

Veriopt Theories

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1 Data-flow Semantics

```

theory IRTreeEval
  imports
    Graph.Values
    Graph.Stamp
begin

```

We define a tree representation of data-flow nodes, as an abstraction of the graph view.

Data-flow trees are evaluated in the context of a method state (currently called *MapState* in the theories for historical reasons).

The method state consists of the values for each method parameter, references to method parameters use an index of the parameter within the parameter list, as such we store a list of parameter values which are looked up at parameter references.

The method state also stores a mapping of node ids to values. The contents of this mapping is calculates during the traversal of the control flow graph.

As a concrete example, as the *SignedDivNode* can have side-effects (during division by zero), it is treated as part of the control-flow, since the data-flow phase is specified to be side-effect free. As a result, the control-flow semantics for *SignedDivNode* calculates the value of a node and maps the node identifier to the value within the method state. The data-flow semantics then just reads the value stored in the method state for the node.

```

type-synonym ID = nat
type-synonym MapState = ID  $\Rightarrow$  Value
type-synonym Params = Value list

```

```

definition new-map-state :: MapState where
  new-map-state = ( $\lambda x.$  UndefVal)

```

1.1 Data-flow Tree Representation

```

datatype IRUnaryOp =
  UnaryAbs
| UnaryNeg
| UnaryNot
| UnaryLogicNegation
| UnaryNarrow (ir-inputBits: nat) (ir-resultBits: nat)
| UnarySignExtend (ir-inputBits: nat) (ir-resultBits: nat)
| UnaryZeroExtend (ir-inputBits: nat) (ir-resultBits: nat)

```

```

datatype IRBinaryOp =
  BinAdd

```

```

| BinMul
| BinSub
| BinAnd
| BinOr
| BinXor
| BinLeftShift
| BinRightShift
| BinURightShift
| BinIntegerEquals
| BinIntegerLessThan
| BinIntegerBelow

datatype (discs-sels) IRExpr =
  UnaryExpr (ir-uop: IRUnaryOp) (ir-value: IRExpr)
| BinaryExpr (ir-op: IRBinaryOp) (ir-x: IRExpr) (ir-y: IRExpr)
| ConditionalExpr (ir-condition: IRExpr) (ir-trueValue: IRExpr) (ir-falseValue:
IRExpr)

| ParameterExpr (ir-index: nat) (ir-stamp: Stamp)

| LeafExpr (ir-nid: ID) (ir-stamp: Stamp)

| ConstantExpr (ir-const: Value)
| ConstantVar (ir-name: string)
| VariableExpr (ir-name: string) (ir-stamp: Stamp)

fun is-ground :: IRExpr ⇒ bool where
  is-ground (UnaryExpr op e) = is-ground e |
  is-ground (BinaryExpr op e1 e2) = (is-ground e1 ∧ is-ground e2) |
  is-ground (ConditionalExpr b e1 e2) = (is-ground b ∧ is-ground e1 ∧ is-ground
e2) |
  is-ground (ParameterExpr i s) = True |
  is-ground (LeafExpr n s) = True |
  is-ground (ConstantExpr v) = True |
  is-ground (ConstantVar name) = False |
  is-ground (VariableExpr name s) = False

typedef GroundExpr = { e :: IRExpr . is-ground e }
using is-ground.simps(6) by blast

fun stamp-unary :: IRUnaryOp ⇒ Stamp ⇒ Stamp where
  stamp-unary op (IntegerStamp b lo hi) = unrestricted-stamp (IntegerStamp b lo
hi) |

  stamp-unary op - = IllegalStamp

definition fixed-32 :: IRBinaryOp set where

```

```

fixed-32 = {BinIntegerEquals, BinIntegerLessThan, BinIntegerBelow}

fun stamp-binary :: IRBinaryOp ⇒ Stamp ⇒ Stamp ⇒ Stamp where
  stamp-binary op (IntegerStamp b1 lo1 hi1) (IntegerStamp b2 lo2 hi2) =
    (case op ∈ fixed-32 of True ⇒ unrestricted-stamp (IntegerStamp 32 lo1 hi1) |
     False ⇒
      (if (b1 = b2) then unrestricted-stamp (IntegerStamp b1 lo1 hi1) else Illegal-
       Stamp)) |

  stamp-binary op - - = IllegalStamp

fun stamp-expr :: IRExpr ⇒ Stamp where
  stamp-expr (UnaryExpr op x) = stamp-unary op (stamp-expr x) |
  stamp-expr (BinaryExpr bop x y) = stamp-binary bop (stamp-expr x) (stamp-expr
  y) |
  stamp-expr (ConstantExpr val) = constantAsStamp val |
  stamp-expr (LeafExpr i s) = s |
  stamp-expr (ParameterExpr i s) = s |
  stamp-expr (ConditionalExpr c t f) = meet (stamp-expr t) (stamp-expr f)

export-code stamp-unary stamp-binary stamp-expr

```

1.2 Data-flow Tree Evaluation

```

fun unary-eval :: IRUnaryOp ⇒ Value ⇒ Value where
  unary-eval UnaryAbs v = intval-abs v |
  unary-eval UnaryNeg v = intval-negate v |
  unary-eval UnaryNot v = intval-not v |
  unary-eval UnaryLogicNegation v = intval-logic-negation v |
  unary-eval op v1 =.UndefVal

fun bin-eval :: IRBinaryOp ⇒ Value ⇒ Value ⇒ Value where
  bin-eval BinAdd v1 v2 = intval-add v1 v2 |
  bin-eval BinMul v1 v2 = intval-mul v1 v2 |
  bin-eval BinSub v1 v2 = intval-sub v1 v2 |
  bin-eval BinAnd v1 v2 = intval-and v1 v2 |
  bin-eval BinOr v1 v2 = intval-or v1 v2 |
  bin-eval BinXor v1 v2 = intval-xor v1 v2 |
  bin-eval BinLeftShift v1 v2 = intval-left-shift v1 v2 |
  bin-eval BinRightShift v1 v2 = intval-right-shift v1 v2 |
  bin-eval BinURightShift v1 v2 = intval-uright-shift v1 v2 |
  bin-eval BinIntegerEquals v1 v2 = intval-equals v1 v2 |
  bin-eval BinIntegerLessThan v1 v2 = intval-less-than v1 v2 |
  bin-eval BinIntegerBelow v1 v2 = intval-below v1 v2

```

```

inductive not-undef-or-fail :: Value ⇒ Value ⇒ bool where
  [[value ≠.UndefVal]] ⇒ not-undef-or-fail value value

```

notation (*latex output*)
not-undef-or-fail ($- = -$)

inductive

evaltree :: *MapState* \Rightarrow *Params* \Rightarrow *IRExpr* \Rightarrow *Value* \Rightarrow *bool* ($[-,-] \vdash - \mapsto -$ 55)

for *m p* **where**

ConstantExpr:

$\llbracket \text{valid-value } c \text{ (constantAsStamp } c) \rrbracket$
 $\implies [m,p] \vdash (\text{ConstantExpr } c) \mapsto c \mid$

ParameterExpr:

$\llbracket i < \text{length } p; \text{valid-value } (p!i) \text{ } s \rrbracket$
 $\implies [m,p] \vdash (\text{ParameterExpr } i \text{ } s) \mapsto p!i \mid$

ConditionalExpr:

$\llbracket [m,p] \vdash ce \mapsto \text{cond};$
 $\text{branch} = (\text{if val-to-bool cond then } te \text{ else } fe);$
 $[m,p] \vdash \text{branch} \mapsto v;$
 $v \neq \text{UndefVal} \rrbracket$
 $\implies [m,p] \vdash (\text{ConditionalExpr } ce \text{ } te \text{ } fe) \mapsto v \mid$

UnaryExpr:

$\llbracket [m,p] \vdash xe \mapsto v;$
 $\text{result} = (\text{unary-eval op } v);$
 $\text{result} \neq \text{UndefVal} \rrbracket$
 $\implies [m,p] \vdash (\text{UnaryExpr op } xe) \mapsto \text{result} \mid$

BinaryExpr:

$\llbracket [m,p] \vdash xe \mapsto x;$
 $[m,p] \vdash ye \mapsto y;$
 $\text{result} = (\text{bin-eval op } x \text{ } y);$
 $\text{result} \neq \text{UndefVal} \rrbracket$
 $\implies [m,p] \vdash (\text{BinaryExpr op } xe \text{ } ye) \mapsto \text{result} \mid$

LeafExpr:

$\llbracket \text{val} = m \text{ } n;$
 $\text{valid-value val } s \rrbracket$
 $\implies [m,p] \vdash \text{LeafExpr } n \text{ } s \mapsto \text{val}$

code-pred (*modes*: $i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$ as *evalT*)

[*show-steps,show-mode-inference,show-intermediate-results*]

evaltree .

inductive

evaltrees :: *MapState* \Rightarrow *Params* \Rightarrow *IRExpr list* \Rightarrow *Value list* \Rightarrow *bool* ($[-,-] \vdash - \mapsto_L$

- 55)

for *m p* **where**

```

EvalNil:
[m,p] ⊢ [] ↦L [] |

EvalCons:
[[m,p] ⊢ x ↦ xval;
 [m,p] ⊢ yy ↦L yyval]
⇒ [m,p] ⊢ (x#yy) ↦L (xval#yyval)

code-pred (modes: i ⇒ i ⇒ i ⇒ o ⇒ bool as evalTs)
  evaltrees .

definition sq-param0 :: IRExp where
  sq-param0 = BinaryExpr BinMul
    (ParameterExpr 0 (IntegerStamp 32 (- 2147483648) 2147483647))
    (ParameterExpr 0 (IntegerStamp 32 (- 2147483648) 2147483647))

values {v. evaltree new-map-state [IntVal32 5] sq-param0 v}

declare evaltree.intros [intro]
declare evaltrees.intros [intro]

```

1.3 Data-flow Tree Refinement

We define the induced semantic equivalence relation between expressions. Note that syntactic equality implies semantic equivalence, but not vice versa.

definition *equiv-exprs* :: IRExp ⇒ IRExp ⇒ bool (- ≐ - 55) **where**
 (e1 ≐ e2) = (∀ m p v. (([m,p] ⊢ e1 ↦ v) ⟷ ([m,p] ⊢ e2 ↦ v)))

We also prove that this is a total equivalence relation (*equivp equiv-exprs*) (HOL.Equiv_Relations), so that we can reuse standard results about equivalence relations.

lemma *equivp equiv-exprs*
apply (auto simp add: equivp-def equiv-exprs-def)
by (metis equiv-exprs-def)+

We define a refinement ordering over IRExp and show that it is a preorder. Note that it is asymmetric because e2 may refer to fewer variables than e1.

instantiation IRExp :: preorder **begin**

notation *less-eq* (**infix** ⊑ 65)

definition
le-expr-def [simp]:
 (e2 ≤ e1) ⟷ (∀ m p v. (([m,p] ⊢ e1 ↦ v) ⟶ ([m,p] ⊢ e2 ↦ v)))

definition

lt-expr-def [*simp*]:
 $(e_1 < e_2) \longleftrightarrow (e_1 \leq e_2 \wedge \neg (e_1 \dot{=} e_2))$

instance proof

fix $x\ y\ z :: IRExpr$
show $x < y \longleftrightarrow x \leq y \wedge \neg (y \leq x)$ **by** (*simp add: equiv-exprs-def; auto*)
show $x \leq x$ **by** *simp*
show $x \leq y \implies y \leq z \implies x \leq z$ **by** *simp*
qed

end

abbreviation (**output**) *Refines* :: $IRExpr \Rightarrow IRExpr \Rightarrow bool$ (**infix** \sqsupseteq 64)
where $e_1 \sqsupseteq e_2 \equiv (e_2 \leq e_1)$

end

1.4 Data-flow Tree Theorems

theory *IRTreeEvalThms*

imports

IRTreeEval

begin

1.4.1 Deterministic Data-flow Evaluation

lemma *evalDet*:

$[m, p] \vdash e \mapsto v_1 \implies$
 $[m, p] \vdash e \mapsto v_2 \implies$
 $v_1 = v_2$
apply (*induction arbitrary: v2 rule: evaltree.induct*)
by (*elim EvalTreeE; auto*)+

lemma *evalAllDet*:

$[m, p] \vdash e \mapsto_L v1 \implies$
 $[m, p] \vdash e \mapsto_L v2 \implies$
 $v1 = v2$
apply (*induction arbitrary: v2 rule: evaltrees.induct*)
apply (*elim EvalTreeE; auto*)
using *evalDet* **by** *force*

1.4.2 Evaluation Results are Valid

A valid value cannot be *UndefVal*.

lemma *valid-not-undef*:

assumes *a1*: *valid-value val s*
assumes *a2*: $s \neq VoidStamp$

```

shows  $val \neq \text{UndefVal}$ 
apply (rule valid-value.elims(1)[of  $val\ s\ \text{True}$ ])
using a1 a2 by auto

lemma valid-VoidStamp[elim]:
  shows valid-value val VoidStamp  $\implies$ 
     $val = \text{UndefVal}$ 
  using valid-value.simps by metis

lemma valid-ObjStamp[elim]:
  shows valid-value val (ObjectStamp klass exact nonNull alwaysNull)  $\implies$ 
     $(\exists v. val = \text{ObjRef } v)$ 
  using valid-value.simps by (metis val-to-bool.cases)

lemma valid-int32[elim]:
  shows valid-value val (IntegerStamp 32 l h)  $\implies$ 
     $(\exists v. val = \text{IntVal32 } v)$ 
  apply (rule val-to-bool.cases[of  $val$ ])
  using Value.distinct by simp+

lemma valid-int64[elim]:
  shows valid-value val (IntegerStamp 64 l h)  $\implies$ 
     $(\exists v. val = \text{IntVal64 } v)$ 
  apply (rule val-to-bool.cases[of  $val$ ])
  using Value.distinct by simp+

lemmas valid-value-elim =
  valid-VoidStamp
  valid-ObjStamp
  valid-int32
  valid-int64

lemma evaltree-not-undef:
  fixes  $m\ p\ e\ v$ 
  shows  $([m,p] \vdash e \mapsto v) \implies v \neq \text{UndefVal}$ 
  apply (induction rule: evaltree.induct)
  using valid-not-undef by auto

lemma leafint32:
  assumes  $ev: [m,p] \vdash \text{LeafExpr } i\ (\text{IntegerStamp } 32\ lo\ hi) \mapsto val$ 
  shows  $\exists v. val = (\text{IntVal32 } v)$ 

proof –
  have valid-value val (IntegerStamp 32 lo hi)
    using  $ev$  by (rule LeafExprE; simp)
  then show ?thesis by auto

```


qed

lemma *leafint64*:

assumes *ev*: $[m,p] \vdash \text{LeafExpr } i \text{ (IntegerStamp 64 lo hi)} \mapsto \text{val}$
shows $\exists v. \text{val} = (\text{IntVal64 } v)$

proof –

have *valid-value* *val* (IntegerStamp 64 lo hi)
using *ev* **by** (rule LeafExprE; simp)
then show ?thesis **by** auto

qed

lemma *default-stamp* [simp]: *default-stamp* = IntegerStamp 32 (–2147483648)
2147483647

using *default-stamp-def* **by** auto

lemma *valid32* [simp]:

assumes *valid-value* *val* (IntegerStamp 32 lo hi)
shows $\exists v. (\text{val} = (\text{IntVal32 } v) \wedge \text{lo} \leq \text{sint } v \wedge \text{sint } v \leq \text{hi})$
using *assms valid-int32* **by** force

lemma *valid64* [simp]:

assumes *valid-value* *val* (IntegerStamp 64 lo hi)
shows $\exists v. (\text{val} = (\text{IntVal64 } v) \wedge \text{lo} \leq \text{sint } v \wedge \text{sint } v \leq \text{hi})$
using *assms valid-int64* **by** force

lemma *valid32or64*:

assumes *valid-value* *x* (IntegerStamp *b* lo hi)
shows $(\exists v1. (x = \text{IntVal32 } v1)) \vee (\exists v2. (x = \text{IntVal64 } v2))$
using *valid32 valid64 assms valid-value.elims(2)* **by** blast

lemma *valid32or64-both*:

assumes *valid-value* *x* (IntegerStamp *b* lox hix)
and *valid-value* *y* (IntegerStamp *b* loy hiy)
shows $(\exists v1 v2. x = \text{IntVal32 } v1 \wedge y = \text{IntVal32 } v2) \vee (\exists v3 v4. x = \text{IntVal64 } v3 \wedge y = \text{IntVal64 } v4)$
using *assms valid32or64 valid32 valid-value.elims(2) valid-value.simps(1)* **by** metis

1.4.3 Example Data-flow Optimisations

lemma *a0a-helper* [simp]:

assumes *a*: *valid-value* *v* (IntegerStamp 32 lo hi)
shows *intval-add* *v* (IntVal32 0) = *v*

proof –

obtain *v32* :: *int32* **where** *v* = (IntVal32 *v32*) **using** *a valid32* **by** blast
then show ?thesis **by** simp

qed

```

lemma a0a: (BinaryExpr BinAdd (LeafExpr 1 default-stamp) (ConstantExpr (IntVal32
0)))
  ≥ (LeafExpr 1 default-stamp)
by (auto simp add: evaltree.LeafExpr)

```

```

lemma xyx-y-helper [simp]:
  assumes valid-value x (IntegerStamp 32 lox hix)
  assumes valid-value y (IntegerStamp 32 loy hiy)
  shows intval-add x (intval-sub y x) = y
proof –
  obtain x32 :: int32 where x: x = (IntVal32 x32) using assms valid32 by blast
  obtain y32 :: int32 where y: y = (IntVal32 y32) using assms valid32 by blast
  show ?thesis using x y by simp
qed

```

```

lemma xyx-y:
  (BinaryExpr BinAdd
    (LeafExpr x (IntegerStamp 32 lox hix))
    (BinaryExpr BinSub
      (LeafExpr y (IntegerStamp 32 loy hiy))
      (LeafExpr x (IntegerStamp 32 lox hix))))
  ≥ (LeafExpr y (IntegerStamp 32 loy hiy))
by (auto simp add: LeafExpr)

```

1.4.4 Monotonicity of Expression Refinement

We prove that each subexpression position is monotonic. That is, optimizing a subexpression anywhere deep inside a top-level expression also optimizes that top-level expression.

Note that we might also be able to do this via reusing Isabelle’s ‘mono’ operator (HOL.Orderings theory), proving instantiations like ‘mono (UnaryExpr op)’, but it is not obvious how to do this for both arguments of the binary expressions.

```

lemma mono-unary:
  assumes e ≥ e'
  shows (UnaryExpr op e) ≥ (UnaryExpr op e')
  using UnaryExpr assms by auto

```

```

lemma mono-binary:
  assumes x ≥ x'
  assumes y ≥ y'
  shows (BinaryExpr op x y) ≥ (BinaryExpr op x' y')
  using BinaryExpr assms by auto

```

```

lemma never-void:

```

```

assumes  $[m, p] \vdash x \mapsto xv$ 
assumes valid-value  $xv$  (stamp-expr  $xe$ )
shows stamp-expr  $xe \neq \text{VoidStamp}$ 
using valid-value.simps
using assms(2) by force

lemma stamp32:
   $\exists v . xv = \text{IntVal32 } v \longleftrightarrow \text{valid-value } xv \ (\text{IntegerStamp } 32 \text{ lo } hi)$ 
using valid-int32
by (metis (full-types) Value.inject(1) zero-neq-one)

lemma stamp64:
   $\exists v . xv = \text{IntVal64 } v \longleftrightarrow \text{valid-value } xv \ (\text{IntegerStamp } 64 \text{ lo } hi)$ 
using valid-int64
by (metis (full-types) Value.inject(2) zero-neq-one)

lemma stamprange:
   $\text{valid-value } v \ s \longrightarrow (\exists b \text{ lo } hi. (s = \text{IntegerStamp } b \text{ lo } hi) \wedge (b = 32 \vee b = 64))$ 
using valid-value.elims stamp32 stamp64
by (smt (verit, del-insts))

lemma compatible-trans:
   $\text{compatible } x \ y \wedge \text{compatible } y \ z \Longrightarrow \text{compatible } x \ z$ 
by (smt (verit, best) compatible.elims(2) compatible.simps(1))

lemma compatible-refl:
   $\text{compatible } x \ y \Longrightarrow \text{compatible } y \ x$ 
using compatible.elims(2) by fastforce

lemma mono-conditional:
  assumes  $ce \geq ce'$ 
  assumes  $te \geq te'$ 
  assumes  $fe \geq fe'$ 
  shows  $(\text{ConditionalExpr } ce \ te \ fe) \geq (\text{ConditionalExpr } ce' \ te' \ fe')$ 
proof (simp only: le-expr-def; (rule allI) $+$ ; rule impI)
  fix  $m \ p \ v$ 
  assume  $a: [m, p] \vdash \text{ConditionalExpr } ce \ te \ fe \mapsto v$ 
  then obtain  $cond$  where  $ce: [m, p] \vdash ce \mapsto cond$  by auto
  then have  $ce': [m, p] \vdash ce' \mapsto cond$  using assms by auto

  define  $branch$  where  $b: branch = (\text{if val-to-bool } cond \text{ then } te \text{ else } fe)$ 
  define  $branch'$  where  $b': branch' = (\text{if val-to-bool } cond \text{ then } te' \text{ else } fe')$ 
  then have  $beval: [m, p] \vdash branch \mapsto v$  using  $a \ b \ ce \ \text{evalDet}$  by blast

  from  $beval$  have  $[m, p] \vdash branch' \mapsto v$  using assms  $b \ b'$  by auto
  then show  $[m, p] \vdash \text{ConditionalExpr } ce' \ te' \ fe' \mapsto v$ 
    using ConditionalExpr  $ce' \ b'$ 

```

```

    using a by blast
qed

```

```

end

```

2 Tree to Graph

```

theory TreeToGraph
  imports
    Semantics.IRTreeEval
    Graph.IRGraph
begin

```

2.1 Subgraph to Data-flow Tree

```

fun find-node-and-stamp :: IRGraph  $\Rightarrow$  (IRNode  $\times$  Stamp)  $\Rightarrow$  ID option where
  find-node-and-stamp g (n,s) =
    find ( $\lambda i.$  kind g i = n  $\wedge$  stamp g i = s) (sorted-list-of-set(ids g))

export-code find-node-and-stamp

```

```

fun is-preevaluated :: IRNode  $\Rightarrow$  bool where
  is-preevaluated (InvokeNode n - - - -) = True |
  is-preevaluated (InvokeWithExceptionNode n - - - -) = True |
  is-preevaluated (NewInstanceNode n - -) = True |
  is-preevaluated (LoadFieldNode n - -) = True |
  is-preevaluated (SignedDivNode n - - - -) = True |
  is-preevaluated (SignedRemNode n - - - -) = True |
  is-preevaluated (ValuePhiNode n -) = True |
  is-preevaluated - = False

```

```

inductive
  rep :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  IRExpr  $\Rightarrow$  bool (-  $\vdash$  -  $\simeq$  - 55)
for g where

