GraalVM Stamp Theory

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Abstract

The GraalVM compiler uses stamps to track type and range information during program analysis. Type information is recorded by using distinct subclasses of the abstract base class `Stamp`, i.e. `IntegerStamp` is used to represent an integer type. Each subclass introduces facilities for tracking range information. Every subclass of the `Stamp` class forms a lattice, together with an arbitrary top and bottom element each sublattice forms a lattice of all stamps. This Isabelle/HOL theory models stamps as instantiations of a lattice.

Contents

1 Stamps: Type and Range Information

theory *StampLattice*
imports
  Values
  HOL.Lattices
begin

1.1 Void Stamp

The VoidStamp represents a type with no associated values. The VoidStamp lattice is therefore a simple single element lattice.

datatype *void* =
  VoidStamp

instantiation *void* :: *order*
begin

definition *less-eq-void* :: *void* ⇒ *void* ⇒ *bool* **where**
  less-eq-void a b = *True*

definition *less-void* :: *void* ⇒ *void* ⇒ *bool* **where**
  less-void a b = *False*

instance
  apply *standard*
    apply (*simp add*: *less-eq-void-def less-void-def*)
   apply (*simp add*: *less-eq-void-def*)
  apply (*simp add*: *less-eq-void-def*)
  by (*metis* (*full-types*) *void.exhaust*)

end

instantiation *void* :: *semilattice-inf*
begin

definition *inf-void* :: *void* ⇒ *void* ⇒ *void* **where**
  inf-void a b = *VoidStamp*

instance
  apply *standard*
   apply (*simp add*: *less-eq-void-def*)
  apply (*simp add*: *less-eq-void-def*)
  by (*metis* (*mono-tags*) *void.exhaust*)

end

instantiation *void* :: *semilattice-sup*
begin

definition *sup-void* :: *void* ⇒ *void* ⇒ *void* **where**
sup-void a b = *VoidStamp*

instance
apply *standard*
apply (*simp add*: *less-eq-void-def*)
apply (*simp add*: *less-eq-void-def*)
by (*metis* (*mono-tags*) *void.exhaust*)

end

instantiation *void* :: *bounded-lattice*
begin

definition *bot-void* :: *void* **where**
bot-void = *VoidStamp*

definition *top-void* :: *void* **where**
top-void = *VoidStamp*

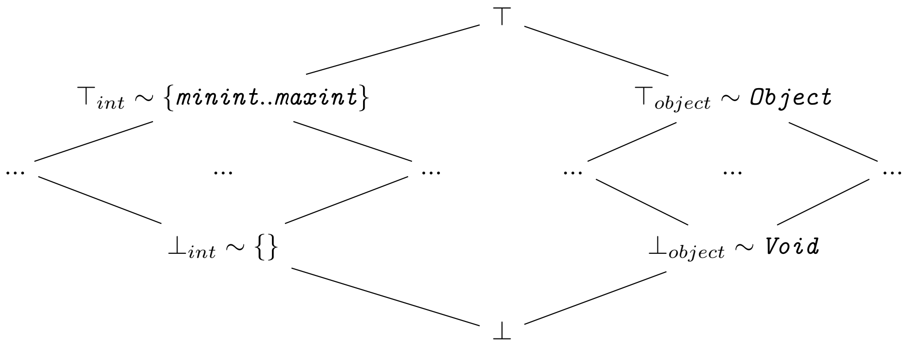
instance
apply *standard*
apply (*simp add*: *less-eq-void-def*)
by (*simp add*: *less-eq-void-def*)

end

Definition of the stamp type

datatype *stamp* =
intstamp int64 int64 — Type: Integer; Range: Lower Bound & Upper Bound

1.2 Stamp Lattice



1.2.1 Stamp Order

Defines an ordering on the stamp type.

One stamp is less than another if the valid values for the stamp are a strict subset of the other stamp.

instantiation *stamp* :: *order*
begin

fun *less-eq-stamp* :: *stamp* ⇒ *stamp* ⇒ *bool* **where**
  less-eq-stamp (*intstamp l1 u1*) (*intstamp l2 u2*) = ({*l1..u1*} ⊆ {*l2..u2*})

fun *less-stamp* :: *stamp* ⇒ *stamp* ⇒ *bool* **where**
  less-stamp (*intstamp l1 u1*) (*intstamp l2 u2*) = ({*l1..u1*} ⊂ {*l2..u2*})

lemma *less-le-not-le*:
  fixes *x y* :: *stamp*
  shows $(x < y) = (x \leq y \wedge \neg\ y \leq x)$
  using *less-eq-stamp.simps less-stamp.simps*
  using *stamp.exhaust subset-not-subset-eq* **by** *metis*

lemma *order-refl*:
  fixes *x* :: *stamp*
  shows $x \leq x$
  using *less-eq-stamp.simps less-stamp.simps*
  using *dual-order.refl stamp.exhaust* **by** *metis*

