

# Veriopt

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## Abstract

The Veriopt project aims to prove the optimization pass of the GraalVM compiler. The GraalVM compiler includes a sophisticated Intermediate Representation (IR) in the form of a sea-of-nodes based graph structure. We first define the IR graph structure in the Isabelle/HOL interactive theorem prover. We subsequently give the evaluation of the structure a semantics based on the current understanding of the purpose of each IR graph node. Optimization phases are then encoded including the static analysis passes required for an optimization. Each optimization phase is proved to be correct by proving that a bisimulation exists between the unoptimized and optimized graphs. The following document has been automatically generated from the Isabelle/HOL source to provide a very comprehensive definition of the semantics and optimizations introduced by the Veriopt project.

## Contents

<b>1</b>	<b>Runtime Values and Arithmetic</b>	<b>3</b>
<b>2</b>	<b>Nodes</b>	<b>9</b>
2.1	Types of Nodes . . . . .	9
2.2	Hierarchy of Nodes . . . . .	17
<b>3</b>	<b>Stamp Typing</b>	<b>24</b>
<b>4</b>	<b>Graph Representation</b>	<b>27</b>
4.0.1	Example Graphs . . . . .	32
4.1	Control-flow Graph Traversal . . . . .	32
4.2	Structural Graph Comparison . . . . .	34
<b>5</b>	<b>Data-flow Semantics</b>	<b>36</b>
5.1	Data-flow Tree Representation . . . . .	36
5.2	Data-flow Tree Evaluation . . . . .	38
5.3	Data-flow Tree Refinement . . . . .	40
5.4	Data-flow Tree Theorems . . . . .	41
5.4.1	Deterministic Data-flow Evaluation . . . . .	41
5.4.2	Evaluation Results are Valid . . . . .	42
5.4.3	Example Data-flow Optimisations . . . . .	44
5.4.4	Monotonicity of Expression Refinement . . . . .	44
<b>6</b>	<b>Tree to Graph</b>	<b>46</b>
6.1	Subgraph to Data-flow Tree . . . . .	46
6.2	Data-flow Tree to Subgraph . . . . .	50
6.3	Lift Data-flow Tree Semantics . . . . .	54
6.4	Graph Refinement . . . . .	54
6.5	Maximal Sharing . . . . .	54
6.6	Formedness Properties . . . . .	54
6.7	Dynamic Frames . . . . .	56
6.8	Tree to Graph Theorems . . . . .	68
6.8.1	Extraction and Evaluation of Expression Trees is Deterministic. . . . .	68
6.8.2	Monotonicity of Graph Refinement . . . . .	74
6.8.3	Lift Data-flow Tree Refinement to Graph Refinement . . . . .	77
6.8.4	Term Graph Reconstruction . . . . .	93
<b>7</b>	<b>Control-flow Semantics</b>	<b>101</b>
7.1	Object Heap . . . . .	101
7.2	Intraprocedural Semantics . . . . .	102
7.3	Interprocedural Semantics . . . . .	104
7.4	Big-step Execution . . . . .	105

7.4.1	Heap Testing . . . . .	106
7.5	Control-flow Semantics Theorems . . . . .	107
7.5.1	Control-flow Step is Deterministic . . . . .	107
<b>8</b>	<b>Proof Infrastructure</b>	<b>112</b>
8.1	Bisimulation . . . . .	112
8.2	Graph Rewriting . . . . .	113
8.3	Stuttering . . . . .	117
8.4	Evaluation Stamp Theorems . . . . .	118
8.4.1	Evaluated Value Satisfies Stamps . . . . .	118
<b>9</b>	<b>Optization DSLs</b>	<b>130</b>
9.1	Canonicalization DSL . . . . .	132
<b>10</b>	<b>Canonicalization Phase</b>	<b>134</b>
10.1	Conditional Expression . . . . .	135
<b>11</b>	<b>Conditional Elimination Phase</b>	<b>137</b>
11.1	Individual Elimination Rules . . . . .	137
11.2	Control-flow Graph Traversal . . . . .	147

# 1 Runtime Values and Arithmetic

```
theory Values
imports
  HOL-Library.Word
  HOL-Library.Signed-Division
  HOL-Library.Float
  HOL-Library.LaTeXsugar
begin
```

In order to properly implement the IR semantics we first introduce a type that represents runtime values. These runtime values represent the full range of primitive types currently allowed by our semantics, ranging from basic integer types to object references and arrays.

Note that Java supports 64, 32, 16, 8 signed ints, plus 1 bit (boolean) ints, but during calculations the smaller sizes are expanded to 32 bits, so here we model just 32 and 64 bit values.

An object reference is an option type where the *None* object reference points to the static fields. This is examined more closely in our definition of the heap.

```
type-synonym int64 = 64 word — long
type-synonym int32 = 32 word — int
type-synonym int16 = 16 word — short
type-synonym int8 = 8 word — char
type-synonym int1 = 1 word — boolean
```

```
type-synonym objref = nat option
```

```
datatype (discs-sels) Value =
  UndefVal |
  IntVal32 32 word |
  IntVal64 64 word |

  ObjRef objref |
  ObjStr string
```

```
fun wf-bool :: Value  $\Rightarrow$  bool where
  wf-bool (IntVal32 v) = (v = 0  $\vee$  v = 1) |
  wf-bool - = False
```

```
fun val-to-bool :: Value  $\Rightarrow$  bool where
  val-to-bool (IntVal32 val) = (if val = 0 then False else True) |
  val-to-bool (IntVal64 val) = (if val = 0 then False else True) |
  val-to-bool v = False
```

```
fun bool-to-val :: bool  $\Rightarrow$  Value where
  bool-to-val True = (IntVal32 1) |
```

```

    bool-to-val False = (IntVal32 0)

value sint(word-of-int (1) :: int1)

fun is-int-val :: Value ⇒ bool where
    is-int-val (IntVal32 v) = True |
    is-int-val (IntVal64 v) = True |
    is-int-val - = False

```

We need to introduce arithmetic operations which agree with the JVM.

Within the JVM, bytecode arithmetic operations are performed on 32 or 64 bit integers, unboxing where appropriate.

The following collection of intval functions correspond to the JVM arithmetic operations. We merge the 32 and 64 bit operations into a single function, even though the stamp of each IRNode tells us exactly what the bit widths will be. These merged functions know to make it easier to do the instantiation of Value as 'plus', etc. It might be worse for reasoning, because it could cause more case analysis, but this does not seem to be a problem in practice.

```

fun intval-add :: Value ⇒ Value ⇒ Value where
    intval-add (IntVal32 v1) (IntVal32 v2) = (IntVal32 (v1+v2)) |
    intval-add (IntVal64 v1) (IntVal64 v2) = (IntVal64 (v1+v2)) |
    intval-add - - = UndefVal

```

```

instantiation Value :: ab-semigroup-add
begin

```

```

definition plus-Value :: Value ⇒ Value ⇒ Value where
    plus-Value = intval-add

```

```

print-locale! ab-semigroup-add

```

```

instance proof
    fix a b c :: Value
    show a + b + c = a + (b + c)
        apply (simp add: plus-Value-def)
        apply (induction a; induction b; induction c; auto)
        done
    show a + b = b + a
        apply (simp add: plus-Value-def)
        apply (induction a; induction b; auto)
        done
qed
end

```

```

fun intval-sub :: Value ⇒ Value ⇒ Value where
    intval-sub (IntVal32 v1) (IntVal32 v2) = (IntVal32 (v1-v2)) |

```

```

    intval-sub (IntVal64 v1) (IntVal64 v2) = (IntVal64 (v1-v2)) |
    intval-sub - - = UndefVal

```

```

instantiation Value :: minus
begin

```

```

definition minus-Value :: Value ⇒ Value ⇒ Value where
    minus-Value = intval-sub

```

```

instance proof qed
end

```

```

fun intval-mul :: Value ⇒ Value ⇒ Value where
    intval-mul (IntVal32 v1) (IntVal32 v2) = (IntVal32 (v1*v2)) |
    intval-mul (IntVal64 v1) (IntVal64 v2) = (IntVal64 (v1*v2)) |
    intval-mul - - = UndefVal

```

```

instantiation Value :: times
begin

```

```

definition times-Value :: Value ⇒ Value ⇒ Value where
    times-Value = intval-mul

```

```

instance proof qed
end

```

```

fun intval-div :: Value ⇒ Value ⇒ Value where
    intval-div (IntVal32 v1) (IntVal32 v2) = (IntVal32 (word-of-int((sint v1) sdiv
(sint v2)))) |
    intval-div (IntVal64 v1) (IntVal64 v2) = (IntVal64 (word-of-int((sint v1) sdiv
(sint v2)))) |
    intval-div - - = UndefVal

```

```

instantiation Value :: divide
begin

```

```

definition divide-Value :: Value ⇒ Value ⇒ Value where
    divide-Value = intval-div

```

```

instance proof qed
end

```

```

fun intval-mod :: Value ⇒ Value ⇒ Value where
    intval-mod (IntVal32 v1) (IntVal32 v2) = (IntVal32 (word-of-int((sint v1) smod
(sint v2)))) |

```

```

    intval-mod (IntVal64 v1) (IntVal64 v2) = (IntVal64 (word-of-int((sint v1) smod
(sint v2)))) |
    intval-mod - - = UndefVal

```

```

instantiation Value :: modulo
begin

```

```

definition modulo-Value :: Value ⇒ Value ⇒ Value where
    modulo-Value = intval-mod

```

```

instance proof qed
end

```

```

context
    includes bit-operations-syntax
begin

```

```

fun intval-and :: Value ⇒ Value ⇒ Value (infix &&* 64) where
    intval-and (IntVal32 v1) (IntVal32 v2) = (IntVal32 (v1 AND v2)) |
    intval-and (IntVal64 v1) (IntVal64 v2) = (IntVal64 (v1 AND v2)) |
    intval-and - - = UndefVal

```

```

fun intval-or :: Value ⇒ Value ⇒ Value (infix ||* 59) where
    intval-or (IntVal32 v1) (IntVal32 v2) = (IntVal32 (v1 OR v2)) |
    intval-or (IntVal64 v1) (IntVal64 v2) = (IntVal64 (v1 OR v2)) |
    intval-or - - = UndefVal

```

```

fun intval-xor :: Value ⇒ Value ⇒ Value (infix ^* 59) where
    intval-xor (IntVal32 v1) (IntVal32 v2) = (IntVal32 (v1 XOR v2)) |
    intval-xor (IntVal64 v1) (IntVal64 v2) = (IntVal64 (v1 XOR v2)) |
    intval-xor - - = UndefVal

```

```

fun intval-equals :: Value ⇒ Value ⇒ Value where
    intval-equals (IntVal32 v1) (IntVal32 v2) = bool-to-val (v1 = v2) |
    intval-equals (IntVal64 v1) (IntVal64 v2) = bool-to-val (v1 = v2) |
    intval-equals - - = UndefVal

```

```

fun intval-less-than :: Value ⇒ Value ⇒ Value where
    intval-less-than (IntVal32 v1) (IntVal32 v2) = bool-to-val (v1 < v2) |
    intval-less-than (IntVal64 v1) (IntVal64 v2) = bool-to-val (v1 < v2) |
    intval-less-than - - = UndefVal

```

```

fun intval-below :: Value ⇒ Value ⇒ Value where
    intval-below (IntVal32 v1) (IntVal32 v2) = bool-to-val (v1 < v2) |
    intval-below (IntVal64 v1) (IntVal64 v2) = bool-to-val (v1 < v2) |

```

*intval-below* - = *UndefVal*

**fun** *intval-not* :: *Value*  $\Rightarrow$  *Value* **where**  
*intval-not* (*IntVal32* *v*) = (*IntVal32* (*NOT* *v*)) |  
*intval-not* (*IntVal64* *v*) = (*IntVal64* (*NOT* *v*)) |  
*intval-not* - = *UndefVal*

**fun** *intval-negate* :: *Value*  $\Rightarrow$  *Value* **where**  
*intval-negate* (*IntVal32* *v*) = *IntVal32* ( $-$  *v*) |  
*intval-negate* (*IntVal64* *v*) = *IntVal64* ( $-$  *v*) |  
*intval-negate* - = *UndefVal*

**fun** *intval-abs* :: *Value*  $\Rightarrow$  *Value* **where**  
*intval-abs* (*IntVal32* *v*) = (if (*v*) <<sub>s</sub> 0 then (*IntVal32* ( $-$  *v*)) else (*IntVal32* *v*)) |  
*intval-abs* (*IntVal64* *v*) = (if (*v*) <<sub>s</sub> 0 then (*IntVal64* ( $-$  *v*)) else (*IntVal64* *v*)) |  
*intval-abs* - = *UndefVal*

**fun** *intval-conditional* :: *Value*  $\Rightarrow$  *Value*  $\Rightarrow$  *Value*  $\Rightarrow$  *Value* **where**  
*intval-conditional* *cond* *tv* *fv* = (if (*val-to-bool* *cond*) then *tv* else *fv*)

**fun** *intval-logic-negation* :: *Value*  $\Rightarrow$  *Value* **where**  
*intval-logic-negation* (*IntVal32* *v*) = (if *v* = 0 then (*IntVal32* 1) else (*IntVal32* 0)) |  
*intval-logic-negation* (*IntVal64* *v*) = (if *v* = 0 then (*IntVal64* 1) else (*IntVal64* 0)) |  
*intval-logic-negation* - = *UndefVal*

**definition** *shiffl* (**infix** << 75) **where**

*shiffl* *w* *n* = (*push-bit* *n*) *w*

**lemma** *shiffl-power*[*simp*]: (*x*::('a::len) *word*) \* ( $2^j$ ) = *x* << *j*  
**unfolding** *shiffl-def* **apply** (*induction* *j*)  
**apply** *simp* **unfolding** *funpow-Suc-right*  
**by** (*metis* (*no-types*, *opaque-lifting*) *push-bit-eq-mult*)

**lemma** (*x*::('a::len) *word*) \* (( $2^j$ ) + 1) = *x* << *j* + *x*  
**by** (*simp* *add: distrib-left*)

**lemma** (*x*::('a::len) *word*) \* (( $2^j$ ) - 1) = *x* << *j* - *x*  
**by** (*simp* *add: right-diff-distrib*)

**lemma** (*x*::('a::len) *word*) \* (( $2^j$ ) + ( $2^k$ )) = *x* << *j* + *x* << *k*  
**by** (*simp* *add: distrib-left*)

**lemma** (*x*::('a::len) *word*) \* (( $2^j$ ) - ( $2^k$ )) = *x* << *j* - *x* << *k*  
**by** (*simp* *add: right-diff-distrib*)



**definition** *shiftr* (**infix** >>> 75) **where**

*shiftr w n* = (*drop-bit n*) *w*

**value** (255 :: 8 word) >>> (2 :: nat)

**definition** *signed-shiftr* :: 'a :: len word  $\Rightarrow$  nat  $\Rightarrow$  'a :: len word (**infix** >> 75)  
**where**

*signed-shiftr w n* = *word-of-int* ((*sint w*) *div* (2  $^$  n))

**value** (128 :: 8 word) >> 2

**fun** *intval-left-shift* :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value **where**

*intval-left-shift* (*IntVal32 v1*) (*IntVal32 v2*) = *IntVal32* (*v1* << *unat* (*v2* AND 0x1f)) |  
*intval-left-shift* (*IntVal64 v1*) (*IntVal64 v2*) = *IntVal64* (*v1* << *unat* (*v2* AND 0x3f)) |  
*intval-left-shift* - - = *UndefVal*

**fun** *intval-right-shift* :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value **where**

*intval-right-shift* (*IntVal32 v1*) (*IntVal32 v2*) = *IntVal32* (*v1* >> *unat* (*v2* AND 0x1f)) |  
*intval-right-shift* (*IntVal64 v1*) (*IntVal64 v2*) = *IntVal64* (*v1* >> *unat* (*v2* AND 0x3f)) |  
*intval-right-shift* - - = *UndefVal*

**fun** *intval-uright-shift* :: Value  $\Rightarrow$  Value  $\Rightarrow$  Value **where**

*intval-uright-shift* (*IntVal32 v1*) (*IntVal32 v2*) = *IntVal32* (*v1* >>> *unat* (*v2* AND 0x1f)) |  
*intval-uright-shift* (*IntVal64 v1*) (*IntVal64 v2*) = *IntVal64* (*v1* >>> *unat* (*v2* AND 0x3f)) |  
*intval-uright-shift* - - = *UndefVal*

**end**

**lemma** *intval-add-sym*:

**shows** *intval-add a b* = *intval-add b a*

**by** (*induction a*; *induction b*; *auto*)

```

code-deps intval-add
code-thms intval-add

```

```

lemma intval-add (IntVal32 ( $2^{31}-1$ )) (IntVal32 ( $2^{31}-1$ )) = IntVal32 ( $-2$ )
  by eval
lemma intval-add (IntVal64 ( $2^{31}-1$ )) (IntVal64 ( $2^{31}-1$ )) = IntVal64 4294967294
  by eval

end

```

## 2 Nodes

### 2.1 Types of Nodes

```

theory IRNodes
  imports
    Values
  begin

```

The GraalVM IR is represented using a graph data structure. Here we define the nodes that are contained within the graph. Each node represents a Node subclass in the GraalVM compiler, the node classes have annotated fields to indicate input and successor edges.

We represent these classes with each IRNode constructor explicitly labelling a reference to the node IDs that it stores as inputs and successors.

The `inputs_of` and `successors_of` functions partition those labelled references into input edges and successor edges of a node.

To identify each Node, we use a simple natural number index. Zero is always the start node in a graph. For human readability, within nodes we write INPUT (or special case thereof) instead of ID for input edges, and SUCC instead of ID for control-flow successor edges. Optional edges are handled as "INPUT option" etc.

```

type-synonym ID = nat
type-synonym INPUT = ID
type-synonym INPUT-ASSOC = ID
type-synonym INPUT-STATE = ID
type-synonym INPUT-GUARD = ID
type-synonym INPUT-COND = ID
type-synonym INPUT-EXT = ID
type-synonym SUCC = ID

```

```

datatype (discs-sels) IRNode =
  AbsNode (ir-value: INPUT)
  | AddNode (ir-x: INPUT) (ir-y: INPUT)

```

- | *AndNode* (*ir-x*: *INPUT*) (*ir-y*: *INPUT*)
- | *BeginNode* (*ir-next*: *SUCC*)
- | *BytecodeExceptionNode* (*ir-arguments*: *INPUT list*) (*ir-stateAfter-opt*: *INPUT-STATE option*) (*ir-next*: *SUCC*)
- | *ConditionalNode* (*ir-condition*: *INPUT-COND*) (*ir-trueValue*: *INPUT*) (*ir-falseValue*: *INPUT*)
- | *ConstantNode* (*ir-const*: *Value*)
- | *DynamicNewArrayNode* (*ir-elementType*: *INPUT*) (*ir-length*: *INPUT*) (*ir-voidClass-opt*: *INPUT option*) (*ir-stateBefore-opt*: *INPUT-STATE option*) (*ir-next*: *SUCC*)
- | *EndNode*
- | *ExceptionObjectNode* (*ir-stateAfter-opt*: *INPUT-STATE option*) (*ir-next*: *SUCC*)
- | *FrameState* (*ir-monitorIds*: *INPUT-ASSOC list*) (*ir-outerFrameState-opt*: *INPUT-STATE option*) (*ir-values-opt*: *INPUT list option*) (*ir-virtualObjectMappings-opt*: *INPUT-STATE list option*)
- | *IfNode* (*ir-condition*: *INPUT-COND*) (*ir-trueSuccessor*: *SUCC*) (*ir-falseSuccessor*: *SUCC*)
- | *IntegerBelowNode* (*ir-x*: *INPUT*) (*ir-y*: *INPUT*)
- | *IntegerEqualsNode* (*ir-x*: *INPUT*) (*ir-y*: *INPUT*)
- | *IntegerLessThanNode* (*ir-x*: *INPUT*) (*ir-y*: *INPUT*)
- | *InvokeNode* (*ir-nid*: *ID*) (*ir-callTarget*: *INPUT-EXT*) (*ir-classInit-opt*: *INPUT option*) (*ir-stateDuring-opt*: *INPUT-STATE option*) (*ir-stateAfter-opt*: *INPUT-STATE option*) (*ir-next*: *SUCC*)
- | *InvokeWithExceptionNode* (*ir-nid*: *ID*) (*ir-callTarget*: *INPUT-EXT*) (*ir-classInit-opt*: *INPUT option*) (*ir-stateDuring-opt*: *INPUT-STATE option*) (*ir-stateAfter-opt*: *INPUT-STATE option*) (*ir-next*: *SUCC*) (*ir-exceptionEdge*: *SUCC*)
- | *IsNullNode* (*ir-value*: *INPUT*)
- | *KillingBeginNode* (*ir-next*: *SUCC*)
- | *LeftShiftNode* (*ir-x*: *INPUT*) (*ir-y*: *INPUT*)
- | *LoadFieldNode* (*ir-nid*: *ID*) (*ir-field*: *string*) (*ir-object-opt*: *INPUT option*) (*ir-next*: *SUCC*)
- | *LogicNegationNode* (*ir-value*: *INPUT-COND*)
- | *LoopBeginNode* (*ir-ends*: *INPUT-ASSOC list*) (*ir-overflowGuard-opt*: *INPUT-GUARD option*) (*ir-stateAfter-opt*: *INPUT-STATE option*) (*ir-next*: *SUCC*)
- | *LoopEndNode* (*ir-loopBegin*: *INPUT-ASSOC*)
- | *LoopExitNode* (*ir-loopBegin*: *INPUT-ASSOC*) (*ir-stateAfter-opt*: *INPUT-STATE option*) (*ir-next*: *SUCC*)
- | *MergeNode* (*ir-ends*: *INPUT-ASSOC list*) (*ir-stateAfter-opt*: *INPUT-STATE option*) (*ir-next*: *SUCC*)
- | *MethodCallTargetNode* (*ir-targetMethod*: *string*) (*ir-arguments*: *INPUT list*)
- | *MulNode* (*ir-x*: *INPUT*) (*ir-y*: *INPUT*)
- | *NarrowNode* (*ir-inputBits*: *nat*) (*ir-resultBits*: *nat*) (*ir-value*: *INPUT*)
- | *NegateNode* (*ir-value*: *INPUT*)
- | *NewArrayNode* (*ir-length*: *INPUT*) (*ir-stateBefore-opt*: *INPUT-STATE option*) (*ir-next*: *SUCC*)
- | *NewInstanceNode* (*ir-nid*: *ID*) (*ir-instanceClass*: *string*) (*ir-stateBefore-opt*: *INPUT-STATE option*) (*ir-next*: *SUCC*)
- | *NotNode* (*ir-value*: *INPUT*)
- | *OrNode* (*ir-x*: *INPUT*) (*ir-y*: *INPUT*)

```

| ParameterNode (ir-index: nat)
| PiNode (ir-object: INPUT) (ir-guard-opt: INPUT-GUARD option)
| ReturnNode (ir-result-opt: INPUT option) (ir-memoryMap-opt: INPUT-EXT
option)
| RightShiftNode (ir-x: INPUT) (ir-y: INPUT)
| ShortCircuitOrNode (ir-x: INPUT-COND) (ir-y: INPUT-COND)
| SignExtendNode (ir-inputBits: nat) (ir-resultBits: nat) (ir-value: INPUT)
| SignedDivNode (ir-nid: ID) (ir-x: INPUT) (ir-y: INPUT) (ir-zeroCheck-opt: IN-
PUT-GUARD option) (ir-stateBefore-opt: INPUT-STATE option) (ir-next: SUCC)

| SignedRemNode (ir-nid: ID) (ir-x: INPUT) (ir-y: INPUT) (ir-zeroCheck-opt:
INPUT-GUARD option) (ir-stateBefore-opt: INPUT-STATE option) (ir-next: SUCC)

| StartNode (ir-stateAfter-opt: INPUT-STATE option) (ir-next: SUCC)
| StoreFieldNode (ir-nid: ID) (ir-field: string) (ir-value: INPUT) (ir-stateAfter-opt:
INPUT-STATE option) (ir-object-opt: INPUT option) (ir-next: SUCC)
| SubNode (ir-x: INPUT) (ir-y: INPUT)
| UnsignedRightShiftNode (ir-x: INPUT) (ir-y: INPUT)
| UnwindNode (ir-exception: INPUT)
| ValuePhiNode (ir-nid: ID) (ir-values: INPUT list) (ir-merge: INPUT-ASSOC)
| ValueProxyNode (ir-value: INPUT) (ir-loopExit: INPUT-ASSOC)
| XorNode (ir-x: INPUT) (ir-y: INPUT)
| ZeroExtendNode (ir-inputBits: nat) (ir-resultBits: nat) (ir-value: INPUT)
| NoNode

| RefNode (ir-ref: ID)

```

```

fun opt-to-list :: 'a option ⇒ 'a list where
  opt-to-list None = [] |
  opt-to-list (Some v) = [v]

```

```

fun opt-list-to-list :: 'a list option ⇒ 'a list where
  opt-list-to-list None = [] |
  opt-list-to-list (Some x) = x

```

The following functions, `inputs_of` and `successors_of`, are automatically generated from the GraalVM compiler. Their purpose is to partition the node edges into input or successor edges.

```

fun inputs-of :: IRNode ⇒ ID list where
  inputs-of-AbsNode:
  inputs-of (AbsNode value) = [value] |
  inputs-of-AddNode:
  inputs-of (AddNode x y) = [x, y] |
  inputs-of-AndNode:
  inputs-of (AndNode x y) = [x, y] |

```

*inputs-of-BeginNode:*  
*inputs-of (BeginNode next) = [] |*  
*inputs-of-BytecodeExceptionNode:*  
*inputs-of (BytecodeExceptionNode arguments stateAfter next) = arguments @*  
*(opt-to-list stateAfter) |*  
*inputs-of-ConditionalNode:*  
*inputs-of (ConditionalNode condition trueValue falseValue) = [condition, true-*  
*Value, falseValue] |*  
*inputs-of-ConstantNode:*  
*inputs-of (ConstantNode const) = [] |*  
*inputs-of-DynamicNewArrayNode:*  
*inputs-of (DynamicNewArrayNode elementType length0 voidClass stateBefore*  
*next) = [elementType, length0] @ (opt-to-list voidClass) @ (opt-to-list stateBefore)*  
*|*  
*inputs-of-EndNode:*  
*inputs-of (EndNode) = [] |*  
*inputs-of-ExceptionObjectNode:*  
*inputs-of (ExceptionObjectNode stateAfter next) = (opt-to-list stateAfter) |*  
*inputs-of-FrameState:*  
*inputs-of (FrameState monitorIds outerFrameState values virtualObjectMappings)*  
*= monitorIds @ (opt-to-list outerFrameState) @ (opt-list-to-list values) @ (opt-list-to-list*  
*virtualObjectMappings) |*  
*inputs-of-IfNode:*  
*inputs-of (IfNode condition trueSuccessor falseSuccessor) = [condition] |*  
*inputs-of-IntegerBelowNode:*  
*inputs-of (IntegerBelowNode x y) = [x, y] |*  
*inputs-of-IntegerEqualsNode:*  
*inputs-of (IntegerEqualsNode x y) = [x, y] |*  
*inputs-of-IntegerLessThanNode:*  
*inputs-of (IntegerLessThanNode x y) = [x, y] |*  
*inputs-of-InvokeNode:*  
*inputs-of (InvokeNode nid0 callTarget classInit stateDuring stateAfter next)*  
*= callTarget # (opt-to-list classInit) @ (opt-to-list stateDuring) @ (opt-to-list*  
*stateAfter) |*  
*inputs-of-InvokeWithExceptionNode:*  
*inputs-of (InvokeWithExceptionNode nid0 callTarget classInit stateDuring stateAfter*  
*next exceptionEdge) = callTarget # (opt-to-list classInit) @ (opt-to-list stateDur-*  
*ing) @ (opt-to-list stateAfter) |*  
*inputs-of-IsNullNode:*  
*inputs-of (IsNullNode value) = [value] |*  
*inputs-of-KillingBeginNode:*  
*inputs-of (KillingBeginNode next) = [] |*  
*inputs-of-LeftShiftNode:*  
*inputs-of (LeftShiftNode x y) = [x, y] |*  
*inputs-of-LoadFieldNode:*  
*inputs-of (LoadFieldNode nid0 field object next) = (opt-to-list object) |*  
*inputs-of-LogicNegationNode:*  
*inputs-of (LogicNegationNode value) = [value] |*  
*inputs-of-LoopBeginNode:*

*inputs-of* (*LoopBeginNode* *ends overflowGuard stateAfter next*) = *ends* @ (*opt-to-list overflowGuard*) @ (*opt-to-list stateAfter*) |  
*inputs-of-LoopEndNode*:  
*inputs-of* (*LoopEndNode loopBegin*) = [*loopBegin*] |  
*inputs-of-LoopExitNode*:  
*inputs-of* (*LoopExitNode loopBegin stateAfter next*) = *loopBegin* # (*opt-to-list stateAfter*) |  
*inputs-of-MergeNode*:  
*inputs-of* (*MergeNode ends stateAfter next*) = *ends* @ (*opt-to-list stateAfter*) |  
*inputs-of-MethodCallTargetNode*:  
*inputs-of* (*MethodCallTargetNode targetMethod arguments*) = *arguments* |  
*inputs-of-MulNode*:  
*inputs-of* (*MulNode x y*) = [*x, y*] |  
*inputs-of-NarrowNode*:  
*inputs-of* (*NarrowNode inputBits resultBits value*) = [*value*] |  
*inputs-of-NegateNode*:  
*inputs-of* (*NegateNode value*) = [*value*] |  
*inputs-of-NewArrayNode*:  
*inputs-of* (*NewArrayNode length0 stateBefore next*) = *length0* # (*opt-to-list stateBefore*) |  
*inputs-of-NewInstanceNode*:  
*inputs-of* (*NewInstanceNode nid0 instanceClass stateBefore next*) = (*opt-to-list stateBefore*) |  
*inputs-of-NotNode*:  
*inputs-of* (*NotNode value*) = [*value*] |  
*inputs-of-OrNode*:  
*inputs-of* (*OrNode x y*) = [*x, y*] |  
*inputs-of-ParameterNode*:  
*inputs-of* (*ParameterNode index*) = [] |  
*inputs-of-PiNode*:  
*inputs-of* (*PiNode object guard*) = *object* # (*opt-to-list guard*) |  
*inputs-of-ReturnNode*:  
*inputs-of* (*ReturnNode result memoryMap*) = (*opt-to-list result*) @ (*opt-to-list memoryMap*) |  
*inputs-of-RightShiftNode*:  
*inputs-of* (*RightShiftNode x y*) = [*x, y*] |  
*inputs-of-ShortCircuitOrNode*:  
*inputs-of* (*ShortCircuitOrNode x y*) = [*x, y*] |  
*inputs-of-SignExtendNode*:  
*inputs-of* (*SignExtendNode inputBits resultBits value*) = [*value*] |  
*inputs-of-SignedDivNode*:  
*inputs-of* (*SignedDivNode nid0 x y zeroCheck stateBefore next*) = [*x, y*] @ (*opt-to-list zeroCheck*) @ (*opt-to-list stateBefore*) |  
*inputs-of-SignedRemNode*:  
*inputs-of* (*SignedRemNode nid0 x y zeroCheck stateBefore next*) = [*x, y*] @ (*opt-to-list zeroCheck*) @ (*opt-to-list stateBefore*) |  
*inputs-of-StartNode*:  
*inputs-of* (*StartNode stateAfter next*) = (*opt-to-list stateAfter*) |  
*inputs-of-StoreFieldNode*:

*inputs-of* (*StoreFieldNode* *nid0* *field* *value* *stateAfter* *object* *next*) = *value* #  
 (*opt-to-list* *stateAfter*) @ (*opt-to-list* *object*) |  
*inputs-of-SubNode*:  
*inputs-of* (*SubNode* *x* *y*) = [*x*, *y*] |  
*inputs-of-UnsignedRightShiftNode*:  
*inputs-of* (*UnsignedRightShiftNode* *x* *y*) = [*x*, *y*] |  
*inputs-of-UnwindNode*:  
*inputs-of* (*UnwindNode* *exception*) = [*exception*] |  
*inputs-of-ValuePhiNode*:  
*inputs-of* (*ValuePhiNode* *nid0* *values* *merge*) = *merge* # *values* |  
*inputs-of-ValueProxyNode*:  
*inputs-of* (*ValueProxyNode* *value* *loopExit*) = [*value*, *loopExit*] |  
*inputs-of-XorNode*:  
*inputs-of* (*XorNode* *x* *y*) = [*x*, *y*] |  
*inputs-of-ZeroExtendNode*:  
*inputs-of* (*ZeroExtendNode* *inputBits* *resultBits* *value*) = [*value*] |  
*inputs-of-NoNode*: *inputs-of* (*NoNode*) = [] |  
  
*inputs-of-RefNode*: *inputs-of* (*RefNode* *ref*) = [*ref*]

**fun** *successors-of* :: *IRNode*  $\Rightarrow$  *ID list* **where**

*successors-of-AbsNode*:  
*successors-of* (*AbsNode* *value*) = [] |  
*successors-of-AddNode*:  
*successors-of* (*AddNode* *x* *y*) = [] |  
*successors-of-AndNode*:  
*successors-of* (*AndNode* *x* *y*) = [] |  
*successors-of-BeginNode*:  
*successors-of* (*BeginNode* *next*) = [*next*] |  
*successors-of-BytecodeExceptionNode*:  
*successors-of* (*BytecodeExceptionNode* *arguments* *stateAfter* *next*) = [*next*] |  
*successors-of-ConditionalNode*:  
*successors-of* (*ConditionalNode* *condition* *trueValue* *falseValue*) = [] |  
*successors-of-ConstantNode*:  
*successors-of* (*ConstantNode* *const*) = [] |  
*successors-of-DynamicNewArrayNode*:  
*successors-of* (*DynamicNewArrayNode* *elementType* *length0* *voidClass* *stateBefore* *next*) = [*next*] |  
*successors-of-EndNode*:  
*successors-of* (*EndNode*) = [] |  
*successors-of-ExceptionObjectNode*:  
*successors-of* (*ExceptionObjectNode* *stateAfter* *next*) = [*next*] |  
*successors-of-FrameState*:  
*successors-of* (*FrameState* *monitorIds* *outerFrameState* *values* *virtualObjectMappings*) = [] |  
*successors-of-IfNode*:  
*successors-of* (*IfNode* *condition* *trueSuccessor* *falseSuccessor*) = [*trueSuccessor*,

*falseSuccessor*] |  
*successors-of-IntegerBelowNode*:  
*successors-of* (*IntegerBelowNode* *x y*) = [] |  
*successors-of-IntegerEqualsNode*:  
*successors-of* (*IntegerEqualsNode* *x y*) = [] |  
*successors-of-IntegerLessThanNode*:  
*successors-of* (*IntegerLessThanNode* *x y*) = [] |  
*successors-of-InvokeNode*:  
*successors-of* (*InvokeNode* *nid0 callTarget classInit stateDuring stateAfter next*)  
= [*next*] |  
*successors-of-InvokeWithExceptionNode*:  
*successors-of* (*InvokeWithExceptionNode* *nid0 callTarget classInit stateDuring*  
*stateAfter next exceptionEdge*) = [*next*, *exceptionEdge*] |  
*successors-of-IsNullNode*:  
*successors-of* (*IsNullNode* *value*) = [] |  
*successors-of-KillingBeginNode*:  
*successors-of* (*KillingBeginNode* *next*) = [*next*] |  
*successors-of-LeftShiftNode*:  
*successors-of* (*LeftShiftNode* *x y*) = [] |  
*successors-of-LoadFieldNode*:  
*successors-of* (*LoadFieldNode* *nid0 field object next*) = [*next*] |  
*successors-of-LogicNegationNode*:  
*successors-of* (*LogicNegationNode* *value*) = [] |  
*successors-of-LoopBeginNode*:  
*successors-of* (*LoopBeginNode* *ends overflowGuard stateAfter next*) = [*next*] |  
*successors-of-LoopEndNode*:  
*successors-of* (*LoopEndNode* *loopBegin*) = [] |  
*successors-of-LoopExitNode*:  
*successors-of* (*LoopExitNode* *loopBegin stateAfter next*) = [*next*] |  
*successors-of-MergeNode*:  
*successors-of* (*MergeNode* *ends stateAfter next*) = [*next*] |  
*successors-of-MethodCallTargetNode*:  
*successors-of* (*MethodCallTargetNode* *targetMethod arguments*) = [] |  
*successors-of-MulNode*:  
*successors-of* (*MulNode* *x y*) = [] |  
*successors-of-NarrowNode*:  
*successors-of* (*NarrowNode* *inputBits resultBits value*) = [] |  
*successors-of-NegateNode*:  
*successors-of* (*NegateNode* *value*) = [] |  
*successors-of-NewArrayNode*:  
*successors-of* (*NewArrayNode* *length0 stateBefore next*) = [*next*] |  
*successors-of-NewInstanceNode*:  
*successors-of* (*NewInstanceNode* *nid0 instanceClass stateBefore next*) = [*next*] |  
*successors-of-NotNode*:  
*successors-of* (*NotNode* *value*) = [] |  
*successors-of-OrNode*:  
*successors-of* (*OrNode* *x y*) = [] |  
*successors-of-ParameterNode*:  
*successors-of* (*ParameterNode* *index*) = [] |



*successors-of-PiNode:*  
*successors-of (PiNode object guard) = [] |*  
*successors-of-ReturnNode:*  
*successors-of (ReturnNode result memoryMap) = [] |*  
*successors-of-RightShiftNode:*  
*successors-of (RightShiftNode x y) = [] |*  
*successors-of-ShortCircuitOrNode:*  
*successors-of (ShortCircuitOrNode x y) = [] |*  
*successors-of-SignExtendNode:*  
*successors-of (SignExtendNode inputBits resultBits value) = [] |*  
*successors-of-SignedDivNode:*  
*successors-of (SignedDivNode nid0 x y zeroCheck stateBefore next) = [next] |*  
*successors-of-SignedRemNode:*  
*successors-of (SignedRemNode nid0 x y zeroCheck stateBefore next) = [next] |*  
*successors-of-StartNode:*  
*successors-of (StartNode stateAfter next) = [next] |*  
*successors-of-StoreFieldNode:*  
*successors-of (StoreFieldNode nid0 field value stateAfter object next) = [next] |*  
*successors-of-SubNode:*  
*successors-of (SubNode x y) = [] |*  
*successors-of-UnsignedRightShiftNode:*  
*successors-of (UnsignedRightShiftNode x y) = [] |*  
*successors-of-UnwindNode:*  
*successors-of (UnwindNode exception) = [] |*  
*successors-of-ValuePhiNode:*  
*successors-of (ValuePhiNode nid0 values merge) = [] |*  
*successors-of-ValueProxyNode:*  
*successors-of (ValueProxyNode value loopExit) = [] |*  
*successors-of-XorNode:*  
*successors-of (XorNode x y) = [] |*  
*successors-of-ZeroExtendNode:*  
*successors-of (ZeroExtendNode inputBits resultBits value) = [] |*  
*successors-of-NoNode: successors-of (NoNode) = [] |*

*successors-of-RefNode: successors-of (RefNode ref) = [ref]*

**lemma** *inputs-of (FrameState x (Some y) (Some z) None) = x @ [y] @ z*  
**unfolding** *inputs-of-FrameState by simp*  
**lemma** *successors-of (FrameState x (Some y) (Some z) None) = []*  
**unfolding** *inputs-of-FrameState by simp*

**lemma** *inputs-of (IfNode c t f) = [c]*  
**unfolding** *inputs-of-IfNode by simp*  
**lemma** *successors-of (IfNode c t f) = [t, f]*  
**unfolding** *successors-of-IfNode by simp*

```

lemma inputs-of (EndNode) = []  $\wedge$  successors-of (EndNode) = []
  unfolding inputs-of-EndNode successors-of-EndNode by simp

end

```

## 2.2 Hierarchy of Nodes

```

theory IRNodeHierarchy
imports IRNodes
begin

```

It is helpful to introduce a node hierarchy into our formalization. Often the GraalVM compiler relies on explicit type checks to determine which operations to perform on a given node, we try to mimic the same functionality by using a suite of predicate functions over the *IRNode* class to determine inheritance.

As one would expect, the function *is*<ClassName>Type will be true if the node parameter is a subclass of the *ClassName* within the GraalVM compiler.

These functions have been automatically generated from the compiler.

```

fun is-EndNode :: IRNode  $\Rightarrow$  bool where
  is-EndNode EndNode = True |
  is-EndNode - = False

```

```

fun is-VirtualState :: IRNode  $\Rightarrow$  bool where
  is-VirtualState n = ((is-FrameState n))

```

```

fun is-BinaryArithmeticNode :: IRNode  $\Rightarrow$  bool where
  is-BinaryArithmeticNode n = ((is-AddNode n)  $\vee$  (is-AndNode n)  $\vee$  (is-MulNode
n)  $\vee$  (is-OrNode n)  $\vee$  (is-SubNode n)  $\vee$  (is-XorNode n))

```

```

fun is-ShiftNode :: IRNode  $\Rightarrow$  bool where
  is-ShiftNode n = ((is-LeftShiftNode n)  $\vee$  (is-RightShiftNode n)  $\vee$  (is-UnsignedRightShiftNode
n))

```

```

fun is-BinaryNode :: IRNode  $\Rightarrow$  bool where
  is-BinaryNode n = ((is-BinaryArithmeticNode n)  $\vee$  (is-ShiftNode n))

```

```

fun is-AbstractLocalNode :: IRNode  $\Rightarrow$  bool where
  is-AbstractLocalNode n = ((is-ParameterNode n))

```

```

fun is-IntegerConvertNode :: IRNode  $\Rightarrow$  bool where
  is-IntegerConvertNode n = ((is-NarrowNode n)  $\vee$  (is-SignExtendNode n)  $\vee$ 
(is-ZeroExtendNode n))

```

```

fun is-UnaryArithmeticNode :: IRNode  $\Rightarrow$  bool where

```

*is-UnaryArithmeticNode* *n* = ((*is-AbsNode* *n*) ∨ (*is-NegateNode* *n*) ∨ (*is-NotNode* *n*))

**fun** *is-UnaryNode* :: *IRNode* ⇒ *bool* **where**  
*is-UnaryNode* *n* = ((*is-IntegerConvertNode* *n*) ∨ (*is-UnaryArithmeticNode* *n*))

**fun** *is-PhiNode* :: *IRNode* ⇒ *bool* **where**  
*is-PhiNode* *n* = ((*is-ValuePhiNode* *n*))

**fun** *is-FloatingGuardedNode* :: *IRNode* ⇒ *bool* **where**  
*is-FloatingGuardedNode* *n* = ((*is-PiNode* *n*))

**fun** *is-UnaryOpLogicNode* :: *IRNode* ⇒ *bool* **where**  
*is-UnaryOpLogicNode* *n* = ((*is-IsNullNode* *n*))

**fun** *is-IntegerLowerThanNode* :: *IRNode* ⇒ *bool* **where**  
*is-IntegerLowerThanNode* *n* = ((*is-IntegerBelowNode* *n*) ∨ (*is-IntegerLessThanNode* *n*))

**fun** *is-CompareNode* :: *IRNode* ⇒ *bool* **where**  
*is-CompareNode* *n* = ((*is-IntegerEqualsNode* *n*) ∨ (*is-IntegerLowerThanNode* *n*))

**fun** *is-BinaryOpLogicNode* :: *IRNode* ⇒ *bool* **where**  
*is-BinaryOpLogicNode* *n* = ((*is-CompareNode* *n*))

**fun** *is-LogicNode* :: *IRNode* ⇒ *bool* **where**  
*is-LogicNode* *n* = ((*is-BinaryOpLogicNode* *n*) ∨ (*is-LogicNegationNode* *n*) ∨ (*is-ShortCircuitOrNode* *n*) ∨ (*is-UnaryOpLogicNode* *n*))

**fun** *is-ProxyNode* :: *IRNode* ⇒ *bool* **where**  
*is-ProxyNode* *n* = ((*is-ValueProxyNode* *n*))

**fun** *is-FloatingNode* :: *IRNode* ⇒ *bool* **where**  
*is-FloatingNode* *n* = ((*is-AbstractLocalNode* *n*) ∨ (*is-BinaryNode* *n*) ∨ (*is-ConditionalNode* *n*) ∨ (*is-ConstantNode* *n*) ∨ (*is-FloatingGuardedNode* *n*) ∨ (*is-LogicNode* *n*) ∨ (*is-PhiNode* *n*) ∨ (*is-ProxyNode* *n*) ∨ (*is-UnaryNode* *n*))

**fun** *is-AccessFieldNode* :: *IRNode* ⇒ *bool* **where**  
*is-AccessFieldNode* *n* = ((*is-LoadFieldNode* *n*) ∨ (*is-StoreFieldNode* *n*))

**fun** *is-AbstractNewArrayNode* :: *IRNode* ⇒ *bool* **where**  
*is-AbstractNewArrayNode* *n* = ((*is-DynamicNewArrayNode* *n*) ∨ (*is-NewArrayNode* *n*))

**fun** *is-AbstractNewObjectNode* :: *IRNode* ⇒ *bool* **where**  
*is-AbstractNewObjectNode* *n* = ((*is-AbstractNewArrayNode* *n*) ∨ (*is-NewInstanceNode* *n*))

**fun** *is-IntegerDivRemNode* :: *IRNode* ⇒ *bool* **where**

```

is-IntegerDivRemNode n = ((is-SignedDivNode n) ∨ (is-SignedRemNode n))

fun is-FixedBinaryNode :: IRNode ⇒ bool where
  is-FixedBinaryNode n = ((is-IntegerDivRemNode n))

fun is-DeoptimizingFixedWithNextNode :: IRNode ⇒ bool where
  is-DeoptimizingFixedWithNextNode n = ((is-AbstractNewObjectNode n) ∨ (is-FixedBinaryNode n))

fun is-AbstractMemoryCheckpoint :: IRNode ⇒ bool where
  is-AbstractMemoryCheckpoint n = ((is-BytecodeExceptionNode n) ∨ (is-InvokeNode n))

fun is-AbstractStateSplit :: IRNode ⇒ bool where
  is-AbstractStateSplit n = ((is-AbstractMemoryCheckpoint n))

fun is-AbstractMergeNode :: IRNode ⇒ bool where
  is-AbstractMergeNode n = ((is-LoopBeginNode n) ∨ (is-MergeNode n))

fun is-BeginStateSplitNode :: IRNode ⇒ bool where
  is-BeginStateSplitNode n = ((is-AbstractMergeNode n) ∨ (is-ExceptionObjectNode n) ∨ (is-LoopExitNode n) ∨ (is-StartNode n))

fun is-AbstractBeginNode :: IRNode ⇒ bool where
  is-AbstractBeginNode n = ((is-BeginNode n) ∨ (is-BeginStateSplitNode n) ∨ (is-KillingBeginNode n))

fun is-FixedWithNextNode :: IRNode ⇒ bool where
  is-FixedWithNextNode n = ((is-AbstractBeginNode n) ∨ (is-AbstractStateSplit n) ∨ (is-AccessFieldNode n) ∨ (is-DeoptimizingFixedWithNextNode n))

fun is-WithExceptionNode :: IRNode ⇒ bool where
  is-WithExceptionNode n = ((is-InvokeWithExceptionNode n))

fun is-ControlSplitNode :: IRNode ⇒ bool where
  is-ControlSplitNode n = ((is-IfNode n) ∨ (is-WithExceptionNode n))

fun is-ControlSinkNode :: IRNode ⇒ bool where
  is-ControlSinkNode n = ((is-ReturnNode n) ∨ (is-UnwindNode n))

fun is-AbstractEndNode :: IRNode ⇒ bool where
  is-AbstractEndNode n = ((is-EndNode n) ∨ (is-LoopEndNode n))

fun is-FixedNode :: IRNode ⇒ bool where
  is-FixedNode n = ((is-AbstractEndNode n) ∨ (is-ControlSinkNode n) ∨ (is-ControlSplitNode n) ∨ (is-FixedWithNextNode n))

fun is-CallTargetNode :: IRNode ⇒ bool where
  is-CallTargetNode n = ((is-MethodCallTargetNode n))