```

```

  ConstantNode:
   $\llbracket \text{kind } g \text{ } n = \text{ConstantNode } c \rrbracket$ 
 $\implies g \vdash n \simeq (\text{ConstantExpr } c) \mid$ 

```

```

  ParameterNode:
   $\llbracket \text{kind } g \text{ } n = \text{ParameterNode } i; \text{ stamp } g \text{ } n = s \rrbracket$ 
 $\implies g \vdash n \simeq (\text{ParameterExpr } i \text{ } s) \mid$ 

```

```

  ConditionalNode:
   $\llbracket \text{kind } g \text{ } n = \text{ConditionalNode } c \text{ } t \text{ } f; \rrbracket$ 

```

$g \vdash c \simeq ce;$
 $g \vdash t \simeq te;$
 $g \vdash f \simeq fe]$
 $\implies g \vdash n \simeq (\text{ConditionalExpr } ce \ te \ fe) \mid$

AbsNode:
 $\llbracket kind \ g \ n = \text{AbsNode } x;$
 $g \vdash x \simeq xe]$
 $\implies g \vdash n \simeq (\text{UnaryExpr } \text{UnaryAbs } xe) \mid$

NotNode:
 $\llbracket kind \ g \ n = \text{NotNode } x;$
 $g \vdash x \simeq xe]$
 $\implies g \vdash n \simeq (\text{UnaryExpr } \text{UnaryNot } xe) \mid$

NegateNode:
 $\llbracket kind \ g \ n = \text{NegateNode } x;$
 $g \vdash x \simeq xe]$
 $\implies g \vdash n \simeq (\text{UnaryExpr } \text{UnaryNeg } xe) \mid$

LogicNegationNode:
 $\llbracket kind \ g \ n = \text{LogicNegationNode } x;$
 $g \vdash x \simeq xe]$
 $\implies g \vdash n \simeq (\text{UnaryExpr } \text{UnaryLogicNegation } xe) \mid$

AddNode:
 $\llbracket kind \ g \ n = \text{AddNode } x \ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye]$
 $\implies g \vdash n \simeq (\text{BinaryExpr } \text{BinAdd } xe \ ye) \mid$

MulNode:
 $\llbracket kind \ g \ n = \text{MulNode } x \ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye]$
 $\implies g \vdash n \simeq (\text{BinaryExpr } \text{BinMul } xe \ ye) \mid$

SubNode:
 $\llbracket kind \ g \ n = \text{SubNode } x \ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye]$
 $\implies g \vdash n \simeq (\text{BinaryExpr } \text{BinSub } xe \ ye) \mid$

AndNode:
 $\llbracket kind \ g \ n = \text{AndNode } x \ y;$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye]$

$$\implies g \vdash n \simeq (\text{BinaryExpr BinAnd } xe \ ye) \mid$$

OrNode:

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{OrNode } x \ y; \\ & \quad g \vdash x \simeq xe; \\ & \quad g \vdash y \simeq ye \rrbracket \\ & \implies g \vdash n \simeq (\text{BinaryExpr BinOr } xe \ ye) \mid \end{aligned}$$

XorNode:

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{XorNode } x \ y; \\ & \quad g \vdash x \simeq xe; \\ & \quad g \vdash y \simeq ye \rrbracket \\ & \implies g \vdash n \simeq (\text{BinaryExpr BinXor } xe \ ye) \mid \end{aligned}$$

LeftShiftNode:

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{LeftShiftNode } x \ y; \\ & \quad g \vdash x \simeq xe; \\ & \quad g \vdash y \simeq ye \rrbracket \\ & \implies g \vdash n \simeq (\text{BinaryExpr BinLeftShift } xe \ ye) \mid \end{aligned}$$

RightShiftNode:

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{RightShiftNode } x \ y; \\ & \quad g \vdash x \simeq xe; \\ & \quad g \vdash y \simeq ye \rrbracket \\ & \implies g \vdash n \simeq (\text{BinaryExpr BinRightShift } xe \ ye) \mid \end{aligned}$$

UnsignedRightShiftNode:

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{UnsignedRightShiftNode } x \ y; \\ & \quad g \vdash x \simeq xe; \\ & \quad g \vdash y \simeq ye \rrbracket \\ & \implies g \vdash n \simeq (\text{BinaryExpr BinURightShift } xe \ ye) \mid \end{aligned}$$

IntegerBelowNode:

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{IntegerBelowNode } x \ y; \\ & \quad g \vdash x \simeq xe; \\ & \quad g \vdash y \simeq ye \rrbracket \\ & \implies g \vdash n \simeq (\text{BinaryExpr BinIntegerBelow } xe \ ye) \mid \end{aligned}$$

IntegerEqualsNode:

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{IntegerEqualsNode } x \ y; \\ & \quad g \vdash x \simeq xe; \\ & \quad g \vdash y \simeq ye \rrbracket \\ & \implies g \vdash n \simeq (\text{BinaryExpr BinIntegerEquals } xe \ ye) \mid \end{aligned}$$

IntegerLessThanNode:

$$\begin{aligned} & \llbracket \text{kind } g \ n = \text{IntegerLessThanNode } x \ y; \\ & \quad g \vdash x \simeq xe; \\ & \quad g \vdash y \simeq ye \rrbracket \\ & \implies g \vdash n \simeq (\text{BinaryExpr BinIntegerLessThan } xe \ ye) \mid \end{aligned}$$

NarrowNode:

$\llbracket \text{kind } g \ n = \text{NarrowNode } \text{inputBits } \text{resultBits } x; \\ g \vdash x \simeq xe \rrbracket \\ \implies g \vdash n \simeq (\text{UnaryExpr } (\text{UnaryNarrow } \text{inputBits } \text{resultBits}) \ xe) \mid$

SignExtendNode:

$\llbracket \text{kind } g \ n = \text{SignExtendNode } \text{inputBits } \text{resultBits } x; \\ g \vdash x \simeq xe \rrbracket \\ \implies g \vdash n \simeq (\text{UnaryExpr } (\text{UnarySignExtend } \text{inputBits } \text{resultBits}) \ xe) \mid$

ZeroExtendNode:

$\llbracket \text{kind } g \ n = \text{ZeroExtendNode } \text{inputBits } \text{resultBits } x; \\ g \vdash x \simeq xe \rrbracket \\ \implies g \vdash n \simeq (\text{UnaryExpr } (\text{UnaryZeroExtend } \text{inputBits } \text{resultBits}) \ xe) \mid$

LeafNode:

$\llbracket \text{is-preevaluated } (\text{kind } g \ n); \\ \text{stamp } g \ n = s \rrbracket \\ \implies g \vdash n \simeq (\text{LeafExpr } n \ s) \mid$

RefNode:

$\llbracket \text{kind } g \ n = \text{RefNode } n'; \\ g \vdash n' \simeq e \rrbracket \\ \implies g \vdash n \simeq e$

code-pred (*modes*: $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$ as *exprE*) *rep* .

inductive

replist :: $\text{IRGraph} \Rightarrow \text{ID list} \Rightarrow \text{IRExpr list} \Rightarrow \text{bool}$ ($- \vdash - \simeq_L - 55$)
for *g* **where**

RepNil:

$g \vdash [] \simeq_L [] \mid$

RepCons:

$\llbracket g \vdash x \simeq xe; \\ g \vdash xs \simeq_L xse \rrbracket \\ \implies g \vdash x \# xs \simeq_L xe \# xse$

code-pred (*modes*: $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$ as *exprListE*) *replist* .

definition *wf-term-graph* :: $\text{MapState} \Rightarrow \text{Params} \Rightarrow \text{IRGraph} \Rightarrow \text{ID} \Rightarrow \text{bool}$ **where**

wf-term-graph *m p g n* = $(\exists e. (g \vdash n \simeq e) \wedge (\exists v. ([m, p] \vdash e \mapsto v)))$

values $\{t. \text{eg2-sq} \vdash 4 \simeq t\}$

2.2 Data-flow Tree to Subgraph

fun *unary-node* :: *IRUnaryOp* \Rightarrow *ID* \Rightarrow *IRNode* **where**
unary-node *UnaryAbs* *v* = *AbsNode* *v* |
unary-node *UnaryNot* *v* = *NotNode* *v* |
unary-node *UnaryNeg* *v* = *NegateNode* *v* |
unary-node *UnaryLogicNegation* *v* = *LogicNegationNode* *v* |
unary-node (*UnaryNarrow* *ib* *rb*) *v* = *NarrowNode* *ib* *rb* *v* |
unary-node (*UnarySignExtend* *ib* *rb*) *v* = *SignExtendNode* *ib* *rb* *v* |
unary-node (*UnaryZeroExtend* *ib* *rb*) *v* = *ZeroExtendNode* *ib* *rb* *v*

fun *bin-node* :: *IRBinaryOp* \Rightarrow *ID* \Rightarrow *ID* \Rightarrow *IRNode* **where**
bin-node *BinAdd* *x* *y* = *AddNode* *x* *y* |
bin-node *BinMul* *x* *y* = *MulNode* *x* *y* |
bin-node *BinSub* *x* *y* = *SubNode* *x* *y* |
bin-node *BinAnd* *x* *y* = *AndNode* *x* *y* |
bin-node *BinOr* *x* *y* = *OrNode* *x* *y* |
bin-node *BinXor* *x* *y* = *XorNode* *x* *y* |
bin-node *BinLeftShift* *x* *y* = *LeftShiftNode* *x* *y* |
bin-node *BinRightShift* *x* *y* = *RightShiftNode* *x* *y* |
bin-node *BinURightShift* *x* *y* = *UnsignedRightShiftNode* *x* *y* |
bin-node *BinIntegerEquals* *x* *y* = *IntegerEqualsNode* *x* *y* |
bin-node *BinIntegerLessThan* *x* *y* = *IntegerLessThanNode* *x* *y* |
bin-node *BinIntegerBelow* *x* *y* = *IntegerBelowNode* *x* *y*

fun *choose-32-64* :: *int* \Rightarrow *int64* \Rightarrow *Value* **where**
choose-32-64 *bits* *val* =
 (if *bits* = 32
 then (*IntVal32* (*ucast* *val*))
 else (*IntVal64* (*val*)))

inductive *fresh-id* :: *IRGraph* \Rightarrow *ID* \Rightarrow *bool* **where**
n \notin *ids* *g* \Longrightarrow *fresh-id* *g* *n*

code-pred *fresh-id* .

fun *get-fresh-id* :: *IRGraph* \Rightarrow *ID* **where**
get-fresh-id *g* = *last*(*sorted-list-of-set*(*ids* *g*)) + 1

export-code *get-fresh-id*

value *get-fresh-id* *eg2-sq*

value *get-fresh-id* (*add-node* 6 (*ParameterNode* 2, *default-stamp*) *eg2-sq*)

inductive

unrep :: *IRGraph* \Rightarrow *IRExpr* \Rightarrow (*IRGraph* \times *ID*) \Rightarrow *bool* (*-* \oplus *-* \rightsquigarrow *-* 55)
where

ConstantNodeSame:

$\llbracket \text{find-node-and-stamp } g \text{ (ConstantNode } c, \text{ constantAsStamp } c) = \text{Some } n \rrbracket$
 $\implies g \oplus (\text{ConstantExpr } c) \rightsquigarrow (g, n) \mid$

ConstantNodeNew:

$\llbracket \text{find-node-and-stamp } g \text{ (ConstantNode } c, \text{ constantAsStamp } c) = \text{None};$
n = *get-fresh-id* *g*;
g' = *add-node* *n* (*ConstantNode* *c*, *constantAsStamp* *c*) *g* \rrbracket
 $\implies g \oplus (\text{ConstantExpr } c) \rightsquigarrow (g', n) \mid$

ParameterNodeSame:

$\llbracket \text{find-node-and-stamp } g \text{ (ParameterNode } i, s) = \text{Some } n \rrbracket$
 $\implies g \oplus (\text{ParameterExpr } i \text{ } s) \rightsquigarrow (g, n) \mid$

ParameterNodeNew:

$\llbracket \text{find-node-and-stamp } g \text{ (ParameterNode } i, s) = \text{None};$
n = *get-fresh-id* *g*;
g' = *add-node* *n* (*ParameterNode* *i*, *s*) *g* \rrbracket
 $\implies g \oplus (\text{ParameterExpr } i \text{ } s) \rightsquigarrow (g', n) \mid$

ConditionalNodeSame:

$\llbracket g \oplus ce \rightsquigarrow (g2, c);$
g2 \oplus *te* \rightsquigarrow (*g3*, *t*);
g3 \oplus *fe* \rightsquigarrow (*g4*, *f*);
s' = *meet* (*stamp* *g4* *t*) (*stamp* *g4* *f*);
find-node-and-stamp *g4* (*ConditionalNode* *c* *t* *f*, *s'*) = *Some* *n* \rrbracket
 $\implies g \oplus (\text{ConditionalExpr } ce \text{ } te \text{ } fe) \rightsquigarrow (g4, n) \mid$

ConditionalNodeNew:

$\llbracket g \oplus ce \rightsquigarrow (g2, c);$
g2 \oplus *te* \rightsquigarrow (*g3*, *t*);
g3 \oplus *fe* \rightsquigarrow (*g4*, *f*);
s' = *meet* (*stamp* *g4* *t*) (*stamp* *g4* *f*);
find-node-and-stamp *g4* (*ConditionalNode* *c* *t* *f*, *s'*) = *None*;
n = *get-fresh-id* *g4*;
g' = *add-node* *n* (*ConditionalNode* *c* *t* *f*, *s'*) *g4* \rrbracket
 $\implies g \oplus (\text{ConditionalExpr } ce \text{ } te \text{ } fe) \rightsquigarrow (g', n) \mid$

UnaryNodeSame:

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$
 $s' = \text{stamp-unary op (stamp } g2 \text{ } x);$
 $\text{find-node-and-stamp } g2 \text{ (unary-node op } x, s') = \text{Some } n \rrbracket$
 $\implies g \oplus (\text{UnaryExpr op } xe) \rightsquigarrow (g2, n) \mid$

UnaryNodeNew:

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$
 $s' = \text{stamp-unary op (stamp } g2 \text{ } x);$
 $\text{find-node-and-stamp } g2 \text{ (unary-node op } x, s') = \text{None};$
 $n = \text{get-fresh-id } g2;$
 $g' = \text{add-node } n \text{ (unary-node op } x, s') \text{ } g2 \rrbracket$
 $\implies g \oplus (\text{UnaryExpr op } xe) \rightsquigarrow (g', n) \mid$

BinaryNodeSame:

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$
 $g2 \oplus ye \rightsquigarrow (g3, y);$
 $s' = \text{stamp-binary op (stamp } g3 \text{ } x) \text{ (stamp } g3 \text{ } y);$
 $\text{find-node-and-stamp } g3 \text{ (bin-node op } x \text{ } y, s') = \text{Some } n \rrbracket$
 $\implies g \oplus (\text{BinaryExpr op } xe \text{ } ye) \rightsquigarrow (g3, n) \mid$

BinaryNodeNew:

$\llbracket g \oplus xe \rightsquigarrow (g2, x);$
 $g2 \oplus ye \rightsquigarrow (g3, y);$
 $s' = \text{stamp-binary op (stamp } g3 \text{ } x) \text{ (stamp } g3 \text{ } y);$
 $\text{find-node-and-stamp } g3 \text{ (bin-node op } x \text{ } y, s') = \text{None};$
 $n = \text{get-fresh-id } g3;$
 $g' = \text{add-node } n \text{ (bin-node op } x \text{ } y, s') \text{ } g3 \rrbracket$
 $\implies g \oplus (\text{BinaryExpr op } xe \text{ } ye) \rightsquigarrow (g', n) \mid$

AllLeafNodes:

$\llbracket \text{stamp } g \text{ } n = s;$
 $\text{is-preevaluated (kind } g \text{ } n) \rrbracket$
 $\implies g \oplus (\text{LeafExpr } n \text{ } s) \rightsquigarrow (g, n)$

code-pred (*modes*: $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$ as *unrepE*)
unrep .

$$\frac{\text{find-node-and-stamp } g \text{ (ConstantNode } c, \text{ constantAsStamp } c) = \text{Some } n}{g \oplus \text{ConstantExpr } c \rightsquigarrow (g, n)}$$

$$\frac{\begin{array}{l} \text{find-node-and-stamp } g \text{ (ConstantNode } c, \text{ constantAsStamp } c) = \text{None} \\ n = \text{get-fresh-id } g \\ g' = \text{add-node } n \text{ (ConstantNode } c, \text{ constantAsStamp } c) \text{ } g \end{array}}{g \oplus \text{ConstantExpr } c \rightsquigarrow (g', n)}$$

$$\frac{\text{find-node-and-stamp } g \text{ (ParameterNode } i, s) = \text{Some } n}{g \oplus \text{ParameterExpr } i \text{ } s \rightsquigarrow (g, n)}$$

$$\frac{\begin{array}{l} \text{find-node-and-stamp } g \text{ (ParameterNode } i, s) = \text{None} \\ n = \text{get-fresh-id } g \quad g' = \text{add-node } n \text{ (ParameterNode } i, s) \text{ } g \end{array}}{g \oplus \text{ParameterExpr } i \text{ } s \rightsquigarrow (g', n)}$$

$$\frac{\begin{array}{l} g \oplus ce \rightsquigarrow (g2, c) \quad g2 \oplus te \rightsquigarrow (g3, t) \\ g3 \oplus fe \rightsquigarrow (g4, f) \quad s' = \text{meet (stamp } g4 \text{ } t) \text{ (stamp } g4 \text{ } f) \\ \text{find-node-and-stamp } g4 \text{ (ConditionalNode } c \text{ } t \text{ } f, s') = \text{Some } n \end{array}}{g \oplus \text{ConditionalExpr } ce \text{ } te \text{ } fe \rightsquigarrow (g4, n)}$$

$$\frac{\begin{array}{l} g \oplus ce \rightsquigarrow (g2, c) \quad g2 \oplus te \rightsquigarrow (g3, t) \\ g3 \oplus fe \rightsquigarrow (g4, f) \quad s' = \text{meet (stamp } g4 \text{ } t) \text{ (stamp } g4 \text{ } f) \\ \text{find-node-and-stamp } g4 \text{ (ConditionalNode } c \text{ } t \text{ } f, s') = \text{None} \\ n = \text{get-fresh-id } g4 \quad g' = \text{add-node } n \text{ (ConditionalNode } c \text{ } t \text{ } f, s') \text{ } g4 \end{array}}{g \oplus \text{ConditionalExpr } ce \text{ } te \text{ } fe \rightsquigarrow (g', n)}$$

$$\frac{\begin{array}{l} g \oplus xe \rightsquigarrow (g2, x) \\ g2 \oplus ye \rightsquigarrow (g3, y) \quad s' = \text{stamp-binary op (stamp } g3 \text{ } x) \text{ (stamp } g3 \text{ } y) \\ \text{find-node-and-stamp } g3 \text{ (bin-node op } x \text{ } y, s') = \text{Some } n \end{array}}{g \oplus \text{BinaryExpr op } xe \text{ } ye \rightsquigarrow (g3, n)}$$

$$\frac{\begin{array}{l} g \oplus xe \rightsquigarrow (g2, x) \\ g2 \oplus ye \rightsquigarrow (g3, y) \quad s' = \text{stamp-binary op (stamp } g3 \text{ } x) \text{ (stamp } g3 \text{ } y) \\ \text{find-node-and-stamp } g3 \text{ (bin-node op } x \text{ } y, s') = \text{None} \\ n = \text{get-fresh-id } g3 \quad g' = \text{add-node } n \text{ (bin-node op } x \text{ } y, s') \text{ } g3 \end{array}}{g \oplus \text{BinaryExpr op } xe \text{ } ye \rightsquigarrow (g', n)}$$

$$\frac{\begin{array}{l} g \oplus xe \rightsquigarrow (g2, x) \quad s' = \text{stamp-unary op (stamp } g2 \text{ } x) \\ \text{find-node-and-stamp } g2 \text{ (unary-node op } x, s') = \text{Some } n \end{array}}{g \oplus \text{UnaryExpr op } xe \rightsquigarrow (g2, n)}$$

$$\frac{\begin{array}{l} g \oplus xe \rightsquigarrow (g2, x) \quad s' = \text{stamp-unary op (stamp } g2 \text{ } x) \\ \text{find-node-and-stamp } g2 \text{ (unary-node op } x, s') = \text{None} \\ n = \text{get-fresh-id } g2 \quad g' = \text{add-node } n \text{ (unary-node op } x, s') \text{ } g2 \end{array}}{g \oplus \text{UnaryExpr op } xe \rightsquigarrow (g', n)}$$

$$\frac{\text{stamp } g \text{ } n = s \quad \text{is-preevaluated (kind } g \text{ } n)}{g \oplus \text{LeafExpr } n \text{ } s \rightsquigarrow (g, n)}$$

values $\{(n, g) . (eg2\text{-}sq \oplus sq\text{-}param0 \rightsquigarrow (g, n))\}$

2.3 Lift Data-flow Tree Semantics

definition *encodeeval* :: *IRGraph* \Rightarrow *MapState* \Rightarrow *Params* \Rightarrow *ID* \Rightarrow *Value* \Rightarrow *bool*
 $([_, _, _] \vdash - \mapsto - \ 50)$
where
encodeeval *g m p n v* = $(\exists e. (g \vdash n \simeq e) \wedge ([m, p] \vdash e \mapsto v))$

2.4 Graph Refinement

definition *graph-represents-expression* :: *IRGraph* \Rightarrow *ID* \Rightarrow *IRExpr* \Rightarrow *bool*
 $(- \vdash - \leq - \ 50)$
where
 $(g \vdash n \leq e) = (\exists e'. (g \vdash n \simeq e') \wedge (e' \leq e))$

definition *graph-refinement* :: *IRGraph* \Rightarrow *IRGraph* \Rightarrow *bool* **where**
graph-refinement *g1 g2* =
 $((ids\ g_1 \subseteq ids\ g_2) \wedge$
 $(\forall n. n \in ids\ g_1 \longrightarrow (\forall e. (g_1 \vdash n \simeq e) \longrightarrow (g_2 \vdash n \leq e))))$

lemma *graph-refinement*:

graph-refinement *g1 g2* $\implies (\forall n\ m\ p\ v. n \in ids\ g1 \longrightarrow ([g1, m, p] \vdash n \mapsto v) \longrightarrow$
 $([g2, m, p] \vdash n \mapsto v))$
by (*meson encodeeval-def graph-refinement-def graph-represents-expression-def le-expr-def*)