lemma *order-trans*:
  fixes *x y z* :: *stamp*
  shows $x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$
proof –
  fix *x* :: *stamp* **and** *y* :: *stamp* **and** *z* :: *stamp*
  assume $x \leq y$
  assume $y \leq z$
  obtain *l1 u1* **where** *xdef*: *x* = *intstamp l1 u1*
    using *stamp.exhaust*
    by *blast*
  obtain *l2 u2* **where** *ydef*: *y* = *intstamp l2 u2*
    using *stamp.exhaust*
    by *blast*
  obtain *l3 u3* **where** *zdef*: *z* = *intstamp l3 u3*
    using *stamp.exhaust*
    by *blast*
  have *s1*: {*l1..u1*} ≤ {*l2..u2*}
    using ‹$x \leq y$› *less-eq-stamp.simps xdef ydef* **by** *blast*
  have *s2*: {*l2..u2*} ≤ {*l3..u3*}
    using ‹$y \leq z$› *less-eq-stamp.simps ydef zdef* **by** *blast*
  from *s1 s2* **have** {*l1..u1*} ≤ {*l3..u3*}
    by (*meson dual-order.trans*)
  then show $x \leq z$

```
    using less-eq-stamp.simps
    using xdef zdef by presburger
qed

lemma antisym:
  fixes x y :: stamp
  shows x ≤ y ⟹ y ≤ x ⟹ x = y
proof –
  fix x :: stamp
  fix y :: stamp
  assume xlessy: x ≤ y
  assume ylessx: y ≤ x
  obtain l1 u1 where xdef: x = intstamp l1 u1
    using stamp.exhaust by blast
  obtain l2 u2 where ydef: y = intstamp l2 u2
    using stamp.exhaust by blast

  from xlessy have s1: {l1..u1} ⊆ {l2..u2}
    using less-eq-stamp.simps
    using xdef ydef by blast
  from ylessx have s2: {l2..u2} ⊆ {l1..u1}
    using less-eq-stamp.simps
    using xdef ydef by blast
  have {l1..u1} ⊆ {l2..u2} ⟹ {l2..u2} ⊆ {l1..u1} ⟹ {l1..u1} = {l2..u2}
    by fastforce
  then have s3: {l1..u1} = {l2..u2} ⟹ (l1 = l2) ∧ (u1 = u2)


    sorry
  then have (l1 = l2) ∧ (u1 = u2) ⟹ x = y
    using xdef ydef by fastforce
  then show x = y
    using s1 s2 s3 by fastforce
qed

instance
  apply standard
  using less-le-not-le apply simp
  using order-refl apply simp
  using order-trans apply simp
  using antisym by simp
end
```

1.2.2 Stamp Join

Defines the *join* operation for stamps.

For any two stamps, the *join* is defined as the intersection of the valid values for the stamp.

instantiation *stamp* :: *semilattice-inf*
begin

notation *inf* (**infix** ⊓ *65*)

fun *inf-stamp* :: *stamp* ⇒ *stamp* ⇒ *stamp* **where**
  inf-stamp (*intstamp l1 u1*) (*intstamp l2 u2*) = *intstamp* (*max l1 l2*) (*min u1 u2*)

lemma *inf-le1*:
  fixes *x y* :: *stamp*
  shows (*x* ⊓ *y*) ≤ *x*
proof −
  fix *x* :: *stamp*
  fix *y* :: *stamp*
  obtain *l1 u1* **where** *xdef*: *x* = *intstamp l1 u1*
    using *stamp.exhaust* **by** *blast*
  obtain *l2 u2* **where** *ydef*: *y* = *intstamp l2 u2*
    using *stamp.exhaust* **by** *blast*
  have *joindef*: *x* ⊓ *y* = *intstamp* (*max l1 l2*) (*min u1 u2*)
    (**is** *?join* = *intstamp ?l3 ?u3*)
    using *inf-stamp.simps xdef ydef*
    by *force*
  have *leq*: {*?l3..?u3*} ⊆ {*l1..u1*}
    by *force*
  have (*x* ⊓ *y*) ≤ *x* = ({*?l3..?u3*} ⊆ {*l1..u1*})
    using *xdef joindef inf-stamp.simps*
    by *force*
  then show (*x* ⊓ *y*) ≤ *x*
    using *leq*
    by *fastforce*
qed

lemma *inf-le2*:
  fixes *x y* :: *stamp*
  shows (*x* ⊓ *y*) ≤ *y*
proof −
  fix *x* :: *stamp*
  fix *y* :: *stamp*
  obtain *l1 u1* **where** *xdef*: *x* = *intstamp l1 u1*
    using *stamp.exhaust* **by** *blast*
  obtain *l2 u2* **where** *ydef*: *y* = *intstamp l2 u2*
    using *stamp.exhaust* **by** *blast*
  have *joindef*: *x* ⊓ *y* = *intstamp* (*max l1 l2*) (*min u1 u2*)
    (**is** *?join* = *intstamp ?l3 ?u3*)
    using *inf-stamp.simps xdef ydef*
    by *force*
  have *leq*: {*?l3..?u3*} ⊆ {*l2..u2*}
    by *force*
  have (*x* ⊓ *y*) ≤ *y* = ({*?l3..?u3*} ⊆ {*l2..u2*})

```
    using ydef joindef
    by force
  then show (x ⊓ y) ≤ y
    using leq
    by fastforce
qed

lemma inf-greatest:
  fixes x y z :: stamp
  shows x ≤ y ⟹ x ≤ z ⟹ x ≤ (y ⊓ z)
proof –
  fix x y z :: stamp
  assume xlessy: x ≤ y
  assume xlessz: x ≤ z
  obtain l1 u1 where xdef: x = intstamp l1 u1
    using stamp.exhaust by blast
  obtain l2 u2 where ydef: y = intstamp l2 u2
    using stamp.exhaust by blast
  obtain l3 u3 where zdef: z = intstamp l3 u3
    using stamp.exhaust by blast
  obtain l4 u4 where yzdef: y ⊓ z = intstamp l4 u4
    by (meson inf-stamp.elims)
  have max4: l4 = max l2 l3
    using yzdef ydef zdef inf-stamp.simps by simp
  have min4: u4 = min u2 u3
    using yzdef ydef zdef inf-stamp.simps by simp
  have {l1..u1} ⊆ {l2..u2}
    using xlessy xdef ydef
    using less-eq-stamp.simps by blast
  have {l1..u1} ⊆ {l3..u3}
    using xlessz xdef zdef
    using less-eq-stamp.simps by blast
  have leq: {l1..u1} ⊆ {l4..u4}
    using ‹{l1..u1} ⊆ {l2..u2}› ‹{l1..u1} ⊆ {l3..u3}› max4 min4 by auto
  have x ≤ (y ⊓ z) = ({l1..u1} ⊆ {l4..u4})
    by (simp add: xdef yzdef)
  then show x ≤ (y ⊓ z)
    using leq
    by fastforce
qed

instance
  apply standard
  using inf-le1 apply simp
  using inf-le2 apply simp
  using inf-greatest by simp
end
```

1.2.3 Stamp Meet

Defines the *meet* operation for stamps.