```

```

fun is-ValueNode :: IRNode  $\Rightarrow$  bool where
  is-ValueNode n = ((is-CallTargetNode n)  $\vee$  (is-FixedNode n)  $\vee$  (is-FloatingNode
n))

fun is-Node :: IRNode  $\Rightarrow$  bool where
  is-Node n = ((is-ValueNode n)  $\vee$  (is-VirtualState n))

fun is-MemoryKill :: IRNode  $\Rightarrow$  bool where
  is-MemoryKill n = ((is-AbstractMemoryCheckpoint n))

fun is-NarrowableArithmeticNode :: IRNode  $\Rightarrow$  bool where
  is-NarrowableArithmeticNode n = ((is-AbsNode n)  $\vee$  (is-AddNode n)  $\vee$  (is-AndNode
n)  $\vee$  (is-MulNode n)  $\vee$  (is-NegateNode n)  $\vee$  (is-NotNode n)  $\vee$  (is-OrNode n)  $\vee$ 
(is-ShiftNode n)  $\vee$  (is-SubNode n)  $\vee$  (is-XorNode n))

fun is-AnchoringNode :: IRNode  $\Rightarrow$  bool where
  is-AnchoringNode n = ((is-AbstractBeginNode n))

fun is-DeoptBefore :: IRNode  $\Rightarrow$  bool where
  is-DeoptBefore n = ((is-DeoptimizingFixedWithNextNode n))

fun is-IndirectCanonicalization :: IRNode  $\Rightarrow$  bool where
  is-IndirectCanonicalization n = ((is-LogicNode n))

fun is-IterableNodeType :: IRNode  $\Rightarrow$  bool where
  is-IterableNodeType n = ((is-AbstractBeginNode n)  $\vee$  (is-AbstractMergeNode n)  $\vee$ 
(is-FrameState n)  $\vee$  (is-IfNode n)  $\vee$  (is-IntegerDivRemNode n)  $\vee$  (is-InvokeWithExceptionNode
n)  $\vee$  (is-LoopBeginNode n)  $\vee$  (is-LoopExitNode n)  $\vee$  (is-MethodCallTargetNode n)
 $\vee$  (is-ParameterNode n)  $\vee$  (is-ReturnNode n)  $\vee$  (is-ShortCircuitOrNode n))

fun is-Invoke :: IRNode  $\Rightarrow$  bool where
  is-Invoke n = ((is-InvokeNode n)  $\vee$  (is-InvokeWithExceptionNode n))

fun is-Proxy :: IRNode  $\Rightarrow$  bool where
  is-Proxy n = ((is-ProxyNode n))

fun is-ValueProxy :: IRNode  $\Rightarrow$  bool where
  is-ValueProxy n = ((is-PiNode n)  $\vee$  (is-ValueProxyNode n))

fun is-ValueNodeInterface :: IRNode  $\Rightarrow$  bool where
  is-ValueNodeInterface n = ((is-ValueNode n))

fun is-ArrayLengthProvider :: IRNode  $\Rightarrow$  bool where
  is-ArrayLengthProvider n = ((is-AbstractNewArrayNode n)  $\vee$  (is-ConstantNode
n))

fun is-StampInverter :: IRNode  $\Rightarrow$  bool where
  is-StampInverter n = ((is-IntegerConvertNode n)  $\vee$  (is-NegateNode n)  $\vee$  (is-NotNode
n))

```

*n*))

**fun** *is-GuardingNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
*is-GuardingNode* *n* = ((*is-AbstractBeginNode* *n*))

**fun** *is-SingleMemoryKill* :: *IRNode*  $\Rightarrow$  *bool* **where**  
*is-SingleMemoryKill* *n* = ((*is-BytecodeExceptionNode* *n*)  $\vee$  (*is-ExceptionObjectNode* *n*)  $\vee$  (*is-InvokeNode* *n*)  $\vee$  (*is-InvokeWithExceptionNode* *n*)  $\vee$  (*is-KillingBeginNode* *n*)  $\vee$  (*is-StartNode* *n*))

**fun** *is-LIRLowerable* :: *IRNode*  $\Rightarrow$  *bool* **where**  
*is-LIRLowerable* *n* = ((*is-AbstractBeginNode* *n*)  $\vee$  (*is-AbstractEndNode* *n*)  $\vee$  (*is-AbstractMergeNode* *n*)  $\vee$  (*is-BinaryOpLogicNode* *n*)  $\vee$  (*is-CallTargetNode* *n*)  $\vee$  (*is-ConditionalNode* *n*)  $\vee$  (*is-ConstantNode* *n*)  $\vee$  (*is-IfNode* *n*)  $\vee$  (*is-InvokeNode* *n*)  $\vee$  (*is-InvokeWithExceptionNode* *n*)  $\vee$  (*is-IsNullNode* *n*)  $\vee$  (*is-LoopBeginNode* *n*)  $\vee$  (*is-PiNode* *n*)  $\vee$  (*is-ReturnNode* *n*)  $\vee$  (*is-SignedDivNode* *n*)  $\vee$  (*is-SignedRemNode* *n*)  $\vee$  (*is-UnaryOpLogicNode* *n*)  $\vee$  (*is-UnwindNode* *n*))

**fun** *is-GuardedNode* :: *IRNode*  $\Rightarrow$  *bool* **where**  
*is-GuardedNode* *n* = ((*is-FloatingGuardedNode* *n*))

**fun** *is-ArithmeticLIRLowerable* :: *IRNode*  $\Rightarrow$  *bool* **where**  
*is-ArithmeticLIRLowerable* *n* = ((*is-AbsNode* *n*)  $\vee$  (*is-BinaryArithmeticNode* *n*)  $\vee$  (*is-IntegerConvertNode* *n*)  $\vee$  (*is-NotNode* *n*)  $\vee$  (*is-ShiftNode* *n*)  $\vee$  (*is-UnaryArithmeticNode* *n*))

**fun** *is-SwitchFoldable* :: *IRNode*  $\Rightarrow$  *bool* **where**  
*is-SwitchFoldable* *n* = ((*is-IfNode* *n*))

**fun** *is-VirtualizableAllocation* :: *IRNode*  $\Rightarrow$  *bool* **where**  
*is-VirtualizableAllocation* *n* = ((*is-NewArrayNode* *n*)  $\vee$  (*is-NewInstanceNode* *n*))

**fun** *is-Unary* :: *IRNode*  $\Rightarrow$  *bool* **where**  
*is-Unary* *n* = ((*is-LoadFieldNode* *n*)  $\vee$  (*is-LogicNegationNode* *n*)  $\vee$  (*is-UnaryNode* *n*)  $\vee$  (*is-UnaryOpLogicNode* *n*))

**fun** *is-FixedNodeInterface* :: *IRNode*  $\Rightarrow$  *bool* **where**  
*is-FixedNodeInterface* *n* = ((*is-FixedNode* *n*))

**fun** *is-BinaryCommutative* :: *IRNode*  $\Rightarrow$  *bool* **where**  
*is-BinaryCommutative* *n* = ((*is-AddNode* *n*)  $\vee$  (*is-AndNode* *n*)  $\vee$  (*is-IntegerEqualsNode* *n*)  $\vee$  (*is-MulNode* *n*)  $\vee$  (*is-OrNode* *n*)  $\vee$  (*is-XorNode* *n*))

**fun** *is-Canonicalizable* :: *IRNode*  $\Rightarrow$  *bool* **where**  
*is-Canonicalizable* *n* = ((*is-BytecodeExceptionNode* *n*)  $\vee$  (*is-ConditionalNode* *n*)  $\vee$  (*is-DynamicNewArrayNode* *n*)  $\vee$  (*is-PhiNode* *n*)  $\vee$  (*is-PiNode* *n*)  $\vee$  (*is-ProxyNode* *n*)  $\vee$  (*is-StoreFieldNode* *n*)  $\vee$  (*is-ValueProxyNode* *n*))

**fun** *is-UncheckedInterfaceProvider* :: *IRNode*  $\Rightarrow$  *bool* **where**

```

    is-UncheckedInterfaceProvider n = ((is-InvokeNode n) ∨ (is-InvokeWithExceptionNode
n) ∨ (is-LoadFieldNode n) ∨ (is-ParameterNode n))

```

```

fun is-Binary :: IRNode ⇒ bool where
    is-Binary n = ((is-BinaryArithmeticNode n) ∨ (is-BinaryNode n) ∨ (is-BinaryOpLogicNode
n) ∨ (is-CompareNode n) ∨ (is-FixedBinaryNode n) ∨ (is-ShortCircuitOrNode n))

```

```

fun is-ArithmeticOperation :: IRNode ⇒ bool where
    is-ArithmeticOperation n = ((is-BinaryArithmeticNode n) ∨ (is-IntegerConvertNode
n) ∨ (is-ShiftNode n) ∨ (is-UnaryArithmeticNode n))

```

```

fun is-ValueNumberable :: IRNode ⇒ bool where
    is-ValueNumberable n = ((is-FloatingNode n) ∨ (is-ProxyNode n))

```

```

fun is-Lowerable :: IRNode ⇒ bool where
    is-Lowerable n = ((is-AbstractNewObjectNode n) ∨ (is-AccessFieldNode n) ∨
(is-BytecodeExceptionNode n) ∨ (is-ExceptionObjectNode n) ∨ (is-IntegerDivRemNode
n) ∨ (is-UnwindNode n))

```

```

fun is-Virtualizable :: IRNode ⇒ bool where
    is-Virtualizable n = ((is-IsNullNode n) ∨ (is-LoadFieldNode n) ∨ (is-PiNode n)
∨ (is-StoreFieldNode n) ∨ (is-ValueProxyNode n))

```

```

fun is-Simplifiable :: IRNode ⇒ bool where
    is-Simplifiable n = ((is-AbstractMergeNode n) ∨ (is-BeginNode n) ∨ (is-IfNode
n) ∨ (is-LoopExitNode n) ∨ (is-MethodCallTargetNode n) ∨ (is-NewArrayNode n))

```

```

fun is-StateSplit :: IRNode ⇒ bool where
    is-StateSplit n = ((is-AbstractStateSplit n) ∨ (is-BeginStateSplitNode n) ∨ (is-StoreFieldNode
n))

```

```

fun is-ConvertNode :: IRNode ⇒ bool where
    is-ConvertNode n = ((is-IntegerConvertNode n))

```

```

fun is-sequential-node :: IRNode ⇒ bool where
    is-sequential-node (StartNode -) = True |
    is-sequential-node (BeginNode -) = True |
    is-sequential-node (KillingBeginNode -) = True |
    is-sequential-node (LoopBeginNode - - -) = True |
    is-sequential-node (LoopExitNode - -) = True |
    is-sequential-node (MergeNode - -) = True |
    is-sequential-node (RefNode -) = True |
    is-sequential-node - = False

```

The following convenience function is useful in determining if two IRNodes are of the same type irregardless of their edges. It will return true if both the node parameters are the same node class.

```

fun is-same-ir-node-type :: IRNode ⇒ IRNode ⇒ bool where

```

```

is-same-ir-node-type n1 n2 = (
  ((is-AbsNode n1) ∧ (is-AbsNode n2)) ∨
  ((is-AddNode n1) ∧ (is-AddNode n2)) ∨
  ((is-AndNode n1) ∧ (is-AndNode n2)) ∨
  ((is-BEGINNode n1) ∧ (is-BEGINNode n2)) ∨
  ((is-BytecodeExceptionNode n1) ∧ (is-BytecodeExceptionNode n2)) ∨
  ((is-ConditionalNode n1) ∧ (is-ConditionalNode n2)) ∨
  ((is-ConstantNode n1) ∧ (is-ConstantNode n2)) ∨
  ((is-DynamicNewArrayNode n1) ∧ (is-DynamicNewArrayNode n2)) ∨
  ((is-EndNode n1) ∧ (is-EndNode n2)) ∨
  ((is-ExceptionObjectNode n1) ∧ (is-ExceptionObjectNode n2)) ∨
  ((is-FrameState n1) ∧ (is-FrameState n2)) ∨
  ((is-IfNode n1) ∧ (is-IfNode n2)) ∨
  ((is-IntegerBelowNode n1) ∧ (is-IntegerBelowNode n2)) ∨
  ((is-IntegerEqualsNode n1) ∧ (is-IntegerEqualsNode n2)) ∨
  ((is-IntegerLessThanNode n1) ∧ (is-IntegerLessThanNode n2)) ∨
  ((is-InvokeNode n1) ∧ (is-InvokeNode n2)) ∨
  ((is-InvokeWithExceptionNode n1) ∧ (is-InvokeWithExceptionNode n2)) ∨
  ((is-IsNullNode n1) ∧ (is-IsNullNode n2)) ∨
  ((is-Killing-BEGINNode n1) ∧ (is-Killing-BEGINNode n2)) ∨
  ((is-LoadFieldNode n1) ∧ (is-LoadFieldNode n2)) ∨
  ((is-LogicNegationNode n1) ∧ (is-LogicNegationNode n2)) ∨
  ((is-Loop-BEGINNode n1) ∧ (is-Loop-BEGINNode n2)) ∨
  ((is-Loop-EndNode n1) ∧ (is-Loop-EndNode n2)) ∨
  ((is-Loop-ExitNode n1) ∧ (is-Loop-ExitNode n2)) ∨
  ((is-MergeNode n1) ∧ (is-MergeNode n2)) ∨
  ((is-MethodCallTargetNode n1) ∧ (is-MethodCallTargetNode n2)) ∨
  ((is-MulNode n1) ∧ (is-MulNode n2)) ∨
  ((is-NegateNode n1) ∧ (is-NegateNode n2)) ∨
  ((is-NewArrayNode n1) ∧ (is-NewArrayNode n2)) ∨
  ((is-NewInstanceNode n1) ∧ (is-NewInstanceNode n2)) ∨
  ((is-NotNode n1) ∧ (is-NotNode n2)) ∨
  ((is-OrNode n1) ∧ (is-OrNode n2)) ∨
  ((is-ParameterNode n1) ∧ (is-ParameterNode n2)) ∨
  ((is-PiNode n1) ∧ (is-PiNode n2)) ∨
  ((is-ReturnNode n1) ∧ (is-ReturnNode n2)) ∨
  ((is-ShortCircuitOrNode n1) ∧ (is-ShortCircuitOrNode n2)) ∨
  ((is-SignedDivNode n1) ∧ (is-SignedDivNode n2)) ∨
  ((is-StartNode n1) ∧ (is-StartNode n2)) ∨
  ((is-StoreFieldNode n1) ∧ (is-StoreFieldNode n2)) ∨
  ((is-SubNode n1) ∧ (is-SubNode n2)) ∨
  ((is-UnwindNode n1) ∧ (is-UnwindNode n2)) ∨
  ((is-ValuePhiNode n1) ∧ (is-ValuePhiNode n2)) ∨
  ((is-ValueProxyNode n1) ∧ (is-ValueProxyNode n2)) ∨
  ((is-XorNode n1) ∧ (is-XorNode n2)))

```

end



### 3 Stamp Typing

```
theory Stamp
imports Values
begin
```

The GraalVM compiler uses the Stamp class to store range and type information for a given node in the IR graph. We model the Stamp class as a datatype, Stamp, and provide a number of functions on the datatype which correspond to the class methods within the compiler.

Stamp information is used in a variety of ways in optimizations, and so, we additionally provide a number of lemmas which help to prove future optimizations.

```
datatype Stamp =
  VoidStamp
| IntegerStamp (stp-bits: nat) (stp-lower: int) (stp-upper: int)
| KlassPointerStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
| MethodCountersPointerStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
| MethodPointersStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
| ObjectStamp (stp-type: string) (stp-exactType: bool) (stp-nonNull: bool) (stp-alwaysNull:
bool)
| RawPointerStamp (stp-nonNull: bool) (stp-alwaysNull: bool)
| IllegalStamp
```

```
fun bit-bounds :: nat  $\Rightarrow$  (int  $\times$  int) where
  bit-bounds bits = (((2  $\wedge$  bits) div 2) * -1, ((2  $\wedge$  bits) div 2) - 1)
```

— A stamp which includes the full range of the type

```
fun unrestricted-stamp :: Stamp  $\Rightarrow$  Stamp where
  unrestricted-stamp VoidStamp = VoidStamp |
  unrestricted-stamp (IntegerStamp bits lower upper) = (IntegerStamp bits (fst
(bit-bounds bits)) (snd (bit-bounds bits))) |

  unrestricted-stamp (KlassPointerStamp nonNull alwaysNull) = (KlassPointerStamp
False False) |
  unrestricted-stamp (MethodCountersPointerStamp nonNull alwaysNull) = (MethodCountersPointerStamp
False False) |
  unrestricted-stamp (MethodPointersStamp nonNull alwaysNull) = (MethodPointersStamp
False False) |
  unrestricted-stamp (ObjectStamp type exactType nonNull alwaysNull) = (ObjectStamp
"" False False False) |
  unrestricted-stamp - = IllegalStamp
```

```
fun is-stamp-unrestricted :: Stamp  $\Rightarrow$  bool where
```

*is-stamp-unrestricted*  $s = (s = \text{unrestricted-stamp } s)$

— A stamp which provides type information but has an empty range of values

```
fun empty-stamp :: Stamp  $\Rightarrow$  Stamp where
  empty-stamp VoidStamp = VoidStamp |
  empty-stamp (IntegerStamp bits lower upper) = (IntegerStamp bits (snd (bit-bounds
bits)) (fst (bit-bounds bits))) |

  empty-stamp (KlassPointerStamp nonNull alwaysNull) = (KlassPointerStamp
nonNull alwaysNull) |
  empty-stamp (MethodCountersPointerStamp nonNull alwaysNull) = (MethodCountersPointerStamp
nonNull alwaysNull) |
  empty-stamp (MethodPointersStamp nonNull alwaysNull) = (MethodPointersStamp
nonNull alwaysNull) |
  empty-stamp (ObjectStamp type exactType nonNull alwaysNull) = (ObjectStamp
  "" True True False) |
  empty-stamp stamp = IllegalStamp

fun is-stamp-empty :: Stamp  $\Rightarrow$  bool where
  is-stamp-empty (IntegerStamp b lower upper) = (upper < lower) |

  is-stamp-empty x = False
```

— Calculate the meet stamp of two stamps

```
fun meet :: Stamp  $\Rightarrow$  Stamp  $\Rightarrow$  Stamp where
  meet VoidStamp VoidStamp = VoidStamp |
  meet (IntegerStamp b1 l1 u1) (IntegerStamp b2 l2 u2) = (
    if b1  $\neq$  b2 then IllegalStamp else
    (IntegerStamp b1 (min l1 l2) (max u1 u2))
  ) |

  meet (KlassPointerStamp nn1 an1) (KlassPointerStamp nn2 an2) = (
    KlassPointerStamp (nn1  $\wedge$  nn2) (an1  $\wedge$  an2)
  ) |
  meet (MethodCountersPointerStamp nn1 an1) (MethodCountersPointerStamp
nn2 an2) = (
    MethodCountersPointerStamp (nn1  $\wedge$  nn2) (an1  $\wedge$  an2)
  ) |
  meet (MethodPointersStamp nn1 an1) (MethodPointersStamp nn2 an2) = (
    MethodPointersStamp (nn1  $\wedge$  nn2) (an1  $\wedge$  an2)
  ) |
  meet s1 s2 = IllegalStamp
```

— Calculate the join stamp of two stamps

```
fun join :: Stamp  $\Rightarrow$  Stamp  $\Rightarrow$  Stamp where
  join VoidStamp VoidStamp = VoidStamp |
  join (IntegerStamp b1 l1 u1) (IntegerStamp b2 l2 u2) = (
    if b1  $\neq$  b2 then IllegalStamp else
    (IntegerStamp b1 (max l1 l2) (min u1 u2))
  )
```

```

) |
join (KlassPointerStamp nn1 an1) (KlassPointerStamp nn2 an2) = (
  if ((nn1 ∨ nn2) ∧ (an1 ∨ an2))
  then (empty-stamp (KlassPointerStamp nn1 an1))
  else (KlassPointerStamp (nn1 ∨ nn2) (an1 ∨ an2))
) |
join (MethodCountersPointerStamp nn1 an1) (MethodCountersPointerStamp nn2
an2) = (
  if ((nn1 ∨ nn2) ∧ (an1 ∨ an2))
  then (empty-stamp (MethodCountersPointerStamp nn1 an1))
  else (MethodCountersPointerStamp (nn1 ∨ nn2) (an1 ∨ an2))
) |
join (MethodPointersStamp nn1 an1) (MethodPointersStamp nn2 an2) = (
  if ((nn1 ∨ nn2) ∧ (an1 ∨ an2))
  then (empty-stamp (MethodPointersStamp nn1 an1))
  else (MethodPointersStamp (nn1 ∨ nn2) (an1 ∨ an2))
) |
join s1 s2 = IllegalStamp

```

— In certain circumstances a stamp provides enough information to evaluate a value as a stamp, the `asConstant` function converts the stamp to a value where one can be inferred.

```

fun asConstant :: Stamp ⇒ Value where
  asConstant (IntegerStamp b l h) = (if l = h then IntVal64 (word-of-int l) else
UndefVal) |
  asConstant - = UndefVal

```

— Determine if two stamps never have value overlaps i.e. their join is empty

```

fun alwaysDistinct :: Stamp ⇒ Stamp ⇒ bool where
  alwaysDistinct stamp1 stamp2 = is-stamp-empty (join stamp1 stamp2)

```

— Determine if two stamps must always be the same value i.e. two equal constants

```

fun neverDistinct :: Stamp ⇒ Stamp ⇒ bool where
  neverDistinct stamp1 stamp2 = (asConstant stamp1 = asConstant stamp2 ∧
asConstant stamp1 ≠ UndefVal)

```

```

fun constantAsStamp :: Value ⇒ Stamp where
  constantAsStamp (IntVal32 v) = (IntegerStamp (nat 32) (sint v) (sint v)) |
  constantAsStamp (IntVal64 v) = (IntegerStamp (nat 64) (sint v) (sint v)) |

  constantAsStamp - = IllegalStamp

```

— Define when a runtime value is valid for a stamp

```

fun valid-value :: Value ⇒ Stamp ⇒ bool where
  valid-value (IntVal32 v) (IntegerStamp b l h) = (b=32 ∧ (sint v ≥ l) ∧ (sint v ≤
h)) |
  valid-value (IntVal64 v) (IntegerStamp b l h) = (b=64 ∧ (sint v ≥ l) ∧ (sint v ≤

```

$h)) \mid$

$valid\_value (ObjRef\ ref) (ObjectStamp\ klass\ exact\ nonNull\ alwaysNull) = False \mid$   
 $valid\_value\ stamp\ val = False$

**fun** *compatible* :: *Stamp*  $\Rightarrow$  *Stamp*  $\Rightarrow$  *bool* **where**  
     *compatible* (*IntegerStamp* *b1* - -) (*IntegerStamp* *b2* - -) = (*b1* = *b2*)  $\mid$   
     *compatible* (*VoidStamp*) (*VoidStamp*) = *True*  $\mid$   
     *compatible* - - = *False*

**fun** *stamp-under* :: *Stamp*  $\Rightarrow$  *Stamp*  $\Rightarrow$  *bool* **where**  
     *stamp-under* *x y* = ((*stpi-upper* *x*) < (*stpi-lower* *y*))

— The most common type of stamp within the compiler (apart from the *VoidStamp*) is a 32 bit integer stamp with an unrestricted range. We use *default-stamp* as it is a frequently used stamp.

**definition** *default-stamp* :: *Stamp* **where**  
     *default-stamp* = (*unrestricted-stamp* (*IntegerStamp* 32 0 0))

**end**

## 4 Graph Representation

**theory** *IRGraph*  
     **imports**  
         *IRNodeHierarchy*  
         *Stamp*  
         *HOL-Library.FSet*  
         *HOL.Relation*

**begin**

This theory defines the main Graal data structure - an entire IR Graph.

*IRGraph* is defined as a partial map with a finite domain. The finite domain is required to be able to generate code and produce an interpreter.

**typedef** *IRGraph* = {*g* :: *ID*  $\rightarrow$  (*IRNode*  $\times$  *Stamp*) . *finite* (*dom* *g*)}

**proof** –

**have** *finite*(*dom*(*Map.empty*))  $\wedge$  *ran* *Map.empty* = {} **by** *auto*  
     **then show** *?thesis*  
         **by** *fastforce*

**qed**

**setup-lifting** *type-definition-IRGraph*

**lift-definition** *ids* :: *IRGraph*  $\Rightarrow$  *ID* *set*  
     **is**  $\lambda g. \{nid \in dom\ g . \nexists s. g\ nid = (Some\ (NoNode,\ s))\}$  .

**fun** *with-default* :: 'c  $\Rightarrow$  ('b  $\Rightarrow$  'c)  $\Rightarrow$  (('a  $\rightarrow$  'b)  $\Rightarrow$  'a  $\Rightarrow$  'c) **where**  
*with-default* def conv = ( $\lambda m$  k.  
 (case m k of None  $\Rightarrow$  def | Some v  $\Rightarrow$  conv v))

**lift-definition** *kind* :: IRGraph  $\Rightarrow$  (ID  $\Rightarrow$  IRNode)  
**is** *with-default* NoNode fst .

**lift-definition** *stamp* :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  Stamp  
**is** *with-default* IllegalStamp snd .

**lift-definition** *add-node* :: ID  $\Rightarrow$  (IRNode  $\times$  Stamp)  $\Rightarrow$  IRGraph  $\Rightarrow$  IRGraph  
**is**  $\lambda nid$  k g. if fst k = NoNode then g else g(nid  $\mapsto$  k) **by** simp

**lift-definition** *remove-node* :: ID  $\Rightarrow$  IRGraph  $\Rightarrow$  IRGraph  
**is**  $\lambda nid$  g. g(nid := None) **by** simp

**lift-definition** *replace-node* :: ID  $\Rightarrow$  (IRNode  $\times$  Stamp)  $\Rightarrow$  IRGraph  $\Rightarrow$  IRGraph  
**is**  $\lambda nid$  k g. if fst k = NoNode then g else g(nid  $\mapsto$  k) **by** simp

**lift-definition** *as-list* :: IRGraph  $\Rightarrow$  (ID  $\times$  IRNode  $\times$  Stamp) list  
**is**  $\lambda g$ . map ( $\lambda k$ . (k, the (g k))) (sorted-list-of-set (dom g)) .

**fun** *no-node* :: (ID  $\times$  (IRNode  $\times$  Stamp)) list  $\Rightarrow$  (ID  $\times$  (IRNode  $\times$  Stamp)) list  
**where**  
*no-node* g = filter ( $\lambda n$ . fst (snd n)  $\neq$  NoNode) g

**lift-definition** *irgraph* :: (ID  $\times$  (IRNode  $\times$  Stamp)) list  $\Rightarrow$  IRGraph  
**is** map-of  $\circ$  *no-node*  
**by** (simp add: finite-dom-map-of)

**definition** *as-set* :: IRGraph  $\Rightarrow$  (ID  $\times$  (IRNode  $\times$  Stamp)) set **where**  
*as-set* g = {(n, kind g n, stamp g n) | n . n  $\in$  ids g}

**definition** *true-ids* :: IRGraph  $\Rightarrow$  ID set **where**  
*true-ids* g = ids g - {n  $\in$  ids g .  $\exists n'$  . kind g n = RefNode n'}

**definition** *domain-subtraction* :: 'a set  $\Rightarrow$  ('a  $\times$  'b) set  $\Rightarrow$  ('a  $\times$  'b) set  
**(infix  $\leq$  30) where**  
*domain-subtraction* s r = {(x, y) . (x, y)  $\in$  r  $\wedge$  x  $\notin$  s}

**notation** (*latex*)  
*domain-subtraction* (-  $\triangleleft$  -)

**code-datatype** *irgraph*

**fun** *filter-none* **where**  
*filter-none* g = {nid  $\in$  dom g .  $\nexists s$ . g nid = (Some (NoNode, s))}

**lemma** *no-node-clears*:

*res* = *no-node xs*  $\longrightarrow$   $(\forall x \in \text{set } \text{res}. \text{fst } (\text{snd } x) \neq \text{NoNode})$   
**by** *simp*

**lemma** *dom-eq*:

**assumes**  $\forall x \in \text{set } \text{xs}. \text{fst } (\text{snd } x) \neq \text{NoNode}$   
**shows** *filter-none* (*map-of xs*) = *dom* (*map-of xs*)  
**unfolding** *filter-none.simps* **using** *assms map-of-SomeD*  
**by** *fastforce*

**lemma** *fil-eq*:

*filter-none* (*map-of* (*no-node xs*)) = *set* (*map fst* (*no-node xs*))  
**using** *no-node-clears*  
**by** (*metis dom-eq dom-map-of-conv-image-fst list.set-map*)

**lemma** *irgraph[code]*: *ids* (*irgraph m*) = *set* (*map fst* (*no-node m*))

**unfolding** *irgraph-def ids-def* **using** *fil-eq*  
**by** (*smt Rep-IRGraph comp-apply eq-onp-same-args filter-none.simps ids.abs-eq*  
*ids-def irgraph.abs-eq irgraph.rep-eq irgraph-def mem-Collect-eq*)

**lemma** *[code]*: *Rep-IRGraph* (*irgraph m*) = *map-of* (*no-node m*)

**using** *Abs-IRGraph-inverse*  
**by** (*simp add: irgraph.rep-eq*)

— Get the inputs set of a given node ID

**fun** *inputs* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *ID set* **where**

*inputs g nid* = *set* (*inputs-of* (*kind g nid*))

— Get the successor set of a given node ID

**fun** *succ* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *ID set* **where**

*succ g nid* = *set* (*successors-of* (*kind g nid*))

— Gives a relation between node IDs - between a node and its input nodes

**fun** *input-edges* :: *IRGraph*  $\Rightarrow$  *ID rel* **where**

*input-edges g* =  $(\bigcup i \in \text{ids } g. \{(i,j) | j. j \in (\text{inputs } g i)\})$

— Find all the nodes in the graph that have nid as an input - the usages of nid

**fun** *usages* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *ID set* **where**

*usages g nid* =  $\{j. j \in \text{ids } g \wedge (j, \text{nid}) \in \text{input-edges } g\}$

**fun** *successor-edges* :: *IRGraph*  $\Rightarrow$  *ID rel* **where**

*successor-edges g* =  $(\bigcup i \in \text{ids } g. \{(i,j) | j. j \in (\text{succ } g i)\})$

**fun** *predecessors* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *ID set* **where**

*predecessors g nid* =  $\{j. j \in \text{ids } g \wedge (j, \text{nid}) \in \text{successor-edges } g\}$

**fun** *nodes-of* :: *IRGraph*  $\Rightarrow$  (*IRNode*  $\Rightarrow$  *bool*)  $\Rightarrow$  *ID set* **where**

*nodes-of g sel* =  $\{\text{nid} \in \text{ids } g. \text{sel } (\text{kind } g \text{ nid})\}$

**fun** *edge* :: (*IRNode*  $\Rightarrow$  'a)  $\Rightarrow$  *ID*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  'a **where**

*edge sel nid g* = *sel* (*kind g nid*)

**fun** *filtered-inputs* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  (*IRNode*  $\Rightarrow$  *bool*)  $\Rightarrow$  *ID list* **where**

*filtered-inputs g nid f* = *filter* (*f*  $\circ$  (*kind g*)) (*inputs-of* (*kind g nid*))

```

fun filtered-successors :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  (IRNode  $\Rightarrow$  bool)  $\Rightarrow$  ID list where
  filtered-successors g nid f = filter (f  $\circ$  (kind g)) (successors-of (kind g nid))
fun filtered-usages :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  (IRNode  $\Rightarrow$  bool)  $\Rightarrow$  ID set where
  filtered-usages g nid f = {n  $\in$  (usages g nid). f (kind g n)}

fun is-empty :: IRGraph  $\Rightarrow$  bool where
  is-empty g = (ids g = {})

fun any-usage :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID where
  any-usage g nid = hd (sorted-list-of-set (usages g nid))

lemma ids-some[simp]: x  $\in$  ids g  $\longleftrightarrow$  kind g x  $\neq$  NoNode
proof –
  have that: x  $\in$  ids g  $\longrightarrow$  kind g x  $\neq$  NoNode
  using ids.rep-eq kind.rep-eq by force
  have kind g x  $\neq$  NoNode  $\longrightarrow$  x  $\in$  ids g
  unfolding with-default.simps kind-def ids-def
  by (cases Rep-IRGraph g x = None; auto)
  from this that show ?thesis by auto
qed

lemma not-in-g:
  assumes nid  $\notin$  ids g
  shows kind g nid = NoNode
  using asms ids-some by blast

lemma valid-creation[simp]:
  finite (dom g)  $\longleftrightarrow$  Rep-IRGraph (Abs-IRGraph g) = g
  using Abs-IRGraph-inverse by (metis Rep-IRGraph mem-Collect-eq)

lemma [simp]: finite (ids g)
  using Rep-IRGraph ids.rep-eq by simp

lemma [simp]: finite (ids (irgraph g))
  by (simp add: finite-dom-map-of)

lemma [simp]: finite (dom g)  $\longrightarrow$  ids (Abs-IRGraph g) = {nid  $\in$  dom g .  $\nexists$  s. g
  nid = Some (NoNode, s)}
  using ids.rep-eq by simp

lemma [simp]: finite (dom g)  $\longrightarrow$  kind (Abs-IRGraph g) = ( $\lambda$ x . (case g x of None
 $\Rightarrow$  NoNode | Some n  $\Rightarrow$  fst n))
  by (simp add: kind.rep-eq)

lemma [simp]: finite (dom g)  $\longrightarrow$  stamp (Abs-IRGraph g) = ( $\lambda$ x . (case g x of
  None  $\Rightarrow$  IllegalStamp | Some n  $\Rightarrow$  snd n))
  using stamp.abs-eq stamp.rep-eq by auto

lemma [simp]: ids (irgraph g) = set (map fst (no-node g))

```

```

using irgraph by auto

lemma [simp]: kind (irgraph g) = ( $\lambda$ nid. (case (map-of (no-node g)) nid of None
 $\Rightarrow$  NoNode | Some n  $\Rightarrow$  fst n))
using irgraph.rep-eq kind.transfer kind.rep-eq by auto

lemma [simp]: stamp (irgraph g) = ( $\lambda$ nid. (case (map-of (no-node g)) nid of None
 $\Rightarrow$  IllegalStamp | Some n  $\Rightarrow$  snd n))
using irgraph.rep-eq stamp.transfer stamp.rep-eq by auto

lemma map-of-upd: (map-of g)(k  $\mapsto$  v) = (map-of ((k, v) # g))
by simp

lemma [code]: replace-node nid k (irgraph g) = (irgraph ( ((nid, k) # g)))
proof (cases fst k = NoNode)
  case True
    then show ?thesis
    by (metis (mono-tags, lifting) Rep-IRGraph-inject filter.simps(2) irgraph.abs-eq
no-node.simps replace-node.rep-eq snd-conv)
  next
    case False
    then show ?thesis unfolding irgraph-def replace-node-def no-node.simps
    by (smt (verit, best) Rep-IRGraph comp-apply eq-onp-same-args filter.simps(2)
id-def irgraph.rep-eq map-fun-apply map-of-upd mem-Collect-eq no-node.elims re-
place-node.abs-eq replace-node-def snd-eqD)
  qed

lemma [code]: add-node nid k (irgraph g) = (irgraph (((nid, k) # g)))
by (smt (z3) Rep-IRGraph-inject add-node.rep-eq filter.simps(2) irgraph.rep-eq
map-of-upd no-node.simps snd-conv)

lemma add-node-lookup:
  gup = add-node nid (k, s) g  $\longrightarrow$ 
    (if k  $\neq$  NoNode then kind gup nid = k  $\wedge$  stamp gup nid = s else kind gup nid
    = kind g nid)
proof (cases k = NoNode)
  case True
    then show ?thesis
    by (simp add: add-node.rep-eq kind.rep-eq)
  next
    case False
    then show ?thesis
    by (simp add: kind.rep-eq add-node.rep-eq stamp.rep-eq)
  qed

lemma remove-node-lookup:
  gup = remove-node nid g  $\longrightarrow$  kind gup nid = NoNode  $\wedge$  stamp gup nid =
IllegalStamp

```



**by** (*simp add: kind.rep-eq remove-node.rep-eq stamp.rep-eq*)

**lemma** *replace-node-lookup*[*simp*]:

*gup = replace-node nid (k, s) g*  $\wedge k \neq \text{NoNode} \longrightarrow \text{kind } \text{gup } \text{nid} = k \wedge \text{stamp } \text{gup } \text{nid} = s$

**by** (*simp add: replace-node.rep-eq kind.rep-eq stamp.rep-eq*)

**lemma** *replace-node-unchanged*:

*gup = replace-node nid (k, s) g*  $\longrightarrow (\forall n \in (\text{ids } g - \{\text{nid}\}) . n \in \text{ids } g \wedge n \in \text{ids } \text{gup} \wedge \text{kind } g \text{ } n = \text{kind } \text{gup } n)$

**by** (*simp add: kind.rep-eq replace-node.rep-eq*)

#### 4.0.1 Example Graphs

Example 1: empty graph (just a start and end node)

**definition** *start-end-graph*:: *IRGraph* **where**

*start-end-graph* = *irgraph* [(0, *StartNode* *None* 1, *VoidStamp*), (1, *ReturnNode* *None* *None*, *VoidStamp*)]

Example 2: public static int sq(int x) return x \* x;

[1 P(0)] / [0 Start] [4 \*] | / V / [5 Return]

**definition** *eg2-sq* :: *IRGraph* **where**

*eg2-sq* = *irgraph* [  
 (0, *StartNode* *None* 5, *VoidStamp*),  
 (1, *ParameterNode* 0, *default-stamp*),  
 (4, *MulNode* 1 1, *default-stamp*),  
 (5, *ReturnNode* (*Some* 4) *None*, *default-stamp*)  
 ]

**value** *input-edges* *eg2-sq*

**value** *usages* *eg2-sq* 1

**end**

### 4.1 Control-flow Graph Traversal

**theory**

*Traversal*

**imports**

*IRGraph*

**begin**

**type-synonym** *Seen* = *ID set*

*nextEdge* helps determine which node to traverse next by returning the first

successor edge that isn't in the set of already visited nodes. If there is not an appropriate successor, None is returned instead.

```
fun nextEdge :: Seen ⇒ ID ⇒ IRGraph ⇒ ID option where
  nextEdge seen nid g =
    (let nids = (filter (λnid'. nid' ∉ seen) (successors-of (kind g nid))) in
     (if length nids > 0 then Some (hd nids) else None))
```

pred determines which node, if any, acts as the predecessor of another.

Merge nodes represent a special case where-in the predecessor exists as an input edge of the merge node, to simplify the traversal we treat only the first input end node as the predecessor, ignoring that multiple nodes may act as a successor.

For all other nodes, the predecessor is the first element of the predecessors set. Note that in a well-formed graph there should only be one element in the predecessor set.

```
fun pred :: IRGraph ⇒ ID ⇒ ID option where
  pred g nid = (case kind g nid of
    (MergeNode ends -) ⇒ Some (hd ends) |
    - ⇒
      (if IRGraph.predecessors g nid = {}
       then None else
        Some (hd (sorted-list-of-set (IRGraph.predecessors g nid))))
  )
```

Here we try to implement a generic fork of the control-flow traversal algorithm that was initially implemented for the ConditionalElimination phase

**type-synonym** 'a TraversalState = (ID × Seen × 'a)

**inductive** Step

```
:: ('a TraversalState ⇒ 'a) ⇒ IRGraph ⇒ 'a TraversalState ⇒ 'a TraversalState
option ⇒ bool
```

**for** sa g **where**

— Hit a BeginNode with an IfNode predecessor which represents the start of a basic block for the IfNode. 1. nid' will be the successor of the begin node. 2. Find the first and only predecessor. 3. Extract condition from the preceding IfNode. 4. Negate condition if the begin node is second branch (we've taken the else branch of the condition) 5. Add the condition or the negated condition to stack 6. Perform any stamp updates based on the condition using the registerNewCondition function and place them on the top of the stack of stamp information

```
[[kind g nid = BeginNode nid';
```

```
  nid ∉ seen;
  seen' = {nid} ∪ seen;
```

```
  Some ifcond = pred g nid;
  kind g ifcond = IfNode cond t f;
```

$$\begin{aligned} & \text{analysis}' = \text{sa } (nid, \text{seen}, \text{analysis}) \\ \implies & \text{Step sa } g \text{ } (nid, \text{seen}, \text{analysis}) \text{ } (\text{Some } (nid', \text{seen}', \text{analysis}')) \mid \end{aligned}$$

— Hit an EndNode 1.  $nid'$  will be the usage of EndNode 2. pop the conditions and stamp stack

$$\begin{aligned} & \llbracket \text{kind } g \text{ } nid = \text{EndNode}; \\ & \text{nid} \notin \text{seen}; \\ & \text{seen}' = \{nid\} \cup \text{seen}; \\ & \text{nid}' = \text{any-usage } g \text{ } nid; \\ & \text{analysis}' = \text{sa } (nid, \text{seen}, \text{analysis}) \\ \implies & \text{Step sa } g \text{ } (nid, \text{seen}, \text{analysis}) \text{ } (\text{Some } (nid', \text{seen}', \text{analysis}')) \mid \end{aligned}$$

— We can find a successor edge that is not in seen, go there

$$\begin{aligned} & \llbracket \neg(\text{is-EndNode } (kind \text{ } g \text{ } nid)); \\ & \neg(\text{is-BEGINNode } (kind \text{ } g \text{ } nid)); \\ & \text{nid} \notin \text{seen}; \\ & \text{seen}' = \{nid\} \cup \text{seen}; \\ & \text{Some } nid' = \text{nextEdge seen}' \text{ } nid \text{ } g; \\ & \text{analysis}' = \text{sa } (nid, \text{seen}, \text{analysis}) \\ \implies & \text{Step sa } g \text{ } (nid, \text{seen}, \text{analysis}) \text{ } (\text{Some } (nid', \text{seen}', \text{analysis}')) \mid \end{aligned}$$

— We can cannot find a successor edge that is not in seen, give back None

$$\begin{aligned} & \llbracket \neg(\text{is-EndNode } (kind \text{ } g \text{ } nid)); \\ & \neg(\text{is-BEGINNode } (kind \text{ } g \text{ } nid)); \\ & \text{nid} \notin \text{seen}; \\ & \text{seen}' = \{nid\} \cup \text{seen}; \\ & \text{None} = \text{nextEdge seen}' \text{ } nid \text{ } g \\ \implies & \text{Step sa } g \text{ } (nid, \text{seen}, \text{analysis}) \text{ } \text{None} \mid \end{aligned}$$

— We've already seen this node, give back None

$$\llbracket nid \in \text{seen} \rrbracket \implies \text{Step sa } g \text{ } (nid, \text{seen}, \text{analysis}) \text{ } \text{None}$$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$ ) *Step* .

**end**

## 4.2 Structural Graph Comparison

**theory**

*Comparison*

```

imports
  IRGraph
begin

```

We introduce a form of structural graph comparison that is able to assert structural equivalence of graphs which differ in zero or more reference node chains for any given nodes.

```

fun find-ref-nodes :: IRGraph  $\Rightarrow$  (ID  $\rightarrow$  ID) where
find-ref-nodes g = map-of
  (map ( $\lambda n.$  (n, ir-ref (kind g n))) (filter ( $\lambda id.$  is-RefNode (kind g id)) (sorted-list-of-set
  (ids g))))

```

```

fun replace-ref-nodes :: IRGraph  $\Rightarrow$  (ID  $\rightarrow$  ID)  $\Rightarrow$  ID list  $\Rightarrow$  ID list where
replace-ref-nodes g m xs = map ( $\lambda id.$  (case (m id) of Some other  $\Rightarrow$  other | None
 $\Rightarrow$  id)) xs

```

```

fun find-next :: ID list  $\Rightarrow$  ID set  $\Rightarrow$  ID option where
find-next to-see seen = (let l = (filter ( $\lambda nid.$  nid  $\notin$  seen) to-see)
  in (case l of []  $\Rightarrow$  None | xs  $\Rightarrow$  Some (hd xs)))

```

```

inductive reachables :: IRGraph  $\Rightarrow$  ID list  $\Rightarrow$  ID set  $\Rightarrow$  ID set  $\Rightarrow$  bool where
reachables g [] {} {} |
[[None = find-next to-see seen]  $\implies$  reachables g to-see seen seen |
[Some n = find-next to-see seen;
  node = kind g n;
  new = (inputs-of node) @ (successors-of node);
  reachables g (to-see @ new) ({n}  $\cup$  seen) seen']  $\implies$  reachables g to-see seen
seen'

```

```

code-pred (modes: i  $\Rightarrow$  i  $\Rightarrow$  i  $\Rightarrow$  o  $\Rightarrow$  bool) [show-steps,show-mode-inference,show-intermediate-results]
reachables .

```

```

inductive nodeEq :: (ID  $\rightarrow$  ID)  $\Rightarrow$  IRGraph  $\Rightarrow$  ID  $\Rightarrow$  IRGraph  $\Rightarrow$  ID  $\Rightarrow$  bool
where
[[ kind g1 n1 = RefNode ref; nodeEq m g1 ref g2 n2 ]  $\implies$  nodeEq m g1 n1 g2 n2 |
[ x = kind g1 n1;
  y = kind g2 n2;
  is-same-ir-node-type x y;
  replace-ref-nodes g1 m (successors-of x) = successors-of y;
  replace-ref-nodes g1 m (inputs-of x) = inputs-of y ]
 $\implies$  nodeEq m g1 n1 g2 n2

```

```

code-pred [show-modes] nodeEq .

```

```

fun diffNodesGraph :: IRGraph  $\Rightarrow$  IRGraph  $\Rightarrow$  ID set where
diffNodesGraph g1 g2 = (let refNodes = find-ref-nodes g1 in
  { n . n  $\in$  Predicate.the (reachables-i-i-i-o g1 [0] {})  $\wedge$  (case refNodes n of Some
  -  $\Rightarrow$  False | -  $\Rightarrow$  True)  $\wedge$   $\neg$ (nodeEq refNodes g1 n g2 n)})

```

```
fun diffNodesInfo :: IRGraph ⇒ IRGraph ⇒ (ID × IRNode × IRNode) set where
  diffNodesInfo g1 g2 = {(nid, kind g1 nid, kind g2 nid) | nid . nid ∈ diffNodesGraph
    g1 g2}
```

```
fun eqGraph :: IRGraph ⇒ IRGraph ⇒ bool where
  eqGraph isabelle-graph graal-graph = ((diffNodesGraph isabelle-graph graal-graph)
    = {})
```

```
end
```

## 5 Data-flow Semantics

```
theory IRTreeEval
  imports
    Graph.Values
    Graph.Stamp
  begin
```

We define a tree representation of data-flow nodes, as an abstraction of the graph view.