2.5 Maximal Sharing

definition *maximal-sharing*:

maximal-sharing *g* = $(\forall n_1\ n_2. n_1 \in true\text{-}ids\ g \wedge n_2 \in true\text{-}ids\ g \longrightarrow$
 $(\forall e. (g \vdash n_1 \simeq e) \wedge (g \vdash n_2 \simeq e) \wedge (stamp\ g\ n_1 = stamp\ g\ n_2) \longrightarrow n_1 =$
 $n_2))$

end

2.6 Formedness Properties

theory *Form*

imports

Semantics.TreeToGraph

begin

definition *wf-start* **where**

wf-start *g* = $(0 \in ids\ g \wedge$
 $is\text{-}StartNode\ (kind\ g\ 0))$

definition *wf-closed* **where**

wf-closed *g* =
 $(\forall n \in ids\ g .$

$$\begin{aligned} &inputs\ g\ n \subseteq ids\ g \wedge \\ &succ\ g\ n \subseteq ids\ g \wedge \\ &kind\ g\ n \neq NoNode) \end{aligned}$$

definition *wf-phs* **where**

$$\begin{aligned} wf-phs\ g = & \\ &(\forall\ n \in ids\ g. \\ &\quad is-PhiNode\ (kind\ g\ n) \longrightarrow \\ &\quad length\ (ir-values\ (kind\ g\ n)) \\ &= length\ (ir-ends \\ &\quad (kind\ g\ (ir-merge\ (kind\ g\ n)))))) \end{aligned}$$

definition *wf-ends* **where**

$$\begin{aligned} wf-ends\ g = & \\ &(\forall\ n \in ids\ g . \\ &\quad is-AbstractEndNode\ (kind\ g\ n) \longrightarrow \\ &\quad card\ (usages\ g\ n) > 0) \end{aligned}$$

fun *wf-graph* :: *IRGraph* \Rightarrow *bool* **where**

$$wf-graph\ g = (wf-start\ g \wedge wf-closed\ g \wedge wf-phs\ g \wedge wf-ends\ g)$$

lemmas *wf-folds* =

$$\begin{aligned} &wf-graph.simps \\ &wf-start-def \\ &wf-closed-def \\ &wf-phs-def \\ &wf-ends-def \end{aligned}$$

fun *wf-stamps* :: *IRGraph* \Rightarrow *bool* **where**

$$\begin{aligned} wf-stamps\ g = &(\forall\ n \in ids\ g . \\ &(\forall\ v\ m\ p\ e . (g \vdash n \simeq e) \wedge ([m, p] \vdash e \mapsto v) \longrightarrow valid-value\ v\ (stamp-expr\ e))) \end{aligned}$$

fun *wf-stamp* :: *IRGraph* \Rightarrow (*ID* \Rightarrow *Stamp*) \Rightarrow *bool* **where**

$$\begin{aligned} wf-stamp\ g\ s = &(\forall\ n \in ids\ g . \\ &(\forall\ v\ m\ p\ e . (g \vdash n \simeq e) \wedge ([m, p] \vdash e \mapsto v) \longrightarrow valid-value\ v\ (s\ n))) \end{aligned}$$

lemma *wf-empty*: *wf-graph start-end-graph*

unfolding *start-end-graph-def wf-folds by simp*

lemma *wf-eg2-sq*: *wf-graph eg2-sq*

unfolding *eg2-sq-def wf-folds by simp*

fun *wf-logic-node-inputs* :: *IRGraph* \Rightarrow *ID* \Rightarrow *bool* **where**

$$\begin{aligned} wf-logic-node-inputs\ g\ n = & \\ &(\forall\ inp \in set\ (inputs-of\ (kind\ g\ n)) . (\forall\ v\ m\ p . ([g, m, p] \vdash inp \mapsto v) \longrightarrow wf-bool \\ &v)) \end{aligned}$$

fun *wf-values* :: *IRGraph* \Rightarrow *bool* **where**

$$wf-values\ g = (\forall\ n \in ids\ g .$$

$$\begin{aligned}
& (\forall v m p . ([g, m, p] \vdash n \mapsto v) \longrightarrow \\
& \quad (is-LogicNode (kind g n) \longrightarrow \\
& \quad \quad wf-bool v \wedge wf-logic-node-inputs g n)))
\end{aligned}$$

end

2.7 Dynamic Frames

This theory defines two operators, 'unchanged' and 'changeonly', that are useful for specifying which nodes in an IRGraph can change. The dynamic framing idea originates from 'Dynamic Frames' in software verification, started by Ioannis T. Kassios in "Dynamic frames: Support for framing, dependencies and sharing without restrictions", In FM 2006.

theory *IRGraphFrames*

imports

Form

Semantics.IRTreeEval

begin

fun *unchanged* :: *ID set* \Rightarrow *IRGraph* \Rightarrow *IRGraph* \Rightarrow *bool* **where**

unchanged ns g1 g2 = $(\forall n . n \in ns \longrightarrow$
 $(n \in ids\ g1 \wedge n \in ids\ g2 \wedge kind\ g1\ n = kind\ g2\ n \wedge stamp\ g1\ n = stamp\ g2\ n))$

fun *changeonly* :: *ID set* \Rightarrow *IRGraph* \Rightarrow *IRGraph* \Rightarrow *bool* **where**

changeonly ns g1 g2 = $(\forall n . n \in ids\ g1 \wedge n \notin ns \longrightarrow$
 $(n \in ids\ g1 \wedge n \in ids\ g2 \wedge kind\ g1\ n = kind\ g2\ n \wedge stamp\ g1\ n = stamp\ g2\ n))$

lemma *node-unchanged*:

assumes *unchanged ns g1 g2*

assumes *nid* \in *ns*

shows *kind g1 nid* = *kind g2 nid*

using *assms* **by** *auto*

lemma *other-node-unchanged*:

assumes *changeonly ns g1 g2*

assumes *nid* \in *ids g1*

assumes *nid* \notin *ns*

shows *kind g1 nid* = *kind g2 nid*

using *assms*

using *changeonly.simps* **by** *blast*

Some notation for input nodes used

inductive *eval-uses*:: *IRGraph* \Rightarrow *ID* \Rightarrow *ID* \Rightarrow *bool*

for *g* **where**

use0: *nid* \in *ids g*

$\implies eval-uses\ g\ nid\ nid$ |

```

use-inp:  $nid' \in inputs\ g\ n$ 
 $\implies eval-uses\ g\ nid\ nid' \mid$ 

use-trans:  $\llbracket eval-uses\ g\ nid\ nid';$ 
 $eval-uses\ g\ nid'\ nid'' \rrbracket$ 
 $\implies eval-uses\ g\ nid\ nid''$ 

fun eval-usages :: IRGraph  $\Rightarrow$  ID  $\Rightarrow$  ID set where
  eval-usages g nid = {n  $\in$  ids g . eval-uses g nid n}

lemma eval-usages-self:
  assumes nid  $\in$  ids g
  shows nid  $\in$  eval-usages g nid
  using assms eval-usages.simps eval-uses.intros(1)
  by (simp add: ids.rep-eq)

lemma not-in-g-inputs:
  assumes nid  $\notin$  ids g
  shows inputs g nid = {}
proof –
  have k: kind g nid = NoNode using assms not-in-g by blast
  then show ?thesis by (simp add: k)
qed

lemma child-member:
  assumes n = kind g nid
  assumes n  $\neq$  NoNode
  assumes List.member (inputs-of n) child
  shows child  $\in$  inputs g nid
  unfolding inputs.simps using assms
  by (metis in-set-member)

lemma child-member-in:
  assumes nid  $\in$  ids g
  assumes List.member (inputs-of (kind g nid)) child
  shows child  $\in$  inputs g nid
  unfolding inputs.simps using assms
  by (metis child-member ids-some inputs.elims)

lemma inp-in-g:
  assumes n  $\in$  inputs g nid
  shows nid  $\in$  ids g
proof –
  have inputs g nid  $\neq$  {}
  using assms

```

```

    by (metis empty-iff empty-set)
  then have kind g nid  $\neq$  NoNode
    using not-in-g-inputs
    using ids-some by blast
  then show ?thesis
    using not-in-g
    by metis
qed

```

```

lemma inp-in-g-wf:
  assumes wf-graph g
  assumes  $n \in \text{inputs } g \text{ nid}$ 
  shows  $n \in \text{ids } g$ 
  using assms unfolding wf-folds
  using inp-in-g by blast

```

```

lemma kind-unchanged:
  assumes nid  $\in \text{ids } g1$ 
  assumes unchanged (eval-usages g1 nid) g1 g2
  shows kind g1 nid = kind g2 nid
proof -
  show ?thesis
    using assms eval-usages-self
    using unchanged.simps by blast
qed

```

```

lemma stamp-unchanged:
  assumes nid  $\in \text{ids } g1$ 
  assumes unchanged (eval-usages g1 nid) g1 g2
  shows stamp g1 nid = stamp g2 nid
  by (meson assms(1) assms(2) eval-usages-self unchanged.elims(2))

```

```

lemma child-unchanged:
  assumes child  $\in \text{inputs } g1 \text{ nid}$ 
  assumes unchanged (eval-usages g1 nid) g1 g2
  shows unchanged (eval-usages g1 child) g1 g2
  by (smt assms(1) assms(2) eval-usages.simps mem-Collect-eq
      unchanged.simps use-inp use-trans)

```

```

lemma eval-usages:
  assumes us = eval-usages g nid
  assumes nid'  $\in \text{ids } g$ 
  shows eval-uses g nid nid'  $\longleftrightarrow$  nid'  $\in \text{us}$  (is ?P  $\longleftrightarrow$  ?Q)
  using assms eval-usages.simps
  by (simp add: ids.rep-eq)

```

```

lemma inputs-are-uses:

```



```

assumes  $nid' \in inputs\ g\ nid$ 
shows  $eval\text{-}uses\ g\ nid\ nid'$ 
by (metis assms use-inp)

lemma inputs-are-usages:
assumes  $nid' \in inputs\ g\ nid$ 
assumes  $nid' \in ids\ g$ 
shows  $nid' \in eval\text{-}usages\ g\ nid$ 
using assms(1) assms(2) eval-usages inputs-are-uses by blast

lemma inputs-of-are-usages:
assumes  $List.member\ (inputs\text{-}of\ (kind\ g\ nid))\ nid'$ 
assumes  $nid' \in ids\ g$ 
shows  $nid' \in eval\text{-}usages\ g\ nid$ 
by (metis assms(1) assms(2) in-set-member inputs.elims inputs-are-usages)

lemma usage-includes-inputs:
assumes  $us = eval\text{-}usages\ g\ nid$ 
assumes  $ls = inputs\ g\ nid$ 
assumes  $ls \subseteq ids\ g$ 
shows  $ls \subseteq us$ 
using inputs-are-usages eval-usages
using assms(1) assms(2) assms(3) by blast

lemma elim-inp-set:
assumes  $k = kind\ g\ nid$ 
assumes  $k \neq NoNode$ 
assumes  $child \in set\ (inputs\text{-}of\ k)$ 
shows  $child \in inputs\ g\ nid$ 
using assms by auto

lemma encode-in-ids:
assumes  $g \vdash nid \simeq e$ 
shows  $nid \in ids\ g$ 
using assms
apply (induction rule: rep.induct)
apply simp+
by fastforce+

lemma eval-in-ids:
assumes  $[g, m, p] \vdash nid \mapsto v$ 
shows  $nid \in ids\ g$ 
using assms using encodeeval-def encode-in-ids
by auto

lemma transitive-kind-same:
assumes unchanged (eval-usages  $g1\ nid$ )  $g1\ g2$ 
shows  $\forall\ nid' \in (eval\text{-}usages\ g1\ nid) . kind\ g1\ nid' = kind\ g2\ nid'$ 
using assms

```

```

by (meson unchanged.elims(1))

theorem stay-same-encoding:
  assumes nc: unchanged (eval-usages g1 nid) g1 g2
  assumes g1: g1 ⊢ nid ≃ e
  assumes wf: wf-graph g1
  shows g2 ⊢ nid ≃ e
proof -
  have dom: nid ∈ ids g1
  using g1 encode-in-ids by simp
  show ?thesis
using g1 nc wf dom proof (induction e rule: rep.induct)
  case (ConstantNode n c)
  then have kind g2 n = ConstantNode c
  using dom nc kind-unchanged
  by metis
  then show ?case using rep.ConstantNode
  by presburger
next
  case (ParameterNode n i s)
  then have kind g2 n = ParameterNode i
  by (metis kind-unchanged)
  then show ?case
  by (metis ParameterNode.hyps(2) ParameterNode.premis(1) ParameterNode.premis(3)
rep.ParameterNode stamp-unchanged)
next
  case (ConditionalNode n c t f ce te fe)
  then have kind g2 n = ConditionalNode c t f
  by (metis kind-unchanged)
  have c ∈ eval-usages g1 n ∧ t ∈ eval-usages g1 n ∧ f ∈ eval-usages g1 n
  using inputs-of-ConditionalNode
  by (metis ConditionalNode.hyps(1) ConditionalNode.hyps(2) ConditionalNode.hyps(3) ConditionalNode.hyps(4) encode-in-ids inputs.simps inputs-are-usages
list.set-intros(1) set-subset-Cons subset-code(1))
  then show ?case using transitive-kind-same
  by (metis ConditionalNode.hyps(1) ConditionalNode.premis(1) IRNodes.inputs-of-ConditionalNode
⟨kind g2 n = ConditionalNode c t f⟩ child-unchanged inputs.simps list.set-intros(1)
local.ConditionalNode(5) local.ConditionalNode(6) local.ConditionalNode(7) local.ConditionalNode(9)
rep.ConditionalNode set-subset-Cons subset-code(1) unchanged.elims(2))
next
  case (AbsNode n x xe)
  then have kind g2 n = AbsNode x
  using kind-unchanged
  by metis
  then have x ∈ eval-usages g1 n
  using inputs-of-AbsNode
  by (metis AbsNode.hyps(1) AbsNode.hyps(2) encode-in-ids inputs.simps inputs-are-usages list.set-intros(1))
  then show ?case

```

```

    by (metis AbsNode.IH AbsNode.hyps(1) AbsNode.prem(1) AbsNode.prem(3)
      IRNodes.inputs-of-AbsNode ⟨kind g2 n = AbsNode x⟩ child-member-in child-unchanged
      local.wf member-rec(1) rep.AbsNode unchanged.simps)
  next
    case (NotNode n x xe)
    then have kind g2 n = NotNode x
      using kind-unchanged
      by metis
    then have x ∈ eval-usages g1 n
      using inputs-of-NotNode
      by (metis NotNode.hyps(1) NotNode.hyps(2) encode-in-ids inputs.simps in-
        puts-are-usages list.set-intros(1))
    then show ?case
      by (metis NotNode.IH NotNode.hyps(1) NotNode.prem(1) NotNode.prem(3)
        IRNodes.inputs-of-NotNode ⟨kind g2 n = NotNode x⟩ child-member-in child-unchanged
        local.wf member-rec(1) rep.NotNode unchanged.simps)
  next
    case (NegateNode n x xe)
    then have kind g2 n = NegateNode x
      using kind-unchanged by metis
    then have x ∈ eval-usages g1 n
      using inputs-of-NegateNode
      by (metis NegateNode.hyps(1) NegateNode.hyps(2) encode-in-ids inputs.simps
        inputs-are-usages list.set-intros(1))
    then show ?case
      by (metis IRNodes.inputs-of-NegateNode NegateNode.IH NegateNode.hyps(1)
        NegateNode.prem(1) NegateNode.prem(3) ⟨kind g2 n = NegateNode x⟩ child-member-in
        child-unchanged local.wf member-rec(1) rep.NegateNode unchanged.elims(1))
  next
    case (LogicNegationNode n x xe)
    then have kind g2 n = LogicNegationNode x
      using kind-unchanged by metis
    then have x ∈ eval-usages g1 n
      using inputs-of-LogicNegationNode inputs-of-are-usages
      by (metis LogicNegationNode.hyps(1) LogicNegationNode.hyps(2) encode-in-ids
        member-rec(1))
    then show ?case
      by (metis IRNodes.inputs-of-LogicNegationNode LogicNegationNode.IH Logic-
        NegationNode.hyps(1) LogicNegationNode.hyps(2) LogicNegationNode.prem(1) ⟨kind
        g2 n = LogicNegationNode x⟩ child-unchanged encode-in-ids inputs.simps list.set-intros(1)
        local.wf rep.LogicNegationNode)
  next
    case (AddNode n x y xe ye)
    then have kind g2 n = AddNode x y
      using kind-unchanged by metis
    then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
      using inputs-of-LogicNegationNode inputs-of-are-usages
      by (metis AddNode.hyps(1) AddNode.hyps(2) AddNode.hyps(3) IRNodes.inputs-of-AddNode
        encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)

```

```

then show ?case
  by (metis AddNode.IH(1) AddNode.IH(2) AddNode.hyps(1) AddNode.hyps(2)
    AddNode.hyps(3) AddNode.premis(1) IRNodes.inputs-of-AddNode ⟨kind g2 n = AddNode
    x y⟩ child-unchanged encode-in-ids in-set-member inputs.simps local.wf member-rec(1)
    rep.AddNode)
next
  case (MulNode n x y xe ye)
  then have kind g2 n = MulNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
  by (metis MulNode.hyps(1) MulNode.hyps(2) MulNode.hyps(3) IRNodes.inputs-of-MulNode
    encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using MulNode inputs-of-MulNode
    by (metis ⟨kind g2 n = MulNode x y⟩ child-unchanged inputs.simps list.set-intros(1)
    rep.MulNode set-subset-Cons subset-iff unchanged.elims(2))
next
  case (SubNode n x y xe ye)
  then have kind g2 n = SubNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
  by (metis SubNode.hyps(1) SubNode.hyps(2) SubNode.hyps(3) IRNodes.inputs-of-SubNode
    encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using SubNode inputs-of-SubNode
    by (metis ⟨kind g2 n = SubNode x y⟩ child-member child-unchanged encode-in-ids
    ids-some member-rec(1) rep.SubNode)
next
  case (AndNode n x y xe ye)
  then have kind g2 n = AndNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-LogicNegationNode inputs-of-are-usages
  by (metis AndNode.hyps(1) AndNode.hyps(2) AndNode.hyps(3) IRNodes.inputs-of-AndNode
    encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using AndNode inputs-of-AndNode
    by (metis ⟨kind g2 n = AndNode x y⟩ child-unchanged inputs.simps list.set-intros(1)
    rep.AndNode set-subset-Cons subset-iff unchanged.elims(2))
next
  case (OrNode n x y xe ye)
  then have kind g2 n = OrNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-OrNode inputs-of-are-usages
  by (metis OrNode.hyps(1) OrNode.hyps(2) OrNode.hyps(3) IRNodes.inputs-of-OrNode
    encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using OrNode inputs-of-OrNode
    by (metis ⟨kind g2 n = OrNode x y⟩ child-member child-unchanged encode-in-ids
    ids-some member-rec(1) rep.OrNode)

```

```

next
  case (XorNode n x y xe ye)
  then have kind g2 n = XorNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-XorNode inputs-of-are-usages
  by (metis XorNode.hyps(1) XorNode.hyps(2) XorNode.hyps(3) IRNodes.inputs-of-XorNode
    encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using XorNode inputs-of-XorNode
    by (metis ⟨kind g2 n = XorNode x y⟩ child-member child-unchanged encode-in-ids
      ids-some member-rec(1) rep.XorNode)
next
  case (LeftShiftNode n x y xe ye)
  then have kind g2 n = LeftShiftNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-XorNode inputs-of-are-usages
  by (metis LeftShiftNode.hyps(1) LeftShiftNode.hyps(2) LeftShiftNode.hyps(3)
    IRNodes.inputs-of-LeftShiftNode encode-in-ids in-mono inputs.simps inputs-are-usages
    list.set-intros(1) set-subset-Cons)
  then show ?case using LeftShiftNode inputs-of-LeftShiftNode
    by (metis ⟨kind g2 n = LeftShiftNode x y⟩ child-member child-unchanged
      encode-in-ids ids-some member-rec(1) rep.LeftShiftNode)
next
  case (RightShiftNode n x y xe ye)
  then have kind g2 n = RightShiftNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-RightShiftNode inputs-of-are-usages
  by (metis RightShiftNode.hyps(1) RightShiftNode.hyps(2) RightShiftNode.hyps(3)
    IRNodes.inputs-of-RightShiftNode encode-in-ids in-mono inputs.simps inputs-are-usages
    list.set-intros(1) set-subset-Cons)
  then show ?case using RightShiftNode inputs-of-RightShiftNode
    by (metis ⟨kind g2 n = RightShiftNode x y⟩ child-member child-unchanged
      encode-in-ids ids-some member-rec(1) rep.RightShiftNode)
next
  case (UnsignedRightShiftNode n x y xe ye)
  then have kind g2 n = UnsignedRightShiftNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-UnsignedRightShiftNode inputs-of-are-usages
  by (metis UnsignedRightShiftNode.hyps(1) UnsignedRightShiftNode.hyps(2) Un-
    signedRightShiftNode.hyps(3) IRNodes.inputs-of-UnsignedRightShiftNode encode-in-ids
    in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using UnsignedRightShiftNode inputs-of-UnsignedRightShiftNode
    by (metis ⟨kind g2 n = UnsignedRightShiftNode x y⟩ child-member child-unchanged
      encode-in-ids ids-some member-rec(1) rep.UnsignedRightShiftNode)
next
  case (IntegerBelowNode n x y xe ye)

```

```

then have kind g2 n = IntegerBelowNode x y
  using kind-unchanged by metis
then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
  using inputs-of-IntegerBelowNode inputs-of-are-usages
  by (metis IntegerBelowNode.hyps(1) IntegerBelowNode.hyps(2) IntegerBelowNode.hyps(3) IRNodes.inputs-of-IntegerBelowNode encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
  then show ?case using IntegerBelowNode inputs-of-IntegerBelowNode
    by (metis ⟨kind g2 n = IntegerBelowNode x y⟩ child-member child-unchanged encode-in-ids ids-some member-rec(1) rep.IntegerBelowNode)
next
  case (IntegerEqualsNode n x y xe ye)
  then have kind g2 n = IntegerEqualsNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-IntegerEqualsNode inputs-of-are-usages
    by (metis IntegerEqualsNode.hyps(1) IntegerEqualsNode.hyps(2) IntegerEqualsNode.hyps(3) IRNodes.inputs-of-IntegerEqualsNode encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
    then show ?case using IntegerEqualsNode inputs-of-IntegerEqualsNode
      by (metis ⟨kind g2 n = IntegerEqualsNode x y⟩ child-member child-unchanged encode-in-ids ids-some member-rec(1) rep.IntegerEqualsNode)
next
  case (IntegerLessThanNode n x y xe ye)
  then have kind g2 n = IntegerLessThanNode x y
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n ∧ y ∈ eval-usages g1 n
    using inputs-of-IntegerLessThanNode inputs-of-are-usages
    by (metis IntegerLessThanNode.hyps(1) IntegerLessThanNode.hyps(2) IntegerLessThanNode.hyps(3) IRNodes.inputs-of-IntegerLessThanNode encode-in-ids in-mono inputs.simps inputs-are-usages list.set-intros(1) set-subset-Cons)
    then show ?case using IntegerLessThanNode inputs-of-IntegerLessThanNode
      by (metis ⟨kind g2 n = IntegerLessThanNode x y⟩ child-member child-unchanged encode-in-ids ids-some member-rec(1) rep.IntegerLessThanNode)
next
  case (NarrowNode n ib rb x xe)
  then have kind g2 n = NarrowNode ib rb x
    using kind-unchanged by metis
  then have x ∈ eval-usages g1 n
    using inputs-of-NarrowNode inputs-of-are-usages
    by (metis NarrowNode.hyps(1) NarrowNode.hyps(2) IRNodes.inputs-of-NarrowNode encode-in-ids inputs.simps inputs-are-usages list.set-intros(1))
    then show ?case using NarrowNode inputs-of-NarrowNode
      by (metis ⟨kind g2 n = NarrowNode ib rb x⟩ child-unchanged inputs.elims list.set-intros(1) rep.NarrowNode unchanged.simps)
next
  case (SignExtendNode n ib rb x xe)
  then have kind g2 n = SignExtendNode ib rb x
    using kind-unchanged by metis