For any two stamps, the *meet* is defined as the union of the valid values for the stamp.

instantiation *stamp* :: *semilattice-sup*
begin

notation *sup* (**infix** ⊔ *65*)

fun *sup-stamp* :: *stamp* ⇒ *stamp* ⇒ *stamp* **where**
  sup-stamp (*intstamp l1 u1*) (*intstamp l2 u2*) = *intstamp* (*min l1 l2*) (*max u1 u2*)

lemma *sup-ge1*:
  fixes *x y* :: *stamp*
  shows *x* ≤ *x* ⊔ *y*
proof −
  fix *x* :: *stamp*
  fix *y* :: *stamp*
  obtain *l1 u1* **where** *xdef*: *x* = *intstamp l1 u1*
    using *stamp.exhaust* **by** *blast*
  obtain *l2 u2* **where** *ydef*: *y* = *intstamp l2 u2*
    using *stamp.exhaust* **by** *blast*
  have *joindef*: *x* ⊔ *y* = *intstamp* (*min l1 l2*) (*max u1 u2*)
    (**is** *?join* = *intstamp ?l3 ?u3*)
    using *inf-stamp.simps xdef ydef*
    by *force*
  have *leq*: {*l1..u1*} ⊆ {*?l3..?u3*}
    by *simp*
  have *x* ≤ *x* ⊔ *y* = ({*l1..u1*} ⊆ {*?l3..?u3*})
    using *xdef joindef inf-stamp.simps*
    by *force*
  then show *x* ≤ *x* ⊔ *y*
    using *leq*
    by *fastforce*
qed

lemma *sup-ge2*:
  fixes *x y* :: *stamp*
  shows *y* ≤ *x* ⊔ *y*
proof −
  fix *x* :: *stamp*
  fix *y* :: *stamp*
  obtain *l1 u1* **where** *xdef*: *x* = *intstamp l1 u1*
    using *stamp.exhaust* **by** *blast*
  obtain *l2 u2* **where** *ydef*: *y* = *intstamp l2 u2*
    using *stamp.exhaust* **by** *blast*
  have *joindef*: *x* ⊔ *y* = *intstamp* (*min l1 l2*) (*max u1 u2*)
    (**is** *?join* = *intstamp ?l3 ?u3*)

using *inf-stamp.simps xdef ydef*
by *force*
have *leq*: $\{l2..u2\} \subseteq \{?l3..?u3\}$ (**is** *?subset-thesis*)
by *simp*
have *?thesis* = (*?subset-thesis*)
using *ydef joindef sup-stamp.simps less-eq-stamp.simps*
by (*metis StampLattice.sup-ge1 max.commute min.commute sup-stamp.elims*)
then show *?thesis*
using *leq*
by *fastforce*
qed

lemma *sup-least*:
fixes *x y z* :: *stamp*
shows $y \leq x \Longrightarrow z \leq x \Longrightarrow ((y \sqcup z) \leq x)$
proof −
fix *x y z* :: *stamp*
assume *xlessy*: $y \leq x$
assume *xlessz*: $z \leq x$
obtain *l1 u1* **where** *xdef*: *x* = *intstamp l1 u1*
using *stamp.exhaust* **by** *blast*
obtain *l2 u2* **where** *ydef*: *y* = *intstamp l2 u2*
using *stamp.exhaust* **by** *blast*
obtain *l3 u3* **where** *zdef*: *z* = *intstamp l3 u3*
using *stamp.exhaust* **by** *blast*
have *yzdef*: $y \sqcup z$ = *intstamp* (*min l2 l3*) (*max u2 u3*)
(**is** *?meet* = *intstamp ?l4 ?u4*)
using *sup-stamp.simps*
by (*simp add*: *ydef zdef*)
have *s1*: $\{l2..u2\} \subseteq \{l1..u1\}$
using *xlessy xdef ydef*
using *less-eq-stamp.simps* **by** *blast*
have *s2*: $\{l3..u3\} \subseteq \{l1..u1\}$
using *xlessz xdef zdef*
using *less-eq-stamp.simps* **by** *blast*
have *leq*: $\{?l4..?u4\} \subseteq \{l1..u1\}$ (**is** *?subset-thesis*)
using *s1 s2* **unfolding** *atLeastatMost-subset-iff*

by (*metis* (*no-types*, *opaque-lifting*) *inf.orderE inf-stamp.simps max.bounded-iff max.cobounded2 min.bounded-iff min.cobounded2 stamp.inject xdef xlessy xlessz ydef zdef*)
have $(y \sqcup z \leq x)$ = *?subset-thesis*
using *yzdef xdef less-eq-stamp.simps*
by *simp*
then show $(y \sqcup z \leq x)$
using *leq* **by** *fastforce*
qed

instance

apply *standard*
using *sup-ge1* **apply** *simp*
using *sup-ge2* **apply** *simp*
using *sup-least* **by** *simp*
end

1.2.4 Stamp Bounds

Defines the top and bottom elements of the stamp lattice.

This poses an interesting question as our stamp type is a union of the various *Stamp* subclasses, e.g. *IntegerStamp*, *ObjectStamp*, etc.

