase.ML
```

**end**

## 8.3 Canonicalization DSL

```
theory Canonicalization
  imports
    Markup
    Phase
    HOL-Eisbach.Eisbach
  keywords
    phase :: thy-decl and
    terminating :: quasi-command and
    print-phases :: diag and
    export-phases :: thy-decl and
    optimization :: thy-goal-defn
begin
```

**print-methods**

```
ML <
datatype 'a Rewrite =
  Transform of 'a * 'a |
  Conditional of 'a * 'a * term |
  Sequential of 'a Rewrite * 'a Rewrite |
  Transitive of 'a Rewrite
```

```
type rewrite = {
  name: binding,
  rewrite: term Rewrite,
  proofs: thm list,
  code: thm list,
  source: term
}
```

```
structure RewriteRule : Rule =
struct
type T = rewrite;
```

(\*

```

fun pretty-rewrite ctxt (Transform (from, to)) =
  Pretty.block [
    Syntax.pretty-term ctxt from,
    Pretty.str  $\mapsto$  ,
    Syntax.pretty-term ctxt to
  ]
| pretty-rewrite ctxt (Conditional (from, to, cond)) =
  Pretty.block [
    Syntax.pretty-term ctxt from,
    Pretty.str  $\mapsto$  ,
    Syntax.pretty-term ctxt to,
    Pretty.str when ,
    Syntax.pretty-term ctxt cond
  ]
| pretty-rewrite - - = Pretty.str not implemented*)

fun pretty-thm ctxt thm =
  (Proof-Context.pretty-fact ctxt (, [thm]))

fun pretty ctxt obligations t =
  let
    val is-skipped = Thm-Deps.has-skip-proof (#proofs t);

    val warning = (if is-skipped
      then [Pretty.str (proof skipped), Pretty.brk 0]
      else []);

    val obligations = (if obligations
      then [Pretty.big-list
        obligations:
        (map (pretty-thm ctxt) (#proofs t)),
        Pretty.brk 0]
      else []);

    fun pretty-bind binding =
      Pretty.markup
        (Position.markup (Binding.pos-of binding) Markup.position)
        [Pretty.str (Binding.name-of binding)];

    in
      Pretty.block ([
        pretty-bind (#name t), Pretty.str : ,
        Syntax.pretty-term ctxt (#source t), Pretty.fbrk
      ] @ obligations @ warning)
    end
  end

structure RewritePhase = DSL-Phase(RewriteRule);

```

```

val - =
  Outer-Syntax.command command-keyword⟨phase⟩ enter an optimization phase
  (Parse.binding --| Parse.$$$ terminating -- Parse.const --| Parse.begin
   >> (Toplevel.begin-main-target true o RewritePhase.setup));

fun print-phases print-obligations ctxt =
  let
    val thy = Proof-Context.theory-of ctxt;
    fun print phase = RewritePhase.pretty print-obligations phase ctxt
  in
    map print (RewritePhase.phases thy)
  end

fun print-optimizations print-obligations thy =
  print-phases print-obligations thy |> Pretty.writeln-chunks

val - =
  Outer-Syntax.command command-keyword⟨print-phases⟩
  print debug information for optimizations
  (Parse.opt-bang >>
   (fn b => Toplevel.keep ((print-optimizations b) o Toplevel.context-of)));

fun export-phases thy name =
  let
    val state = Toplevel.theory-tolevel thy;
    val ctxt = Toplevel.context-of state;
    val content = Pretty.string-of (Pretty.chunks (print-phases false ctxt));
    val cleaned = YXML.content-of content;

    val filename = Path.explode (name^".rules");
    val directory = Path.explode optimizations;
    val path = Path.binding (
      Path.append directory filename,
      Position.none);
    val thy' = thy |> Generated-Files.add-files (path, content);

    val - = Export.export thy' path [YXML.parse cleaned];

    val - = writeln (Export.message thy' (Path.basic optimizations));
  in
    thy'
  end

val - =
  Outer-Syntax.command command-keyword⟨export-phases⟩
  export information about encoded optimizations
  (Parse.text >>
   (fn name => Toplevel.theory (fn state => export-phases state name)))

```



>

ML-file *rewrites.ML*

### 8.3.1 Semantic Preservation Obligation

```
fun rewrite-preservation :: IRExp Rewrite  $\Rightarrow$  bool where
  rewrite-preservation (Transform x y) = (y  $\leq$  x) |
  rewrite-preservation (Conditional x y cond) = (cond  $\longrightarrow$  (y  $\leq$  x)) |
  rewrite-preservation (Sequential x y) = (rewrite-preservation x  $\wedge$  rewrite-preservation
y) |
  rewrite-preservation (Transitive x) = rewrite-preservation x
```

### 8.3.2 Termination Obligation

```
fun rewrite-termination :: IRExp Rewrite  $\Rightarrow$  (IRExp  $\Rightarrow$  nat)  $\Rightarrow$  bool where
  rewrite-termination (Transform x y) trm = (trm x > trm y) |
  rewrite-termination (Conditional x y cond) trm = (cond  $\longrightarrow$  (trm x > trm y)) |
  rewrite-termination (Sequential x y) trm = (rewrite-termination x trm  $\wedge$  rewrite-termination
y trm) |
  rewrite-termination (Transitive x) trm = rewrite-termination x trm
```

```
fun intval :: Value Rewrite  $\Rightarrow$  bool where
  intval (Transform x y) = (x  $\neq$  UndefVal  $\wedge$  y  $\neq$  UndefVal  $\longrightarrow$  x = y) |
  intval (Conditional x y cond) = (cond  $\longrightarrow$  (x = y)) |
  intval (Sequential x y) = (intval x  $\wedge$  intval y) |
  intval (Transitive x) = intval x
```

### 8.3.3 Standard Termination Measure

```
fun size :: IRExp  $\Rightarrow$  nat where
  unary-size:
  size (UnaryExpr op e) = (size e) + 2 |

  bin-const-size:
  size (BinaryExpr op x (ConstantExpr cy)) = (size x) + 2 |
  bin-size:
  size (BinaryExpr op x y) = (size x) + (size y) + 2 |
  cond-size:
  size (ConditionalExpr cond t f) = (size cond) + (size t) + (size f) + 2 |
  const-size:
  size (ConstantExpr c) = 1 |
  param-size:
  size (ParameterExpr ind s) = 2 |
  leaf-size:
  size (LeafExpr nid s) = 2 |
  size (ConstantVar c) = 2 |
  size (VariableExpr x s) = 2
```

### 8.3.4 Automated Tactics

**named-theorems** *size-simps size simplification rules*

```
method unfold-optimization =
  (unfold rewrite-preservation.simps, unfold rewrite-termination.simps,
   unfold intval.simps,
   rule conjE, simp, simp del: le-expr-def, force?)
| (unfold rewrite-preservation.simps, unfold rewrite-termination.simps,
   rule conjE, simp, simp del: le-expr-def, force?)
```

```
method unfold-size =
  (((unfold size.simps, simp add: size-simps del: le-expr-def)?
   ; (simp add: size-simps del: le-expr-def)?
   ; (auto simp: size-simps)?
   ; (unfold size.simps)?)[1])
```

**print-methods**

```
ML <
  structure System : RewriteSystem =
    struct
      val preservation = @{const rewrite-preservation};
      val termination = @{const rewrite-termination};
      val intval = @{const intval};
    end

  structure DSL = DSL-Rewrites(System);

  val - =
    Outer-Syntax.local-theory-to-proof command-keyword <optimization>
      define an optimization and open proof obligation
      (Parse-Spec.thm-name : -- Parse.term
       >> DSL.rewrite-cmd);
>

end
```

## 9 Canonicalization Optimizations

```
theory Common
imports
  OptimizationDSL.Canonicalization
  Semantics.IRTreeEvalThms
begin

lemma size-pos[size-simps]: 0 < size y
apply (induction y; auto?)
```

**by** (*smt* (*z3*) *add-2-eq-Suc'* *add-is-0* *not-gr0* *size.elims* *size.simps*(12) *size.simps*(13) *size.simps*(14) *size.simps*(15) *zero-neq-numeral* *zero-neq-one*)

**lemma** *size-non-add*[*size-simps*]: *size* (*BinaryExpr* *op* *a* *b*) = *size* *a* + *size* *b* + 2  
 $\longleftrightarrow \neg(\text{is-ConstantExpr } b)$   
**by** (*induction* *b*; *induction* *op*; *auto* *simp*: *is-ConstantExpr-def*)

**lemma** *size-non-const*[*size-simps*]:  
 $\neg \text{is-ConstantExpr } y \implies 1 < \text{size } y$   
**using** *size-pos* **apply** (*induction* *y*; *auto*)  
**by** (*metis* *Suc-lessI* *add-is-1* *is-ConstantExpr-def* *le-less* *linorder-not-le* *n-not-Suc-n* *numeral-2-eq-2* *pos2* *size.simps*(2) *size-non-add*)

**lemma** *size-binary-const*[*size-simps*]:  
 $\text{size} (\text{BinaryExpr } \text{op } a \ b) = \text{size } a + 2 \longleftrightarrow (\text{is-ConstantExpr } b)$   
**by** (*induction* *b*; *auto* *simp*: *is-ConstantExpr-def* *size-pos*)

**lemma** *size-flip-binary*[*size-simps*]:  
 $\neg(\text{is-ConstantExpr } y) \longrightarrow \text{size} (\text{BinaryExpr } \text{op} (\text{ConstantExpr } x) \ y) > \text{size} (\text{BinaryExpr } \text{op } y (\text{ConstantExpr } x))$   
**by** (*metis* *add-Suc* *not-less-eq* *order-less-asym* *plus-1-eq-Suc* *size.simps*(11) *size.simps*(2) *size-non-add*)

**lemma** *size-binary-lhs-a*[*size-simps*]:  
 $\text{size} (\text{BinaryExpr } \text{op} (\text{BinaryExpr } \text{op}' \ a \ b) \ c) > \text{size } a$   
**by** (*metis* *add-lessD1* *less-add-same-cancel1* *pos2* *size-binary-const* *size-non-add*)

**lemma** *size-binary-lhs-b*[*size-simps*]:  
 $\text{size} (\text{BinaryExpr } \text{op} (\text{BinaryExpr } \text{op}' \ a \ b) \ c) > \text{size } b$   
**by** (*metis* *IRExpr.disc*(42) *One-nat-def* *add.left-commute* *add.right-neutral* *is-ConstantExpr-def* *less-add-Suc2* *numeral-2-eq-2* *plus-1-eq-Suc* *size.simps*(11) *size-binary-const* *size-non-add* *size-non-const* *trans-less-add1*)

**lemma** *size-binary-lhs-c*[*size-simps*]:  
 $\text{size} (\text{BinaryExpr } \text{op} (\text{BinaryExpr } \text{op}' \ a \ b) \ c) > \text{size } c$   
**by** (*metis* *IRExpr.disc*(42) *add.left-commute* *add.right-neutral* *is-ConstantExpr-def* *less-Suc-eq* *numeral-2-eq-2* *plus-1-eq-Suc* *size.simps*(11) *size-non-add* *size-non-const* *trans-less-add2*)

**lemma** *size-binary-rhs-a*[*size-simps*]:  
 $\text{size} (\text{BinaryExpr } \text{op } c (\text{BinaryExpr } \text{op}' \ a \ b)) > \text{size } a$   
**by** (*smt* (*verit*, *best*) *less-Suc-eq* *less-add-Suc2* *less-add-same-cancel1* *linorder-neqE-nat* *not-add-less1* *order-less-trans* *pos2* *size.simps*(4) *size-binary-const* *size-non-add*)

**lemma** *size-binary-rhs-b*[*size-simps*]:  
 $\text{size} (\text{BinaryExpr } \text{op } c (\text{BinaryExpr } \text{op}' \ a \ b)) > \text{size } b$   
**by** (*metis* *add.left-commute* *add.right-neutral* *is-ConstantExpr-def* *lessI* *numeral-2-eq-2* *plus-1-eq-Suc* *size.simps*(11) *size.simps*(4) *size-non-add* *trans-less-add2*)

```

lemma size-binary-rhs-c[size-simps]:
  size (BinaryExpr op c (BinaryExpr op' a b)) > size c
by simp

lemma size-binary-lhs[size-simps]:
  size (BinaryExpr op x y) > size x
by (metis One-nat-def Suc-eq-plus1 add-Suc-right less-add-Suc1 numeral-2-eq-2
size-binary-const size-non-add)

lemma size-binary-rhs[size-simps]:
  size (BinaryExpr op x y) > size y
by (metis IRExpr.disc(42) add-strict-increasing is-ConstantExpr-def linorder-not-le
not-add-less1 size.simps(11) size-non-add size-non-const size-pos)

lemmas arith[size-simps] = Suc-leI add-strict-increasing order-less-trans trans-less-add2

definition well-formed-equal :: Value  $\Rightarrow$  Value  $\Rightarrow$  bool
  (infix  $\approx$  50) where
    well-formed-equal v1 v2 = (v1  $\neq$  UndefVal  $\longrightarrow$  v1 = v2)

lemma well-formed-equal-defn [simp]:
  well-formed-equal v1 v2 = (v1  $\neq$  UndefVal  $\longrightarrow$  v1 = v2)
  unfolding well-formed-equal-def by simp

end

## 9.1 AddNode Phase

theory AddPhase
  imports
    Common
begin

phase AddNode
  terminating size
begin

lemma binadd-commute:
  assumes bin-eval BinAdd x y  $\neq$  UndefVal
  shows bin-eval BinAdd x y = bin-eval BinAdd y x
  using assms intval-add-sym by simp

optimization AddShiftConstantRight: ((const v) + y)  $\mapsto$  y + (const v) when
   $\neg$ (is-ConstantExpr y)
  using size-non-const
  apply (metis add-2-eq-Suc' less-Suc-eq plus-1-eq-Suc size.simps(11) size-non-add)

```

```

unfolding le-expr-def
apply (rule impI)
subgoal premises 1
  apply (rule allI impI)+

  subgoal premises 2 for m p va
    apply (rule BinaryExprE[OF 2])
  subgoal premises 3 for x ya
    apply (rule BinaryExpr)
    using 3 apply simp
    using 3 apply simp
    using 3 binadd-commute apply auto
  done
done
done
done

optimization AddShiftConstantRight2:  $((\text{const } v) + y) \mapsto y + (\text{const } v)$  when
 $\neg(\text{is-ConstantExpr } y)$ 
  unfolding le-expr-def
  apply (auto simp: intval-add-sym)

using size-non-const
by (metis add-2-eq-Suc' lessI plus-1-eq-Suc size.simps(11) size-non-add)

lemma is-neutral-0 [simp]:
  assumes 1: intval-add (IntVal b x) (IntVal b 0)  $\neq$  UndefVal
  shows intval-add (IntVal b x) (IntVal b 0) = (new-int b x)
  using 1 by auto

optimization AddNeutral:  $(e + (\text{const } (\text{IntVal } 32\ 0))) \mapsto e$ 
  unfolding le-expr-def apply auto
  using is-neutral-0 eval-unused-bits-zero
  by (smt (verit) add-cancel-left-right intval-add.elims val-to-bool.simps(1))

ML-val  $\langle @\{term \langle x = y \rangle\} \rangle$ 

lemma NeutralLeftSubVal:
  assumes e1 = new-int b ival
  shows val[(e1 - e2) + e2]  $\approx$  e1
  apply simp using assms by (cases e1; cases e2; auto)

```

**optimization** *RedundantSubAdd*:  $((e_1 - e_2) + e_2) \mapsto e_1$   
**apply** *auto using eval-unused-bits-zero NeutralLeftSubVal*  
**unfolding** *well-formed-equal-defn*  
**by** (*smt (verit) evalDet intval-sub.elims new-int.elims*)

**lemma** *allE2*:  $(\forall x y. P x y) \implies (P a b \implies R) \implies R$   
**by** *simp*

**lemma** *just-goal2*:  
**assumes** *1*:  $(\forall a b. (intval\text{-}add (intval\text{-}sub a b) b \neq UndefinedVal \wedge a \neq UndefinedVal$   
 $\longrightarrow$   
 $intval\text{-}add (intval\text{-}sub a b) b = a))$   
**shows**  $(BinaryExpr\ BinAdd (BinaryExpr\ BinSub\ e_1\ e_2)\ e_2) \geq e_1$   
**unfolding** *le-expr-def unfold-binary bin-eval.simps*  
**by** (*metis 1 evalDet evaltree-not-undef*)

**optimization** *RedundantSubAdd2*:  $e_2 + (e_1 - e_2) \mapsto e_1$   
**apply** (*metis add.commute add-less-cancel-right less-add-Suc2 plus-1-eq-Suc size-binary-const*  
*size-non-add trans-less-add2*)  
**by** (*smt (verit, del-insts) BinaryExpr BinaryExprE RedundantSubAdd(1) bi-*  
*nadd-commute le-expr-def rewrite-preservation.simps(1)*)

**lemma** *AddToSubHelperLowLevel*:  
**shows**  $intval\text{-}add (intval\text{-}negate\ e)\ y = intval\text{-}sub\ y\ e$  (*is ?x = ?y*)  
**by** (*induction y; induction e; auto*)

**print-phases**

**lemma** *val-redundant-add-sub*:  
**assumes**  $a = new\text{-}int\ bb\ ival$   
**assumes**  $val[b + a] \neq UndefinedVal$   
**shows**  $val[(b + a) - b] = a$   
**using** *assms apply (cases a; cases b; auto)*  
**by** *presburger*

**lemma** *val-add-right-negate-to-sub*:

```

assumes  $val[x + e] \neq \text{UndefVal}$ 
shows  $val[x + (-e)] = val[x - e]$ 
using assms by (cases x; cases e; auto)

```