Data-flow trees are evaluated in the context of a method state (currently called MapState in the theories for historical reasons).

The method state consists of the values for each method parameter, references to method parameters use an index of the parameter within the parameter list, as such we store a list of parameter values which are looked up at parameter references.

The method state also stores a mapping of node ids to values. The contents of this mapping is calculates during the traversal of the control flow graph.

As a concrete example, as the *SignedDivNode* can have side-effects (during division by zero), it is treated as part of the control-flow, since the data-flow phase is specified to be side-effect free. As a result, the control-flow semantics for *SignedDivNode* calculates the value of a node and maps the node identifier to the value within the method state. The data-flow semantics then just reads the value stored in the method state for the node.

```
type-synonym ID = nat
type-synonym MapState = ID ⇒ Value
type-synonym Params = Value list
```

```
definition new-map-state :: MapState where
  new-map-state = (λx..UndefVal)
```

### 5.1 Data-flow Tree Representation

```
datatype IRUnaryOp =
```

```

    UnaryAbs
  | UnaryNeg
  | UnaryNot
  | UnaryLogicNegation
  | UnaryNarrow (ir-inputBits: nat) (ir-resultBits: nat)
  | UnarySignExtend (ir-inputBits: nat) (ir-resultBits: nat)
  | UnaryZeroExtend (ir-inputBits: nat) (ir-resultBits: nat)

datatype IRBinaryOp =
  BinAdd
  | BinMul
  | BinSub
  | BinAnd
  | BinOr
  | BinXor
  | BinLeftShift
  | BinRightShift
  | BinURightShift
  | BinIntegerEquals
  | BinIntegerLessThan
  | BinIntegerBelow

datatype (discs-sels) IRExpr =
  UnaryExpr (ir-uop: IRUnaryOp) (ir-value: IRExpr)
  | BinaryExpr (ir-op: IRBinaryOp) (ir-x: IRExpr) (ir-y: IRExpr)
  | ConditionalExpr (ir-condition: IRExpr) (ir-trueValue: IRExpr) (ir-falseValue:
IRExpr)

  | ParameterExpr (ir-index: nat) (ir-stamp: Stamp)

  | LeafExpr (ir-nid: ID) (ir-stamp: Stamp)

  | ConstantExpr (ir-const: Value)
  | ConstantVar (ir-name: string)
  | VariableExpr (ir-name: string) (ir-stamp: Stamp)

fun is-ground :: IRExpr ⇒ bool where
  is-ground (UnaryExpr op e) = is-ground e |
  is-ground (BinaryExpr op e1 e2) = (is-ground e1 ∧ is-ground e2) |
  is-ground (ConditionalExpr b e1 e2) = (is-ground b ∧ is-ground e1 ∧ is-ground
e2) |
  is-ground (ParameterExpr i s) = True |
  is-ground (LeafExpr n s) = True |
  is-ground (ConstantExpr v) = True |
  is-ground (ConstantVar name) = False |
  is-ground (VariableExpr name s) = False

typedef GroundExpr = { e :: IRExpr . is-ground e }

```

```

using is-ground.simps(6) by blast

fun stamp-unary :: IRUnaryOp ⇒ Stamp ⇒ Stamp where
  stamp-unary op (IntegerStamp b lo hi) = unrestricted-stamp (IntegerStamp b lo
hi) |

  stamp-unary op - = IllegalStamp

definition fixed-32 :: IRBinaryOp set where
  fixed-32 = {BinIntegerEquals, BinIntegerLessThan, BinIntegerBelow}

fun stamp-binary :: IRBinaryOp ⇒ Stamp ⇒ Stamp ⇒ Stamp where
  stamp-binary op (IntegerStamp b1 lo1 hi1) (IntegerStamp b2 lo2 hi2) =
    (case op ∈ fixed-32 of True ⇒ unrestricted-stamp (IntegerStamp 32 lo1 hi1) |
```

*False* ⇒

```

    (if (b1 = b2) then unrestricted-stamp (IntegerStamp b1 lo1 hi1) else Illegal-
Stamp)) |

  stamp-binary op - - = IllegalStamp

fun stamp-expr :: IRExpr ⇒ Stamp where
  stamp-expr (UnaryExpr op x) = stamp-unary op (stamp-expr x) |
  stamp-expr (BinaryExpr bop x y) = stamp-binary bop (stamp-expr x) (stamp-expr
y) |
  stamp-expr (ConstantExpr val) = constantAsStamp val |
  stamp-expr (LeafExpr i s) = s |
  stamp-expr (ParameterExpr i s) = s |
  stamp-expr (ConditionalExpr c t f) = meet (stamp-expr t) (stamp-expr f)

export-code stamp-unary stamp-binary stamp-expr

```

## 5.2 Data-flow Tree Evaluation

```

fun unary-eval :: IRUnaryOp ⇒ Value ⇒ Value where
  unary-eval UnaryAbs v = intval-abs v |
  unary-eval UnaryNeg v = intval-negate v |
  unary-eval UnaryNot v = intval-not v |
  unary-eval UnaryLogicNegation v = intval-logic-negation v |
  unary-eval op v1 = UndefVal

fun bin-eval :: IRBinaryOp ⇒ Value ⇒ Value ⇒ Value where
  bin-eval BinAdd v1 v2 = intval-add v1 v2 |
  bin-eval BinMul v1 v2 = intval-mul v1 v2 |
  bin-eval BinSub v1 v2 = intval-sub v1 v2 |
  bin-eval BinAnd v1 v2 = intval-and v1 v2 |
  bin-eval BinOr v1 v2 = intval-or v1 v2 |
  bin-eval BinXor v1 v2 = intval-xor v1 v2 |
  bin-eval BinLeftShift v1 v2 = intval-left-shift v1 v2 |

```

$\text{bin-eval BinRightShift } v1 \ v2 = \text{intval-right-shift } v1 \ v2 \mid$   
 $\text{bin-eval BinURightShift } v1 \ v2 = \text{intval-uright-shift } v1 \ v2 \mid$   
 $\text{bin-eval BinIntegerEquals } v1 \ v2 = \text{intval-equals } v1 \ v2 \mid$   
 $\text{bin-eval BinIntegerLessThan } v1 \ v2 = \text{intval-less-than } v1 \ v2 \mid$   
 $\text{bin-eval BinIntegerBelow } v1 \ v2 = \text{intval-below } v1 \ v2$

**inductive** *not-undef-or-fail* :: *Value*  $\Rightarrow$  *Value*  $\Rightarrow$  *bool* **where**  
 $\llbracket \text{value} \neq \text{UndefVal} \rrbracket \implies \text{not-undef-or-fail value value}$

**notation** (*latex output*)  
 $\text{not-undef-or-fail } (- = -)$

**inductive**  
 $\text{evaltree} :: \text{MapState} \Rightarrow \text{Params} \Rightarrow \text{IRExpr} \Rightarrow \text{Value} \Rightarrow \text{bool} \ ([-, -] \vdash - \mapsto - \ 55)$   
**for** *m p* **where**

*ConstantExpr*:  
 $\llbracket \text{valid-value } c \ (\text{constantAsStamp } c) \rrbracket$   
 $\implies [m, p] \vdash (\text{ConstantExpr } c) \mapsto c \mid$

*ParameterExpr*:  
 $\llbracket i < \text{length } p; \text{valid-value } (p!i) \ s \rrbracket$   
 $\implies [m, p] \vdash (\text{ParameterExpr } i \ s) \mapsto p!i \mid$

*ConditionalExpr*:  
 $\llbracket [m, p] \vdash ce \mapsto \text{cond};$   
 $\text{branch} = (\text{if val-to-bool cond then te else fe});$   
 $[m, p] \vdash \text{branch} \mapsto v;$   
 $v \neq \text{UndefVal} \rrbracket$   
 $\implies [m, p] \vdash (\text{ConditionalExpr } ce \ te \ fe) \mapsto v \mid$

*UnaryExpr*:  
 $\llbracket [m, p] \vdash xe \mapsto v;$   
 $\text{result} = (\text{unary-eval } op \ v);$   
 $\text{result} \neq \text{UndefVal} \rrbracket$   
 $\implies [m, p] \vdash (\text{UnaryExpr } op \ xe) \mapsto \text{result} \mid$

*BinaryExpr*:  
 $\llbracket [m, p] \vdash xe \mapsto x;$   
 $[m, p] \vdash ye \mapsto y;$   
 $\text{result} = (\text{bin-eval } op \ x \ y);$   
 $\text{result} \neq \text{UndefVal} \rrbracket$   
 $\implies [m, p] \vdash (\text{BinaryExpr } op \ xe \ ye) \mapsto \text{result} \mid$

*LeafExpr*:  
 $\llbracket \text{val} = m \ n;$   
 $\text{valid-value val } s \rrbracket$



$\implies [m,p] \vdash \text{LeafExpr } n \ s \mapsto \text{val}$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$  as *evalT*)  
 $[show-steps, show-mode-inference, show-intermediate-results]$   
*evaltree* .

**inductive**

*evaltrees* :: *MapState*  $\Rightarrow$  *Params*  $\Rightarrow$  *IRExpr list*  $\Rightarrow$  *Value list*  $\Rightarrow$  *bool* ( $[-,] \vdash - \mapsto_L$   
- 55)

**for** *m p* **where**

*EvalNil*:

$[m,p] \vdash [] \mapsto_L []$  |

*EvalCons*:

$[[m,p] \vdash x \mapsto \text{xval};$

$[m,p] \vdash yy \mapsto_L \text{yyval}]$

$\implies [m,p] \vdash (x\#yy) \mapsto_L (\text{xval}\#\text{yyval})$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$  as *evalTs*)  
*evaltrees* .

**definition** *sq-param0* :: *IRExpr* **where**

*sq-param0* = *BinaryExpr BinMul*

(*ParameterExpr* 0 (*IntegerStamp* 32 (− 2147483648) 2147483647))

(*ParameterExpr* 0 (*IntegerStamp* 32 (− 2147483648) 2147483647))

**values** {*v*. *evaltree new-map-state* [*IntVal32* 5] *sq-param0 v*}

**declare** *evaltree.intros* [*intro*]

**declare** *evaltrees.intros* [*intro*]

### 5.3 Data-flow Tree Refinement

We define the induced semantic equivalence relation between expressions. Note that syntactic equality implies semantic equivalence, but not vice versa.

**definition** *equiv-exprs* :: *IRExpr*  $\Rightarrow$  *IRExpr*  $\Rightarrow$  *bool* ( $- \doteq -$  55) **where**

( $e1 \doteq e2$ ) = ( $\forall \ m \ p \ v. ([m,p] \vdash e1 \mapsto v) \longleftrightarrow ([m,p] \vdash e2 \mapsto v)$ )

We also prove that this is a total equivalence relation (*equivp equiv-exprs*) (HOL.Equiv\_Relations), so that we can reuse standard results about equivalence relations.

**lemma** *equivp equiv-exprs*

**apply** (*auto simp add: equivp-def equiv-exprs-def*)

**by** (*metis equiv-exprs-def*)+

We define a refinement ordering over  $IRExpr$  and show that it is a preorder. Note that it is asymmetric because  $e_2$  may refer to fewer variables than  $e_1$ .

**instantiation**  $IRExpr :: preorder$  **begin**

**notation**  $less-eq$  (**infix**  $\sqsubseteq$  65)

**definition**

$le-expr-def$  [*simp*]:

$(e_2 \leq e_1) \longleftrightarrow (\forall m p v. ([m,p] \vdash e_1 \mapsto v) \longrightarrow ([m,p] \vdash e_2 \mapsto v))$

**definition**

$lt-expr-def$  [*simp*]:

$(e_1 < e_2) \longleftrightarrow (e_1 \leq e_2 \wedge \neg (e_1 \doteq e_2))$

**instance proof**

**fix**  $x y z :: IRExpr$

**show**  $x < y \longleftrightarrow x \leq y \wedge \neg (y \leq x)$  **by** (*simp add: equiv-exprs-def; auto*)

**show**  $x \leq x$  **by** *simp*

**show**  $x \leq y \implies y \leq z \implies x \leq z$  **by** *simp*

**qed**

**end**

**abbreviation** (**output**)  $Refines :: IRExpr \Rightarrow IRExpr \Rightarrow bool$  (**infix**  $\sqsupseteq$  64)

**where**  $e_1 \sqsupseteq e_2 \equiv (e_2 \leq e_1)$

**end**

## 5.4 Data-flow Tree Theorems

**theory**  $IRTreeEvalThms$

**imports**

$IRTreeEval$

**begin**

### 5.4.1 Deterministic Data-flow Evaluation

**lemma**  $evalDet$ :

$[m,p] \vdash e \mapsto v_1 \implies$

$[m,p] \vdash e \mapsto v_2 \implies$

$v_1 = v_2$

**apply** (*induction arbitrary: v2 rule: evaltree.induct*)

**by** (*elim EvalTreeE; auto*)+

**lemma**  $evalAllDet$ :

$[m,p] \vdash e \mapsto_L v1 \implies$

$[m,p] \vdash e \mapsto_L v2 \implies$

$v1 = v2$

**apply** (*induction arbitrary: v2 rule: evaltrees.induct*)

```

apply (elim EvalTreeE; auto)
using evalDet by force

```

### 5.4.2 Evaluation Results are Valid

A valid value cannot be *UndefVal*.

```

lemma valid-not-undef:
  assumes a1: valid-value val s
  assumes a2: s ≠ VoidStamp
  shows val ≠ UndefVal
  apply (rule valid-value.elims(1)[of val s True])
  using a1 a2 by auto

```

```

lemma valid-VoidStamp[elim]:
  shows valid-value val VoidStamp  $\implies$ 
    val = UndefVal
  using valid-value.simps by metis

```

```

lemma valid-ObjStamp[elim]:
  shows valid-value val (ObjectStamp klass exact nonNull alwaysNull)  $\implies$ 
    ( $\exists v. val = ObjRef v$ )
  using valid-value.simps by (metis val-to-bool.cases)

```

```

lemma valid-int32[elim]:
  shows valid-value val (IntegerStamp 32 l h)  $\implies$ 
    ( $\exists v. val = IntVal32 v$ )
  apply (rule val-to-bool.cases[of val])
  using Value.distinct by simp+

```

```

lemma valid-int64[elim]:
  shows valid-value val (IntegerStamp 64 l h)  $\implies$ 
    ( $\exists v. val = IntVal64 v$ )
  apply (rule val-to-bool.cases[of val])
  using Value.distinct by simp+

```

```

lemmas valid-value-elim =
  valid-VoidStamp
  valid-ObjStamp
  valid-int32
  valid-int64

```

```

lemma evaltree-not-undef:
  fixes m p e v
  shows ( $[m,p] \vdash e \mapsto v$ )  $\implies v \neq UndefVal$ 
  apply (induction rule: evaltree.induct)
  using valid-not-undef by auto

```

**lemma** *leafint32*:

**assumes** *ev*:  $[m,p] \vdash \text{LeafExpr } i \ (\text{IntegerStamp } 32 \text{ lo } hi) \mapsto \text{val}$   
**shows**  $\exists v. \text{val} = (\text{IntVal32 } v)$

**proof** –

**have** *valid-value* *val* (*IntegerStamp* 32 lo hi)  
**using** *ev* **by** (rule *LeafExprE*; *simp*)  
**then show** *?thesis* **by** *auto*

**qed**

**lemma** *leafint64*:

**assumes** *ev*:  $[m,p] \vdash \text{LeafExpr } i \ (\text{IntegerStamp } 64 \text{ lo } hi) \mapsto \text{val}$   
**shows**  $\exists v. \text{val} = (\text{IntVal64 } v)$

**proof** –

**have** *valid-value* *val* (*IntegerStamp* 64 lo hi)  
**using** *ev* **by** (rule *LeafExprE*; *simp*)  
**then show** *?thesis* **by** *auto*

**qed**

**lemma** *default-stamp* [*simp*]: *default-stamp* = *IntegerStamp* 32 (–2147483648)  
2147483647

**using** *default-stamp-def* **by** *auto*

**lemma** *valid32* [*simp*]:

**assumes** *valid-value* *val* (*IntegerStamp* 32 lo hi)  
**shows**  $\exists v. (\text{val} = (\text{IntVal32 } v) \wedge \text{lo} \leq \text{sint } v \wedge \text{sint } v \leq \text{hi})$   
**using** *assms valid-int32* **by** *force*

**lemma** *valid64* [*simp*]:

**assumes** *valid-value* *val* (*IntegerStamp* 64 lo hi)  
**shows**  $\exists v. (\text{val} = (\text{IntVal64 } v) \wedge \text{lo} \leq \text{sint } v \wedge \text{sint } v \leq \text{hi})$   
**using** *assms valid-int64* **by** *force*

**lemma** *valid32or64*:

**assumes** *valid-value* *x* (*IntegerStamp* *b* lo hi)  
**shows**  $(\exists v1. (x = \text{IntVal32 } v1)) \vee (\exists v2. (x = \text{IntVal64 } v2))$   
**using** *valid32 valid64 assms valid-value.elims(2)* **by** *blast*

**lemma** *valid32or64-both*:

**assumes** *valid-value* *x* (*IntegerStamp* *b* lox hix)  
**and** *valid-value* *y* (*IntegerStamp* *b* loy hiy)  
**shows**  $(\exists v1 v2. x = \text{IntVal32 } v1 \wedge y = \text{IntVal32 } v2) \vee (\exists v3 v4. x = \text{IntVal64 } v3 \wedge y = \text{IntVal64 } v4)$   
**using** *assms valid32or64 valid32 valid-value.elims(2) valid-value.simps(1)* **by** *metis*

### 5.4.3 Example Data-flow Optimisations

```

lemma a0a-helper [simp]:
  assumes a: valid-value v (IntegerStamp 32 lo hi)
  shows intval-add v (IntVal32 0) = v
proof –
  obtain v32 :: int32 where v = (IntVal32 v32) using a valid32 by blast
  then show ?thesis by simp
qed

lemma a0a: (BinaryExpr BinAdd (LeafExpr 1 default-stamp) (ConstantExpr (IntVal32
0)))
  ≥ (LeafExpr 1 default-stamp)
by (auto simp add: evaltree.LeanExpr)

```

```

lemma xyx-y-helper [simp]:
  assumes valid-value x (IntegerStamp 32 lox hix)
  assumes valid-value y (IntegerStamp 32 loy hiy)
  shows intval-add x (intval-sub y x) = y
proof –
  obtain x32 :: int32 where x: x = (IntVal32 x32) using assms valid32 by blast
  obtain y32 :: int32 where y: y = (IntVal32 y32) using assms valid32 by blast
  show ?thesis using x y by simp
qed

```

```

lemma xyx-y:
  (BinaryExpr BinAdd
    (LeafExpr x (IntegerStamp 32 lox hix))
    (BinaryExpr BinSub
      (LeafExpr y (IntegerStamp 32 loy hiy))
      (LeafExpr x (IntegerStamp 32 lox hix))))
  ≥ (LeafExpr y (IntegerStamp 32 loy hiy))
by (auto simp add: LeafExpr)

```

### 5.4.4 Monotonicity of Expression Refinement

We prove that each subexpression position is monotonic. That is, optimizing a subexpression anywhere deep inside a top-level expression also optimizes that top-level expression.

Note that we might also be able to do this via reusing Isabelle’s ‘mono’ operator (HOL.Orderings theory), proving instantiations like ‘mono (UnaryExpr op)’, but it is not obvious how to do this for both arguments of the binary expressions.

```

lemma mono-unary:
  assumes e ≥ e'
  shows (UnaryExpr op e) ≥ (UnaryExpr op e')

```

```

using UnaryExpr assms by auto

lemma mono-binary:
  assumes  $x \geq x'$ 
  assumes  $y \geq y'$ 
  shows  $(BinaryExpr\ op\ x\ y) \geq (BinaryExpr\ op\ x'\ y')$ 
  using BinaryExpr assms by auto

lemma never-void:
  assumes  $[m, p] \vdash x \mapsto xv$ 
  assumes valid-value  $xv$  (stamp-expr  $xe$ )
  shows stamp-expr  $xe \neq VoidStamp$ 
  using valid-value.simps
  using assms(2) by force

lemma stamp32:
   $\exists v . xv = IntVal32\ v \longleftrightarrow \text{valid-value } xv\ (IntegerStamp\ 32\ lo\ hi)$ 
  using valid-int32
  by (metis (full-types) Value.inject(1) zero-neq-one)

lemma stamp64:
   $\exists v . xv = IntVal64\ v \longleftrightarrow \text{valid-value } xv\ (IntegerStamp\ 64\ lo\ hi)$ 
  using valid-int64
  by (metis (full-types) Value.inject(2) zero-neq-one)

lemma stamprange:
  valid-value  $v\ s \longrightarrow (\exists b\ lo\ hi. (s = IntegerStamp\ b\ lo\ hi) \wedge (b = 32 \vee b = 64))$ 
  using valid-value.elims stamp32 stamp64
  by (smt (verit, del-insts))

lemma compatible-trans:
  compatible  $x\ y \wedge \text{compatible } y\ z \implies \text{compatible } x\ z$ 
  by (smt (verit, best) compatible.elims(2) compatible.simps(1))

lemma compatible-refl:
  compatible  $x\ y \implies \text{compatible } y\ x$ 
  using compatible.elims(2) by fastforce

lemma mono-conditional:
  assumes  $ce \geq ce'$ 
  assumes  $te \geq te'$ 
  assumes  $fe \geq fe'$ 
  shows  $(ConditionalExpr\ ce\ te\ fe) \geq (ConditionalExpr\ ce'\ te'\ fe')$ 
proof (simp only: le-expr-def; (rule allI) $+$ ; rule impI)
  fix  $m\ p\ v$ 
  assume  $a: [m, p] \vdash ConditionalExpr\ ce\ te\ fe \mapsto v$ 
  then obtain cond where  $ce: [m, p] \vdash ce \mapsto cond$  by auto

```

```

then have ce': [m,p] ⊢ ce' ↦ cond using assms by auto

define branch where b: branch = (if val-to-bool cond then te else fe)
define branch' where b': branch' = (if val-to-bool cond then te' else fe')
then have beval: [m,p] ⊢ branch ↦ v using a b ce evalDet by blast

from beval have [m,p] ⊢ branch' ↦ v using assms b b' by auto
then show [m,p] ⊢ ConditionalExpr ce' te' fe' ↦ v
  using ConditionalExpr ce' b'
  using a by blast
qed

end

```

## 6 Tree to Graph

```

theory TreeToGraph
  imports
    Semantics.IRTreeEval
    Graph.IRGraph
begin

```

### 6.1 Subgraph to Data-flow Tree

```

fun find-node-and-stamp :: IRGraph ⇒ (IRNode × Stamp) ⇒ ID option where
  find-node-and-stamp g (n,s) =
    find (λi. kind g i = n ∧ stamp g i = s) (sorted-list-of-set(ids g))

export-code find-node-and-stamp

```

```

fun is-preevaluated :: IRNode ⇒ bool where
  is-preevaluated (InvokeNode n - - - -) = True |
  is-preevaluated (InvokeWithExceptionNode n - - - -) = True |
  is-preevaluated (NewInstanceNode n - -) = True |
  is-preevaluated (LoadFieldNode n - -) = True |
  is-preevaluated (SignedDivNode n - - - -) = True |
  is-preevaluated (SignedRemNode n - - - -) = True |
  is-preevaluated (ValuePhiNode n -) = True |
  is-preevaluated - = False

```

```

inductive
  rep :: IRGraph ⇒ ID ⇒ IRExpr ⇒ bool (- ⊢ - ≃ - 55)
  for g where

  ConstantNode:
    [[kind g n = ConstantNode c]]

```

$$\implies g \vdash n \simeq (\text{ConstantExpr } c) \mid$$

*ParameterNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{ParameterNode } i; \\ & \quad \text{stamp } g \ n = s \rrbracket \\ & \implies g \vdash n \simeq (\text{ParameterExpr } i \ s) \mid \end{aligned}$$

*ConditionalNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{ConditionalNode } c \ t \ f; \\ & \quad g \vdash c \simeq ce; \\ & \quad g \vdash t \simeq te; \\ & \quad g \vdash f \simeq fe \rrbracket \\ & \implies g \vdash n \simeq (\text{ConditionalExpr } ce \ te \ fe) \mid \end{aligned}$$

*AbsNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{AbsNode } x; \\ & \quad g \vdash x \simeq xe \rrbracket \\ & \implies g \vdash n \simeq (\text{UnaryExpr } \text{UnaryAbs } xe) \mid \end{aligned}$$

*NotNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{NotNode } x; \\ & \quad g \vdash x \simeq xe \rrbracket \\ & \implies g \vdash n \simeq (\text{UnaryExpr } \text{UnaryNot } xe) \mid \end{aligned}$$

*NegateNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{NegateNode } x; \\ & \quad g \vdash x \simeq xe \rrbracket \\ & \implies g \vdash n \simeq (\text{UnaryExpr } \text{UnaryNeg } xe) \mid \end{aligned}$$

*LogicNegationNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{LogicNegationNode } x; \\ & \quad g \vdash x \simeq xe \rrbracket \\ & \implies g \vdash n \simeq (\text{UnaryExpr } \text{UnaryLogicNegation } xe) \mid \end{aligned}$$

*AddNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{AddNode } x \ y; \\ & \quad g \vdash x \simeq xe; \\ & \quad g \vdash y \simeq ye \rrbracket \\ & \implies g \vdash n \simeq (\text{BinaryExpr } \text{BinAdd } xe \ ye) \mid \end{aligned}$$

*MulNode:*

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{MulNode } x \ y; \\ & \quad g \vdash x \simeq xe; \\ & \quad g \vdash y \simeq ye \rrbracket \\ & \implies g \vdash n \simeq (\text{BinaryExpr } \text{BinMul } xe \ ye) \mid \end{aligned}$$

*SubNode:*



$\llbracket \text{kind } g \ n = \text{SubNode } x \ y;$   
 $g \vdash x \simeq xe;$   
 $g \vdash y \simeq ye \rrbracket$   
 $\implies g \vdash n \simeq (\text{BinaryExpr BinSub } xe \ ye) \mid$

*AndNode:*  
 $\llbracket \text{kind } g \ n = \text{AndNode } x \ y;$   
 $g \vdash x \simeq xe;$   
 $g \vdash y \simeq ye \rrbracket$   
 $\implies g \vdash n \simeq (\text{BinaryExpr BinAnd } xe \ ye) \mid$

*OrNode:*  
 $\llbracket \text{kind } g \ n = \text{OrNode } x \ y;$   
 $g \vdash x \simeq xe;$   
 $g \vdash y \simeq ye \rrbracket$   
 $\implies g \vdash n \simeq (\text{BinaryExpr BinOr } xe \ ye) \mid$

*XorNode:*  
 $\llbracket \text{kind } g \ n = \text{XorNode } x \ y;$   
 $g \vdash x \simeq xe;$   
 $g \vdash y \simeq ye \rrbracket$   
 $\implies g \vdash n \simeq (\text{BinaryExpr BinXor } xe \ ye) \mid$

*LeftShiftNode:*  
 $\llbracket \text{kind } g \ n = \text{LeftShiftNode } x \ y;$   
 $g \vdash x \simeq xe;$   
 $g \vdash y \simeq ye \rrbracket$   
 $\implies g \vdash n \simeq (\text{BinaryExpr BinLeftShift } xe \ ye) \mid$

*RightShiftNode:*  
 $\llbracket \text{kind } g \ n = \text{RightShiftNode } x \ y;$   
 $g \vdash x \simeq xe;$   
 $g \vdash y \simeq ye \rrbracket$   
 $\implies g \vdash n \simeq (\text{BinaryExpr BinRightShift } xe \ ye) \mid$

*UnsignedRightShiftNode:*  
 $\llbracket \text{kind } g \ n = \text{UnsignedRightShiftNode } x \ y;$   
 $g \vdash x \simeq xe;$   
 $g \vdash y \simeq ye \rrbracket$   
 $\implies g \vdash n \simeq (\text{BinaryExpr BinURightShift } xe \ ye) \mid$

*IntegerBelowNode:*  
 $\llbracket \text{kind } g \ n = \text{IntegerBelowNode } x \ y;$   
 $g \vdash x \simeq xe;$   
 $g \vdash y \simeq ye \rrbracket$   
 $\implies g \vdash n \simeq (\text{BinaryExpr BinIntegerBelow } xe \ ye) \mid$

*IntegerEqualsNode:*  
 $\llbracket \text{kind } g \ n = \text{IntegerEqualsNode } x \ y;$

$g \vdash x \simeq xe;$   
 $g \vdash y \simeq ye$   
 $\implies g \vdash n \simeq (\text{BinaryExpr BinIntegerEquals } xe \ ye) \mid$

*IntegerLessThanNode:*

$\llbracket \text{kind } g \ n = \text{IntegerLessThanNode } x \ y;$   
 $g \vdash x \simeq xe;$   
 $g \vdash y \simeq ye$   
 $\implies g \vdash n \simeq (\text{BinaryExpr BinIntegerLessThan } xe \ ye) \mid$

*NarrowNode:*

$\llbracket \text{kind } g \ n = \text{NarrowNode inputBits resultBits } x;$   
 $g \vdash x \simeq xe$   
 $\implies g \vdash n \simeq (\text{UnaryExpr (UnaryNarrow inputBits resultBits) } xe) \mid$

*SignExtendNode:*

$\llbracket \text{kind } g \ n = \text{SignExtendNode inputBits resultBits } x;$   
 $g \vdash x \simeq xe$   
 $\implies g \vdash n \simeq (\text{UnaryExpr (UnarySignExtend inputBits resultBits) } xe) \mid$

*ZeroExtendNode:*

$\llbracket \text{kind } g \ n = \text{ZeroExtendNode inputBits resultBits } x;$   
 $g \vdash x \simeq xe$   
 $\implies g \vdash n \simeq (\text{UnaryExpr (UnaryZeroExtend inputBits resultBits) } xe) \mid$

*LeafNode:*

$\llbracket \text{is-preevaluated (kind } g \ n);$   
 $\text{stamp } g \ n = s$   
 $\implies g \vdash n \simeq (\text{LeafExpr } n \ s) \mid$

*RefNode:*

$\llbracket \text{kind } g \ n = \text{RefNode } n';$   
 $g \vdash n' \simeq e$   
 $\implies g \vdash n \simeq e$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool as exprE}$ ) *rep* .

**inductive**

$\text{replist} :: \text{IRGraph} \Rightarrow \text{ID list} \Rightarrow \text{IRExpr list} \Rightarrow \text{bool} \ (- \vdash - \simeq_L - \ 55)$   
**for** *g* **where**

*RepNil:*

$g \vdash [] \simeq_L [] \mid$

*RepCons:*

$$\begin{aligned} & \llbracket g \vdash x \simeq xe; \\ & \quad g \vdash xs \simeq_L xse \rrbracket \\ & \implies g \vdash x \# xs \simeq_L xe \# xse \end{aligned}$$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$  as *exprListE*) *replist* .

**definition** *wf-term-graph* :: *MapState*  $\Rightarrow$  *Params*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *bool* **where**  
*wf-term-graph* *m p g n* = ( $\exists e. (g \vdash n \simeq e) \wedge (\exists v. ([m, p] \vdash e \mapsto v))$ )

**values** {*t. eg2-sq*  $\vdash 4 \simeq t$ }

## 6.2 Data-flow Tree to Subgraph

**fun** *unary-node* :: *IRUnaryOp*  $\Rightarrow$  *ID*  $\Rightarrow$  *IRNode* **where**

*unary-node* *UnaryAbs* *v* = *AbsNode* *v* |  
*unary-node* *UnaryNot* *v* = *NotNode* *v* |  
*unary-node* *UnaryNeg* *v* = *NegateNode* *v* |  
*unary-node* *UnaryLogicNegation* *v* = *LogicNegationNode* *v* |  
*unary-node* (*UnaryNarrow* *ib rb*) *v* = *NarrowNode* *ib rb v* |  
*unary-node* (*UnarySignExtend* *ib rb*) *v* = *SignExtendNode* *ib rb v* |  
*unary-node* (*UnaryZeroExtend* *ib rb*) *v* = *ZeroExtendNode* *ib rb v*

**fun** *bin-node* :: *IRBinaryOp*  $\Rightarrow$  *ID*  $\Rightarrow$  *ID*  $\Rightarrow$  *IRNode* **where**

*bin-node* *BinAdd* *x y* = *AddNode* *x y* |  
*bin-node* *BinMul* *x y* = *MulNode* *x y* |  
*bin-node* *BinSub* *x y* = *SubNode* *x y* |  
*bin-node* *BinAnd* *x y* = *AndNode* *x y* |  
*bin-node* *BinOr* *x y* = *OrNode* *x y* |  
*bin-node* *BinXor* *x y* = *XorNode* *x y* |  
*bin-node* *BinLeftShift* *x y* = *LeftShiftNode* *x y* |  
*bin-node* *BinRightShift* *x y* = *RightShiftNode* *x y* |  
*bin-node* *BinURightShift* *x y* = *UnsignedRightShiftNode* *x y* |  
*bin-node* *BinIntegerEquals* *x y* = *IntegerEqualsNode* *x y* |  
*bin-node* *BinIntegerLessThan* *x y* = *IntegerLessThanNode* *x y* |  
*bin-node* *BinIntegerBelow* *x y* = *IntegerBelowNode* *x y*

**fun** *choose-32-64* :: *int*  $\Rightarrow$  *int64*  $\Rightarrow$  *Value* **where**

*choose-32-64* *bits val* =  
 (if *bits* = 32  
 then (*IntVal32* (*ucast* *val*))  
 else (*IntVal64* (*val*)))

**inductive** *fresh-id* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *bool* **where**

$n \notin \text{ids } g \implies \text{fresh-id } g \ n$

**code-pred** *fresh-id* .

**fun** *get-fresh-id* :: *IRGraph*  $\Rightarrow$  *ID* **where**

*get-fresh-id* *g* = *last*(*sorted-list-of-set*(*ids* *g*)) + 1

**export-code** *get-fresh-id*

**value** *get-fresh-id* *eg2-sq*

**value** *get-fresh-id* (*add-node* 6 (*ParameterNode* 2, *default-stamp*) *eg2-sq*)

**inductive**

*unrep* :: *IRGraph*  $\Rightarrow$  *IRExpr*  $\Rightarrow$  (*IRGraph*  $\times$  *ID*)  $\Rightarrow$  *bool* ( $- \oplus - \rightsquigarrow -$  55)  
**where**

*ConstantNodeSame*:

$\llbracket \text{find-node-and-stamp } g \ (\text{ConstantNode } c, \text{constantAsStamp } c) = \text{Some } n \rrbracket$   
 $\implies g \oplus (\text{ConstantExpr } c) \rightsquigarrow (g, n) \mid$

*ConstantNodeNew*:

$\llbracket \text{find-node-and-stamp } g \ (\text{ConstantNode } c, \text{constantAsStamp } c) = \text{None};$   
 $n = \text{get-fresh-id } g;$   
 $g' = \text{add-node } n \ (\text{ConstantNode } c, \text{constantAsStamp } c) \ g \rrbracket$   
 $\implies g \oplus (\text{ConstantExpr } c) \rightsquigarrow (g', n) \mid$

*ParameterNodeSame*:

$\llbracket \text{find-node-and-stamp } g \ (\text{ParameterNode } i, s) = \text{Some } n \rrbracket$   
 $\implies g \oplus (\text{ParameterExpr } i \ s) \rightsquigarrow (g, n) \mid$

*ParameterNodeNew*:

$\llbracket \text{find-node-and-stamp } g \ (\text{ParameterNode } i, s) = \text{None};$   
 $n = \text{get-fresh-id } g;$   
 $g' = \text{add-node } n \ (\text{ParameterNode } i, s) \ g \rrbracket$   
 $\implies g \oplus (\text{ParameterExpr } i \ s) \rightsquigarrow (g', n) \mid$

*ConditionalNodeSame*:

$\llbracket g \oplus ce \rightsquigarrow (g2, c);$   
 $g2 \oplus te \rightsquigarrow (g3, t);$   
 $g3 \oplus fe \rightsquigarrow (g4, f);$   
 $s' = \text{meet } (\text{stamp } g4 \ t) \ (\text{stamp } g4 \ f);$   
 $\text{find-node-and-stamp } g4 \ (\text{ConditionalNode } c \ t \ f, s') = \text{Some } n \rrbracket$   
 $\implies g \oplus (\text{ConditionalExpr } ce \ te \ fe) \rightsquigarrow (g4, n) \mid$

*ConditionalNodeNew*:

$\llbracket g \oplus ce \rightsquigarrow (g2, c);$

$g2 \oplus te \rightsquigarrow (g3, t);$   
 $g3 \oplus fe \rightsquigarrow (g4, f);$   
 $s' = \text{meet } (\text{stamp } g4 \ t) (\text{stamp } g4 \ f);$   
 $\text{find-node-and-stamp } g4 \ (\text{ConditionalNode } c \ t \ f, s') = \text{None};$   
 $n = \text{get-fresh-id } g4;$   
 $g' = \text{add-node } n \ (\text{ConditionalNode } c \ t \ f, s') \ g4$   
 $\implies g \oplus (\text{ConditionalExpr } ce \ te \ fe) \rightsquigarrow (g', n) \mid$

*UnaryNodeSame:*

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$   
 $s' = \text{stamp-unary } op \ (\text{stamp } g2 \ x);$   
 $\text{find-node-and-stamp } g2 \ (\text{unary-node } op \ x, s') = \text{Some } n \rrbracket$   
 $\implies g \oplus (\text{UnaryExpr } op \ xe) \rightsquigarrow (g2, n) \mid$

*UnaryNodeNew:*

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$   
 $s' = \text{stamp-unary } op \ (\text{stamp } g2 \ x);$   
 $\text{find-node-and-stamp } g2 \ (\text{unary-node } op \ x, s') = \text{None};$   
 $n = \text{get-fresh-id } g2;$   
 $g' = \text{add-node } n \ (\text{unary-node } op \ x, s') \ g2$   
 $\implies g \oplus (\text{UnaryExpr } op \ xe) \rightsquigarrow (g', n) \mid$

*BinaryNodeSame:*

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$   
 $g2 \oplus ye \rightsquigarrow (g3, y);$   
 $s' = \text{stamp-binary } op \ (\text{stamp } g3 \ x) (\text{stamp } g3 \ y);$   
 $\text{find-node-and-stamp } g3 \ (\text{bin-node } op \ x \ y, s') = \text{Some } n \rrbracket$   
 $\implies g \oplus (\text{BinaryExpr } op \ xe \ ye) \rightsquigarrow (g3, n) \mid$

*BinaryNodeNew:*

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$   
 $g2 \oplus ye \rightsquigarrow (g3, y);$   
 $s' = \text{stamp-binary } op \ (\text{stamp } g3 \ x) (\text{stamp } g3 \ y);$   
 $\text{find-node-and-stamp } g3 \ (\text{bin-node } op \ x \ y, s') = \text{None};$   
 $n = \text{get-fresh-id } g3;$   
 $g' = \text{add-node } n \ (\text{bin-node } op \ x \ y, s') \ g3$   
 $\implies g \oplus (\text{BinaryExpr } op \ xe \ ye) \rightsquigarrow (g', n) \mid$

*AllLeafNodes:*

$\llbracket \text{stamp } g \ n = s;$   
 $\text{is-preevaluated } (\text{kind } g \ n) \rrbracket$   
 $\implies g \oplus (\text{LeafExpr } n \ s) \rightsquigarrow (g, n)$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$  as *unrepE*)  
*unrep* .

$$\frac{\text{find-node-and-stamp } g \text{ (ConstantNode } c, \text{ constantAsStamp } c) = \text{Some } n}{g \oplus \text{ConstantExpr } c \rightsquigarrow (g, n)}$$

$$\frac{\begin{array}{l} \text{find-node-and-stamp } g \text{ (ConstantNode } c, \text{ constantAsStamp } c) = \text{None} \\ n = \text{get-fresh-id } g \\ g' = \text{add-node } n \text{ (ConstantNode } c, \text{ constantAsStamp } c) \end{array} g}{g \oplus \text{ConstantExpr } c \rightsquigarrow (g', n)}$$

$$\frac{\text{find-node-and-stamp } g \text{ (ParameterNode } i, s) = \text{Some } n}{g \oplus \text{ParameterExpr } i \text{ } s \rightsquigarrow (g, n)}$$

$$\frac{\begin{array}{l} \text{find-node-and-stamp } g \text{ (ParameterNode } i, s) = \text{None} \\ n = \text{get-fresh-id } g \quad g' = \text{add-node } n \text{ (ParameterNode } i, s) \end{array} g}{g \oplus \text{ParameterExpr } i \text{ } s \rightsquigarrow (g', n)}$$

$$\frac{\begin{array}{l} g \oplus ce \rightsquigarrow (g2, c) \quad g2 \oplus te \rightsquigarrow (g3, t) \\ g3 \oplus fe \rightsquigarrow (g4, f) \quad s' = \text{meet (stamp } g4 \text{ } t) \text{ (stamp } g4 \text{ } f) \\ \text{find-node-and-stamp } g4 \text{ (ConditionalNode } c \text{ } t \text{ } f, s') = \text{Some } n \end{array}}{g \oplus \text{ConditionalExpr } ce \text{ } te \text{ } fe \rightsquigarrow (g4, n)}$$

$$\frac{\begin{array}{l} g \oplus ce \rightsquigarrow (g2, c) \quad g2 \oplus te \rightsquigarrow (g3, t) \\ g3 \oplus fe \rightsquigarrow (g4, f) \quad s' = \text{meet (stamp } g4 \text{ } t) \text{ (stamp } g4 \text{ } f) \\ \text{find-node-and-stamp } g4 \text{ (ConditionalNode } c \text{ } t \text{ } f, s') = \text{None} \\ n = \text{get-fresh-id } g4 \quad g' = \text{add-node } n \text{ (ConditionalNode } c \text{ } t \text{ } f, s') \end{array} g4}{g \oplus \text{ConditionalExpr } ce \text{ } te \text{ } fe \rightsquigarrow (g', n)}$$

$$\frac{\begin{array}{l} g \oplus xe \rightsquigarrow (g2, x) \\ g2 \oplus ye \rightsquigarrow (g3, y) \quad s' = \text{stamp-binary op (stamp } g3 \text{ } x) \text{ (stamp } g3 \text{ } y) \\ \text{find-node-and-stamp } g3 \text{ (bin-node op } x \text{ } y, s') = \text{Some } n \end{array}}{g \oplus \text{BinaryExpr op } xe \text{ } ye \rightsquigarrow (g3, n)}$$

$$\frac{\begin{array}{l} g \oplus xe \rightsquigarrow (g2, x) \\ g2 \oplus ye \rightsquigarrow (g3, y) \quad s' = \text{stamp-binary op (stamp } g3 \text{ } x) \text{ (stamp } g3 \text{ } y) \\ \text{find-node-and-stamp } g3 \text{ (bin-node op } x \text{ } y, s') = \text{None} \\ n = \text{get-fresh-id } g3 \quad g' = \text{add-node } n \text{ (bin-node op } x \text{ } y, s') \end{array} g3}{g \oplus \text{BinaryExpr op } xe \text{ } ye \rightsquigarrow (g', n)}$$

$$\frac{\begin{array}{l} g \oplus xe \rightsquigarrow (g2, x) \quad s' = \text{stamp-unary op (stamp } g2 \text{ } x) \\ \text{find-node-and-stamp } g2 \text{ (unary-node op } x, s') = \text{Some } n \end{array}}{g \oplus \text{UnaryExpr op } xe \rightsquigarrow (g2, n)}$$

$$\frac{\begin{array}{l} g \oplus xe \rightsquigarrow (g2, x) \quad s' = \text{stamp-unary op (stamp } g2 \text{ } x) \\ \text{find-node-and-stamp } g2 \text{ (unary-node op } x, s') = \text{None} \\ n = \text{get-fresh-id } g2 \quad g' = \text{add-node } n \text{ (unary-node op } x, s') \end{array} g2}{g \oplus \text{UnaryExpr op } xe \rightsquigarrow (g', n)}$$

$$\frac{\text{stamp } g \text{ } n = s \quad \text{is-preevaluated (kind } g \text{ } n)}{g \oplus \text{LeafExpr } n \text{ } s \rightsquigarrow (g, n)}$$

values  $\{(n, g) . (eg2\text{-}sq \oplus sq\text{-}param0 \rightsquigarrow (g, n))\}$

### 6.3 Lift Data-flow Tree Semantics

**definition** *encodeeval* :: *IRGraph*  $\Rightarrow$  *MapState*  $\Rightarrow$  *Params*  $\Rightarrow$  *ID*  $\Rightarrow$  *Value*  $\Rightarrow$  *bool*  
 $([\cdot, \cdot, \cdot] \vdash \cdot \mapsto \cdot \ 50)$   
**where**  
*encodeeval* *g m p n v* =  $(\exists e. (g \vdash n \simeq e) \wedge ([m, p] \vdash e \mapsto v))$

### 6.4 Graph Refinement

**definition** *graph-represents-expression* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *IRExpr*  $\Rightarrow$  *bool*  
 $(\vdash \cdot \preceq \cdot \ 50)$   
**where**  
 $(g \vdash n \preceq e) = (\exists e'. (g \vdash n \simeq e') \wedge (e' \leq e))$

**definition** *graph-refinement* :: *IRGraph*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *bool* **where**  
*graph-refinement* *g1 g2* =  
 $((ids\ g_1 \subseteq ids\ g_2) \wedge$   
 $(\forall n. n \in ids\ g_1 \longrightarrow (\forall e. (g_1 \vdash n \simeq e) \longrightarrow (g_2 \vdash n \preceq e))))$

**lemma** *graph-refinement*:

*graph-refinement* *g1 g2*  $\implies (\forall n\ m\ p\ v. n \in ids\ g1 \longrightarrow ([g1, m, p] \vdash n \mapsto v) \longrightarrow$   
 $([g2, m, p] \vdash n \mapsto v))$   
**by** (*meson encodeeval-def graph-refinement-def graph-represents-expression-def le-expr-def*)

### 6.5 Maximal Sharing

**definition** *maximal-sharing*:

*maximal-sharing* *g* =  $(\forall n_1\ n_2. n_1 \in true\text{-}ids\ g \wedge n_2 \in true\text{-}ids\ g \longrightarrow$   
 $(\forall e. (g \vdash n_1 \simeq e) \wedge (g \vdash n_2 \simeq e) \wedge (stamp\ g\ n_1 = stamp\ g\ n_2) \longrightarrow n_1 =$   
 $n_2))$

**end**

### 6.6 Formedness Properties

**theory** *Form*

**imports**

*Semantics.TreeToGraph*

**begin**

**definition** *wf-start* **where**

*wf-start* *g* =  $(0 \in ids\ g \wedge$   
 $is\text{-}StartNode\ (kind\ g\ 0))$

**definition** *wf-closed* **where**

*wf-closed* *g* =  
 $(\forall n \in ids\ g .$

$$\begin{aligned} &inputs\ g\ n \subseteq ids\ g \wedge \\ &succ\ g\ n \subseteq ids\ g \wedge \\ &kind\ g\ n \neq NoNode) \end{aligned}$$

**definition** *wf-phs* **where**

$$\begin{aligned} wf-phs\ g = & \\ &(\forall\ n \in ids\ g. \\ &\quad is-PhiNode\ (kind\ g\ n) \longrightarrow \\ &\quad length\ (ir-values\ (kind\ g\ n)) \\ &= length\ (ir-ends \\ &\quad (kind\ g\ (ir-merge\ (kind\ g\ n)))))) \end{aligned}$$

**definition** *wf-ends* **where**

$$\begin{aligned} wf-ends\ g = & \\ &(\forall\ n \in ids\ g . \\ &\quad is-AbstractEndNode\ (kind\ g\ n) \longrightarrow \\ &\quad card\ (usages\ g\ n) > 0) \end{aligned}$$

**fun** *wf-graph* :: *IRGraph*  $\Rightarrow$  *bool* **where**

$$wf-graph\ g = (wf-start\ g \wedge wf-closed\ g \wedge wf-phs\ g \wedge wf-ends\ g)$$

**lemmas** *wf-folds* =

$$\begin{aligned} &wf-graph.simps \\ &wf-start-def \\ &wf-closed-def \\ &wf-phs-def \\ &wf-ends-def \end{aligned}$$

**fun** *wf-stamps* :: *IRGraph*  $\Rightarrow$  *bool* **where**

$$\begin{aligned} wf-stamps\ g = &(\forall\ n \in ids\ g . \\ &(\forall\ v\ m\ p\ e . (g \vdash n \simeq e) \wedge ([m, p] \vdash e \mapsto v) \longrightarrow valid-value\ v\ (stamp-expr\ e))) \end{aligned}$$

**fun** *wf-stamp* :: *IRGraph*  $\Rightarrow$  (*ID*  $\Rightarrow$  *Stamp*)  $\Rightarrow$  *bool* **where**

$$\begin{aligned} wf-stamp\ g\ s = &(\forall\ n \in ids\ g . \\ &(\forall\ v\ m\ p\ e . (g \vdash n \simeq e) \wedge ([m, p] \vdash e \mapsto v) \longrightarrow valid-value\ v\ (s\ n))) \end{aligned}$$

**lemma** *wf-empty*: *wf-graph start-end-graph*

**unfolding** *start-end-graph-def wf-folds by simp*

**lemma** *wf-eg2-sq*: *wf-graph eg2-sq*

**unfolding** *eg2-sq-def wf-folds by simp*

**fun** *wf-logic-node-inputs* :: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *bool* **where**

$$\begin{aligned} wf-logic-node-inputs\ g\ n = & \\ &(\forall\ inp \in set\ (inputs-of\ (kind\ g\ n)) . (\forall\ v\ m\ p . ([g, m, p] \vdash inp \mapsto v) \longrightarrow wf-bool \\ &v)) \end{aligned}$$

**fun** *wf-values* :: *IRGraph*  $\Rightarrow$  *bool* **where**

$$wf-values\ g = (\forall\ n \in ids\ g .$$



$$\begin{aligned}
& (\forall v m p . ([g, m, p] \vdash n \mapsto v) \longrightarrow \\
& \quad (is-LogicNode (kind g n) \longrightarrow \\
& \quad \quad wf-bool v \wedge wf-logic-node-inputs g n)))
\end{aligned}$$

**end**

## 6.7 Dynamic Frames

This theory defines two operators, 'unchanged' and 'changeonly', that are useful for specifying which nodes in an IRGraph can change. The dynamic framing idea originates from 'Dynamic Frames' in software verification, started by Ioannis T. Kassios in "Dynamic frames: Support for framing, dependencies and sharing without restrictions", In FM 2006.