Each subclass should preferably have its own unique top and bottom element, i.e. An *IntegerStamp* would have the top element of the full range of integers allowed by the bit width and a bottom of a range with no integers. While the *ObjectStamp* should have *Object* as the top and *Void* as the bottom element.

instantiation *stamp* :: *bounded-lattice*
begin

notation *bot* ($\perp$ *50*)
notation *top* ($\top$ *50*)

definition *width-min* :: *nat* $\Rightarrow$ *int64* **where**
$\quad$ *width-min bits* $= -(2\widehat{\ }(bits-1))$

definition *width-max* :: *nat* $\Rightarrow$ *int64* **where**
$\quad$ *width-max bits* $= (2\widehat{\ }(bits-1)) - 1$

value (*sint* (*width-min 64*), *sint* (*width-max 64*))
value *max-word*::*int64*

lemma
$\quad$ **assumes** $x =$ *width-min 64*
$\quad$ **assumes** $y =$ *width-max 64*
$\quad$ **shows** *sint* $x <$ *sint* y
$\quad$ **using** *assms* **unfolding** *width-min-def width-max-def* **by** *simp*

Note that this definition is valid for unsigned integers only.

The bottom and top element for signed integers would be (- 9223372036854775808, 9223372036854775807).

For unsigned we have (0, 18446744073709551615).

For Java we are likely to be more concerned with signed integers. To use the appropriate bottom and top for signed integers we would need to change our definition of less_eq from l1..u1 <= l2..u2 to sint l1..sint u1 <= sint l2..sint u2

We may still find an unsigned integer stamp useful. I plan to investigate the Java code to see if this is useful and then apply the changes to switch to signed integers.

definition *bot-stamp* = *intstamp* (−*1*) *0*
definition *top-stamp* = *intstamp* *0* (−*1*)

lemma *bot-least*:
  fixes *a* :: *stamp*
  shows (⊥) ≤ *a*
proof −
  obtain *min max* **where** *bot-def*:⊥ = *intstamp max min*
    using *bot-stamp-def*
    by *force*
  have *min* < *max*
    using *bot-def*
    unfolding *bot-stamp-def width-min-def width-max-def*
    using *word-gt-0* **by** *fastforce*
  then have {*max..min*} = {}
    using *bot-def*
    unfolding *bot-stamp-def width-min-def width-max-def*
    by *auto*
  then show *?thesis*
    unfolding *bot-stamp-def*
    using *less-eq-stamp.simps*
    by (*simp add*: *stamp.induct*)
qed

lemma *top-greatest*:
  fixes *a* :: *stamp*
  shows *a* ≤ (⊤)
proof −
  obtain *min max* **where** *top-def*:⊤ = *intstamp min max*
    using *top-stamp-def*
    by *force*
  have *max-is-max*: ¬(∃ *n*. *n* > *max*)
    by (*metis stamp.inject top-def top-stamp-def word-order.extremum-strict*)
  have *min-is-min*: ¬(∃ *n*. *n* < *min*)
   by (*metis not-less-iff-gr-or-eq stamp.inject top-def top-stamp-def word-coorder.not-eq-extremum*)
  have ¬(∃ *l u*. {*min..max*} < {*l..u*})
    using *max-is-max min-is-min*
    by (*metis atLeastatMost-psubset-iff not-less*)
  then show *?thesis*
    unfolding *top-stamp-def*
    using *less-eq-stamp.simps*
    using *less-eq-stamp.elims*(*3*) **by** *fastforce*
qed

instance
  apply *standard*
  using *bot-least* **apply** *simp*

using *top-greatest* **by** *simp*
end

1.3 Java Stamp Methods

The following are methods from the Java Stamp class, they are the methods primarily used for optimizations.

definition *is-unrestricted* :: *stamp* ⇒ *bool* **where**
is-unrestricted s = (⊤ = *s*)

fun *is-empty* :: *stamp* ⇒ *bool* **where**
is-empty s = (⊥ = *s*)

fun *as-constant* :: *stamp* ⇒ *Value option* **where**
as-constant (*intstamp l u*) = (*if* (*card* {*l*..*u*}) = *1*
then Some (*IntVal64* (*SOME x*. *x* ∈ {*l*..*u*}))
else None)

definition *always-distinct* :: *stamp* ⇒ *stamp* ⇒ *bool* **where**
always-distinct stamp1 stamp2 = (⊥ = (*stamp1* ⊓ *stamp2*))

definition *never-distinct* :: *stamp* ⇒ *stamp* ⇒ *bool* **where**
never-distinct stamp1 stamp2 =
(*as-constant stamp1* = *as-constant stamp2* ∧ *as-constant stamp1* ≠ *None*)

1.4 Mapping to Values

fun *valid-value* :: *stamp* => *Value* => *bool* **where**
valid-value (*intstamp l u*) (*IntVal64 v*) = (*v* ∈ {*l*..*u*}) |
valid-value (*intstamp l u*) - = *False*

The *valid-value* function is used to map a stamp instance to the values that are allowed by the stamp.

It would be nice if there was a slightly more integrated way to perform this mapping as it requires some infrastructure to prove some fairly simple properties.

lemma *bottom-range-empty*:
¬(*valid-value* (⊥) *v*)
unfolding *bot-stamp-def*
using *valid-value.elims*(*2*) **by** *fastforce*

lemma *join-values*:
assumes *joined* = *x-stamp* ⊓ *y-stamp*
shows *valid-value joined x* ⟷ (*valid-value x-stamp x* ∧ *valid-value y-stamp x*)
proof (*cases x*)
case *UndefVal*
then show *?thesis*
using *valid-value.elims*(*2*) **by** *blast*

next
  case (*IntVal32 x2*)
  then show *?thesis*
    using *valid-value.elims*(*2*) **by** *blast*
next
  case (*IntVal64 x3*)
  obtain *lx ux* **where** *xdef*: *x-stamp* = *intstamp lx ux*
    using *stamp.exhaust* **by** *blast*
  obtain *ly uy* **where** *ydef*: *y-stamp* = *intstamp ly uy*
    using *stamp.exhaust* **by** *blast*
  obtain *v* **where** *x* = *IntVal64 v*
    using *IntVal64* **by** *blast*
  have *joined* = *intstamp* (*max lx ly*) (*min ux uy*)
    (**is** *joined* = *intstamp ?lj ?uj*)
    by (*simp add*: *xdef ydef assms*)
  then have *valid-value joined* (*IntVal64 v*) = (*v* ∈ {*?lj*..*?uj*})
    by *simp*
  then show *?thesis*
    using ‹*x* = *IntVal64 v*› *xdef ydef* **by** *force*
next
  case (*ObjRef x5*)
  then show *?thesis*
    using *valid-value.elims*(*2*) **by** *blast*
next
  case (*ObjStr x6*)
  then show *?thesis*
    using *valid-value.elims*(*2*) **by** *blast*
qed