```

lemma exp-add-left-negate-to-sub:
 $exp[-e + y] \geq exp[y - e]$ 
apply (cases e; cases y; auto)
using AddToSubHelperLowLevel by auto

```

Optimisations

```

optimization RedundantAddSub:  $(b + a) - b \mapsto a$ 
apply auto
by (smt (verit) evalDet intval-add.elims new-int.elims val-redundant-add-sub
eval-unused-bits-zero)

```

```

optimization AddRightNegateToSub:  $x + -e \mapsto x - e$ 
apply (metis Nat.add-0-right add-2-eq-Suc' add-less-mono1 add-mono-thms-linordered-field(2)
less-SucI not-less-less-Suc-eq size-binary-const size-non-add size-pos)
using AddToSubHelperLowLevel intval-add-sym by auto

```

```

optimization AddLeftNegateToSub:  $-e + y \mapsto y - e$ 
defer
using exp-add-left-negate-to-sub apply blast
by (smt (verit, best) One-nat-def add.commute add-Suc-right is-ConstantExpr-def
less-add-Suc2
numeral-2-eq-2 plus-1-eq-Suc size.simps(1) size.simps(11) size-binary-const
size-non-add)

```

**end**

**end**

## 9.2 AndNode Phase

```

theory AndPhase
imports
  Common
  Proofs.StampEvalThms
begin

```

```

context stamp-mask
begin

```

```

lemma AndRightFallthrough: (((and (not ( $\downarrow$  x)) ( $\uparrow$  y)) = 0))  $\longrightarrow$  exp[x & y]  $\geq$ 
exp[y]
  apply simp apply (rule impI; (rule allI)+)
  apply (rule impI)
  subgoal premises p for m p v
  proof –
    obtain xv where xv: [m, p]  $\vdash$  x  $\mapsto$  xv
    using p(2) by blast
    obtain yv where yv: [m, p]  $\vdash$  y  $\mapsto$  yv
    using p(2) by blast
    have v = val[xv & yv]
    using p(2) xv yv
    by (metis BinaryExprE bin-eval.simps(4) evalDet)
    then have v = yv
    using p(1) not-down-up-mask-and-zero-implies-zero
    by (smt (verit) eval-unused-bits-zero intval-and.elims new-int.elims new-int-bin.elims
p(2) unfold-binary xv yv)
    then show ?thesis using yv by simp
  qed
done

```

```

lemma AndLeftFallthrough: (((and (not ( $\downarrow$  y)) ( $\uparrow$  x)) = 0))  $\longrightarrow$  exp[x & y]  $\geq$ 
exp[x]
  apply simp apply (rule impI; (rule allI)+)
  apply (rule impI)
  subgoal premises p for m p v
  proof –
    obtain xv where xv: [m, p]  $\vdash$  x  $\mapsto$  xv
    using p(2) by blast
    obtain yv where yv: [m, p]  $\vdash$  y  $\mapsto$  yv
    using p(2) by blast
    have v = val[xv & yv]
    using p(2) xv yv
    by (metis BinaryExprE bin-eval.simps(4) evalDet)
    then have v = xv
    using p(1) not-down-up-mask-and-zero-implies-zero
    by (smt (verit) and commute eval-unused-bits-zero intval-and.elims new-int.simps
new-int-bin.simps p(2) unfold-binary xv yv)
    then show ?thesis using xv by simp
  qed
done
end

```

```

phase AndNode
  terminating size
begin

```

```

lemma bin-and-nots:

```



$(\sim x \ \& \ \sim y) = (\sim (x \mid y))$   
**by** *simp*

**lemma** *bin-and-neutral*:  
 $(x \ \& \ \sim False) = x$   
**by** *simp*

**lemma** *val-and-equal*:  
**assumes**  $x = \text{new-int } b \ v$   
**and**  $\text{val}[x \ \& \ x] \neq \text{UndefVal}$   
**shows**  $\text{val}[x \ \& \ x] = x$   
**using** *assms* **by** (*cases x; auto*)

**lemma** *val-and-nots*:  
 $\text{val}[\sim x \ \& \ \sim y] = \text{val}[\sim (x \mid y)]$   
**apply** (*cases x; cases y; auto*) **by** (*simp add: take-bit-not-take-bit*)

**lemma** *val-and-neutral*:  
**assumes**  $x = \text{new-int } b \ v$   
**and**  $\text{val}[x \ \& \ \sim (\text{new-int } b' \ 0)] \neq \text{UndefVal}$   
**shows**  $\text{val}[x \ \& \ \sim (\text{new-int } b' \ 0)] = x$   
**using** *assms* **apply** (*cases x; auto*) **apply** (*simp add: take-bit-eq-mask*)  
**by** *presburger*

**lemma** *val-and-zero*:  
**assumes**  $x = \text{new-int } b \ v$   
**shows**  $\text{val}[x \ \& \ (\text{IntVal } b \ 0)] = \text{IntVal } b \ 0$   
**using** *assms* **by** (*cases x; auto*)

**lemma** *exp-and-equal*:  
 $\text{exp}[x \ \& \ x] \geq \text{exp}[x]$   
**apply** *auto* **using** *val-and-equal eval-unused-bits-zero*  
**by** (*smt (verit) evalDet intval-and.elims new-int.elims*)

**lemma** *exp-and-nots*:  
 $\text{exp}[\sim x \ \& \ \sim y] \geq \text{exp}[\sim (x \mid y)]$   
**apply** (*cases x; cases y; auto*) **using** *val-and-nots*  
**by** *fastforce+*

**lemma** *exp-sign-extend*:  
**assumes**  $e = (1 << \text{In}) - 1$   
**shows**  $\text{BinaryExpr } \text{BinAnd } (\text{UnaryExpr } (\text{UnarySignExtend } \text{In } \text{Out}) \ x)$   
 $\quad (\text{ConstantExpr } (\text{new-int } b \ e))$

```

      ≥ (UnaryExpr (UnaryZeroExtend In Out) x)
apply auto
subgoal premises p for m p va
proof -
  obtain va where va: [m,p] ⊢ x ↦ va
  using p(2) by auto
  then have va ≠ UndefVal
  by (simp add: evaltree-not-undef)
  then have 1: intval-and (intval-sign-extend In Out va) (IntVal b (take-bit b
e)) ≠ UndefVal
  using evalDet p(1) p(2) va by blast
  then have 2: intval-sign-extend In Out va ≠ UndefVal
  by auto
  then have 21: (0::nat) < b
  using eval-bits-1-64 p(4) by blast
  then have 3: b ⊆ (64::nat)
  using eval-bits-1-64 p(4) by blast
  then have 4: - ((2::int) ^ b div (2::int)) ⊆ sint (signed-take-bit (b - Suc
(0::nat)) (take-bit b e))
  by (simp add: 21 int-power-div-base signed-take-bit-int-greater-eq-minus-exp-word)
  then have 5: sint (signed-take-bit (b - Suc (0::nat)) (take-bit b e)) < (2::int)
^ b div (2::int)
  by (simp add: 21 3 Suc-le-lessD int-power-div-base signed-take-bit-int-less-exp-word)
  then have 6: [m,p] ⊢ UnaryExpr (UnaryZeroExtend In Out)
    x ↦ intval-and (intval-sign-extend In Out va) (IntVal b (take-bit b e))
  apply (cases va; simp)
  apply (simp add: ⟨(va::Value) ≠ UndefVal⟩) defer
  subgoal premises p for x3
  proof -
    have va = ObjRef x3
    using p(1) by auto
    then have sint (signed-take-bit (b - Suc (0::nat)) (take-bit b e)) <
(2::int) ^ b div (2::int)
    by (simp add: 5)
    then show ?thesis
    using 2 intval-sign-extend.simps(3) p(1) by blast
  qed

subgoal premises p for x4
proof -
  have sg1: va = ObjStr x4
  using 2 p(1) by auto
  then have sint (signed-take-bit (b - Suc (0::nat)) (take-bit b e)) <
(2::int) ^ b div (2::int)
  by (simp add: 5)
  then show ?thesis
  using 1 sg1 by auto
qed

```

```

    subgoal premises p for x21 x22
    proof -
      have sgg1: va = IntVal x21 x22
      by (simp add: p(1))
      then have sgg2: sint (signed-take-bit (b - Suc (0::nat)) (take-bit b e))
    < (2::int) ^ b div (2::int)
      by (simp add: 5)
      then show ?thesis
      sorry
    qed
  done
then show ?thesis
  by (metis evalDet p(2) va)
qed
done

```

```

lemma val-and-commute[simp]:
  val[x & y] = val[y & x]
  apply (cases x; cases y; auto)
  by (simp add: word-bw-comms(1))

```

Optimisations

```

optimization AndEqual: x & x  $\mapsto$  x
  using exp-and-equal by blast

```

```

optimization AndShiftConstantRight: ((const x) & y)  $\mapsto$  y & (const x)
  when  $\neg$ (is-ConstantExpr y)
  using size-flip-binary by auto

```

```

optimization AndNots: ( $\sim$ x) & ( $\sim$ y)  $\mapsto$   $\sim$ (x | y)
  defer using exp-and-nots
  apply presburger
  by (metis add-2-eq-Suc' less-SucI less-add-Suc1 not-less-eq size-binary-const size-non-add)

```

```

optimization AndSignExtend: BinaryExpr BinAnd (UnaryExpr (UnarySignExtend
In Out) (x))

```

```

  (const (new-int b e))
 $\mapsto$  (UnaryExpr (UnaryZeroExtend In Out) (x))
  when (e = (1 << In) - 1)

```

```

  using exp-sign-extend by simp

```

```

optimization AndNeutral: (x &  $\sim$ (const (IntVal b 0)))  $\mapsto$  x

```

```

when (wf-stamp x ∧ stamp-expr x = IntegerStamp b lo hi)
apply auto
by (smt (verit) Value.sel(1) eval-unused-bits-zero intval-and.elims intval-word.simps

new-int.simps new-int-bin.simps take-bit-eq-mask)

```

```

optimization AndRightFallThrough: (x & y) ⟶ y
when (((and (not (IRExpr-down x)) (IRExpr-up y)) = 0))
by (simp add: IRExpr-down-def IRExpr-up-def)

```

```

optimization AndLeftFallThrough: (x & y) ⟶ x
when (((and (not (IRExpr-down y)) (IRExpr-up x)) = 0))
by (simp add: IRExpr-down-def IRExpr-up-def)

```

**end**

**end**

### 9.3 Experimental AndNode Phase

**theory** NewAnd

**imports**

Common

Graph.Long

**begin**

```

lemma bin-distribute-and-over-or:
bin[z & (x | y)] = bin[(z & x) | (z & y)]
by (smt (verit, best) bit-and-iff bit-eqI bit-or-iff)

```

```

lemma intval-distribute-and-over-or:
val[z & (x | y)] = val[(z & x) | (z & y)]
apply (cases x; cases y; cases z; auto)
using bin-distribute-and-over-or by blast+

```

```

lemma exp-distribute-and-over-or:
exp[z & (x | y)] ≥ exp[(z & x) | (z & y)]
apply simp using intval-distribute-and-over-or
using BinaryExpr bin-eval.simps(4,5)
using intval-or.simps(1) unfolding new-int-bin.simps new-int.simps apply auto
by (metis bin-eval.simps(4) bin-eval.simps(5) intval-or.simps(2) intval-or.simps(5))

```

```

lemma intval-and-commute:
val[x & y] = val[y & x]
by (cases x; cases y; auto simp: and.commute)

```

**lemma** *intval-or-commute*:  
 $val[x \mid y] = val[y \mid x]$   
**by** (*cases x; cases y; auto simp: or.commute*)

**lemma** *intval-xor-commute*:  
 $val[x \oplus y] = val[y \oplus x]$   
**by** (*cases x; cases y; auto simp: xor.commute*)

**lemma** *exp-and-commute*:  
 $exp[x \& z] \geq exp[z \& x]$   
**apply** *simp using intval-and-commute* **by** *auto*

**lemma** *exp-or-commute*:  
 $exp[x \mid y] \geq exp[y \mid x]$   
**apply** *simp using intval-or-commute* **by** *auto*

**lemma** *exp-xor-commute*:  
 $exp[x \oplus y] \geq exp[y \oplus x]$   
**apply** *simp using intval-xor-commute* **by** *auto*

**lemma** *bin-eliminate-y*:  
**assumes**  $bin[y \& z] = 0$   
**shows**  $bin[(x \mid y) \& z] = bin[x \& z]$   
**using** *assms*  
**by** (*simp add: and.commute bin-distribute-and-over-or*)

**lemma** *intval-eliminate-y*:  
**assumes**  $val[y \& z] = IntVal\ b\ 0$   
**shows**  $val[(x \mid y) \& z] = val[x \& z]$   
**using** *assms bin-eliminate-y* **by** (*cases x; cases y; cases z; auto*)

**lemma** *intval-and-associative*:  
 $val[(x \& y) \& z] = val[x \& (y \& z)]$   
**apply** (*cases x; cases y; cases z; auto*)  
**by** (*simp add: and.assoc*)+

**lemma** *intval-or-associative*:  
 $val[(x \mid y) \mid z] = val[x \mid (y \mid z)]$   
**apply** (*cases x; cases y; cases z; auto*)  
**by** (*simp add: or.assoc*)+

**lemma** *intval-xor-associative*:  
 $val[(x \oplus y) \oplus z] = val[x \oplus (y \oplus z)]$   
**apply** (*cases x; cases y; cases z; auto*)  
**by** (*simp add: xor.assoc*)+

**lemma** *exp-and-associative*:  
 $exp[(x \& y) \& z] \geq exp[x \& (y \& z)]$

**apply simp using intval-and-associative by fastforce**

**lemma exp-or-associative:**  
 $\text{exp}[(x \mid y) \mid z] \geq \text{exp}[x \mid (y \mid z)]$   
**apply simp using intval-or-associative by fastforce**

**lemma exp-xor-associative:**  
 $\text{exp}[(x \oplus y) \oplus z] \geq \text{exp}[x \oplus (y \oplus z)]$   
**apply simp using intval-xor-associative by fastforce**

**lemma intval-and-absorb-or:**  
**assumes**  $\exists b \ v. x = \text{new-int } b \ v$   
**assumes**  $\text{val}[x \ \& \ (x \mid y)] \neq \text{UndefVal}$   
**shows**  $\text{val}[x \ \& \ (x \mid y)] = \text{val}[x]$   
**using** *assms* **apply** (*cases x; cases y; auto*)  
**by** (*metis (mono-tags, lifting) intval-and.simps(5)*)

**lemma intval-or-absorb-and:**  
**assumes**  $\exists b \ v. x = \text{new-int } b \ v$   
**assumes**  $\text{val}[x \mid (x \ \& \ y)] \neq \text{UndefVal}$   
**shows**  $\text{val}[x \mid (x \ \& \ y)] = \text{val}[x]$   
**using** *assms* **apply** (*cases x; cases y; auto*)  
**by** (*metis (mono-tags, lifting) intval-or.simps(5)*)

**lemma exp-and-absorb-or:**  
 $\text{exp}[x \ \& \ (x \mid y)] \geq \text{exp}[x]$   
**apply auto using intval-and-absorb-or eval-unused-bits-zero**  
**by** (*smt (verit) evalDet intval-or.elims new-int.elims*)

**lemma exp-or-absorb-and:**  
 $\text{exp}[x \mid (x \ \& \ y)] \geq \text{exp}[x]$   
**apply auto using intval-or-absorb-and eval-unused-bits-zero**  
**by** (*smt (verit) evalDet intval-or.elims new-int.elims*)

**lemma**  
**assumes**  $y = 0$   
**shows**  $x + y = \text{or } x \ y$   
**using** *assms*  
**by** *simp*

**lemma no-overlap-or:**  
**assumes**  $\text{and } x \ y = 0$   
**shows**  $x + y = \text{or } x \ y$   
**using** *assms*  
**by** (*metis bit-and-iff bit-xor-iff disjunctive-add xor-self-eq*)

**context** *stamp-mask*  
**begin**

**lemma** *intval-up-and-zero-implies-zero*:  
**assumes**  $\text{and } (\uparrow x) (\uparrow y) = 0$   
**assumes**  $[m, p] \vdash x \mapsto xv$   
**assumes**  $[m, p] \vdash y \mapsto yv$   
**assumes**  $\text{val}[xv \ \& \ yv] \neq \text{UndefVal}$   
**shows**  $\exists b. \text{val}[xv \ \& \ yv] = \text{new-int } b \ 0$   
**using** *assms* **apply** (*cases xv*; *cases yv*; *auto*)  
**using** *up-mask-and-zero-implies-zero*  
**apply** (*smt (verit, best) take-bit-and take-bit-of-0*)  
**by** *presburger*

**lemma** *exp-eliminate-y*:  
 $\text{and } (\uparrow y) (\uparrow z) = 0 \longrightarrow \text{BinaryExpr BinAnd } (\text{BinaryExpr BinOr } x \ y) \ z \geq \text{BinaryExpr BinAnd } x \ z$   
**apply** *simp* **apply** (*rule impI*; *rule allI*; *rule allI*; *rule allI*)  
**subgoal** **premises** *p* **for** *m p v* **apply** (*rule impI*) **subgoal** **premises** *e*  
**proof** –  
**obtain** *xv* **where** *xv*:  $[m, p] \vdash x \mapsto xv$   
**using** *e* **by** *auto*  
**obtain** *yv* **where** *yv*:  $[m, p] \vdash y \mapsto yv$   
**using** *e* **by** *auto*  
**obtain** *zv* **where** *zv*:  $[m, p] \vdash z \mapsto zv$   
**using** *e* **by** *auto*  
**have** *lhs*:  $v = \text{val}[(xv \mid yv) \ \& \ zv]$   
**using** *xv yv zv*  
**by** (*smt (verit, best) BinaryExprE bin-eval.simps(4) bin-eval.simps(5) e evalDet*)  
**then** **have**  $v = \text{val}[(xv \ \& \ zv) \mid (yv \ \& \ zv)]$   
**by** (*simp add: intval-and-commute intval-distribute-and-over-or*)  
**also** **have**  $\exists b. \text{val}[yv \ \& \ zv] = \text{new-int } b \ 0$   
**using** *intval-up-and-zero-implies-zero*  
**by** (*metis calculation e intval-or.simps(5) p unfold-binary yv zv*)  
**ultimately** **have** *rhs*:  $v = \text{val}[xv \ \& \ zv]$   
**using** *intval-eliminate-y lhs* **by** *force*  
**from** *lhs rhs* **show** *?thesis*  
**by** (*metis BinaryExpr BinaryExprE bin-eval.simps(4) e xv zv*)  
**qed**  
**done**  
**done**