**theory** *IRGraphFrames*

**imports**

*Form*

*Semantics.IRTreeEval*

**begin**

**fun** *unchanged* :: *ID set*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *bool* **where**

*unchanged ns g1 g2* =  $(\forall n . n \in ns \longrightarrow$   
 $(n \in ids\ g1 \wedge n \in ids\ g2 \wedge kind\ g1\ n = kind\ g2\ n \wedge stamp\ g1\ n = stamp\ g2\ n))$

**fun** *changeonly* :: *ID set*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *IRGraph*  $\Rightarrow$  *bool* **where**

*changeonly ns g1 g2* =  $(\forall n . n \in ids\ g1 \wedge n \notin ns \longrightarrow$   
 $(n \in ids\ g1 \wedge n \in ids\ g2 \wedge kind\ g1\ n = kind\ g2\ n \wedge stamp\ g1\ n = stamp\ g2\ n))$

**lemma** *node-unchanged*:

**assumes** *unchanged ns g1 g2*

**assumes** *nid*  $\in$  *ns*

**shows** *kind g1 nid* = *kind g2 nid*

**using** *assms* **by** *auto*

**lemma** *other-node-unchanged*:

**assumes** *changeonly ns g1 g2*

**assumes** *nid*  $\in$  *ids g1*

**assumes** *nid*  $\notin$  *ns*

**shows** *kind g1 nid* = *kind g2 nid*

**using** *assms*

**using** *changeonly.simps* **by** *blast*

Some notation for input nodes used

**inductive** *eval-uses*:: *IRGraph*  $\Rightarrow$  *ID*  $\Rightarrow$  *ID*  $\Rightarrow$  *bool*

**for** *g* **where**

*use0*: *nid*  $\in$  *ids g*

$\implies eval-uses\ g\ nid\ nid$  |

```

use-inp:  $nid' \in inputs\ g\ n$ 
 $\implies eval\text{-}uses\ g\ nid\ nid' \mid$ 

use-trans:  $\llbracket eval\text{-}uses\ g\ nid\ nid';$ 
 $eval\text{-}uses\ g\ nid'\ nid'' \rrbracket$ 
 $\implies eval\text{-}uses\ g\ nid\ nid''$ 

fun eval-usages :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID set where
  eval-usages g nid = {n  $\in$  ids g . eval-uses g nid n}

lemma eval-usages-self:
  assumes nid  $\in$  ids g
  shows nid  $\in$  eval-usages g nid
  using assms eval-usages.simps eval-uses.intros(1)
  by (simp add: ids.rep-eq)

lemma not-in-g-inputs:
  assumes nid  $\notin$  ids g
  shows inputs g nid = {}
proof –
  have k: kind g nid = NoNode using assms not-in-g by blast
  then show ?thesis by (simp add: k)
qed

lemma child-member:
  assumes n = kind g nid
  assumes n  $\neq$  NoNode
  assumes List.member (inputs-of n) child
  shows child  $\in$  inputs g nid
  unfolding inputs.simps using assms
  by (metis in-set-member)

lemma child-member-in:
  assumes nid  $\in$  ids g
  assumes List.member (inputs-of (kind g nid)) child
  shows child  $\in$  inputs g nid
  unfolding inputs.simps using assms
  by (metis child-member ids-some inputs.elims)

lemma inp-in-g:
  assumes n  $\in$  inputs g nid
  shows nid  $\in$  ids g
proof –
  have inputs g nid  $\neq$  {}
  using assms

```

```

    by (metis empty-iff empty-set)
  then have kind g nid  $\neq$  NoNode
    using not-in-g-inputs
    using ids-some by blast
  then show ?thesis
    using not-in-g
    by metis
qed

```

```

lemma inp-in-g-wf:
  assumes wf-graph g
  assumes  $n \in \text{inputs } g \text{ nid}$ 
  shows  $n \in \text{ids } g$ 
  using assms unfolding wf-folds
  using inp-in-g by blast

```

```

lemma kind-unchanged:
  assumes  $\text{nid} \in \text{ids } g1$ 
  assumes unchanged (eval-usages g1 nid) g1 g2
  shows  $\text{kind } g1 \text{ nid} = \text{kind } g2 \text{ nid}$ 
proof -
  show ?thesis
    using assms eval-usages-self
    using unchanged.simps by blast
qed

```

```

lemma stamp-unchanged:
  assumes  $\text{nid} \in \text{ids } g1$ 
  assumes unchanged (eval-usages g1 nid) g1 g2
  shows  $\text{stamp } g1 \text{ nid} = \text{stamp } g2 \text{ nid}$ 
  by (meson assms(1) assms(2) eval-usages-self unchanged.elims(2))

```

```

lemma child-unchanged:
  assumes  $\text{child} \in \text{inputs } g1 \text{ nid}$ 
  assumes unchanged (eval-usages g1 nid) g1 g2
  shows unchanged (eval-usages g1 child) g1 g2
  by (smt assms(1) assms(2) eval-usages.simps mem-Collect-eq
      unchanged.simps use-inp use-trans)

```

```

lemma eval-usages:
  assumes  $us = \text{eval-usages } g \text{ nid}$ 
  assumes  $\text{nid}' \in \text{ids } g$ 
  shows  $\text{eval-uses } g \text{ nid } \text{nid}' \longleftrightarrow \text{nid}' \in us$  (is ?P  $\longleftrightarrow$  ?Q)
  using assms eval-usages.simps
  by (simp add: ids.rep-eq)

```

```

lemma inputs-are-uses:

```

```

assumes  $nid' \in inputs\ g\ nid$ 
shows  $eval\text{-}uses\ g\ nid\ nid'$ 
by (metis assms use-inp)

lemma inputs-are-usages:
assumes  $nid' \in inputs\ g\ nid$ 
assumes  $nid' \in ids\ g$ 
shows  $nid' \in eval\text{-}usages\ g\ nid$ 
using assms(1) assms(2) eval-usages inputs-are-uses by blast

lemma inputs-of-are-usages:
assumes  $List.member\ (inputs\text{-}of\ (kind\ g\ nid))\ nid'$ 
assumes  $nid' \in ids\ g$ 
shows  $nid' \in eval\text{-}usages\ g\ nid$ 
by (metis assms(1) assms(2) in-set-member inputs.elims inputs-are-usages)

lemma usage-includes-inputs:
assumes  $us = eval\text{-}usages\ g\ nid$ 
assumes  $ls = inputs\ g\ nid$ 
assumes  $ls \subseteq ids\ g$ 
shows  $ls \subseteq us$ 
using inputs-are-usages eval-usages
using assms(1) assms(2) assms(3) by blast

lemma elim-inp-set:
assumes  $k = kind\ g\ nid$ 
assumes  $k \neq NoNode$ 
assumes  $child \in set\ (inputs\text{-}of\ k)$ 
shows  $child \in inputs\ g\ nid$ 
using assms by auto

lemma encode-in-ids:
assumes  $g \vdash nid \simeq e$ 
shows  $nid \in ids\ g$ 
using assms
apply (induction rule: rep.induct)
apply simp+
by fastforce+

lemma eval-in-ids:
assumes  $[g, m, p] \vdash nid \mapsto v$ 
shows  $nid \in ids\ g$ 
using assms using encodeeval-def encode-in-ids
by auto

lemma transitive-kind-same:
assumes unchanged (eval-usages  $g1\ nid$ )  $g1\ g2$ 
shows  $\forall\ nid' \in (eval\text{-}usages\ g1\ nid) . kind\ g1\ nid' = kind\ g2\ nid'$ 
using assms

```

```

by (meson unchanged.elims(1))

theorem stay-same-encoding:
  assumes nc: unchanged (eval-usages g1 nid) g1 g2
  assumes g1: g1 ⊢ nid ≃ e
  assumes wf: wf-graph g1
  shows g2 ⊢ nid ≃ e
proof -
  have dom: nid ∈ ids g1
  using g1 encode-in-ids by simp
  show ?thesis
using g1 nc wf dom proof (induction e rule: rep.induct)
  case (ConstantNode n c)
  then have kind g2 n = ConstantNode c
  using dom nc kind-unchanged
  by metis
  then show ?case using rep.ConstantNode
  by presburger
next
  case (ParameterNode n i s)
  then have kind g2 n = ParameterNode i
  by (metis kind-unchanged)
  then show ?case
  by (metis ParameterNode.hyps(2) ParameterNode.premis(1) ParameterNode.premis(3)
  rep.ParameterNode stamp-unchanged)
next
  case (ConditionalNode n c t f ce te fe)
  then have kind g2 n = ConditionalNode c t f
  by (metis kind-unchanged)
  have c ∈ eval-usages g1 n ∧ t ∈ eval-usages g1 n ∧ f ∈ eval-usages g1 n
  using inputs-of-ConditionalNode
  by (metis ConditionalNode.hyps(1) ConditionalNode.hyps(2) ConditionalNode.hyps(3)
  ConditionalNode.hyps(4) encode-in-ids inputs.simps inputs-are-usages
  list.set-intros(1) set-subset-Cons subset-code(1))
  then show ?case using transitive-kind-same
  by (metis ConditionalNode.hyps(1) ConditionalNode.premis(1) IRNodes.inputs-of-ConditionalNode
  ⟨kind g2 n = ConditionalNode c t f⟩ child-unchanged inputs.simps list.set-intros(1)
  local.ConditionalNode(5) local.ConditionalNode(6) local.ConditionalNode(7) local.ConditionalNode(9)
  rep.ConditionalNode set-subset-Cons subset-code(1) unchanged.elims(2))
next
  case (AbsNode n x xe)
  then have kind g2 n = AbsNode x
  using kind-unchanged
  by metis
  then have x ∈ eval-usages g1 n
  using inputs-of-AbsNode
  by (metis AbsNode.hyps(1) AbsNode.hyps(2) encode-in-ids inputs.simps inputs-are-usages
  list.set-intros(1))
  then show ?case

```

```

    by (metis AbsNode.IH AbsNode.hyps(1) AbsNode.prem(1) AbsNode.prem(3)
      IRNodes.inputs-of-AbsNode ⟨kind g2 n = AbsNode x⟩ child-member-in child-unchanged
      local.wf member-rec(1) rep.AbsNode unchanged.simps)
next
  case (NotNode n x xe)
  then have kind g2 n = NotNode x
    using kind-unchanged
    by metis
  then have x ∈ eval-usages g1 n
    using inputs-of-NotNode
    by (metis NotNode.hyps(1) NotNode.hyps(2) encode-in-ids inputs.simps in-
      puts-are-usages list.set-intros(1))
  then show ?case
    by (metis NotNode.IH NotNode.hyps(1) NotNode.prem(1) NotNode.prem(3)
      IRNodes.inputs-of-NotNode ⟨kind g2 n = NotNode x⟩ child-member-in child-unchanged
      local.wf member-rec(1) rep.NotNode unchanged.simps)
next
  case (NegateNode n x xe)
  then have kind g2 n = NegateNode x
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n
    using inputs-of-NegateNode
    by (metis NegateNode.hyps(1) NegateNode.hyps(2) encode-in-ids inputs.simps
      inputs-are-usages list.set-intros(1))
  then show ?case
    by (metis IRNodes.inputs-of-NegateNode NegateNode.IH NegateNode.hyps(1)
      NegateNode.prem(1) NegateNode.prem(3) ⟨kind g2 n = NegateNode x⟩ child-member-in
      child-unchanged local.wf member-rec(1) rep.NegateNode unchanged.elims(1))
next
  case (LogicNegationNode n x xe)
  then have kind g2 n = LogicNegationNode x
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
    by (metis LogicNegationNode.hyps(1) LogicNegationNode.hyps(2) encode-in-ids
      member-rec(1))
  then show ?case
    by (metis IRNodes.inputs-of-LogicNegationNode LogicNegationNode.IH Logic-
      NegationNode.hyps(1) LogicNegationNode.hyps(2) LogicNegationNode.prem(1) ⟨kind
      g2 n = LogicNegationNode x⟩ child-unchanged encode-in-ids inputs.simps list.set-intros(1)
      local.wf rep.LogicNegationNode)
next
  case (AddNode n x y xe ye)
  then have kind g2 n = AddNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
    by (metis AddNode.hyps(1) AddNode.hyps(2) AddNode.hyps(3) IRNodes.inputs-of-AddNode
      encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)

```

```

then show ?case
  by (metis AddNode.IH(1) AddNode.IH(2) AddNode.hyps(1) AddNode.hyps(2)
    AddNode.hyps(3) AddNode.premis(1) IRNodes.inputs-of-AddNode ⟨kind g2 n = AddNode
    x y⟩ child-unchanged encode-in-ids in-set-member inputs.simps local.wf member-rec(1)
    rep.AddNode)
next
  case (MulNode n x y xe ye)
  then have kind g2 n = MulNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
  by (metis MulNode.hyps(1) MulNode.hyps(2) MulNode.hyps(3) IRNodes.inputs-of-MulNode
    encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using MulNode inputs-of-MulNode
    by (metis ⟨kind g2 n = MulNode x y⟩ child-unchanged inputs.simps list.set-intros(1)
    rep.MulNode set-subset-Cons subset-iff unchanged.elims(2))
next
  case (SubNode n x y xe ye)
  then have kind g2 n = SubNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
  by (metis SubNode.hyps(1) SubNode.hyps(2) SubNode.hyps(3) IRNodes.inputs-of-SubNode
    encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using SubNode inputs-of-SubNode
    by (metis ⟨kind g2 n = SubNode x y⟩ child-member child-unchanged encode-in-ids
    ids-some member-rec(1) rep.SubNode)
next
  case (AndNode n x y xe ye)
  then have kind g2 n = AndNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
  by (metis AndNode.hyps(1) AndNode.hyps(2) AndNode.hyps(3) IRNodes.inputs-of-AndNode
    encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using AndNode inputs-of-AndNode
    by (metis ⟨kind g2 n = AndNode x y⟩ child-unchanged inputs.simps list.set-intros(1)
    rep.AndNode set-subset-Cons subset-iff unchanged.elims(2))
next
  case (OrNode n x y xe ye)
  then have kind g2 n = OrNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-OrNode inputs-of-are-usages
  by (metis OrNode.hyps(1) OrNode.hyps(2) OrNode.hyps(3) IRNodes.inputs-of-OrNode
    encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using OrNode inputs-of-OrNode
    by (metis ⟨kind g2 n = OrNode x y⟩ child-member child-unchanged encode-in-ids
    ids-some member-rec(1) rep.OrNode)

```

```

next
  case (XorNode n x y xe ye)
  then have kind g2 n = XorNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-XorNode inputs-of-are-usages
  by (metis XorNode.hyps(1) XorNode.hyps(2) XorNode.hyps(3) IRNodes.inputs-of-XorNode
    encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using XorNode inputs-of-XorNode
    by (metis ⟨kind g2 n = XorNode x y⟩ child-member child-unchanged encode-in-ids
      ids-some member-rec(1) rep.XorNode)
next
  case (LeftShiftNode n x y xe ye)
  then have kind g2 n = LeftShiftNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-XorNode inputs-of-are-usages
  by (metis LeftShiftNode.hyps(1) LeftShiftNode.hyps(2) LeftShiftNode.hyps(3)
    IRNodes.inputs-of-LeftShiftNode encode-in-ids in-mono inputs.simps inputs-are-usages
    list.set-intros(1) set-subset-Cons)
  then show ?case using LeftShiftNode inputs-of-LeftShiftNode
    by (metis ⟨kind g2 n = LeftShiftNode x y⟩ child-member child-unchanged
      encode-in-ids ids-some member-rec(1) rep.LeftShiftNode)
next
  case (RightShiftNode n x y xe ye)
  then have kind g2 n = RightShiftNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-RightShiftNode inputs-of-are-usages
  by (metis RightShiftNode.hyps(1) RightShiftNode.hyps(2) RightShiftNode.hyps(3)
    IRNodes.inputs-of-RightShiftNode encode-in-ids in-mono inputs.simps inputs-are-usages
    list.set-intros(1) set-subset-Cons)
  then show ?case using RightShiftNode inputs-of-RightShiftNode
    by (metis ⟨kind g2 n = RightShiftNode x y⟩ child-member child-unchanged
      encode-in-ids ids-some member-rec(1) rep.RightShiftNode)
next
  case (UnsignedRightShiftNode n x y xe ye)
  then have kind g2 n = UnsignedRightShiftNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-UnsignedRightShiftNode inputs-of-are-usages
  by (metis UnsignedRightShiftNode.hyps(1) UnsignedRightShiftNode.hyps(2) Un-
    signedRightShiftNode.hyps(3) IRNodes.inputs-of-UnsignedRightShiftNode encode-in-ids
    in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using UnsignedRightShiftNode inputs-of-UnsignedRightShiftNode
    by (metis ⟨kind g2 n = UnsignedRightShiftNode x y⟩ child-member child-unchanged
      encode-in-ids ids-some member-rec(1) rep.UnsignedRightShiftNode)
next
  case (IntegerBelowNode n x y xe ye)

```



```

then have kind g2 n = IntegerBelowNode x y
  using kind-unchanged by metis
then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
  using inputs-of-IntegerBelowNode inputs-of-are-usages
  by (metis IntegerBelowNode.hyps(1) IntegerBelowNode.hyps(2) IntegerBelowNode.hyps(3) IRNodes.inputs-of-IntegerBelowNode encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using IntegerBelowNode inputs-of-IntegerBelowNode
    by (metis ⟨kind g2 n = IntegerBelowNode x y⟩ child-member child-unchanged encode-in-ids ids-some member-rec(1) rep.IntegerBelowNode)
next
  case (IntegerEqualsNode n x y xe ye)
  then have kind g2 n = IntegerEqualsNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-IntegerEqualsNode inputs-of-are-usages
    by (metis IntegerEqualsNode.hyps(1) IntegerEqualsNode.hyps(2) IntegerEqualsNode.hyps(3) IRNodes.inputs-of-IntegerEqualsNode encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
    then show ?case using IntegerEqualsNode inputs-of-IntegerEqualsNode
      by (metis ⟨kind g2 n = IntegerEqualsNode x y⟩ child-member child-unchanged encode-in-ids ids-some member-rec(1) rep.IntegerEqualsNode)
next
  case (IntegerLessThanNode n x y xe ye)
  then have kind g2 n = IntegerLessThanNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-IntegerLessThanNode inputs-of-are-usages
    by (metis IntegerLessThanNode.hyps(1) IntegerLessThanNode.hyps(2) IntegerLessThanNode.hyps(3) IRNodes.inputs-of-IntegerLessThanNode encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
    then show ?case using IntegerLessThanNode inputs-of-IntegerLessThanNode
      by (metis ⟨kind g2 n = IntegerLessThanNode x y⟩ child-member child-unchanged encode-in-ids ids-some member-rec(1) rep.IntegerLessThanNode)
next
  case (NarrowNode n ib rb x xe)
  then have kind g2 n = NarrowNode ib rb x
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n
    using inputs-of-NarrowNode inputs-of-are-usages
    by (metis NarrowNode.hyps(1) NarrowNode.hyps(2) IRNodes.inputs-of-NarrowNode encode-in-ids inputs.simps inputs-are-usages list.set-intros(1))
    then show ?case using NarrowNode inputs-of-NarrowNode
      by (metis ⟨kind g2 n = NarrowNode ib rb x⟩ child-unchanged inputs.elims list.set-intros(1) rep.NarrowNode unchanged.simps)
next
  case (SignExtendNode n ib rb x xe)
  then have kind g2 n = SignExtendNode ib rb x
    using kind-unchanged by metis

```

```

then have  $x \in \text{eval-usages } g1 \ n$ 
  using  $\text{inputs-of-SignExtendNode inputs-of-are-usages}$ 
  by ( $\text{metis SignExtendNode.hyps(1) SignExtendNode.hyps(2) encode-in-ids inputs.simps inputs-are-usages list.set-intros(1)}$ )
  then show ?case using  $\text{SignExtendNode inputs-of-SignExtendNode}$ 
  by ( $\text{metis } \langle \text{kind } g2 \ n = \text{SignExtendNode ib rb } x \rangle \text{ child-member-in child-unchanged in-set-member list.set-intros(1) rep.SignExtendNode unchanged.elims(2)}$ )
next
  case ( $\text{ZeroExtendNode } n \text{ ib rb } x \text{ xe}$ )
  then have  $\text{kind } g2 \ n = \text{ZeroExtendNode ib rb } x$ 
  using  $\text{kind-unchanged by metis}$ 
  then have  $x \in \text{eval-usages } g1 \ n$ 
  using  $\text{inputs-of-ZeroExtendNode inputs-of-are-usages}$ 
  by ( $\text{metis ZeroExtendNode.hyps(1) ZeroExtendNode.hyps(2) IRNodes.inputs-of-ZeroExtendNode encode-in-ids inputs.simps inputs-are-usages list.set-intros(1)}$ )
  then show ?case using  $\text{ZeroExtendNode inputs-of-ZeroExtendNode}$ 
  by ( $\text{metis } \langle \text{kind } g2 \ n = \text{ZeroExtendNode ib rb } x \rangle \text{ child-member-in child-unchanged member-rec(1) rep.ZeroExtendNode unchanged.simps}$ )
next
  case ( $\text{LeafNode } n \ s$ )
  then show ?case
  by ( $\text{metis kind-unchanged rep.LeafNode stamp-unchanged}$ )
next
  case ( $\text{RefNode } n \ n'$ )
  then have  $\text{kind } g2 \ n = \text{RefNode } n'$ 
  using  $\text{kind-unchanged by metis}$ 
  then have  $n' \in \text{eval-usages } g1 \ n$ 
  by ( $\text{metis IRNodes.inputs-of-RefNode RefNode.hyps(1) RefNode.hyps(2) encode-in-ids inputs.elims inputs-are-usages list.set-intros(1)}$ )
  then show ?case
  by ( $\text{metis IRNodes.inputs-of-RefNode RefNode.IH RefNode.hyps(1) RefNode.hyps(2) RefNode.premis(1) } \langle \text{kind } g2 \ n = \text{RefNode } n' \rangle \text{ child-unchanged encode-in-ids inputs.elims list.set-intros(1) local.wf rep.RefNode}$ )
qed
qed

```

**theorem** *stay-same:*

```

assumes  $nc: \text{unchanged (eval-usages } g1 \ nid) \ g1 \ g2$ 
assumes  $g1: [g1, m, p] \vdash nid \mapsto v1$ 
assumes  $wf: \text{wf-graph } g1$ 
shows  $[g2, m, p] \vdash nid \mapsto v1$ 
proof –
  have  $nid: nid \in \text{ids } g1$ 
  using  $g1 \text{ eval-in-ids by simp}$ 
  then have  $nid \in \text{eval-usages } g1 \ nid$ 
  using  $\text{eval-usages-self by blast}$ 
  then have  $\text{kind-same: kind } g1 \ nid = \text{kind } g2 \ nid$ 

```

```

    using nc node-unchanged by blast
  obtain e where e: (g1 ⊢ nid ≃ e) ∧ ([m,p] ⊢ e ↦ v1)
    using encodeeval-def g1
    by auto
  then have val: [m,p] ⊢ e ↦ v1
    using g1 encodeeval-def
    by simp
  then show ?thesis using e nid nc
    unfolding encodeeval-def
  proof (induct e v1 arbitrary: nid rule: evaltree.induct)
    case (ConstantExpr c)
    then show ?case
      by (meson local.wf stay-same-encoding)
  next
    case (ParameterExpr i s)
    have g2 ⊢ nid ≃ ParameterExpr i s
      using stay-same-encoding ParameterExpr
      by (meson local.wf)
    then show ?case using evaltree.ParameterExpr
      by (meson ParameterExpr.hyps)
  next
    case (ConditionalExpr ce cond branch te fe v)
    then have g2 ⊢ nid ≃ ConditionalExpr ce te fe
      using ConditionalExpr.prem1 ConditionalExpr.prem3 local.wf stay-same-encoding
      by presburger
    then show ?case
      by (meson ConditionalExpr.prem1 ConditionalExpr.prem3 local.wf
        stay-same-encoding)
  next
    case (UnaryExpr xe v op)
    then show ?case
      using local.wf stay-same-encoding by blast
  next
    case (BinaryExpr xe x ye y op)
    then show ?case
      using local.wf stay-same-encoding by blast
  next
    case (LeafExpr val nid s)
    then show ?case
      by (metis local.wf stay-same-encoding)
qed
qed

```

```

lemma add-changed:
  assumes gup = add-node new k g
  shows changeonly {new} g gup
  using assms unfolding add-node-def changeonly.simps
  using add-node.rep-eq add-node-def kind.rep-eq stamp.rep-eq by simp

```

```

lemma disjoint-change:
  assumes changeonly change g gup
  assumes nochange = ids g - change
  shows unchanged nochange g gup
  using assms unfolding changeonly.simps unchanged.simps
  by blast

lemma add-node-unchanged:
  assumes new ∉ ids g
  assumes nid ∈ ids g
  assumes gup = add-node new k g
  assumes wf-graph g
  shows unchanged (eval-usages g nid) g gup
proof –
  have new ∉ (eval-usages g nid) using assms
  using eval-usages.simps by blast
  then have changeonly {new} g gup
  using assms add-changed by blast
  then show ?thesis using assms add-node-def disjoint-change
  using Diff-insert-absorb by auto
qed

lemma eval-uses-imp:
   $((nid' \in ids\ g \wedge nid = nid') \vee$ 
     $nid' \in inputs\ g\ nid \vee (\exists nid'' . eval-uses\ g\ nid\ nid'' \wedge eval-uses\ g\ nid''\ nid'))$ 
     $\longleftrightarrow eval-uses\ g\ nid\ nid'$ 
  using use0 use-inp use-trans
  by (meson eval-uses.simps)

lemma wf-use-ids:
  assumes wf-graph g
  assumes nid ∈ ids g
  assumes eval-uses g nid nid'
  shows nid' ∈ ids g
  using assms(3)
proof (induction rule: eval-uses.induct)
  case use0
  then show ?case by simp
next
  case use-inp
  then show ?case
  using assms(1) inp-in-g-wf by blast
next
  case use-trans
  then show ?case by blast
qed

```

```

lemma no-external-use:
  assumes wf-graph g
  assumes nid' ∉ ids g
  assumes nid ∈ ids g
  shows  $\neg(\text{eval-uses } g \text{ } nid \text{ } nid')$ 
proof –
  have 0: nid ≠ nid'
    using assms by blast
  have inp: nid' ∉ inputs g nid
    using assms
    using inp-in-g-wf by blast
  have rec-0: ∄ n . n ∈ ids g ∧ n = nid'
    using assms by blast
  have rec-inp: ∄ n . n ∈ ids g ∧ n ∈ inputs g nid'
    using assms(2) inp-in-g by blast
  have rec: ∄ nid'' . eval-uses g nid nid'' ∧ eval-uses g nid'' nid'
    using wf-use-ids assms(1) assms(2) assms(3) by blast
  from inp 0 rec show ?thesis
    using eval-uses-imp by blast
qed

end

```

## 6.8 Tree to Graph Theorems

```

theory TreeToGraphThms
imports
  TreeToGraph
  IRTreeEvalThms
  IRGraphFrames
  HOL-Eisbach.Eisbach
  HOL-Eisbach.Eisbach-Tools
begin

```

### 6.8.1 Extraction and Evaluation of Expression Trees is Deterministic.

First, we prove some extra rules that relate each type of `IRNode` to the corresponding `IRExpr` type that 'rep' will produce. These are very helpful for proving that 'rep' is deterministic.

**named-theorems** *rep*

```

lemma rep-constant [rep]:
  g ⊢ n ≃ e  $\implies$ 
  kind g n = ConstantNode c  $\implies$ 
  e = ConstantExpr c
  by (induction rule: rep.induct; auto)

```

```

lemma rep-parameter [rep]:

```

$g \vdash n \simeq e \implies$   
 $kind\ g\ n = ParameterNode\ i \implies$   
 $(\exists s. e = ParameterExpr\ i\ s)$   
**by** (induction rule: *rep.induct*; *auto*)

**lemma** *rep-conditional* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = ConditionalNode\ c\ t\ f \implies$   
 $(\exists ce\ te\ fe. e = ConditionalExpr\ ce\ te\ fe)$   
**by** (induction rule: *rep.induct*; *auto*)

**lemma** *rep-abs* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = AbsNode\ x \implies$   
 $(\exists xe. e = UnaryExpr\ UnaryAbs\ xe)$   
**by** (induction rule: *rep.induct*; *auto*)

**lemma** *rep-not* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = NotNode\ x \implies$   
 $(\exists xe. e = UnaryExpr\ UnaryNot\ xe)$   
**by** (induction rule: *rep.induct*; *auto*)

**lemma** *rep-negate* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = NegateNode\ x \implies$   
 $(\exists xe. e = UnaryExpr\ UnaryNeg\ xe)$   
**by** (induction rule: *rep.induct*; *auto*)

**lemma** *rep-logicnegation* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = LogicNegationNode\ x \implies$   
 $(\exists xe. e = UnaryExpr\ UnaryLogicNegation\ xe)$   
**by** (induction rule: *rep.induct*; *auto*)

**lemma** *rep-add* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = AddNode\ x\ y \implies$   
 $(\exists xe\ ye. e = BinaryExpr\ BinAdd\ xe\ ye)$   
**by** (induction rule: *rep.induct*; *auto*)

**lemma** *rep-sub* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = SubNode\ x\ y \implies$   
 $(\exists xe\ ye. e = BinaryExpr\ BinSub\ xe\ ye)$   
**by** (induction rule: *rep.induct*; *auto*)

**lemma** *rep-mul* [*rep*]:  
 $g \vdash n \simeq e \implies$

$kind\ g\ n = MulNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinMul\ xe\ ye)$   
**by** (induction rule: rep.induct; auto)

**lemma** rep-and [rep]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = AndNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinAnd\ xe\ ye)$   
**by** (induction rule: rep.induct; auto)

**lemma** rep-or [rep]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = OrNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinOr\ xe\ ye)$   
**by** (induction rule: rep.induct; auto)

**lemma** rep-xor [rep]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = XorNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinXor\ xe\ ye)$   
**by** (induction rule: rep.induct; auto)

**lemma** rep-left-shift [rep]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = LeftShiftNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinLeftShift\ xe\ ye)$   
**by** (induction rule: rep.induct; auto)

**lemma** rep-right-shift [rep]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = RightShiftNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinRightShift\ xe\ ye)$   
**by** (induction rule: rep.induct; auto)

**lemma** rep-unsigned-right-shift [rep]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = UnsignedRightShiftNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinURightShift\ xe\ ye)$   
**by** (induction rule: rep.induct; auto)

**lemma** rep-integer-below [rep]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = IntegerBelowNode\ x\ y \implies$   
 $(\exists\ xe\ ye. e = BinaryExpr\ BinIntegerBelow\ xe\ ye)$   
**by** (induction rule: rep.induct; auto)

**lemma** rep-integer-equals [rep]:  
 $g \vdash n \simeq e \implies$   
 $kind\ g\ n = IntegerEqualsNode\ x\ y \implies$

( $\exists xe ye. e = \text{BinaryExpr BinIntegerEquals } xe ye$ )  
**by** (induction rule: *rep.induct*; auto)

**lemma** *rep-integer-less-than* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $\text{kind } g \ n = \text{IntegerLessThanNode } x \ y \implies$   
( $\exists xe ye. e = \text{BinaryExpr BinIntegerLessThan } xe ye$ )  
**by** (induction rule: *rep.induct*; auto)

**lemma** *rep-narrow* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $\text{kind } g \ n = \text{NarrowNode } ib \ rb \ x \implies$   
( $\exists x. e = \text{UnaryExpr } (\text{UnaryNarrow } ib \ rb) \ x$ )  
**by** (induction rule: *rep.induct*; auto)

**lemma** *rep-sign-extend* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $\text{kind } g \ n = \text{SignExtendNode } ib \ rb \ x \implies$   
( $\exists x. e = \text{UnaryExpr } (\text{UnarySignExtend } ib \ rb) \ x$ )  
**by** (induction rule: *rep.induct*; auto)

**lemma** *rep-zero-extend* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $\text{kind } g \ n = \text{ZeroExtendNode } ib \ rb \ x \implies$   
( $\exists x. e = \text{UnaryExpr } (\text{UnaryZeroExtend } ib \ rb) \ x$ )  
**by** (induction rule: *rep.induct*; auto)

**lemma** *rep-load-field* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $\text{is-preevaluated } (\text{kind } g \ n) \implies$   
( $\exists s. e = \text{LeafExpr } n \ s$ )  
**by** (induction rule: *rep.induct*; auto)

**lemma** *rep-ref* [*rep*]:  
 $g \vdash n \simeq e \implies$   
 $\text{kind } g \ n = \text{RefNode } n' \implies$   
 $g \vdash n' \simeq e$   
**by** (induction rule: *rep.induct*; auto)

**method** *solve-det* **uses** *node* =  
(match *node* **in** *kind* - - = *node* - **for** *node*  $\implies$   
 <match *rep* **in** *r*: -  $\implies$  - = *node* -  $\implies$  -  $\implies$   
 <match *IRNode.inject* **in** *i*: (*node* - = *node* -) = -  $\implies$   
 <match *RepE* **in** *e*: -  $\implies$  ( $\bigwedge x. - = \text{node } x \implies -$ )  $\implies$  -  $\implies$   
 <match *IRNode.distinct* **in** *d*: *node* -  $\neq$  *RefNode* -  $\implies$   
 <metis *i e r d*>>>> |  
 match *node* **in** *kind* - - = *node* - - **for** *node*  $\implies$   
 <match *rep* **in** *r*: -  $\implies$  - = *node* - -  $\implies$  -  $\implies$



```

    <match IRNode.inject in i: (node - - = node - -) = - =>
      <match RepE in e: - ==> (Λx y. - = node x y ==> -) ==> - =>
        <match IRNode.distinct in d: node - - ≠ RefNode - =>
          <metis i e r d>>>> |
match node in kind - - = node - - - for node =>
  <match rep in r: - ==> - = node - - - ==> - =>
    <match IRNode.inject in i: (node - - - = node - - -) = - =>
      <match RepE in e: - ==> (Λx y z. - = node x y z ==> -) ==> - =>
        <match IRNode.distinct in d: node - - - ≠ RefNode - =>
          <metis i e r d>>>> |
match node in kind - - = node - - - for node =>
  <match rep in r: - ==> - = node - - - ==> - =>
    <match IRNode.inject in i: (node - - - = node - - -) = - =>
      <match RepE in e: - ==> (Λx. - = node - - x ==> -) ==> - =>
        <match IRNode.distinct in d: node - - - ≠ RefNode - =>
          <metis i e r d>>>>)

```

Now we can prove that 'rep' and 'eval', and their list versions, are deterministic.

**lemma** *repDet*:

**shows**  $(g \vdash n \simeq e_1) \implies (g \vdash n \simeq e_2) \implies e_1 = e_2$

**proof** (*induction arbitrary: e<sub>2</sub> rule: rep.induct*)

**case** (*ConstantNode n c*)

**then show** ?case **using** *rep-constant* **by** *auto*

**next**

**case** (*ParameterNode n i s*)

**then show** ?case

**by** (*metis IRNode.disc(2685) ParameterNodeE is-RefNode-def rep-parameter*)

**next**

**case** (*ConditionalNode n c t f ce te fe*)

**then show** ?case

**using** *IRNode.distinct(593)*

**using** *IRNode.inject(6) ConditionalNodeE rep-conditional*

**by** *metis*

**next**

**case** (*AbsNode n x xe*)

**then show** ?case

**by** (*solve-det node: AbsNode*)

**next**

**case** (*NotNode n x xe*)

**then show** ?case

**by** (*solve-det node: NotNode*)

**next**

**case** (*NegateNode n x xe*)

**then show** ?case

**by** (*solve-det node: NegateNode*)

**next**

**case** (*LogicNegationNode n x xe*)

**then show** ?case

```

    by (solve-det node: LogicNegationNode)
next
  case (AddNode n x y xe ye)
  then show ?case
    by (solve-det node: AddNode)
next
  case (MulNode n x y xe ye)
  then show ?case
    by (solve-det node: MulNode)
next
  case (SubNode n x y xe ye)
  then show ?case
    by (solve-det node: SubNode)
next
  case (AndNode n x y xe ye)
  then show ?case
    by (solve-det node: AndNode)
next
  case (OrNode n x y xe ye)
  then show ?case
    by (solve-det node: OrNode)
next
  case (XorNode n x y xe ye)
  then show ?case
    by (solve-det node: XorNode)
next
  case (LeftShiftNode n x y xe ye)
  then show ?case
    by (solve-det node: LeftShiftNode)
next
  case (RightShiftNode n x y xe ye)
  then show ?case
    by (solve-det node: RightShiftNode)
next
  case (UnsignedRightShiftNode n x y xe ye)
  then show ?case
    by (solve-det node: UnsignedRightShiftNode)
next
  case (IntegerBelowNode n x y xe ye)
  then show ?case
    by (solve-det node: IntegerBelowNode)
next
  case (IntegerEqualsNode n x y xe ye)
  then show ?case
    by (solve-det node: IntegerEqualsNode)
next
  case (IntegerLessThanNode n x y xe ye)
  then show ?case
    by (solve-det node: IntegerLessThanNode)

```

```

next
  case (NarrowNode n x xe)
  then show ?case
    by (metis IRNode.distinct(2203) IRNode.inject(28) NarrowNodeE rep-narrow)
next
  case (SignExtendNode n x xe)
  then show ?case
    by (metis IRNode.distinct(2599) IRNode.inject(39) SignExtendNodeE rep-sign-extend)
next
  case (ZeroExtendNode n x xe)
  then show ?case
    by (metis IRNode.distinct(2753) IRNode.inject(50) ZeroExtendNodeE rep-zero-extend)
next
  case (LeafNode n s)
  then show ?case using rep-load-field LeafNodeE
    by (metis is-preevaluated.simps(53))
next
  case (RefNode n^)
  then show ?case
    using rep-ref by blast
qed

```

```

lemma repAllDet:
  g ⊢ xs ≃L e1 ⟹
  g ⊢ xs ≃L e2 ⟹
  e1 = e2
proof (induction arbitrary: e2 rule: replist.induct)
  case RepNil
  then show ?case
    using replist.cases by auto
next
  case (RepCons x xe xs xse)
  then show ?case
    by (metis list.distinct(1) list.sel(1) list.sel(3) repDet replist.cases)
qed

```

```

lemma encodeEvalDet:
  [g,m,p] ⊢ e ↦ v1 ⟹
  [g,m,p] ⊢ e ↦ v2 ⟹
  v1 = v2
by (metis encodeeval-def evalDet repDet)

```

```

lemma graphDet: ([g,m,p] ⊢ n ↦ v1) ∧ ([g,m,p] ⊢ n ↦ v2) ⟹ v1 = v2
using encodeEvalDet by blast

```

## 6.8.2 Monotonicity of Graph Refinement

Lift refinement monotonicity to graph level. Hopefully these shouldn't really be required.