lemma *disjoint-empty*:
  fixes *x-stamp y-stamp* :: *stamp*
  assumes ⊥ = *x-stamp* ⊓ *y-stamp*
  shows ¬(*valid-value x-stamp x* ∧ *valid-value y-stamp x*)
  using *assms bottom-range-empty join-values*
  by *blast*

experiment begin

A possible equivalent alternative to the definition of less_eq

fun *less-eq-alt* :: *'a*::*ord* × *'a* ⇒ *'a* × *'a* ⇒ *bool* **where**
  less-eq-alt (*l1*, *u1*) (*l2*, *u2*) = ((¬ *l1* ≤ *u1*) ∨ *l2* ≤ *l1* ∧ *u1* ≤ *u2*)

Proof equivalence

lemma
  fixes *l1 l2 u1 u2* :: *int*
  assumes *l1* ≤ *u1* ∧ *l2* ≤ *u2*
  shows {*l1*..*u1*} ⊆ {*l2*..*u2*} = ((*l1* ≥ *l2*) ∧ (*u1* ≤ *u2*))
  by (*simp add*: *assms*)

lemma
  fixes $l1\ l2\ u1\ u2 :: int$
  shows $\{l1..u1\} \subseteq \{l2..u2\} = less\text{-}eq\text{-}alt\ (l1,\ u1)\ (l2,\ u2)$
  by *simp*
end

1.5 Generic Integer Stamp

Experimental definition of integer stamps generically, restricting the datatype to only allow valid ranges and the bottom integer element (max_int..min_int).

lemma
  assumes $(x::int) > 0$
  shows $(2\ \hat{}\ x)/2 = (2\ \hat{}\ (x - 1))$
  sorry

definition $max\text{-}signed\text{-}int :: 'a::len\ word$ **where**
  $max\text{-}signed\text{-}int = (2\ \hat{}\ (LENGTH('a) - 1)) - 1$

definition $min\text{-}signed\text{-}int :: 'a::len\ word$ **where**
  $min\text{-}signed\text{-}int = -(2\ \hat{}\ (LENGTH('a) - 1))$

definition $int\text{-}bottom :: 'a::len\ word \times 'a\ word$ **where**
  $int\text{-}bottom = (max\text{-}signed\text{-}int,\ min\text{-}signed\text{-}int)$

definition $int\text{-}top :: 'a::len\ word \times 'a\ word$ **where**
  $int\text{-}top = (min\text{-}signed\text{-}int,\ max\text{-}signed\text{-}int)$

lemma
  fixes $x :: 'a::len\ word$
  shows $sint\ x \leq sint\ (((2\ \hat{}\ (LENGTH('a) - 1)) - 1)::'a\ word)$
  using *sint-greater-eq* **sorry**

value $sint\ (0::1\ word)$
value $sint\ (1::1\ word)$
value $sint\ (((2\ \hat{}\ 0) - 1)::1\ word)$

value $sint\ (((2\ \hat{}\ 31) - 1)::32\ word)$

lemma *max-signed*:
  fixes $a :: 'a::len\ word$
  shows $sint\ a \leq sint\ (max\text{-}signed\text{-}int::'a\ word)$
proof $(cases\ sint\ a = sint\ (max\text{-}signed\text{-}int::'a\ word))$
  case *True*
  then show *?thesis* **by** *simp*
next

```
  case False
  have sint a < sint (max-signed-int::'a word)
    using False unfolding max-signed-int-def sorry
  then show ?thesis by simp
qed

lemma min-signed:
  fixes a :: 'a::len word
  shows sint a ≥ sint (min-signed-int::'a word)
  sorry

value max-signed-int :: 32 word
value int-bottom::(32 word × 32 word)
value sint (2147483647::32 word)
value sint (2147483648::32 word)



typedef (overloaded) ('a::len) intstamp =
  {bounds :: ('a word, 'a word) prod . ((fst bounds) ≤s (snd bounds) ∨ bounds =
int-bottom)}
proof −
  show ?thesis
    by (smt (z3) mem-Collect-eq prod.sel(1) prod.sel(2) signed-minus-1 sint-0)
qed

setup-lifting type-definition-intstamp

lift-definition lower :: ('a::len) intstamp ⇒ 'a word
  is prod.fst ∘ Rep-intstamp .

lift-definition upper :: ('a::len) intstamp ⇒ 'a word
  is prod.snd ∘ Rep-intstamp .

lift-definition lower-int :: ('a::len) intstamp ⇒ int
  is sint ∘ prod.fst .

lift-definition upper-int :: ('a::len) intstamp ⇒ int
  is sint ∘ prod.snd .

lift-definition range :: ('a::len) intstamp ⇒ int set
  is λ (l, u). {sint l..sint u} .

lift-definition bounds :: ('a::len) intstamp ⇒ ('a word × 'a word)
  is Rep-intstamp .

lift-definition is-bottom :: ('a::len) intstamp ⇒ bool
  is λ x. x = int-bottom .
```

```
lift-definition from-bounds :: ('a::len word × 'a word) ⇒ 'a intstamp
  is Abs-intstamp .


instantiation intstamp :: (len) order
begin

definition less-eq-intstamp :: 'a intstamp ⇒ 'a intstamp ⇒ bool where
  less-eq-intstamp s1 s2 = (range s1 ⊆ range s2)

definition less-intstamp :: 'a intstamp ⇒ 'a intstamp ⇒ bool where
  less-intstamp s1 s2 = (range s1 ⊂ range s2)


value int-bottom::(1 word × 1 word)
value sint (0::1 word)
value sint (1::1 word)

value int-bottom::(2 word × 2 word)
value sint (1::2 word)
value sint (2::2 word)
value sint ((2 ^(LENGTH(32) − 1) − 1)::32 word) > sint ((− (2 ^(LENGTH(32)
− 1)))::32 word)

lemma bottom-is-bottom:
  assumes is-bottom s
  shows s ≤ a
proof −
  have boundsdef: bounds s = int-bottom
    by (metis assms bounds.transfer is-bottom.rep-eq)
  obtain min max where bounds s = (max, min)
    by fastforce
  then have max ≠ min
   by (metis boundsdef dual-order.eq-iff fst-conv int-bottom-def less-minus-one-simps(1)
max-signed min-signed not-less sint-0 sint-n1 snd-conv)
  then have sint min < sint max
    unfolding boundsdef int-bottom-def
    using max-signed
   by (metis ‹bounds s = (max, min)› boundsdef int-bottom-def order.not-eq-order-implies-strict
prod.sel(1) signed-word-eqI)
  then have range s = {}
    unfolding range-def bounds-def
    by (simp add: ‹bounds s = (max, min)› bounds.transfer)
  then show ?thesis
    by (simp add: StampLattice.less-eq-intstamp-def)
qed

lemma bounds-has-value:
  fixes x y :: int
```