```

lemma leadingZeroBounds:
  fixes  $x :: 'a::len$  word
  assumes  $n = \text{numberOfLeadingZeros } x$ 
  shows  $0 \leq n \wedge n \leq \text{Nat.size } x$ 
  using assms unfolding numberOfLeadingZeros-def
  by (simp add: MaxOrNeg-def highestOneBit-def nat-le-iff)

lemma above-nth-not-set:
  fixes  $x :: \text{int64}$ 
  assumes  $n = 64 - \text{numberOfLeadingZeros } x$ 
  shows  $j > n \longrightarrow \neg(\text{bit } x \ j)$ 
  using assms unfolding numberOfLeadingZeros-def
  by (smt (verit, ccfv-SIG) highestOneBit-def int-nat-eq int-ops(6) less-imp-of-nat-less
max-set-bit size64 zerosAboveHighestOne)

no-notation LogicNegationNotation (!-)

lemma zero-horner:
  horner-sum of-bool 2 (map ( $\lambda x. \text{False}$ )  $xs$ ) = 0
  apply (induction xs) apply simp
  by force

lemma zero-map:
  assumes  $j \leq n$ 
  assumes  $\forall i. j \leq i \longrightarrow \neg(f \ i)$ 
  shows  $\text{map } f \ [0..<n] = \text{map } f \ [0..<j] @ \text{map } (\lambda x. \text{False}) \ [j..<n]$ 
  apply (insert assms)
  by (smt (verit, del-Insts) add-diff-inverse-nat atLeastLessThan-iff bot-nat-0.extremum
leD map-append map-eq-conv set-upt upt-add-eq-append)

lemma map-join-horner:
  assumes  $\text{map } f \ [0..<n] = \text{map } f \ [0..<j] @ \text{map } (\lambda x. \text{False}) \ [j..<n]$ 
  shows  $\text{horner-sum of-bool } (2::'a::len \text{ word}) \ (\text{map } f \ [0..<n]) = \text{horner-sum of-bool}$ 
  2 (map  $f$   $[0..<j]$ )
  proof –
    have  $\text{horner-sum of-bool } (2::'a::len \text{ word}) \ (\text{map } f \ [0..<n]) = \text{horner-sum of-bool}$ 
    2 (map  $f$   $[0..<j]$ ) +  $2 \wedge \text{length } [0..<j] * \text{horner-sum of-bool } 2 \ (\text{map } f \ [j..<n])$ 
    using horner-sum-append
    by (smt (verit) assms diff-le-self diff-zero le-add-same-cancel2 length-append
length-map length-upt map-append upt-add-eq-append)
    also have  $\dots = \text{horner-sum of-bool } 2 \ (\text{map } f \ [0..<j]) + 2 \wedge \text{length } [0..<j] * \text{horner-sum of-bool } 2 \ (\text{map } (\lambda x. \text{False}) \ [j..<n])$ 
    using assms
    by (metis calculation horner-sum-append length-map)
    also have  $\dots = \text{horner-sum of-bool } 2 \ (\text{map } f \ [0..<j])$ 
    using zero-horner
    using mult-not-zero by auto
  finally show ?thesis by simp

```



qed

**lemma** *split-horner*:

assumes  $j \leq n$   
 assumes  $\forall i. j \leq i \longrightarrow \neg(f\ i)$   
 shows *horner-sum of-bool* ( $2::'a::len\ word$ ) ( $map\ f\ [0..<n]$ ) = *horner-sum of-bool*  
 $2\ (map\ f\ [0..<j])$   
 apply (rule *map-join-horner*)  
 apply (rule *zero-map*)  
 using *assms* by *auto*

**lemma** *transfer-map*:

assumes  $\forall i. i < n \longrightarrow f\ i = f'\ i$   
 shows  $(map\ f\ [0..<n]) = (map\ f'\ [0..<n])$   
 using *assms* by *simp*

**lemma** *transfer-horner*:

assumes  $\forall i. i < n \longrightarrow f\ i = f'\ i$   
 shows *horner-sum of-bool* ( $2::'a::len\ word$ ) ( $map\ f\ [0..<n]$ ) = *horner-sum of-bool*  
 $2\ (map\ f'\ [0..<n])$   
 using *assms* using *transfer-map*  
 by (smt (verit, best))

**lemma** *L1*:

assumes  $n = 64 - \text{numberOfLeadingZeros } (\uparrow z)$   
 assumes  $[m, p] \vdash z \mapsto \text{IntVal } b\ zv$   
 shows  $and\ v\ zv = and\ (v\ mod\ 2^{\wedge}n)\ zv$   
**proof** –  
 have  $nle: n \leq 64$   
 using *assms*  
 using *diff-le-self* by *blast*  
 also have  $and\ v\ zv = \text{horner-sum of-bool } 2\ (map\ (bit\ (and\ v\ zv))\ [0..<64])$   
 using *horner-sum-bit-eq-take-bit size64*  
 by (metis *size-word.rep-eq take-bit-length-eq*)  
 also have  $\dots = \text{horner-sum of-bool } 2\ (map\ (\lambda i. bit\ (and\ v\ zv)\ i)\ [0..<64])$   
 by *blast*  
 also have  $\dots = \text{horner-sum of-bool } 2\ (map\ (\lambda i. ((bit\ v\ i) \wedge (bit\ zv\ i)))\ [0..<64])$   
 using *bit-and-iff* by *metis*  
 also have  $\dots = \text{horner-sum of-bool } 2\ (map\ (\lambda i. ((bit\ v\ i) \wedge (bit\ zv\ i)))\ [0..<n])$   
**proof** –  
 have  $\forall i. i \geq n \longrightarrow \neg(bit\ zv\ i)$   
 using *above-nth-not-set assms(1)*  
 using *assms(2) not-may-implies-false*  
 by (smt (verit, ccfv-SIG) *One-nat-def diff-less int-ops(6) leadingZerosAddHighestOne linorder-not-le nat-int-comparison(2) not-numeral-le-zero size64 zero-less-Suc zerosAboveHighestOne*)  
 then have  $\forall i. i \geq n \longrightarrow \neg((bit\ v\ i) \wedge (bit\ zv\ i))$   
 by *auto*  
 then show *?thesis* using *nle split-horner*

```

    by (metis (no-types, lifting))
  qed
  also have ... = horner-sum of-bool 2 (map (λi. ((bit (v mod 2n) i) ∧ (bit zv
i))) [0..n])
  proof -
    have ∀ i. i < n ⟶ bit (v mod 2n) i = bit v i
    by (metis bit-take-bit-iff take-bit-eq-mod)
    then have ∀ i. i < n ⟶ ((bit v i) ∧ (bit zv i)) = ((bit (v mod 2n) i) ∧ (bit
zv i))
    by force
    then show ?thesis
    by (rule transfer-horner)
  qed
  also have ... = horner-sum of-bool 2 (map (λi. ((bit (v mod 2n) i) ∧ (bit zv
i))) [0..64])
  proof -
    have ∀ i. i ≥ n ⟶ ¬(bit zv i)
    using above-nth-not-set assms(1)
    using assms(2) not-may-implies-false
    by (smt (verit, ccfv-SIG) One-nat-def diff-less int-ops(6) leadingZerosAddHigh-
estOne linorder-not-le nat-int-comparison(2) not-numeral-le-zero size64 zero-less-Suc
zerosAboveHighestOne)
    then show ?thesis
    by (metis (no-types, lifting) assms(1) diff-le-self split-horner)
  qed
  also have ... = horner-sum of-bool 2 (map (bit (and (v mod 2n) zv)) [0..64])
  by (meson bit-and-iff)
  also have ... = and (v mod 2n) zv
  using horner-sum-bit-eq-take-bit size64
  by (metis size-word.rep-eq take-bit-length-eq)
  finally show ?thesis
  using ⟨and (v::64 word) (zv::64 word) = horner-sum of-bool (2::64 word)
(map (bit (and v zv)) [0::nat..64::nat])⟩ ⟨horner-sum of-bool (2::64 word) (map
(λi::nat. bit ((v::64 word) mod (2::64 word) ^ (n::nat)) i ∧ bit (zv::64 word) i)
[0::nat..64::nat]) = horner-sum of-bool (2::64 word) (map (bit (and (v mod (2::64
word) ^ n) zv)) [0::nat..64::nat])⟩ ⟨horner-sum of-bool (2::64 word) (map (λi::nat.
bit ((v::64 word) mod (2::64 word) ^ (n::nat)) i ∧ bit (zv::64 word) i) [0::nat..n])
= horner-sum of-bool (2::64 word) (map (λi::nat. bit (v mod (2::64 word) ^ n) i
∧ bit zv i) [0::nat..64::nat])⟩ ⟨horner-sum of-bool (2::64 word) (map (λi::nat. bit
(v::64 word) i ∧ bit (zv::64 word) i) [0::nat..64::nat]) = horner-sum of-bool (2::64
word) (map (λi::nat. bit v i ∧ bit zv i) [0::nat..n::nat])⟩ ⟨horner-sum of-bool (2::64
word) (map (λi::nat. bit (v::64 word) i ∧ bit (zv::64 word) i) [0::nat..n::nat]) =
horner-sum of-bool (2::64 word) (map (λi::nat. bit (v mod (2::64 word) ^ n) i ∧
bit zv i) [0::nat..n])⟩ ⟨horner-sum of-bool (2::64 word) (map (bit (and ((v::64
word) mod (2::64 word) ^ (n::nat)) (zv::64 word))) [0::nat..64::nat]) = and (v
mod (2::64 word) ^ n) zv⟩ ⟨horner-sum of-bool (2::64 word) (map (bit (and (v::64
word) (zv::64 word))) [0::nat..64::nat]) = horner-sum of-bool (2::64 word) (map
(λi::nat. bit v i ∧ bit zv i) [0::nat..64::nat])⟩ by presburger
  qed

```

**lemma** *up-mask-upper-bound*:

**assumes**  $[m, p] \vdash x \mapsto \text{IntVal } b \ xv$

**shows**  $xv \leq (\uparrow x)$

**using** *assms*

**by** (*metis* (*no-types*, *lifting*) *and.idem* *and.right-neutral* *bit.conj-cancel-left* *bit.conj-disj-distrib*(1) *bit.double-compl* *ucast-id* *up-spec* *word-and-le1* *word-not-dist*(2))

**lemma** *L2*:

**assumes**  $\text{numberOfLeadingZeros } (\uparrow z) + \text{numberOfTrailingZeros } (\uparrow y) \geq 64$

**assumes**  $n = 64 - \text{numberOfLeadingZeros } (\uparrow z)$

**assumes**  $[m, p] \vdash z \mapsto \text{IntVal } b \ zv$

**assumes**  $[m, p] \vdash y \mapsto \text{IntVal } b \ yv$

**shows**  $yv \bmod 2^{\wedge n} = 0$

**proof** –

**have**  $yv \bmod 2^{\wedge n} = \text{horner-sum of-bool } 2 \ (\text{map } (\text{bit } yv) \ [0..<n])$

**by** (*simp* *add.horner-sum-bit-eq-take-bit* *take-bit-eq-mod*)

**also have**  $\dots \leq \text{horner-sum of-bool } 2 \ (\text{map } (\text{bit } (\uparrow y)) \ [0..<n])$

**using** *up-mask-upper-bound* *assms*(4)

**by** (*metis* (*no-types*, *opaque-lifting*) *and.right-neutral* *bit.conj-cancel-right* *bit.conj-disj-distrib*(1) *bit.double-compl* *horner-sum-bit-eq-take-bit* *take-bit-and* *ucast-id* *up-spec* *word-and-le1* *word-not-dist*(2))

**also have**  $\text{horner-sum of-bool } 2 \ (\text{map } (\text{bit } (\uparrow y)) \ [0..<n]) = \text{horner-sum of-bool } 2 \ (\text{map } (\lambda x. \text{False}) \ [0..<n])$

**proof** –

**have**  $\forall i < n. \neg(\text{bit } (\uparrow y) \ i)$

**using** *assms*(1,2) *zerosBelowLowestOne*

**by** (*metis* *add.commute* *add-diff-inverse-nat* *add-lessD1* *leD* *le-diff-conv* *numberOfTrailingZeros-def*)

**then show** *?thesis*

**by** (*metis* (*full-types*) *transfer-map*)

**qed**

**also have**  $\text{horner-sum of-bool } 2 \ (\text{map } (\lambda x. \text{False}) \ [0..<n]) = 0$

**using** *zero-horner*

**by** *blast*

**finally show** *?thesis*

**by** *auto*

**qed**

**thm-oracles** *L1 L2*

**lemma** *unfold-binary-width-add*:

**shows**  $([m, p] \vdash \text{BinaryExpr BinAdd } xe \ ye \mapsto \text{IntVal } b \ val) = (\exists \ x \ y.$

$([m, p] \vdash xe \mapsto \text{IntVal } b \ x) \wedge$

$([m, p] \vdash ye \mapsto \text{IntVal } b \ y) \wedge$

$(\text{IntVal } b \ val = \text{bin-eval BinAdd } (\text{IntVal } b \ x) \ (\text{IntVal } b \ y)) \wedge$

$(\text{IntVal } b \ val \neq \text{UndefVal})$

$) \text{ (is ?L = ?R)}$

**proof** (*intro* *iffI*)

```

assume  $\mathcal{I}$ : ?L
show ?R apply (rule evaltree.cases[OF  $\mathcal{I}$ ])
  apply force+ apply auto[1]
  apply (smt (verit) intval-add.elims intval-bits.simps)
  by blast
next
assume R: ?R
then obtain  $x\ y$  where  $[m,p] \vdash xe \mapsto \text{IntVal } b\ x$ 
  and  $[m,p] \vdash ye \mapsto \text{IntVal } b\ y$ 
  and  $\text{new-int } b\ \text{val} = \text{bin-eval BinAdd } (\text{IntVal } b\ x)\ (\text{IntVal } b\ y)$ 
  and  $\text{new-int } b\ \text{val} \neq \text{UndefVal}$ 
  by auto
then show ?L
  using R by blast
qed

lemma unfold-binary-width-and:
shows  $([m,p] \vdash \text{BinaryExpr BinAnd } xe\ ye \mapsto \text{IntVal } b\ \text{val}) = (\exists\ x\ y.$ 
   $(([m,p] \vdash xe \mapsto \text{IntVal } b\ x) \wedge$ 
   $([m,p] \vdash ye \mapsto \text{IntVal } b\ y) \wedge$ 
   $(\text{IntVal } b\ \text{val} = \text{bin-eval BinAnd } (\text{IntVal } b\ x)\ (\text{IntVal } b\ y)) \wedge$ 
   $(\text{IntVal } b\ \text{val} \neq \text{UndefVal})$ 
   $))$  (is ?L = ?R)
proof (intro iffI)
assume  $\mathcal{I}$ : ?L
show ?R apply (rule evaltree.cases[OF  $\mathcal{I}$ ])
  apply force+ apply auto[1] using intval-and.elims intval-bits.simps
  apply (smt (verit) new-int.simps new-int-bin.simps take-bit-and)
  by blast
next
assume R: ?R
then obtain  $x\ y$  where  $[m,p] \vdash xe \mapsto \text{IntVal } b\ x$ 
  and  $[m,p] \vdash ye \mapsto \text{IntVal } b\ y$ 
  and  $\text{new-int } b\ \text{val} = \text{bin-eval BinAnd } (\text{IntVal } b\ x)\ (\text{IntVal } b\ y)$ 
  and  $\text{new-int } b\ \text{val} \neq \text{UndefVal}$ 
  by auto
then show ?L
  using R by blast
qed