**lemma** *mono-abs*:

**assumes**  $\text{kind } g1 \ n = \text{AbsNode } x \wedge \text{kind } g2 \ n = \text{AbsNode } x$   
**assumes**  $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$   
**assumes**  $xe1 \geq xe2$   
**assumes**  $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$   
**shows**  $e1 \geq e2$   
**by** (*metis* *AbsNode* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *mono-unary repDet*)

**lemma** *mono-not*:

**assumes**  $\text{kind } g1 \ n = \text{NotNode } x \wedge \text{kind } g2 \ n = \text{NotNode } x$   
**assumes**  $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$   
**assumes**  $xe1 \geq xe2$   
**assumes**  $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$   
**shows**  $e1 \geq e2$   
**by** (*metis* *NotNode* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *mono-unary repDet*)

**lemma** *mono-negate*:

**assumes**  $\text{kind } g1 \ n = \text{NegateNode } x \wedge \text{kind } g2 \ n = \text{NegateNode } x$   
**assumes**  $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$   
**assumes**  $xe1 \geq xe2$   
**assumes**  $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$   
**shows**  $e1 \geq e2$   
**by** (*metis* *NegateNode* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *mono-unary repDet*)

**lemma** *mono-logic-negation*:

**assumes**  $\text{kind } g1 \ n = \text{LogicNegationNode } x \wedge \text{kind } g2 \ n = \text{LogicNegationNode } x$   
**assumes**  $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$   
**assumes**  $xe1 \geq xe2$   
**assumes**  $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$   
**shows**  $e1 \geq e2$   
**by** (*metis* *LogicNegationNode* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *mono-unary repDet*)

**lemma** *mono-narrow*:

**assumes**  $\text{kind } g1 \ n = \text{NarrowNode } ib \ rb \ x \wedge \text{kind } g2 \ n = \text{NarrowNode } ib \ rb \ x$   
**assumes**  $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$   
**assumes**  $xe1 \geq xe2$   
**assumes**  $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$   
**shows**  $e1 \geq e2$   
**using** *assms* *mono-unary repDet* *NarrowNode*  
**by** *metis*

**lemma** *mono-sign-extend*:

**assumes**  $\text{kind } g1 \ n = \text{SignExtendNode } ib \ rb \ x \wedge \text{kind } g2 \ n = \text{SignExtendNode } ib \ rb \ x$   
**assumes**  $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$   
**assumes**  $xe1 \geq xe2$   
**assumes**  $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$   
**shows**  $e1 \geq e2$

by (metis SignExtendNode assms(1) assms(2) assms(3) assms(4) mono-unary repDet)

**lemma** *mono-zero-extend*:

assumes kind g1 n = ZeroExtendNode ib rb x  $\wedge$  kind g2 n = ZeroExtendNode ib rb x  
 assumes (g1  $\vdash$  x  $\simeq$  xe1)  $\wedge$  (g2  $\vdash$  x  $\simeq$  xe2)  
 assumes xe1  $\geq$  xe2  
 assumes (g1  $\vdash$  n  $\simeq$  e1)  $\wedge$  (g2  $\vdash$  n  $\simeq$  e2)  
 shows e1  $\geq$  e2  
 using assms mono-unary repDet ZeroExtendNode  
 by metis

**lemma** *mono-conditional-graph*:

assumes kind g1 n = ConditionalNode c t f  $\wedge$  kind g2 n = ConditionalNode c t f  
 assumes (g1  $\vdash$  c  $\simeq$  ce1)  $\wedge$  (g2  $\vdash$  c  $\simeq$  ce2)  
 assumes (g1  $\vdash$  t  $\simeq$  te1)  $\wedge$  (g2  $\vdash$  t  $\simeq$  te2)  
 assumes (g1  $\vdash$  f  $\simeq$  fe1)  $\wedge$  (g2  $\vdash$  f  $\simeq$  fe2)  
 assumes ce1  $\geq$  ce2  $\wedge$  te1  $\geq$  te2  $\wedge$  fe1  $\geq$  fe2  
 assumes (g1  $\vdash$  n  $\simeq$  e1)  $\wedge$  (g2  $\vdash$  n  $\simeq$  e2)  
 shows e1  $\geq$  e2  
 using ConditionalNodeE IRNode.inject(6) assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) mono-conditional repDet rep-conditional  
 by (smt (verit, best) ConditionalNode)

**lemma** *mono-add*:

assumes kind g1 n = AddNode x y  $\wedge$  kind g2 n = AddNode x y  
 assumes (g1  $\vdash$  x  $\simeq$  xe1)  $\wedge$  (g2  $\vdash$  x  $\simeq$  xe2)  
 assumes (g1  $\vdash$  y  $\simeq$  ye1)  $\wedge$  (g2  $\vdash$  y  $\simeq$  ye2)  
 assumes xe1  $\geq$  xe2  $\wedge$  ye1  $\geq$  ye2  
 assumes (g1  $\vdash$  n  $\simeq$  e1)  $\wedge$  (g2  $\vdash$  n  $\simeq$  e2)  
 shows e1  $\geq$  e2  
 using mono-binary assms AddNodeE IRNode.inject(2) repDet rep-add  
 by (metis IRNode.distinct(205))

**lemma** *mono-mul*:

assumes kind g1 n = MulNode x y  $\wedge$  kind g2 n = MulNode x y  
 assumes (g1  $\vdash$  x  $\simeq$  xe1)  $\wedge$  (g2  $\vdash$  x  $\simeq$  xe2)  
 assumes (g1  $\vdash$  y  $\simeq$  ye1)  $\wedge$  (g2  $\vdash$  y  $\simeq$  ye2)  
 assumes xe1  $\geq$  xe2  $\wedge$  ye1  $\geq$  ye2  
 assumes (g1  $\vdash$  n  $\simeq$  e1)  $\wedge$  (g2  $\vdash$  n  $\simeq$  e2)  
 shows e1  $\geq$  e2  
 using mono-binary assms IRNode.inject(27) MulNodeE repDet rep-mul  
 by (smt (verit, best) MulNode)

**lemma** *term-graph-evaluation*:

(g  $\vdash$  n  $\leq$  e)  $\implies$  ( $\forall$  m p v . ([m,p]  $\vdash$  e  $\mapsto$  v)  $\longrightarrow$  ([g,m,p]  $\vdash$  n  $\mapsto$  v))  
 unfolding graph-represents-expression-def apply auto

```

by (meson encodeeval-def)

lemma encodes-contains:
  g ⊢ n ≃ e ⇒
  kind g n ≠ NoNode
  apply (induction rule: rep.induct)
  apply (match IRNode.distinct in e: ?n ≠ NoNode ⇒
    ⟨presburger add: e⟩)+
  apply force
  by fastforce

lemma no-encoding:
  assumes n ∉ ids g
  shows ¬(g ⊢ n ≃ e)
  using assms apply simp apply (rule notI) by (induction e; simp add: en-
codes-contains)

lemma not-excluded-keep-type:
  assumes n ∈ ids g1
  assumes n ∉ excluded
  assumes (excluded ⊆ as-set g1) ⊆ as-set g2
  shows kind g1 n = kind g2 n ∧ stamp g1 n = stamp g2 n
  using assms unfolding as-set-def domain-subtraction-def by blast

method metis-node-eq-unary for node :: 'a ⇒ IRNode =
  (match IRNode.inject in i: (node - = node -) = - ⇒
    ⟨metis i⟩)
method metis-node-eq-binary for node :: 'a ⇒ 'a ⇒ IRNode =
  (match IRNode.inject in i: (node - - = node - -) = - ⇒
    ⟨metis i⟩)
method metis-node-eq-ternary for node :: 'a ⇒ 'a ⇒ 'a ⇒ IRNode =
  (match IRNode.inject in i: (node - - - = node - - -) = - ⇒
    ⟨metis i⟩)



### 6.8.3 Lift Data-flow Tree Refinement to Graph Refinement



theorem graph-semantics-preservation:
  assumes a: e1' ≥ e2'
  assumes b: ({n'} ⊆ as-set g1) ⊆ as-set g2
  assumes c: g1 ⊢ n' ≃ e1'
  assumes d: g2 ⊢ n' ≃ e2'
  shows graph-refinement g1 g2
  unfolding graph-refinement-def apply rule
  apply (metis b d ids-some no-encoding not-excluded-keep-type singleton-iff sub-
setI)
  apply (rule allI) apply (rule impI) apply (rule allI) apply (rule impI)
  unfolding graph-represents-expression-def
proof -
  fix n e1

```

```

assume  $e: n \in \text{ids } g1$ 
assume  $f: (g1 \vdash n \simeq e1)$ 

show  $\exists e2. (g2 \vdash n \simeq e2) \wedge e1 \geq e2$ 
proof ( $\text{cases } n = n'$ )
  case True
    have  $g: e1 = e1'$  using  $c \ f \ \text{True} \ \text{repDet} \ \text{by} \ \text{simp}$ 
    have  $h: (g2 \vdash n \simeq e2') \wedge e1' \geq e2'$ 
      using  $\text{True} \ a \ d \ \text{by} \ \text{blast}$ 
    then show  $?thesis$ 
      using  $g \ \text{by} \ \text{blast}$ 
  next
    case False
    have  $n \notin \{n'\}$ 
      using False by simp
    then have  $i: \text{kind } g1 \ n = \text{kind } g2 \ n \wedge \text{stamp } g1 \ n = \text{stamp } g2 \ n$ 
      using not-excluded-keep-type
      using  $b \ e \ \text{by} \ \text{presburger}$ 
    show  $?thesis$  using  $f \ i$ 
    proof (induction  $e1$ )
      case (ConstantNode  $n \ c$ )
      then show  $?case$ 
        by (metis eq-refl rep.ConstantNode)
    next
      case (ParameterNode  $n \ i \ s$ )
      then show  $?case$ 
        by (metis eq-refl rep.ParameterNode)
    next
      case (ConditionalNode  $n \ c \ t \ f \ ce1 \ te1 \ fe1$ )
      have  $k: g1 \vdash n \simeq \text{ConditionalExpr } ce1 \ te1 \ fe1$  using  $f \ \text{ConditionalNode}$ 
        by (simp add: ConditionalNode.hyps(2) rep.ConditionalNode)
      obtain  $cn \ tn \ fn$  where  $l: \text{kind } g1 \ n = \text{ConditionalNode } cn \ tn \ fn$ 
        using ConditionalNode.hyps(1) by blast
      then have  $mc: g1 \vdash cn \simeq ce1$ 
        using ConditionalNode.hyps(1) ConditionalNode.hyps(2) by fastforce
      from  $l$  have  $mt: g1 \vdash tn \simeq te1$ 
        using ConditionalNode.hyps(1) ConditionalNode.hyps(3) by fastforce
      from  $l$  have  $mf: g1 \vdash fn \simeq fe1$ 
        using ConditionalNode.hyps(1) ConditionalNode.hyps(4) by fastforce
      then show  $?case$ 
      proof –
        have  $g1 \vdash cn \simeq ce1$  using  $mc \ \text{by} \ \text{simp}$ 
        have  $g1 \vdash tn \simeq te1$  using  $mt \ \text{by} \ \text{simp}$ 
        have  $g1 \vdash fn \simeq fe1$  using  $mf \ \text{by} \ \text{simp}$ 
        have  $cer: \exists ce2. (g2 \vdash cn \simeq ce2) \wedge ce1 \geq ce2$ 
          using ConditionalNode
          using  $a \ b \ c \ d \ l \ \text{no-encoding not-excluded-keep-type repDet singletonD}$ 
          by (metis-node-eq-ternary ConditionalNode)
        have  $ter: \exists te2. (g2 \vdash tn \simeq te2) \wedge te1 \geq te2$ 

```

```

      using ConditionalNode a b c d l no-encoding not-excluded-keep-type repDet
singletonD
      by (metis-node-eq-ternary ConditionalNode)
      have  $\exists fe2. (g2 \vdash fn \simeq fe2) \wedge fe1 \geq fe2$ 
      using ConditionalNode a b c d l no-encoding not-excluded-keep-type repDet
singletonD
      by (metis-node-eq-ternary ConditionalNode)
      then have  $\exists ce2 te2 fe2. (g2 \vdash n \simeq ConditionalExpr ce2 te2 fe2) \wedge$ 
ConditionalExpr ce1 te1 fe1  $\geq ConditionalExpr ce2 te2 fe2$ 
      using ConditionalNode.premis l rep.ConditionalNode cer ter
      by (smt (verit) mono-conditional)
      then show ?thesis
      by meson
qed
next
case (AbsNode n x xe1)
have k:  $g1 \vdash n \simeq UnaryExpr UnaryAbs xe1$  using f AbsNode
  by (simp add: AbsNode.hyps(2) rep.AbsNode)
obtain xn where l: kind g1 n = AbsNode xn
  using AbsNode.hyps(1) by blast
then have m:  $g1 \vdash xn \simeq xe1$ 
  using AbsNode.hyps(1) AbsNode.hyps(2) by fastforce
then show ?case
proof (cases xn = n')
case True
  then have n:  $xe1 = e1'$  using c m repDet by simp
  then have ev:  $g2 \vdash n \simeq UnaryExpr UnaryAbs e2'$  using AbsNode.hyps(1)
l m n
    using AbsNode.premis True d rep.AbsNode by simp
  then have r:  $UnaryExpr UnaryAbs e1' \geq UnaryExpr UnaryAbs e2'$ 
    by (meson a mono-unary)
  then show ?thesis using ev r
    by (metis n)
next
case False
have  $g1 \vdash xn \simeq xe1$  using m by simp
have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
  using AbsNode
using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
  by (metis-node-eq-unary AbsNode)
  then have  $\exists xe2. (g2 \vdash n \simeq UnaryExpr UnaryAbs xe2) \wedge UnaryExpr$ 
UnaryAbs xe1  $\geq UnaryExpr UnaryAbs xe2$ 
  by (metis AbsNode.premis l mono-unary rep.AbsNode)
  then show ?thesis
  by meson
qed
next
case (NotNode n x xe1)
have k:  $g1 \vdash n \simeq UnaryExpr UnaryNot xe1$  using f NotNode

```



```

    by (simp add: NotNode.hyps(2) rep.NotNode)
  obtain  $xn$  where  $l$ :  $kind\ g1\ n = NotNode\ xn$ 
    using NotNode.hyps(1) by blast
  then have  $m$ :  $g1 \vdash xn \simeq xe1$ 
    using NotNode.hyps(1) NotNode.hyps(2) by fastforce
  then show ?case
  proof (cases  $xn = n'$ )
    case True
    then have  $n$ :  $xe1 = e1'$  using  $c\ m\ repDet$  by simp
    then have  $ev$ :  $g2 \vdash n \simeq UnaryExpr\ UnaryNot\ e2'$  using NotNode.hyps(1)
  l m n
    using NotNode.prem1 True  $d\ rep.NotNode$  by simp
  then have  $r$ :  $UnaryExpr\ UnaryNot\ e1' \geq UnaryExpr\ UnaryNot\ e2'$ 
    by (meson a mono-unary)
  then show ?thesis using  $ev\ r$ 
    by (metis  $n$ )
  next
  case False
  have  $g1 \vdash xn \simeq xe1$  using  $m$  by simp
  have  $\exists\ xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using NotNode
    using False  $i\ b\ l\ not-excluded-keep-type\ singletonD\ no-encoding$ 
    by (metis node-eq-unary NotNode)
  then have  $\exists\ xe2. (g2 \vdash n \simeq UnaryExpr\ UnaryNot\ xe2) \wedge UnaryExpr$ 
     $UnaryNot\ xe1 \geq UnaryExpr\ UnaryNot\ xe2$ 
    by (metis NotNode.prem1  $l\ mono-unary\ rep.NotNode$ )
  then show ?thesis
    by meson
  qed
next
case (NegateNode  $n\ x\ xe1$ )
have  $k$ :  $g1 \vdash n \simeq UnaryExpr\ UnaryNeg\ xe1$  using  $f\ NegateNode$ 
  by (simp add: NegateNode.hyps(2) rep.NegateNode)
obtain  $xn$  where  $l$ :  $kind\ g1\ n = NegateNode\ xn$ 
  using NegateNode.hyps(1) by blast
then have  $m$ :  $g1 \vdash xn \simeq xe1$ 
  using NegateNode.hyps(1) NegateNode.hyps(2) by fastforce
then show ?case
proof (cases  $xn = n'$ )
  case True
  then have  $n$ :  $xe1 = e1'$  using  $c\ m\ repDet$  by simp
  then have  $ev$ :  $g2 \vdash n \simeq UnaryExpr\ UnaryNeg\ e2'$  using NegateNode.hyps(1)
  l m n
    using NegateNode.prem1 True  $d\ rep.NegateNode$  by simp
  then have  $r$ :  $UnaryExpr\ UnaryNeg\ e1' \geq UnaryExpr\ UnaryNeg\ e2'$ 
    by (meson a mono-unary)
  then show ?thesis using  $ev\ r$ 
    by (metis  $n$ )
  next

```

```

    case False
    have  $g1 \vdash xn \simeq xe1$  using  $m$  by simp
    have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using NegateNode
      using False i b l not-excluded-keep-type singletonD no-encoding
      by (metis-node-eq-unary NegateNode)
    then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr UnaryNeg } xe2) \wedge \text{UnaryExpr}$ 
       $\text{UnaryNeg } xe1 \geq \text{UnaryExpr UnaryNeg } xe2$ 
      by (metis NegateNode.prem1 mono-unary rep.NegateNode)
    then show ?thesis
      by meson
  qed
next
case (LogicNegationNode n x xe1)
  have  $k: g1 \vdash n \simeq \text{UnaryExpr UnaryLogicNegation } xe1$  using  $f$  LogicNegationNode
    by (simp add: LogicNegationNode.hyps(2) rep.LogicNegationNode)
  obtain  $xn$  where  $l: \text{kind } g1 \ n = \text{LogicNegationNode } xn$ 
    using LogicNegationNode.hyps(1) by blast
  then have  $m: g1 \vdash xn \simeq xe1$ 
    using LogicNegationNode.hyps(1) LogicNegationNode.hyps(2) by fastforce
  then show ?case
  proof (cases  $xn = n'$ )
    case True
    then have  $n: xe1 = e1'$  using  $c \ m \ repDet$  by simp
    then have  $ev: g2 \vdash n \simeq \text{UnaryExpr UnaryLogicNegation } e2'$  using
      LogicNegationNode.hyps(1)  $l \ m \ n$ 
      using LogicNegationNode.prem1 True  $d \ rep.LogicNegationNode$  by simp
    then have  $r: \text{UnaryExpr UnaryLogicNegation } e1' \geq \text{UnaryExpr UnaryLogicNegation } e2'$ 
      by (meson a mono-unary)
    then show ?thesis using  $ev \ r$ 
      by (metis  $n$ )
  next
  case False
  have  $g1 \vdash xn \simeq xe1$  using  $m$  by simp
  have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using LogicNegationNode
    using False i b l not-excluded-keep-type singletonD no-encoding
    by (metis-node-eq-unary LogicNegationNode)
  then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr UnaryLogicNegation } xe2) \wedge$ 
     $\text{UnaryExpr UnaryLogicNegation } xe1 \geq \text{UnaryExpr UnaryLogicNegation } xe2$ 
    by (metis LogicNegationNode.prem1 mono-unary rep.LogicNegationNode)
  then show ?thesis
    by meson
  qed
next
case (AddNode n x y xe1 ye1)
  have  $k: g1 \vdash n \simeq \text{BinaryExpr BinAdd } xe1 \ ye1$  using  $f \ AddNode$ 

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    by (simp add: AddNode.hyps(2) rep.AddNode)
  obtain  $xn\ yn$  where  $l$ : kind  $g1\ n = AddNode\ xn\ yn$ 
    using AddNode.hyps(1) by blast
  then have  $mx$ :  $g1 \vdash xn \simeq xe1$ 
    using AddNode.hyps(1) AddNode.hyps(2) by fastforce
  from  $l$  have  $my$ :  $g1 \vdash yn \simeq ye1$ 
    using AddNode.hyps(1) AddNode.hyps(3) by fastforce
  then show ?case
  proof -
    have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
    have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
    have  $xer$ :  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using AddNode
      using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary AddNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using AddNode
      using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary AddNode)
    then have  $\exists xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinAdd\ xe2\ ye2) \wedge BinaryExpr$ 
       $BinAdd\ xe1\ ye1 \geq BinaryExpr\ BinAdd\ xe2\ ye2$ 
      by (metis AddNode.prem1 l mono-binary rep.AddNode xer)
    then show ?thesis
      by meson
  qed
next
case (MulNode  $n\ x\ y\ xe1\ ye1$ )
have  $k$ :  $g1 \vdash n \simeq BinaryExpr\ BinMul\ xe1\ ye1$  using  $f$  MulNode
  by (simp add: MulNode.hyps(2) rep.MulNode)
obtain  $xn\ yn$  where  $l$ : kind  $g1\ n = MulNode\ xn\ yn$ 
  using MulNode.hyps(1) by blast
then have  $mx$ :  $g1 \vdash xn \simeq xe1$ 
  using MulNode.hyps(1) MulNode.hyps(2) by fastforce
from  $l$  have  $my$ :  $g1 \vdash yn \simeq ye1$ 
  using MulNode.hyps(1) MulNode.hyps(3) by fastforce
then show ?case
  proof -
    have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
    have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
    have  $xer$ :  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using MulNode
      using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary MulNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using MulNode
      using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary MulNode)
    then have  $\exists xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinMul\ xe2\ ye2) \wedge BinaryExpr$ 
       $BinMul\ xe1\ ye1 \geq BinaryExpr\ BinMul\ xe2\ ye2$ 

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      by (metis MulNode.premis l mono-binary rep.MulNode xer)
    then show ?thesis
      by meson
  qed
next
case (SubNode n x y xe1 ye1)
have k:  $g1 \vdash n \simeq \text{BinaryExpr BinSub } xe1 \text{ ye1}$  using f SubNode
  by (simp add: SubNode.hyps(2) rep.SubNode)
obtain xn yn where l: kind g1 n = SubNode xn yn
  using SubNode.hyps(1) by blast
then have mx:  $g1 \vdash xn \simeq xe1$ 
  using SubNode.hyps(1) SubNode.hyps(2) by fastforce
from l have my:  $g1 \vdash yn \simeq ye1$ 
  using SubNode.hyps(1) SubNode.hyps(3) by fastforce
then show ?case
proof -
  have  $g1 \vdash xn \simeq xe1$  using mx by simp
  have  $g1 \vdash yn \simeq ye1$  using my by simp
  have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using SubNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary SubNode)
  have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
  using SubNode a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary SubNode)
  then have  $\exists xe2 ye2. (g2 \vdash n \simeq \text{BinaryExpr BinSub } xe2 \text{ ye2}) \wedge \text{BinaryExpr BinSub } xe1 \text{ ye1} \geq \text{BinaryExpr BinSub } xe2 \text{ ye2}$ 
    by (metis SubNode.premis l mono-binary rep.SubNode xer)
  then show ?thesis
    by meson
  qed
next
case (AndNode n x y xe1 ye1)
have k:  $g1 \vdash n \simeq \text{BinaryExpr BinAnd } xe1 \text{ ye1}$  using f AndNode
  by (simp add: AndNode.hyps(2) rep.AndNode)
obtain xn yn where l: kind g1 n = AndNode xn yn
  using AndNode.hyps(1) by blast
then have mx:  $g1 \vdash xn \simeq xe1$ 
  using AndNode.hyps(1) AndNode.hyps(2) by fastforce
from l have my:  $g1 \vdash yn \simeq ye1$ 
  using AndNode.hyps(1) AndNode.hyps(3) by fastforce
then show ?case
proof -
  have  $g1 \vdash xn \simeq xe1$  using mx by simp
  have  $g1 \vdash yn \simeq ye1$  using my by simp
  have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using AndNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary AndNode)

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    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using AndNode a b c d l no-encoding not-excluded-keep-type repDet
singletonD
    by (metis-node-eq-binary AndNode)
    then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr BinAnd xe2 ye2) \wedge BinaryExpr$ 
BinAnd xe1 ye1  $\geq BinaryExpr BinAnd xe2 ye2$ 
      by (metis AndNode.premis l mono-binary rep.AndNode xer)
    then show ?thesis
      by meson
  qed
next
case (OrNode n x y xe1 ye1)
have k:  $g1 \vdash n \simeq BinaryExpr BinOr xe1 ye1$  using f OrNode
  by (simp add: OrNode.hyps(2) rep.OrNode)
obtain xn yn where l: kind g1 n = OrNode xn yn
  using OrNode.hyps(1) by blast
then have mx:  $g1 \vdash xn \simeq xe1$ 
  using OrNode.hyps(1) OrNode.hyps(2) by fastforce
from l have my:  $g1 \vdash yn \simeq ye1$ 
  using OrNode.hyps(1) OrNode.hyps(3) by fastforce
then show ?case
proof -
  have  $g1 \vdash xn \simeq xe1$  using mx by simp
  have  $g1 \vdash yn \simeq ye1$  using my by simp
  have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using OrNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary OrNode)
  have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
  using OrNode a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary OrNode)
  then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr BinOr xe2 ye2) \wedge BinaryExpr$ 
BinOr xe1 ye1  $\geq BinaryExpr BinOr xe2 ye2$ 
    by (metis OrNode.premis l mono-binary rep.OrNode xer)
  then show ?thesis
    by meson
  qed
next
case (XorNode n x y xe1 ye1)
have k:  $g1 \vdash n \simeq BinaryExpr BinXor xe1 ye1$  using f XorNode
  by (simp add: XorNode.hyps(2) rep.XorNode)
obtain xn yn where l: kind g1 n = XorNode xn yn
  using XorNode.hyps(1) by blast
then have mx:  $g1 \vdash xn \simeq xe1$ 
  using XorNode.hyps(1) XorNode.hyps(2) by fastforce
from l have my:  $g1 \vdash yn \simeq ye1$ 
  using XorNode.hyps(1) XorNode.hyps(3) by fastforce
then show ?case
proof -

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    have  $g1 \vdash xn \simeq xe1$  using  $mx$  by  $simp$ 
    have  $g1 \vdash yn \simeq ye1$  using  $my$  by  $simp$ 
    have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using  $XorNode$ 
      using  $a\ b\ c\ d\ l\ no\text{-}encoding\ not\text{-}excluded\text{-}keep\text{-}type\ repDet\ singletonD$ 
      by  $(metis\ node\ eq\ binary\ XorNode)$ 
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using  $XorNode\ a\ b\ c\ d\ l\ no\text{-}encoding\ not\text{-}excluded\text{-}keep\text{-}type\ repDet$ 
     $singletonD$ 
      by  $(metis\ node\ eq\ binary\ XorNode)$ 
    then have  $\exists xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinXor\ xe2\ ye2) \wedge BinaryExpr$ 
     $BinXor\ xe1\ ye1 \geq BinaryExpr\ BinXor\ xe2\ ye2$ 
      by  $(metis\ XorNode.prem\ l\ mono\text{-}binary\ rep.XorNode\ xer)$ 
    then show  $?thesis$ 
      by  $meson$ 
  qed
next
  case  $(LeftShiftNode\ n\ x\ y\ xe1\ ye1)$ 
  have  $k: g1 \vdash n \simeq BinaryExpr\ BinLeftShift\ xe1\ ye1$  using  $f\ LeftShiftNode$ 
    by  $(simp\ add: LeftShiftNode.hyps(2)\ rep.LeftShiftNode)$ 
  obtain  $xn\ yn$  where  $l: kind\ g1\ n = LeftShiftNode\ xn\ yn$ 
    using  $LeftShiftNode.hyps(1)$  by  $blast$ 
  then have  $mx: g1 \vdash xn \simeq xe1$ 
    using  $LeftShiftNode.hyps(1)\ LeftShiftNode.hyps(2)$  by  $fastforce$ 
  from  $l$  have  $my: g1 \vdash yn \simeq ye1$ 
    using  $LeftShiftNode.hyps(1)\ LeftShiftNode.hyps(3)$  by  $fastforce$ 
  then show  $?case$ 
  proof –
    have  $g1 \vdash xn \simeq xe1$  using  $mx$  by  $simp$ 
    have  $g1 \vdash yn \simeq ye1$  using  $my$  by  $simp$ 
    have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using  $LeftShiftNode$ 
      using  $a\ b\ c\ d\ l\ no\text{-}encoding\ not\text{-}excluded\text{-}keep\text{-}type\ repDet\ singletonD$ 
      by  $(metis\ node\ eq\ binary\ LeftShiftNode)$ 
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using  $LeftShiftNode\ a\ b\ c\ d\ l\ no\text{-}encoding\ not\text{-}excluded\text{-}keep\text{-}type\ repDet$ 
     $singletonD$ 
      by  $(metis\ node\ eq\ binary\ LeftShiftNode)$ 
    then have  $\exists xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinLeftShift\ xe2\ ye2) \wedge$ 
     $BinaryExpr\ BinLeftShift\ xe1\ ye1 \geq BinaryExpr\ BinLeftShift\ xe2\ ye2$ 
      by  $(metis\ LeftShiftNode.prem\ l\ mono\text{-}binary\ rep.LeftShiftNode\ xer)$ 
    then show  $?thesis$ 
      by  $meson$ 
  qed
next
  case  $(RightShiftNode\ n\ x\ y\ xe1\ ye1)$ 
  have  $k: g1 \vdash n \simeq BinaryExpr\ BinRightShift\ xe1\ ye1$  using  $f\ RightShiftNode$ 
    by  $(simp\ add: RightShiftNode.hyps(2)\ rep.RightShiftNode)$ 
  obtain  $xn\ yn$  where  $l: kind\ g1\ n = RightShiftNode\ xn\ yn$ 

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    using RightShiftNode.hyps(1) by blast
  then have mx:  $g1 \vdash xn \simeq xe1$ 
    using RightShiftNode.hyps(1) RightShiftNode.hyps(2) by fastforce
  from l have my:  $g1 \vdash yn \simeq ye1$ 
    using RightShiftNode.hyps(1) RightShiftNode.hyps(3) by fastforce
  then show ?case
  proof -
    have  $g1 \vdash xn \simeq xe1$  using mx by simp
    have  $g1 \vdash yn \simeq ye1$  using my by simp
    have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using RightShiftNode
      using a b c d l no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary RightShiftNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using RightShiftNode a b c d l no-encoding not-excluded-keep-type repDet
      singletonD
      by (metis-node-eq-binary RightShiftNode)
    then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr BinRightShift xe2 ye2) \wedge$ 
       $BinaryExpr BinRightShift xe1 ye1 \geq BinaryExpr BinRightShift xe2 ye2$ 
      by (metis RightShiftNode.premis l mono-binary rep.RightShiftNode xer)
    then show ?thesis
      by meson
  qed
next
case (UnsignedRightShiftNode n x y xe1 ye1)
  have k:  $g1 \vdash n \simeq BinaryExpr BinURightShift xe1 ye1$  using f UnsignedRight-
  ShiftNode
    by (simp add: UnsignedRightShiftNode.hyps(2) rep.UnsignedRightShiftNode)
  obtain xn yn where l: kind g1 n = UnsignedRightShiftNode xn yn
    using UnsignedRightShiftNode.hyps(1) by blast
  then have mx:  $g1 \vdash xn \simeq xe1$ 
    using UnsignedRightShiftNode.hyps(1) UnsignedRightShiftNode.hyps(2) by
  fastforce
  from l have my:  $g1 \vdash yn \simeq ye1$ 
    using UnsignedRightShiftNode.hyps(1) UnsignedRightShiftNode.hyps(3) by
  fastforce
  then show ?case
  proof -
    have  $g1 \vdash xn \simeq xe1$  using mx by simp
    have  $g1 \vdash yn \simeq ye1$  using my by simp
    have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using UnsignedRightShiftNode
      using a b c d l no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary UnsignedRightShiftNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using UnsignedRightShiftNode a b c d l no-encoding not-excluded-keep-type
      repDet singletonD
      by (metis-node-eq-binary UnsignedRightShiftNode)
    then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr BinURightShift xe2 ye2) \wedge$ 

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BinaryExpr BinURightShift xe1 ye1 ≥ BinaryExpr BinURightShift xe2 ye2
  by (metis UnsignedRightShiftNode.prem1 mono-binary rep.UnsignedRightShiftNode
xer)
  then show ?thesis
    by meson
qed
next
case (IntegerBelowNode n x y xe1 ye1)
  have k: g1 ⊢ n ≃ BinaryExpr BinIntegerBelow xe1 ye1 using f IntegerBe-
lowNode
  by (simp add: IntegerBelowNode.hyps(2) rep.IntegerBelowNode)
  obtain xn yn where l: kind g1 n = IntegerBelowNode xn yn
  using IntegerBelowNode.hyps(1) by blast
  then have mx: g1 ⊢ xn ≃ xe1
  using IntegerBelowNode.hyps(1) IntegerBelowNode.hyps(2) by fastforce
  from l have my: g1 ⊢ yn ≃ ye1
  using IntegerBelowNode.hyps(1) IntegerBelowNode.hyps(3) by fastforce
  then show ?case
  proof -
    have g1 ⊢ xn ≃ xe1 using mx by simp
    have g1 ⊢ yn ≃ ye1 using my by simp
    have xer: ∃ xe2. (g2 ⊢ xn ≃ xe2) ∧ xe1 ≥ xe2
      using IntegerBelowNode
      using a b c d l no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary IntegerBelowNode)
    have ∃ ye2. (g2 ⊢ yn ≃ ye2) ∧ ye1 ≥ ye2
      using IntegerBelowNode a b c d l no-encoding not-excluded-keep-type repDet
singletonD
      by (metis-node-eq-binary IntegerBelowNode)
    then have ∃ xe2 ye2. (g2 ⊢ n ≃ BinaryExpr BinIntegerBelow xe2 ye2) ∧
BinaryExpr BinIntegerBelow xe1 ye1 ≥ BinaryExpr BinIntegerBelow xe2 ye2
      by (metis IntegerBelowNode.prem1 mono-binary rep.IntegerBelowNode
xer)
    then show ?thesis
      by meson
  qed
next
case (IntegerEqualsNode n x y xe1 ye1)
  have k: g1 ⊢ n ≃ BinaryExpr BinIntegerEquals xe1 ye1 using f IntegerEqual-
sNode
  by (simp add: IntegerEqualsNode.hyps(2) rep.IntegerEqualsNode)
  obtain xn yn where l: kind g1 n = IntegerEqualsNode xn yn
  using IntegerEqualsNode.hyps(1) by blast
  then have mx: g1 ⊢ xn ≃ xe1
  using IntegerEqualsNode.hyps(1) IntegerEqualsNode.hyps(2) by fastforce
  from l have my: g1 ⊢ yn ≃ ye1
  using IntegerEqualsNode.hyps(1) IntegerEqualsNode.hyps(3) by fastforce
  then show ?case
  proof -

```



```

    have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
    have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
    have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using IntegerEqualsNode
      using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary IntegerEqualsNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using IntegerEqualsNode  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type
repDet singletonD
      by (metis-node-eq-binary IntegerEqualsNode)
    then have  $\exists xe2\ ye2. (g2 \vdash n \simeq \text{BinaryExpr BinIntegerEquals } xe2\ ye2) \wedge$ 
BinaryExpr BinIntegerEquals  $xe1\ ye1 \geq \text{BinaryExpr BinIntegerEquals } xe2\ ye2$ 
      by (metis IntegerEqualsNode.premis l mono-binary rep.IntegerEqualsNode
xer)
    then show ?thesis
      by meson
  qed
next
case (IntegerLessThanNode  $n\ x\ y\ xe1\ ye1$ )
  have  $k: g1 \vdash n \simeq \text{BinaryExpr BinIntegerLessThan } xe1\ ye1$  using  $f$  IntegerLessThanNode
  by (simp add: IntegerLessThanNode.hyps(2) rep.IntegerLessThanNode)
  obtain  $xn\ yn$  where  $l: \text{kind } g1\ n = \text{IntegerLessThanNode } xn\ yn$ 
  using IntegerLessThanNode.hyps(1) by blast
  then have  $mx: g1 \vdash xn \simeq xe1$ 
  using IntegerLessThanNode.hyps(1) IntegerLessThanNode.hyps(2) by fast-force
  from  $l$  have  $my: g1 \vdash yn \simeq ye1$ 
  using IntegerLessThanNode.hyps(1) IntegerLessThanNode.hyps(3) by fast-force
  then show ?case
  proof -
    have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
    have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
    have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using IntegerLessThanNode
      using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary IntegerLessThanNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using IntegerLessThanNode  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type
repDet singletonD
      by (metis-node-eq-binary IntegerLessThanNode)
    then have  $\exists xe2\ ye2. (g2 \vdash n \simeq \text{BinaryExpr BinIntegerLessThan } xe2\ ye2)$ 
 $\wedge \text{BinaryExpr BinIntegerLessThan } xe1\ ye1 \geq \text{BinaryExpr BinIntegerLessThan } xe2\ ye2$ 
      by (metis IntegerLessThanNode.premis l mono-binary rep.IntegerLessThanNode
xer)
    then show ?thesis
      by meson

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    qed
  next
    case (NarrowNode n inputBits resultBits x xe1)
    have k:  $g1 \vdash n \simeq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe1$  using
  f NarrowNode
    by (simp add: NarrowNode.hyps(2) rep.NarrowNode)
    obtain xn where l: kind  $g1$   $n = \text{NarrowNode inputBits resultBits xn}$ 
    using NarrowNode.hyps(1) by blast
    then have m:  $g1 \vdash xn \simeq xe1$ 
    using NarrowNode.hyps(1) NarrowNode.hyps(2)
    by auto
    then show ?case
    proof (cases  $xn = n'$ )
      case True
        then have n:  $xe1 = e1'$  using c m repDet by simp
        then have ev:  $g2 \vdash n \simeq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) e2'$ 
      using NarrowNode.hyps(1) l m n
        using NarrowNode.premis True d rep.NarrowNode by simp
        then have r:  $\text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) e1' \geq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) e2'$ 
        by (meson a mono-unary)
        then show ?thesis using ev r
        by (metis n)
      next
        case False
        have  $g1 \vdash xn \simeq xe1$  using m by simp
        have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
        using NarrowNode
        using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
        by (metis node-eq-ternary NarrowNode)
        then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe2) \wedge \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe1 \geq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe2$ 
        by (metis NarrowNode.premis l mono-unary rep.NarrowNode)
        then show ?thesis
        by meson
    qed
  next
    case (SignExtendNode n inputBits resultBits x xe1)
    have k:  $g1 \vdash n \simeq \text{UnaryExpr } (\text{UnarySignExtend inputBits resultBits}) xe1$ 
  using f SignExtendNode
    by (simp add: SignExtendNode.hyps(2) rep.SignExtendNode)
    obtain xn where l: kind  $g1$   $n = \text{SignExtendNode inputBits resultBits xn}$ 
    using SignExtendNode.hyps(1) by blast
    then have m:  $g1 \vdash xn \simeq xe1$ 
    using SignExtendNode.hyps(1) SignExtendNode.hyps(2)
    by auto
    then show ?case
    proof (cases  $xn = n'$ )

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    case True
    then have n:  $xe1 = e1'$  using c m repDet by simp
    then have ev:  $g2 \vdash n \simeq \text{UnaryExpr } (\text{UnarySignExtend inputBits resultBits})$ 
    e2' using SignExtendNode.hyps(1) l m n
    using SignExtendNode.premis True d rep.SignExtendNode by simp
    then have r:  $\text{UnaryExpr } (\text{UnarySignExtend inputBits resultBits}) e1' \geq$ 
     $\text{UnaryExpr } (\text{UnarySignExtend inputBits resultBits}) e2'$ 
    by (meson a mono-unary)
    then show ?thesis using ev r
    by (metis n)
  next
  case False
  have  $g1 \vdash xn \simeq xe1$  using m by simp
  have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
  using SignExtendNode
  using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
  by (metis-node-eq-ternary SignExtendNode)
  then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr } (\text{UnarySignExtend inputBits result-}$ 
   $\text{Bits}) xe2) \wedge \text{UnaryExpr } (\text{UnarySignExtend inputBits resultBits}) xe1 \geq \text{UnaryExpr } (\text{UnarySignExtend inputBits resultBits}) xe2$ 
  by (metis SignExtendNode.premis l mono-unary rep.SignExtendNode)
  then show ?thesis
  by meson
qed
next
case (ZeroExtendNode n inputBits resultBits x xe1)
have k:  $g1 \vdash n \simeq \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) xe1$ 
using f ZeroExtendNode
by (simp add: ZeroExtendNode.hyps(2) rep.ZeroExtendNode)
obtain xn where l: kind  $g1$  n = ZeroExtendNode inputBits resultBits xn
using ZeroExtendNode.hyps(1) by blast
then have m:  $g1 \vdash xn \simeq xe1$ 
using ZeroExtendNode.hyps(1) ZeroExtendNode.hyps(2)
by auto
then show ?case
proof (cases xn = n')
case True
then have n:  $xe1 = e1'$  using c m repDet by simp
then have ev:  $g2 \vdash n \simeq \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits})$ 
e2' using ZeroExtendNode.hyps(1) l m n
using ZeroExtendNode.premis True d rep.ZeroExtendNode by simp
then have r:  $\text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) e1' \geq$ 
 $\text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) e2'$ 
by (meson a mono-unary)
then show ?thesis using ev r
by (metis n)
next
case False
have  $g1 \vdash xn \simeq xe1$  using m by simp

```

```

    have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using ZeroExtendNode
    using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
    by (metis-node-eq-ternary ZeroExtendNode)
    then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) xe2) \wedge \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) xe1 \geq \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) xe2$ 
    by (metis ZeroExtendNode.premis l mono-unary rep.ZeroExtendNode)
    then show ?thesis
    by meson
  qed
next
  case (LeafNode n s)
  then show ?case
  by (metis eq-refl rep.LeafNode)
next
  case (RefNode n')
  then show ?case
  by (metis a b c d no-encoding not-excluded-keep-type rep.RefNode repDet singletonD)
  qed
  qed
  qed

```

**lemma** *graph-semantic-preservation-subscript:*  
 assumes  $a: e_1' \geq e_2'$   
 assumes  $b: (\{n\} \trianglelefteq \text{as-set } g_1) \subseteq \text{as-set } g_2$   
 assumes  $c: g_1 \vdash n \simeq e_1'$   
 assumes  $d: g_2 \vdash n \simeq e_2'$   
 shows *graph-refinement*  $g_1 g_2$   
 using *graph-semantic-preservation* *assms* by *simp*

**lemma** *tree-to-graph-rewriting:*  
 $e_1 \geq e_2$   
 $\wedge (g_1 \vdash n \simeq e_1) \wedge \text{maximal-sharing } g_1$   
 $\wedge (\{n\} \trianglelefteq \text{as-set } g_1) \subseteq \text{as-set } g_2$   
 $\wedge (g_2 \vdash n \simeq e_2) \wedge \text{maximal-sharing } g_2$   
 $\implies \text{graph-refinement } g_1 g_2$   
 using *graph-semantic-preservation*  
 by *auto*

**declare**  $[[\text{simp-trace}]]$   
**lemma** *equal-refines:*  
 fixes  $e1 e2 :: \text{IRExpr}$   
 assumes  $e1 = e2$   
 shows  $e1 \geq e2$   
 using *assms*  
 by *simp*

**declare**  $[[simp\text{-}trace=false]]$

**lemma** *eval-contains-id* $[simp]$ :  $g1 \vdash n \simeq e \implies n \in ids\ g1$   
**using** *no-encoding* **by** *blast*

**lemma** *subset-kind* $[simp]$ :  $as\text{-}set\ g1 \subseteq as\text{-}set\ g2 \implies g1 \vdash n \simeq e \implies kind\ g1\ n = kind\ g2\ n$   
**using** *eval-contains-id* **unfolding** *as-set-def*  
**by** *blast*

**lemma** *subset-stamp* $[simp]$ :  $as\text{-}set\ g1 \subseteq as\text{-}set\ g2 \implies g1 \vdash n \simeq e \implies stamp\ g1\ n = stamp\ g2\ n$   
**using** *eval-contains-id* **unfolding** *as-set-def*  
**by** *blast*

**method** *solve-subset-eval* **uses** *as-set eval =*  
*(metis eval as-set subset-kind subset-stamp |*  
*metis eval as-set subset-kind)*

**lemma** *subset-implies-evals*:  
**assumes**  $as\text{-}set\ g1 \subseteq as\text{-}set\ g2$   
**assumes**  $(g1 \vdash n \simeq e)$   
**shows**  $(g2 \vdash n \simeq e)$   
**using** *assms(2)*  
**apply** (*induction e*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: ConstantNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: ParameterNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: ConditionalNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: AbsNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: NotNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: NegateNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: LogicNegationNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: AddNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: MulNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: SubNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: AndNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: OrNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: XorNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: LeftShiftNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: RightShiftNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: UnsignedRightShiftNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: IntegerBelowNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: IntegerEqualsNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: IntegerLessThanNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: NarrowNode*)  
**apply** (*solve-subset-eval as-set: assms(1) eval: SignExtendNode*)

```

    apply (solve-subset-eval as-set: assms(1) eval: ZeroExtendNode)
    apply (solve-subset-eval as-set: assms(1) eval: LeafNode)
    by (solve-subset-eval as-set: assms(1) eval: RefNode)

lemma subset-refines:
  assumes as-set g1  $\subseteq$  as-set g2
  shows graph-refinement g1 g2
proof -
  have ids g1  $\subseteq$  ids g2 using assms unfolding as-set-def
  by blast
  then show ?thesis unfolding graph-refinement-def apply rule
  apply (rule allI) apply (rule impI) apply (rule allI) apply (rule impI)
  unfolding graph-represents-expression-def
  proof -
    fix n e1
    assume 1:n  $\in$  ids g1
    assume 2:g1  $\vdash$  n  $\simeq$  e1

    show  $\exists e2. (g2 \vdash n \simeq e2) \wedge e1 \geq e2$ 
    using assms 1 2 using subset-implies-evals
    by (meson equal-refines)
  qed
qed

```

```

lemma graph-construction:
  e1  $\geq$  e2
   $\wedge$  as-set g1  $\subseteq$  as-set g2
   $\wedge$  (g2  $\vdash$  n  $\simeq$  e2)
   $\implies$  (g2  $\vdash$  n  $\trianglelefteq$  e1)  $\wedge$  graph-refinement g1 g2
  using subset-refines
  by (meson encodeeval-def graph-represents-expression-def le-expr-def)

```

## 6.8.4 Term Graph Reconstruction

```

lemma find-exists-kind:
  assumes find-node-and-stamp g (node, s) = Some nid
  shows kind g nid = node
  using assms unfolding find-node-and-stamp.simps
  by (metis (mono-tags, lifting) find-Some-iff)

```

```

lemma find-exists-stamp:
  assumes find-node-and-stamp g (node, s) = Some nid
  shows stamp g nid = s
  using assms unfolding find-node-and-stamp.simps
  by (metis (mono-tags, lifting) find-Some-iff)

```

```

lemma find-new-kind:
  assumes g' = add-node nid (node, s) g
  assumes node  $\neq$  NoNode

```

```

shows kind g' nid = node
using assms
using add-node-lookup by presburger

lemma find-new-stamp:
  assumes g' = add-node nid (node, s) g
  assumes node ≠ NoNode
  shows stamp g' nid = s
  using assms
  using add-node-lookup by presburger

lemma sorted-bottom:
  assumes finite xs
  assumes x ∈ xs
  shows x ≤ last(sorted-list-of-set(xs::nat set))
  using assms
  using sorted2-simps(2) sorted-list-of-set(2)
  by (smt (verit, del-insts) Diff-iff Max-ge Max-in empty-iff list.set(1) snoc-eq-iff-butlast
sorted-insort-is-snoc sorted-list-of-set(1) sorted-list-of-set.fold-insort-key.infinite sorted-list-of-set.fold-insort-ke

lemma fresh: finite xs ⇒ last(sorted-list-of-set(xs::nat set)) + 1 ∉ xs
  using sorted-bottom
  using not-le by auto

lemma fresh-ids:
  assumes n = get-fresh-id g
  shows n ∉ ids g
proof -
  have finite (ids g) using Rep-IRGraph by auto
  then show ?thesis
    using assms fresh unfolding get-fresh-id.simps
    by blast
qed

lemma graph-unchanged-rep-unchanged:
  assumes ∀ n ∈ ids g. kind g n = kind g' n
  assumes ∀ n ∈ ids g. stamp g n = stamp g' n
  shows (g ⊢ n ≃ e) ⟶ (g' ⊢ n ≃ e)
  apply (rule impI) subgoal premises e using e assms
  apply (induction n e)
  apply (metis no-encoding rep.ConstantNode)
  apply (metis no-encoding rep.ParameterNode)
  apply (metis no-encoding rep.ConditionalNode)
  apply (metis no-encoding rep.AbsNode)
  apply (metis no-encoding rep.NotNode)
  apply (metis no-encoding rep.NegateNode)
  apply (metis no-encoding rep.LogicNegationNode)
  apply (metis no-encoding rep.AddNode)
  apply (metis no-encoding rep.MulNode)

```

```

    apply (metis no-encoding rep.SubNode)
    apply (metis no-encoding rep.AndNode)
    apply (metis no-encoding rep.OrNode)
    apply (metis no-encoding rep.XorNode)
    apply (metis no-encoding rep.LeftShiftNode)
    apply (metis no-encoding rep.RightShiftNode)
    apply (metis no-encoding rep.UnsignedRightShiftNode)
    apply (metis no-encoding rep.IntegerBelowNode)
    apply (metis no-encoding rep.IntegerEqualsNode)
    apply (metis no-encoding rep.IntegerLessThanNode)
    apply (metis no-encoding rep.NarrowNode)
    apply (metis no-encoding rep.SignExtendNode)
    apply (metis no-encoding rep.ZeroExtendNode)
    apply (metis no-encoding rep.LeafNode)
    by (metis no-encoding rep.RefNode)
done

```

**lemma** *fresh-node-subset*:

```

  assumes  $n \notin \text{ids } g$ 
  assumes  $g' = \text{add-node } n (k, s) g$ 
  shows  $\text{as-set } g \subseteq \text{as-set } g'$ 
  using assms
  by (smt (verit, del-Insts) Collect-mono-iff Diff-idemp Diff-insert-absorb add-changed
    as-set-def disjoint-change unchanged.simps)

```

**lemma** *unrep-subset*:

```

  assumes  $(g \oplus e \rightsquigarrow (g', n))$ 
  shows  $\text{as-set } g \subseteq \text{as-set } g'$ 
  using assms proof (induction  $g \ e \ (g', n)$  arbitrary:  $g' \ n$ )
  case (ConstantNodeSame  $g \ c \ n$ )
  then show ?case by blast
next
  case (ConstantNodeNew  $g \ c \ n \ g'$ )
  then show ?case using fresh-ids fresh-node-subset
    by presburger
next
  case (ParameterNodeSame  $g \ i \ s \ n$ )
  then show ?case by blast
next
  case (ParameterNodeNew  $g \ i \ s \ n \ g'$ )
  then show ?case using fresh-ids fresh-node-subset
    by presburger
next
  case (ConditionalNodeSame  $g \ ce \ g2 \ c \ te \ g3 \ t \ fe \ g4 \ f \ s' \ n$ )
  then show ?case by blast
next
  case (ConditionalNodeNew  $g \ ce \ g2 \ c \ te \ g3 \ t \ fe \ g4 \ f \ s' \ n \ g'$ )
  then show ?case using fresh-ids fresh-node-subset
    by (meson subset-trans)

```



```

next
  case (UnaryNodeSame g xe g2 x s' op n)
  then show ?case by blast
next
  case (UnaryNodeNew g xe g2 x s' op n g')
  then show ?case using fresh-ids fresh-node-subset
    by (meson subset-trans)
next
  case (BinaryNodeSame g xe g2 x ye g3 y s' op n)
  then show ?case by blast
next
  case (BinaryNodeNew g xe g2 x ye g3 y s' op n g')
  then show ?case using fresh-ids fresh-node-subset
    by (meson subset-trans)
next
  case (AllLeafNodes g n s)
  then show ?case by blast
qed

lemma fresh-node-preserves-other-nodes:
  assumes n' = get-fresh-id g
  assumes g' = add-node n' (k, s) g
  shows  $\forall n \in \text{ids } g. (g \vdash n \simeq e) \longrightarrow (g' \vdash n \simeq e)$ 
  using assms
  by (smt (verit, ccfv-SIG) Diff-idemp Diff-insert-absorb add-changed disjoint-change
    fresh-ids graph-unchanged-rep-unchanged unchanged.elims(2))

lemma found-node-preserves-other-nodes:
  assumes find-node-and-stamp g (k, s) = Some n
  shows  $\forall n \in \text{ids } g. (g \vdash n \simeq e) \longleftrightarrow (g \vdash n \simeq e)$ 
  using assms
  by blast

lemma unrep-ids-subset[simp]:
  assumes  $g \oplus e \rightsquigarrow (g', n)$ 
  shows  $\text{ids } g \subseteq \text{ids } g'$ 
  using assms unrep-subset
  by (meson graph-refinement-def subset-refines)

lemma unrep-unchanged:
  assumes  $g \oplus e \rightsquigarrow (g', n)$ 
  shows  $\forall n \in \text{ids } g. \forall e. (g \vdash n \simeq e) \longrightarrow (g' \vdash n \simeq e)$ 
  using assms unrep-subset fresh-node-preserves-other-nodes
  by (meson subset-implies-evals)

theorem term-graph-reconstruction:
   $g \oplus e \rightsquigarrow (g', n) \implies (g' \vdash n \simeq e) \wedge \text{as-set } g \subseteq \text{as-set } g'$ 
  subgoal premises e apply (rule conjI) defer
    using e unrep-subset apply blast using e