assumes $x < y$
shows $card\ \{x..y\} > 0$
using *assms* **by** *auto*

lemma *bounds-has-no-value*:
fixes $x\ y :: int$
assumes $x < y$
shows $card\ \{y..x\} = 0$
using *assms* **by** *auto*

lemma *bottom-unique*:
fixes $a\ s :: 'a\ intstamp$
assumes *is-bottom s*
shows $a \leq s \longleftrightarrow is\text{-}bottom\ a$
proof −
have $\forall x.\ sint\ (fst\ (bounds\ x)) \leq sint\ (snd\ (bounds\ x)) \lor is\text{-}bottom\ x$
unfolding *bounds-def is-bottom-def*
using *Rep-intstamp*
using *word-sle-eq* **by** *auto*
then have $\forall x.\ (card\ (range\ x)) > 0 \lor is\text{-}bottom\ x$
unfolding *range-def* **using** *bounds-has-value*
by (*simp add*: *bounds.transfer case-prod-beta*)
obtain *min max* **where** *boundsdef*: $bounds\ s = (max, min)$
by *fastforce*
have *nooverlap*: $sint\ min < sint\ max$
using *max-signed*
by (*metis assms bounds.transfer boundsdef fst-conv int-bottom-def is-bottom.rep-eq min-signed order.not-eq-order-implies-strict signed-word-eqI sint-0 snd-conv verit-la-disequality zero-neq-one*)
have $range\ s = \{sint\ max..sint\ min\}$
by (*simp add*: *bounds.transfer boundsdef range.rep-eq*)
then have $card\ (range\ s) = 0$
using *nooverlap bounds-has-no-value* **by** *simp*
then have $\forall x.\ (card\ (range\ x)) > 0 \longrightarrow s < x$
using ‹$StampLattice.range\ s = \{sint\ max..sint\ min\}$› *atLeastatMost-empty less-intstamp-def* **by** *auto*
then show *?thesis*
by (*meson* ‹$\forall x.\ 0 < card\ (StampLattice.range\ x) \lor is\text{-}bottom\ x$› *bottom-is-bottom leD less-eq-intstamp-def less-intstamp-def*)
qed

lemma *bottom-antisym*:
assumes *is-bottom x*
shows $x \leq y \Longrightarrow y \leq x \Longrightarrow x = y$
using *assms* **proof** (*cases is-bottom y*)
case *True*
then show *?thesis*
by (*metis Rep-intstamp-inverse assms is-bottom.rep-eq*)

```
next
  case False
  assume y ≤ x
  have ¬(y ≤ x)
    using bottom-unique False assms
    by simp
  then show ?thesis
    using ‹y ≤ x› by auto
qed

lemma int-antisym:
  fixes x y :: 'a intstamp
  shows x ≤ y ⟹ y ≤ x ⟹ x = y
proof −
  fix x :: 'a intstamp
  fix y :: 'a intstamp
  assume xlessy: x ≤ y
  assume ylessx: y ≤ x
  obtain l1 u1 where xdef: bounds x = (l1, u1)
    by fastforce
  obtain l2 u2 where ydef: bounds y = (l2, u2)
    by fastforce

  from xlessy have s1: {sint l1..sint u1} ⊆ {sint l2..sint u2} (is ?xlessy)
    using xdef ydef unfolding bounds-def range-def less-eq-intstamp-def
    by simp
  from ylessx have s2: {sint l2..sint u2} ⊆ {sint l1..sint u1} (is ?ylessx)
    using xdef ydef unfolding bounds-def range-def less-eq-intstamp-def
    by simp
  show x = y proof (cases is-bottom x)
    case True
    then show ?thesis using bottom-antisym xlessy ylessx
      by simp
  next
    case False
    then show ?thesis sorry
  qed
qed

instance
  apply standard
     apply (simp add: less-eq-intstamp-def less-intstamp-def less-le-not-le)
  apply blast
  using less-eq-intstamp-def apply force
  using less-eq-intstamp-def apply force
  by (simp add: int-antisym)
end

value take-bit LENGTH(63) 20::int
```

value *take-bit LENGTH*(*63*) ((−*20*)::*int*)
value *bit* (*20*::*int64*) (*63*::*nat*)
value *bit* ((−*20*)::*int64*) (*63*::*nat*)

value ((−*20*)::*int64*) < (*20*::*int64*)