lemma mod-dist-over-add-right:
fixes  $a\ b\ c :: \text{int64}$ 
fixes  $n :: \text{nat}$ 
assumes 1:  $0 < n$ 
assumes 2:  $n < 64$ 
shows  $(a + b \bmod 2^n) \bmod 2^n = (a + b) \bmod 2^n$ 
using mod-dist-over-add
by (simp add: 1 2 add.commute)

```

**lemma** *numberOfLeadingZeros-range*:  
 $0 \leq \text{numberOfLeadingZeros } n \wedge \text{numberOfLeadingZeros } n \leq \text{Nat.size } n$   
**unfolding** *numberOfLeadingZeros-def highestOneBit-def* **using** *max-set-bit*  
**by** (*simp add: highestOneBit-def leadingZeroBounds numberOfLeadingZeros-def*)

**lemma** *improved-opt*:  
**assumes**  $\text{numberOfLeadingZeros } (\uparrow z) + \text{numberOfTrailingZeros } (\uparrow y) \geq 64$   
**shows**  $\text{exp}[(x + y) \& z] \geq \text{exp}[x \& z]$   
**apply** *simp apply ((rule allI)+; rule impI)*  
**subgoal premises** *eval* **for** *m p v*  
**proof** –  
**obtain** *n* **where** *n*:  $n = 64 - \text{numberOfLeadingZeros } (\uparrow z)$   
**by** *simp*  
**obtain** *b val* **where** *val*:  $[m, p] \vdash \text{exp}[(x + y) \& z] \mapsto \text{IntVal } b \text{ val}$   
**by** (*metis BinaryExprE bin-eval-new-int eval new-int.simps*)  
**then obtain** *xv yv* **where** *addv*:  $[m, p] \vdash \text{exp}[x + y] \mapsto \text{IntVal } b (xv + yv)$   
**apply** (*subst (asm) unfold-binary-width-and*) **by** (*metis add.right-neutral*)  
**then obtain** *yv* **where** *yv*:  $[m, p] \vdash y \mapsto \text{IntVal } b \text{ yv}$   
**apply** (*subst (asm) unfold-binary-width-add*) **by** *blast*  
**from** *addv* **obtain** *xv* **where** *xv*:  $[m, p] \vdash x \mapsto \text{IntVal } b \text{ xv}$   
**apply** (*subst (asm) unfold-binary-width-add*) **by** *blast*  
**from** *val* **obtain** *zv* **where** *zv*:  $[m, p] \vdash z \mapsto \text{IntVal } b \text{ zv}$   
**apply** (*subst (asm) unfold-binary-width-and*) **by** *blast*  
**have** *addv*:  $[m, p] \vdash \text{exp}[x + y] \mapsto \text{new-int } b (xv + yv)$   
**apply** (*rule evaltree.BinaryExpr*)  
**using** *xv* **apply** *simp*  
**using** *yv* **apply** *simp*  
**by** *simp+*  
**have** *lhs*:  $[m, p] \vdash \text{exp}[(x + y) \& z] \mapsto \text{new-int } b (\text{and } (xv + yv) \text{ zv})$   
**apply** (*rule evaltree.BinaryExpr*)  
**using** *addv* **apply** *simp*  
**using** *zv* **apply** *simp*  
**using** *addv* **apply** *auto[1]*  
**by** *simp*  
**have** *rhs*:  $[m, p] \vdash \text{exp}[x \& z] \mapsto \text{new-int } b (\text{and } xv \text{ zv})$   
**apply** (*rule evaltree.BinaryExpr*)  
**using** *xv* **apply** *simp*  
**using** *zv* **apply** *simp*  
**apply** *force*  
**by** *simp*  
**then show** *?thesis*  
**proof** (*cases numberOfLeadingZeros*  $(\uparrow z) > 0$ )  
**case** *True*  
**have** *n-bounds*:  $0 \leq n \wedge n < 64$   
**using** *diff-le-self n numberOfLeadingZeros-range*  
**by** (*simp add: True*)  
**have** *and*  $(xv + yv) \text{ zv} = \text{and } ((xv + yv) \bmod 2^n) \text{ zv}$   
**using** *L1 n zv* **by** *blast*  
**also have**  $\dots = \text{and } ((xv + (yv \bmod 2^n)) \bmod 2^n) \text{ zv}$

```

    using mod-dist-over-add-right n-bounds
    by (metis take-bit-0 take-bit-eq-mod zero-less-iff-neq-zero)
  also have ... = and (((xv mod 2^n) + (yv mod 2^n)) mod 2^n) zv
    by (metis bits-mod-by-1 mod-dist-over-add n-bounds order-le-imp-less-or-eq
power-0)
  also have ... = and ((xv mod 2^n) mod 2^n) zv
    using L2 n zv yv
    using assms by auto
  also have ... = and (xv mod 2^n) zv
    using mod-mod-trivial
  by (smt (verit, best) and.idem take-bit-eq-mask take-bit-eq-mod word-bw-assocs(1))
  also have ... = and xv zv
    using L1 n zv by metis
  finally show ?thesis
    using eval lhs rhs
    by (metis evalDet)
next
case False
then have numberOfLeadingZeros (↑z) = 0
  by simp
then have numberOfTrailingZeros (↑y) ≥ 64
  using assms(1)
  by fastforce
then have yv = 0
  using yv
  by (metis (no-types, lifting) L1 L2 add-diff-cancel-left' and.comm-neutral
and.idem bit.compl-zero bit.conj-cancel-right bit.conj-disj-distrib(1) bit.double-compl
less-imp-diff-less linorder-not-le word-not-dist(2))
  then show ?thesis
    by (metis add.right-neutral eval evalDet lhs rhs)
qed
qed
done

thm-oracles improved-opt

```

end

```

phase NewAnd
  terminating size
begin

```

```

optimization redundant-lhs-y-or: ((x | y) & z) ⟶ x & z
  when (((and (IExpr-up y) (IExpr-up z)) = 0))

```

```

apply (simp add: IRExp-up-def)
using simple-mask.exp-eliminate-y by blast

optimization redundant-lhs-x-or:  $((x \mid y) \& z) \mapsto y \& z$ 
                                 $\text{when } (((\text{and } (\text{IRExp-up } x) (\text{IRExp-up } z)) = 0))$ 
apply (simp add: IRExp-up-def)
using simple-mask.exp-eliminate-y
by (meson exp-or-commute mono-binary order-refl order-trans)

optimization redundant-rhs-y-or:  $(z \& (x \mid y)) \mapsto z \& x$ 
                                 $\text{when } (((\text{and } (\text{IRExp-up } y) (\text{IRExp-up } z)) = 0))$ 
apply (simp add: IRExp-up-def)
using simple-mask.exp-eliminate-y
by (meson exp-and-commute order.trans)

optimization redundant-rhs-x-or:  $(z \& (x \mid y)) \mapsto z \& y$ 
                                 $\text{when } (((\text{and } (\text{IRExp-up } x) (\text{IRExp-up } z)) = 0))$ 
apply (simp add: IRExp-up-def)
using simple-mask.exp-eliminate-y
by (meson dual-order.trans exp-and-commute exp-or-commute mono-binary order-refl)

end

end

```

## 9.4 ConditionalNode Phase

```

theory ConditionalPhase
imports
  Common
  Proofs.StampEvalThms
begin

phase ConditionalNode
  terminating size
begin

lemma negates:  $\exists v b. e = \text{IntVal } b \ v \wedge b > 0 \implies \text{val-to-bool } (\text{val}[e]) \longleftrightarrow$ 
 $\neg(\text{val-to-bool } (\text{val}[\text{!}e]))$ 
  unfolding intval-logic-negation.simps
  by (metis (mono-tags, lifting) intval-logic-negation.simps(1) logic-negate-def new-int.simps
    of-bool-eq(2) one-neq-zero take-bit-of-0 take-bit-of-1 val-to-bool.simps(1))

lemma negation-condition-intval:
  assumes  $e = \text{IntVal } b \ ie$ 
  assumes  $0 < b$ 

```

**shows**  $\text{val}[(!e) \text{ ? } x : y] = \text{val}[e \text{ ? } y : x]$   
**using** *assms* **by** (*cases* *e*; *auto simp: negates logic-negate-def*)

**lemma** *negation-preserve-eval*:  
**assumes**  $[m, p] \vdash \text{exp}[!e] \mapsto v$   
**shows**  $\exists v'. ([m, p] \vdash \text{exp}[e] \mapsto v') \wedge v = \text{val}[!v']$   
**using** *assms* **by** *auto*

**lemma** *negation-preserve-eval-intval*:  
**assumes**  $[m, p] \vdash \text{exp}[!e] \mapsto v$   
**shows**  $\exists v' b vv. ([m, p] \vdash \text{exp}[e] \mapsto v') \wedge v' = \text{IntVal } b \text{ } vv \wedge b > 0$   
**using** *assms*  
**by** (*metis eval-bits-1-64 intval-logic-negation.elims negation-preserve-eval unfold-unary*)

**optimization** *NegateConditionFlipBranches*:  $((!e) \text{ ? } x : y) \mapsto (e \text{ ? } y : x)$   
**apply** *simp* **using** *negation-condition-intval negation-preserve-eval-intval*  
**by** (*smt (z3) ConditionalExpr ConditionalExprE evalDet negates negation-preserve-eval*)

**optimization** *DefaultTrueBranch*:  $(\text{true} \text{ ? } x : y) \mapsto x$  .

**optimization** *DefaultFalseBranch*:  $(\text{false} \text{ ? } x : y) \mapsto y$  .

**optimization** *ConditionalEqualBranches*:  $(e \text{ ? } x : x) \mapsto x$  .

**optimization** *condition-bounds-x*:  $((u < v) \text{ ? } x : y) \mapsto x$   
*when* (*stamp-under (stamp-expr u) (stamp-expr v)  $\wedge$  wf-stamp u  $\wedge$  wf-stamp v*)  
**using** *stamp-under-defn* **by** *auto*

**optimization** *condition-bounds-y*:  $((u < v) \text{ ? } x : y) \mapsto y$   
*when* (*stamp-under (stamp-expr v) (stamp-expr u)  $\wedge$  wf-stamp u  $\wedge$  wf-stamp v*)  
**using** *stamp-under-defn-inverse* **by** *auto*

**lemma** *val-optimise-integer-test*:  
**assumes**  $\exists v. x = \text{IntVal } 32 \text{ } v$   
**shows**  $\text{val}[(x \ \& \ (\text{IntVal } 32 \text{ } 1)) \text{ eq } (\text{IntVal } 32 \text{ } 0)) \text{ ? } (\text{IntVal } 32 \text{ } 0) : (\text{IntVal } 32 \text{ } 1)]$   
 $=$   
 $\text{val}[x \ \& \ \text{IntVal } 32 \text{ } 1]$   
**using** *assms* **apply** *auto*  
**apply** (*metis (full-types) bool-to-val.simps(2) val-to-bool.simps(1)*)  
**by** (*metis (mono-tags, lifting) and-one-eq bool-to-val.simps(1) even-iff-mod-2-eq-zero odd-iff-mod-2-eq-one val-to-bool.simps(1)*)

**optimization** *ConditionalEliminateKnownLess*:  $((x < y) \text{ ? } x : y) \mapsto x$   
*when* (*stamp-under (stamp-expr x) (stamp-expr y)*  
 $\wedge$  *wf-stamp x  $\wedge$  wf-stamp y*)



**using** *stamp-under-defn* **by** *auto*

**optimization** *ConditionalEqualsRHS*:  $((x \text{ eq } y) \text{ ? } x : y) \mapsto y$   
**apply** *auto*  
**by** (*smt* (*verit*) *Value.inject*(1) *bool-to-val.simps*(2) *bool-to-val-bin.simps* *evalDet*  
*intval-equals.elims* *val-to-bool.elims*(1))

**optimization** *normalizeX*:  $((x \text{ eq } \text{const } (\text{IntVal } 32 \ 0)) \text{ ? } (\text{const } (\text{IntVal } 32 \ 0)) : (\text{const } (\text{IntVal } 32 \ 1))) \mapsto x$   
*when* (*IRExpr-up* *x* = 1)  $\wedge$  *stamp-expr* *x* = *IntegerStamp*

*b* 0 1  
**apply** *auto*  
**subgoal** *premises* *p* **for** *m* *p* *v* *xa*  
**proof** –  
**obtain** *xa* **where** *xa*: [*m*,*p*]  $\vdash x \mapsto xa$   
**using** *p* **by** *blast*  
**have** 3: [*m*,*p*]  $\vdash$  *if* *val-to-bool* (*intval-equals* *xa* (*IntVal* (32::nat) (0::64 word)))  
*then* *ConstantExpr* (*IntVal* (32::nat) (0::64 word))  
*else* *ConstantExpr* (*IntVal* (32::nat) (1::64 word))  $\mapsto v$   
**using** *evalDet* *p*(3) *p*(5) *xa* **by** *blast*  
**then** *have* 4: *xa* = *IntVal* 32 0 | *xa* = *IntVal* 32 1  
**sorry**  
**then** *have* 6: *v* = *xa*  
**sorry**  
**then** *show* ?*thesis*  
**using** *xa* **by** *auto*  
**qed**  
**done**

**optimization** *normalizeX2*:  $((x \text{ eq } (\text{const } (\text{IntVal } 32 \ 1))) \text{ ? } (\text{const } (\text{IntVal } 32 \ 1)) : (\text{const } (\text{IntVal } 32 \ 0))) \mapsto x$   
*when* (*x* = *ConstantExpr* (*IntVal* 32 0) | (*x* = *ConstantExpr* (*IntVal* 32 1))) .

**optimization** *flipX*:  $((x \text{ eq } (\text{const } (\text{IntVal } 32 \ 0))) \text{ ? } (\text{const } (\text{IntVal } 32 \ 1)) : (\text{const } (\text{IntVal } 32 \ 0))) \mapsto x \oplus (\text{const } (\text{IntVal } 32 \ 1))$   
*when* (*x* = *ConstantExpr* (*IntVal* 32 0) | (*x* = *ConstantExpr* (*IntVal* 32 1))) .

**optimization** *flipX2*:  $((x \text{ eq } (\text{const } (\text{IntVal } 32 \ 1))) \text{ ? } (\text{const } (\text{IntVal } 32 \ 0)) : (\text{const } (\text{IntVal } 32 \ 1))) \mapsto x \oplus (\text{const } (\text{IntVal } 32 \ 1))$

when  $(x = \text{ConstantExpr } (\text{IntVal } 32\ 0) \mid (x = \text{ConstantExpr } (\text{IntVal } 32\ 1)))$  .

**lemma** *stamp-of-default*:

**assumes** *stamp-expr*  $x = \text{default-stamp}$   
**assumes** *wf-stamp*  $x$   
**shows**  $([m, p] \vdash x \mapsto v) \longrightarrow (\exists vv. v = \text{IntVal } 32\ vv)$   
**using** *assms*  
**by** (*metis default-stamp valid-value-elim3 wf-stamp-def*)

**optimization** *OptimiseIntegerTest*:

$((x \ \& \ (\text{const } (\text{IntVal } 32\ 1))) \text{ eq } (\text{const } (\text{IntVal } 32\ 0))) ?$   
 $(\text{const } (\text{IntVal } 32\ 0)) : (\text{const } (\text{IntVal } 32\ 1))) \longmapsto$   
 $x \ \& \ (\text{const } (\text{IntVal } 32\ 1))$   
 when  $(\text{stamp-expr } x = \text{default-stamp} \wedge \text{wf-stamp } x)$   
**apply** *simp* **apply** (*rule impI*; (*rule allI*)+; *rule impI*)  
**subgoal** **premises** *eval* **for**  $m\ p\ v$   
**proof** –  
**obtain**  $xv$  **where**  $xv: [m, p] \vdash x \mapsto xv$   
**using** *eval* **by** *fast*  
**then** **have**  $x32: \exists v. xv = \text{IntVal } 32\ v$   
**using** *stamp-of-default eval* **by** *auto*  
**obtain**  $lhs$  **where**  $lhs: [m, p] \vdash \text{exp}[((x \ \& \ (\text{const } (\text{IntVal } 32\ 1))) \text{ eq } (\text{const } (\text{IntVal } 32\ 0))) ?$   
 $(\text{const } (\text{IntVal } 32\ 0)) : (\text{const } (\text{IntVal } 32\ 1))) \mapsto lhs$   
**using** *eval(2)* **by** *auto*  
**then** **have**  $lhsV: lhs = \text{val}[((xv \ \& \ (\text{IntVal } 32\ 1)) \text{ eq } (\text{IntVal } 32\ 0)) ? (\text{IntVal } 32\ 0) : (\text{IntVal } 32\ 1))]$   
**using**  $xv$  *evaltree.BinaryExpr evaltree.ConstantExpr evaltree.ConditionalExpr*  
**by** (*smt (verit) ConditionalExprE ConstantExprE bin-eval.simps(11) bin-eval.simps(4) evalDet intval-conditional.simps unfold-binary*)  
**obtain**  $rhs$  **where**  $rhs: [m, p] \vdash \text{exp}[x \ \& \ (\text{const } (\text{IntVal } 32\ 1))] \mapsto rhs$   
**using** *eval(2)* **by** *blast*  
**then** **have**  $rhsV: rhs = \text{val}[xv \ \& \ \text{IntVal } 32\ 1]$   
**by** (*metis BinaryExprE ConstantExprE bin-eval.simps(4) evalDet xv*)  
**have**  $lhs = rhs$  **using** *val-optimize-integer-test x32*  
**using**  $lhsV\ rhsV$  **by** *presburger*  
**then** **show** *?thesis*  
**by** (*metis eval(2) evalDet lhs rhs*)  
**qed**  
**done**

**optimization** *opt-optimize-integer-test-2*:

$((x \ \& \ (\text{const } (\text{IntVal } 32\ 1))) \text{ eq } (\text{const } (\text{IntVal } 32\ 0))) ?$   
 $(\text{const } (\text{IntVal } 32\ 0)) : (\text{const } (\text{IntVal } 32\ 1))) \longmapsto$   
 $x$   
 when  $(x = \text{ConstantExpr } (\text{IntVal } 32\ 0) \mid (x = \text{ConstantExpr } (\text{IntVal } 32\ 1)))$  .

**end**

**end**

## 9.5 MulNode Phase

**theory** *MulPhase*

**imports**

*Common*

*Proofs.StampEvalThms*

**begin**

**fun** *mul-size* :: *IRExpr*  $\Rightarrow$  *nat* **where**

*mul-size* (*UnaryExpr* *op* *e*) = (*mul-size* *e*) + 2 |

*mul-size* (*BinaryExpr* *BinMul* *x* *y*) = ((*mul-size* *x*) + (*mul-size* *y*) + 2) \* 2 |

*mul-size* (*BinaryExpr* *op* *x* *y*) = (*mul-size* *x*) + (*mul-size* *y*) + 2 |

*mul-size* (*ConditionalExpr* *cond* *t* *f*) = (*mul-size* *cond*) + (*mul-size* *t*) + (*mul-size* *f*) + 2 |

*mul-size* (*ConstantExpr* *c*) = 1 |

*mul-size* (*ParameterExpr* *ind* *s*) = 2 |

*mul-size* (*LeafExpr* *nid* *s*) = 2 |

*mul-size* (*ConstantVar* *c*) = 2 |

*mul-size* (*VariableExpr* *x* *s*) = 2

**phase** *MulNode*

**terminating** *mul-size*

**begin**

**lemma** *bin-eliminate-redundant-negative*:

*uminus* (*x* :: 'a::len word) \* *uminus* (*y* :: 'a::len word) = *x* \* *y*

**by** *simp*

**lemma** *bin-multiply-identity*:

(*x* :: 'a::len word) \* 1 = *x*

**by** *simp*

**lemma** *bin-multiply-eliminate*:

(*x* :: 'a::len word) \* 0 = 0

**by** *simp*

**lemma** *bin-multiply-negative*:

```

(x :: 'a::len word) * uminus 1 = uminus x
by simp

```

```

lemma bin-multiply-power-2:
(x :: 'a::len word) * (2^j) = x << j
by simp

```

```

lemma take-bit64[simp]:
  fixes w :: int64
  shows take-bit 64 w = w
proof -
  have Nat.size w = 64
  by (simp add: size64)
  then show ?thesis
  by (metis lt2p-lem mask-eq-iff take-bit-eq-mask verit-comp-simplify1(2) wsst-TYs(3))
qed

```

```

lemma testt:
  fixes a :: nat
  fixes b c :: 64 word
  shows take-bit a (take-bit a (b) * take-bit a (c)) =
    take-bit a (b * c)
by (smt (verit, ccfv-SIG) take-bit-mult take-bit-of-int unsigned-take-bit-eq word-mult-def)

```

```

lemma val-eliminate-redundant-negative:
  assumes val[-x * -y] ≠ UndefVal
  shows val[-x * -y] = val[x * y]
  using assms apply (cases x; cases y; auto)
  using testt by auto

```

```

lemma val-multiply-neutral:
  assumes x = new-int b v
  shows val[x * (IntVal b 1)] = val[x]
  using assms by force

```

```

lemma val-multiply-zero:
  assumes x = new-int b v
  shows val[x * (IntVal b 0)] = IntVal b 0
  using assms by simp

```

```

lemma val-multiply-negative:
  assumes x = new-int b v
  shows val[x * intval-negate (IntVal b 1)] = intval-negate x
  using assms

```

```

by (smt (verit) Value.disc(1) Value.inject(1) add.inverse-neutral intval-negate.simps(1)

      is-IntVal-def mask-0 mask-eq-take-bit-minus-one new-int.elims of-bool-eq(2)
take-bit-dist-neg
      take-bit-of-1 val-eliminate-redundant-negative val-multiply-neutral val-multiply-zero

      verit-minus-simplify(4) zero-neq-one)

```

**lemma** *val-MulPower2*:

```

fixes i :: 64 word
assumes y = IntVal 64 (2 ^ unat(i))
and 0 < i
and i < 64
and val[x * y] ≠ UndefVal
shows val[x * y] = val[x << IntVal 64 i]
using assms apply (cases x; cases y; auto)
subgoal premises p for x2
proof –
  have 63: (63 :: int64) = mask 6
    by eval
  then have (2::int) ^ 6 = 64
    by eval
  then have uint i < (2::int) ^ 6
    by (metis linorder-not-less lt2p-lem of-int-numeral p(4) size64 word-2p-lem
word-of-int-2p wsst-TYs(3))
  then have and i (mask 6) = i
    using mask-eq-iff by blast
  then show x2 << unat i = x2 << unat (and i (63::64 word))
    unfolding 63
    by force
qed
by presburger

```

**lemma** *val-MulPower2Add1*:

```

fixes i :: 64 word
assumes y = IntVal 64 ((2 ^ unat(i)) + 1)
and 0 < i
and i < 64
and val-to-bool(val[IntVal 64 0 < x])
and val-to-bool(val[IntVal 64 0 < y])
shows val[x * y] = val[(x << IntVal 64 i) + x]
using assms apply (cases x; cases y; auto)
subgoal premises p for x2
proof –
  have 63: (63 :: int64) = mask 6
    by eval
  then have (2::int) ^ 6 = 64

```

```

    by eval
  then have and i (mask 6) = i
    using mask-eq-iff by (simp add: less-mask-eq p(6))
  then have  $x2 * ((2::64 \text{ word}) \wedge \text{unat } i + (1::64 \text{ word})) = (x2 * ((2::64 \text{ word}) \wedge \text{unat } i)) + x2$ 
    by (simp add: distrib-left)
  then show  $x2 * ((2::64 \text{ word}) \wedge \text{unat } i + (1::64 \text{ word})) = x2 << \text{unat } (\text{and } i (63::64 \text{ word})) + x2$ 
    by (simp add: 63 and (i::64 word) (mask (6::nat)) = i)
  qed
  using val-to-bool.simps(2) by presburger

```

**lemma** *val-MulPower2Sub1:*

```

  fixes i :: 64 word
  assumes y = IntVal 64 ((2 ^ unat(i)) - 1)
  and 0 < i
  and i < 64
  and val-to-bool(val[IntVal 64 0 < x])
  and val-to-bool(val[IntVal 64 0 < y])
  shows val[x * y] = val[(x << IntVal 64 i) - x]
  using assms apply (cases x; cases y; auto)
  subgoal premises p for x2
  proof -
    have 63: (63 :: int64) = mask 6
      by eval
    then have (2::int) ^ 6 = 64
      by eval
    then have and i (mask 6) = i
      using mask-eq-iff by (simp add: less-mask-eq p(6))
    then have  $x2 * ((2::64 \text{ word}) \wedge \text{unat } i - (1::64 \text{ word})) = (x2 * ((2::64 \text{ word}) \wedge \text{unat } i)) - x2$ 
      by (simp add: right-diff-distrib')
    then show  $x2 * ((2::64 \text{ word}) \wedge \text{unat } i - (1::64 \text{ word})) = x2 << \text{unat } (\text{and } i (63::64 \text{ word})) - x2$ 
      by (simp add: 63 and (i::64 word) (mask (6::nat)) = i)
    qed
  using val-to-bool.simps(2) by presburger

```

**lemma** *val-distribute-multiplication:*

```

  assumes x = new-int 64 xx ∧ q = new-int 64 qq ∧ a = new-int 64 aa
  shows val[x * (q + a)] = val[(x * q) + (x * a)]
  apply (cases x; cases q; cases a; auto) using distrib-left assms by auto

```

**lemma** *val-MulPower2AddPower2:*

```

  fixes i j :: 64 word

```

```

assumes  $y = \text{IntVal } 64 \ ((2 \wedge \text{unat}(i)) + (2 \wedge \text{unat}(j)))$ 
and  $0 < i$ 
and  $0 < j$ 
and  $i < 64$ 
and  $j < 64$ 
and  $x = \text{new-int } 64 \ xx$ 
shows  $\text{val}[x * y] = \text{val}[(x << \text{IntVal } 64 \ i) + (x << \text{IntVal } 64 \ j)]$ 
using assms
proof –
  have  $63 :: \text{int}64 = \text{mask } 6$ 
  by eval
  then have  $(2 :: \text{int}) \wedge 6 = 64$ 
  by eval
  then have  $n : \text{IntVal } 64 \ ((2 \wedge \text{unat}(i)) + (2 \wedge \text{unat}(j))) =$ 
     $\text{val}[(\text{IntVal } 64 \ (2 \wedge \text{unat}(i))) + (\text{IntVal } 64 \ (2 \wedge \text{unat}(j)))]$ 

    using assms by (cases i; cases j; auto)
  then have  $1 : \text{val}[x * ((\text{IntVal } 64 \ (2 \wedge \text{unat}(i))) + (\text{IntVal } 64 \ (2 \wedge \text{unat}(j))))] =$ 
     $\text{val}[(x * \text{IntVal } 64 \ (2 \wedge \text{unat}(i))) + (x * \text{IntVal } 64 \ (2 \wedge \text{unat}(j)))]$ 

    using assms val-distribute-multiplication val-MulPower2 by simp
  then have  $2 : \text{val}[(x * \text{IntVal } 64 \ (2 \wedge \text{unat}(i)))] = \text{val}[x << \text{IntVal } 64 \ i]$ 
  using assms val-MulPower2
  using Value.distinct(1) intval-mul.simps(1) new-int.simps new-int-bin.simps
  by (smt (verit))
  then show ?thesis
  using  $1 \text{ Value.distinct}(1) \text{ assms}(1) \text{ assms}(3) \text{ assms}(5) \text{ assms}(6) \text{ intval-mul.simps}(1)$ 
n
     $\text{new-int.simps new-int-bin.elims val-MulPower2}$ 
  by (smt (verit, del-insts))
qed

thm-oracles val-MulPower2AddPower2

lemma exp-multiply-zero-64:
   $\text{exp}[x * (\text{const } (\text{IntVal } 64 \ 0))] \geq \text{ConstantExpr } (\text{IntVal } 64 \ 0)$ 
  using val-multiply-zero apply auto
  using Value.inject(1) constantAsStamp.simps(1) int-signed-value-bounds intval-mul.elims
    mult-zero-right new-int.simps new-int-bin.simps nle-le numeral-eq-Suc
take-bit-of-0
    unfold-const valid-stamp.simps(1) valid-value.simps(1) zero-less-Suc wf-value-def
  by (smt (verit))

lemma exp-multiply-neutral:
   $\text{exp}[x * (\text{const } (\text{IntVal } b \ 1))] \geq x$ 
  using val-multiply-neutral apply auto
  by (smt (verit) Value.inject(1) eval-unused-bits-zero intval-mul.elims mult.right-neutral)

```

*new-int.elims new-int-bin.elims)*

**thm-oracles** *exp-multiply-neutral*

**lemma** *exp-MulPower2:*

**fixes**  $i :: 64 \text{ word}$   
**assumes**  $y = \text{ConstantExpr } (\text{IntVal } 64 \ (2 \wedge \text{unat}(i)))$   
**and**  $0 < i$   
**and**  $i < 64$   
**and**  $\text{exp}[x > (\text{const IntVal } b \ 0)]$   
**and**  $\text{exp}[y > (\text{const IntVal } b \ 0)]$   
**shows**  $\text{exp}[x * y] \geq \text{exp}[x << \text{ConstantExpr } (\text{IntVal } 64 \ i)]$   
**using** *assms apply simp*  
**by** (*metis ConstantExprE equiv-exprs-def unfold-binary*)

**lemma** *exp-MulPower2Add1:*

**fixes**  $i :: 64 \text{ word}$   
**assumes**  $y = \text{ConstantExpr } (\text{IntVal } 64 \ ((2 \wedge \text{unat}(i)) + 1))$   
**and**  $0 < i$   
**and**  $i < 64$   
**and**  $\text{exp}[x > (\text{const IntVal } b \ 0)]$   
**and**  $\text{exp}[y > (\text{const IntVal } b \ 0)]$   
**shows**  $\text{exp}[x * y] \geq \text{exp}[(x << \text{ConstantExpr } (\text{IntVal } 64 \ i)) + x]$   
**using** *assms apply simp*  
**by** (*metis (no-types, lifting) ConstantExprE equiv-exprs-def unfold-binary*)

**lemma** *exp-MulPower2Sub1:*

**fixes**  $i :: 64 \text{ word}$   
**assumes**  $y = \text{ConstantExpr } (\text{IntVal } 64 \ ((2 \wedge \text{unat}(i)) - 1))$   
**and**  $0 < i$   
**and**  $i < 64$   
**and**  $\text{exp}[x > (\text{const IntVal } b \ 0)]$   
**and**  $\text{exp}[y > (\text{const IntVal } b \ 0)]$   
**shows**  $\text{exp}[x * y] \geq \text{exp}[(x << \text{ConstantExpr } (\text{IntVal } 64 \ i)) - x]$   
**using** *assms apply simp*  
**by** (*metis (no-types, lifting) ConstantExprE equiv-exprs-def unfold-binary*)

**lemma** *exp-MulPower2AddPower2:*

**fixes**  $i \ j :: 64 \text{ word}$   
**assumes**  $y = \text{ConstantExpr } (\text{IntVal } 64 \ ((2 \wedge \text{unat}(i)) + (2 \wedge \text{unat}(j))))$   
**and**  $0 < i$   
**and**  $0 < j$   
**and**  $i < 64$   
**and**  $j < 64$   
**and**  $\text{exp}[x > (\text{const IntVal } b \ 0)]$   
**and**  $\text{exp}[y > (\text{const IntVal } b \ 0)]$   
**shows**  $\text{exp}[x * y] \geq \text{exp}[(x << \text{ConstantExpr } (\text{IntVal } 64 \ i)) + (x << \text{ConstantExpr } (\text{IntVal } 64 \ j))]$



```

using assms apply simp
by (metis (no-types, lifting) ConstantExprE equiv-exprs-def unfold-binary)

```

```

lemma greaterConstant:
  fixes a b :: 64 word
  assumes a > b
  and y = ConstantExpr (IntVal 64 a)
  and x = ConstantExpr (IntVal 64 b)
  shows exp[y > x]
  apply auto
  sorry

```

```

lemma exp-distribute-multiplication:
  shows exp[(x * q) + (x * a)] ≥ exp[x * (q + a)]
  sorry

```

Optimisations

```

optimization EliminateRedundantNegative:  $-x * -y \mapsto x * y$ 
  using mul-size.simps apply auto[1]
  using val-eliminate-redundant-negative bin-eval.simps(2)
  by (metis BinaryExpr)

```

```

optimization MulNeutral:  $x * \text{ConstantExpr } (\text{IntVal } b \ 1) \mapsto x$ 
  using exp-multiply-neutral by blast

```

```

optimization MulEliminator:  $x * \text{ConstantExpr } (\text{IntVal } b \ 0) \mapsto \text{const } (\text{IntVal } b \ 0)$ 
  apply auto
  by (smt (verit) Value.inject(1) constantAsStamp.simps(1) int-signed-value-bounds
intval-mul.elims
    mult-zero-right new-int.simps new-int-bin.simps take-bit-of-0 unfold-const
valid-stamp.simps(1) valid-value.simps(1) val-multiply-zero)

```

```

optimization MulNegate:  $x * -(\text{const } (\text{IntVal } b \ 1)) \mapsto -x$ 
  apply auto using val-multiply-negative wf-value-def
  by (smt (verit) Value.distinct(1) Value.sel(1) add.inverse-inverse intval-mul.elims
intval-negate.simps(1) mask-eq-take-bit-minus-one new-int.simps new-int-bin.simps
    take-bit-dist-neg unary-eval.simps(2) unfold-unary val-multiply-negative
val-eliminate-redundant-negative)

```

```

fun isNonZero :: Stamp ⇒ bool where
  isNonZero (IntegerStamp b lo hi) = (lo > 0) |
  isNonZero - = False

```

```

lemma isNonZero-defn:
  assumes isNonZero (stamp-expr x)
  assumes wf-stamp x
  shows ( $[m, p] \vdash x \mapsto v \longrightarrow (\exists vv\ b. (v = \text{IntVal } b\ vv \wedge \text{val-to-bool val}[(\text{IntVal } b\ 0) < v]))$ )
  apply (rule impI) subgoal premises eval
proof –
  obtain b lo hi where xstamp: stamp-expr x = IntegerStamp b lo hi
    using assms
    by (meson isNonZero.elims(2))
  then obtain vv where vdef: v = IntVal b vv
    by (metis assms(2) eval valid-int wf-stamp-def)
  have lo > 0
    using assms(1) xstamp by force
  then have signed-above: int-signed-value b vv > 0
    using assms unfolding wf-stamp-def
    using eval vdef xstamp by fastforce
  have take-bit b vv = vv
    using eval eval-unused-bits-zero vdef by auto
  then have vv > 0
    using signed-above
    by (metis bit-take-bit-iff int-signed-value.simps not-less-zero signed-eq-0-iff signed-take-bit-eq-if-positive take-bit-0 take-bit-of-0 verit-comp-simplify1(1) word-gt-0)
  then show ?thesis
    using vdef using signed-above
    by simp
qed
done

optimization MulPower2: x * y  $\mapsto x << \text{const } (\text{IntVal } 64\ i)$ 
  when (i > 0  $\wedge$ 
    64 > i  $\wedge$ 
    y = exp[const (IntVal 64 (2  $\wedge$  unat(i)))]])

  defer
  apply simp apply (rule impI; (rule allI) $^+$ ; rule impI)
  subgoal premises eval for m p v
proof –
  obtain xv where xv: [m, p]  $\vdash x \mapsto xv$ 
    using eval(2) by blast
  then obtain xvv where xvv: xv = IntVal 64 xvv
    using eval
    using ConstantExprE bin-eval.simps(2) evalDet intval-bits.simps intval-mul.elims new-int-bin.simps unfold-binary
    by (smt (verit))
  obtain yv where yv: [m, p]  $\vdash y \mapsto yv$ 
    using eval(1) eval(2) by blast
  then have lhs: [m, p]  $\vdash \text{exp}[x * y] \mapsto \text{val}[xv * yv]$ 
    by (metis bin-eval.simps(2) eval(1) eval(2) evalDet unfold-binary xv)
  have  $[m, p] \vdash \text{exp}[\text{const } (\text{IntVal } 64\ i)] \mapsto \text{val}[(\text{IntVal } 64\ i)]$ 

```