```

```

proof (induction  $g\ e\ (g',\ n)$  arbitrary:  $g'\ n$ )
  case (ConstantNodeSame  $g'\ c\ n$ )
    then have  $\text{kind } g'\ n = \text{ConstantNode } c$ 
      using find-exists-kind local.ConstantNodeSame by blast
    then show  $?case$  using ConstantNode by blast
next
  case (ConstantNodeNew  $g\ c$ )
    then show  $?case$ 
      using ConstantNode IRNode.distinct(683) add-node-lookup by presburger
next
  case (ParameterNodeSame  $i\ s$ )
    then show  $?case$ 
      by (metis ParameterNode find-exists-kind find-exists-stamp)
next
  case (ParameterNodeNew  $g\ i\ s$ )
    then show  $?case$ 
      by (metis IRNode.distinct(2447) ParameterNode add-node-lookup)
next
  case (ConditionalNodeSame  $g\ ce\ g2\ c\ te\ g3\ t\ fe\ g4\ f\ s'\ n$ )
    then have  $k: \text{kind } g4\ n = \text{ConditionalNode } c\ t\ f$ 
      using find-exists-kind by blast
    have  $c: g4 \vdash c \simeq ce$  using local.ConditionalNodeSame unrep-unchanged
      using no-encoding by blast
    have  $t: g4 \vdash t \simeq te$  using local.ConditionalNodeSame unrep-unchanged
      using no-encoding by blast
    have  $f: g4 \vdash f \simeq fe$  using local.ConditionalNodeSame unrep-unchanged
      using no-encoding by blast
    then show  $?case$  using  $c\ t\ f$ 
      using ConditionalNode k by blast
next
  case (ConditionalNodeNew  $g\ ce\ g2\ c\ te\ g3\ t\ fe\ g4\ f\ s'\ n\ g'$ )
    moreover have  $\text{ConditionalNode } c\ t\ f \neq \text{NoNode}$ 
      using unary-node.elims by blast
    ultimately have  $k: \text{kind } g'\ n = \text{ConditionalNode } c\ t\ f$ 
      using find-new-kind local.ConditionalNodeNew
      by presburger
    then have  $c: g' \vdash c \simeq ce$  using local.ConditionalNodeNew unrep-unchanged
      using no-encoding
      by (metis ConditionalNodeNew.hyps(9) fresh-node-preserves-other-nodes)
    then have  $t: g' \vdash t \simeq te$  using local.ConditionalNodeNew unrep-unchanged
      using no-encoding fresh-node-preserves-other-nodes
      by metis
    then have  $f: g' \vdash f \simeq fe$  using local.ConditionalNodeNew unrep-unchanged
      using no-encoding fresh-node-preserves-other-nodes
      by metis
    then show  $?case$  using  $c\ t\ f$ 
      using ConditionalNode k by blast
next
  case (UnaryNodeSame  $g\ xe\ g'\ x\ s'\ op\ n$ )

```

```

then have k: kind g' n = unary-node op x
  using find-exists-kind local.UnaryNodeSame by blast
then have g' ⊢ x ≃ xe using local.UnaryNodeSame by blast
then show ?case using k
  apply (cases op)
  using AbsNode unary-node.simps(1) apply presburger
  using NegateNode unary-node.simps(3) apply presburger
  using NotNode unary-node.simps(2) apply presburger
  using LogicNegationNode unary-node.simps(4) apply presburger
  using NarrowNode unary-node.simps(5) apply presburger
  using SignExtendNode unary-node.simps(6) apply presburger
  using ZeroExtendNode unary-node.simps(7) by presburger
next
case (UnaryNodeNew g xe g2 x s' op n g')
moreover have unary-node op x ≠ NoNode
  using unary-node.elims by blast
ultimately have k: kind g' n = unary-node op x
  using find-new-kind local.UnaryNodeNew
  by presburger
have x ∈ ids g2 using local.UnaryNodeNew
  using eval-contains-id by blast
then have x ≠ n using local.UnaryNodeNew(5) fresh-ids by blast
have g' ⊢ x ≃ xe using local.UnaryNodeNew fresh-node-preserved-other-nodes
  using ⟨x ∈ ids g2⟩ by blast
then show ?case using k
  apply (cases op)
  using AbsNode unary-node.simps(1) apply presburger
  using NegateNode unary-node.simps(3) apply presburger
  using NotNode unary-node.simps(2) apply presburger
  using LogicNegationNode unary-node.simps(4) apply presburger
  using NarrowNode unary-node.simps(5) apply presburger
  using SignExtendNode unary-node.simps(6) apply presburger
  using ZeroExtendNode unary-node.simps(7) by presburger
next
case (BinaryNodeSame g xe g2 x ye g3 y s' op n)
then have k: kind g3 n = bin-node op x y
  using find-exists-kind by blast
have x: g3 ⊢ x ≃ xe using local.BinaryNodeSame unrep-unchanged
  using no-encoding by blast
have y: g3 ⊢ y ≃ ye using local.BinaryNodeSame unrep-unchanged
  using no-encoding by blast
then show ?case using x y k apply (cases op)
  using AddNode bin-node.simps(1) apply presburger
  using MulNode bin-node.simps(2) apply presburger
  using SubNode bin-node.simps(3) apply presburger
  using AndNode bin-node.simps(4) apply presburger
  using OrNode bin-node.simps(5) apply presburger
  using XorNode bin-node.simps(6) apply presburger
  using LeftShiftNode bin-node.simps(7) apply presburger

```

```

    using RightShiftNode bin-node.simps(8) apply presburger
    using UnsignedRightShiftNode bin-node.simps(9) apply presburger
    using IntegerEqualsNode bin-node.simps(10) apply presburger
    using IntegerLessThanNode bin-node.simps(11) apply presburger
    using IntegerBelowNode bin-node.simps(12) by presburger
next
case (BinaryNodeNew g xe g2 x ye g3 y s' op n g')
moreover have bin-node op x y  $\neq$  NoNode
    using bin-node.elims by blast
ultimately have k: kind g' n = bin-node op x y
    using find-new-kind local.BinaryNodeNew
    by presburger
then have k: kind g' n = bin-node op x y
    using find-exists-kind by blast
have x: g'  $\vdash$  x  $\simeq$  xe using local.BinaryNodeNew unrep-unchanged
    using no-encoding
    by (meson fresh-node-preserves-other-nodes)
have y: g'  $\vdash$  y  $\simeq$  ye using local.BinaryNodeNew unrep-unchanged
    using no-encoding
    by (meson fresh-node-preserves-other-nodes)
then show ?case using x y k apply (cases op)
    using AddNode bin-node.simps(1) apply presburger
    using MulNode bin-node.simps(2) apply presburger
    using SubNode bin-node.simps(3) apply presburger
    using AndNode bin-node.simps(4) apply presburger
    using OrNode bin-node.simps(5) apply presburger
    using XorNode bin-node.simps(6) apply presburger
    using LeftShiftNode bin-node.simps(7) apply presburger
    using RightShiftNode bin-node.simps(8) apply presburger
    using UnsignedRightShiftNode bin-node.simps(9) apply presburger
    using IntegerEqualsNode bin-node.simps(10) apply presburger
    using IntegerLessThanNode bin-node.simps(11) apply presburger
    using IntegerBelowNode bin-node.simps(12) by presburger
next
case (AllLeafNodes g n s)
then show ?case using rep.LeafNode by blast
qed
done

```

```

lemma ref-refinement:
  assumes g  $\vdash$  n  $\simeq$  e1
  assumes kind g n' = RefNode n
  shows g  $\vdash$  n'  $\trianglelefteq$  e1
  using assms RefNode
  by (meson equal-refines graph-represents-expression-def)

```

```

lemma unrep-refines:
  assumes g  $\oplus$  e  $\rightsquigarrow$  (g', n)
  shows graph-refinement g g'

```

```

using assms
using graph-refinement-def subset-refines unrep-subset by blast

lemma add-new-node-refines:
  assumes  $n \notin \text{ids } g$ 
  assumes  $g' = \text{add-node } n \ (k, s) \ g$ 
  shows graph-refinement  $g \ g'$ 
  using assms unfolding graph-refinement
  using fresh-node-subset subset-refines by presburger

lemma add-node-as-set:
  assumes  $g' = \text{add-node } n \ (k, s) \ g$ 
  shows  $(\{n\} \trianglelefteq \text{as-set } g) \subseteq \text{as-set } g'$ 
  using assms unfolding as-set-def domain-subtraction-def
  using add-changed
  by (smt (z3) case-prodE changeonly.simps mem-Collect-eq prod.sel(1) subsetI)

theorem refined-insert:
  assumes  $e_1 \geq e_2$ 
  assumes  $g_1 \oplus e_2 \rightsquigarrow (g_2, n')$ 
  shows  $(g_2 \vdash n' \trianglelefteq e_1) \wedge \text{graph-refinement } g_1 \ g_2$ 
  using assms
  using graph-construction term-graph-reconstruction by blast

lemma ids-finite: finite (ids  $g$ )
  using Rep-IRGraph ids.rep-eq by simp

lemma unwrap-sorted: set (sorted-list-of-set (ids  $g$ )) = ids  $g$ 
  using Rep-IRGraph set-sorted-list-of-set ids-finite
  by blast

lemma find-none:
  assumes find-node-and-stamp  $g \ (k, s) = \text{None}$ 
  shows  $\forall n \in \text{ids } g. \text{kind } g \ n \neq k \vee \text{stamp } g \ n \neq s$ 
proof –
  have  $(\nexists n. n \in \text{ids } g \wedge (\text{kind } g \ n = k \wedge \text{stamp } g \ n = s))$ 
    using assms unfolding find-node-and-stamp.simps using find-None-iff unwrap-sorted
    by (metis (mono-tags, lifting))
  then show ?thesis
    by blast
qed

```

```

method ref-represents uses node =
  (metis IRNode.distinct(2755) RefNode dual-order.refl find-new-kind fresh-node-subset
node subset-implies-evals)

```

```

end

```

## 7 Control-flow Semantics

```

theory IRStepObj
  imports
    TreeToGraph
begin

```

### 7.1 Object Heap

The heap model we introduce maps field references to object instances to runtime values. We use the  $H[f][p]$  heap representation. See *\cite{heap-reps-2011}*. We also introduce the `DynamicHeap` type which allocates new object references sequentially storing the next free object reference as 'Free'.

*heapdef*

```

type-synonym ('a, 'b) Heap = 'a  $\Rightarrow$  'b  $\Rightarrow$  Value
type-synonym Free = nat
type-synonym ('a, 'b) DynamicHeap = ('a, 'b) Heap  $\times$  Free

fun h-load-field :: 'a  $\Rightarrow$  'b  $\Rightarrow$  ('a, 'b) DynamicHeap  $\Rightarrow$  Value where
  h-load-field f r (h, n) = h f r

fun h-store-field :: 'a  $\Rightarrow$  'b  $\Rightarrow$  Value  $\Rightarrow$  ('a, 'b) DynamicHeap  $\Rightarrow$  ('a, 'b)
DynamicHeap where
  h-store-field f r v (h, n) = (h(f := ((h f)(r := v))), n)

fun h-new-inst :: ('a, 'b) DynamicHeap  $\Rightarrow$  ('a, 'b) DynamicHeap  $\times$  Value
where
  h-new-inst (h, n) = ((h, n+1), (ObjRef (Some n)))

type-synonym FieldRefHeap = (string, objref) DynamicHeap

```

```

definition new-heap :: ('a, 'b) DynamicHeap where
  new-heap = (( $\lambda f. \lambda p. \text{UndefVal}$ ), 0)

```

## 7.2 Intraprocedural Semantics

```

fun find-index :: 'a ⇒ 'a list ⇒ nat where
  find-index - [] = 0 |
  find-index v (x # xs) = (if (x=v) then 0 else find-index v xs + 1)

fun phi-list :: IRGraph ⇒ ID ⇒ ID list where
  phi-list g n =
    (filter (λx.(is-PhiNode (kind g x)))
     (sorted-list-of-set (usages g n)))

fun input-index :: IRGraph ⇒ ID ⇒ ID ⇒ nat where
  input-index g n n' = find-index n' (inputs-of (kind g n))

fun phi-inputs :: IRGraph ⇒ nat ⇒ ID list ⇒ ID list where
  phi-inputs g i nodes = (map (λn. (inputs-of (kind g n))!(i + 1)) nodes)

fun set-phis :: ID list ⇒ Value list ⇒ MapState ⇒ MapState where
  set-phis [] [] m = m |
  set-phis (n # xs) (v # vs) m = (set-phis xs vs (m(n := v))) |
  set-phis [] (v # vs) m = m |
  set-phis (x # xs) [] m = m

```

Intraprocedural semantics are given as a small-step semantics.

Within the context of a graph, the configuration triple, (ID, MethodState, Heap), is related to the subsequent configuration.

```

inductive step :: IRGraph ⇒ Params ⇒ (ID × MapState × FieldRefHeap) ⇒ (ID
× MapState × FieldRefHeap) ⇒ bool
  (¬, - ⊢ - → - 55) for g p where

```

*SequentialNode:*

```

[[is-sequential-node (kind g nid);
  nid' = (successors-of (kind g nid))!0]]
⇒ g, p ⊢ (nid, m, h) → (nid', m, h) |

```

*IfNode:*

```

[[kind g nid = (IfNode cond tb fb);
  g ⊢ cond ≃ condE;
  [m, p] ⊢ condE ↦ val;
  nid' = (if val-to-bool val then tb else fb)]]
⇒ g, p ⊢ (nid, m, h) → (nid', m, h) |

```

*EndNodes:*

```

[[is-AbstractEndNode (kind g nid);
  merge = any-usage g nid;
  is-AbstractMergeNode (kind g merge);

  i = find-index nid (inputs-of (kind g merge));
  phis = (phi-list g merge);

```

$inps = (\text{phi-inputs } g \ i \ phis);$   
 $g \vdash inps \simeq_L inpsE;$   
 $[m, p] \vdash inpsE \mapsto_L vs;$

$m' = \text{set-phs } phis \ vs \ m$   
 $\implies g, p \vdash (nid, m, h) \rightarrow (\text{merge}, m', h) \mid$

*NewInstanceNode:*

$\llbracket \text{kind } g \ nid = (\text{NewInstanceNode } nid \ f \ \text{obj } nid');$   
 $(h', \text{ref}) = h\text{-new-inst } h;$   
 $m' = m(nid := \text{ref}) \rrbracket$   
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m', h') \mid$

*LoadFieldNode:*

$\llbracket \text{kind } g \ nid = (\text{LoadFieldNode } nid \ f \ (\text{Some } \text{obj}) \ nid');$   
 $g \vdash \text{obj} \simeq \text{obj}E;$   
 $[m, p] \vdash \text{obj}E \mapsto \text{ObjRef } \text{ref};$   
 $h\text{-load-field } f \ \text{ref } h = v;$   
 $m' = m(nid := v) \rrbracket$   
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m', h) \mid$

*SignedDivNode:*

$\llbracket \text{kind } g \ nid = (\text{SignedDivNode } nid \ x \ y \ \text{zero } sb \ \text{nxt});$   
 $g \vdash x \simeq xe;$   
 $g \vdash y \simeq ye;$   
 $[m, p] \vdash xe \mapsto v1;$   
 $[m, p] \vdash ye \mapsto v2;$   
 $v = (\text{intval-div } v1 \ v2);$   
 $m' = m(nid := v) \rrbracket$   
 $\implies g, p \vdash (nid, m, h) \rightarrow (\text{nxt}, m', h) \mid$

*SignedRemNode:*

$\llbracket \text{kind } g \ nid = (\text{SignedRemNode } nid \ x \ y \ \text{zero } sb \ \text{nxt});$   
 $g \vdash x \simeq xe;$   
 $g \vdash y \simeq ye;$   
 $[m, p] \vdash xe \mapsto v1;$   
 $[m, p] \vdash ye \mapsto v2;$   
 $v = (\text{intval-mod } v1 \ v2);$   
 $m' = m(nid := v) \rrbracket$   
 $\implies g, p \vdash (nid, m, h) \rightarrow (\text{nxt}, m', h) \mid$

*StaticLoadFieldNode:*

$\llbracket \text{kind } g \ nid = (\text{LoadFieldNode } nid \ f \ \text{None } nid');$   
 $h\text{-load-field } f \ \text{None } h = v;$   
 $m' = m(nid := v) \rrbracket$   
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m', h) \mid$

*StoreFieldNode:*



$$\begin{aligned}
& \llbracket \text{kind } g \text{ nid} = (\text{StoreFieldNode } \text{nid } f \text{ newval} - (\text{Some } \text{obj}) \text{ nid}') \rrbracket; \\
& g \vdash \text{newval} \simeq \text{newval}E; \\
& g \vdash \text{obj} \simeq \text{obj}E; \\
& [m, p] \vdash \text{newval}E \mapsto \text{val}; \\
& [m, p] \vdash \text{obj}E \mapsto \text{ObjRef } \text{ref}; \\
& h' = h\text{-store-field } f \text{ ref val } h; \\
& m' = m(\text{nid} := \text{val}) \\
& \implies g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h') \mid
\end{aligned}$$

*StaticStoreFieldNode:*

$$\begin{aligned}
& \llbracket \text{kind } g \text{ nid} = (\text{StoreFieldNode } \text{nid } f \text{ newval} - \text{None } \text{nid}') \rrbracket; \\
& g \vdash \text{newval} \simeq \text{newval}E; \\
& [m, p] \vdash \text{newval}E \mapsto \text{val}; \\
& h' = h\text{-store-field } f \text{ None val } h; \\
& m' = m(\text{nid} := \text{val}) \\
& \implies g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h')
\end{aligned}$$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow i * i * i \Rightarrow o * o * o \Rightarrow \text{bool}$ ) *step* .

### 7.3 Interprocedural Semantics

**type-synonym** *Signature* = *string*

**type-synonym** *Program* = *Signature*  $\rightarrow$  *IRGraph*

**inductive** *step-top* :: *Program*  $\Rightarrow$  (*IRGraph*  $\times$  *ID*  $\times$  *MapState*  $\times$  *Params*) *list*  $\times$  *FieldRefHeap*  $\Rightarrow$  (*IRGraph*  $\times$  *ID*  $\times$  *MapState*  $\times$  *Params*) *list*  $\times$  *FieldRefHeap*  $\Rightarrow$  *bool*

( $\vdash - \longrightarrow -$  55)

**for** *P* **where**

*Lift*:

$$\begin{aligned}
& \llbracket g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h') \rrbracket \\
& \implies P \vdash ((g, \text{nid}, m, p) \# \text{stk}, h) \longrightarrow ((g, \text{nid}', m', p) \# \text{stk}, h') \mid
\end{aligned}$$

*InvokeNodeStep*:

$\llbracket \text{is-Invoke } (\text{kind } g \text{ nid}) \rrbracket;$

$$\begin{aligned}
& \text{callTarget} = \text{ir-callTarget } (\text{kind } g \text{ nid}); \\
& \text{kind } g \text{ callTarget} = (\text{MethodCallTargetNode } \text{targetMethod } \text{arguments}); \\
& \text{Some } \text{targetGraph} = P \text{ targetMethod}; \\
& m' = \text{new-map-state}; \\
& g \vdash \text{arguments} \simeq_L \text{args}E; \\
& [m, p] \vdash \text{args}E \mapsto_L p \\
& \implies P \vdash ((g, \text{nid}, m, p) \# \text{stk}, h) \longrightarrow ((\text{targetGraph}, 0, m', p') \# (g, \text{nid}, m, p) \# \text{stk}, h)
\end{aligned}$$

|

*ReturnNode*:

$\llbracket \text{kind } g \text{ nid} = (\text{ReturnNode } (\text{Some } \text{expr}) -) \rrbracket;$

$g \vdash \text{expr} \simeq e;$   
 $[m, p] \vdash e \mapsto v;$

$cm' = cm(cnid := v);$   
 $cnid' = (\text{successors-of } (kind \text{ cg } cnid))!0$   
 $\implies P \vdash ((g, nid, m, p) \# (cg, cnid, cm, cp) \# stk, h) \longrightarrow ((cg, cnid', cm', cp) \# stk, h) \mid$

*ReturnNodeVoid:*

$\llbracket kind \text{ g } nid = (\text{ReturnNode } None \text{ -});$   
 $cm' = cm(cnid := (\text{ObjRef } (\text{Some } (2048))));$

$cnid' = (\text{successors-of } (kind \text{ cg } cnid))!0$   
 $\implies P \vdash ((g, nid, m, p) \# (cg, cnid, cm, cp) \# stk, h) \longrightarrow ((cg, cnid', cm', cp) \# stk, h) \mid$

*UnwindNode:*

$\llbracket kind \text{ g } nid = (\text{UnwindNode } exception);$

$g \vdash \text{exception} \simeq \text{exceptionE};$   
 $[m, p] \vdash \text{exceptionE} \mapsto e;$

$kind \text{ cg } cnid = (\text{InvokeWithExceptionNode } \text{ - - - - - } exEdge);$

$cm' = cm(cnid := e)$   
 $\implies P \vdash ((g, nid, m, p) \# (cg, cnid, cm, cp) \# stk, h) \longrightarrow ((cg, exEdge, cm', cp) \# stk, h)$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$ ) *step-top* .

## 7.4 Big-step Execution

**type-synonym** *Trace* = (*IRGraph*  $\times$  *ID*  $\times$  *MapState*  $\times$  *Params*) *list*

**fun** *has-return* :: *MapState*  $\Rightarrow$  *bool* **where**  
*has-return* *m* = (*m* 0  $\neq$  *UndefVal*)

**inductive** *exec* :: *Program*

$\Rightarrow (IRGraph \times ID \times MapState \times Params) \text{ list} \times FieldRefHeap$   
 $\Rightarrow Trace$   
 $\Rightarrow (IRGraph \times ID \times MapState \times Params) \text{ list} \times FieldRefHeap$   
 $\Rightarrow Trace$   
 $\Rightarrow \text{bool}$

(-  $\vdash$  -  $\mid$  -  $\longrightarrow$  \* -  $\mid$  -)

**for** *P*

**where**

$\llbracket P \vdash (((g, nid, m, p) \# xs), h) \longrightarrow (((g', nid', m', p') \# ys), h');$   
 $\neg(\text{has-return } m');$

$l' = (l \text{ @ } [(g, nid, m, p)]);$

*exec* *P* (((*g'*, *nid'*, *m'*, *p'*) # *ys*), *h'*) *l'* *next-state* *l''*

$\Rightarrow \text{exec } P \ ((g, \text{nid}, m, p) \# xs), h) \ l \ \text{next-state } l''$   
 $\mid$   
 $\llbracket P \vdash ((g, \text{nid}, m, p) \# xs), h) \longrightarrow ((g', \text{nid}', m', p') \# ys), h' \rrbracket;$   
 $\text{has-return } m';$   
 $l' = (l \ @ \ [(g, \text{nid}, m, p)])$   
 $\Rightarrow \text{exec } P \ ((g, \text{nid}, m, p) \# xs), h) \ l \ ((g', \text{nid}', m', p') \# ys), h' \ l'$   
**code-pred** (modes:  $i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow o \Rightarrow \text{bool}$  as *Exec*) *exec* .

**inductive** *exec-debug* :: *Program*  
 $\Rightarrow (IRGraph \times ID \times MapState \times Params) \ \text{list} \times FieldRefHeap$   
 $\Rightarrow \text{nat}$   
 $\Rightarrow (IRGraph \times ID \times MapState \times Params) \ \text{list} \times FieldRefHeap$   
 $\Rightarrow \text{bool}$   
 $(\vdash \longrightarrow * - * -)$   
**where**  
 $\llbracket n > 0; \mid$   
 $p \vdash s \longrightarrow s';$   
 $\text{exec-debug } p \ s' \ (n - 1) \ s' \rrbracket$   
 $\Rightarrow \text{exec-debug } p \ s \ n \ s'' \mid$   
 $\llbracket n = 0 \rrbracket$   
 $\Rightarrow \text{exec-debug } p \ s \ n \ s$   
**code-pred** (modes:  $i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$ ) *exec-debug* .

### 7.4.1 Heap Testing

**definition** *p3* :: *Params* **where**  
 $p3 = [IntVal32 \ 3]$

**values**  $\{(prod.fst(prod.snd \ (prod.snd \ (hd \ (prod.fst \ res)))) \ 0$   
 $\mid res. (\lambda x. \text{Some } eg2\text{-sq}) \vdash ([ (eg2\text{-sq}, 0, \text{new-map-state}, p3), (eg2\text{-sq}, 0, \text{new-map-state}, p3)],$   
 $\text{new-heap}) \rightarrow * 2 * \ res\}$

**definition** *field-sq* :: *string* **where**  
 $\text{field-sq} = "sq"$

**definition** *eg3-sq* :: *IRGraph* **where**  
 $eg3\text{-sq} = \text{irgraph} \ [$   
 $(0, \text{StartNode } None \ 4, \text{VoidStamp}),$   
 $(1, \text{ParameterNode } 0, \text{default-stamp}),$   
 $(3, \text{MulNode } 1 \ 1, \text{default-stamp}),$   
 $(4, \text{StoreFieldNode } 4 \ \text{field-sq} \ 3 \ None \ None \ 5, \text{VoidStamp}),$   
 $(5, \text{ReturnNode } (\text{Some } 3) \ None, \text{default-stamp})$   
 $\ ]$

**values** {*h-load-field field-sq None (prod.snd res)*  
 | *res. (λx. Some eg3-sq) ⊢ ([ (eg3-sq, 0, new-map-state, p3), (eg3-sq, 0, new-map-state, p3)], new-heap) →\*3\* res*}

**definition** *eg4-sq :: IRGraph where*

*eg4-sq = irgraph [*  
 (*0, StartNode None 4, VoidStamp*),  
 (*1, ParameterNode 0, default-stamp*),  
 (*3, MulNode 1 1, default-stamp*),  
 (*4, NewInstanceNode 4 "obj-class" None 5, ObjectStamp "obj-class" True True*  
*True*),  
 (*5, StoreFieldNode 5 field-sq 3 None (Some 4) 6, VoidStamp*),  
 (*6, ReturnNode (Some 3) None, default-stamp*)  
*]*

**values** {*h-load-field field-sq (Some 0) (prod.snd res) | res.*  
*(λx. Some eg4-sq) ⊢ ([ (eg4-sq, 0, new-map-state, p3), (eg4-sq, 0, new-map-state, p3)], new-heap) →\*4\* res*}

**end**

## 7.5 Control-flow Semantics Theorems

**theory** *IRStepThms*

**imports**

*IRStepObj*

*TreeToGraphThms*

**begin**

We prove that within the same graph, a configuration triple will always transition to the same subsequent configuration. Therefore, our step semantics is deterministic.

### 7.5.1 Control-flow Step is Deterministic

**theorem** *stepDet:*

*(g, p ⊢ (nid,m,h) → next) ⇒*  
*(∀ next'. ((g, p ⊢ (nid,m,h) → next') ⇒ next = next'))*

**proof** (*induction rule: step.induct*)

**case** (*SequentialNode nid next m h*)

**have** *notif: ¬(is-IfNode (kind g nid))*

**using** *SequentialNode.hyps(1) is-sequential-node.simps*

**by** (*metis is-IfNode-def*)

**have** *notend: ¬(is-AbstractEndNode (kind g nid))*

**using** *SequentialNode.hyps(1) is-sequential-node.simps*

**by** (*metis is-AbstractEndNode.simps is-EndNode.elims(2) is-LoopEndNode-def*)

**have** *notnew: ¬(is-NewInstanceNode (kind g nid))*

```

    using SequentialNode.hyps(1) is-sequential-node.simps
    by (metis is-NewInstanceNode-def)
  have notload:  $\neg(is\text{-LoadFieldNode } (kind\ g\ nid))$ 
    using SequentialNode.hyps(1) is-sequential-node.simps
    by (metis is-LoadFieldNode-def)
  have notstore:  $\neg(is\text{-StoreFieldNode } (kind\ g\ nid))$ 
    using SequentialNode.hyps(1) is-sequential-node.simps
    by (metis is-StoreFieldNode-def)
  have notdivrem:  $\neg(is\text{-IntegerDivRemNode } (kind\ g\ nid))$ 
    using SequentialNode.hyps(1) is-sequential-node.simps is-SignedDivNode-def
    is-SignedRemNode-def
    by (metis is-IntegerDivRemNode.simps)
  from notif notend notnew notload notstore notdivrem
  show ?case using SequentialNode.step.cases
    by (smt (z3) IRNode.disc(1028) IRNode.disc(2270) IRNode.discI(31) Pair-inject
    is-sequential-node.simps(18) is-sequential-node.simps(43) is-sequential-node.simps(44))
next
case (IfNode nid cond tb fb m val next h)
then have notseq:  $\neg(is\text{-sequential-node } (kind\ g\ nid))$ 
    using is-sequential-node.simps is-AbstractMergeNode.simps
    by (simp add: IfNode.hyps(1))
have notend:  $\neg(is\text{-AbstractEndNode } (kind\ g\ nid))$ 
    using is-AbstractEndNode.simps
    by (simp add: IfNode.hyps(1))
have notdivrem:  $\neg(is\text{-IntegerDivRemNode } (kind\ g\ nid))$ 
    using is-AbstractEndNode.simps
    by (simp add: IfNode.hyps(1))
from notseq notend notdivrem show ?case using IfNode.repDet evalDet IRN-
ode.distinct IRNode.inject(11) Pair-inject step.simps
    by (smt (z3) IRNode.distinct IRNode.inject(12) Pair-inject step.simps)
next
case (EndNodes nid merge i phis inputs m vs m' h)
have notseq:  $\neg(is\text{-sequential-node } (kind\ g\ nid))$ 
    using EndNodes.hyps(1) is-AbstractEndNode.simps is-sequential-node.simps
    by (metis is-EndNode.elims(2) is-LoopEndNode-def)
have notif:  $\neg(is\text{-IfNode } (kind\ g\ nid))$ 
    using EndNodes.hyps(1) is-IfNode-def is-AbstractEndNode.elims
    by (metis IRNode.distinct-disc(1058) is-EndNode.simps(12))
have notref:  $\neg(is\text{-RefNode } (kind\ g\ nid))$ 
    using EndNodes.hyps(1) is-sequential-node.simps
    using IRNode.disc(1899) IRNode.distinct(1473) is-AbstractEndNode.simps
    is-EndNode.elims(2) is-LoopEndNode-def is-RefNode-def
    by metis
have notnew:  $\neg(is\text{-NewInstanceNode } (kind\ g\ nid))$ 
    using EndNodes.hyps(1) is-AbstractEndNode.simps
    using IRNode.distinct-disc(1442) is-EndNode.simps(29) is-NewInstanceNode-def
    by (metis IRNode.distinct-disc(1901) is-EndNode.simps(32))
have notload:  $\neg(is\text{-LoadFieldNode } (kind\ g\ nid))$ 
    using EndNodes.hyps(1) is-AbstractEndNode.simps

```

```

    using is-LoadFieldName-def
    by (metis IRNode.distinct-disc(1706) is-EndNode.simps(21))
  have notstore:  $\neg$ (is-StoreFieldName (kind g nid))
    using EndNodes.hyps(1) is-AbstractEndNode.simps is-StoreFieldName-def
    by (metis IRNode.distinct-disc(1926) is-EndNode.simps(44))
  have notdivrem:  $\neg$ (is-IntegerDivRemNode (kind g nid))
    using EndNodes.hyps(1) is-AbstractEndNode.simps is-SignedDivNode-def is-SignedRemNode-def
    using IRNode.distinct-disc(1498) IRNode.distinct-disc(1500) is-IntegerDivRemNode.simps
    is-EndNode.simps(36) is-EndNode.simps(37)
  by auto
  from notseq notif notref notnew notload notstore notdivrem
  show ?case using EndNodes.repAllDet evalAllDet
    by (smt (z3) is-IfNode-def is-LoadFieldName-def is-NewInstanceNode-def is-RefNode-def
    is-StoreFieldName-def is-SignedDivNode-def is-SignedRemNode-def Pair-inject is-IntegerDivRemNode.elims(3)
    step.cases)
next
case (NewInstanceNode nid f obj nxt h' ref h m' m)
then have notseq:  $\neg$ (is-sequential-node (kind g nid))
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notend:  $\neg$ (is-AbstractEndNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notif:  $\neg$ (is-IfNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notref:  $\neg$ (is-RefNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notload:  $\neg$ (is-LoadFieldName (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notstore:  $\neg$ (is-StoreFieldName (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notdivrem:  $\neg$ (is-IntegerDivRemNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
from notseq notend notif notref notload notstore notdivrem
show ?case using NewInstanceNode.step.cases
  by (smt (z3) IRNode.disc(1028) IRNode.disc(2270) IRNode.discI(11) IRN-
ode.distinct(2311) IRNode.distinct(2313) IRNode.inject(31) Pair-inject)
next
case (LoadFieldName nid f obj nxt m ref h v m')
then have notseq:  $\neg$ (is-sequential-node (kind g nid))
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: LoadFieldName.hyps(1))
have notend:  $\neg$ (is-AbstractEndNode (kind g nid))
  using is-AbstractEndNode.simps

```

```

    by (simp add: LoadFieldNode.hyps(1))
  have notdivrem: ¬(is-IntegerDivRemNode (kind g nid))
    using is-AbstractEndNode.simps
    by (simp add: LoadFieldNode.hyps(1))
  from notseq notend notdivrem
  show ?case using LoadFieldNode step.cases repDet evalDet
    by (smt (z3) IRNode.distinct(1051) IRNode.distinct(1721) IRNode.distinct(1739)
IRNode.distinct(1741) IRNode.distinct(1745) IRNode.inject(20) Pair-inject Value.inject(3)
option.distinct(1) option.inject)
next
  case (StaticLoadFieldNode nid f nrt h v m' m)
  then have notseq: ¬(is-sequential-node (kind g nid))
    using is-sequential-node.simps is-AbstractMergeNode.simps
    by (simp add: StaticLoadFieldNode.hyps(1))
  have notend: ¬(is-AbstractEndNode (kind g nid))
    using is-AbstractEndNode.simps
    by (simp add: StaticLoadFieldNode.hyps(1))
  have notdivrem: ¬(is-IntegerDivRemNode (kind g nid))
    by (simp add: StaticLoadFieldNode.hyps(1))
  from notseq notend notdivrem
  show ?case using StaticLoadFieldNode step.cases
    by (smt (z3) IRNode.distinct(1051) IRNode.distinct(1721) IRNode.distinct(1739)
IRNode.distinct(1741) IRNode.distinct(1745) IRNode.inject(20) Pair-inject option.distinct(1))
next
  case (StoreFieldNode nid f newval uu obj nrt m val ref h' h m')
  then have notseq: ¬(is-sequential-node (kind g nid))
    using is-sequential-node.simps is-AbstractMergeNode.simps
    by (simp add: StoreFieldNode.hyps(1))
  have notend: ¬(is-AbstractEndNode (kind g nid))
    using is-AbstractEndNode.simps
    by (simp add: StoreFieldNode.hyps(1))
  have notdivrem: ¬(is-IntegerDivRemNode (kind g nid))
    by (simp add: StoreFieldNode.hyps(1))
  from notseq notend notdivrem
  show ?case using StoreFieldNode step.cases repDet evalDet
    by (smt (z3) IRNode.distinct(1097) IRNode.distinct(1745) IRNode.distinct(2317)
IRNode.distinct(2605) IRNode.distinct(2627) IRNode.inject(43) Pair-inject Value.inject(3)
option.distinct(1) option.inject)
next
  case (StaticStoreFieldNode nid f newval uv nrt m val h' h m')
  then have notseq: ¬(is-sequential-node (kind g nid))
    using is-sequential-node.simps is-AbstractMergeNode.simps
    by (simp add: StaticStoreFieldNode.hyps(1))
  have notend: ¬(is-AbstractEndNode (kind g nid))
    using is-AbstractEndNode.simps
    by (simp add: StaticStoreFieldNode.hyps(1))
  have notdivrem: ¬(is-IntegerDivRemNode (kind g nid))
    by (simp add: StaticStoreFieldNode.hyps(1))
  from notseq notend notdivrem

```

```

show ?case using StoreFieldNode step.cases repDet evalDet
by (smt (z3) IRNode.distinct(1097) IRNode.distinct(1745) IRNode.distinct(2317)
IRNode.distinct(2605) IRNode.distinct(2627) IRNode.inject(43) Pair-inject Static-
StoreFieldNode.hyps(1) StaticStoreFieldNode.hyps(2) StaticStoreFieldNode.hyps(3)
StaticStoreFieldNode.hyps(4) StaticStoreFieldNode.hyps(5) option.distinct(1))
next
case (SignedDivNode nid x y zero sb nxt m v1 v2 v m' h)
then have notseq:  $\neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$ 
using is-sequential-node.simps is-AbstractMergeNode.simps
by (simp add: SignedDivNode.hyps(1))
have notend:  $\neg(\text{is-AbstractEndNode } (\text{kind } g \text{ nid}))$ 
using is-AbstractEndNode.simps
by (simp add: SignedDivNode.hyps(1))
from notseq notend
show ?case using SignedDivNode step.cases repDet evalDet
by (smt (z3) IRNode.distinct(1091) IRNode.distinct(1739) IRNode.distinct(2311)
IRNode.distinct(2601) IRNode.distinct(2605) IRNode.inject(40) Pair-inject)
next
case (SignedRemNode nid x y zero sb nxt m v1 v2 v m' h)
then have notseq:  $\neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$ 
using is-sequential-node.simps is-AbstractMergeNode.simps
by (simp add: SignedRemNode.hyps(1))
have notend:  $\neg(\text{is-AbstractEndNode } (\text{kind } g \text{ nid}))$ 
using is-AbstractEndNode.simps
by (simp add: SignedRemNode.hyps(1))
from notseq notend
show ?case using SignedRemNode step.cases repDet evalDet
by (smt (z3) IRNode.distinct(1093) IRNode.distinct(1741) IRNode.distinct(2313)
IRNode.distinct(2601) IRNode.distinct(2627) IRNode.inject(41) Pair-inject)
qed

```

**lemma** stepRefNode:

$\llbracket \text{kind } g \text{ nid} = \text{RefNode nid}' \rrbracket \implies g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m, h)$

**using** SequentialNode

**by** (metis IRNodes.successors-of-RefNode is-sequential-node.simps(7) nth-Cons-0)

**lemma** IfNodeStepCases:

**assumes** kind g nid = IfNode cond tb fb

**assumes**  $g \vdash \text{cond} \simeq \text{condE}$

**assumes**  $[m, p] \vdash \text{condE} \mapsto v$

**assumes**  $g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m, h)$

**shows**  $\text{nid}' \in \{\text{tb}, \text{fb}\}$

**using** step.IfNode repDet stepDet assms

**by** (metis insert-iff old.prod.inject)

**lemma** IfNodeSeq:

**shows** kind g nid = IfNode cond tb fb  $\longrightarrow \neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$

**unfolding** is-sequential-node.simps

**using** is-sequential-node.simps(18) **by** presburger



```

lemma IfNodeCond:
  assumes kind g nid = IfNode cond tb fb
  assumes g, p ⊢ (nid, m, h) → (nid', m, h)
  shows  $\exists \text{ condE } v. ((g \vdash \text{cond} \simeq \text{condE}) \wedge ([m, p] \vdash \text{condE} \mapsto v))$ 
  using assms(2,1) by (induct (nid,m,h) (nid',m,h) rule: step.induct; auto)

lemma step-in-ids:
  assumes g, p ⊢ (nid, m, h) → (nid', m', h')
  shows nid ∈ ids g
  using assms apply (induct (nid, m, h) (nid', m', h') rule: step.induct)
  using is-sequential-node.simps(45) not-in-g
  apply simp
  apply (metis is-sequential-node.simps(53))
  using ids-some
  using IRNode.distinct(1113) apply presburger
  using EndNodes(1) is-AbstractEndNode.simps is-EndNode.simps(45) ids-some
  apply (metis IRNode.disc(1218) is-EndNode.simps(52))
  by simp+

end

```

## 8 Proof Infrastructure

### 8.1 Bisimulation

```

theory Bisimulation
imports
  Stuttering
begin

```

```

inductive weak-bisimilar :: ID ⇒ IRGraph ⇒ IRGraph ⇒ bool
  (- . - ~ -) for nid where
     $\llbracket \forall P'. (g \text{ m } p \text{ h} \vdash \text{nid} \rightsquigarrow P') \longrightarrow (\exists Q'. (g' \text{ m } p \text{ h} \vdash \text{nid} \rightsquigarrow Q') \wedge P' = Q');$ 
     $\forall Q'. (g' \text{ m } p \text{ h} \vdash \text{nid} \rightsquigarrow Q') \longrightarrow (\exists P'. (g \text{ m } p \text{ h} \vdash \text{nid} \rightsquigarrow P') \wedge P' = Q') \rrbracket$ 
     $\implies \text{nid} . g \sim g'$ 

```

A strong bisimulation between no-op transitions

```

inductive strong-noop-bisimilar :: ID ⇒ IRGraph ⇒ IRGraph ⇒ bool
  (- | - ~ -) for nid where
     $\llbracket \forall P'. (g, p \vdash (\text{nid}, m, h) \rightarrow P') \longrightarrow (\exists Q'. (g', p \vdash (\text{nid}, m, h) \rightarrow Q') \wedge P' = Q');$ 
     $\forall Q'. (g', p \vdash (\text{nid}, m, h) \rightarrow Q') \longrightarrow (\exists P'. (g, p \vdash (\text{nid}, m, h) \rightarrow P') \wedge P' = Q') \rrbracket$ 
     $\implies \text{nid} \mid g \sim g'$ 

```

```

lemma lockstep-strong-bisimulation:

```

```

assumes  $g' = \text{replace-node } nid \text{ node } g$ 
assumes  $g, p \vdash (nid, m, h) \rightarrow (nid', m, h)$ 
assumes  $g', p \vdash (nid, m, h) \rightarrow (nid', m, h)$ 
shows  $nid \mid g \sim g'$ 
using  $\text{assms}(2) \text{ assms}(3) \text{ stepDet strong-noop-bisimilar.simps}$  by  $\text{metis}$ 

lemma no-step-bisimulation:
assumes  $\forall m p h nid' m' h'. \neg(g, p \vdash (nid, m, h) \rightarrow (nid', m', h'))$ 
assumes  $\forall m p h nid' m' h'. \neg(g', p \vdash (nid, m, h) \rightarrow (nid', m', h'))$ 
shows  $nid \mid g \sim g'$ 
using  $\text{assms}$ 
by ( $\text{simp add: assms}(1) \text{ assms}(2) \text{ strong-noop-bisimilar.intros}$ )

end

```

## 8.2 Graph Rewriting

```

theory
  Rewrites
imports
  Semantics.IRGraphFrames
  Stuttering
begin

fun replace-usages ::  $ID \Rightarrow ID \Rightarrow IRGraph \Rightarrow IRGraph$  where
  replace-usages  $nid \ nid' \ g = \text{replace-node } nid \ (\text{RefNode } nid', \text{stamp } g \ nid') \ g$ 

lemma replace-usages-effect:
assumes  $g' = \text{replace-usages } nid \ nid' \ g$ 
shows  $\text{kind } g' \ nid = \text{RefNode } nid'$ 
using  $\text{assms replace-node-lookup replace-usages.simps}$ 
by ( $\text{metis IRNode.distinct}(2755)$ )

lemma replace-usages-changeonly:
assumes  $nid \in \text{ids } g$ 
assumes  $g' = \text{replace-usages } nid \ nid' \ g$ 
shows  $\text{changeonly } \{nid\} \ g \ g'$ 
using  $\text{assms unfolding replace-usages.simps}$ 
by ( $\text{metis add-changed add-node-def replace-node-def}$ )

lemma replace-usages-unchanged:
assumes  $nid \in \text{ids } g$ 
assumes  $g' = \text{replace-usages } nid \ nid' \ g$ 
shows  $\text{unchanged } (\text{ids } g - \{nid\}) \ g \ g'$ 
using  $\text{assms unfolding replace-usages.simps}$ 
using  $\text{assms}(2) \text{ disjoint-change replace-usages-changeonly}$  by  $\text{presburger}$ 

```

```

fun nextNid :: IRGraph ⇒ ID where
  nextNid g = (Max (ids g)) + 1

lemma max-plus-one:
  fixes c :: ID set
  shows  $\llbracket \text{finite } c; c \neq \{\} \rrbracket \implies (\text{Max } c) + 1 \notin c$ 
  by (meson Max-gr-iff less-add-one less-irrefl)

lemma ids-finite:
  finite (ids g)
  by simp

lemma nextNidNotIn:
  ids g ≠ {} ⟶ nextNid g ∉ ids g
  unfolding nextNid.simps
  using ids-finite max-plus-one by blast

fun constantCondition :: bool ⇒ ID ⇒ IRNode ⇒ IRGraph ⇒ IRGraph where
  constantCondition val nid (IfNode cond t f) g =
    replace-node nid (IfNode (nextNid g) t f, stamp g nid)
      (add-node (nextNid g) ((ConstantNode (bool-to-val val)), constantAsStamp
        (bool-to-val val)) g) |
    constantCondition cond nid - g = g

lemma constantConditionTrue:
  assumes kind g ifcond = IfNode cond t f
  assumes g' = constantCondition True ifcond (kind g ifcond) g
  shows g', p ⊢ (ifcond, m, h) → (t, m, h)
proof –
  have ifn: ∧ c t f. IfNode c t f ≠ NoNode
    by simp
  then have if': kind g' ifcond = IfNode (nextNid g) t f
    using assms(1) assms(2) constantCondition.simps(1) replace-node-lookup
    by presburger
  have truedef: bool-to-val True = (IntVal32 1)
    by auto
  from ifn have ifcond ≠ (nextNid g)
    by (metis assms(1) emptyE ids-some nextNidNotIn)
  moreover have  $\bigwedge c. \text{ConstantNode } c \neq \text{NoNode}$  by simp
  ultimately have kind g' (nextNid g) = ConstantNode (bool-to-val True)
    using add-changed add-node-def assms(1) assms(2) constantCondition.simps(1)
    not-in-g other-node-unchanged replace-node-def replace-node-lookup singletonD
    by (smt (z3) DiffI add-node-lookup replace-node-unchanged)
  then have c': kind g' (nextNid g) = ConstantNode (IntVal32 1)
    using truedef by simp
  have valid-value (IntVal32 1) (constantAsStamp (IntVal32 1))
    unfolding constantAsStamp.simps valid-value.simps
    using nat-numeral by blast
  then have  $[g', m, p] \vdash \text{nextNid } g \mapsto \text{IntVal32 } 1$ 

```

**using** *ConstantExpr ConstantNode Value.distinct(1) <kind g' (nextNid g) = ConstantNode (bool-to-val True)> encodeeval-def truedef*  
**by** *metis*  
**from** *if' c' show ?thesis using IfNode*  
**by** (*metis (no-types, opaque-lifting) val-to-bool.simps(1) <[g',m,p] ⊢ nextNid g ↦ IntVal32 1> encodeeval-def zero-neq-one*)  
**qed**