```

```

then have  $x \in \text{eval-usages } g1 \ n$ 
  using  $\text{inputs-of-SignExtendNode inputs-of-are-usages}$ 
  by ( $\text{metis SignExtendNode.hyps(1) SignExtendNode.hyps(2) encode-in-ids inputs.simps inputs-are-usages list.set-intros(1)}$ )
  then show  $?case$  using  $\text{SignExtendNode inputs-of-SignExtendNode}$ 
  by ( $\text{metis } \langle \text{kind } g2 \ n = \text{SignExtendNode } ib \ rb \ x \rangle \text{ child-member-in child-unchanged in-set-member list.set-intros(1) rep.SignExtendNode unchanged.elims(2)}$ )
next
  case ( $\text{ZeroExtendNode } n \ ib \ rb \ x \ xe$ )
  then have  $\text{kind } g2 \ n = \text{ZeroExtendNode } ib \ rb \ x$ 
  using  $\text{kind-unchanged}$  by  $\text{metis}$ 
  then have  $x \in \text{eval-usages } g1 \ n$ 
  using  $\text{inputs-of-ZeroExtendNode inputs-of-are-usages}$ 
  by ( $\text{metis ZeroExtendNode.hyps(1) ZeroExtendNode.hyps(2) IRNodes.inputs-of-ZeroExtendNode encode-in-ids inputs.simps inputs-are-usages list.set-intros(1)}$ )
  then show  $?case$  using  $\text{ZeroExtendNode inputs-of-ZeroExtendNode}$ 
  by ( $\text{metis } \langle \text{kind } g2 \ n = \text{ZeroExtendNode } ib \ rb \ x \rangle \text{ child-member-in child-unchanged member-rec(1) rep.ZeroExtendNode unchanged.simps}$ )
next
  case ( $\text{LeafNode } n \ s$ )
  then show  $?case$ 
  by ( $\text{metis kind-unchanged rep.LeafNode stamp-unchanged}$ )
next
  case ( $\text{RefNode } n \ n'$ )
  then have  $\text{kind } g2 \ n = \text{RefNode } n'$ 
  using  $\text{kind-unchanged}$  by  $\text{metis}$ 
  then have  $n' \in \text{eval-usages } g1 \ n$ 
  by ( $\text{metis IRNodes.inputs-of-RefNode RefNode.hyps(1) RefNode.hyps(2) encode-in-ids inputs.elims inputs-are-usages list.set-intros(1)}$ )
  then show  $?case$ 
  by ( $\text{metis IRNodes.inputs-of-RefNode RefNode.IH RefNode.hyps(1) RefNode.hyps(2) RefNode.premis(1) } \langle \text{kind } g2 \ n = \text{RefNode } n' \rangle \text{ child-unchanged encode-in-ids inputs.elims list.set-intros(1) local.wf rep.RefNode}$ )
qed
qed

```

theorem *stay-same:*

```

assumes  $nc: \text{unchanged } (\text{eval-usages } g1 \ nid) \ g1 \ g2$ 
assumes  $g1: [g1, m, p] \vdash nid \mapsto v1$ 
assumes  $wf: \text{wf-graph } g1$ 
shows  $[g2, m, p] \vdash nid \mapsto v1$ 
proof –
  have  $nid: nid \in \text{ids } g1$ 
  using  $g1 \text{ eval-in-ids}$  by  $\text{simp}$ 
  then have  $nid \in \text{eval-usages } g1 \ nid$ 
  using  $\text{eval-usages-self}$  by  $\text{blast}$ 
  then have  $\text{kind-same: kind } g1 \ nid = \text{kind } g2 \ nid$ 

```

```

    using nc node-unchanged by blast
  obtain e where e: (g1 ⊢ nid ≃ e) ∧ ([m,p] ⊢ e ↦ v1)
    using encodeeval-def g1
    by auto
  then have val: [m,p] ⊢ e ↦ v1
    using g1 encodeeval-def
    by simp
  then show ?thesis using e nid nc
    unfolding encodeeval-def
  proof (induct e v1 arbitrary: nid rule: evaltree.induct)
    case (ConstantExpr c)
    then show ?case
      by (meson local.wf stay-same-encoding)
  next
    case (ParameterExpr i s)
    have g2 ⊢ nid ≃ ParameterExpr i s
      using stay-same-encoding ParameterExpr
      by (meson local.wf)
    then show ?case using evaltree.ParameterExpr
      by (meson ParameterExpr.hyps)
  next
    case (ConditionalExpr ce cond branch te fe v)
    then have g2 ⊢ nid ≃ ConditionalExpr ce te fe
      using ConditionalExpr.prem1 ConditionalExpr.prem3 local.wf stay-same-encoding
      by presburger
    then show ?case
      by (meson ConditionalExpr.prem1 ConditionalExpr.prem3 local.wf
        stay-same-encoding)
  next
    case (UnaryExpr xe v op)
    then show ?case
      using local.wf stay-same-encoding by blast
  next
    case (BinaryExpr xe x ye y op)
    then show ?case
      using local.wf stay-same-encoding by blast
  next
    case (LeafExpr val nid s)
    then show ?case
      by (metis local.wf stay-same-encoding)
qed
qed

```

```

lemma add-changed:
  assumes gup = add-node new k g
  shows changeonly {new} g gup
  using assms unfolding add-node-def changeonly.simps
  using add-node.rep-eq add-node-def kind.rep-eq stamp.rep-eq by simp

```



```

lemma disjoint-change:
  assumes changeonly change g gup
  assumes nochange = ids g - change
  shows unchanged nochange g gup
  using assms unfolding changeonly.simps unchanged.simps
  by blast

lemma add-node-unchanged:
  assumes new ∉ ids g
  assumes nid ∈ ids g
  assumes gup = add-node new k g
  assumes wf-graph g
  shows unchanged (eval-usages g nid) g gup
proof –
  have new ∉ (eval-usages g nid) using assms
    using eval-usages.simps by blast
  then have changeonly {new} g gup
    using assms add-changed by blast
  then show ?thesis using assms add-node-def disjoint-change
    using Diff-insert-absorb by auto
qed

lemma eval-uses-imp:
   $((nid' \in ids\ g \wedge nid = nid') \vee$ 
     $nid' \in inputs\ g\ nid \vee (\exists nid'' . eval-uses\ g\ nid\ nid'' \wedge eval-uses\ g\ nid''\ nid'))$ 
     $\longleftrightarrow eval-uses\ g\ nid\ nid'$ 
  using use0 use-inp use-trans
  by (meson eval-uses.simps)

lemma wf-use-ids:
  assumes wf-graph g
  assumes nid ∈ ids g
  assumes eval-uses g nid nid'
  shows nid' ∈ ids g
  using assms(3)
proof (induction rule: eval-uses.induct)
  case use0
    then show ?case by simp
  next
    case use-inp
    then show ?case
      using assms(1) inp-in-g-wf by blast
  next
    case use-trans
    then show ?case by blast
qed

```

```

lemma no-external-use:
  assumes wf-graph g
  assumes nid' ∉ ids g
  assumes nid ∈ ids g
  shows  $\neg(\text{eval-uses } g \text{ nid nid'})$ 
proof –
  have 0: nid ≠ nid'
    using assms by blast
  have inp: nid' ∉ inputs g nid
    using assms
    using inp-in-g-wf by blast
  have rec-0: ∄ n . n ∈ ids g ∧ n = nid'
    using assms by blast
  have rec-inp: ∄ n . n ∈ ids g ∧ n ∈ inputs g nid'
    using assms(2) inp-in-g by blast
  have rec: ∄ nid'' . eval-uses g nid nid'' ∧ eval-uses g nid'' nid'
    using wf-use-ids assms(1) assms(2) assms(3) by blast
  from inp 0 rec show ?thesis
    using eval-uses-imp by blast
qed

end

```

2.8 Tree to Graph Theorems

```

theory TreeToGraphThms
imports
  TreeToGraph
  IRTreeEvalThms
  IRGraphFrames
  HOL-Eisbach.Eisbach
  HOL-Eisbach.Eisbach-Tools
begin

```

2.8.1 Extraction and Evaluation of Expression Trees is Deterministic.

First, we prove some extra rules that relate each type of `IRNode` to the corresponding `IRExpr` type that 'rep' will produce. These are very helpful for proving that 'rep' is deterministic.

named-theorems *rep*

```

lemma rep-constant [rep]:
   $g \vdash n \simeq e \implies$ 
   $\text{kind } g \text{ } n = \text{ConstantNode } c \implies$ 
   $e = \text{ConstantExpr } c$ 
  by (induction rule: rep.induct; auto)

```

```

lemma rep-parameter [rep]:

```

$g \vdash n \simeq e \implies$
 $kind\ g\ n = ParameterNode\ i \implies$
 $(\exists s. e = ParameterExpr\ i\ s)$
by (induction rule: *rep.induct*; *auto*)

lemma *rep-conditional* [*rep*]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = ConditionalNode\ c\ t\ f \implies$
 $(\exists ce\ te\ fe. e = ConditionalExpr\ ce\ te\ fe)$
by (induction rule: *rep.induct*; *auto*)

lemma *rep-abs* [*rep*]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = AbsNode\ x \implies$
 $(\exists xe. e = UnaryExpr\ UnaryAbs\ xe)$
by (induction rule: *rep.induct*; *auto*)

lemma *rep-not* [*rep*]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = NotNode\ x \implies$
 $(\exists xe. e = UnaryExpr\ UnaryNot\ xe)$
by (induction rule: *rep.induct*; *auto*)

lemma *rep-negate* [*rep*]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = NegateNode\ x \implies$
 $(\exists xe. e = UnaryExpr\ UnaryNeg\ xe)$
by (induction rule: *rep.induct*; *auto*)

lemma *rep-logicnegation* [*rep*]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = LogicNegationNode\ x \implies$
 $(\exists xe. e = UnaryExpr\ UnaryLogicNegation\ xe)$
by (induction rule: *rep.induct*; *auto*)

lemma *rep-add* [*rep*]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = AddNode\ x\ y \implies$
 $(\exists xe\ ye. e = BinaryExpr\ BinAdd\ xe\ ye)$
by (induction rule: *rep.induct*; *auto*)

lemma *rep-sub* [*rep*]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = SubNode\ x\ y \implies$
 $(\exists xe\ ye. e = BinaryExpr\ BinSub\ xe\ ye)$
by (induction rule: *rep.induct*; *auto*)

lemma *rep-mul* [*rep*]:
 $g \vdash n \simeq e \implies$

$kind\ g\ n = MulNode\ x\ y \implies$
 $(\exists\ xe\ ye. e = BinaryExpr\ BinMul\ xe\ ye)$
by (induction rule: rep.induct; auto)

lemma rep-and [rep]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = AndNode\ x\ y \implies$
 $(\exists\ xe\ ye. e = BinaryExpr\ BinAnd\ xe\ ye)$
by (induction rule: rep.induct; auto)

lemma rep-or [rep]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = OrNode\ x\ y \implies$
 $(\exists\ xe\ ye. e = BinaryExpr\ BinOr\ xe\ ye)$
by (induction rule: rep.induct; auto)

lemma rep-xor [rep]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = XorNode\ x\ y \implies$
 $(\exists\ xe\ ye. e = BinaryExpr\ BinXor\ xe\ ye)$
by (induction rule: rep.induct; auto)

lemma rep-left-shift [rep]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = LeftShiftNode\ x\ y \implies$
 $(\exists\ xe\ ye. e = BinaryExpr\ BinLeftShift\ xe\ ye)$
by (induction rule: rep.induct; auto)

lemma rep-right-shift [rep]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = RightShiftNode\ x\ y \implies$
 $(\exists\ xe\ ye. e = BinaryExpr\ BinRightShift\ xe\ ye)$
by (induction rule: rep.induct; auto)

lemma rep-unsigned-right-shift [rep]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = UnsignedRightShiftNode\ x\ y \implies$
 $(\exists\ xe\ ye. e = BinaryExpr\ BinURightShift\ xe\ ye)$
by (induction rule: rep.induct; auto)

lemma rep-integer-below [rep]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = IntegerBelowNode\ x\ y \implies$
 $(\exists\ xe\ ye. e = BinaryExpr\ BinIntegerBelow\ xe\ ye)$
by (induction rule: rep.induct; auto)

lemma rep-integer-equals [rep]:
 $g \vdash n \simeq e \implies$
 $kind\ g\ n = IntegerEqualsNode\ x\ y \implies$

($\exists xe ye. e = \text{BinaryExpr BinIntegerEquals } xe ye$)
by (induction rule: *rep.induct*; auto)

lemma *rep-integer-less-than* [*rep*]:
 $g \vdash n \simeq e \implies$
 $\text{kind } g \ n = \text{IntegerLessThanNode } x \ y \implies$
($\exists xe ye. e = \text{BinaryExpr BinIntegerLessThan } xe ye$)
by (induction rule: *rep.induct*; auto)

lemma *rep-narrow* [*rep*]:
 $g \vdash n \simeq e \implies$
 $\text{kind } g \ n = \text{NarrowNode } ib \ rb \ x \implies$
($\exists x. e = \text{UnaryExpr } (\text{UnaryNarrow } ib \ rb) \ x$)
by (induction rule: *rep.induct*; auto)

lemma *rep-sign-extend* [*rep*]:
 $g \vdash n \simeq e \implies$
 $\text{kind } g \ n = \text{SignExtendNode } ib \ rb \ x \implies$
($\exists x. e = \text{UnaryExpr } (\text{UnarySignExtend } ib \ rb) \ x$)
by (induction rule: *rep.induct*; auto)

lemma *rep-zero-extend* [*rep*]:
 $g \vdash n \simeq e \implies$
 $\text{kind } g \ n = \text{ZeroExtendNode } ib \ rb \ x \implies$
($\exists x. e = \text{UnaryExpr } (\text{UnaryZeroExtend } ib \ rb) \ x$)
by (induction rule: *rep.induct*; auto)

lemma *rep-load-field* [*rep*]:
 $g \vdash n \simeq e \implies$
 $\text{is-preevaluated } (\text{kind } g \ n) \implies$
($\exists s. e = \text{LeafExpr } n \ s$)
by (induction rule: *rep.induct*; auto)

lemma *rep-ref* [*rep*]:
 $g \vdash n \simeq e \implies$
 $\text{kind } g \ n = \text{RefNode } n' \implies$
 $g \vdash n' \simeq e$
by (induction rule: *rep.induct*; auto)

method *solve-det* **uses** *node* =
(match *node* **in** *kind* - - = *node* - **for** *node* \implies
 <match *rep* **in** *r*: - \implies - = *node* - \implies - \implies
 <match *IRNode.inject* **in** *i*: (*node* - = *node* -) = - \implies
 <match *RepE* **in** *e*: - \implies ($\bigwedge x. - = \text{node } x \implies -$) \implies - \implies
 <match *IRNode.distinct* **in** *d*: *node* - \neq *RefNode* - \implies
 <metis *i e r d*>>>> |
 match *node* **in** *kind* - - = *node* - - **for** *node* \implies
 <match *rep* **in** *r*: - \implies - = *node* - - \implies - \implies

```

    <match IRNode.inject in i: (node - - = node - -) = - =>
      <match RepE in e: - => (Λx y. - = node x y => -) => - =>
        <match IRNode.distinct in d: node - - ≠ RefNode - =>
          <metis i e r d>>>> |
  match node in kind - - = node - - - for node =>
    <match rep in r: - => - = node - - - => - =>
      <match IRNode.inject in i: (node - - - = node - - -) = - =>
        <match RepE in e: - => (Λx y z. - = node x y z => -) => - =>
          <match IRNode.distinct in d: node - - - ≠ RefNode - =>
            <metis i e r d>>>> |
  match node in kind - - = node - - - for node =>
    <match rep in r: - => - = node - - - => - =>
      <match IRNode.inject in i: (node - - - = node - - -) = - =>
        <match RepE in e: - => (Λx. - = node - - x => -) => - =>
          <match IRNode.distinct in d: node - - - ≠ RefNode - =>
            <metis i e r d>>>>)

```

Now we can prove that 'rep' and 'eval', and their list versions, are deterministic.

lemma *repDet*:

shows $(g \vdash n \simeq e_1) \implies (g \vdash n \simeq e_2) \implies e_1 = e_2$

proof (*induction arbitrary: e₂ rule: rep.induct*)

case (*ConstantNode n c*)

then show ?case **using** *rep-constant* **by** *auto*

next

case (*ParameterNode n i s*)

then show ?case

by (*metis IRNode.disc(2685) ParameterNodeE is-RefNode-def rep-parameter*)

next

case (*ConditionalNode n c t f ce te fe*)

then show ?case

using *IRNode.distinct(593)*

using *IRNode.inject(6) ConditionalNodeE rep-conditional*

by *metis*

next

case (*AbsNode n x xe*)

then show ?case

by (*solve-det node: AbsNode*)

next

case (*NotNode n x xe*)

then show ?case

by (*solve-det node: NotNode*)

next

case (*NegateNode n x xe*)

then show ?case

by (*solve-det node: NegateNode*)

next

case (*LogicNegationNode n x xe*)

then show ?case

```

    by (solve-det node: LogicNegationNode)
next
  case (AddNode n x y xe ye)
  then show ?case
    by (solve-det node: AddNode)
next
  case (MulNode n x y xe ye)
  then show ?case
    by (solve-det node: MulNode)
next
  case (SubNode n x y xe ye)
  then show ?case
    by (solve-det node: SubNode)
next
  case (AndNode n x y xe ye)
  then show ?case
    by (solve-det node: AndNode)
next
  case (OrNode n x y xe ye)
  then show ?case
    by (solve-det node: OrNode)
next
  case (XorNode n x y xe ye)
  then show ?case
    by (solve-det node: XorNode)
next
  case (LeftShiftNode n x y xe ye)
  then show ?case
    by (solve-det node: LeftShiftNode)
next
  case (RightShiftNode n x y xe ye)
  then show ?case
    by (solve-det node: RightShiftNode)
next
  case (UnsignedRightShiftNode n x y xe ye)
  then show ?case
    by (solve-det node: UnsignedRightShiftNode)
next
  case (IntegerBelowNode n x y xe ye)
  then show ?case
    by (solve-det node: IntegerBelowNode)
next
  case (IntegerEqualsNode n x y xe ye)
  then show ?case
    by (solve-det node: IntegerEqualsNode)
next
  case (IntegerLessThanNode n x y xe ye)
  then show ?case
    by (solve-det node: IntegerLessThanNode)

```

```

next
  case (NarrowNode n x xe)
  then show ?case
    by (metis IRNode.distinct(2203) IRNode.inject(28) NarrowNodeE rep-narrow)
next
  case (SignExtendNode n x xe)
  then show ?case
    by (metis IRNode.distinct(2599) IRNode.inject(39) SignExtendNodeE rep-sign-extend)
next
  case (ZeroExtendNode n x xe)
  then show ?case
    by (metis IRNode.distinct(2753) IRNode.inject(50) ZeroExtendNodeE rep-zero-extend)
next
  case (LeafNode n s)
  then show ?case using rep-load-field LeafNodeE
    by (metis is-preevaluated.simps(53))
next
  case (RefNode n^)
  then show ?case
    using rep-ref by blast
qed

```

```

lemma repAllDet:
  g ⊢ xs ≃L e1 ⟹
  g ⊢ xs ≃L e2 ⟹
  e1 = e2
proof (induction arbitrary: e2 rule: replist.induct)
  case RepNil
  then show ?case
    using replist.cases by auto
next
  case (RepCons x xe xs xse)
  then show ?case
    by (metis list.distinct(1) list.sel(1) list.sel(3) repDet replist.cases)
qed

```

```

lemma encodeEvalDet:
  [g,m,p] ⊢ e ↦ v1 ⟹
  [g,m,p] ⊢ e ↦ v2 ⟹
  v1 = v2
  by (metis encodeeval-def evalDet repDet)

```

```

lemma graphDet: ([g,m,p] ⊢ n ↦ v1) ∧ ([g,m,p] ⊢ n ↦ v2) ⟹ v1 = v2
  using encodeEvalDet by blast

```

2.8.2 Monotonicity of Graph Refinement

Lift refinement monotonicity to graph level. Hopefully these shouldn't really be required.

lemma *mono-abs*:

assumes $kind\ g1\ n = AbsNode\ x \wedge kind\ g2\ n = AbsNode\ x$
assumes $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$
assumes $xe1 \geq xe2$
assumes $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$
shows $e1 \geq e2$
by (*metis* *AbsNode* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *mono-unary repDet*)

lemma *mono-not*:

assumes $kind\ g1\ n = NotNode\ x \wedge kind\ g2\ n = NotNode\ x$
assumes $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$
assumes $xe1 \geq xe2$
assumes $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$
shows $e1 \geq e2$
by (*metis* *NotNode* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *mono-unary repDet*)

lemma *mono-negate*:

assumes $kind\ g1\ n = NegateNode\ x \wedge kind\ g2\ n = NegateNode\ x$
assumes $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$
assumes $xe1 \geq xe2$
assumes $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$
shows $e1 \geq e2$
by (*metis* *NegateNode* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *mono-unary repDet*)

lemma *mono-logic-negation*:

assumes $kind\ g1\ n = LogicNegationNode\ x \wedge kind\ g2\ n = LogicNegationNode\ x$
assumes $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$
assumes $xe1 \geq xe2$
assumes $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$
shows $e1 \geq e2$
by (*metis* *LogicNegationNode* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *mono-unary repDet*)

lemma *mono-narrow*:

assumes $kind\ g1\ n = NarrowNode\ ib\ rb\ x \wedge kind\ g2\ n = NarrowNode\ ib\ rb\ x$
assumes $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$
assumes $xe1 \geq xe2$
assumes $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$
shows $e1 \geq e2$
using *assms* *mono-unary repDet* *NarrowNode*
by *metis*

lemma *mono-sign-extend*:

assumes $kind\ g1\ n = SignExtendNode\ ib\ rb\ x \wedge kind\ g2\ n = SignExtendNode\ ib\ rb\ x$
assumes $(g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)$
assumes $xe1 \geq xe2$
assumes $(g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)$
shows $e1 \geq e2$

by (metis SignExtendNode assms(1) assms(2) assms(3) assms(4) mono-unary repDet)

lemma *mono-zero-extend*:

assumes kind g1 n = ZeroExtendNode ib rb x \wedge kind g2 n = ZeroExtendNode ib rb x
 assumes (g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)
 assumes xe1 \geq xe2
 assumes (g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)
 shows e1 \geq e2
 using assms mono-unary repDet ZeroExtendNode
 by metis

lemma *mono-conditional-graph*:

assumes kind g1 n = ConditionalNode c t f \wedge kind g2 n = ConditionalNode c t f
 assumes (g1 \vdash c \simeq ce1) \wedge (g2 \vdash c \simeq ce2)
 assumes (g1 \vdash t \simeq te1) \wedge (g2 \vdash t \simeq te2)
 assumes (g1 \vdash f \simeq fe1) \wedge (g2 \vdash f \simeq fe2)
 assumes ce1 \geq ce2 \wedge te1 \geq te2 \wedge fe1 \geq fe2
 assumes (g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)
 shows e1 \geq e2
 using ConditionalNodeE IRNode.inject(6) assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) mono-conditional repDet rep-conditional
 by (smt (verit, best) ConditionalNode)

lemma *mono-add*:

assumes kind g1 n = AddNode x y \wedge kind g2 n = AddNode x y
 assumes (g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)
 assumes (g1 \vdash y \simeq ye1) \wedge (g2 \vdash y \simeq ye2)
 assumes xe1 \geq xe2 \wedge ye1 \geq ye2
 assumes (g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)
 shows e1 \geq e2
 using mono-binary assms AddNodeE IRNode.inject(2) repDet rep-add
 by (metis IRNode.distinct(205))

lemma *mono-mul*:

assumes kind g1 n = MulNode x y \wedge kind g2 n = MulNode x y
 assumes (g1 \vdash x \simeq xe1) \wedge (g2 \vdash x \simeq xe2)
 assumes (g1 \vdash y \simeq ye1) \wedge (g2 \vdash y \simeq ye2)
 assumes xe1 \geq xe2 \wedge ye1 \geq ye2
 assumes (g1 \vdash n \simeq e1) \wedge (g2 \vdash n \simeq e2)
 shows e1 \geq e2
 using mono-binary assms IRNode.inject(27) MulNodeE repDet rep-mul
 by (smt (verit, best) MulNode)

lemma *term-graph-evaluation*:

(g \vdash n \leq e) \implies (\forall m p v . ([m,p] \vdash e \mapsto v) \longrightarrow ([g,m,p] \vdash n \mapsto v))
 unfolding graph-represents-expression-def apply auto

```

by (meson encodeeval-def)

lemma encodes-contains:
  g ⊢ n ≃ e ⇒
  kind g n ≠ NoNode
  apply (induction rule: rep.induct)
  apply (match IRNode.distinct in e: ?n ≠ NoNode ⇒
    ⟨presburger add: e⟩) +
  apply force
  by fastforce

lemma no-encoding:
  assumes n ∉ ids g
  shows ¬(g ⊢ n ≃ e)
  using assms apply simp apply (rule notI) by (induction e; simp add: en-
codes-contains)

lemma not-excluded-keep-type:
  assumes n ∈ ids g1
  assumes n ∉ excluded
  assumes (excluded ⊆ as-set g1) ⊆ as-set g2
  shows kind g1 n = kind g2 n ∧ stamp g1 n = stamp g2 n
  using assms unfolding as-set-def domain-subtraction-def by blast

method metis-node-eq-unary for node :: 'a ⇒ IRNode =
  (match IRNode.inject in i: (node - = node -) = - ⇒
    ⟨metis i⟩)
method metis-node-eq-binary for node :: 'a ⇒ 'a ⇒ IRNode =
  (match IRNode.inject in i: (node - - = node - -) = - ⇒
    ⟨metis i⟩)
method metis-node-eq-ternary for node :: 'a ⇒ 'a ⇒ 'a ⇒ IRNode =
  (match IRNode.inject in i: (node - - - = node - - -) = - ⇒
    ⟨metis i⟩)



### 2.8.3 Lift Data-flow Tree Refinement to Graph Refinement



theorem graph-semantics-preservation:
  assumes a: e1' ≥ e2'
  assumes b: ({n'} ⊆ as-set g1) ⊆ as-set g2
  assumes c: g1 ⊢ n' ≃ e1'
  assumes d: g2 ⊢ n' ≃ e2'
  shows graph-refinement g1 g2
  unfolding graph-refinement-def apply rule
  apply (metis b d ids-some no-encoding not-excluded-keep-type singleton-iff sub-
setI)
  apply (rule allI) apply (rule impI) apply (rule allI) apply (rule impI)
  unfolding graph-represents-expression-def
proof -
  fix n e1

```

```

assume  $e: n \in \text{ids } g1$ 
assume  $f: (g1 \vdash n \simeq e1)$ 

show  $\exists e2. (g2 \vdash n \simeq e2) \wedge e1 \geq e2$ 
proof ( $\text{cases } n = n'$ )
  case True
    have  $g: e1 = e1'$  using  $c \ f \ \text{True} \ \text{repDet} \ \text{by} \ \text{simp}$ 
    have  $h: (g2 \vdash n \simeq e2') \wedge e1' \geq e2'$ 
      using  $\text{True} \ a \ d \ \text{by} \ \text{blast}$ 
    then show  $?thesis$ 
      using  $g \ \text{by} \ \text{blast}$ 
  next
    case False
    have  $n \notin \{n'\}$ 
      using False by simp
    then have  $i: \text{kind } g1 \ n = \text{kind } g2 \ n \wedge \text{stamp } g1 \ n = \text{stamp } g2 \ n$ 
      using not-excluded-keep-type
      using  $b \ e \ \text{by} \ \text{presburger}$ 
    show  $?thesis$  using  $f \ i$ 
    proof (induction  $e1$ )
      case (ConstantNode  $n \ c$ )
        then show  $?case$ 
          by (metis eq-refl rep.ConstantNode)
      next
        case (ParameterNode  $n \ i \ s$ )
        then show  $?case$ 
          by (metis eq-refl rep.ParameterNode)
      next
        case (ConditionalNode  $n \ c \ t \ f \ ce1 \ te1 \ fe1$ )
        have  $k: g1 \vdash n \simeq \text{ConditionalExpr } ce1 \ te1 \ fe1$  using  $f \ \text{ConditionalNode}$ 
          by (simp add: ConditionalNode.hyps(2) rep.ConditionalNode)
        obtain  $cn \ tn \ fn$  where  $l: \text{kind } g1 \ n = \text{ConditionalNode } cn \ tn \ fn$ 
          using ConditionalNode.hyps(1) by blast
        then have  $mc: g1 \vdash cn \simeq ce1$ 
          using ConditionalNode.hyps(1) ConditionalNode.hyps(2) by fastforce
        from  $l$  have  $mt: g1 \vdash tn \simeq te1$ 
          using ConditionalNode.hyps(1) ConditionalNode.hyps(3) by fastforce
        from  $l$  have  $mf: g1 \vdash fn \simeq fe1$ 
          using ConditionalNode.hyps(1) ConditionalNode.hyps(4) by fastforce
        then show  $?case$ 
        proof –
          have  $g1 \vdash cn \simeq ce1$  using  $mc \ \text{by} \ \text{simp}$ 
          have  $g1 \vdash tn \simeq te1$  using  $mt \ \text{by} \ \text{simp}$ 
          have  $g1 \vdash fn \simeq fe1$  using  $mf \ \text{by} \ \text{simp}$ 
          have  $cer: \exists ce2. (g2 \vdash cn \simeq ce2) \wedge ce1 \geq ce2$ 
            using ConditionalNode
            using  $a \ b \ c \ d \ l \ \text{no-encoding} \ \text{not-excluded-keep-type} \ \text{repDet} \ \text{singletonD}$ 
            by (metis-node-eq-ternary ConditionalNode)
          have  $ter: \exists te2. (g2 \vdash tn \simeq te2) \wedge te1 \geq te2$ 

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    using ConditionalNode a b c d l no-encoding not-excluded-keep-type repDet
singletonD
    by (metis-node-eq-ternary ConditionalNode)
    have  $\exists fe2. (g2 \vdash fn \simeq fe2) \wedge fe1 \geq fe2$ 
    using ConditionalNode a b c d l no-encoding not-excluded-keep-type repDet
singletonD
    by (metis-node-eq-ternary ConditionalNode)
    then have  $\exists ce2 te2 fe2. (g2 \vdash n \simeq ConditionalExpr\ ce2\ te2\ fe2) \wedge$ 
ConditionalExpr ce1 te1 fe1  $\geq ConditionalExpr\ ce2\ te2\ fe2$ 
    using ConditionalNode.premis l rep.ConditionalNode cer ter
    by (smt (verit) mono-conditional)
    then show ?thesis
    by meson
qed
next
case (AbsNode n x xe1)
have k:  $g1 \vdash n \simeq UnaryExpr\ UnaryAbs\ xe1$  using f AbsNode
    by (simp add: AbsNode.hyps(2) rep.AbsNode)
obtain xn where l: kind g1 n = AbsNode xn
    using AbsNode.hyps(1) by blast
then have m:  $g1 \vdash xn \simeq xe1$ 
    using AbsNode.hyps(1) AbsNode.hyps(2) by fastforce
then show ?case
proof (cases xn = n')
case True
    then have n:  $xe1 = e1'$  using c m repDet by simp
    then have ev:  $g2 \vdash n \simeq UnaryExpr\ UnaryAbs\ e2'$  using AbsNode.hyps(1)
l m n
        using AbsNode.premis True d rep.AbsNode by simp
    then have r:  $UnaryExpr\ UnaryAbs\ e1' \geq UnaryExpr\ UnaryAbs\ e2'$ 
        by (meson a mono-unary)
    then show ?thesis using ev r
        by (metis n)
next
case False
have  $g1 \vdash xn \simeq xe1$  using m by simp
have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using AbsNode
using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
    by (metis-node-eq-unary AbsNode)
    then have  $\exists xe2. (g2 \vdash n \simeq UnaryExpr\ UnaryAbs\ xe2) \wedge UnaryExpr$ 
UnaryAbs xe1  $\geq UnaryExpr\ UnaryAbs\ xe2$ 
    by (metis AbsNode.premis l mono-unary rep.AbsNode)
    then show ?thesis
    by meson
qed
next
case (NotNode n x xe1)
have k:  $g1 \vdash n \simeq UnaryExpr\ UnaryNot\ xe1$  using f NotNode

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    by (simp add: NotNode.hyps(2) rep.NotNode)
  obtain  $xn$  where  $l$ :  $kind\ g1\ n = NotNode\ xn$ 
    using NotNode.hyps(1) by blast
  then have  $m$ :  $g1 \vdash xn \simeq xe1$ 
    using NotNode.hyps(1) NotNode.hyps(2) by fastforce
  then show ?case
  proof (cases  $xn = n'$ )
    case True
      then have  $n$ :  $xe1 = e1'$  using  $c\ m\ repDet$  by simp
      then have  $ev$ :  $g2 \vdash n \simeq UnaryExpr\ UnaryNot\ e2'$  using NotNode.hyps(1)
    l m n
      using NotNode.premis True  $d\ rep.NotNode$  by simp
      then have  $r$ :  $UnaryExpr\ UnaryNot\ e1' \geq UnaryExpr\ UnaryNot\ e2'$ 
        by (meson  $a\ mono-unary$ )
      then show ?thesis using  $ev\ r$ 
        by (metis  $n$ )
    next
      case False
        have  $g1 \vdash xn \simeq xe1$  using  $m$  by simp
        have  $\exists\ xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
          using NotNode
          using False  $i\ b\ l\ not-excluded-keep-type\ singletonD\ no-encoding$ 
          by (metis-node-eq-unary NotNode)
        then have  $\exists\ xe2. (g2 \vdash n \simeq UnaryExpr\ UnaryNot\ xe2) \wedge UnaryExpr$ 
          UnaryNot  $xe1 \geq UnaryExpr\ UnaryNot\ xe2$ 
          by (metis NotNode.premis  $l\ mono-unary\ rep.NotNode$ )
        then show ?thesis
          by meson
      qed
    next
      case (NegateNode  $n\ x\ xe1$ )
        have  $k$ :  $g1 \vdash n \simeq UnaryExpr\ UnaryNeg\ xe1$  using  $f\ NegateNode$ 
          by (simp add: NegateNode.hyps(2) rep.NegateNode)
        obtain  $xn$  where  $l$ :  $kind\ g1\ n = NegateNode\ xn$ 
          using NegateNode.hyps(1) by blast
        then have  $m$ :  $g1 \vdash xn \simeq xe1$ 
          using NegateNode.hyps(1) NegateNode.hyps(2) by fastforce
        then show ?case
        proof (cases  $xn = n'$ )
          case True
            then have  $n$ :  $xe1 = e1'$  using  $c\ m\ repDet$  by simp
            then have  $ev$ :  $g2 \vdash n \simeq UnaryExpr\ UnaryNeg\ e2'$  using NegateNode.hyps(1)
          l m n
            using NegateNode.premis True  $d\ rep.NegateNode$  by simp
            then have  $r$ :  $UnaryExpr\ UnaryNeg\ e1' \geq UnaryExpr\ UnaryNeg\ e2'$ 
              by (meson  $a\ mono-unary$ )
            then show ?thesis using  $ev\ r$ 
              by (metis  $n$ )
          next

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    case False
    have  $g1 \vdash xn \simeq xe1$  using  $m$  by simp
    have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using NegateNode
      using False i b l not-excluded-keep-type singletonD no-encoding
      by (metis-node-eq-unary NegateNode)
      then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr UnaryNeg } xe2) \wedge \text{UnaryExpr}$ 
        UnaryNeg } xe1 \geq \text{UnaryExpr UnaryNeg } xe2
      by (metis NegateNode.prem s l mono-unary rep.NegateNode)
      then show ?thesis
      by meson
  qed
next
  case (LogicNegationNode n x xe1)
  have  $k: g1 \vdash n \simeq \text{UnaryExpr UnaryLogicNegation } xe1$  using  $f$  LogicNegationNode
    by (simp add: LogicNegationNode.hyps(2) rep.LogicNegationNode)
  obtain  $xn$  where  $l: \text{kind } g1 \ n = \text{LogicNegationNode } xn$ 
    using LogicNegationNode.hyps(1) by blast
  then have  $m: g1 \vdash xn \simeq xe1$ 
    using LogicNegationNode.hyps(1) LogicNegationNode.hyps(2) by fastforce
  then show ?case
  proof (cases  $xn = n'$ )
    case True
    then have  $n: xe1 = e1'$  using  $c \ m \ repDet$  by simp
    then have  $ev: g2 \vdash n \simeq \text{UnaryExpr UnaryLogicNegation } e2'$  using
      LogicNegationNode.hyps(1) l m n
    using LogicNegationNode.prem s True d rep.LogicNegationNode by simp
    then have  $r: \text{UnaryExpr UnaryLogicNegation } e1' \geq \text{UnaryExpr UnaryLogicNegation } e2'$ 
    by (meson a mono-unary)
    then show ?thesis using  $ev \ r$ 
    by (metis n)
  next
  case False
  have  $g1 \vdash xn \simeq xe1$  using  $m$  by simp
  have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using LogicNegationNode
    using False i b l not-excluded-keep-type singletonD no-encoding
    by (metis-node-eq-unary LogicNegationNode)
    then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr UnaryLogicNegation } xe2) \wedge$ 
      UnaryExpr UnaryLogicNegation } xe1 \geq \text{UnaryExpr UnaryLogicNegation } xe2
    by (metis LogicNegationNode.prem s l mono-unary rep.LogicNegationNode)
    then show ?thesis
    by meson
  qed
next
  case (AddNode n x y xe1 ye1)
  have  $k: g1 \vdash n \simeq \text{BinaryExpr BinAdd } xe1 \ ye1$  using  $f$  AddNode

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    by (simp add: AddNode.hyps(2) rep.AddNode)
  obtain  $xn\ yn$  where  $l$ : kind  $g1\ n = AddNode\ xn\ yn$ 
    using AddNode.hyps(1) by blast
  then have  $mx$ :  $g1 \vdash xn \simeq xe1$ 
    using AddNode.hyps(1) AddNode.hyps(2) by fastforce
  from  $l$  have  $my$ :  $g1 \vdash yn \simeq ye1$ 
    using AddNode.hyps(1) AddNode.hyps(3) by fastforce
  then show ?case
  proof -
    have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
    have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
    have  $xer$ :  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using AddNode
      using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary AddNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using AddNode
      using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary AddNode)
    then have  $\exists xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinAdd\ xe2\ ye2) \wedge BinaryExpr$ 
       $BinAdd\ xe1\ ye1 \geq BinaryExpr\ BinAdd\ xe2\ ye2$ 
      by (metis AddNode.prem1 l mono-binary rep.AddNode xer)
    then show ?thesis
      by meson
  qed
next
case (MulNode  $n\ x\ y\ xe1\ ye1$ )
have  $k$ :  $g1 \vdash n \simeq BinaryExpr\ BinMul\ xe1\ ye1$  using  $f$  MulNode
  by (simp add: MulNode.hyps(2) rep.MulNode)
obtain  $xn\ yn$  where  $l$ : kind  $g1\ n = MulNode\ xn\ yn$ 
  using MulNode.hyps(1) by blast
then have  $mx$ :  $g1 \vdash xn \simeq xe1$ 
  using MulNode.hyps(1) MulNode.hyps(2) by fastforce
from  $l$  have  $my$ :  $g1 \vdash yn \simeq ye1$ 
  using MulNode.hyps(1) MulNode.hyps(3) by fastforce
then show ?case
proof -
  have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
  have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
  have  $xer$ :  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using MulNode
    using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary MulNode)
  have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
    using MulNode
    using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary MulNode)
  then have  $\exists xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinMul\ xe2\ ye2) \wedge BinaryExpr$ 
     $BinMul\ xe1\ ye1 \geq BinaryExpr\ BinMul\ xe2\ ye2$ 

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    by (metis MulNode.premis l mono-binary rep.MulNode xer)
  then show ?thesis
    by meson
qed
next
case (SubNode n x y xe1 ye1)
have k:  $g1 \vdash n \simeq \text{BinaryExpr BinSub } xe1 \text{ ye1}$  using f SubNode
  by (simp add: SubNode.hyps(2) rep.SubNode)
obtain xn yn where l: kind g1 n = SubNode xn yn
  using SubNode.hyps(1) by blast
then have mx:  $g1 \vdash xn \simeq xe1$ 
  using SubNode.hyps(1) SubNode.hyps(2) by fastforce
from l have my:  $g1 \vdash yn \simeq ye1$ 
  using SubNode.hyps(1) SubNode.hyps(3) by fastforce
then show ?case
proof -
  have  $g1 \vdash xn \simeq xe1$  using mx by simp
  have  $g1 \vdash yn \simeq ye1$  using my by simp
  have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using SubNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary SubNode)
  have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
  using SubNode a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary SubNode)
  then have  $\exists xe2 ye2. (g2 \vdash n \simeq \text{BinaryExpr BinSub } xe2 \text{ ye2}) \wedge \text{BinaryExpr BinSub } xe1 \text{ ye1} \geq \text{BinaryExpr BinSub } xe2 \text{ ye2}$ 
    by (metis SubNode.premis l mono-binary rep.SubNode xer)
  then show ?thesis
    by meson
qed
next
case (AndNode n x y xe1 ye1)
have k:  $g1 \vdash n \simeq \text{BinaryExpr BinAnd } xe1 \text{ ye1}$  using f AndNode
  by (simp add: AndNode.hyps(2) rep.AndNode)
obtain xn yn where l: kind g1 n = AndNode xn yn
  using AndNode.hyps(1) by blast
then have mx:  $g1 \vdash xn \simeq xe1$ 
  using AndNode.hyps(1) AndNode.hyps(2) by fastforce
from l have my:  $g1 \vdash yn \simeq ye1$ 
  using AndNode.hyps(1) AndNode.hyps(3) by fastforce
then show ?case
proof -
  have  $g1 \vdash xn \simeq xe1$  using mx by simp
  have  $g1 \vdash yn \simeq ye1$  using my by simp
  have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using AndNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary AndNode)

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    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using AndNode a b c d l no-encoding not-excluded-keep-type repDet
singletonD
    by (metis-node-eq-binary AndNode)
    then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr BinAnd xe2 ye2) \wedge BinaryExpr$ 
BinAnd xe1 ye1  $\geq BinaryExpr BinAnd xe2 ye2$ 
      by (metis AndNode.premis l mono-binary rep.AndNode xer)
    then show ?thesis
      by meson
  qed
next
case (OrNode n x y xe1 ye1)
have k:  $g1 \vdash n \simeq BinaryExpr BinOr xe1 ye1$  using f OrNode
  by (simp add: OrNode.hyps(2) rep.OrNode)
obtain xn yn where l: kind g1 n = OrNode xn yn
  using OrNode.hyps(1) by blast
then have mx:  $g1 \vdash xn \simeq xe1$ 
  using OrNode.hyps(1) OrNode.hyps(2) by fastforce
from l have my:  $g1 \vdash yn \simeq ye1$ 
  using OrNode.hyps(1) OrNode.hyps(3) by fastforce
then show ?case
proof -
  have  $g1 \vdash xn \simeq xe1$  using mx by simp
  have  $g1 \vdash yn \simeq ye1$  using my by simp
  have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using OrNode
    using a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary OrNode)
  have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
  using OrNode a b c d l no-encoding not-excluded-keep-type repDet singletonD
    by (metis-node-eq-binary OrNode)
  then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr BinOr xe2 ye2) \wedge BinaryExpr$ 
BinOr xe1 ye1  $\geq BinaryExpr BinOr xe2 ye2$ 
    by (metis OrNode.premis l mono-binary rep.OrNode xer)
  then show ?thesis
    by meson
  qed
next
case (XorNode n x y xe1 ye1)
have k:  $g1 \vdash n \simeq BinaryExpr BinXor xe1 ye1$  using f XorNode
  by (simp add: XorNode.hyps(2) rep.XorNode)
obtain xn yn where l: kind g1 n = XorNode xn yn
  using XorNode.hyps(1) by blast
then have mx:  $g1 \vdash xn \simeq xe1$ 
  using XorNode.hyps(1) XorNode.hyps(2) by fastforce
from l have my:  $g1 \vdash yn \simeq ye1$ 
  using XorNode.hyps(1) XorNode.hyps(3) by fastforce
then show ?case
proof -

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    have  $g1 \vdash xn \simeq xe1$  using  $mx$  by  $simp$ 
    have  $g1 \vdash yn \simeq ye1$  using  $my$  by  $simp$ 
    have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using  $XorNode$ 
      using  $a\ b\ c\ d\ l\ no\text{-}encoding\ not\text{-}excluded\text{-}keep\text{-}type\ repDet\ singletonD$ 
      by  $(metis\ node\ eq\ binary\ XorNode)$ 
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using  $XorNode\ a\ b\ c\ d\ l\ no\text{-}encoding\ not\text{-}excluded\text{-}keep\text{-}type\ repDet$ 
     $singletonD$ 
      by  $(metis\ node\ eq\ binary\ XorNode)$ 
    then have  $\exists xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinXor\ xe2\ ye2) \wedge BinaryExpr$ 
     $BinXor\ xe1\ ye1 \geq BinaryExpr\ BinXor\ xe2\ ye2$ 
      by  $(metis\ XorNode.prem\ l\ mono\text{-}binary\ rep.XorNode\ xer)$ 
    then show  $?thesis$ 
      by  $meson$ 
  qed
next
  case  $(LeftShiftNode\ n\ x\ y\ xe1\ ye1)$ 
  have  $k: g1 \vdash n \simeq BinaryExpr\ BinLeftShift\ xe1\ ye1$  using  $f\ LeftShiftNode$ 
    by  $(simp\ add: LeftShiftNode.hyps(2)\ rep.LeftShiftNode)$ 
  obtain  $xn\ yn$  where  $l: kind\ g1\ n = LeftShiftNode\ xn\ yn$ 
    using  $LeftShiftNode.hyps(1)$  by  $blast$ 
  then have  $mx: g1 \vdash xn \simeq xe1$ 
    using  $LeftShiftNode.hyps(1)\ LeftShiftNode.hyps(2)$  by  $fastforce$ 
  from  $l$  have  $my: g1 \vdash yn \simeq ye1$ 
    using  $LeftShiftNode.hyps(1)\ LeftShiftNode.hyps(3)$  by  $fastforce$ 
  then show  $?case$ 
  proof –
    have  $g1 \vdash xn \simeq xe1$  using  $mx$  by  $simp$ 
    have  $g1 \vdash yn \simeq ye1$  using  $my$  by  $simp$ 
    have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using  $LeftShiftNode$ 
      using  $a\ b\ c\ d\ l\ no\text{-}encoding\ not\text{-}excluded\text{-}keep\text{-}type\ repDet\ singletonD$ 
      by  $(metis\ node\ eq\ binary\ LeftShiftNode)$ 
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using  $LeftShiftNode\ a\ b\ c\ d\ l\ no\text{-}encoding\ not\text{-}excluded\text{-}keep\text{-}type\ repDet$ 
     $singletonD$ 
      by  $(metis\ node\ eq\ binary\ LeftShiftNode)$ 
    then have  $\exists xe2\ ye2. (g2 \vdash n \simeq BinaryExpr\ BinLeftShift\ xe2\ ye2) \wedge$ 
     $BinaryExpr\ BinLeftShift\ xe1\ ye1 \geq BinaryExpr\ BinLeftShift\ xe2\ ye2$ 
      by  $(metis\ LeftShiftNode.prem\ l\ mono\text{-}binary\ rep.LeftShiftNode\ xer)$ 
    then show  $?thesis$ 
      by  $meson$ 
  qed
next
  case  $(RightShiftNode\ n\ x\ y\ xe1\ ye1)$ 
  have  $k: g1 \vdash n \simeq BinaryExpr\ BinRightShift\ xe1\ ye1$  using  $f\ RightShiftNode$ 
    by  $(simp\ add: RightShiftNode.hyps(2)\ rep.RightShiftNode)$ 
  obtain  $xn\ yn$  where  $l: kind\ g1\ n = RightShiftNode\ xn\ yn$ 

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    using RightShiftNode.hyps(1) by blast
  then have mx:  $g1 \vdash xn \simeq xe1$ 
    using RightShiftNode.hyps(1) RightShiftNode.hyps(2) by fastforce
  from l have my:  $g1 \vdash yn \simeq ye1$ 
    using RightShiftNode.hyps(1) RightShiftNode.hyps(3) by fastforce
  then show ?case
  proof -
    have  $g1 \vdash xn \simeq xe1$  using mx by simp
    have  $g1 \vdash yn \simeq ye1$  using my by simp
    have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using RightShiftNode
      using a b c d l no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary RightShiftNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using RightShiftNode a b c d l no-encoding not-excluded-keep-type repDet
      singletonD
      by (metis-node-eq-binary RightShiftNode)
    then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr BinRightShift xe2 ye2) \wedge$ 
       $BinaryExpr BinRightShift xe1 ye1 \geq BinaryExpr BinRightShift xe2 ye2$ 
      by (metis RightShiftNode.premis l mono-binary rep.RightShiftNode xer)
    then show ?thesis
      by meson
  qed
next
case (UnsignedRightShiftNode n x y xe1 ye1)
  have k:  $g1 \vdash n \simeq BinaryExpr BinURightShift xe1 ye1$  using f UnsignedRight-
  ShiftNode
    by (simp add: UnsignedRightShiftNode.hyps(2) rep.UnsignedRightShiftNode)
  obtain xn yn where l: kind g1 n = UnsignedRightShiftNode xn yn
    using UnsignedRightShiftNode.hyps(1) by blast
  then have mx:  $g1 \vdash xn \simeq xe1$ 
    using UnsignedRightShiftNode.hyps(1) UnsignedRightShiftNode.hyps(2) by
  fastforce
  from l have my:  $g1 \vdash yn \simeq ye1$ 
    using UnsignedRightShiftNode.hyps(1) UnsignedRightShiftNode.hyps(3) by
  fastforce
  then show ?case
  proof -
    have  $g1 \vdash xn \simeq xe1$  using mx by simp
    have  $g1 \vdash yn \simeq ye1$  using my by simp
    have xer:  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using UnsignedRightShiftNode
      using a b c d l no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary UnsignedRightShiftNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using UnsignedRightShiftNode a b c d l no-encoding not-excluded-keep-type
      repDet singletonD
      by (metis-node-eq-binary UnsignedRightShiftNode)
    then have  $\exists xe2 ye2. (g2 \vdash n \simeq BinaryExpr BinURightShift xe2 ye2) \wedge$ 