value *take-bit LENGTH*(*63*) ((−*20*)::*int*)

lift-definition *smax* :: *'a*::*len word* ⇒ *'a word* ⇒ *'a word*
  is λ *a b*. (*if* (*sint a*) ≤ (*sint b*) *then b else a*) **.**

lift-definition *smin* :: *'a*::*len word* ⇒ *'a word* ⇒ *'a word*
  is λ *a b*. (*if* (*sint a*) ≤ (*sint b*) *then a else b*) **.**

instantiation *intstamp* :: (*len*) *semilattice-inf*
begin

notation *inf* (**infix** ⊓ *65*)

definition *join-bounds* :: *'a intstamp* ⇒ *'a intstamp* ⇒ (*'a word* × *'a word*) **where**
  join-bounds s1 s2 = (*smax* (*lower s1*) (*lower s2*), *smin* (*upper s1*) (*upper s2*))

definition *join-or-bottom* :: *'a intstamp* ⇒ *'a intstamp* ⇒ (*'a word* × *'a word*) **where**
  join-or-bottom s1 s2 = (*let bound* = (*join-bounds s1 s2*) *in*
    if sint (*fst bound*) ≥ *sint* (*snd bound*) *then int-bottom else bound*)

definition *inf-intstamp* :: *'a intstamp* ⇒ *'a intstamp* ⇒ *'a intstamp* **where**
  inf-intstamp s1 s2 = *from-bounds* (*join-or-bottom s1 s2*)

lemma *always-valid*:
  fixes *s1 s2* :: *'a intstamp*
  shows *Rep-intstamp* (*from-bounds* (*join-or-bottom s1 s2*)) = *join-or-bottom s1 s2*
  unfolding *join-or-bottom-def join-bounds-def from-bounds-def*
  using *Abs-intstamp-inverse*
  by (*smt* (*z3*) *from-bounds.transfer from-bounds-def mem-Collect-eq word-sle-eq*)

lemma *invalid-join*:
  fixes *s1 s2* :: *'a intstamp*
  assumes *bound* = *join-bounds s1 s2*
  assumes *sint* (*fst bound*) ≥ *sint* (*snd bound*)
  shows *from-bounds int-bottom* = *s1* ⊓ *s2*
  using *assms*(*1*) *assms*(*2*) *inf-intstamp-def join-or-bottom-def* **by** *presburger*

lemma *unfold-bounds*:
  bounds x = (*lower x*, *upper x*)
  by (*simp add*: *bounds.transfer lower.rep-eq upper.rep-eq*)

```
lemma int-inf-le1:
  fixes x y :: 'a intstamp
  shows (x ⊓ y) ≤ x
proof (cases is-bottom (x ⊓ y))
  case True
  then show ?thesis
    by (simp add: bottom-is-bottom)
next
  case False
  then show ?thesis
  using False proof −
  obtain l1 u1 where xdef: lower x = l1 ∧ upper x = u1
    by fastforce
  obtain l2 u2 where ydef: lower y = l2 ∧ upper y = u2
    by fastforce
  have joindef: x ⊓ y = from-bounds ((smax l1 l2, smin u1 u2))
    (is x ⊓ y = from-bounds (?l3, ?u3))
    using False
     by (smt (z3) StampLattice.inf-intstamp-def StampLattice.join-bounds-def al-
ways-valid is-bottom.rep-eq join-or-bottom-def xdef ydef)
  have leq: {sint ?l3..sint ?u3} ⊆ {sint l1..sint u1}
    by (smt (z3) atLeastatMost-subset-iff smax.transfer smin.transfer)
  have (x ⊓ y) ≤ x = ({sint ?l3..sint ?u3} ⊆ {sint l1..sint u1})
    using xdef joindef range-def less-eq-intstamp-def
     by (smt (z3) False StampLattice.always-valid StampLattice.join-or-bottom-def
bounds.abs-eq case-prod-conv inf-intstamp-def is-bottom.rep-eq join-bounds-def range.rep-eq
unfold-bounds ydef)
  then show (x ⊓ y) ≤ x
    using leq
    by fastforce
qed
qed

lemma int-inf-le2:
  fixes x y :: 'a intstamp
  shows (x ⊓ y) ≤ y
proof (cases is-bottom (x ⊓ y))
  case True
  then show ?thesis
    by (simp add: bottom-is-bottom)
next
  case False
  then show ?thesis
  using False proof −
  obtain l1 u1 where xdef: lower x = l1 ∧ upper x = u1
    by fastforce
  obtain l2 u2 where ydef: lower y = l2 ∧ upper y = u2
    by fastforce
  have joindef: x ⊓ y = from-bounds ((smax l1 l2, smin u1 u2))
```

(**is** $x \sqcap y$ = *from-bounds* (*?l3*, *?u3*))
  using *False*
   by (*smt* (*z3*) *StampLattice.inf-intstamp-def StampLattice.join-bounds-def always-valid is-bottom.rep-eq join-or-bottom-def xdef ydef*)
 have *leq*: {*sint ?l3..sint ?u3*} ⊆ {*sint l1..sint u1*}
  by (*smt* (*z3*) *atLeastatMost-subset-iff smax.transfer smin.transfer*)
 have $(x \sqcap y) \leq y$ = ({*sint ?l3..sint ?u3*} ⊆ {*sint l2..sint u2*})
  using *xdef joindef range-def less-eq-intstamp-def*
   by (*smt* (*z3*) *False StampLattice.always-valid StampLattice.join-or-bottom-def bounds.abs-eq case-prod-conv inf-intstamp-def is-bottom.rep-eq join-bounds-def range.rep-eq unfold-bounds ydef*)
 then show $(x \sqcap y) \leq y$
  using *leq*
  by (*smt* (*z3*) *atLeastatMost-subset-iff smax.transfer smin.transfer*)
qed
qed