**lemma** *constantConditionFalse:*

**assumes** *kind g ifcond = IfNode cond t f*  
**assumes** *g' = constantCondition False ifcond (kind g ifcond) g*  
**shows** *g', p ⊢ (ifcond, m, h) → (f, m, h)*  
**proof** –  
**have** *ifn: ∧ c t f. IfNode c t f ≠ NoNode*  
**by** *simp*  
**then have** *if': kind g' ifcond = IfNode (nextNid g) t f*  
**by** (*metis assms(1) assms(2) constantCondition.simps(1) replace-node-lookup*)  
**have** *falsedef: bool-to-val False = (IntVal32 0)*  
**by** *auto*  
**from** *ifn have ifcond ≠ (nextNid g)*  
**by** (*metis assms(1) equals0D ids-some nextNidNotIn*)  
**moreover have** *∧ c. ConstantNode c ≠ NoNode* **by** *simp*  
**ultimately have** *kind g' (nextNid g) = ConstantNode (bool-to-val False)*  
**by** (*smt (z3) add-changed add-node-def assms(1) assms(2) constantCondition.simps(1) not-in-g other-node-unchanged replace-node-def replace-node-lookup singletonD*)  
**then have** *c': kind g' (nextNid g) = ConstantNode (IntVal32 0)*  
**using** *falsedef* **by** *simp*  
**have** *valid-value (IntVal32 0) (constantAsStamp (IntVal32 0))*  
**unfolding** *constantAsStamp.simps valid-value.simps*  
**using** *nat-numeral* **by** *blast*  
**then have** *[g', m, p] ⊢ nextNid g ↦ IntVal32 0*  
**by** (*metis ConstantExpr ConstantNode <kind g' (nextNid g) = ConstantNode (bool-to-val False)> encodeeval-def falsedef*)  
**from** *if' c' show ?thesis using IfNode*  
**by** (*metis (no-types, opaque-lifting) val-to-bool.simps(1) <[g',m,p] ⊢ nextNid g ↦ IntVal32 0> encodeeval-def*)  
**qed**

**lemma** *diff-forall:*

**assumes** *∀ n ∈ ids g - {nid}. cond n*  
**shows** *∀ n. n ∈ ids g ∧ n ∉ {nid} → cond n*  
**by** (*meson Diff-iff assms*)

**lemma** *replace-node-changeonly:*

**assumes** *g' = replace-node nid node g*  
**shows** *changeonly {nid} g g'*  
**using** *assms replace-node-unchanged*  
**unfolding** *changeonly.simps* **using** *diff-forall*

```

    by (metis add-changed add-node-def changeonly.simps replace-node-def)

lemma add-node-changeonly:
  assumes  $g' = \text{add-node } \textit{nid} \textit{ node } g$ 
  shows  $\textit{changeonly } \{\textit{nid}\} g g'$ 
  by (metis Rep-IRGraph-inverse add-node.rep-eq assms replace-node.rep-eq replace-node-changeonly)

lemma constantConditionNoEffect:
  assumes  $\neg(\textit{is-IfNode } (\textit{kind } g \textit{ nid}))$ 
  shows  $g = \textit{constantCondition } b \textit{ nid } (\textit{kind } g \textit{ nid}) g$ 
  using assms apply (cases  $\textit{kind } g \textit{ nid}$ )
  using constantCondition.simps
  apply presburger+
  apply (metis  $\textit{is-IfNode-def}$ )
  using constantCondition.simps
  by presburger+

lemma constantConditionIfNode:
  assumes  $\textit{kind } g \textit{ nid} = \textit{IfNode } \textit{cond } t f$ 
  shows  $\textit{constantCondition } \textit{val } \textit{nid } (\textit{kind } g \textit{ nid}) g =$ 
     $\textit{replace-node } \textit{nid } (\textit{IfNode } (\textit{nextNid } g) t f, \textit{stamp } g \textit{ nid})$ 
     $(\textit{add-node } (\textit{nextNid } g) ((\textit{ConstantNode } (\textit{bool-to-val } \textit{val})), \textit{constantAsStamp}$ 
     $(\textit{bool-to-val } \textit{val})) g)$ 
  using constantCondition.simps
  by (simp add: assms)

lemma constantCondition-changeonly:
  assumes  $\textit{nid} \in \textit{ids } g$ 
  assumes  $g' = \textit{constantCondition } b \textit{ nid } (\textit{kind } g \textit{ nid}) g$ 
  shows  $\textit{changeonly } \{\textit{nid}\} g g'$ 
proof (cases  $\textit{is-IfNode } (\textit{kind } g \textit{ nid})$ )
  case True
  have  $\textit{nextNid } g \notin \textit{ids } g$ 
  using nextNidNotIn by (metis emptyE)
  then show ?thesis using assms
  using replace-node-changeonly add-node-changeonly unfolding changeonly.simps
  using True constantCondition.simps(1)  $\textit{is-IfNode-def}$ 
  by (metis (no-types, lifting) insert-iff)
next
  case False
  have  $g = g'$ 
  using constantConditionNoEffect
  using False assms(2) by blast
  then show ?thesis by simp
qed

```

```

lemma constantConditionNoIf:

```

```

assumes  $\forall \text{ cond } t f. \text{ kind } g \text{ ifcond} \neq \text{ IfNode cond } t f$ 
assumes  $g' = \text{constantCondition val ifcond (kind } g \text{ ifcond) } g$ 
shows  $\exists \text{ nid}' . (g \text{ m } p \text{ h} \vdash \text{ifcond} \rightsquigarrow \text{nid}') \longleftrightarrow (g' \text{ m } p \text{ h} \vdash \text{ifcond} \rightsquigarrow \text{nid}')$ 
proof -
  have  $g' = g$ 
    using  $\text{assms}(2) \text{ assms}(1)$ 
    using  $\text{constantConditionNoEffect}$ 
    by  $(\text{metis IRNode.collapse}(11))$ 
  then show  $?thesis$  by  $\text{simp}$ 
qed

```

```

lemma  $\text{constantConditionValid}$ :
  assumes  $\text{kind } g \text{ ifcond} = \text{IfNode cond } t f$ 
  assumes  $[g, m, p] \vdash \text{cond} \mapsto v$ 
  assumes  $\text{const} = \text{val-to-bool } v$ 
  assumes  $g' = \text{constantCondition const ifcond (kind } g \text{ ifcond) } g$ 
  shows  $\exists \text{ nid}' . (g \text{ m } p \text{ h} \vdash \text{ifcond} \rightsquigarrow \text{nid}') \longleftrightarrow (g' \text{ m } p \text{ h} \vdash \text{ifcond} \rightsquigarrow \text{nid}')$ 
proof  $(\text{cases const})$ 
  case  $\text{True}$ 
    have  $\text{ifstep}: g, p \vdash (\text{ifcond}, m, h) \rightarrow (t, m, h)$ 
      by  $(\text{meson IfNode True assms}(1) \text{ assms}(2) \text{ assms}(3) \text{ encodeeval-def})$ 
    have  $\text{ifstep}': g', p \vdash (\text{ifcond}, m, h) \rightarrow (t, m, h)$ 
      using  $\text{constantConditionTrue}$ 
      using  $\text{True assms}(1) \text{ assms}(4)$  by  $\text{presburger}$ 
    from  $\text{ifstep ifstep' show ?thesis}$ 
    using  $\text{StutterStep}$  by  $\text{blast}$ 
  next
    case  $\text{False}$ 
      have  $\text{ifstep}: g, p \vdash (\text{ifcond}, m, h) \rightarrow (f, m, h)$ 
        by  $(\text{meson IfNode False assms}(1) \text{ assms}(2) \text{ assms}(3) \text{ encodeeval-def})$ 
      have  $\text{ifstep}': g', p \vdash (\text{ifcond}, m, h) \rightarrow (f, m, h)$ 
        using  $\text{constantConditionFalse}$ 
        using  $\text{False assms}(1) \text{ assms}(4)$  by  $\text{presburger}$ 
      from  $\text{ifstep ifstep' show ?thesis}$ 
      using  $\text{StutterStep}$  by  $\text{blast}$ 
  qed
end

```

### 8.3 Stuttering

```

theory  $\text{Stuttering}$ 
  imports
     $\text{Semantics.IRStepThms}$ 
begin

```

```

inductive  $\text{stutter}:: \text{IRGraph} \Rightarrow \text{MapState} \Rightarrow \text{Params} \Rightarrow \text{FieldRefHeap} \Rightarrow \text{ID} \Rightarrow$ 
 $\text{ID} \Rightarrow \text{bool} \text{ (- - - } \vdash \text{ - } \rightsquigarrow \text{ - } 55)$ 
  for  $g \text{ m } p \text{ h}$  where

```

*StutterStep:*  
 $\llbracket g, p \vdash (nid, m, h) \rightarrow (nid', m, h) \rrbracket$   
 $\implies g \ m \ p \ h \vdash nid \rightsquigarrow nid' \mid$

*Transitive:*  
 $\llbracket g, p \vdash (nid, m, h) \rightarrow (nid'', m, h);$   
 $g \ m \ p \ h \vdash nid'' \rightsquigarrow nid \rrbracket$   
 $\implies g \ m \ p \ h \vdash nid \rightsquigarrow nid'$

**lemma** *stuttering-successor:*  
**assumes**  $(g, p \vdash (nid, m, h) \rightarrow (nid', m, h))$   
**shows**  $\{P'. (g \ m \ p \ h \vdash nid \rightsquigarrow P')\} = \{nid'\} \cup \{nid''. (g \ m \ p \ h \vdash nid' \rightsquigarrow nid'')\}$   
**proof** –  
**have** *nextin:*  $nid' \in \{P'. (g \ m \ p \ h \vdash nid \rightsquigarrow P')\}$   
**using** *assms StutterStep by blast*  
**have** *nextsubset:*  $\{nid''. (g \ m \ p \ h \vdash nid' \rightsquigarrow nid'')\} \subseteq \{P'. (g \ m \ p \ h \vdash nid \rightsquigarrow P')\}$   
**by** (*metis Collect-mono assms stutter.Transitive*)  
**have**  $\forall n \in \{P'. (g \ m \ p \ h \vdash nid \rightsquigarrow P')\}. n = nid' \vee n \in \{nid''. (g \ m \ p \ h \vdash nid' \rightsquigarrow nid'')\}$   
**using** *stepDet*  
**by** (*metis (no-types, lifting) Pair-inject assms mem-Collect-eq stutter.simps*)  
**then show** *?thesis*  
**using** *insert-absorb mk-disjoint-insert nextin nextsubset by auto*  
**qed**  
**end**

## 8.4 Evaluation Stamp Theorems

**theory** *StampEvalThms*  
**imports** *Semantics.IRTreeEvalThms*  
**begin**

### 8.4.1 Evaluated Value Satisfies Stamps

**lemma** *size32:* *size*  $v = 32$  **for**  $v :: 32 \text{ word}$   
**using** *size-word.rep-eq*  
**using** *One-nat-def add.right-neutral add-Suc-right len-of-numeral-defs(2) len-of-numeral-defs(3)*  
*mult.right-neutral mult-Suc-right numeral-2-eq-2 numeral-Bit0*  
**by** (*smt (verit, del-insts) mult.commute*)

**lemma** *size64:* *size*  $v = 64$  **for**  $v :: 64 \text{ word}$   
**using** *size-word.rep-eq*  
**using** *One-nat-def add.right-neutral add-Suc-right len-of-numeral-defs(2) len-of-numeral-defs(3)*  
*mult.right-neutral mult-Suc-right numeral-2-eq-2 numeral-Bit0*  
**by** (*smt (verit, del-insts) mult.commute*)

**lemma** *signed-int-bottom32:*  $-(((2::\text{int}) \wedge 31)) \leq \text{sint } (v::\text{int32})$   
**using** *sint-range-size size32*

**by** (*smt* (*verit*, *ccfv-SIG*) *One-nat-def Suc-pred add-Suc add-Suc-right eval-nat-numeral*(3)  
*nat.inject numeral-2-eq-2 numeral-Bit0 numeral-Bit1 zero-less-numeral*)

**lemma** *signed-int-top32*:  $(2 \wedge 31) - 1 \geq \text{sint } (v::\text{int32})$   
**using** *sint-range-size size32*  
**by** (*smt* (*verit*, *ccfv-SIG*) *One-nat-def Suc-pred add-Suc add-Suc-right eval-nat-numeral*(3)  
*nat.inject numeral-2-eq-2 numeral-Bit0 numeral-Bit1 zero-less-numeral*)

**lemma** *lower-bounds-equiv32*:  $-(((2::\text{int}) \wedge 31)) = (2::\text{int}) \wedge 32 \text{ div } 2 * - 1$   
**by** *fastforce*

**lemma** *upper-bounds-equiv32*:  $(2::\text{int}) \wedge 31 = (2::\text{int}) \wedge 32 \text{ div } 2$   
**by** *simp*

**lemma** *bit-bounds-min32*:  $((\text{fst } (\text{bit-bounds } 32))) \leq (\text{sint } (v::\text{int32}))$   
**unfolding** *bit-bounds.simps fst-def* **using** *signed-int-bottom32 lower-bounds-equiv32*  
**by** *auto*

**lemma** *bit-bounds-max32*:  $((\text{snd } (\text{bit-bounds } 32))) \geq (\text{sint } (v::\text{int32}))$   
**unfolding** *bit-bounds.simps fst-def* **using** *signed-int-top32 upper-bounds-equiv32*  
**by** *auto*

**lemma** *signed-int-bottom64*:  $-(((2::\text{int}) \wedge 63)) \leq \text{sint } (v::\text{int64})$   
**using** *sint-range-size size64*  
**by** (*smt* (*verit*, *ccfv-SIG*) *One-nat-def Suc-pred add-Suc add-Suc-right eval-nat-numeral*(3)  
*nat.inject numeral-2-eq-2 numeral-Bit0 numeral-Bit1 zero-less-numeral*)

**lemma** *signed-int-top64*:  $(2 \wedge 63) - 1 \geq \text{sint } (v::\text{int64})$   
**using** *sint-range-size size64*  
**by** (*smt* (*verit*, *ccfv-SIG*) *One-nat-def Suc-pred add-Suc add-Suc-right eval-nat-numeral*(3)  
*nat.inject numeral-2-eq-2 numeral-Bit0 numeral-Bit1 zero-less-numeral*)

**lemma** *lower-bounds-equiv64*:  $-(((2::\text{int}) \wedge 63)) = (2::\text{int}) \wedge 64 \text{ div } 2 * - 1$   
**by** *fastforce*

**lemma** *upper-bounds-equiv64*:  $(2::\text{int}) \wedge 63 = (2::\text{int}) \wedge 64 \text{ div } 2$   
**by** *simp*

**lemma** *bit-bounds-min64*:  $((\text{fst } (\text{bit-bounds } 64))) \leq (\text{sint } (v::\text{int64}))$   
**unfolding** *bit-bounds.simps fst-def* **using** *signed-int-bottom64 lower-bounds-equiv64*  
**by** *auto*

**lemma** *bit-bounds-max64*:  $((\text{snd } (\text{bit-bounds } 64))) \geq (\text{sint } (v::\text{int64}))$   
**unfolding** *bit-bounds.simps fst-def* **using** *signed-int-top64 upper-bounds-equiv64*  
**by** *auto*

**lemma** *unrestricted-32bit-always-valid*:  
*valid-value* (*IntVal32* *v*) (*unrestricted-stamp* (*IntegerStamp* 32 *lo hi*))  
**using** *valid-value.simps*(1) *bit-bounds-min32 bit-bounds-max32*



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using unrestricted-stamp.simps(2) by presburger

lemma unrestricted-64bit-always-valid:
  valid-value (IntVal64 v) (unrestricted-stamp (IntegerStamp 64 lo hi))
using valid-value.simps(2) bit-bounds-min64 bit-bounds-max64
using unrestricted-stamp.simps(2) by presburger

lemma unary-undef: val = UndefVal  $\implies$  unary-eval op val = UndefVal
by (cases op; auto)

lemma unary-obj: val = ObjRef x  $\implies$  unary-eval op val = UndefVal
by (cases op; auto)

lemma unary-eval-implies-valid-value:
  assumes [m,p]  $\vdash$  expr  $\mapsto$  val
  assumes result = unary-eval op val
  assumes result  $\neq$  UndefVal
  assumes valid-value val (stamp-expr expr)
  shows valid-value result (stamp-expr (UnaryExpr op expr))
proof –
  have is-IntVal:  $\exists$  x y. result = IntVal32 x  $\vee$  result = IntVal64 y
    using assms(2,3) apply (cases op; auto; cases val; auto)
    apply metis
    by metis
  then have is-IntegerStamp (stamp-expr expr)
    using assms(2,3,4) apply (cases (stamp-expr expr); auto)
    using valid-VoidStamp unary-undef apply simp
    using valid-VoidStamp unary-undef apply simp
    using valid-ObjStamp unary-obj apply fastforce
    using valid-ObjStamp unary-obj by fastforce
  then obtain b lo hi where stamp-expr-def: stamp-expr expr = (IntegerStamp b
lo hi)
    using is-IntegerStamp-def by auto
  then have stamp-expr (UnaryExpr op expr) = unrestricted-stamp (IntegerStamp
b lo hi)
    using stamp-expr.simps(1) stamp-unary.simps(1) by presburger
  from stamp-expr-def have bit32: b = 32  $\implies$   $\exists$  x. result = IntVal32 x
    using assms(2,3,4) by (cases op; auto; cases val; auto)
  from stamp-expr-def have bit64: b = 64  $\implies$   $\exists$  x. result = IntVal64 x
    using assms(2,3,4) by (cases op; auto; cases val; auto)

  show ?thesis using valid-value.simps(1,2)
    unrestricted-32bit-always-valid unrestricted-64bit-always-valid stamp-expr-def
    bit32 bit64
    by (metis  $\langle$ stamp-expr (UnaryExpr op expr) = unrestricted-stamp (IntegerStamp
b lo hi) $\rangle$  assms(4) valid32or64-both)
qed

lemma binary-undef: v1 = UndefVal  $\vee$  v2 = UndefVal  $\implies$  bin-eval op v1 v2 =

```

*UndefVal*  
**by** (*cases op; auto*)

**lemma** *binary-obj*:  $v1 = \text{ObjRef } x \vee v2 = \text{ObjRef } y \implies \text{bin-eval } op \ v1 \ v2 = \text{UndefVal}$   
**by** (*cases op; auto*)

**lemma** *binary-eval-bits-equal*:  
**assumes**  $\text{result} = \text{bin-eval } op \ val1 \ val2$   
**assumes**  $\text{result} \neq \text{UndefVal}$   
**assumes**  $\text{valid-value } val1 \ (\text{IntegerStamp } b1 \ lo1 \ hi1)$   
**assumes**  $\text{valid-value } val2 \ (\text{IntegerStamp } b2 \ lo2 \ hi2)$   
**shows**  $b1 = b2$   
**using** *assms*  
**by** (*cases op; cases val1; cases val2; auto*)

**lemma** *binary-eval-values*:  
**assumes**  $\exists x \ y. \text{result} = \text{IntVal32 } x \vee \text{result} = \text{IntVal64 } y$   
**assumes**  $\text{result} = \text{bin-eval } op \ val1 \ val2$   
**shows**  $\exists x32 \ x64 \ y32 \ y64. \ val1 = \text{IntVal32 } x32 \wedge \text{val2} = \text{IntVal32 } y32 \vee \text{val1} = \text{IntVal64 } x64 \wedge \text{val2} = \text{IntVal64 } y64$   
**using** *assms* **apply** (*cases result*)  
**apply** *simp* **apply** (*cases op; cases val1; cases val2; auto*)  
**apply** (*cases op; cases val1; cases val2; auto*) **by** *auto+*

**lemma** *binary-eval-implies-valid-value*:  
**assumes**  $[m, p] \vdash \text{expr1} \mapsto \text{val1}$   
**assumes**  $[m, p] \vdash \text{expr2} \mapsto \text{val2}$   
**assumes**  $\text{result} = \text{bin-eval } op \ val1 \ val2$   
**assumes**  $\text{result} \neq \text{UndefVal}$   
**assumes**  $\text{valid-value } val1 \ (\text{stamp-expr } \text{expr1})$   
**assumes**  $\text{valid-value } val2 \ (\text{stamp-expr } \text{expr2})$   
**shows**  $\text{valid-value } \text{result} \ (\text{stamp-expr } (\text{BinaryExpr } op \ \text{expr1 } \text{expr2}))$   
**proof** –  
**have** *is-IntVal*:  $\exists x \ y. \text{result} = \text{IntVal32 } x \vee \text{result} = \text{IntVal64 } y$   
**using** *assms*(1,2,3,4) **apply** (*cases op; auto; cases val1; auto; cases val2; auto*)  
**by** (*meson Values.bool-to-val.elims*)+  
**then have** *expr1-intstamp*:  $\text{is-IntegerStamp } (\text{stamp-expr } \text{expr1})$   
**using** *assms*(1,3,4,5) **apply** (*cases (stamp-expr expr1); auto simp: valid-VoidStamp binary-undef*)  
**using** *valid-ObjStamp binary-obj* **apply** (*metis assms*(4))  
**using** *valid-ObjStamp binary-obj* **by** (*metis assms*(4))  
**from** *is-IntVal* **have** *expr2-intstamp*:  $\text{is-IntegerStamp } (\text{stamp-expr } \text{expr2})$   
**using** *assms*(2,3,4,6) **apply** (*cases (stamp-expr expr2); auto simp: valid-VoidStamp binary-undef*)  
**using** *valid-ObjStamp binary-obj* **apply** (*metis assms*(4))  
**using** *valid-ObjStamp binary-obj* **by** (*metis assms*(4))  
**from** *expr1-intstamp* **obtain**  $b1 \ lo1 \ hi1$  **where** *stamp-expr1-def*:  $\text{stamp-expr } \text{expr1} = (\text{IntegerStamp } b1 \ lo1 \ hi1)$

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    using is-IntegerStamp-def by auto
    from expr2-intstamp obtain b2 lo2 hi2 where stamp-expr2-def: stamp-expr
    expr2 = (IntegerStamp b2 lo2 hi2)
    using is-IntegerStamp-def by auto

    have  $\exists x32\ x64\ y32\ y64. (val1 = IntVal32\ x32 \wedge val2 = IntVal32\ y32) \vee (val1$ 
    = IntVal64 x64  $\wedge val2 = IntVal64\ y64)$ 
    using is-IntVal assms(3) binary-eval-values
    by presburger

    have b1 = b2
    using assms(3,4,5,6) stamp-expr1-def stamp-expr2-def
    using binary-eval-bits-equal
    by auto
    then have stamp-def: stamp-expr (BinaryExpr op expr1 expr2) =
    (case op  $\in$  fixed-32 of True  $\Rightarrow$  unrestricted-stamp (IntegerStamp 32 lo1 hi1)|
    False  $\Rightarrow$  unrestricted-stamp (IntegerStamp b1 lo1 hi1))
    using stamp-expr.simps(2) stamp-binary.simps(1)
    using stamp-expr1-def stamp-expr2-def by presburger
    from stamp-expr1-def have bit32: b1 = 32  $\Longrightarrow \exists x. result = IntVal32\ x$ 
    using assms apply (cases op; cases val1; cases val2; auto)
    by (meson Values.bool-to-val.elims)+
    from stamp-expr1-def have bit64: b1 = 64  $\wedge op \notin$  fixed-32  $\Longrightarrow \exists x\ y. result =$ 
    IntVal64 x
    using assms apply (cases op; cases val1; cases val2; simp)
    using fixed-32-def by auto+
    from stamp-expr1-def have fixed: op  $\in$  fixed-32  $\Longrightarrow \exists x\ y. result = IntVal32\ x$ 
    using assms unfolding fixed-32-def apply (cases op; auto)
    apply (cases val1; cases val2; auto)
    using bit32 apply fastforce
    apply (meson Values.bool-to-val.elims)
    apply (cases val1; cases val2; auto)
    using bit32 apply fastforce
    apply (meson Values.bool-to-val.elims)
    apply (cases val1; cases val2; auto)
    using bit32 apply fastforce
    by (meson Values.bool-to-val.elims)

    show ?thesis apply (cases op  $\in$  fixed-32) defer using valid-value.simps(1,2)
    unrestricted-32bit-always-valid unrestricted-64bit-always-valid stamp-expr1-def
    bit32 bit64 stamp-def apply auto
    using  $\langle \exists x32\ x64\ y32\ y64. val1 = IntVal32\ x32 \wedge val2 = IntVal32\ y32 \vee val1$ 
    = IntVal64 x64  $\wedge val2 = IntVal64\ y64 \rangle$  assms(5) apply auto[1]
    using fixed by force
  qed

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lemma stamp-meet-is-valid:
  assumes valid-value val stamp1  $\vee$  valid-value val stamp2
  assumes meet stamp1 stamp2  $\neq$  IllegalStamp

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shows valid-value val (meet stamp1 stamp2)
using assms proof (cases stamp1)
case VoidStamp
then show ?thesis
  by (metis Stamp.exhaust assms(1) assms(2) meet.simps(1) meet.simps(37)
meet.simps(44) meet.simps(51) meet.simps(58) meet.simps(65) meet.simps(66) meet.simps(67))
next
case (IntegerStamp b lo hi)
obtain b2 lo2 hi2 where stamp2-def: stamp2 = IntegerStamp b2 lo2 hi2
by (metis IntegerStamp assms(2) meet.simps(45) meet.simps(52) meet.simps(59)
meet.simps(6) meet.simps(65) meet.simps(66) meet.simps(67) unrestricted-stamp.cases)
then have b = b2 using meet.simps(2) assms(2)
  by (metis IntegerStamp)
then have meet-def: meet stamp1 stamp2 = (IntegerStamp b (min lo lo2) (max
hi hi2))
  by (simp add: IntegerStamp stamp2-def)
then show ?thesis proof (cases b = 32)
case True
then obtain x where val-def: val = IntVal32 x
  using IntegerStamp assms(1) valid32
  using ⟨b = b2⟩ stamp2-def by blast
have min: sint x ≥ min lo lo2
  using val-def
  using IntegerStamp assms(1)
  using stamp2-def by force
have max: sint x ≤ max hi hi2
  using val-def
  using IntegerStamp assms(1)
  using stamp2-def by force
from min max show ?thesis
  by (simp add: True meet-def val-def)
next
case False
then have bit64: b = 64
  using assms(1) IntegerStamp valid-value.simps
  valid32or64-both
  by (metis ⟨b = b2⟩ stamp2-def)
then obtain x where val-def: val = IntVal64 x
  using IntegerStamp assms(1) valid64
  using ⟨b = b2⟩ stamp2-def by blast
have min: sint x ≥ min lo lo2
  using val-def
  using IntegerStamp assms(1)
  using stamp2-def by force
have max: sint x ≤ max hi hi2
  using val-def
  using IntegerStamp assms(1)
  using stamp2-def by force
from min max show ?thesis

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    by (simp add: bit64 meet-def val-def)
  qed
next
  case (KlassPointerStamp x31 x32)
  then show ?thesis using assms valid-value.elims(2)
    by (metis meet.simps(14) valid-value.simps(21))
next
  case (MethodCountersPointerStamp x41 x42)
  then show ?thesis using assms valid-value.elims(2)
    by (metis meet.simps(39) valid-value.simps(22))
next
  case (MethodPointersStamp x51 x52)
  then show ?thesis using assms valid-value.elims(2)
    by (metis meet.simps(40) valid-value.simps(23))
next
  case (ObjectStamp x61 x62 x63 x64)
  then show ?thesis using assms
    using meet.simps(34) by blast
next
  case (RawPointerStamp x71 x72)
  then show ?thesis using assms
    using meet.simps(35) by blast
next
  case IllegalStamp
  then show ?thesis using assms
    using meet.simps(36) by blast
qed

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**lemma** *conditional-eval-implies-valid-value:*

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  assumes [m,p] ⊢ cond ↦ condv
  assumes expr = (if val-to-bool condv then expr1 else expr2)
  assumes [m,p] ⊢ expr ↦ val
  assumes val ≠ UndefVal
  assumes valid-value condv (stamp-expr cond)
  assumes valid-value val (stamp-expr expr)
  assumes compatible (stamp-expr expr1) (stamp-expr expr2)
  shows valid-value val (stamp-expr (ConditionalExpr cond expr1 expr2))
proof -
  have meet (stamp-expr expr1) (stamp-expr expr2) ≠ IllegalStamp
    using assms
    by (metis Stamp.distinct(13) Stamp.distinct(25) compatible.elims(2) meet.simps(1)
        meet.simps(2))
  then show ?thesis using stamp-meet-is-valid using stamp-expr.simps(6)
    using assms(2) assms(6) by presburger
qed

```

**experiment begin**

**lemma** *stamp-implies-valid-value:*

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    assumes  $[m, p] \vdash \text{expr} \mapsto \text{val}$ 
    shows valid-value  $\text{val}$  (stamp-expr  $\text{expr}$ )
    using assms proof (induction  $\text{expr}$   $\text{val}$ )
case (UnaryExpr  $\text{expr}$   $\text{val}$  result op)
  then show ?case using unary-eval-implies-valid-value by simp
next
  case (BinaryExpr  $\text{expr1}$   $\text{val1}$   $\text{expr2}$   $\text{val2}$  result op)
  then show ?case using binary-eval-implies-valid-value by simp
next
  case (ConditionalExpr cond condv  $\text{expr}$   $\text{expr1}$   $\text{expr2}$   $\text{val}$ )
  then show ?case using conditional-eval-implies-valid-value sorry
next
  case (ParameterExpr  $x1$   $x2$ )
  then show ?case by auto
next
  case (LeafExpr  $x1$   $x2$ )
  then show ?case by auto
next
  case (ConstantExpr  $x$ )
  then show ?case by auto
qed

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lemma value-range:
  assumes  $[m, p] \vdash e \mapsto v$ 
  shows  $v \in \{\text{val} . \text{valid-value } \text{val} \text{ (stamp-expr } e)\}$ 
  using assms sorry
end

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lemma upper-bound-32:
  assumes  $\text{val} = \text{IntVal32 } v$ 
  assumes  $\exists l h. s = (\text{IntegerStamp } 32 \ l \ h)$ 
  shows valid-value  $\text{val}$   $s \implies \text{sint } v \leq (\text{stpi-upper } s)$ 
  using assms by force

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lemma upper-bound-64:
  assumes  $\text{val} = \text{IntVal64 } v$ 
  assumes  $\exists l h. s = (\text{IntegerStamp } 64 \ l \ h)$ 
  shows valid-value  $\text{val}$   $s \implies \text{sint } v \leq (\text{stpi-upper } s)$ 
  using assms by force

```

```

lemma lower-bound-32:
  assumes  $\text{val} = \text{IntVal32 } v$ 
  assumes  $\exists l h. s = (\text{IntegerStamp } 32 \ l \ h)$ 
  shows valid-value  $\text{val}$   $s \implies \text{sint } v \geq (\text{stpi-lower } s)$ 
  using assms by force

```

```

lemma lower-bound-64:
  assumes  $\text{val} = \text{IntVal64 } v$ 
  assumes  $\exists l h. s = (\text{IntegerStamp } 64 \ l \ h)$ 

```

shows *valid-value val s*  $\implies$  *sint v*  $\geq$  (*stpi-lower s*)  
 using *assms*  
 by *force*

**lemma** *stamp-under-semantics*:

assumes *stamp-under* (*stamp-expr x*) (*stamp-expr y*)  
 assumes  $[m, p] \vdash (\text{BinaryExpr BinIntegerLessThan } x \ y) \mapsto v$   
 assumes *xvalid*:  $(\forall m \ p \ v. ([m, p] \vdash x \mapsto v) \longrightarrow \text{valid-value } v \ (\text{stamp-expr } x))$   
 assumes *yvalid*:  $(\forall m \ p \ v. ([m, p] \vdash y \mapsto v) \longrightarrow \text{valid-value } v \ (\text{stamp-expr } y))$   
 shows *val-to-bool v*  
**proof** –  
 obtain *xval* where *xval-def*:  $[m, p] \vdash x \mapsto xval$   
 using *assms*(2) by blast  
 obtain *yval* where *yval-def*:  $[m, p] \vdash y \mapsto yval$   
 using *assms*(2) by blast  
 have *is-IntVal32 xval*  $\vee$  *is-IntVal64 xval*  
 by (metis *BinaryExprE Value.exhaust-disc assms*(2) *bin-eval.simps*(11) *binary-obj binary-undef evalDet intval-less-than.simps*(9) *is-ObjRef-def is-ObjStr-def xval-def*)  
 have *is-IntVal32 yval*  $\vee$  *is-IntVal64 yval*  
 by (metis *BinaryExprE Value.exhaust-disc assms*(2) *bin-eval.simps*(11) *binary-obj binary-undef evalDet intval-less-than.simps*(16) *is-ObjRef-def is-ObjStr-def yval-def*)  
 have *is-IntVal32 xval* = *is-IntVal32 yval*  
 by (metis *BinaryExprE Value.collapse*(2)  $\langle \text{is-IntVal32 } xval \vee \text{is-IntVal64 } xval \rangle$   $\langle \text{is-IntVal32 } yval \vee \text{is-IntVal64 } yval \rangle$  *assms*(2) *bin-eval.simps*(11) *evalDet intval-less-than.simps*(12) *intval-less-than.simps*(5) *is-IntVal32-def xval-def yval-def*)  
 have *is-IntVal64 xval* = *is-IntVal64 yval*  
 using  $\langle \text{is-IntVal32 } xval = \text{is-IntVal32 } yval \rangle$   $\langle \text{is-IntVal32 } xval \vee \text{is-IntVal64 } xval \rangle$   $\langle \text{is-IntVal32 } yval \vee \text{is-IntVal64 } yval \rangle$  by blast  
 have  $(\text{intval-less-than } xval \ yval) \neq \text{UndefVal}$   
 using *assms*(2)  
 by (metis *BinaryExprE bin-eval.simps*(11) *evalDet xval-def yval-def*)  
 have *is-IntVal32 xval*  $\implies ((\exists \text{ lo hi. stamp-expr } x = \text{IntegerStamp } 32 \text{ lo hi}) \wedge (\exists \text{ lo hi. stamp-expr } y = \text{IntegerStamp } 32 \text{ lo hi}))$   
 using *assms*(2) *binary-eval-bits-equal valid-value.elims*(2) *xval-def*  
 by (metis *Value.distinct*(9) *Value.distinct-disc*(9)  $\langle \text{is-IntVal32 } yval \vee \text{is-IntVal64 } yval \rangle$   $\langle \text{is-IntVal64 } xval = \text{is-IntVal64 } yval \rangle$  *is-IntVal32-def xvalid yval-def yvalid*)  
 have *is-IntVal64 xval*  $\implies ((\exists \text{ lo hi. stamp-expr } x = \text{IntegerStamp } 64 \text{ lo hi}) \wedge (\exists \text{ lo hi. stamp-expr } y = \text{IntegerStamp } 64 \text{ lo hi}))$   
 by (metis (*full-types*)  $\langle \text{is-IntVal64 } xval = \text{is-IntVal64 } yval \rangle$  *is-IntVal64-def stamp-range valid-value.simps*(2) *xval-def xvalid yval-def yvalid*)  
 have *xvalid*: *valid-value xval* (*stamp-expr x*)  
 using *xvalid xval-def* by auto  
 have *yvalid*: *valid-value yval* (*stamp-expr y*)  
 using *yvalid yval-def* by auto  
 { assume *c*: *is-IntVal32 xval*  
 obtain *xxval* where *x32*: *xval* = *IntVal32 xxval*

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    using c is-IntVal32-def by blast
  obtain yyval where y32: yval = IntVal32 yyval
    using ⟨is-IntVal32 xval = is-IntVal32 yval⟩ c is-IntVal32-def by auto
  have xs: ∃ lo hi. stamp-expr x = IntegerStamp 32 lo hi
    by (simp add: ⟨is-IntVal32 xval ⟹ (∃ lo hi. stamp-expr x = IntegerStamp 32 lo hi)⟩ c)
  have ys: ∃ lo hi. stamp-expr y = IntegerStamp 32 lo hi
    using ⟨is-IntVal32 xval ⟹ (∃ lo hi. stamp-expr x = IntegerStamp 32 lo hi)
  ∧ (∃ lo hi. stamp-expr y = IntegerStamp 32 lo hi)⟩ c by blast
  have sint xxval ≤ stpi-upper (stamp-expr x)
    using upper-bound-32 x32 xs xvalid by presburger
  have stpi-lower (stamp-expr y) ≤ sint yyval
    using lower-bound-32 y32 ys yvalid by presburger
  have stpi-upper (stamp-expr x) < stpi-lower (stamp-expr y)
    using assms(1) unfolding stamp-under.simps
    by auto
  then have xxval < s yyval
    using assms(1) unfolding stamp-under.simps
    using ⟨sint xxval ⊆ stpi-upper (stamp-expr x)⟩ ⟨stpi-lower (stamp-expr y) ⊆
sint yyval⟩ word-sless-alt by fastforce
  then have (intval-less-than xval yval) = IntVal32 1
    by (simp add: x32 y32)
}
note case32 = this
{ assume c: is-IntVal64 xval
  obtain xxval where x64: xval = IntVal64 xxval
    using c is-IntVal64-def by blast
  obtain yyval where y64: yval = IntVal64 yyval
    using ⟨is-IntVal64 xval = is-IntVal64 yval⟩ c is-IntVal64-def by auto
  have xs: ∃ lo hi. stamp-expr x = IntegerStamp 64 lo hi
    by (simp add: ⟨is-IntVal64 xval ⟹ (∃ lo hi. stamp-expr x = IntegerStamp 64 lo hi)⟩ c)
  have ys: ∃ lo hi. stamp-expr y = IntegerStamp 64 lo hi
    using ⟨is-IntVal64 xval ⟹ (∃ lo hi. stamp-expr x = IntegerStamp 64 lo hi)
  ∧ (∃ lo hi. stamp-expr y = IntegerStamp 64 lo hi)⟩ c by blast
  have sint xxval ≤ stpi-upper (stamp-expr x)
    using upper-bound-64 x64 xs xvalid by presburger
  have stpi-lower (stamp-expr y) ≤ sint yyval
    using lower-bound-64 y64 ys yvalid by presburger
  have stpi-upper (stamp-expr x) < stpi-lower (stamp-expr y)
    using assms(1) unfolding stamp-under.simps
    by auto
  then have xxval < s yyval
    using assms(1) unfolding stamp-under.simps
    using ⟨sint xxval ⊆ stpi-upper (stamp-expr x)⟩ ⟨stpi-lower (stamp-expr y) ⊆
sint yyval⟩ word-sless-alt by fastforce
  then have (intval-less-than xval yval) = IntVal32 1
    by (simp add: x64 y64)
}

```



```

note case64 = this
have (intval-less-than xval yval) = IntVal32 1
  using ⟨is-IntVal32 xval ∨ is-IntVal64 xval⟩ case32 case64 by fastforce
then show ?thesis
  by (metis BinaryExprE assms(2) bin-eval.simps(11) evalDet val-to-bool.simps(1)
xval-def yval-def zero-neq-one)
qed

```

**lemma** stamp-under-semantics-inversed:

```

assumes stamp-under (stamp-expr y) (stamp-expr x)
assumes [m, p] ⊢ (BinaryExpr BinIntegerLessThan x y) ↦ v
assumes xvalid: (∀ m p v. ([m, p] ⊢ x ↦ v) ⟶ valid-value v (stamp-expr x))
assumes yvalid: (∀ m p v. ([m, p] ⊢ y ↦ v) ⟶ valid-value v (stamp-expr y))
shows ¬(val-to-bool v)
proof –
  obtain xval where xval-def: [m, p] ⊢ x ↦ xval
    using assms(2) by blast
  obtain yval where yval-def: [m, p] ⊢ y ↦ yval
    using assms(2) by blast
  have is-IntVal32 xval ∨ is-IntVal64 xval
    by (metis is-IntVal32-def is-IntVal64-def xvalid valid-value.elims(2) xval-def)
  have is-IntVal32 yval ∨ is-IntVal64 yval
    by (metis is-IntVal32-def is-IntVal64-def yvalid valid-value.elims(2) yval-def)
  have is-IntVal32 xval = is-IntVal32 yval
    by (metis BinaryExprE Value.collapse(2) ⟨is-IntVal32 xval ∨ is-IntVal64 xval⟩
⟨is-IntVal32 yval ∨ is-IntVal64 yval⟩ assms(2) bin-eval.simps(11) evalDet int-
val-less-than.simps(12) intval-less-than.simps(5) is-IntVal32-def xval-def yval-def)
  have is-IntVal64 xval = is-IntVal64 yval
    using ⟨is-IntVal32 xval = is-IntVal32 yval⟩ ⟨is-IntVal32 xval ∨ is-IntVal64 xval⟩
⟨is-IntVal32 yval ∨ is-IntVal64 yval⟩ by blast
  have (intval-less-than xval yval) ≠ UndefVal
    using assms(2)
    by (metis BinaryExprE bin-eval.simps(11) evalDet xval-def yval-def)
  have is-IntVal32 xval ⟹ ((∃ lo hi. stamp-expr x = IntegerStamp 32 lo hi) ∧
(∃ lo hi. stamp-expr y = IntegerStamp 32 lo hi))
    by (smt (verit) BinaryExprE Value.discI(2) Value.distinct-disc(9) assms(2)
binary-eval-bits-equal xvalid yvalid valid-value.elims(2) xval-def)
  have is-IntVal64 xval ⟹ ((∃ lo hi. stamp-expr x = IntegerStamp 64 lo hi) ∧
(∃ lo hi. stamp-expr y = IntegerStamp 64 lo hi))
    by (smt (verit, best) BinaryExprE ⟨intval-less-than xval yval ≠ UndefVal⟩
assms(2) binary-eval-bits-equal intval-less-than.simps(5) is-IntVal64-def xvalid yvalid
valid-value.elims(2) yval-def)
  have xvalid: valid-value xval (stamp-expr x)
    using xvalid xval-def by auto
  have yvalid: valid-value yval (stamp-expr y)
    using yvalid yval-def by auto
  { assume c: is-IntVal32 xval
    obtain x32 where x32: xval = IntVal32 x32
      using c is-IntVal32-def by blast

```

```

obtain yyval where y32: yval = IntVal32 yyval
  using  $\langle is-IntVal32\ xval = is-IntVal32\ yval \rangle\ c\ is-IntVal32-def$  by auto
have xs:  $\exists\ lo\ hi.\ stamp-expr\ x = IntegerStamp\ 32\ lo\ hi$ 
  by (simp add:  $\langle is-IntVal32\ xval \implies (\exists\ lo\ hi.\ stamp-expr\ x = IntegerStamp\ 32\ lo\ hi) \wedge (\exists\ lo\ hi.\ stamp-expr\ y = IntegerStamp\ 32\ lo\ hi) \rangle\ c$ )
have ys:  $\exists\ lo\ hi.\ stamp-expr\ y = IntegerStamp\ 32\ lo\ hi$ 
  using  $\langle is-IntVal32\ xval \implies (\exists\ lo\ hi.\ stamp-expr\ x = IntegerStamp\ 32\ lo\ hi) \wedge (\exists\ lo\ hi.\ stamp-expr\ y = IntegerStamp\ 32\ lo\ hi) \rangle\ c$  by blast
have sint yyval  $\leq stpi-upper\ (stamp-expr\ y)$ 
  using y32 ys yvalid by force
have stpi-lower (stamp-expr x)  $\leq sint\ xxval$ 
  using x32 xs xvalid by force
have stpi-upper (stamp-expr y)  $< stpi-lower\ (stamp-expr\ x)$ 
  using assms(1) unfolding stamp-under.simps
  by auto
then have yyval  $<_s\ xxval$ 
  using assms(1) unfolding stamp-under.simps
  using  $\langle sint\ yyval \sqsubseteq stpi-upper\ (stamp-expr\ y) \rangle\ \langle stpi-lower\ (stamp-expr\ x) \sqsubseteq sint\ xxval \rangle\ word-sless-alt$  by fastforce
then have (intval-less-than xval yval) = IntVal32 0
  using signed.less-not-sym x32 y32 by fastforce
}
note case32 = this
{ assume c: is-IntVal64 xval
  obtain xxval where x64: xval = IntVal64 xxval
    using c is-IntVal64-def by blast
  obtain yyval where y64: yval = IntVal64 yyval
    using  $\langle is-IntVal64\ xval = is-IntVal64\ yval \rangle\ c\ is-IntVal64-def$  by auto
  have xs:  $\exists\ lo\ hi.\ stamp-expr\ x = IntegerStamp\ 64\ lo\ hi$ 
    by (simp add:  $\langle is-IntVal64\ xval \implies (\exists\ lo\ hi.\ stamp-expr\ x = IntegerStamp\ 64\ lo\ hi) \wedge (\exists\ lo\ hi.\ stamp-expr\ y = IntegerStamp\ 64\ lo\ hi) \rangle\ c$ )
  have ys:  $\exists\ lo\ hi.\ stamp-expr\ y = IntegerStamp\ 64\ lo\ hi$ 
    using  $\langle is-IntVal64\ xval \implies (\exists\ lo\ hi.\ stamp-expr\ x = IntegerStamp\ 64\ lo\ hi) \wedge (\exists\ lo\ hi.\ stamp-expr\ y = IntegerStamp\ 64\ lo\ hi) \rangle\ c$  by blast
  have sint yyval  $\leq stpi-upper\ (stamp-expr\ y)$ 
    using y64 ys yvalid by force
  have stpi-lower (stamp-expr x)  $\leq sint\ xxval$ 
    using x64 xs xvalid by force
  have stpi-upper (stamp-expr y)  $< stpi-lower\ (stamp-expr\ x)$ 
    using assms(1) unfolding stamp-under.simps
    by auto
  then have yyval  $<_s\ xxval$ 
    using assms(1) unfolding stamp-under.simps
    using  $\langle sint\ yyval \sqsubseteq stpi-upper\ (stamp-expr\ y) \rangle\ \langle stpi-lower\ (stamp-expr\ x) \sqsubseteq sint\ xxval \rangle\ word-sless-alt$  by fastforce
  then have (intval-less-than xval yval) = IntVal32 0
    using signed.less-imp-triv x64 y64 by fastforce
}
note case64 = this

```

```

have (intval-less-than xval yval) = IntVal32 0
  using <is-IntVal32 xval ∨ is-IntVal64 xval> case32 case64 by fastforce
then show ?thesis
  by (metis BinaryExprE assms(2) bin-eval.simps(11) evalDet val-to-bool.simps(1)
xval-def yval-def)
qed

end

```

## 9 Optimization DSLs

```

theory Markup
  imports Semantics.IRTreeEval Snippets.Snipping
begin

```

```

datatype 'a Rewrite =
  Transform 'a 'a (- ⟶ - 10) |
  Conditional 'a 'a bool (- ⟶ - when - 70) |
  Sequential 'a Rewrite 'a Rewrite |
  Transitive 'a Rewrite

```

```

datatype 'a ExtraNotation =
  ConditionalNotation 'a 'a 'a (- ? - : -) |
  EqualsNotation 'a 'a (- eq -) |
  ConstantNotation 'a (const - 120) |
  TrueNotation (true) |
  FalseNotation (false)