```

```

BinaryExpr BinURightShift xe1 ye1 ≥ BinaryExpr BinURightShift xe2 ye2
  by (metis UnsignedRightShiftNode.prem1 mono-binary rep.UnsignedRightShiftNode
xer)
    then show ?thesis
      by meson
    qed
  next
    case (IntegerBelowNode n x y xe1 ye1)
    have k: g1 ⊢ n ≃ BinaryExpr BinIntegerBelow xe1 ye1 using f IntegerBe-
lowNode
      by (simp add: IntegerBelowNode.hyps(2) rep.IntegerBelowNode)
    obtain xn yn where l: kind g1 n = IntegerBelowNode xn yn
      using IntegerBelowNode.hyps(1) by blast
    then have mx: g1 ⊢ xn ≃ xe1
      using IntegerBelowNode.hyps(1) IntegerBelowNode.hyps(2) by fastforce
    from l have my: g1 ⊢ yn ≃ ye1
      using IntegerBelowNode.hyps(1) IntegerBelowNode.hyps(3) by fastforce
    then show ?case
      proof -
        have g1 ⊢ xn ≃ xe1 using mx by simp
        have g1 ⊢ yn ≃ ye1 using my by simp
        have xer: ∃ xe2. (g2 ⊢ xn ≃ xe2) ∧ xe1 ≥ xe2
          using IntegerBelowNode
          using a b c d l no-encoding not-excluded-keep-type repDet singletonD
          by (metis-node-eq-binary IntegerBelowNode)
        have ye2: (g2 ⊢ yn ≃ ye2) ∧ ye1 ≥ ye2
          using IntegerBelowNode a b c d l no-encoding not-excluded-keep-type repDet
singletonD
          by (metis-node-eq-binary IntegerBelowNode)
        then have ∃ xe2 ye2. (g2 ⊢ n ≃ BinaryExpr BinIntegerBelow xe2 ye2) ∧
BinaryExpr BinIntegerBelow xe1 ye1 ≥ BinaryExpr BinIntegerBelow xe2 ye2
          by (metis IntegerBelowNode.prem1 mono-binary rep.IntegerBelowNode
xer)
        then show ?thesis
          by meson
        qed
      next
        case (IntegerEqualsNode n x y xe1 ye1)
        have k: g1 ⊢ n ≃ BinaryExpr BinIntegerEquals xe1 ye1 using f IntegerEqual-
sNode
          by (simp add: IntegerEqualsNode.hyps(2) rep.IntegerEqualsNode)
        obtain xn yn where l: kind g1 n = IntegerEqualsNode xn yn
          using IntegerEqualsNode.hyps(1) by blast
        then have mx: g1 ⊢ xn ≃ xe1
          using IntegerEqualsNode.hyps(1) IntegerEqualsNode.hyps(2) by fastforce
        from l have my: g1 ⊢ yn ≃ ye1
          using IntegerEqualsNode.hyps(1) IntegerEqualsNode.hyps(3) by fastforce
        then show ?case
          proof -

```

```

    have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
    have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
    have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using IntegerEqualsNode
      using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary IntegerEqualsNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using IntegerEqualsNode a b c d l no-encoding not-excluded-keep-type
repDet singletonD
      by (metis-node-eq-binary IntegerEqualsNode)
    then have  $\exists xe2\ ye2. (g2 \vdash n \simeq \text{BinaryExpr BinIntegerEquals } xe2\ ye2) \wedge$ 
BinaryExpr BinIntegerEquals xe1 ye1  $\geq \text{BinaryExpr BinIntegerEquals } xe2\ ye2$ 
      by (metis IntegerEqualsNode.premis l mono-binary rep.IntegerEqualsNode
xer)
    then show ?thesis
      by meson
  qed
next
case (IntegerLessThanNode n x y xe1 ye1)
  have  $k: g1 \vdash n \simeq \text{BinaryExpr BinIntegerLessThan } xe1\ ye1$  using  $f$  IntegerLessThanNode
  by (simp add: IntegerLessThanNode.hyps(2) rep.IntegerLessThanNode)
  obtain  $xn\ yn$  where  $l: \text{kind } g1\ n = \text{IntegerLessThanNode } xn\ yn$ 
  using IntegerLessThanNode.hyps(1) by blast
  then have  $mx: g1 \vdash xn \simeq xe1$ 
  using IntegerLessThanNode.hyps(1) IntegerLessThanNode.hyps(2) by fast-force
  from  $l$  have  $my: g1 \vdash yn \simeq ye1$ 
  using IntegerLessThanNode.hyps(1) IntegerLessThanNode.hyps(3) by fast-force
  then show ?case
  proof -
    have  $g1 \vdash xn \simeq xe1$  using  $mx$  by simp
    have  $g1 \vdash yn \simeq ye1$  using  $my$  by simp
    have  $xer: \exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using IntegerLessThanNode
      using  $a\ b\ c\ d\ l$  no-encoding not-excluded-keep-type repDet singletonD
      by (metis-node-eq-binary IntegerLessThanNode)
    have  $\exists ye2. (g2 \vdash yn \simeq ye2) \wedge ye1 \geq ye2$ 
      using IntegerLessThanNode a b c d l no-encoding not-excluded-keep-type
repDet singletonD
      by (metis-node-eq-binary IntegerLessThanNode)
    then have  $\exists xe2\ ye2. (g2 \vdash n \simeq \text{BinaryExpr BinIntegerLessThan } xe2\ ye2)$ 
 $\wedge \text{BinaryExpr BinIntegerLessThan } xe1\ ye1 \geq \text{BinaryExpr BinIntegerLessThan } xe2\ ye2$ 
      by (metis IntegerLessThanNode.premis l mono-binary rep.IntegerLessThanNode
xer)
    then show ?thesis
      by meson

```

```

    qed
  next
    case (NarrowNode n inputBits resultBits x xe1)
    have k:  $g1 \vdash n \simeq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe1$  using
  f NarrowNode
    by (simp add: NarrowNode.hyps(2) rep.NarrowNode)
    obtain xn where l: kind  $g1$   $n = \text{NarrowNode inputBits resultBits } xn$ 
    using NarrowNode.hyps(1) by blast
    then have m:  $g1 \vdash xn \simeq xe1$ 
    using NarrowNode.hyps(1) NarrowNode.hyps(2)
    by auto
    then show ?case
    proof (cases  $xn = n'$ )
      case True
        then have n:  $xe1 = e1'$  using c m repDet by simp
        then have ev:  $g2 \vdash n \simeq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) e2'$ 
    using NarrowNode.hyps(1) l m n
      using NarrowNode.premis True d rep.NarrowNode by simp
      then have r:  $\text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) e1' \geq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) e2'$ 
      by (meson a mono-unary)
      then show ?thesis using ev r
      by (metis n)
    next
      case False
        have  $g1 \vdash xn \simeq xe1$  using m by simp
        have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
        using NarrowNode
        using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
        by (metis node-eq-ternary NarrowNode)
        then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe2) \wedge \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe1 \geq \text{UnaryExpr } (\text{UnaryNarrow inputBits resultBits}) xe2$ 
        by (metis NarrowNode.premis l mono-unary rep.NarrowNode)
        then show ?thesis
        by meson
    qed
  next
    case (SignExtendNode n inputBits resultBits x xe1)
    have k:  $g1 \vdash n \simeq \text{UnaryExpr } (\text{UnarySignExtend inputBits resultBits}) xe1$ 
  using f SignExtendNode
    by (simp add: SignExtendNode.hyps(2) rep.SignExtendNode)
    obtain xn where l: kind  $g1$   $n = \text{SignExtendNode inputBits resultBits } xn$ 
    using SignExtendNode.hyps(1) by blast
    then have m:  $g1 \vdash xn \simeq xe1$ 
    using SignExtendNode.hyps(1) SignExtendNode.hyps(2)
    by auto
    then show ?case
    proof (cases  $xn = n'$ )

```

```

      case True
      then have n:  $xe1 = e1'$  using c m repDet by simp
      then have ev:  $g2 \vdash n \simeq \text{UnaryExpr } (\text{UnarySignExtend inputBits resultBits})$ 
e2' using SignExtendNode.hyps(1) l m n
      using SignExtendNode.premis True d rep.SignExtendNode by simp
      then have r:  $\text{UnaryExpr } (\text{UnarySignExtend inputBits resultBits}) e1' \geq$ 
UnaryExpr (UnarySignExtend inputBits resultBits) e2'
      by (meson a mono-unary)
      then show ?thesis using ev r
      by (metis n)
    next
    case False
    have  $g1 \vdash xn \simeq xe1$  using m by simp
    have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
      using SignExtendNode
    using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
      by (metis-node-eq-ternary SignExtendNode)
    then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr } (\text{UnarySignExtend inputBits result-}$ 
Bits)  $xe2) \wedge \text{UnaryExpr } (\text{UnarySignExtend inputBits resultBits}) xe1 \geq \text{UnaryExpr } (\text{UnarySignExtend inputBits resultBits}) xe2$ 
      by (metis SignExtendNode.premis l mono-unary rep.SignExtendNode)
    then show ?thesis
      by meson
  qed
next
case (ZeroExtendNode n inputBits resultBits x xe1)
  have k:  $g1 \vdash n \simeq \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) xe1$ 
using f ZeroExtendNode
  by (simp add: ZeroExtendNode.hyps(2) rep.ZeroExtendNode)
  obtain xn where l: kind  $g1$  n = ZeroExtendNode inputBits resultBits xn
  using ZeroExtendNode.hyps(1) by blast
  then have m:  $g1 \vdash xn \simeq xe1$ 
  using ZeroExtendNode.hyps(1) ZeroExtendNode.hyps(2)
  by auto
  then show ?case
  proof (cases xn = n')
    case True
    then have n:  $xe1 = e1'$  using c m repDet by simp
    then have ev:  $g2 \vdash n \simeq \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits})$ 
e2' using ZeroExtendNode.hyps(1) l m n
    using ZeroExtendNode.premis True d rep.ZeroExtendNode by simp
    then have r:  $\text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) e1' \geq$ 
UnaryExpr (UnaryZeroExtend inputBits resultBits) e2'
    by (meson a mono-unary)
    then show ?thesis using ev r
    by (metis n)
  next
  case False
  have  $g1 \vdash xn \simeq xe1$  using m by simp

```



```

    have  $\exists xe2. (g2 \vdash xn \simeq xe2) \wedge xe1 \geq xe2$ 
    using ZeroExtendNode
    using False b encodes-contains l not-excluded-keep-type not-in-g singleton-iff
    by (metis-node-eq-ternary ZeroExtendNode)
    then have  $\exists xe2. (g2 \vdash n \simeq \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) xe2) \wedge \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) xe1 \geq \text{UnaryExpr } (\text{UnaryZeroExtend inputBits resultBits}) xe2$ 
    by (metis ZeroExtendNode.premis l mono-unary rep.ZeroExtendNode)
    then show ?thesis
    by meson
  qed
next
  case (LeafNode n s)
  then show ?case
  by (metis eq-refl rep.LeafNode)
next
  case (RefNode n')
  then show ?case
  by (metis a b c d no-encoding not-excluded-keep-type rep.RefNode repDet singletonD)
  qed
  qed
  qed

```

lemma *graph-antics-preservation-subscript*:

```

  assumes a:  $e_1' \geq e_2'$ 
  assumes b:  $(\{n\} \trianglelefteq \text{as-set } g_1) \subseteq \text{as-set } g_2$ 
  assumes c:  $g_1 \vdash n \simeq e_1'$ 
  assumes d:  $g_2 \vdash n \simeq e_2'$ 
  shows graph-refinement  $g_1 g_2$ 
  using graph-antics-preservation assms by simp

```

lemma *tree-to-graph-rewriting*:

```

   $e_1 \geq e_2$ 
   $\wedge (g_1 \vdash n \simeq e_1) \wedge \text{maximal-sharing } g_1$ 
   $\wedge (\{n\} \trianglelefteq \text{as-set } g_1) \subseteq \text{as-set } g_2$ 
   $\wedge (g_2 \vdash n \simeq e_2) \wedge \text{maximal-sharing } g_2$ 
 $\implies \text{graph-refinement } g_1 g_2$ 
  using graph-antics-preservation
  by auto

```

declare $[[\text{simp-trace}]]$

lemma *equal-refines*:

```

  fixes e1 e2 :: IRExp
  assumes e1 = e2
  shows  $e1 \geq e2$ 
  using assms
  by simp

```

declare $[[simp\text{-}trace=false]]$

lemma *eval-contains-id* $[simp]$: $g1 \vdash n \simeq e \implies n \in ids\ g1$
using *no-encoding* **by** *blast*

lemma *subset-kind* $[simp]$: $as\text{-}set\ g1 \subseteq as\text{-}set\ g2 \implies g1 \vdash n \simeq e \implies kind\ g1\ n = kind\ g2\ n$
using *eval-contains-id* **unfolding** *as-set-def*
by *blast*

lemma *subset-stamp* $[simp]$: $as\text{-}set\ g1 \subseteq as\text{-}set\ g2 \implies g1 \vdash n \simeq e \implies stamp\ g1\ n = stamp\ g2\ n$
using *eval-contains-id* **unfolding** *as-set-def*
by *blast*

method *solve-subset-eval* **uses** *as-set eval* =
(*metis eval as-set subset-kind subset-stamp* |
metis eval as-set subset-kind)

lemma *subset-implies-evals*:
assumes $as\text{-}set\ g1 \subseteq as\text{-}set\ g2$
assumes $(g1 \vdash n \simeq e)$
shows $(g2 \vdash n \simeq e)$
using *assms(2)*
apply (*induction e*)
apply (*solve-subset-eval as-set: assms(1) eval: ConstantNode*)
apply (*solve-subset-eval as-set: assms(1) eval: ParameterNode*)
apply (*solve-subset-eval as-set: assms(1) eval: ConditionalNode*)
apply (*solve-subset-eval as-set: assms(1) eval: AbsNode*)
apply (*solve-subset-eval as-set: assms(1) eval: NotNode*)
apply (*solve-subset-eval as-set: assms(1) eval: NegateNode*)
apply (*solve-subset-eval as-set: assms(1) eval: LogicNegationNode*)
apply (*solve-subset-eval as-set: assms(1) eval: AddNode*)
apply (*solve-subset-eval as-set: assms(1) eval: MulNode*)
apply (*solve-subset-eval as-set: assms(1) eval: SubNode*)
apply (*solve-subset-eval as-set: assms(1) eval: AndNode*)
apply (*solve-subset-eval as-set: assms(1) eval: OrNode*)
apply (*solve-subset-eval as-set: assms(1) eval: XorNode*)
apply (*solve-subset-eval as-set: assms(1) eval: LeftShiftNode*)
apply (*solve-subset-eval as-set: assms(1) eval: RightShiftNode*)
apply (*solve-subset-eval as-set: assms(1) eval: UnsignedRightShiftNode*)
apply (*solve-subset-eval as-set: assms(1) eval: IntegerBelowNode*)
apply (*solve-subset-eval as-set: assms(1) eval: IntegerEqualsNode*)
apply (*solve-subset-eval as-set: assms(1) eval: IntegerLessThanNode*)
apply (*solve-subset-eval as-set: assms(1) eval: NarrowNode*)
apply (*solve-subset-eval as-set: assms(1) eval: SignExtendNode*)

```

    apply (solve-subset-eval as-set: assms(1) eval: ZeroExtendNode)
    apply (solve-subset-eval as-set: assms(1) eval: LeafNode)
    by (solve-subset-eval as-set: assms(1) eval: RefNode)

lemma subset-refines:
  assumes as-set g1  $\subseteq$  as-set g2
  shows graph-refinement g1 g2
proof -
  have ids g1  $\subseteq$  ids g2 using assms unfolding as-set-def
  by blast
  then show ?thesis unfolding graph-refinement-def apply rule
  apply (rule allI) apply (rule impI) apply (rule allI) apply (rule impI)
  unfolding graph-represents-expression-def
  proof -
    fix n e1
    assume 1:n  $\in$  ids g1
    assume 2:g1  $\vdash$  n  $\simeq$  e1

    show  $\exists e2. (g2 \vdash n \simeq e2) \wedge e1 \geq e2$ 
    using assms 1 2 using subset-implies-evals
    by (meson equal-refines)
  qed
qed

```

```

lemma graph-construction:
  e1  $\geq$  e2
   $\wedge$  as-set g1  $\subseteq$  as-set g2
   $\wedge$  (g2  $\vdash$  n  $\simeq$  e2)
   $\implies$  (g2  $\vdash$  n  $\trianglelefteq$  e1)  $\wedge$  graph-refinement g1 g2
  using subset-refines
  by (meson encodeeval-def graph-represents-expression-def le-expr-def)

```

2.8.4 Term Graph Reconstruction

```

lemma find-exists-kind:
  assumes find-node-and-stamp g (node, s) = Some nid
  shows kind g nid = node
  using assms unfolding find-node-and-stamp.simps
  by (metis (mono-tags, lifting) find-Some-iff)

```

```

lemma find-exists-stamp:
  assumes find-node-and-stamp g (node, s) = Some nid
  shows stamp g nid = s
  using assms unfolding find-node-and-stamp.simps
  by (metis (mono-tags, lifting) find-Some-iff)

```

```

lemma find-new-kind:
  assumes g' = add-node nid (node, s) g
  assumes node  $\neq$  NoNode