```

    by (smt (verit, ccfv-SIG) ConstantExpr constantAsStamp.simps(1) eval-bits-1-64
take-bit64 validStampIntConst wf-value-def valid-value.simps(1) xv xvv)
    then have rhs:  $[m, p] \vdash \text{exp}[x \ll \text{const } (\text{IntVal } 64 \ i)] \mapsto \text{val}[xv \ll (\text{IntVal } 64 \ i)]$ 
    using xv xvv using evaltree.BinaryExpr
    by (metis Value.simps(5) bin-eval.simps(8) intval-left-shift.simps(1) new-int.simps)
    have  $\text{val}[xv * yv] = \text{val}[xv \ll (\text{IntVal } 64 \ i)]$ 
    using val-MulPower2
    by (metis ConstantExprE eval(1) evaltree-not-undef lhs yv)
    then show ?thesis
    by (metis eval(1) eval(2) evalDet lhs rhs)
qed
done

```

end

end

## 9.6 NotNode Phase

```

theory NotPhase
  imports
    Common
begin

phase NotNode
  terminating size
begin

```

```

lemma bin-not-cancel:
   $\text{bin}[\neg(\neg(e))] = \text{bin}[e]$ 
  by auto

```

```

lemma val-not-cancel:
  assumes  $\text{val}[\sim(\text{new-int } b \ v)] \neq \text{UndefVal}$ 
  shows  $\text{val}[\sim(\sim(\text{new-int } b \ v))] = (\text{new-int } b \ v)$ 
  using bin-not-cancel
  by (simp add: take-bit-not-take-bit)

```

```

lemma exp-not-cancel:
  shows  $\text{exp}[\sim(\sim a)] \geq \text{exp}[a]$ 
  using val-not-cancel apply auto
  by (metis eval-unused-bits-zero intval-logic-negation.cases intval-not.simps(1)
    intval-not.simps(2) intval-not.simps(3) intval-not.simps(4) new-int.simps)

```

Optimisations

**optimization** *NotCancel*:  $\text{exp}[\sim(\sim a)] \mapsto a$   
**by** (*metis exp-not-cancel*)

**end**

**end**

## 9.7 OrNode Phase

**theory** *OrPhase*

**imports**

*Common*

**begin**

**context** *stamp-mask*

**begin**

Taking advantage of the truth table of or operations.

#	x	y	$x y$
1	0	0	0
2	0	1	1
3	1	0	1
4	1	1	1

If row 2 never applies, that is,  $\text{canBeZero } x \ \& \ \text{canBeOne } y = 0$ , then  $(x|y) = x$ .

Likewise, if row 3 never applies,  $\text{canBeZero } y \ \& \ \text{canBeOne } x = 0$ , then  $(x|y) = y$ .

**lemma** *OrLeftFallthrough*:

**assumes** (*and* (*not* ( $\downarrow x$ )) ( $\uparrow y$ )) = 0

**shows**  $\text{exp}[x \mid y] \geq \text{exp}[x]$

**using** *assms*

**apply** *simp* **apply** ((*rule allI*) $+$ ; *rule impI*)

**subgoal** **premises** *eval* **for** *m p v*

**proof** –

**obtain** *b vv* **where** *e*:  $[m, p] \vdash \text{exp}[x \mid y] \mapsto \text{IntVal } b \ vv$

**using** *eval*

**by** (*metis BinaryExprE bin-eval-new-int new-int.simps*)

**from** *e* **obtain** *xv* **where** *xv*:  $[m, p] \vdash x \mapsto \text{IntVal } b \ xv$

**apply** (*subst* (*asm*) *unfold-binary-width*)

**by** *force* $+$

**from** *e* **obtain** *yv* **where** *yv*:  $[m, p] \vdash y \mapsto \text{IntVal } b \ yv$

**apply** (*subst* (*asm*) *unfold-binary-width*)

**by** *force* $+$

**have** *vdef*:  $v = \text{intval-or } (\text{IntVal } b \ xv) \ (\text{IntVal } b \ yv)$

```

    using e xv yv
    by (metis bin-eval.simps(5) eval(2) evalDet unfold-binary)
  have  $\forall i. (bit\ xv\ i) \mid (bit\ yv\ i) = (bit\ xv\ i)$ 
    by (metis assms bit-and-iff not-down-up-mask-and-zero-implies-zero xv yv)
  then have  $IntVal\ b\ xv = intval-or\ (IntVal\ b\ xv)\ (IntVal\ b\ yv)$ 
    by (smt (verit, ccv-threshold) and.idem assms bit.conj-disj-distrib eval-unused-bits-zero
    intval-or.simps(1) new-int.simps new-int-bin.simps not-down-up-mask-and-zero-implies-zero
    word-ao-absorbs(3) xv yv)
  then show ?thesis
    using vdef
    using xv by presburger
qed
done

```

lemma *OrRightFallthrough*:

```

assumes (and (not ( $\downarrow y$ )) ( $\uparrow x$ )) = 0
shows  $exp[x \mid y] \geq exp[y]$ 
using assms
apply simp apply ((rule allI)+; rule impI)
subgoal premises eval for m p v
proof -
  obtain b vv where  $e: [m, p] \vdash exp[x \mid y] \mapsto IntVal\ b\ vv$ 
    using eval
    by (metis BinaryExprE bin-eval-new-int new-int.simps)
  from e obtain xv where  $xv: [m, p] \vdash x \mapsto IntVal\ b\ xv$ 
    apply (subst (asm) unfold-binary-width)
    by force+
  from e obtain yv where  $yv: [m, p] \vdash y \mapsto IntVal\ b\ yv$ 
    apply (subst (asm) unfold-binary-width)
    by force+
  have  $vdef: v = intval-or\ (IntVal\ b\ xv)\ (IntVal\ b\ yv)$ 
    using e xv yv
    by (metis bin-eval.simps(5) eval(2) evalDet unfold-binary)
  have  $\forall i. (bit\ xv\ i) \mid (bit\ yv\ i) = (bit\ yv\ i)$ 
    by (metis assms bit-and-iff not-down-up-mask-and-zero-implies-zero xv yv)
  then have  $IntVal\ b\ yv = intval-or\ (IntVal\ b\ xv)\ (IntVal\ b\ yv)$ 
    by (metis (no-types, lifting) assms eval-unused-bits-zero intval-or.simps(1)
    new-int.elims new-int-bin.elims stamp-mask.not-down-up-mask-and-zero-implies-zero
    stamp-mask-axioms word-ao-absorbs(8) xv yv)
  then show ?thesis
    using vdef
    using yv by presburger
qed
done

```

end

```

phase OrNode
terminating size

```

**begin**

**lemma** *bin-or-equal*:

$\text{bin}[x \mid x] = \text{bin}[x]$

**by** *simp*

**lemma** *bin-shift-const-right-helper*:

$x \mid y = y \mid x$

**by** *simp*

**lemma** *bin-or-not-operands*:

$(\sim x \mid \sim y) = (\sim(x \ \& \ y))$

**by** *simp*

**lemma** *val-or-equal*:

**assumes**  $x = \text{new-int } b \ v$

**and**  $(\text{val}[x \mid x] \neq \text{UndefVal})$

**shows**  $\text{val}[x \mid x] = \text{val}[x]$

**apply** (*cases*  $x$ ; *auto*) **using** *bin-or-equal* *assms*

**by** *auto*+

**lemma** *val-elim-redundant-false*:

**assumes**  $x = \text{new-int } b \ v$

**and**  $\text{val}[x \mid \text{false}] \neq \text{UndefVal}$

**shows**  $\text{val}[x \mid \text{false}] = \text{val}[x]$

**using** *assms* **apply** (*cases*  $x$ ; *auto*) **by** *presburger*

**lemma** *val-shift-const-right-helper*:

$\text{val}[x \mid y] = \text{val}[y \mid x]$

**apply** (*cases*  $x$ ; *cases*  $y$ ; *auto*)

**by** (*simp* *add: or.commute*)+

**lemma** *val-or-not-operands*:

$\text{val}[\sim x \mid \sim y] = \text{val}[\sim(x \ \& \ y)]$

**apply** (*cases*  $x$ ; *cases*  $y$ ; *auto*)

**by** (*simp* *add: take-bit-not-take-bit*)

**lemma** *exp-or-equal*:

$\text{exp}[x \mid x] \geq \text{exp}[x]$

**using** *val-or-equal* **apply** *auto*

**by** (*smt* (*verit*, *ccfv-SIG*) *evalDet* *eval-unused-bits-zero* *intval-negate.elims* *int-val-or.simps*(2)

*intval-or.simps*(6) *intval-or.simps*(7) *new-int.simps* *val-or-equal*)

**lemma** *exp-elim-redundant-false*:

$\text{exp}[x \mid \text{false}] \geq \text{exp}[x]$

```

using val-elim-redundant-false apply auto
by (smt (verit) Value.sel(1) eval-unused-bits-zero intval-or.elims new-int.simps
      new-int-bin.simps val-elim-redundant-false)

```

Optimisations

```

optimization OrEqual:  $x \mid x \mapsto x$ 
by (meson exp-or-equal le-expr-def)

```

```

optimization OrShiftConstantRight:  $((\text{const } x) \mid y) \mapsto y \mid (\text{const } x)$  when  $\neg(\text{is-ConstantExpr } y)$ 
using size-flip-binary apply force
apply auto
by (simp add: BinaryExpr unfold-const val-shift-const-right-helper)

```

```

optimization EliminateRedundantFalse:  $x \mid \text{false} \mapsto x$ 
by (meson exp-elim-redundant-false le-expr-def)

```

```

optimization OrNotOperands:  $(\sim x \mid \sim y) \mapsto \sim(x \ \& \ y)$ 
apply (metis add-2-eq-Suc' less-SucI not-add-less1 not-less-eq size-binary-const
      size-non-add)
apply auto using val-or-not-operands
by (metis BinaryExpr UnaryExpr bin-eval.simps(4) intval-not.simps(2) unary-eval.simps(3))

```

```

optimization OrLeftFallthrough:
 $x \mid y \mapsto x$  when  $((\text{and } (\text{not } (\text{IExpr-down } x)) (\text{IExpr-up } y))) = 0$ 
using simple-mask.OrLeftFallthrough by blast

```

```

optimization OrRightFallthrough:
 $x \mid y \mapsto y$  when  $((\text{and } (\text{not } (\text{IExpr-down } y)) (\text{IExpr-up } x))) = 0$ 
using simple-mask.OrRightFallthrough by blast

```

**end**

**end**

## 9.8 SubNode Phase

```

theory SubPhase
imports
  Common
  Proofs.StampEvalThms
begin

phase SubNode
terminating size
begin

```

**lemma** *bin-sub-after-right-add*:  
**shows**  $((x :: ('a :: len) \text{ word}) + (y :: ('a :: len) \text{ word})) - y = x$   
**by** *simp*

**lemma** *sub-self-is-zero*:  
**shows**  $(x :: ('a :: len) \text{ word}) - x = 0$   
**by** *simp*

**lemma** *bin-sub-then-left-add*:  
**shows**  $(x :: ('a :: len) \text{ word}) - (x + (y :: ('a :: len) \text{ word})) = -y$   
**by** *simp*

**lemma** *bin-sub-then-left-sub*:  
**shows**  $(x :: ('a :: len) \text{ word}) - (x - (y :: ('a :: len) \text{ word})) = y$   
**by** *simp*

**lemma** *bin-subtract-zero*:  
**shows**  $(x :: 'a :: len \text{ word}) - (0 :: 'a :: len \text{ word}) = x$   
**by** *simp*

**lemma** *bin-sub-negative-value*:  
**shows**  $(x :: ('a :: len) \text{ word}) - (-(y :: ('a :: len) \text{ word})) = x + y$   
**by** *simp*

**lemma** *bin-sub-self-is-zero*:  
**shows**  $(x :: ('a :: len) \text{ word}) - x = 0$   
**by** *simp*

**lemma** *bin-sub-negative-const*:  
**shows**  $(x :: 'a :: len \text{ word}) - (-(y :: 'a :: len \text{ word})) = x + y$   
**by** *simp*

**lemma** *val-sub-after-right-add-2*:  
**assumes**  $x = \text{new-int } b \ v$   
**assumes**  $\text{val}[(x + y) - y] \neq \text{UndefVal}$   
**shows**  $\text{val}[(x + y) - y] = \text{val}[x]$   
**using** *bin-sub-after-right-add*  
**using** *assms* **apply** (*cases*  $x$ ; *cases*  $y$ ; *auto*)  
**by** (*metis* (*full-types*) *intval-sub.simps*(2))

**lemma** *val-sub-after-left-sub*:  
**assumes**  $\text{val}[(x - y) - x] \neq \text{UndefVal}$   
**shows**  $\text{val}[(x - y) - x] = \text{val}[-y]$   
**using** *assms* **apply** (*cases*  $x$ ; *cases*  $y$ ; *auto*)  
**using** *intval-sub.elims* **by** *fastforce*

**lemma** *val-sub-then-left-sub*:  
**assumes**  $y = \text{new-int } b \ v$



```

assumes  $val[x - (x - y)] \neq \text{UndefVal}$ 
shows  $val[x - (x - y)] = val[y]$ 
using assms apply (cases x; cases y; auto)
by (metis (mono-tags) intval-sub.simps(5))

lemma val-subtract-zero:
assumes  $x = \text{new-int } b \ v$ 
assumes  $\text{intval-sub } x \ (\text{IntVal } b \ 0) \neq \text{UndefVal}$ 
shows  $\text{intval-sub } x \ (\text{IntVal } b \ 0) = val[x]$ 
using assms by (induction x; simp)

lemma val-zero-subtract-value:
assumes  $x = \text{new-int } b \ v$ 
assumes  $\text{intval-sub } (\text{IntVal } b \ 0) \ x \neq \text{UndefVal}$ 
shows  $\text{intval-sub } (\text{IntVal } b \ 0) \ x = val[-x]$ 
using assms by (induction x; simp)

lemma val-sub-then-left-add:
assumes  $val[x - (x + y)] \neq \text{UndefVal}$ 
shows  $val[x - (x + y)] = val[-y]$ 
using assms apply (cases x; cases y; auto)
by (metis (mono-tags, lifting) intval-sub.simps(5))

lemma val-sub-negative-value:
assumes  $val[x - (-y)] \neq \text{UndefVal}$ 
shows  $val[x - (-y)] = val[x + y]$ 
using assms by (cases x; cases y; auto)

lemma val-sub-self-is-zero:
assumes  $x = \text{new-int } b \ v \wedge val[x - x] \neq \text{UndefVal}$ 
shows  $val[x - x] = \text{new-int } b \ 0$ 
using assms by (cases x; auto)

lemma val-sub-negative-const:
assumes  $y = \text{new-int } b \ v \wedge val[x - (-y)] \neq \text{UndefVal}$ 
shows  $val[x - (-y)] = val[x + y]$ 
using assms by (cases x; cases y; auto)

lemma exp-sub-after-right-add:
shows  $\text{exp}[(x + y) - y] \geq \text{exp}[x]$ 
apply auto using val-sub-after-right-add-2
using evalDet eval-unused-bits-zero intval-add.elims new-int.simps
by (smt (verit))

lemma exp-sub-after-right-add2:
shows  $\text{exp}[(x + y) - x] \geq \text{exp}[y]$ 
using exp-sub-after-right-add apply auto
using bin-eval.simps(1) bin-eval.simps(3) intval-add-sym unfold-binary

```

```

by (smt (z3) Value.inject(1) diff-eq-eq evalDet eval-unused-bits-zero intval-add.elims
      intval-sub.elims new-int.simps new-int-bin.simps take-bit-dist-subL)

```

```

lemma exp-sub-negative-value:
  exp[x - (-y)] ≥ exp[x + y]
  apply simp using val-sub-negative-value
  by (smt (verit) bin-eval.simps(1) bin-eval.simps(3) evaltree-not-undef
      unary-eval.simps(2) unfold-binary unfold-unary)

```

```

lemma exp-sub-then-left-sub:
  shows exp[x - (x - y)] ≥ exp[y]
  using val-sub-then-left-sub apply auto
  subgoal premises p for m p xa xaa ya
  proof—
    obtain xa where xa: [m, p] ⊢ x ↦ xa
    using p(2) by blast
    obtain ya where ya: [m, p] ⊢ y ↦ ya
    using p(5) by auto
    obtain xaa where xaa: [m, p] ⊢ x ↦ xaa
    using p(2) by blast
    have 1: val[xa - (xaa - ya)] ≠ UndefVal
    by (metis evalDet p(2) p(3) p(4) p(5) xa xaa ya)
    then have val[xaa - ya] ≠ UndefVal
    by auto
    then have [m,p] ⊢ y ↦ val[xa - (xaa - ya)]
    by (metis 1 Value.exhaust evalDet eval-unused-bits-zero evaltree-not-undef
      intval-sub.simps(6) intval-sub.simps(7) new-int.simps p(5) val-sub-then-left-sub xa
      xaa ya)
    then show ?thesis
    by (metis evalDet p(2) p(4) p(5) xa xaa ya)
  qed
done

```

**thm-oracles** exp-sub-then-left-sub

Optimisations

```

optimization SubAfterAddRight: ((x + y) - y) ↦ x
  using exp-sub-after-right-add by blast

```

```

optimization SubAfterAddLeft: ((x + y) - x) ↦ y
  using exp-sub-after-right-add2 by blast

```