```

**ML-file** <markup.ML>

```

ML <
  structure IRExprTranslator : DSL-TRANSLATION =
  struct
    fun markup DSL-Tokens.Add = @{term BinaryExpr} $ @{term BinAdd}
    | markup DSL-Tokens.Sub = @{term BinaryExpr} $ @{term BinSub}
    | markup DSL-Tokens.Mul = @{term BinaryExpr} $ @{term BinMul}
    | markup DSL-Tokens.And = @{term BinaryExpr} $ @{term BinAnd}
    | markup DSL-Tokens.Abs = @{term UnaryExpr} $ @{term UnaryAbs}
    | markup DSL-Tokens.Less = @{term BinaryExpr} $ @{term BinIntegerLessThan}
    | markup DSL-Tokens.Equals = @{term BinaryExpr} $ @{term BinIntegerEquals}
    | markup DSL-Tokens.Not = @{term UnaryExpr} $ @{term UnaryLogicNegation}
    | markup DSL-Tokens.Negate = @{term UnaryExpr} $ @{term UnaryNeg}
    | markup DSL-Tokens.LeftShift = @{term BinaryExpr} $ @{term BinLeftShift}
    | markup DSL-Tokens.RightShift = @{term BinaryExpr} $ @{term BinRightShift}
    | markup DSL-Tokens.UnsignedRightShift = @{term BinaryExpr} $ @{term Bin-
URightShift}
    | markup DSL-Tokens.Conditional = @{term ConditionalExpr}
    | markup DSL-Tokens.Constant = @{term ConstantExpr}
    | markup DSL-Tokens.TrueConstant = @{term ConstantExpr (IntVal32 1)}
  end

```

```
| markup DSL-Tokens.FalseConstant = @{term ConstantExpr (IntVal32 0)}
end
```

```
structure IntValTranslator : DSL-TRANSLATION =
struct
fun markup DSL-Tokens.Add = @{term intval-add}
| markup DSL-Tokens.Sub = @{term intval-sub}
| markup DSL-Tokens.Mul = @{term intval-mul}
| markup DSL-Tokens.And = @{term intval-and}
| markup DSL-Tokens.Abs = @{term intval-abs}
| markup DSL-Tokens.Less = @{term intval-less-than}
| markup DSL-Tokens.Equals = @{term intval-equals}
| markup DSL-Tokens.Not = @{term intval-logic-negation}
| markup DSL-Tokens.Negate = @{term intval-negate}
| markup DSL-Tokens.LeftShift = @{term intval-left-shift}
| markup DSL-Tokens.RightShift = @{term intval-right-shift}
| markup DSL-Tokens.UnsignedRightShift = @{term intval-uright-shift}
| markup DSL-Tokens.Conditional = @{term intval-conditional}
| markup DSL-Tokens.Constant = @{term IntVal32}
| markup DSL-Tokens.TrueConstant = @{term IntVal32 1}
| markup DSL-Tokens.FalseConstant = @{term IntVal32 0}
end
```

```
structure IRExprMarkup = DSL-Markup(IRExprTranslator);
structure IntValMarkup = DSL-Markup(IntValTranslator);
>
```

*ir expression translation*

```
syntax -expandExpr :: term ⇒ term (exp[-])
parse-translation < [( @{syntax-const -expandExpr} , IRExprMarkup.markup-expr)] >
```

*value expression translation*

```
syntax -expandIntVal :: term ⇒ term (val[-])
parse-translation < [( @{syntax-const -expandIntVal} , IntValMarkup.markup-expr)] >
```

*ir expression example*

```
value exp[(e1 < e2) ? e1 : e2]

ConditionalExpr (BinaryExpr BinIntegerLessThan e1 e2) e1 e2
```

*value expression example*

```
value val[( $e_1 < e_2$ ) ?  $e_1 : e_2$ ]
```

```
intval-conditional (intval-less-than  $e_1$   $e_2$ )  $e_1$   $e_2$ 
```

```
value exp[( $(e_1 - e_2) + (\text{const } (\text{IntVal32 } 0)) + e_2$ )  $\mapsto e_1$  when True]  
value val[( $(e_1 - e_2) + (\text{const } 0) + e_2$ )  $\mapsto e_1$  when True]
```

```
end
```

```
theory Phase
```

```
  imports Main
```

```
begin
```

```
ML-file map.ML
```

```
ML-file phase.ML
```

```
end
```

## 9.1 Canonicalization DSL

```
theory Canonicalization
```

```
  imports
```

```
    Markup
```

```
    Phase
```

```
  keywords
```

```
    phase :: thy-decl and
```

```
    terminating :: quasi-command and
```

```
    print-phases :: diag and
```

```
    optimization :: thy-goal-defn
```

```
begin
```

```
ML <
```

```
datatype 'a Rewrite =
```

```
  Transform of 'a * 'a |
```

```
  Conditional of 'a * 'a * term |
```

```
  Sequential of 'a Rewrite * 'a Rewrite |
```

```
  Transitive of 'a Rewrite
```

```
type rewrite = {name: string, rewrite: term Rewrite}
```

```
structure RewriteRule : Rule =
```

```
struct
```

```
type T = rewrite;
```

```
fun pretty-rewrite ctxt (Transform (from, to)) =
```

```
  Pretty.block [
```

```
    Syntax.pretty-term ctxt from,
```

```
    Pretty.str  $\mapsto$  ,
```

```

      Syntax.pretty-term ctxt to
    ]
| pretty-rewrite ctxt (Conditional (from, to, cond)) =
  Pretty.block [
    Syntax.pretty-term ctxt from,
    Pretty.str  $\mapsto$  ,
    Syntax.pretty-term ctxt to,
    Pretty.str when ,
    Syntax.pretty-term ctxt cond
  ]
| pretty-rewrite - - = Pretty.str not implemented

fun pretty ctxt t =
  Pretty.block [
    Pretty.str ((#name t) ^ ":"),
    pretty-rewrite ctxt (#rewrite t)
  ]
end

structure RewritePhase = DSL-Phase(RewriteRule);

val - =
  Outer-Syntax.command command-keyword⟨phase⟩ enter an optimization phase
  (Parse.binding --| Parse.$$$ terminating -- Parse.const --| Parse.begin
   >> (Toplevel.begin-main-target true o RewritePhase.setup));

fun print-phases ctxt =
  let
    val thy = Proof-Context.theory-of ctxt;
    fun print phase = RewritePhase.pretty phase ctxt
  in
    map print (RewritePhase.phases thy)
  end

fun print-optimizations thy =
  print-phases thy |> Pretty.writeln-chunks

val - =
  Outer-Syntax.command command-keyword⟨print-phases⟩
  print debug information for optimizations
  (Scan.succeed
   (Toplevel.keep (print-optimizations o Toplevel.context-of)));
>

```

**ML-file** rewrites.ML

```

fun rewrite-preservation :: IRExpr Rewrite  $\Rightarrow$  bool where
  rewrite-preservation (Transform x y) = (y  $\leq$  x) |
  rewrite-preservation (Conditional x y cond) = (cond  $\longrightarrow$  (y  $\leq$  x)) |

```

```

rewrite-preservation (Sequential x y) = (rewrite-preservation x ∧ rewrite-preservation
y) |
rewrite-preservation (Transitive x) = rewrite-preservation x

fun rewrite-termination :: IRExp Rewrite ⇒ (IRExp ⇒ nat) ⇒ bool where
rewrite-termination (Transform x y) trm = (trm x > trm y) |
rewrite-termination (Conditional x y cond) trm = (cond ⟶ (trm x > trm y)) |
rewrite-termination (Sequential x y) trm = (rewrite-termination x trm ∧ rewrite-termination
y trm) |
rewrite-termination (Transitive x) trm = rewrite-termination x trm

fun intval :: Value Rewrite ⇒ bool where
intval (Transform x y) = (x ≠ UndefVal ∧ y ≠ UndefVal ⟶ x = y) |
intval (Conditional x y cond) = (cond ⟶ (x = y)) |
intval (Sequential x y) = (intval x ∧ intval y) |
intval (Transitive x) = intval x

ML <
structure System : RewriteSystem =
struct
val preservation = @{const rewrite-preservation};
val termination = @{const rewrite-termination};
val intval = @{const intval};
end

structure DSL = DSL-Rewrites(System);

val - =
Outer-Syntax.local-theory-to-proof command-keyword <optimization>
define an optimization and open proof obligation
(Parse-Spec.thm-name : -- Parse.term
>> DSL.rewrite-cmd);
>

end

```

## 10 Canonicalization Phase

```

theory Common
imports
OptimizationDSL.Canonicalization
HOL-Eisbach.Eisbach
begin

```

```

fun size :: IRExp ⇒ nat where
size (UnaryExpr op e) = (size e) + 1 |
size (BinaryExpr BinAdd x y) = (size x) + ((size y) * 2) |
size (BinaryExpr op x y) = (size x) + (size y) |
size (ConditionalExpr cond t f) = (size cond) + (size t) + (size f) + 2 |

```

```

size (ConstantExpr c) = 1 |
size (ParameterExpr ind s) = 2 |
size (LeafExpr nid s) = 2 |
size (ConstantVar c) = 2 |
size (VariableExpr x s) = 2

method unfold-optimization =
  (unfold rewrite-preservation.simps, unfold rewrite-termination.simps,
   unfold intval.simps,
   rule conjE, simp, simp del: le-expr-def)
| (unfold rewrite-preservation.simps, unfold rewrite-termination.simps,
   rule conjE, simp, simp del: le-expr-def)

end

```

## 10.1 Conditional Expression

```

theory ConditionalPhase
  imports
    Common
    Proofs.StampEvalThms
  begin

  phase Conditional
    terminating size
  begin

  lemma negates: is-IntVal32 e  $\vee$  is-IntVal64 e  $\implies$  val-to-bool (val[e])  $\equiv \neg$ (val-to-bool
    (val[ $\neg$ e]))
  by (smt (verit, best) Value.disc(1) Value.disc(10) Value.disc(4) Value.disc(5)
    Value.disc(6) Value.disc(9) intval-logic-negation.elims val-to-bool.simps(1) val-to-bool.simps(2)
    zero-neq-one)

  optimization negate-condition: (( $\neg$ e) ? x : y)  $\mapsto$  (e ? y : x)
  apply unfold-optimization apply simp using negates
  using ConditionalExprE UnaryExprE intval-logic-negation.elims unary-eval.simps(4)
  val-to-bool.simps(1) val-to-bool.simps(2) zero-neq-one
  apply (smt (verit) ConditionalExpr)
  unfolding size.simps by simp

  optimization const-true: (true ? x : y)  $\mapsto$  x
  apply unfold-optimization
  apply force
  unfolding size.simps by simp

  optimization const-false: (false ? x : y)  $\mapsto$  y
  apply unfold-optimization
  apply force
  unfolding size.simps by simp

```



**optimization** *equal-branches*:  $(e \text{ ? } x : x) \mapsto x$   
 apply *unfold-optimization*  
 apply *force*  
 unfolding *size.simps* by *auto*

**definition** *wff-stamps* :: *bool* **where**  
*wff-stamps* =  $(\forall m \ p \ expr \ val . ([m, p] \vdash expr \mapsto val) \longrightarrow valid\_value \ val \ (stamp\_expr \ expr))$

**definition** *wf-stamp* :: *IRExpr*  $\Rightarrow$  *bool* **where**  
*wf-stamp* *e* =  $(\forall m \ p \ v . ([m, p] \vdash e \mapsto v) \longrightarrow valid\_value \ v \ (stamp\_expr \ e))$

**optimization** *condition-bounds-x*:  $((x < y) \text{ ? } x : y) \mapsto x$   
 when  $(stamp\_under \ (stamp\_expr \ x) \ (stamp\_expr \ y) \wedge wf\_stamp \ x \wedge wf\_stamp \ y)$   
 apply *unfold-optimization*  
 using *stamp-under-semantics*  
 using *wf-stamp-def*  
 apply  $(smt \ (verit, \ best) \ ConditionalExprE \ le\_expr\_def \ stamp\_under.simps)$   
 unfolding *size.simps* by *simp*

**optimization** *condition-bounds-y*:  $((x < y) \text{ ? } x : y) \mapsto y$   
 when  $(stamp\_under \ (stamp\_expr \ y) \ (stamp\_expr \ x) \wedge wf\_stamp \ x \wedge wf\_stamp \ y)$   
 apply *unfold-optimization*  
 using *stamp-under-semantics-inversed*  
 using *wf-stamp-def*  
 apply  $(smt \ (verit, \ best) \ ConditionalExprE \ le\_expr\_def \ stamp\_under.simps)$   
 unfolding *size.simps* by *simp*

**optimization** *b[intval]*:  $((x \text{ eq } y) \text{ ? } x : y) \mapsto y$   
 apply *unfold-optimization*  
 apply  $(smt \ (z3) \ bool\_to\_val.simps(2) \ intval\_equals.elims \ val\_to\_bool.simps(1) \ val\_to\_bool.simps(3))$   
 unfolding *intval.simps*  
 apply  $(smt \ (z3) \ BinaryExprE \ ConditionalExprE \ Value.inject(1) \ Value.inject(2) \ bin\_eval.simps(10) \ bool\_to\_val.simps(2) \ evalDet \ intval\_equals.simps(1) \ intval\_equals.simps(10) \ intval\_equals.simps(12) \ intval\_equals.simps(15) \ intval\_equals.simps(16) \ intval\_equals.simps(2) \ intval\_equals.simps(5) \ intval\_equals.simps(8) \ intval\_equals.simps(9) \ le\_expr\_def \ val\_to\_bool.cases \ val\_to\_bool.elims(2))$   
 unfolding *size.simps* by *auto*

**end**

**end**

## 11 Conditional Elimination Phase

```

theory ConditionalElimination
imports
  Semantics.IRGraphFrames
  Proofs.Stuttering
  Proofs.Rewrites
  Proofs.Bisimulation
begin

```

### 11.1 Individual Elimination Rules

We introduce a `TriState` as in the Graal compiler to represent when static analysis can tell us information about the value of a boolean expression. `Unknown` = No information can be inferred `KnownTrue`/`KnownFalse` = We can infer the expression will always be true or false.

```

datatype TriState = Unknown | KnownTrue | KnownFalse

```

The `implies` relation corresponds to the `LogicNode.implies` method from the compiler which attempts to infer when one logic nodes value can be inferred from a known logic node.

```

inductive implies :: IRGraph ⇒ IRNode ⇒ IRNode ⇒ TriState ⇒ bool
  (- ⊢ - & - ⇔ -) for g where
    eq-imp-less:
      g ⊢ (IntegerEqualsNode x y) & (IntegerLessThanNode x y) ⇔ KnownFalse |
    eq-imp-less-rev:
      g ⊢ (IntegerEqualsNode x y) & (IntegerLessThanNode y x) ⇔ KnownFalse |
    less-imp-rev-less:
      g ⊢ (IntegerLessThanNode x y) & (IntegerLessThanNode y x) ⇔ KnownFalse |
    less-imp-not-eq:
      g ⊢ (IntegerLessThanNode x y) & (IntegerEqualsNode x y) ⇔ KnownFalse |
    less-imp-not-eq-rev:
      g ⊢ (IntegerLessThanNode x y) & (IntegerEqualsNode y x) ⇔ KnownFalse |

    x-imp-x:
      g ⊢ x & x ⇔ KnownTrue |

    negate-false:
      ⟦g ⊢ x & (kind g y) ⇔ KnownTrue⟧ ⇒ g ⊢ x & (LogicNegationNode y) ⇔
KnownFalse |
    negate-true:
      ⟦g ⊢ x & (kind g y) ⇔ KnownFalse⟧ ⇒ g ⊢ x & (LogicNegationNode y) ⇔
KnownTrue

```

Total relation over partial `implies` relation

```

inductive condition-implies :: IRGraph ⇒ IRNode ⇒ IRNode ⇒ TriState ⇒ bool
  (- ⊢ - & - ⇔ -) for g where
    ⟦¬(g ⊢ a & b ⇔ imp)⟧ ⇒ (g ⊢ a & b ⇔ Unknown) |

```

$$\llbracket (g \vdash a \ \& \ b \hookrightarrow \text{imp}) \rrbracket \implies (g \vdash a \ \& \ b \multimap \text{imp})$$

**inductive** *implies-tree* :: *IRExpr*  $\Rightarrow$  *IRExpr*  $\Rightarrow$  *bool*  $\Rightarrow$  *bool*  
 (- & -  $\hookrightarrow$  -) **where**  
*eq-imp-less*:  
 (*BinaryExpr* *BinIntegerEquals* *x y*) & (*BinaryExpr* *BinIntegerLessThan* *x y*)  $\hookrightarrow$  *False* |  
*eq-imp-less-rev*:  
 (*BinaryExpr* *BinIntegerEquals* *x y*) & (*BinaryExpr* *BinIntegerLessThan* *y x*)  $\hookrightarrow$  *False* |  
*less-imp-rev-less*:  
 (*BinaryExpr* *BinIntegerLessThan* *x y*) & (*BinaryExpr* *BinIntegerLessThan* *y x*)  $\hookrightarrow$  *False* |  
*less-imp-not-eq*:  
 (*BinaryExpr* *BinIntegerLessThan* *x y*) & (*BinaryExpr* *BinIntegerEquals* *x y*)  $\hookrightarrow$  *False* |  
*less-imp-not-eq-rev*:  
 (*BinaryExpr* *BinIntegerLessThan* *x y*) & (*BinaryExpr* *BinIntegerEquals* *y x*)  $\hookrightarrow$  *False* |  
  
*x-imp-x*:  
*x* & *x*  $\hookrightarrow$  *True* |  
  
*negate-false*:  
 $\llbracket x \ \& \ y \hookrightarrow \text{True} \rrbracket \implies x \ \& \ (\text{UnaryExpr } \text{UnaryLogicNegation } y) \hookrightarrow \text{False}$  |  
*negate-true*:  
 $\llbracket x \ \& \ y \hookrightarrow \text{False} \rrbracket \implies x \ \& \ (\text{UnaryExpr } \text{UnaryLogicNegation } y) \hookrightarrow \text{True}$

Proofs that the implies relation is correct with respect to the existing evaluation semantics.

**experiment begin**

**lemma** *logic-negate-type*:

**assumes**  $[m, p] \vdash \text{UnaryExpr } \text{UnaryLogicNegation } x \mapsto v$

**assumes**  $v \neq \text{UndefVal}$

**shows**  $\exists v2. [m, p] \vdash x \mapsto \text{IntVal32 } v2$

**proof** –

**obtain** *ve* **where**  $ve: [m, p] \vdash x \mapsto ve$

**using** *assms*(1) **by** *blast*

**then have**  $[m, p] \vdash \text{UnaryExpr } \text{UnaryLogicNegation } x \mapsto \text{unary-eval } \text{UnaryLogicNegation } ve$

**by** (*metis* *UnaryExprE* *assms*(1) *evalDet*)

**then show** *?thesis* **using** *assms* *unary-eval.elims* *evalDet* *ve* *IRUnaryOp.distinct*

**sorry**

**qed**

**lemma** *logic-negation-relation-tree*:

```

assumes  $[m, p] \vdash y \mapsto val$ 
assumes  $[m, p] \vdash \text{UnaryExpr UnaryLogicNegation } y \mapsto invval$ 
assumes  $intval \neq \text{UndefVal}$ 
shows  $val\text{-to-bool } val \longleftrightarrow \neg(val\text{-to-bool } invval)$ 
proof –
  obtain  $v$  where  $invval = \text{unary-eval UnaryLogicNegation } v$ 
  using  $assms(2)$  by blast
  then have  $[m, p] \vdash y \mapsto v$  using  $\text{UnaryExprE } assms(1,2)$  sorry
  then show ?thesis sorry
qed

```

**lemma** *logic-negation-relation*:

```

assumes  $[g, m, p] \vdash y \mapsto val$ 
assumes  $kind\ g\ neg = \text{LogicNegationNode } y$ 
assumes  $[g, m, p] \vdash neg \mapsto invval$ 
assumes  $intval \neq \text{UndefVal}$ 
shows  $val\text{-to-bool } val \longleftrightarrow \neg(val\text{-to-bool } invval)$ 
proof –
  obtain  $yencode$  where  $g \vdash y \simeq yencode$ 
  using  $assms(1)$  encodeeval-def by auto
  then have  $g \vdash neg \simeq \text{UnaryExpr UnaryLogicNegation } yencode$ 
  using  $rep.intros(7)$   $assms(2)$  by simp
  then have  $[m, p] \vdash \text{UnaryExpr UnaryLogicNegation } yencode \mapsto invval$ 
  using  $assms(3)$  encodeeval-def
  by (metis repDet)
  obtain  $v1$  where  $[g, m, p] \vdash y \mapsto \text{IntVal32 } v1$ 
  using  $assms(1,2,3,4)$  using logic-negate-type sorry
  have  $invval = \text{bool-to-val } (\neg(val\text{-to-bool } val))$ 
  using  $assms(1,2,3)$  evalDet unary-eval.simps(4)
  by (smt (verit, ccfv-SIG) UnaryExprE  $\langle [m,p] \vdash \text{UnaryExpr UnaryLogicNegation } yencode \mapsto invval \rangle$ 
   $\langle g \vdash y \simeq yencode \rangle$  bool-to-val.simps(1) bool-to-val.simps(2) encodeeval-def intval-logic-negation.simps(1) logic-negate-type repDet val-to-bool.simps(1))
  have  $val\text{-to-bool } invval \longleftrightarrow \neg(val\text{-to-bool } val)$ 
  using  $\langle invval = \text{bool-to-val } (\neg val\text{-to-bool } val) \rangle$  by force
  then show ?thesis
  by simp
qed
end

```

**lemma** *implies-valid*:

```

assumes  $x \ \&\ y \hookrightarrow imp$ 
assumes  $[m, p] \vdash x \mapsto v1$ 
assumes  $[m, p] \vdash y \mapsto v2$ 
assumes  $v1 \neq \text{UndefVal} \wedge v2 \neq \text{UndefVal}$ 
shows  $(imp \longrightarrow (val\text{-to-bool } v1 \longrightarrow val\text{-to-bool } v2)) \wedge$ 
   $(\neg imp \longrightarrow (val\text{-to-bool } v1 \longrightarrow \neg(val\text{-to-bool } v2)))$ 
   $(is\ (\text{?TP} \longrightarrow \text{?TC}) \wedge (\text{?FP} \longrightarrow \text{?FC}))$ 
apply (intro conjI; rule impI)
proof –

```

```

assume KnownTrue: ?TP
show ?TC
using assms(1) KnownTrue assms(2–) proof (induct x y imp rule: implies-tree.induct)
  case (eq-imp-less x y)
  then show ?case by simp
next
  case (eq-imp-less-rev x y)
  then show ?case by simp
next
  case (less-imp-rev-less x y)
  then show ?case by simp
next
  case (less-imp-not-eq x y)
  then show ?case by simp
next
  case (less-imp-not-eq-rev x y)
  then show ?case by simp
next
  case (x-imp-x)
  then show ?case
    by (metis evalDet)
next
  case (negate-false x1)
  then show ?case using evalDet
    using assms(2,3) by blast
next
  case (negate-true y)
  then show ?case
    sorry
qed
next
assume KnownFalse: ?FP
show ?FC using assms KnownFalse proof (induct x y imp rule: implies-tree.induct)
  case (eq-imp-less x y)
  obtain xval where xval: [m, p] ⊢ x ↦ xval
    using eq-imp-less(1) eq-imp-less.prems(3)
    by blast
  then obtain yval where yval: [m, p] ⊢ y ↦ yval
    using eq-imp-less.prems(3)
    using eq-imp-less.prems(2) by blast
  have equal: [m, p] ⊢ (BinaryExpr BinIntegerEquals x y) ↦ intval-equals xval
yval
    using xval yval evaltree.BinaryExpr
    by (metis BinaryExprE bin-eval.simps(10) eq-imp-less.prems(1) evalDet)
  have lessequal: [m, p] ⊢ (BinaryExpr BinIntegerLessThan x y) ↦ intval-less-than
xval yval
    using xval yval evaltree.BinaryExpr
    by (metis BinaryExprE bin-eval.simps(11) eq-imp-less.prems(2) evalDet)
  have val-to-bool (intval-equals xval yval) ⟶ ¬(val-to-bool (intval-less-than xval

```

```

yval))
  using assms(4) apply (cases xval; cases yval; auto)
  apply (metis (full-types) val-to-bool.simps(1) Values.bool-to-val.simps(2)
signed.less-irrefl)
  by (metis (mono-tags) val-to-bool.simps(1) Values.bool-to-val.elims signed.order.strict-implies-not-eq)
  then show ?case
  using egeval lesseval
  by (metis eq-imp-less.prem(1) eq-imp-less.prem(2) evalDet)
next
case (eq-imp-less-rev x y)
obtain xval where xval: [m, p] ⊢ x ↦ xval
  using eq-imp-less-rev.prem(3)
  using eq-imp-less-rev.prem(2) by blast
obtain yval where yval: [m, p] ⊢ y ↦ yval
  using eq-imp-less-rev.prem(3)
  using eq-imp-less-rev.prem(2) by blast
have egeval: [m, p] ⊢ (BinaryExpr BinIntegerEquals x y) ↦ intval-equals xval
yval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(10) eq-imp-less-rev.prem(1) evalDet)
have lesseval: [m, p] ⊢ (BinaryExpr BinIntegerLessThan y x) ↦ intval-less-than
yval xval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(11) eq-imp-less-rev.prem(2) evalDet)
have val-to-bool (intval-equals xval yval) ⟶ ¬(val-to-bool (intval-less-than yval
xval))
  using assms(4) apply (cases xval; cases yval; auto)
  apply (metis (full-types) val-to-bool.simps(1) Values.bool-to-val.simps(2)
signed.less-irrefl)
  by (metis (full-types) val-to-bool.simps(1) Values.bool-to-val.elims signed.order.strict-implies-not-eq)
  then show ?case
  using egeval lesseval
  by (metis eq-imp-less-rev.prem(1) eq-imp-less-rev.prem(2) evalDet)
next
case (less-imp-rev-less x y)
obtain xval where xval: [m, p] ⊢ x ↦ xval
  using less-imp-rev-less.prem(3)
  using less-imp-rev-less.prem(2) by blast
obtain yval where yval: [m, p] ⊢ y ↦ yval
  using less-imp-rev-less.prem(3)
  using less-imp-rev-less.prem(2) by blast
have lesseval: [m, p] ⊢ (BinaryExpr BinIntegerLessThan x y) ↦ intval-less-than
xval yval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(11) evalDet less-imp-rev-less.prem(1))
have revlesseval: [m, p] ⊢ (BinaryExpr BinIntegerLessThan y x) ↦ int-
val-less-than yval xval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(11) evalDet less-imp-rev-less.prem(2))

```

```

    have val-to-bool (intval-less-than xval yval)  $\longrightarrow$   $\neg$ (val-to-bool (intval-less-than
yval xval))
    using assms(4) apply (cases xval; cases yval; auto)
    apply (metis val-to-bool.simps(1) Values.bool-to-val.elims signed.not-less-iff-gr-or-eq)
    by (metis val-to-bool.simps(1) Values.bool-to-val.elims signed.less-asm^)
    then show ?case
    by (metis evalDet less-imp-rev-less.prem(1) less-imp-rev-less.prem(2) lesseval
revlesseval)
  next
  case (less-imp-not-eq x y)
  obtain xval where xval:  $[m, p] \vdash x \mapsto xval$ 
  using less-imp-not-eq.prem(3)
  using less-imp-not-eq.prem(1) by blast
  obtain yval where yval:  $[m, p] \vdash y \mapsto yval$ 
  using less-imp-not-eq.prem(3)
  using less-imp-not-eq.prem(1) by blast
  have egeval:  $[m, p] \vdash (BinaryExpr BinIntegerEquals x y) \mapsto intval-equals xval$ 
yval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(10) evalDet less-imp-not-eq.prem(2))
  have lesseval:  $[m, p] \vdash (BinaryExpr BinIntegerLessThan x y) \mapsto intval-less-than$ 
xval yval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(11) evalDet less-imp-not-eq.prem(1))
  have val-to-bool (intval-less-than xval yval)  $\longrightarrow$   $\neg$ (val-to-bool (intval-equals xval
yval))
  using assms(4) apply (cases xval; cases yval; auto)
  apply (metis (full-types) bool-to-val.simps(2) signed.less-imp-not-eq val-to-bool.simps(1))
  by (metis (full-types) bool-to-val.simps(2) signed.less-imp-not-eq2 val-to-bool.simps(1))
  then show ?case
  by (metis egeval evalDet less-imp-not-eq.prem(1) less-imp-not-eq.prem(2)
lesseval)
  next
  case (less-imp-not-eq-rev x y)
  obtain xval where xval:  $[m, p] \vdash x \mapsto xval$ 
  using less-imp-not-eq-rev.prem(3)
  using less-imp-not-eq-rev.prem(1) by blast
  obtain yval where yval:  $[m, p] \vdash y \mapsto yval$ 
  using less-imp-not-eq-rev.prem(3)
  using less-imp-not-eq-rev.prem(1) by blast
  have egeval:  $[m, p] \vdash (BinaryExpr BinIntegerEquals y x) \mapsto intval-equals yval$ 
xval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(10) evalDet less-imp-not-eq-rev.prem(2))
  have lesseval:  $[m, p] \vdash (BinaryExpr BinIntegerLessThan x y) \mapsto intval-less-than$ 
xval yval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(11) evalDet less-imp-not-eq-rev.prem(1))
  have val-to-bool (intval-less-than xval yval)  $\longrightarrow$   $\neg$ (val-to-bool (intval-equals yval
xval))

```

```

xval))
  using assms(4) apply (cases xval; cases yval; auto)
  apply (metis (full-types) bool-to-val.simps(2) signed.less-imp-not-eq2 val-to-bool.simps(1))
  by (metis (full-types, opaque-lifting) val-to-bool.simps(1) Values.bool-to-val.elims
signed.dual-order.strict-implies-not-eq)
  then show ?case
  by (metis eqeval evalDet less-imp-not-eq-rev.prem(1) less-imp-not-eq-rev.prem(2)
lesseval)
next
  case (x-imp-x x1)
  then show ?case by simp
next
  case (negate-false x y)
  then show ?case sorry
next
  case (negate-true x1)
  then show ?case by simp
qed
qed

```

```

lemma implies-true-valid:
  assumes  $x \& y \hookrightarrow \text{imp}$ 
  assumes  $\text{imp}$ 
  assumes  $[m, p] \vdash x \mapsto v1$ 
  assumes  $[m, p] \vdash y \mapsto v2$ 
  assumes  $v1 \neq \text{UndefVal} \wedge v2 \neq \text{UndefVal}$ 
  shows  $\text{val-to-bool } v1 \longrightarrow \text{val-to-bool } v2$ 
  using assms implies-valid
  by blast

```

```

lemma implies-false-valid:
  assumes  $x \& y \hookrightarrow \text{imp}$ 
  assumes  $\neg \text{imp}$ 
  assumes  $[m, p] \vdash x \mapsto v1$ 
  assumes  $[m, p] \vdash y \mapsto v2$ 
  assumes  $v1 \neq \text{UndefVal} \wedge v2 \neq \text{UndefVal}$ 
  shows  $\text{val-to-bool } v1 \longrightarrow \neg(\text{val-to-bool } v2)$ 
  using assms implies-valid by blast

```

The following relation corresponds to the `UnaryOpLogicNode.tryFold` and `BinaryOpLogicNode.tryFold` methods and their associated concrete implementations.

The relation determines if a logic operation can be shown true or false through the stamp typing information.

```

inductive tryFold :: IRNode  $\Rightarrow$  (ID  $\Rightarrow$  Stamp)  $\Rightarrow$  bool  $\Rightarrow$  bool
  where
     $\llbracket \text{alwaysDistinct } (\text{stamps } x) (\text{stamps } y) \rrbracket$ 
       $\implies \text{tryFold } (\text{IntegerEqualsNode } x y) \text{ stamps False} \mid$ 
     $\llbracket \text{neverDistinct } (\text{stamps } x) (\text{stamps } y) \rrbracket$ 

```



```

    ==> tryFold (IntegerEqualsNode x y) stamps True |
  [[is-IntegerStamp (stamps x);
    is-IntegerStamp (stamps y);
    stpi-upper (stamps x) < stpi-lower (stamps y)]]
    ==> tryFold (IntegerLessThanNode x y) stamps True |
  [[is-IntegerStamp (stamps x);
    is-IntegerStamp (stamps y);
    stpi-lower (stamps x) ≥ stpi-upper (stamps y)]]
    ==> tryFold (IntegerLessThanNode x y) stamps False

```

Proofs that show that when the stamp lookup function is well-formed, the tryFold relation correctly predicts the output value with respect to our evaluation semantics.

**lemma**

```

  assumes kind g nid = IntegerEqualsNode x y
  assumes [g, m, p] ⊢ nid ↦ v
  assumes v ≠ UndefVal
  assumes ([g, m, p] ⊢ x ↦ xval) ∧ ([g, m, p] ⊢ y ↦ yval)
  shows val-to-bool (intval-equals xval yval) ↔ v = IntVal32 1

```

**proof** –

```

  have v = intval-equals xval yval
    using assms(1, 2, 3, 4) BinaryExprE IntegerEqualsNode bin-eval.simps(7)
    by (smt (verit) bin-eval.simps(10) encodeeval-def evalDet repDet)
  then show ?thesis using intval-equals.simps val-to-bool.simps sorry

```

**qed**

**lemma** tryFoldIntegerEqualsAlwaysDistinct:

```

  assumes wf-stamp g stamps
  assumes kind g nid = (IntegerEqualsNode x y)
  assumes [g, m, p] ⊢ nid ↦ v
  assumes alwaysDistinct (stamps x) (stamps y)
  shows v = IntVal32 0

```

**proof** –

```

  have ∀ val. ¬(valid-value val (join (stamps x) (stamps y)))
    using assms(1,4) unfolding alwaysDistinct.simps
    by (metis is-stamp-empty.elims(2) le-less-trans not-less valid32or64 valid-value.simps(1)
    valid-value.simps(2))
  have ¬(∃ val . ([g, m, p] ⊢ x ↦ val) ∧ ([g, m, p] ⊢ y ↦ val))
    using assms(1,4) unfolding alwaysDistinct.simps wf-stamp.simps encodeeval-def
    sorry
  then show ?thesis sorry

```

**qed**

**lemma** tryFoldIntegerEqualsNeverDistinct:

```

  assumes wf-stamp g stamps
  assumes kind g nid = (IntegerEqualsNode x y)
  assumes [g, m, p] ⊢ nid ↦ v
  assumes neverDistinct (stamps x) (stamps y)
  shows v = IntVal32 1

```

```

using assms IntegerEqualsNodeE sorry

lemma tryFoldIntegerLessThanTrue:
  assumes wf-stamp g stamps
  assumes kind g nid = (IntegerLessThanNode x y)
  assumes [g, m, p] ⊢ nid ↦ v
  assumes stpi-upper (stamps x) < stpi-lower (stamps y)
  shows v = IntVal32 1
proof -
  have stamp-type: is-IntegerStamp (stamps x)
    using assms
  sorry
  obtain xval where xval: [g, m, p] ⊢ x ↦ xval
    using assms(2,3) sorry
  obtain yval where yval: [g, m, p] ⊢ y ↦ yval
    using assms(2,3) sorry
  have is-IntegerStamp (stamps x) ∧ is-IntegerStamp (stamps y)
    using assms(4)
  sorry
  then have val-to-bool (intval-less-than xval yval)
    sorry
  then show ?thesis
    sorry
qed

lemma tryFoldIntegerLessThanFalse:
  assumes wf-stamp g stamps
  assumes kind g nid = (IntegerLessThanNode x y)
  assumes [g, m, p] ⊢ nid ↦ v
  assumes stpi-lower (stamps x) ≥ stpi-upper (stamps y)
  shows v = IntVal32 0
proof -
  have stamp-type: is-IntegerStamp (stamps x)
    using assms
  sorry
  obtain xval where xval: [g, m, p] ⊢ x ↦ xval
    using assms(2,3) sorry
  obtain yval where yval: [g, m, p] ⊢ y ↦ yval
    using assms(2,3) sorry
  have is-IntegerStamp (stamps x) ∧ is-IntegerStamp (stamps y)
    using assms(4)
  sorry
  then have ¬(val-to-bool (intval-less-than xval yval))
    sorry
  then show ?thesis
    sorry
qed

theorem tryFoldProofTrue:

```

```

    assumes wf-stamp g stamps
    assumes tryFold (kind g nid) stamps True
    assumes  $[g, m, p] \vdash \text{nid} \mapsto v$ 
    shows val-to-bool v
    using assms(2) proof (induction kind g nid stamps True rule: tryFold.induct)
case (1 stamps x y)
  then show ?case using tryFoldIntegerEqualsAlwaysDistinct assms sorry
next
  case (2 stamps x y)
  then show ?case using tryFoldIntegerEqualsAlwaysDistinct assms sorry
next
  case (3 stamps x y)
  then show ?case using tryFoldIntegerLessThanTrue assms sorry
next
  case (4 stamps x y)
  then show ?case using tryFoldIntegerLessThanFalse assms sorry
qed

theorem tryFoldProofFalse:
  assumes wf-stamp g stamps
  assumes tryFold (kind g nid) stamps False
  assumes  $[g, m, p] \vdash \text{nid} \mapsto v$ 
  shows  $\neg(\text{val-to-bool } v)$ 
using assms(2) proof (induction kind g nid stamps False rule: tryFold.induct)
case (1 stamps x y)
  then show ?case using tryFoldIntegerEqualsAlwaysDistinct assms sorry
next
  case (2 stamps x y)
  then show ?case using tryFoldIntegerEqualsNeverDistinct assms sorry
next
  case (3 stamps x y)
  then show ?case using tryFoldIntegerLessThanTrue assms sorry
next
  case (4 stamps x y)
  then show ?case using tryFoldIntegerLessThanFalse assms sorry
qed

```

**inductive-cases** *StepE*:

$$g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h)$$

Perform conditional elimination rewrites on the graph for a particular node. In order to determine conditional eliminations appropriately the rule needs two data structures produced by static analysis. The first parameter is the set of IRNodes that we know result in a true value when evaluated. The second parameter is a mapping from node identifiers to the flow-sensitive stamp.

The relation transforms the third parameter to the fifth parameter for a node identifier which represents the fourth parameter.

**inductive** *ConditionalEliminationStep* ::  
 $IRExpr \Rightarrow set \Rightarrow (ID \Rightarrow Stamp) \Rightarrow IRGraph \Rightarrow ID \Rightarrow IRGraph \Rightarrow bool$  **where**  
*impliesTrue*:  
 $\llbracket kind\ g\ ifcond = (IfNode\ cid\ t\ f);$   
 $g \vdash cid \simeq cond;$   
 $\exists\ ce \in conds . (ce \ \&\ cond \hookrightarrow True);$   
 $g' = constantCondition\ True\ ifcond\ (kind\ g\ ifcond)\ g$   
 $\rrbracket \implies ConditionalEliminationStep\ conds\ stamps\ g\ ifcond\ g' \mid$   
*impliesFalse*:  
 $\llbracket kind\ g\ ifcond = (IfNode\ cid\ t\ f);$   
 $g \vdash cid \simeq cond;$   
 $\exists\ ce \in conds . (ce \ \&\ cond \hookrightarrow False);$   
 $g' = constantCondition\ False\ ifcond\ (kind\ g\ ifcond)\ g$   
 $\rrbracket \implies ConditionalEliminationStep\ conds\ stamps\ g\ ifcond\ g' \mid$   
*tryFoldTrue*:  
 $\llbracket kind\ g\ ifcond = (IfNode\ cid\ t\ f);$   
 $cond = kind\ g\ cid;$   
 $tryFold\ (kind\ g\ cid)\ stamps\ True;$   
 $g' = constantCondition\ True\ ifcond\ (kind\ g\ ifcond)\ g$   
 $\rrbracket \implies ConditionalEliminationStep\ conds\ stamps\ g\ ifcond\ g' \mid$   
*tryFoldFalse*:  
 $\llbracket kind\ g\ ifcond = (IfNode\ cid\ t\ f);$   
 $cond = kind\ g\ cid;$   
 $tryFold\ (kind\ g\ cid)\ stamps\ False;$   
 $g' = constantCondition\ False\ ifcond\ (kind\ g\ ifcond)\ g$   
 $\rrbracket \implies ConditionalEliminationStep\ conds\ stamps\ g\ ifcond\ g' \mid$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow bool$ ) *ConditionalEliminationStep* .

**thm** *ConditionalEliminationStep.equation*

## 11.2 Control-flow Graph Traversal

**type-synonym** *Seen* =  $ID\ set$   
**type-synonym** *Condition* =  $IRNode$   
**type-synonym** *Conditions* =  $Condition\ list$   
**type-synonym** *StampFlow* =  $(ID \Rightarrow Stamp)\ list$

*nextEdge* helps determine which node to traverse next by returning the first successor edge that isn't in the set of already visited nodes. If there is not an appropriate successor, *None* is returned instead.

**fun** *nextEdge* ::  $Seen \Rightarrow ID \Rightarrow IRGraph \Rightarrow ID\ option$  **where**  
*nextEdge seen nid g* =

*(let nids = (filter ( $\lambda nid'. nid' \notin seen$ ) (successors-of (kind g nid))) in  
 (if length nids > 0 then Some (hd nids) else None))*

pred determines which node, if any, acts as the predecessor of another.

Merge nodes represent a special case where-in the predecessor exists as an input edge of the merge node, to simplify the traversal we treat only the first input end node as the predecessor, ignoring that multiple nodes may act as a successor.

For all other nodes, the predecessor is the first element of the predecessors set. Note that in a well-formed graph there should only be one element in the predecessor set.

```
fun pred :: IRGraph ⇒ ID ⇒ ID option where
  pred g nid = (case kind g nid of
    (MergeNode ends -) ⇒ Some (hd ends) |
    - ⇒
      (if IRGraph.predecessors g nid = {}
        then None else
        Some (hd (sorted-list-of-set (IRGraph.predecessors g nid)))
      )
  )
```

When the basic block of an if statement is entered, we know that the condition of the preceding if statement must be true. As in the GraalVM compiler, we introduce the registerNewCondition function which roughly corresponds to the ConditionalEliminationPhase.registerNewCondition. This method updates the flow-sensitive stamp information based on the condition which we know must be true.

```
fun clip-upper :: Stamp ⇒ int ⇒ Stamp where
  clip-upper (IntegerStamp b l h) c = (IntegerStamp b l c) |
  clip-upper s c = s
fun clip-lower :: Stamp ⇒ int ⇒ Stamp where
  clip-lower (IntegerStamp b l h) c = (IntegerStamp b c h) |
  clip-lower s c = s
```

```
fun registerNewCondition :: IRGraph ⇒ Condition ⇒ (ID ⇒ Stamp) ⇒ (ID ⇒ Stamp) where
```

```
  registerNewCondition g (IntegerEqualsNode x y) stamps =
    (stamps(x := join (stamps x) (stamps y)))(y := join (stamps x) (stamps y)) |

  registerNewCondition g (IntegerLessThanNode x y) stamps =
    (stamps
      (x := clip-upper (stamps x) (stpi-lower (stamps y)))
      (y := clip-lower (stamps y) (stpi-upper (stamps x)))
    ) |
  registerNewCondition g - stamps = stamps
```

```
fun hdOr :: 'a list ⇒ 'a ⇒ 'a where
```

$$\begin{aligned} \text{hdOr } (x \# xs) \text{ de} &= x \mid \\ \text{hdOr } [] \text{ de} &= \text{de} \end{aligned}$$

The Step relation is a small-step traversal of the graph which handles transitions between individual nodes of the graph.

It relates a pairs of tuple of the current node, the set of seen nodes, the always true stack of IfNode conditions, and the flow-sensitive stamp information.

**inductive Step**

$$\begin{aligned} &:: \text{IRGraph} \Rightarrow (\text{ID} \times \text{Seen} \times \text{Conditions} \times \text{StampFlow}) \Rightarrow (\text{ID} \times \text{Seen} \times \\ &\text{Conditions} \times \text{StampFlow}) \text{ option} \Rightarrow \text{bool} \end{aligned}$$

**for g where**

— Hit a BeginNode with an IfNode predecessor which represents the start of a basic block for the IfNode. 1. nid' will be the successor of the begin node. 2. Find the first and only predecessor. 3. Extract condition from the preceding IfNode. 4. Negate condition if the begin node is second branch (we've taken the else branch of the condition) 5. Add the condition or the negated condition to stack 6. Perform any stamp updates based on the condition using the registerNewCondition function and place them on the top of the stack of stamp information

$$[[\text{kind } g \text{ nid} = \text{BeginNode nid}';$$

$$\begin{aligned} \text{nid} &\notin \text{seen}; \\ \text{seen}' &= \{\text{nid}\} \cup \text{seen}; \end{aligned}$$

$$\begin{aligned} \text{Some ifcond} &= \text{pred } g \text{ nid}; \\ \text{kind } g \text{ ifcond} &= \text{IfNode cond } t \text{ f}; \end{aligned}$$

$$\begin{aligned} i &= \text{find-index nid (successors-of (kind } g \text{ ifcond))}; \\ c &= (\text{if } i = 0 \text{ then kind } g \text{ cond else LogicNegationNode cond}); \\ \text{conds}' &= c \# \text{ conds}; \end{aligned}$$

$$\begin{aligned} \text{flow}' &= \text{registerNewCondition } g \text{ c (hdOr flow (stamp g))}] \\ \implies \text{Step } g (\text{nid}, \text{seen}, \text{conds}, \text{flow}) (\text{Some (nid}', \text{seen}', \text{conds}', \text{flow}' \# \text{flow})) \mid \end{aligned}$$

— Hit an EndNode 1. nid' will be the usage of EndNode 2. pop the conditions and stamp stack

$$[[\text{kind } g \text{ nid} = \text{EndNode};$$

$$\begin{aligned} \text{nid} &\notin \text{seen}; \\ \text{seen}' &= \{\text{nid}\} \cup \text{seen}; \end{aligned}$$

$$\text{nid}' = \text{any-usage } g \text{ nid};$$

$$\begin{aligned} \text{conds}' &= \text{tl conds}; \\ \text{flow}' &= \text{tl flow}] \end{aligned}$$

$$\implies \text{Step } g (\text{nid}, \text{seen}, \text{conds}, \text{flow}) (\text{Some (nid}', \text{seen}', \text{conds}', \text{flow}')) \mid$$

— We can find a successor edge that is not in seen, go there

$$\begin{aligned} &[[\neg(\text{is-EndNode (kind } g \text{ nid)}); \\ &\neg(\text{is-BeginNode (kind } g \text{ nid)}); \end{aligned}$$

$nid \notin seen;$   
 $seen' = \{nid\} \cup seen;$

$Some\ nid' = nextEdge\ seen'\ nid\ g$   
 $\implies Step\ g\ (nid, seen, conds, flow)\ (Some\ (nid', seen', conds, flow))\ |$

— We cannot find a successor edge that is not in seen, give back None  
 $\llbracket \neg(is-EndNode\ (kind\ g\ nid)) \rrbracket;$   
 $\neg(is-BeginNode\ (kind\ g\ nid));$

$nid \notin seen;$   
 $seen' = \{nid\} \cup seen;$

$None = nextEdge\ seen'\ nid\ g$   
 $\implies Step\ g\ (nid, seen, conds, flow)\ None\ |$

— We've already seen this node, give back None  
 $\llbracket nid \in seen \rrbracket \implies Step\ g\ (nid, seen, conds, flow)\ None$

**code-pred** (*modes: i  $\Rightarrow$  i  $\Rightarrow$  o  $\Rightarrow$  bool*) *Step* .

The ConditionalEliminationPhase relation is responsible for combining the individual traversal steps from the Step relation and the optimizations from the ConditionalEliminationStep relation to perform a transformation of the whole graph.

**end**