```

```

shows kind g' nid = node
using assms
using add-node-lookup by presburger

lemma find-new-stamp:
  assumes g' = add-node nid (node, s) g
  assumes node ≠ NoNode
  shows stamp g' nid = s
  using assms
  using add-node-lookup by presburger

lemma sorted-bottom:
  assumes finite xs
  assumes x ∈ xs
  shows x ≤ last(sorted-list-of-set(xs::nat set))
  using assms
  using sorted2-simps(2) sorted-list-of-set(2)
  by (smt (verit, del-insts) Diff-iff Max-ge Max-in empty-iff list.set(1) snoc-eq-iff-butlast
sorted-insort-is-snoc sorted-list-of-set(1) sorted-list-of-set.fold-insort-key.infinite sorted-list-of-set.fold-insort-ke

lemma fresh: finite xs ⇒ last(sorted-list-of-set(xs::nat set)) + 1 ∉ xs
  using sorted-bottom
  using not-le by auto

lemma fresh-ids:
  assumes n = get-fresh-id g
  shows n ∉ ids g
proof -
  have finite (ids g) using Rep-IRGraph by auto
  then show ?thesis
    using assms fresh unfolding get-fresh-id.simps
    by blast
qed

lemma graph-unchanged-rep-unchanged:
  assumes ∀ n ∈ ids g. kind g n = kind g' n
  assumes ∀ n ∈ ids g. stamp g n = stamp g' n
  shows (g ⊢ n ≃ e) ⟶ (g' ⊢ n ≃ e)
  apply (rule impI) subgoal premises e using e assms
    apply (induction n e)
      apply (metis no-encoding rep.ConstantNode)
      apply (metis no-encoding rep.ParameterNode)
      apply (metis no-encoding rep.ConditionalNode)
      apply (metis no-encoding rep.AbsNode)
      apply (metis no-encoding rep.NotNode)
      apply (metis no-encoding rep.NegateNode)
      apply (metis no-encoding rep.LogicNegationNode)
      apply (metis no-encoding rep.AddNode)
      apply (metis no-encoding rep.MulNode)

```

```

    apply (metis no-encoding rep.SubNode)
    apply (metis no-encoding rep.AndNode)
    apply (metis no-encoding rep.OrNode)
    apply (metis no-encoding rep.XorNode)
    apply (metis no-encoding rep.LeftShiftNode)
    apply (metis no-encoding rep.RightShiftNode)
    apply (metis no-encoding rep.UnsignedRightShiftNode)
    apply (metis no-encoding rep.IntegerBelowNode)
    apply (metis no-encoding rep.IntegerEqualsNode)
    apply (metis no-encoding rep.IntegerLessThanNode)
    apply (metis no-encoding rep.NarrowNode)
    apply (metis no-encoding rep.SignExtendNode)
    apply (metis no-encoding rep.ZeroExtendNode)
    apply (metis no-encoding rep.LeafNode)
    by (metis no-encoding rep.RefNode)
done

lemma fresh-node-subset:
  assumes  $n \notin \text{ids } g$ 
  assumes  $g' = \text{add-node } n \ (k, s) \ g$ 
  shows  $\text{as-set } g \subseteq \text{as-set } g'$ 
  using assms
  by (smt (verit, del-Insts) Collect-mono-iff Diff-idemp Diff-insert-absorb add-changed
    as-set-def disjoint-change unchanged.simps)

lemma unrep-subset:
  assumes  $(g \oplus e \rightsquigarrow (g', n))$ 
  shows  $\text{as-set } g \subseteq \text{as-set } g'$ 
  using assms proof (induction  $g \ e \ (g', n)$  arbitrary:  $g' \ n$ )
    case (ConstantNodeSame  $g \ c \ n$ )
    then show ?case by blast
  next
    case (ConstantNodeNew  $g \ c \ n \ g'$ )
    then show ?case using fresh-ids fresh-node-subset
      by presburger
  next
    case (ParameterNodeSame  $g \ i \ s \ n$ )
    then show ?case by blast
  next
    case (ParameterNodeNew  $g \ i \ s \ n \ g'$ )
    then show ?case using fresh-ids fresh-node-subset
      by presburger
  next
    case (ConditionalNodeSame  $g \ ce \ g2 \ c \ te \ g3 \ t \ fe \ g4 \ f \ s' \ n$ )
    then show ?case by blast
  next
    case (ConditionalNodeNew  $g \ ce \ g2 \ c \ te \ g3 \ t \ fe \ g4 \ f \ s' \ n \ g'$ )
    then show ?case using fresh-ids fresh-node-subset
      by (meson subset-trans)

```

```

next
  case (UnaryNodeSame g xe g2 x s' op n)
  then show ?case by blast
next
  case (UnaryNodeNew g xe g2 x s' op n g')
  then show ?case using fresh-ids fresh-node-subset
    by (meson subset-trans)
next
  case (BinaryNodeSame g xe g2 x ye g3 y s' op n)
  then show ?case by blast
next
  case (BinaryNodeNew g xe g2 x ye g3 y s' op n g')
  then show ?case using fresh-ids fresh-node-subset
    by (meson subset-trans)
next
  case (AllLeafNodes g n s)
  then show ?case by blast
qed

lemma fresh-node-preserves-other-nodes:
  assumes n' = get-fresh-id g
  assumes g' = add-node n' (k, s) g
  shows  $\forall n \in \text{ids } g. (g \vdash n \simeq e) \longrightarrow (g' \vdash n \simeq e)$ 
  using assms
  by (smt (verit, ccfv-SIG) Diff-idemp Diff-insert-absorb add-changed disjoint-change
    fresh-ids graph-unchanged-rep-unchanged unchanged.elims(2))

lemma found-node-preserves-other-nodes:
  assumes find-node-and-stamp g (k, s) = Some n
  shows  $\forall n \in \text{ids } g. (g \vdash n \simeq e) \longleftrightarrow (g \vdash n \simeq e)$ 
  using assms
  by blast

lemma unrep-ids-subset[simp]:
  assumes  $g \oplus e \rightsquigarrow (g', n)$ 
  shows  $\text{ids } g \subseteq \text{ids } g'$ 
  using assms unrep-subset
  by (meson graph-refinement-def subset-refines)

lemma unrep-unchanged:
  assumes  $g \oplus e \rightsquigarrow (g', n)$ 
  shows  $\forall n \in \text{ids } g. \forall e. (g \vdash n \simeq e) \longrightarrow (g' \vdash n \simeq e)$ 
  using assms unrep-subset fresh-node-preserves-other-nodes
  by (meson subset-implies-evals)

theorem term-graph-reconstruction:
   $g \oplus e \rightsquigarrow (g', n) \implies (g' \vdash n \simeq e) \wedge \text{as-set } g \subseteq \text{as-set } g'$ 
  subgoal premises e apply (rule conjI) defer
  using e unrep-subset apply blast using e

```

```

proof (induction  $g\ e\ (g',\ n)$  arbitrary:  $g'\ n$ )
  case (ConstantNodeSame  $g'\ c\ n$ )
    then have  $\text{kind } g'\ n = \text{ConstantNode } c$ 
      using find-exists-kind local.ConstantNodeSame by blast
    then show  $?case$  using ConstantNode by blast
next
  case (ConstantNodeNew  $g\ c$ )
    then show  $?case$ 
      using ConstantNode IRNode.distinct(683) add-node-lookup by presburger
next
  case (ParameterNodeSame  $i\ s$ )
    then show  $?case$ 
      by (metis ParameterNode find-exists-kind find-exists-stamp)
next
  case (ParameterNodeNew  $g\ i\ s$ )
    then show  $?case$ 
      by (metis IRNode.distinct(2447) ParameterNode add-node-lookup)
next
  case (ConditionalNodeSame  $g\ ce\ g2\ c\ te\ g3\ t\ fe\ g4\ f\ s'\ n$ )
    then have  $k: \text{kind } g4\ n = \text{ConditionalNode } c\ t\ f$ 
      using find-exists-kind by blast
    have  $c: g4 \vdash c \simeq ce$  using local.ConditionalNodeSame unrep-unchanged
      using no-encoding by blast
    have  $t: g4 \vdash t \simeq te$  using local.ConditionalNodeSame unrep-unchanged
      using no-encoding by blast
    have  $f: g4 \vdash f \simeq fe$  using local.ConditionalNodeSame unrep-unchanged
      using no-encoding by blast
    then show  $?case$  using  $c\ t\ f$ 
      using ConditionalNode  $k$  by blast
next
  case (ConditionalNodeNew  $g\ ce\ g2\ c\ te\ g3\ t\ fe\ g4\ f\ s'\ n\ g'$ )
    moreover have ConditionalNode  $c\ t\ f \neq \text{NoNode}$ 
      using unary-node.elims by blast
    ultimately have  $k: \text{kind } g'\ n = \text{ConditionalNode } c\ t\ f$ 
      using find-new-kind local.ConditionalNodeNew
      by presburger
    then have  $c: g' \vdash c \simeq ce$  using local.ConditionalNodeNew unrep-unchanged
      using no-encoding
      by (metis ConditionalNodeNew.hyps(9) fresh-node-preserves-other-nodes)
    then have  $t: g' \vdash t \simeq te$  using local.ConditionalNodeNew unrep-unchanged
      using no-encoding fresh-node-preserves-other-nodes
      by metis
    then have  $f: g' \vdash f \simeq fe$  using local.ConditionalNodeNew unrep-unchanged
      using no-encoding fresh-node-preserves-other-nodes
      by metis
    then show  $?case$  using  $c\ t\ f$ 
      using ConditionalNode  $k$  by blast
next
  case (UnaryNodeSame  $g\ xe\ g'\ x\ s'\ op\ n$ )

```

```

then have k: kind g' n = unary-node op x
  using find-exists-kind local.UnaryNodeSame by blast
then have g' ⊢ x ≃ xe using local.UnaryNodeSame by blast
then show ?case using k
  apply (cases op)
  using AbsNode unary-node.simps(1) apply presburger
  using NegateNode unary-node.simps(3) apply presburger
  using NotNode unary-node.simps(2) apply presburger
  using LogicNegationNode unary-node.simps(4) apply presburger
  using NarrowNode unary-node.simps(5) apply presburger
  using SignExtendNode unary-node.simps(6) apply presburger
  using ZeroExtendNode unary-node.simps(7) by presburger
next
case (UnaryNodeNew g xe g2 x s' op n g')
moreover have unary-node op x ≠ NoNode
  using unary-node.elims by blast
ultimately have k: kind g' n = unary-node op x
  using find-new-kind local.UnaryNodeNew
  by presburger
have x ∈ ids g2 using local.UnaryNodeNew
  using eval-contains-id by blast
then have x ≠ n using local.UnaryNodeNew(5) fresh-ids by blast
have g' ⊢ x ≃ xe using local.UnaryNodeNew fresh-node-preserved-other-nodes
  using ⟨x ∈ ids g2⟩ by blast
then show ?case using k
  apply (cases op)
  using AbsNode unary-node.simps(1) apply presburger
  using NegateNode unary-node.simps(3) apply presburger
  using NotNode unary-node.simps(2) apply presburger
  using LogicNegationNode unary-node.simps(4) apply presburger
  using NarrowNode unary-node.simps(5) apply presburger
  using SignExtendNode unary-node.simps(6) apply presburger
  using ZeroExtendNode unary-node.simps(7) by presburger
next
case (BinaryNodeSame g xe g2 x ye g3 y s' op n)
then have k: kind g3 n = bin-node op x y
  using find-exists-kind by blast
have x: g3 ⊢ x ≃ xe using local.BinaryNodeSame unrep-unchanged
  using no-encoding by blast
have y: g3 ⊢ y ≃ ye using local.BinaryNodeSame unrep-unchanged
  using no-encoding by blast
then show ?case using x y k apply (cases op)
  using AddNode bin-node.simps(1) apply presburger
  using MulNode bin-node.simps(2) apply presburger
  using SubNode bin-node.simps(3) apply presburger
  using AndNode bin-node.simps(4) apply presburger
  using OrNode bin-node.simps(5) apply presburger
  using XorNode bin-node.simps(6) apply presburger
  using LeftShiftNode bin-node.simps(7) apply presburger

```



```

    using RightShiftNode bin-node.simps(8) apply presburger
    using UnsignedRightShiftNode bin-node.simps(9) apply presburger
    using IntegerEqualsNode bin-node.simps(10) apply presburger
    using IntegerLessThanNode bin-node.simps(11) apply presburger
    using IntegerBelowNode bin-node.simps(12) by presburger
next
case (BinaryNodeNew g xe g2 x ye g3 y s' op n g')
moreover have bin-node op x y  $\neq$  NoNode
    using bin-node.elims by blast
ultimately have k: kind g' n = bin-node op x y
    using find-new-kind local.BinaryNodeNew
    by presburger
then have k: kind g' n = bin-node op x y
    using find-exists-kind by blast
have x: g'  $\vdash$  x  $\simeq$  xe using local.BinaryNodeNew unrep-unchanged
    using no-encoding
    by (meson fresh-node-preserves-other-nodes)
have y: g'  $\vdash$  y  $\simeq$  ye using local.BinaryNodeNew unrep-unchanged
    using no-encoding
    by (meson fresh-node-preserves-other-nodes)
then show ?case using x y k apply (cases op)
    using AddNode bin-node.simps(1) apply presburger
    using MulNode bin-node.simps(2) apply presburger
    using SubNode bin-node.simps(3) apply presburger
    using AndNode bin-node.simps(4) apply presburger
    using OrNode bin-node.simps(5) apply presburger
    using XorNode bin-node.simps(6) apply presburger
    using LeftShiftNode bin-node.simps(7) apply presburger
    using RightShiftNode bin-node.simps(8) apply presburger
    using UnsignedRightShiftNode bin-node.simps(9) apply presburger
    using IntegerEqualsNode bin-node.simps(10) apply presburger
    using IntegerLessThanNode bin-node.simps(11) apply presburger
    using IntegerBelowNode bin-node.simps(12) by presburger
next
case (AllLeafNodes g n s)
then show ?case using rep.LeafNode by blast
qed
done

```

lemma *ref-refinement*:

```

assumes g  $\vdash$  n  $\simeq$  e1
assumes kind g n' = RefNode n
shows g  $\vdash$  n'  $\trianglelefteq$  e1
using assms RefNode
by (meson equal-refines graph-represents-expression-def)

```

lemma *unrep-refines*:

```

assumes g  $\oplus$  e  $\rightsquigarrow$  (g', n)
shows graph-refinement g g'

```

```

using assms
using graph-refinement-def subset-refines unrep-subset by blast

lemma add-new-node-refines:
  assumes  $n \notin \text{ids } g$ 
  assumes  $g' = \text{add-node } n \ (k, s) \ g$ 
  shows graph-refinement  $g \ g'$ 
  using assms unfolding graph-refinement
  using fresh-node-subset subset-refines by presburger

lemma add-node-as-set:
  assumes  $g' = \text{add-node } n \ (k, s) \ g$ 
  shows  $(\{n\} \trianglelefteq \text{as-set } g) \subseteq \text{as-set } g'$ 
  using assms unfolding as-set-def domain-subtraction-def
  using add-changed
  by (smt (z3) case-prodE changeonly.simps mem-Collect-eq prod.sel(1) subsetI)

theorem refined-insert:
  assumes  $e_1 \geq e_2$ 
  assumes  $g_1 \oplus e_2 \rightsquigarrow (g_2, n')$ 
  shows  $(g_2 \vdash n' \trianglelefteq e_1) \wedge \text{graph-refinement } g_1 \ g_2$ 
  using assms
  using graph-construction term-graph-reconstruction by blast

lemma ids-finite: finite (ids  $g$ )
  using Rep-IRGraph ids.rep-eq by simp

lemma unwrap-sorted: set (sorted-list-of-set (ids  $g$ )) = ids  $g$ 
  using Rep-IRGraph set-sorted-list-of-set ids-finite
  by blast

lemma find-none:
  assumes find-node-and-stamp  $g \ (k, s) = \text{None}$ 
  shows  $\forall n \in \text{ids } g. \text{kind } g \ n \neq k \vee \text{stamp } g \ n \neq s$ 
proof –
  have  $(\nexists n. n \in \text{ids } g \wedge (\text{kind } g \ n = k \wedge \text{stamp } g \ n = s))$ 
    using assms unfolding find-node-and-stamp.simps using find-None-iff unwrap-sorted
    by (metis (mono-tags, lifting))
  then show ?thesis
    by blast
qed

```

```

method ref-represents uses node =
  (metis IRNode.distinct(2755) RefNode dual-order.refl find-new-kind fresh-node-subset
node subset-implies-evals)

```

```

end

```

3 Control-flow Semantics

```

theory IRStepObj
  imports
    TreeToGraph
begin

```

3.1 Object Heap

The heap model we introduce maps field references to object instances to runtime values. We use the $H[f][p]$ heap representation. See *\cite{heap-reps-2011}*. We also introduce the `DynamicHeap` type which allocates new object references sequentially storing the next free object reference as 'Free'.

heapdef

```

type-synonym ('a, 'b) Heap = 'a  $\Rightarrow$  'b  $\Rightarrow$  Value
type-synonym Free = nat
type-synonym ('a, 'b) DynamicHeap = ('a, 'b) Heap  $\times$  Free

fun h-load-field :: 'a  $\Rightarrow$  'b  $\Rightarrow$  ('a, 'b) DynamicHeap  $\Rightarrow$  Value where
  h-load-field f r (h, n) = h f r

fun h-store-field :: 'a  $\Rightarrow$  'b  $\Rightarrow$  Value  $\Rightarrow$  ('a, 'b) DynamicHeap  $\Rightarrow$  ('a, 'b)
DynamicHeap where
  h-store-field f r v (h, n) = (h(f := ((h f)(r := v))), n)

fun h-new-inst :: ('a, 'b) DynamicHeap  $\Rightarrow$  ('a, 'b) DynamicHeap  $\times$  Value
where
  h-new-inst (h, n) = ((h, n+1), (ObjRef (Some n)))

type-synonym FieldRefHeap = (string, objref) DynamicHeap

```

```

definition new-heap :: ('a, 'b) DynamicHeap where
  new-heap = (( $\lambda f. \lambda p. \text{UndefVal}$ ), 0)

```

3.2 Intraprocedural Semantics

```

fun find-index :: 'a ⇒ 'a list ⇒ nat where
  find-index - [] = 0 |
  find-index v (x # xs) = (if (x=v) then 0 else find-index v xs + 1)

fun phi-list :: IRGraph ⇒ ID ⇒ ID list where
  phi-list g n =
    (filter (λx.(is-PhiNode (kind g x)))
     (sorted-list-of-set (usages g n)))

fun input-index :: IRGraph ⇒ ID ⇒ ID ⇒ nat where
  input-index g n n' = find-index n' (inputs-of (kind g n))

fun phi-inputs :: IRGraph ⇒ nat ⇒ ID list ⇒ ID list where
  phi-inputs g i nodes = (map (λn. (inputs-of (kind g n))!(i + 1)) nodes)

fun set-phis :: ID list ⇒ Value list ⇒ MapState ⇒ MapState where
  set-phis [] [] m = m |
  set-phis (n # xs) (v # vs) m = (set-phis xs vs (m(n := v))) |
  set-phis [] (v # vs) m = m |
  set-phis (x # xs) [] m = m

```

Intraprocedural semantics are given as a small-step semantics.

Within the context of a graph, the configuration triple, (ID, MethodState, Heap), is related to the subsequent configuration.

```

inductive step :: IRGraph ⇒ Params ⇒ (ID × MapState × FieldRefHeap) ⇒ (ID
× MapState × FieldRefHeap) ⇒ bool
  (¬, - ⊢ - → - 55) for g p where

```

SequentialNode:

```

[[is-sequential-node (kind g nid);
  nid' = (successors-of (kind g nid))!0]]
⇒ g, p ⊢ (nid, m, h) → (nid', m, h) |

```

IfNode:

```

[[kind g nid = (IfNode cond tb fb);
  g ⊢ cond ≃ condE;
  [m, p] ⊢ condE ↦ val;
  nid' = (if val-to-bool val then tb else fb)]]
⇒ g, p ⊢ (nid, m, h) → (nid', m, h) |

```

EndNodes:

```

[[is-AbstractEndNode (kind g nid);
  merge = any-usage g nid;
  is-AbstractMergeNode (kind g merge);

  i = find-index nid (inputs-of (kind g merge));
  phis = (phi-list g merge);

```

$inps = (\text{phi-inputs } g \ i \ phis);$
 $g \vdash inps \simeq_L inpsE;$
 $[m, p] \vdash inpsE \mapsto_L vs;$

$m' = \text{set-phs } phis \ vs \ m$
 $\implies g, p \vdash (nid, m, h) \rightarrow (\text{merge}, m', h) \mid$

NewInstanceNode:

$\llbracket \text{kind } g \ nid = (\text{NewInstanceNode } nid \ f \ \text{obj } nid');$
 $(h', \text{ref}) = h\text{-new-inst } h;$
 $m' = m(nid := \text{ref}) \rrbracket$
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m', h') \mid$

LoadFieldNode:

$\llbracket \text{kind } g \ nid = (\text{LoadFieldNode } nid \ f \ (\text{Some } \text{obj}) \ nid');$
 $g \vdash \text{obj} \simeq \text{obj}E;$
 $[m, p] \vdash \text{obj}E \mapsto \text{ObjRef } \text{ref};$
 $h\text{-load-field } f \ \text{ref } h = v;$
 $m' = m(nid := v) \rrbracket$
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m', h) \mid$

SignedDivNode:

$\llbracket \text{kind } g \ nid = (\text{SignedDivNode } nid \ x \ y \ \text{zero } sb \ \text{nxt});$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye;$
 $[m, p] \vdash xe \mapsto v1;$
 $[m, p] \vdash ye \mapsto v2;$
 $v = (\text{intval-div } v1 \ v2);$
 $m' = m(nid := v) \rrbracket$
 $\implies g, p \vdash (nid, m, h) \rightarrow (\text{nxt}, m', h) \mid$

SignedRemNode:

$\llbracket \text{kind } g \ nid = (\text{SignedRemNode } nid \ x \ y \ \text{zero } sb \ \text{nxt});$
 $g \vdash x \simeq xe;$
 $g \vdash y \simeq ye;$
 $[m, p] \vdash xe \mapsto v1;$
 $[m, p] \vdash ye \mapsto v2;$
 $v = (\text{intval-mod } v1 \ v2);$
 $m' = m(nid := v) \rrbracket$
 $\implies g, p \vdash (nid, m, h) \rightarrow (\text{nxt}, m', h) \mid$

StaticLoadFieldNode:

$\llbracket \text{kind } g \ nid = (\text{LoadFieldNode } nid \ f \ \text{None } nid');$
 $h\text{-load-field } f \ \text{None } h = v;$
 $m' = m(nid := v) \rrbracket$
 $\implies g, p \vdash (nid, m, h) \rightarrow (nid', m', h) \mid$

StoreFieldNode:

$$\begin{aligned}
& \llbracket \text{kind } g \text{ nid} = (\text{StoreFieldNode } \text{nid } f \text{ newval} - (\text{Some } \text{obj}) \text{ nid}') \rrbracket; \\
& g \vdash \text{newval} \simeq \text{newval}E; \\
& g \vdash \text{obj} \simeq \text{obj}E; \\
& [m, p] \vdash \text{newval}E \mapsto \text{val}; \\
& [m, p] \vdash \text{obj}E \mapsto \text{ObjRef } \text{ref}; \\
& h' = h\text{-store-field } f \text{ ref val } h; \\
& m' = m(\text{nid} := \text{val}) \\
& \implies g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h') \mid
\end{aligned}$$

StaticStoreFieldNode:

$$\begin{aligned}
& \llbracket \text{kind } g \text{ nid} = (\text{StoreFieldNode } \text{nid } f \text{ newval} - \text{None } \text{nid}') \rrbracket; \\
& g \vdash \text{newval} \simeq \text{newval}E; \\
& [m, p] \vdash \text{newval}E \mapsto \text{val}; \\
& h' = h\text{-store-field } f \text{ None val } h; \\
& m' = m(\text{nid} := \text{val}) \\
& \implies g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h')
\end{aligned}$$

code-pred (*modes*: $i \Rightarrow i \Rightarrow i * i * i \Rightarrow o * o * o \Rightarrow \text{bool}$) *step* .