lemma
 assumes $x \leq y$
 assumes *is-bottom* y
 shows *is-bottom* x
 using *bottom-is-bottom assms*
 using *bottom-unique* **by** *auto*

lemma *int-inf-greatest*:
 fixes $x\ y$ :: *'a intstamp*
 shows $x \leq y \Longrightarrow x \leq z \Longrightarrow x \leq y \sqcap z$
 sorry

instance
 apply *standard*
  apply (*simp add*: *local.int-inf-le1*)
  apply (*simp add*: *local.int-inf-le2*)
 by (*simp add*: *local.int-inf-greatest*)

end

instantiation *intstamp* :: (*len*) *semilattice-sup*
begin

notation *sup* (**infix** ⊔ *65*)

instance sorry

end

instantiation *intstamp* :: (*len*) *bounded-lattice*
begin

notation *bot* ($\perp$ *50*)
notation *top* ($\top$ *50*)

definition *bot-intstamp* = *int-bottom*
definition *top-intstamp* = *int-top*

instance sorry

end

value *sint* (*0*::*1 word*)
value *sint* (*1*::*1 word*)

datatype *Stamp* =
BottomStamp |
TopStamp |
VoidStamp |

Int8Stamp 8 intstamp |
Int16Stamp 16 intstamp |
Int32Stamp 32 intstamp |
Int64Stamp 64 intstamp

instantiation *Stamp* :: *order*
begin

fun *less-eq-Stamp* :: *Stamp* $\Rightarrow$ *Stamp* $\Rightarrow$ *bool* **where**
less-eq-Stamp BottomStamp - = *True* |
less-eq-Stamp - *TopStamp* = *True* |
less-eq-Stamp VoidStamp VoidStamp = *True* |
less-eq-Stamp (*Int8Stamp v1*) (*Int8Stamp v2*) = (*v1* $\leq$ *v2*) |
less-eq-Stamp (*Int16Stamp v1*) (*Int16Stamp v2*) = (*v1* $\leq$ *v2*) |
less-eq-Stamp (*Int32Stamp v1*) (*Int32Stamp v2*) = (*v1* $\leq$ *v2*) |
less-eq-Stamp (*Int64Stamp v1*) (*Int64Stamp v2*) = (*v1* $\leq$ *v2*) |
less-eq-Stamp - - = *False*

fun *less-Stamp* :: *Stamp* $\Rightarrow$ *Stamp* $\Rightarrow$ *bool* **where**
less-Stamp BottomStamp BottomStamp = *False* |
less-Stamp BottomStamp - = *True* |
less-Stamp TopStamp TopStamp = *False* |
less-Stamp - *TopStamp* = *True* |
less-Stamp VoidStamp VoidStamp = *False* |
less-Stamp (*Int8Stamp v1*) (*Int8Stamp v2*) = (*v1* < *v2*) |
less-Stamp (*Int16Stamp v1*) (*Int16Stamp v2*) = (*v1* < *v2*) |
less-Stamp (*Int32Stamp v1*) (*Int32Stamp v2*) = (*v1* < *v2*) |
less-Stamp (*Int64Stamp v1*) (*Int64Stamp v2*) = (*v1* < *v2*) |
less-Stamp - - = *False*

```
instance
  apply standard sorry
end

instantiation Stamp :: semilattice-inf
begin

notation inf (infix ⊓ 65)

fun inf-Stamp :: Stamp ⇒ Stamp ⇒ Stamp where
  inf-Stamp BottomStamp - = BottomStamp |
  inf-Stamp - BottomStamp = BottomStamp |
  inf-Stamp TopStamp - = TopStamp |
  inf-Stamp - TopStamp = TopStamp |
  inf-Stamp VoidStamp VoidStamp = VoidStamp |
  inf-Stamp (Int8Stamp v1) (Int8Stamp v2) = Int8Stamp (v1 ⊓ v2) |
  inf-Stamp (Int16Stamp v1) (Int16Stamp v2) = Int16Stamp (v1 ⊓ v2) |
  inf-Stamp (Int32Stamp v1) (Int32Stamp v2) = Int32Stamp (v1 ⊓ v2) |
  inf-Stamp (Int64Stamp v1) (Int64Stamp v2) = Int64Stamp (v1 ⊓ v2)

instance
  apply standard sorry
end


instantiation Stamp :: semilattice-sup
begin

notation sup (infix ⊔ 65)

fun sup-Stamp :: Stamp ⇒ Stamp ⇒ Stamp where
  sup-Stamp BottomStamp - = BottomStamp |
  sup-Stamp - BottomStamp = BottomStamp |
  sup-Stamp TopStamp - = TopStamp |
  sup-Stamp - TopStamp = TopStamp |
  sup-Stamp VoidStamp VoidStamp = VoidStamp |
  sup-Stamp (Int8Stamp v1) (Int8Stamp v2) = Int8Stamp (v1 ⊔ v2) |
  sup-Stamp (Int16Stamp v1) (Int16Stamp v2) = Int16Stamp (v1 ⊔ v2) |
  sup-Stamp (Int32Stamp v1) (Int32Stamp v2) = Int32Stamp (v1 ⊔ v2) |
  sup-Stamp (Int64Stamp v1) (Int64Stamp v2) = Int64Stamp (v1 ⊔ v2)

instance
  apply standard sorry
end


instantiation Stamp :: bounded-lattice
begin
```

notation *bot* (⊥ *50*)
notation *top* (⊤ *50*)

definition *top-Stamp* :: *Stamp* **where**
top-Stamp = *TopStamp*
definition *bot-Stamp* :: *Stamp* **where**
bot-Stamp = *BottomStamp*

instance
apply *standard* **sorry**
end

lemma [*code*]: *Rep-intstamp* (*from-bounds* (*l*, *u*)) = (*l*, *u*)
using *Abs-intstamp-inverse from-bounds.rep-eq*
sorry

code-datatype *Abs-intstamp*

end