```

optimization SubAfterSubLeft: ((x - y) - x) ↦ -y
  apply (metis Suc-lessI add-2-eq-Suc' add-less-cancel-right less-trans-Suc not-add-less1
    size-binary-const size-binary-lhs size-binary-rhs size-non-add)
  apply auto
  by (metis evalDet unary-eval.simps(2) unfold-unary val-sub-after-left-sub)

```

**optimization** *SubThenAddLeft*:  $(x - (x + y)) \mapsto -y$   
**apply** *auto*  
**by** (*metis evalDet unary-eval.simps(2) unfold-unary*  
*val-sub-then-left-add*)

**optimization** *SubThenAddRight*:  $(y - (x + y)) \mapsto -x$   
**apply** *auto*  
**by** (*metis evalDet intval-add-sym unary-eval.simps(2) unfold-unary*  
*val-sub-then-left-add*)

**optimization** *SubThenSubLeft*:  $(x - (x - y)) \mapsto y$   
**using** *size-simps* **apply** *simp*  
**using** *exp-sub-then-left-sub* **by** *blast*

**optimization** *SubtractZero*:  $(x - (\text{const IntVal } b \ 0)) \mapsto x$   
**apply** *auto*  
**by** (*smt (verit) add.right-neutral diff-add-cancel eval-unused-bits-zero intval-sub.elims*  
  
*intval-word.simps new-int.simps new-int-bin.simps*)

**thm-oracles** *SubtractZero*

**optimization** *SubNegativeValue*:  $(x - (-y)) \mapsto x + y$   
**apply** (*metis add-2-eq-Suc' less-SucI less-add-Suc1 not-less-eq size-binary-const*  
*size-non-add*)  
**using** *exp-sub-negative-value* **by** *simp*

**thm-oracles** *SubNegativeValue*

**lemma** *negate-idempotent*:  
**assumes**  $x = \text{IntVal } b \ v \wedge \text{take-bit } b \ v = v$   
**shows**  $x = \text{val}[-(-x)]$   
**using** *assms*  
**using** *is-IntVal-def* **by** *force*

**optimization** *ZeroSubtractValue*:  $((\text{const IntVal } b \ 0) - x) \mapsto (-x)$   
*when (wf-stamp x  $\wedge$  stamp-expr x = IntegerStamp b lo*  
*hi  $\wedge$   $\neg(\text{is-ConstantExpr } x)$ )*  
**defer**  
**apply** *auto* **unfolding** *wf-stamp-def*  
**apply** (*smt (verit) diff-0 intval-negate.simps(1) intval-sub.elims intval-word.simps*  
  
*new-int-bin.simps unary-eval.simps(2) unfold-unary*)

```

using add-2-eq-Suc' size.simps(2) size-flip-binary by presburger

optimization SubSelfIsZero:  $(x - x) \mapsto \text{const IntVal } b \ 0 \text{ when}$ 
   $(\text{wf-stamp } x \wedge \text{stamp-expr } x = \text{IntegerStamp } b \text{ lo hi})$ 
  apply simp-all
  apply auto
  using IRExpr.disc(42) One-nat-def size-non-const apply presburger
  by (smt (verit, best) wf-value-def ConstantExpr evalDet eval-bits-1-64 eval-unused-bits-zero
new-int.simps take-bit-of-0 val-sub-self-is-zero validDefIntConst valid-int wf-stamp-def)

end

end

```

## 9.9 XorNode Phase

```

theory XorPhase
  imports
    Common
    Proofs.StampEvalThms
  begin

  phase XorNode
    terminating size
  begin

  lemma bin-xor-self-is-false:
     $\text{bin}[x \oplus x] = 0$ 
    by simp

  lemma bin-xor-commute:
     $\text{bin}[x \oplus y] = \text{bin}[y \oplus x]$ 
    by (simp add: xor.commute)

  lemma bin-eliminate-redundant-false:
     $\text{bin}[x \oplus 0] = \text{bin}[x]$ 
    by simp

  lemma val-xor-self-is-false:
    assumes  $\text{val}[x \oplus x] \neq \text{UndefVal}$ 
    shows  $\text{val-to-bool } (\text{val}[x \oplus x]) = \text{False}$ 
    using assms by (cases x; auto)

  lemma val-xor-self-is-false-2:

```

```

assumes (val[x ⊕ x]) ≠ UndefVal
and x = IntVal 32 v
shows val[x ⊕ x] = bool-to-val False
using assms by (cases x; auto)

lemma val-xor-self-is-false-3:
assumes val[x ⊕ x] ≠ UndefVal ∧ x = IntVal 64 v
shows val[x ⊕ x] = IntVal 64 0
using assms by (cases x; auto)

lemma val-xor-commute:
  val[x ⊕ y] = val[y ⊕ x]
apply (cases x; cases y; auto)
by (simp add: xor.commute)+

lemma val-eliminate-redundant-false:
assumes x = new-int b v
assumes val[x ⊕ (bool-to-val False)] ≠ UndefVal
shows val[x ⊕ (bool-to-val False)] = x
using assms apply (cases x; auto)
by meson

lemma exp-xor-self-is-false:
assumes wf-stamp x ∧ stamp-expr x = default-stamp
shows exp[x ⊕ x] ≥ exp[false]
using assms apply auto unfolding wf-stamp-def
using IntVal0 Value.inject(1) bool-to-val.simps(2) constantAsStamp.simps(1) evalDet

  int-signed-value-bounds new-int.simps unfold-const val-xor-self-is-false-2
valid-int
  valid-stamp.simps(1) valid-value.simps(1) wf-value-def
by (smt (z3) validDefIntConst)

lemma exp-eliminate-redundant-false:
shows exp[x ⊕ false] ≥ exp[x]
using val-eliminate-redundant-false apply auto
subgoal premises p for m p xa
proof –
  obtain xa where xa: [m,p] ⊢ x ↦ xa
  using p(2) by blast
  then have val[xa ⊕ (IntVal 32 0)] ≠ UndefVal
  using evalDet p(2) p(3) by blast
  then have [m,p] ⊢ x ↦ val[xa ⊕ (IntVal 32 0)]
  apply (cases xa; auto) using eval-unused-bits-zero xa by auto
  then show ?thesis
  using evalDet p(2) xa by blast
qed

```

```

done

Optimisations

optimization XorSelfIsFalse:  $(x \oplus x) \mapsto \text{false}$  when
    (wf-stamp  $x \wedge$  stamp-expr  $x = \text{default-stamp}$ )
    using size-non-const apply force
    using exp-xor-self-is-false by auto

optimization XorShiftConstantRight:  $((\text{const } x) \oplus y) \mapsto y \oplus (\text{const } x)$  when
 $\neg(\text{is-ConstantExpr } y)$ 
    using size-flip-binary apply force
    unfolding le-expr-def using val-xor-commute
    by auto

optimization EliminateRedundantFalse:  $(x \oplus \text{false}) \mapsto x$ 
    using exp-eliminate-redundant-false by blast

end

end

```

## 10 Verifying term graph optimizations using Isabelle/HOL

```

theory TreeSnippets
imports
    Canonicalizations.BinaryNode
    Canonicalizations.ConditionalPhase
    Canonicalizations.AddPhase
    Semantics.TreeToGraphThms
    Snippets.Snipping
    HOL-Library.OptionalSugar
begin

— First, we disable undesirable markup.
declare  $[[\text{show-types}=\text{false}, \text{show-sorts}=\text{false}]]$ 
no-notation ConditionalExpr (- ? - : -)

— We want to disable and reduce how aggressive automated tactics are as obligations
are generated in the paper
method unfold-size = —
method unfold-optimization =
    (unfold rewrite-preservation.simps, unfold rewrite-termination.simps,
     rule conjE, simp, simp del: le-expr-def)

```

## 10.1 Markup syntax for common operations

**notation** (*latex*)  
*kind* ( $-\langle\!\langle - \rangle\!\rangle$ )

**notation** (*latex*)  
*valid-value* ( $- \in -$ )

**notation** (*latex*)  
*val-to-bool* (*bool-of*  $-$ )

**notation** (*latex*)  
*constantAsStamp* (*stamp-from-value*  $-$ )

**notation** (*latex*)  
*size* (*trm*( $-$ ))

## 10.2 Representing canonicalization optimizations

We wish to provide an example of the semantics layers at which optimizations can be expressed.

**lemma** *diff-self*:

**fixes**  $x :: \text{int}$

**shows**  $x - x = 0$

**by** *simp*

**lemma** *diff-diff-cancel*:

**fixes**  $x\ y :: \text{int}$

**shows**  $x - (x - y) = y$

**by** *simp*

**thm** *diff-self*

**thm** *diff-diff-cancel*

*algebraic-laws*

$$x - x = 0 \tag{1}$$

$$x - (x - y) = y \tag{2}$$

**lemma** *diff-self-value*:  $\forall v :: 'a :: \text{len word}. v - v = 0$

**by** *simp*

**lemma** *diff-diff-cancel-value*:

$\forall v_1\ v_2 :: 'a :: \text{len word}. v_1 - (v_1 - v_2) = v_2$

**by** *simp*

$$\forall v :: 'a \text{ word}. v - v = (0 :: 'a \text{ word}) \quad (3)$$

$$\forall (v_1 :: 'a \text{ word}) v_2 :: 'a \text{ word}. v_1 - (v_1 - v_2) = v_2 \quad (4)$$

**translations**

$n \leq \text{CONST ConstantExpr (CONST IntVal b n)}$

$x - y \leq \text{CONST BinaryExpr (CONST BinSub) x y}$

**notation** (*ExprRule* **output**)

*Refines* ( $- \mapsto -$ )

**lemma** *diff-self-expr*:

**assumes**  $\forall m p v. [m, p] \vdash \text{exp}[e - e] \mapsto \text{IntVal b } v$

**shows**  $\text{exp}[e - e] \geq \text{exp}[\text{const (IntVal b 0)}]$

**using** *assms* **apply** *simp*

**by** (*metis(full-types) evalDet val-to-bool.simps(1) zero-neq-one*)

**method** *open-eval* = (*simp*; (*rule impI*)?; (*rule allI*)<sup>+</sup>; *rule impI*)

**lemma** *diff-diff-cancel-expr*:

**shows**  $\text{exp}[e_1 - (e_1 - e_2)] \geq \text{exp}[e_2]$

**apply** *open-eval*

**subgoal** **premises** *eval* **for**  $m p v$

**proof** –

**obtain** *v1* **where**  $v1: [m, p] \vdash e_1 \mapsto v1$

**using** *eval* **by** *blast*

**obtain** *v2* **where**  $v2: [m, p] \vdash e_2 \mapsto v2$

**using** *eval* **by** *blast*

**then have**  $e: [m, p] \vdash \text{exp}[e_1 - (e_1 - e_2)] \mapsto \text{val}[v1 - (v1 - v2)]$

**using** *v1 v2 eval*

**by** (*smt (verit, ccfv-SIG) bin-eval.simps(3) evalDet unfold-binary*)

**then have** *notUn*:  $\text{val}[v1 - (v1 - v2)] \neq \text{UndefVal}$

**using** *evaltree-not-undef* **by** *auto*

**then have**  $\text{val}[v1 - (v1 - v2)] = v2$

**apply** (*cases v1*; *cases v2*; *auto simp: notUn*)

**using** *eval-unused-bits-zero v2* **apply** *blast*

**by** (*metis(full-types) intval-sub.simps(5)*)

**then show** *?thesis*

**by** (*metis e eval evalDet v2*)

**qed**

**done**

**thm-oracles** *diff-diff-cancel-expr*



*algebraic-laws-expressions*

$$e - e \mapsto 0 \quad (5)$$

$$e_1 - (e_1 - e_2) \mapsto e_2 \quad (6)$$

**no-translations**

```
n <= CONST ConstantExpr (CONST IntVal b n)
x - y <= CONST BinaryExpr (CONST BinSub) x y
```

**definition** *wf-stamp* :: *IRExpr*  $\Rightarrow$  *bool* **where**

```
wf-stamp e = ( $\forall m\ p\ v.$  ( $[m, p] \vdash e \mapsto v$ )  $\longrightarrow$  valid-value v (stamp-expr e))
```

**lemma** *wf-stamp-eval*:

```
assumes wf-stamp e
assumes stamp-expr e = IntegerStamp b lo hi
shows  $\forall m\ p\ v.$  ( $[m, p] \vdash e \mapsto v$ )  $\longrightarrow$  ( $\exists vv.$   $v = \text{IntVal } b\ vv$ )
using assms unfolding wf-stamp-def
using valid-int-same-bits valid-int
by metis
```

**phase** *SnipPhase*

**terminating** *size*

**begin**

**lemma** *sub-same-val*:

```
assumes val[e - e] = IntVal b v
shows val[e - e] = val[IntVal b 0]
using assms by (cases e; auto)
```

*sub-same-32*

**optimization** *SubIdentity*:

```
e - e  $\mapsto$  ConstantExpr (IntVal b 0)
when ((stamp-expr exp[e - e] = IntegerStamp b lo hi)  $\wedge$  wf-stamp exp[e - e])
```

**using** *IRExpr.disc(42) size.simps(4) size-non-const*

**apply** *simp*

**apply** (*rule impI*) **apply** *simp*

**proof** –

```
assume assms: stamp-binary BinSub (stamp-expr e) (stamp-expr e) = IntegerStamp b lo hi  $\wedge$  wf-stamp exp[e - e]
```

```
have  $\forall m\ p\ v.$  ( $[m, p] \vdash \text{exp}[e - e] \mapsto v$ )  $\longrightarrow$  ( $\exists vv.$   $v = \text{IntVal } b\ vv$ )
```

```
using assms wf-stamp-eval
```

```
by (metis stamp-expr.simps(2))
```

```
then show  $\forall m\ p\ v.$  ( $[m, p] \vdash \text{BinaryExpr BinSub } e\ e \mapsto v$ )  $\longrightarrow$  ( $[m, p] \vdash \text{ConstantExpr (IntVal } b\ 0) \mapsto v$ )
```

```
using wf-value-def
```

```

    by (smt (verit, best) BinaryExprE TreeSnippets.wf-stamp-def assms bin-eval.simps(3)
    constantAsStamp.simps(1) evalDet stamp-expr.simps(2) sub-same-val unfold-const
    valid-stamp.simps(1) valid-value.simps(1))
qed
thm-oracles SubIdentity

RedundantSubtract

optimization RedundantSubtract:
   $e_1 - (e_1 - e_2) \mapsto e_2$ 

using size-simps apply simp
using diff-diff-cancel-expr by presburger
end

```

### 10.3 Representing terms

We wish to show a simple example of expressions represented as terms.

```

ast-example

BinaryExpr BinAdd
  (BinaryExpr BinMul x x)
  (BinaryExpr BinMul x x)

```

Then we need to show the datatypes that compose the example expression.

```

abstract-syntax-tree

datatype IExpr =
  UnaryExpr IRUnaryOp IExpr
| BinaryExpr IRBinaryOp IExpr IExpr
| ConditionalExpr IExpr IExpr IExpr
| ParameterExpr nat Stamp
| LeafExpr nat Stamp
| ConstantExpr Value
| ConstantVar (char list)
| VariableExpr (char list) Stamp

```