3.3 Interprocedural Semantics

type-synonym *Signature* = *string*

type-synonym *Program* = *Signature* \rightarrow *IRGraph*

inductive *step-top* :: *Program* \Rightarrow (*IRGraph* \times *ID* \times *MapState* \times *Params*) *list* \times *FieldRefHeap* \Rightarrow (*IRGraph* \times *ID* \times *MapState* \times *Params*) *list* \times *FieldRefHeap* \Rightarrow *bool*

(- \vdash - \longrightarrow - 55)

for *P* **where**

Lift:

$$\begin{aligned}
& \llbracket g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m', h') \rrbracket \\
& \implies P \vdash ((g, \text{nid}, m, p) \# \text{stk}, h) \longrightarrow ((g, \text{nid}', m', p) \# \text{stk}, h') \mid
\end{aligned}$$

InvokeNodeStep:

$\llbracket \text{is-Invoke } (\text{kind } g \text{ nid}) \rrbracket;$

$$\begin{aligned}
& \text{callTarget} = \text{ir-callTarget } (\text{kind } g \text{ nid}); \\
& \text{kind } g \text{ callTarget} = (\text{MethodCallTargetNode } \text{targetMethod } \text{arguments}); \\
& \text{Some } \text{targetGraph} = P \text{ targetMethod}; \\
& m' = \text{new-map-state}; \\
& g \vdash \text{arguments} \simeq_L \text{args}E; \\
& [m, p] \vdash \text{args}E \mapsto_L p \\
& \implies P \vdash ((g, \text{nid}, m, p) \# \text{stk}, h) \longrightarrow ((\text{targetGraph}, 0, m', p') \# (g, \text{nid}, m, p) \# \text{stk}, h)
\end{aligned}$$

|

ReturnNode:

$\llbracket \text{kind } g \text{ nid} = (\text{ReturnNode } (\text{Some } \text{expr}) -) \rrbracket;$

$g \vdash \text{expr} \simeq e;$
 $[m, p] \vdash e \mapsto v;$

$cm' = cm(cnid := v);$
 $cnid' = (\text{successors-of } (kind \text{ } cg \text{ } cnid))!0$
 $\implies P \vdash ((g, nid, m, p) \# (cg, cnid, cm, cp) \# stk, h) \longrightarrow ((cg, cnid', cm', cp) \# stk, h) \mid$

ReturnNodeVoid:

$\llbracket kind \text{ } g \text{ } nid = (\text{ReturnNode } None \text{ } -);$
 $cm' = cm(cnid := (\text{ObjRef } (\text{Some } (2048))));$

$cnid' = (\text{successors-of } (kind \text{ } cg \text{ } cnid))!0$
 $\implies P \vdash ((g, nid, m, p) \# (cg, cnid, cm, cp) \# stk, h) \longrightarrow ((cg, cnid', cm', cp) \# stk, h) \mid$

UnwindNode:

$\llbracket kind \text{ } g \text{ } nid = (\text{UnwindNode } exception);$

$g \vdash \text{exception} \simeq \text{exceptionE};$
 $[m, p] \vdash \text{exceptionE} \mapsto e;$

$kind \text{ } cg \text{ } cnid = (\text{InvokeWithExceptionNode } - - - - - \text{exEdge});$

$cm' = cm(cnid := e)$
 $\implies P \vdash ((g, nid, m, p) \# (cg, cnid, cm, cp) \# stk, h) \longrightarrow ((cg, exEdge, cm', cp) \# stk, h)$

code-pred (*modes*: $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$) *step-top* .

3.4 Big-step Execution

type-synonym *Trace* = (*IRGraph* \times *ID* \times *MapState* \times *Params*) *list*

fun *has-return* :: *MapState* \Rightarrow *bool* **where**
has-return *m* = (*m* 0 \neq *UndefVal*)

inductive *exec* :: *Program*

\Rightarrow (*IRGraph* \times *ID* \times *MapState* \times *Params*) *list* \times *FieldRefHeap*
 \Rightarrow *Trace*
 \Rightarrow (*IRGraph* \times *ID* \times *MapState* \times *Params*) *list* \times *FieldRefHeap*
 \Rightarrow *Trace*
 \Rightarrow *bool*

(- \vdash - \mid - \longrightarrow * - \mid -)

for *P*

where

$\llbracket P \vdash (((g, nid, m, p) \# xs), h) \longrightarrow (((g', nid', m', p') \# ys), h');$
 $\neg(\text{has-return } m');$

$l' = (l \text{ @ } [(g, nid, m, p)]);$

exec *P* (((*g'*, *nid'*, *m'*, *p'*) $\#$ *ys*), *h'*) *l'* *next-state* *l''*

$\Rightarrow \text{exec } P \ ((g, \text{nid}, m, p) \# xs), h) \ l \ \text{next-state } l''$
 \mid
 $\llbracket P \vdash ((g, \text{nid}, m, p) \# xs), h) \longrightarrow ((g', \text{nid}', m', p') \# ys), h' \rrbracket;$
 $\text{has-return } m';$
 $l' = (l \ @ \ [(g, \text{nid}, m, p)])$
 $\Rightarrow \text{exec } P \ ((g, \text{nid}, m, p) \# xs), h) \ l \ ((g', \text{nid}', m', p') \# ys), h' \ l'$
code-pred (modes: $i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow o \Rightarrow \text{bool}$ as *Exec*) *exec* .

inductive *exec-debug* :: *Program*
 $\Rightarrow (IRGraph \times ID \times MapState \times Params) \ \text{list} \times FieldRefHeap$
 $\Rightarrow \text{nat}$
 $\Rightarrow (IRGraph \times ID \times MapState \times Params) \ \text{list} \times FieldRefHeap$
 $\Rightarrow \text{bool}$
 $(\vdash \longrightarrow * - * -)$
where
 $\llbracket n > 0; \mid$
 $p \vdash s \longrightarrow s';$
 $\text{exec-debug } p \ s' \ (n - 1) \ s' \rrbracket$
 $\Rightarrow \text{exec-debug } p \ s \ n \ s'' \mid$
 $\llbracket n = 0 \rrbracket$
 $\Rightarrow \text{exec-debug } p \ s \ n \ s$
code-pred (modes: $i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$) *exec-debug* .

3.4.1 Heap Testing

definition *p3* :: *Params* **where**
 $p3 = [IntVal32 \ 3]$

values $\{(prod.fst(prod.snd \ (prod.snd \ (hd \ (prod.fst \ res)))) \ 0$
 $\mid res. (\lambda x. \text{Some } eg2\text{-sq}) \vdash ([(eg2\text{-sq}, 0, \text{new-map-state}, p3), (eg2\text{-sq}, 0, \text{new-map-state}, p3)],$
 $\text{new-heap}) \rightarrow * 2 * \ res\}$

definition *field-sq* :: *string* **where**
 $\text{field-sq} = "sq"$

definition *eg3-sq* :: *IRGraph* **where**
 $eg3\text{-sq} = \text{irgraph } [$
 $(0, \text{StartNode } \text{None } 4, \text{VoidStamp}),$
 $(1, \text{ParameterNode } 0, \text{default-stamp}),$
 $(3, \text{MulNode } 1 \ 1, \text{default-stamp}),$
 $(4, \text{StoreFieldNode } 4 \ \text{field-sq } 3 \ \text{None } \text{None } 5, \text{VoidStamp}),$
 $(5, \text{ReturnNode } (\text{Some } 3) \ \text{None}, \text{default-stamp})$
 $]$

values {*h-load-field field-sq None (prod.snd res)*
 | *res. (λx. Some eg3-sq) ⊢ [(eg3-sq, 0, new-map-state, p3), (eg3-sq, 0, new-map-state, p3)], new-heap) →*3* res*}

definition *eg4-sq :: IRGraph where*

eg4-sq = irgraph [
 (*0, StartNode None 4, VoidStamp*),
 (*1, ParameterNode 0, default-stamp*),
 (*3, MulNode 1 1, default-stamp*),
 (*4, NewInstanceNode 4 "obj-class" None 5, ObjectStamp "obj-class" True True*
True),
 (*5, StoreFieldNode 5 field-sq 3 None (Some 4) 6, VoidStamp*),
 (*6, ReturnNode (Some 3) None, default-stamp*)
]

values {*h-load-field field-sq (Some 0) (prod.snd res) | res.*
*(λx. Some eg4-sq) ⊢ [(eg4-sq, 0, new-map-state, p3), (eg4-sq, 0, new-map-state, p3)], new-heap) →*4* res*}

end

3.5 Control-flow Semantics Theorems

theory *IRStepThms*

imports

IRStepObj

TreeToGraphThms

begin

We prove that within the same graph, a configuration triple will always transition to the same subsequent configuration. Therefore, our step semantics is deterministic.

3.5.1 Control-flow Step is Deterministic

theorem *stepDet:*

(g, p ⊢ (nid,m,h) → next) ⇒
(∀ next'. ((g, p ⊢ (nid,m,h) → next') ⇒ next = next'))

proof (*induction rule: step.induct*)

case (*SequentialNode nid next m h*)

have *notif: ¬(is-IfNode (kind g nid))*

using *SequentialNode.hyps(1) is-sequential-node.simps*

by (*metis is-IfNode-def*)

have *notend: ¬(is-AbstractEndNode (kind g nid))*

using *SequentialNode.hyps(1) is-sequential-node.simps*

by (*metis is-AbstractEndNode.simps is-EndNode.elims(2) is-LoopEndNode-def*)

have *notnew: ¬(is-NewInstanceNode (kind g nid))*

```

    using SequentialNode.hyps(1) is-sequential-node.simps
    by (metis is-NewInstanceNode-def)
  have notload:  $\neg(\text{is-LoadFieldNode } (kind\ g\ nid))$ 
    using SequentialNode.hyps(1) is-sequential-node.simps
    by (metis is-LoadFieldNode-def)
  have notstore:  $\neg(\text{is-StoreFieldNode } (kind\ g\ nid))$ 
    using SequentialNode.hyps(1) is-sequential-node.simps
    by (metis is-StoreFieldNode-def)
  have notdivrem:  $\neg(\text{is-IntegerDivRemNode } (kind\ g\ nid))$ 
    using SequentialNode.hyps(1) is-sequential-node.simps is-SignedDivNode-def
    is-SignedRemNode-def
    by (metis is-IntegerDivRemNode.simps)
  from notif notend notnew notload notstore notdivrem
  show ?case using SequentialNode.step.cases
    by (smt (z3) IRNode.disc(1028) IRNode.disc(2270) IRNode.discI(31) Pair-inject
    is-sequential-node.simps(18) is-sequential-node.simps(43) is-sequential-node.simps(44))
next
case (IfNode nid cond tb fb m val next h)
then have notseq:  $\neg(\text{is-sequential-node } (kind\ g\ nid))$ 
    using is-sequential-node.simps is-AbstractMergeNode.simps
    by (simp add: IfNode.hyps(1))
have notend:  $\neg(\text{is-AbstractEndNode } (kind\ g\ nid))$ 
    using is-AbstractEndNode.simps
    by (simp add: IfNode.hyps(1))
have notdivrem:  $\neg(\text{is-IntegerDivRemNode } (kind\ g\ nid))$ 
    using is-AbstractEndNode.simps
    by (simp add: IfNode.hyps(1))
from notseq notend notdivrem show ?case using IfNode.repDet evalDet IRN-
ode.distinct IRNode.inject(11) Pair-inject step.simps
    by (smt (z3) IRNode.distinct IRNode.inject(12) Pair-inject step.simps)
next
case (EndNodes nid merge i phis inputs m vs m' h)
have notseq:  $\neg(\text{is-sequential-node } (kind\ g\ nid))$ 
    using EndNodes.hyps(1) is-AbstractEndNode.simps is-sequential-node.simps
    by (metis is-EndNode.elims(2) is-LoopEndNode-def)
have notif:  $\neg(\text{is-IfNode } (kind\ g\ nid))$ 
    using EndNodes.hyps(1) is-IfNode-def is-AbstractEndNode.elims
    by (metis IRNode.distinct-disc(1058) is-EndNode.simps(12))
have notref:  $\neg(\text{is-RefNode } (kind\ g\ nid))$ 
    using EndNodes.hyps(1) is-sequential-node.simps
    using IRNode.disc(1899) IRNode.distinct(1473) is-AbstractEndNode.simps
    is-EndNode.elims(2) is-LoopEndNode-def is-RefNode-def
    by metis
have notnew:  $\neg(\text{is-NewInstanceNode } (kind\ g\ nid))$ 
    using EndNodes.hyps(1) is-AbstractEndNode.simps
    using IRNode.distinct-disc(1442) is-EndNode.simps(29) is-NewInstanceNode-def
    by (metis IRNode.distinct-disc(1901) is-EndNode.simps(32))
have notload:  $\neg(\text{is-LoadFieldNode } (kind\ g\ nid))$ 
    using EndNodes.hyps(1) is-AbstractEndNode.simps

```

```

    using is-LoadFieldName-def
    by (metis IRNode.distinct-disc(1706) is-EndNode.simps(21))
  have notstore:  $\neg$ (is-StoreFieldName (kind g nid))
    using EndNodes.hyps(1) is-AbstractEndNode.simps is-StoreFieldName-def
    by (metis IRNode.distinct-disc(1926) is-EndNode.simps(44))
  have notdivrem:  $\neg$ (is-IntegerDivRemNode (kind g nid))
    using EndNodes.hyps(1) is-AbstractEndNode.simps is-SignedDivNode-def is-SignedRemNode-def
    using IRNode.distinct-disc(1498) IRNode.distinct-disc(1500) is-IntegerDivRemNode.simps
    is-EndNode.simps(36) is-EndNode.simps(37)
  by auto
  from notseq notif notref notnew notload notstore notdivrem
  show ?case using EndNodes.repAllDet evalAllDet
    by (smt (z3) is-IfNode-def is-LoadFieldName-def is-NewInstanceNode-def is-RefNode-def
    is-StoreFieldName-def is-SignedDivNode-def is-SignedRemNode-def Pair-inject is-IntegerDivRemNode.elims(3)
    step.cases)
next
case (NewInstanceNode nid f obj nxt h' ref h m' m)
then have notseq:  $\neg$ (is-sequential-node (kind g nid))
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notend:  $\neg$ (is-AbstractEndNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notif:  $\neg$ (is-IfNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notref:  $\neg$ (is-RefNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notload:  $\neg$ (is-LoadFieldName (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notstore:  $\neg$ (is-StoreFieldName (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
have notdivrem:  $\neg$ (is-IntegerDivRemNode (kind g nid))
  using is-AbstractMergeNode.simps
  by (simp add: NewInstanceNode.hyps(1))
from notseq notend notif notref notload notstore notdivrem
show ?case using NewInstanceNode step.cases
  by (smt (z3) IRNode.disc(1028) IRNode.disc(2270) IRNode.discI(11) IRN-
ode.distinct(2311) IRNode.distinct(2313) IRNode.inject(31) Pair-inject)
next
case (LoadFieldName nid f obj nxt m ref h v m')
then have notseq:  $\neg$ (is-sequential-node (kind g nid))
  using is-sequential-node.simps is-AbstractMergeNode.simps
  by (simp add: LoadFieldName.hyps(1))
have notend:  $\neg$ (is-AbstractEndNode (kind g nid))
  using is-AbstractEndNode.simps

```

```

    by (simp add: LoadFieldNode.hyps(1))
  have notdivrem:  $\neg(\text{is-IntegerDivRemNode } (\text{kind } g \text{ nid}))$ 
    using is-AbstractEndNode.simps
    by (simp add: LoadFieldNode.hyps(1))
  from notseq notend notdivrem
  show ?case using LoadFieldNode step.cases repDet evalDet
    by (smt (z3) IRNode.distinct(1051) IRNode.distinct(1721) IRNode.distinct(1739)
  IRNode.distinct(1741) IRNode.distinct(1745) IRNode.inject(20) Pair-inject Value.inject(3)
  option.distinct(1) option.inject)
next
  case (StaticLoadFieldNode nid f nxt h v m' m)
  then have notseq:  $\neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$ 
    using is-sequential-node.simps is-AbstractMergeNode.simps
    by (simp add: StaticLoadFieldNode.hyps(1))
  have notend:  $\neg(\text{is-AbstractEndNode } (\text{kind } g \text{ nid}))$ 
    using is-AbstractEndNode.simps
    by (simp add: StaticLoadFieldNode.hyps(1))
  have notdivrem:  $\neg(\text{is-IntegerDivRemNode } (\text{kind } g \text{ nid}))$ 
    by (simp add: StaticLoadFieldNode.hyps(1))
  from notseq notend notdivrem
  show ?case using StaticLoadFieldNode step.cases
    by (smt (z3) IRNode.distinct(1051) IRNode.distinct(1721) IRNode.distinct(1739)
  IRNode.distinct(1741) IRNode.distinct(1745) IRNode.inject(20) Pair-inject option.distinct(1))
next
  case (StoreFieldNode nid f newval uu obj nxt m val ref h' h m')
  then have notseq:  $\neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$ 
    using is-sequential-node.simps is-AbstractMergeNode.simps
    by (simp add: StoreFieldNode.hyps(1))
  have notend:  $\neg(\text{is-AbstractEndNode } (\text{kind } g \text{ nid}))$ 
    using is-AbstractEndNode.simps
    by (simp add: StoreFieldNode.hyps(1))
  have notdivrem:  $\neg(\text{is-IntegerDivRemNode } (\text{kind } g \text{ nid}))$ 
    by (simp add: StoreFieldNode.hyps(1))
  from notseq notend notdivrem
  show ?case using StoreFieldNode step.cases repDet evalDet
    by (smt (z3) IRNode.distinct(1097) IRNode.distinct(1745) IRNode.distinct(2317)
  IRNode.distinct(2605) IRNode.distinct(2627) IRNode.inject(43) Pair-inject Value.inject(3)
  option.distinct(1) option.inject)
next
  case (StaticStoreFieldNode nid f newval uv nxt m val h' h m')
  then have notseq:  $\neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$ 
    using is-sequential-node.simps is-AbstractMergeNode.simps
    by (simp add: StaticStoreFieldNode.hyps(1))
  have notend:  $\neg(\text{is-AbstractEndNode } (\text{kind } g \text{ nid}))$ 
    using is-AbstractEndNode.simps
    by (simp add: StaticStoreFieldNode.hyps(1))
  have notdivrem:  $\neg(\text{is-IntegerDivRemNode } (\text{kind } g \text{ nid}))$ 
    by (simp add: StaticStoreFieldNode.hyps(1))
  from notseq notend notdivrem

```

```

show ?case using StoreFieldNode step.cases repDet evalDet
by (smt (z3) IRNode.distinct(1097) IRNode.distinct(1745) IRNode.distinct(2317)
IRNode.distinct(2605) IRNode.distinct(2627) IRNode.inject(43) Pair-inject Static-
StoreFieldNode.hyps(1) StaticStoreFieldNode.hyps(2) StaticStoreFieldNode.hyps(3)
StaticStoreFieldNode.hyps(4) StaticStoreFieldNode.hyps(5) option.distinct(1))
next
  case (SignedDivNode nid x y zero sb nxt m v1 v2 v m' h)
  then have notseq:  $\neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$ 
    using is-sequential-node.simps is-AbstractMergeNode.simps
    by (simp add: SignedDivNode.hyps(1))
  have notend:  $\neg(\text{is-AbstractEndNode } (\text{kind } g \text{ nid}))$ 
    using is-AbstractEndNode.simps
    by (simp add: SignedDivNode.hyps(1))
  from notseq notend
  show ?case using SignedDivNode step.cases repDet evalDet
    by (smt (z3) IRNode.distinct(1091) IRNode.distinct(1739) IRNode.distinct(2311)
IRNode.distinct(2601) IRNode.distinct(2605) IRNode.inject(40) Pair-inject)
  next
    case (SignedRemNode nid x y zero sb nxt m v1 v2 v m' h)
    then have notseq:  $\neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$ 
      using is-sequential-node.simps is-AbstractMergeNode.simps
      by (simp add: SignedRemNode.hyps(1))
    have notend:  $\neg(\text{is-AbstractEndNode } (\text{kind } g \text{ nid}))$ 
      using is-AbstractEndNode.simps
      by (simp add: SignedRemNode.hyps(1))
    from notseq notend
    show ?case using SignedRemNode step.cases repDet evalDet
      by (smt (z3) IRNode.distinct(1093) IRNode.distinct(1741) IRNode.distinct(2313)
IRNode.distinct(2601) IRNode.distinct(2627) IRNode.inject(41) Pair-inject)
  qed

```

lemma stepRefNode:

$\llbracket \text{kind } g \text{ nid} = \text{RefNode nid}' \rrbracket \implies g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m, h)$

using SequentialNode

by (metis IRNodes.successors-of-RefNode is-sequential-node.simps(7) nth-Cons-0)

lemma IfNodeStepCases:

assumes kind g nid = IfNode cond tb fb

assumes $g \vdash \text{cond} \simeq \text{condE}$

assumes $[m, p] \vdash \text{condE} \mapsto v$

assumes $g, p \vdash (\text{nid}, m, h) \rightarrow (\text{nid}', m, h)$

shows $\text{nid}' \in \{\text{tb}, \text{fb}\}$

using step.IfNode repDet stepDet assms

by (metis insert-iff old.prod.inject)

lemma IfNodeSeq:

shows kind g nid = IfNode cond tb fb $\longrightarrow \neg(\text{is-sequential-node } (\text{kind } g \text{ nid}))$

unfolding is-sequential-node.simps

using is-sequential-node.simps(18) **by** presburger

```

lemma IfNodeCond:
  assumes kind g nid = IfNode cond tb fb
  assumes  $g, p \vdash (nid, m, h) \rightarrow (nid', m, h)$ 
  shows  $\exists \text{ condE } v. ((g \vdash \text{cond} \simeq \text{condE}) \wedge ([m, p] \vdash \text{condE} \mapsto v))$ 
  using assms(2,1) by (induct (nid,m,h) (nid',m,h) rule: step.induct; auto)

lemma step-in-ids:
  assumes  $g, p \vdash (nid, m, h) \rightarrow (nid', m', h')$ 
  shows  $nid \in \text{ids } g$ 
  using assms apply (induct (nid, m, h) (nid', m', h') rule: step.induct)
  using is-sequential-node.simps(45) not-in-g
  apply simp
  apply (metis is-sequential-node.simps(53))
  using ids-some
  using IRNode.distinct(1113) apply presburger
  using EndNodes(1) is-AbstractEndNode.simps is-EndNode.simps(45) ids-some
  apply (metis IRNode.disc(1218) is-EndNode.simps(52))
  by simp+

end

```