```

value

datatype Value = UndefVal
| IntVal nat (64 word)
| ObjRef (nat option)
| ObjStr (char list)

```

## 10.4 Term semantics

The core expression evaluation functions need to be introduced.

*eval*

$unary\text{-}eval :: IRUnaryOp \Rightarrow Value \Rightarrow Value$   
 $bin\text{-}eval :: IRBinaryOp \Rightarrow Value \Rightarrow Value \Rightarrow Value$

We then provide the full semantics of IR expressions.

**no-translations**

$(prop) P \wedge Q \Longrightarrow R \leq (prop) P \Longrightarrow Q \Longrightarrow R$

**translations**

$(prop) P \Longrightarrow Q \Longrightarrow R \leq (prop) P \wedge Q \Longrightarrow R$

*tree-semantics*

$$\frac{[m,p] \vdash xe \mapsto x}{result = unary\text{-}eval\ op\ x \quad result \neq UndefinedVal} \quad \frac{[m,p] \vdash xe \mapsto x \quad [m,p] \vdash ye \mapsto y}{result = bin\text{-}eval\ op\ x\ y \quad result \neq UndefinedVal} \quad \frac{[m,p] \vdash BinaryExpr\ op\ xe\ ye \mapsto result}{[m,p] \vdash ce \mapsto cond \quad branch = (if\ bool\text{-}of\ cond\ then\ te\ else\ fe) \quad [m,p] \vdash branch \mapsto result \quad result \neq UndefinedVal} \quad \frac{[m,p] \vdash ConditionalExpr\ ce\ te\ fe \mapsto result}{wf\text{-}value\ c} \quad \frac{i < |p| \quad p[i] \in s}{[m,p] \vdash ConstantExpr\ c \mapsto c} \quad \frac{val = m\ n \quad val \in s}{[m,p] \vdash ParameterExpr\ i\ s \mapsto p[i]} \quad \frac{[m,p] \vdash LeafExpr\ n\ s \mapsto val}{[m,p] \vdash LeafExpr\ n\ s \mapsto val}$$

**no-translations**

$(prop) P \Longrightarrow Q \Longrightarrow R \leq (prop) P \wedge Q \Longrightarrow R$

**translations**

$(prop) P \wedge Q \Longrightarrow R \leq (prop) P \Longrightarrow Q \Longrightarrow R$

And show that expression evaluation is deterministic.

*tree-evaluation-deterministic*

$[m,p] \vdash e \mapsto v_1 \wedge [m,p] \vdash e \mapsto v_2 \Longrightarrow v_1 = v_2$

We then want to start demonstrating the obligations for optimizations. For this we define refinement over terms.

*expression-refinement*

$$e_1 \sqsupseteq e_2 = (\forall m \ p \ v. [m,p] \vdash e_1 \mapsto v \longrightarrow [m,p] \vdash e_2 \mapsto v)$$

To motivate this definition we show the obligations generated by optimization definitions.

**phase** *SnipPhase*  
**terminating** *size*  
**begin**

*InverseLeftSub*

**optimization** *InverseLeftSub*:  
 $(e_1 - e_2) + e_2 \mapsto e_1$

*InverseLeftSubObligation*

1.  $\text{trm}(e_1) < \text{trm}(\text{BinaryExpr BinAdd} (\text{BinaryExpr BinSub } e_1 \ e_2) \ e_2)$
2.  $\text{BinaryExpr BinAdd} (\text{BinaryExpr BinSub } e_1 \ e_2) \ e_2 \sqsupseteq e_1$

**using** *RedundantSubAdd* **by** *auto*

*InverseRightSub*

**optimization** *InverseRightSub*:  $e_2 + (e_1 - e_2) \mapsto e_1$

*InverseRightSubObligation*

1.  $\text{trm}(e_1) < \text{trm}(\text{BinaryExpr BinAdd } e_2 (\text{BinaryExpr BinSub } e_1 \ e_2))$
2.  $\text{BinaryExpr BinAdd } e_2 (\text{BinaryExpr BinSub } e_1 \ e_2) \sqsupseteq e_1$

**using** *RedundantSubAdd2(2)* *rewrite-termination.simps(1)* **apply** *blast*

**using** *RedundantSubAdd2(1)* *rewrite-preservation.simps(1)* **by** *blast*

**end**

*expression-refinement-monotone*

$$e \sqsupseteq e' \implies \text{UnaryExpr op } e \sqsupseteq \text{UnaryExpr op } e'$$

$$x \sqsupseteq x' \wedge y \sqsupseteq y' \implies \text{BinaryExpr op } x \ y \sqsupseteq \text{BinaryExpr op } x' \ y'$$

$$ce \sqsupseteq ce' \wedge te \sqsupseteq te' \wedge fe \sqsupseteq fe' \implies$$

$$\text{ConditionalExpr } ce \ te \ fe \sqsupseteq \text{ConditionalExpr } ce' \ te' \ fe'$$

phase *SnipPhase*  
 terminating *size*  
 begin

*BinaryFoldConstant*

**optimization** *BinaryFoldConstant*:  $\text{BinaryExpr } op \text{ (const } v1 \text{) (const } v2 \text{)} \mapsto \text{ConstantExpr (bin-eval } op \text{ } v1 \text{ } v2 \text{)}$

*BinaryFoldConstantObligation*

1.  $\text{trm}(\text{ConstantExpr (bin-eval } op \text{ } v1 \text{ } v2 \text{)}) < \text{trm}(\text{BinaryExpr } op \text{ (ConstantExpr } v1 \text{) (ConstantExpr } v2 \text{)})$
2.  $\text{BinaryExpr } op \text{ (ConstantExpr } v1 \text{) (ConstantExpr } v2 \text{)} \sqsubseteq \text{ConstantExpr (bin-eval } op \text{ } v1 \text{ } v2 \text{)}$

using *BinaryFoldConstant(1)* by *auto*

*AddCommuteConstantRight*

**optimization** *AddCommuteConstantRight*:  
 $(\text{const } v) + y \mapsto y + (\text{const } v) \text{ when } \neg(\text{is-ConstantExpr } y)$

*AddCommuteConstantRightObligation*

1.  $\neg \text{is-ConstantExpr } y \longrightarrow \text{trm}(\text{BinaryExpr BinAdd } y \text{ (ConstantExpr } v \text{)}) < \text{trm}(\text{BinaryExpr BinAdd (ConstantExpr } v \text{) } y \text{)}$
2.  $\neg \text{is-ConstantExpr } y \longrightarrow \text{BinaryExpr BinAdd (ConstantExpr } v \text{) } y \sqsubseteq \text{BinaryExpr BinAdd } y \text{ (ConstantExpr } v \text{)}$

using *AddShiftConstantRight* by *auto*

*AddNeutral*

**optimization** *AddNeutral*:  $e + (\text{const (IntVal 32 0)}) \mapsto e$

*AddNeutralObligation*

1.  $\text{trm}(e) < \text{trm}(\text{BinaryExpr BinAdd } e \text{ (ConstantExpr (IntVal 32 0))})$
2.  $\text{BinaryExpr BinAdd } e \text{ (ConstantExpr (IntVal 32 0))} \sqsubseteq e$

apply *auto*

using *AddNeutral(1)* *rewrite-preservation.simps(1)* by *force*

*AddToSub*

**optimization** *AddToSub*:  $-e + y \mapsto y - e$

*AddToSubObligation*

1.  $\text{trm}(\text{BinaryExpr BinSub } y \ e) < \text{trm}(\text{BinaryExpr BinAdd } (\text{UnaryExpr UnaryNeg } e) \ y)$
2.  $\text{BinaryExpr BinAdd } (\text{UnaryExpr UnaryNeg } e) \ y \sqsupseteq \text{BinaryExpr BinSub } y \ e$

**using** *AddLeftNegateToSub* **by** *auto*

**end**

**definition** *trm* **where** *trm* = *size*

**lemma** *trm-defn*[*size-simps*]:

*trm* *x* = *size* *x*  
**by** (*simp* *add*: *trm-def*)

*phase*

**phase** *AddCanonicalizations*  
  **terminating** *trm*  
**begin...end**

**hide-const** (**open**) *Form.wf-stamp*

*phase-example*

**phase** *Conditional*  
  **terminating** *trm*  
**begin**

*phase-example-1*

**optimization** *NegateCond*:  $((!e) \ ? \ x : y) \mapsto (e \ ? \ y : x)$

**apply** (*simp* *add*: *size-simps*)

**using** *ConditionalPhase.NegateConditionFlipBranches*(1) **by** *simp*

*phase-example-2*

**optimization** *TrueCond*:  $(\text{true} \ ? \ x : y) \mapsto x$

**by** (*auto* *simp*: *trm-def*)

*phase-example-3*

**optimization** *FalseCond*:  $(\text{false} \ ? \ x : y) \mapsto y$

by (auto simp: trm-def)

*phase-example-4*

**optimization** *BranchEqual*:  $(e \ ? \ x : x) \mapsto x$

by (auto simp: trm-def)

*phase-example-5*

**optimization** *LessCond*:  $((u < v) \ ? \ x : y) \mapsto x$   
when (stamp-under (stamp-expr u) (stamp-expr v))  
 $\wedge \text{wf-stamp } u \wedge \text{wf-stamp } v$

apply (auto simp: trm-def)

using *ConditionalPhase.condition-bounds-x*(1)

by (metis(full-types) *StampEvalThms.wf-stamp-def TreeSnippets.wf-stamp-def bin-eval.simps*(12)  
*stamp-under-defn*)

*phase-example-6*

**optimization** *condition-bounds-y*:  $((x < y) \ ? \ x : y) \mapsto y$   
when (stamp-under (stamp-expr y) (stamp-expr x))  $\wedge \text{wf-stamp}$   
 $x \wedge \text{wf-stamp } y$

apply (auto simp: trm-def)

using *ConditionalPhase.condition-bounds-y*(1)

by (metis(full-types) *StampEvalThms.wf-stamp-def TreeSnippets.wf-stamp-def bin-eval.simps*(12)  
*stamp-under-defn-inverse*)

*phase-example-7*

**end**

**lemma** *simplified-binary*:  $\neg(\text{is-ConstantExpr } b) \implies \text{size } (\text{BinaryExpr op } a \ b) =$   
 $\text{size } a + \text{size } b + 2$

by (induction b; induction op; auto simp: *is-ConstantExpr-def*)

**thm** *bin-size*

**thm** *bin-const-size*

**thm** *unary-size*

**thm** *size-non-add*

### termination

$$\text{trm}(\text{UnaryExpr } op \ e) = \text{trm}(e) + 2$$

$$\text{trm}(\text{BinaryExpr } op \ x \ (\text{ConstantExpr } cy)) = \text{trm}(x) + 2$$

$$\text{trm}(\text{BinaryExpr } op \ a \ b) = \text{trm}(a) + \text{trm}(b) + 2$$

$$\text{trm}(\text{ConditionalExpr } cond \ t \ f) = \text{trm}(cond) + \text{trm}(t) + \text{trm}(f) + 2$$

$$\text{trm}(\text{ConstantExpr } c) = 1$$

$$\text{trm}(\text{ParameterExpr } ind \ s) = 2$$

$$\text{trm}(\text{LeafExpr } nid \ s) = 2$$

### graph-representation

**typedef** IRGraph =

$$\{g :: ID \rightarrow (IRNode \times Stamp) \cdot \text{finite } (dom \ g)\}$$

### no-translations

$$(prop) \ P \wedge Q \implies R \leq (prop) \ P \implies Q \implies R$$

### translations

$$(prop) \ P \implies Q \implies R \leq (prop) \ P \wedge Q \implies R$$

### graph2tree

$$\frac{g\langle n \rangle = \text{ConstantNode } c \quad g\langle n \rangle = \text{ParameterNode } i \quad \text{stamp } g \ n = s}{g \vdash n \simeq \text{ConstantExpr } c \quad g \vdash n \simeq \text{ParameterExpr } i \ s}$$

$$\frac{g\langle n \rangle = \text{ConditionalNode } c \ t \ f}{g \vdash c \simeq ce \quad g \vdash t \simeq te \quad g \vdash f \simeq fe}$$

$$\frac{g \vdash c \simeq ce \quad g \vdash t \simeq te \quad g \vdash f \simeq fe}{g \vdash n \simeq \text{ConditionalExpr } ce \ te \ fe}$$

$$\frac{g\langle n \rangle = \text{AbsNode } x \quad g \vdash x \simeq xe}{g \vdash n \simeq \text{UnaryExpr } \text{UnaryAbs } xe}$$

$$\frac{g\langle n \rangle = \text{SignExtendNode } inputBits \ resultBits \ x \quad g \vdash x \simeq xe}{g \vdash n \simeq \text{UnaryExpr } (\text{UnarySignExtend } inputBits \ resultBits)}$$

$$\frac{g\langle n \rangle = \text{AddNode } x \ y \quad g \vdash x \simeq xe \quad g \vdash y \simeq ye}{g \vdash n \simeq \text{BinaryExpr } \text{BinAdd } xe \ ye}$$

$$\frac{is\text{-preevaluated } g\langle n \rangle \quad \text{stamp } g \ n = s \quad g\langle n \rangle = \text{RefNode } n' \quad g \vdash n' \simeq e}{g \vdash n \simeq \text{LeafExpr } n \ s \quad g \vdash n \simeq e}$$

### no-translations

$$(prop) \ P \implies Q \implies R \leq (prop) \ P \wedge Q \implies R$$



**translations**

$$(prop) \ P \wedge Q \implies R \leq (prop) \ P \implies Q \implies R$$

*preeval*

*is-preevaluated* (*InvokeNode* *n uu uv uw ux uy*) = *True*  
*is-preevaluated* (*InvokeWithExceptionNode* *n uz va vb vc vd ve*) = *True*  
*is-preevaluated* (*NewInstanceNode* *n vf vg vh*) = *True*  
*is-preevaluated* (*LoadFieldNode* *n vi vj vk*) = *True*  
*is-preevaluated* (*SignedDivNode* *n vl vm vn vo vp*) = *True*  
*is-preevaluated* (*SignedRemNode* *n vq vr vs vt vu*) = *True*  
*is-preevaluated* (*ValuePhiNode* *n vv vw*) = *True*  
*is-preevaluated* (*AbsNode* *v*) = *False*  
*is-preevaluated* (*AddNode* *v va*) = *False*  
*is-preevaluated* (*AndNode* *v va*) = *False*  
*is-preevaluated* (*BeginNode* *v*) = *False*  
*is-preevaluated* (*BytecodeExceptionNode* *v va vb*) = *False*  
*is-preevaluated* (*ConditionalNode* *v va vb*) = *False*  
*is-preevaluated* (*ConstantNode* *v*) = *False*  
*is-preevaluated* (*DynamicNewArrayNode* *v va vb vc vd*) = *False*  
*is-preevaluated* *EndNode* = *False*  
*is-preevaluated* (*ExceptionObjectNode* *v va*) = *False*  
*is-preevaluated* (*FrameState* *v va vb vc*) = *False*  
*is-preevaluated* (*IfNode* *v va vb*) = *False*  
*is-preevaluated* (*IntegerBelowNode* *v va*) = *False*  
*is-preevaluated* (*IntegerEqualsNode* *v va*) = *False*  
*is-preevaluated* (*IntegerLessThanNode* *v va*) = *False*  
*is-preevaluated* (*IsNullNode* *v*) = *False*  
*is-preevaluated* (*KillingBeginNode* *v*) = *False*  
*is-preevaluated* (*LeftShiftNode* *v va*) = *False*  
*is-preevaluated* (*LogicNegationNode* *v*) = *False*  
*is-preevaluated* (*LoopBeginNode* *v va vb vc*) = *False*  
*is-preevaluated* (*LoopEndNode* *v*) = *False*  
*is-preevaluated* (*LoopExitNode* *v va vb*) = *False*  
*is-preevaluated* (*MergeNode* *v va vb*) = *False*  
*is-preevaluated* (*MethodCallTargetNode* *v va*) = *False*  
*is-preevaluated* (*MulNode* *v va*) = *False*  
*is-preevaluated* (*NarrowNode* *v va vb*) = *False*  
*is-preevaluated* (*NegateNode* *v*) = *False*  
*is-preevaluated* (*NewArrayNode* *v va vb*) = *False*  
*is-preevaluated* (*NotNode* *v*) = *False*  
*is-preevaluated* (*OrNode* *v va*) = *False*  
*is-preevaluated* (*ParameterNode* *v*) = *False*  
*is-preevaluated* (*PiNode* *v va*) = *False*  
*is-preevaluated* (*ReturnNode* *v va*) = *False*  
*is-preevaluated* (*RightShiftNode* *v va*) = *False*  
*is-preevaluated* (*ShortCircuitOrNode* *v va*) = *False*  
*is-preevaluated* (*SignExtendNode* *v va vb*) = *False*

*deterministic-representation*

$$g \vdash n \simeq e_1 \wedge g \vdash n \simeq e_2 \implies e_1 = e_2$$

**thm-oracles** *repDet*

*well-formed-term-graph*

$$\exists e. g \vdash n \simeq e \wedge (\exists v. [m,p] \vdash e \mapsto v)$$

*graph-semantics*

$$([g,m,p] \vdash n \mapsto v) = (\exists e. g \vdash n \simeq e \wedge [m,p] \vdash e \mapsto v)$$

*graph-semantics-deterministic*

$$[g,m,p] \vdash n \mapsto v_1 \wedge [g,m,p] \vdash n \mapsto v_2 \implies v_1 = v_2$$

**thm-oracles** *graphDet*

**notation** (*latex*)

*graph-refinement* (*term-graph-refinement* -)

*graph-refinement*

$$\begin{aligned} \text{term-graph-refinement } g_1 \ g_2 = \\ (ids \ g_1 \subseteq ids \ g_2 \wedge \\ (\forall n. n \in ids \ g_1 \longrightarrow (\forall e. g_1 \vdash n \simeq e \longrightarrow g_2 \vdash n \sqsubseteq e))) \end{aligned}$$

**translations**

$n \leq CONST$  as-set  $n$

*graph-semantics-preservation*

$$\begin{aligned} e_1' \sqsupseteq e_2' \wedge \\ \{n\} \triangleleft g_1 \subseteq g_2 \wedge \\ g_1 \vdash n \simeq e_1' \wedge g_2 \vdash n \simeq e_2' \implies \\ \text{term-graph-refinement } g_1 \ g_2 \end{aligned}$$

**thm-oracles** *graph-semantics-preservation-subscript*

### *maximal-sharing*

$\text{maximal-sharing } g =$   
 $(\forall n_1 n_2.$   
 $n_1 \in \text{true-ids } g \wedge n_2 \in \text{true-ids } g \longrightarrow$   
 $(\forall e. g \vdash n_1 \simeq e \wedge$   
 $g \vdash n_2 \simeq e \wedge \text{stamp } g \ n_1 = \text{stamp } g \ n_2 \longrightarrow$   
 $n_1 = n_2))$

### *tree-to-graph-rewriting*

$e_1 \sqsupseteq e_2 \wedge$   
 $g_1 \vdash n \simeq e_1 \wedge$   
 $\text{maximal-sharing } g_1 \wedge$   
 $\{n\} \triangleleft g_1 \subseteq g_2 \wedge$   
 $g_2 \vdash n \simeq e_2 \wedge$   
 $\text{maximal-sharing } g_2 \implies$   
 $\text{term-graph-refinement } g_1 \ g_2$

## **thm-oracles** *tree-to-graph-rewriting*

### *term-graph-refines-term*

$(g \vdash n \sqsubseteq e) = (\exists e'. g \vdash n \simeq e' \wedge e \sqsupseteq e')$

### *term-graph-evaluation*

$g \vdash n \sqsubseteq e \implies \forall m \ p \ v. [m, p] \vdash e \mapsto v \longrightarrow [g, m, p] \vdash n \mapsto v$

### *graph-construction*

$e_1 \sqsupseteq e_2 \wedge g_1 \subseteq g_2 \wedge g_2 \vdash n \simeq e_2 \implies$   
 $g_2 \vdash n \sqsubseteq e_1 \wedge \text{term-graph-refinement } g_1 \ g_2$

## **thm-oracles** *graph-construction*

### *term-graph-reconstruction*

$g \oplus e \rightsquigarrow (g', n) \implies g' \vdash n \simeq e \wedge g \subseteq g'$

*refined-insert*

$$\begin{array}{l} e_1 \sqsupseteq e_2 \wedge g_1 \oplus e_2 \rightsquigarrow (g_2, n') \implies \\ g_2 \vdash n' \sqsubseteq e_1 \wedge \textit{term-graph-refinement } g_1 \ g_2 \end{array}$$

**end**