

# Veriopt Theories

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## 1 Conditional Elimination Phase

```
theory ConditionalElimination
  imports
    Semantics.IRTreeEvalThms
    Proofs.Rewrites
    Proofs.Bisimulation
begin
```

### 1.1 Individual Elimination Rules

The set of rules used for determining whether a condition  $q1::'a$  implies another condition  $q2::'a$  or its negation. These rules are used for conditional elimination.

```
inductive impliesx :: IRExpr  $\Rightarrow$  IRExpr  $\Rightarrow$  bool (-  $\Rightarrow$  -) and
  impliesnot :: IRExpr  $\Rightarrow$  IRExpr  $\Rightarrow$  bool (-  $\Rightarrow$   $\neg$  -) where
  q-imp-q:
    q  $\Rightarrow$  q |
  eq-impliesnot-less:
    (BinaryExpr BinIntegerEquals x y)  $\Rightarrow$   $\neg$  (BinaryExpr BinIntegerLessThan x y) |
  eq-impliesnot-less-rev:
    (BinaryExpr BinIntegerEquals x y)  $\Rightarrow$   $\neg$  (BinaryExpr BinIntegerLessThan y x) |
  less-impliesnot-rev-less:
    (BinaryExpr BinIntegerLessThan x y)  $\Rightarrow$   $\neg$  (BinaryExpr BinIntegerLessThan y x)
  |
  less-impliesnot-eq:
    (BinaryExpr BinIntegerLessThan x y)  $\Rightarrow$   $\neg$  (BinaryExpr BinIntegerEquals x y) |
  less-impliesnot-eq-rev:
    (BinaryExpr BinIntegerLessThan x y)  $\Rightarrow$   $\neg$  (BinaryExpr BinIntegerEquals y x) |
  negate-true:
```

$\llbracket x \Rightarrow \neg y \rrbracket \Longrightarrow x \Rightarrow (\text{UnaryExpr } \text{UnaryLogicNegation } y) \mid$   
*negate-false:*  
 $\llbracket x \Rightarrow y \rrbracket \Longrightarrow x \Rightarrow \neg (\text{UnaryExpr } \text{UnaryLogicNegation } y)$

The relation  $q1::IRExpr \Rightarrow q2::IRExpr$  indicates that the implication  $(q1::bool) \longrightarrow (q2::bool)$  is known true (i.e. universally valid), and the relation  $q1::IRExpr \Rightarrow \neg q2::IRExpr$  indicates that the implication  $(q1::bool) \longrightarrow (q2::bool)$  is known false (i.e.  $(q1::bool) \longrightarrow \neg (q2::bool)$  is universally valid). If neither  $q1::IRExpr \Rightarrow q2::IRExpr$  nor  $q1::IRExpr \Rightarrow \neg q2::IRExpr$  then the status is unknown. Only the known true and known false cases can be used for conditional elimination.

**fun** *implies-valid* :: *IRExpr*  $\Rightarrow$  *IRExpr*  $\Rightarrow$  *bool* (**infix**  $\rightarrow$  50) **where**  
*implies-valid* *q1* *q2* =  
 $(\forall m\ p\ v1\ v2. ([m, p] \vdash q1 \mapsto v1) \wedge ([m, p] \vdash q2 \mapsto v2) \longrightarrow$   
 $(\text{val-to-bool } v1 \longrightarrow \text{val-to-bool } v2))$

**fun** *impliesnot-valid* :: *IRExpr*  $\Rightarrow$  *IRExpr*  $\Rightarrow$  *bool* (**infix**  $\rightarrow$  50) **where**  
*impliesnot-valid* *q1* *q2* =  
 $(\forall m\ p\ v1\ v2. ([m, p] \vdash q1 \mapsto v1) \wedge ([m, p] \vdash q2 \mapsto v2) \longrightarrow$   
 $(\text{val-to-bool } v1 \longrightarrow \neg \text{val-to-bool } v2))$

The relation  $(q1::IRExpr) \rightarrow (q2::IRExpr)$  means  $(q1::bool) \longrightarrow (q2::bool)$  is universally valid, and the relation  $(q1::IRExpr) \rightarrow (q2::IRExpr)$  means  $(q1::bool) \longrightarrow \neg (q2::bool)$  is universally valid.

**lemma** *eq-impliesnot-less-helper*:  
 $v1 = v2 \longrightarrow \neg(\text{int-signed-value } b\ v1 < \text{int-signed-value } b\ v2)$   
**by** *force*

**lemma** *eq-impliesnot-less-val*:  
 $\text{val-to-bool}(\text{intval-equals } v1\ v2) \longrightarrow \neg \text{val-to-bool}(\text{intval-less-than } v1\ v2)$   
**using** *eq-impliesnot-less-helper* *bool-to-val.simps* *bool-to-val-bin.simps* *intval-equals.simps*  
*intval-less-than.elims* *val-to-bool.elims* *val-to-bool.simps*  
**by** (*smt* (*verit*))

**lemma** *eq-impliesnot-less-rev-val*:  
 $\text{val-to-bool}(\text{intval-equals } v1\ v2) \longrightarrow \neg \text{val-to-bool}(\text{intval-less-than } v2\ v1)$

**proof** –  
**have** *a*:  $\text{intval-equals } v1\ v2 = \text{intval-equals } v2\ v1$   
**using** *bool-to-val-bin.simps* *intval-equals.simps* *intval-equals.elims*  
**by** (*smt* (*verit*))  
**show** *?thesis* **using** *a* *eq-impliesnot-less-val* **by** *presburger*  
**qed**

**lemma** *less-impliesnot-rev-less-val*:  
 $\text{val-to-bool}(\text{intval-less-than } v1\ v2) \longrightarrow \neg \text{val-to-bool}(\text{intval-less-than } v2\ v1)$   
**by** (*smt* (*verit*, *del-insts*) *Value.exhaust* *Value.inject*(1) *bool-to-val.simps*(2)  
*bool-to-val-bin.simps* *intval-less-than.simps*(1) *intval-less-than.simps*(5)  
*intval-less-than.simps*(6) *intval-less-than.simps*(7) *val-to-bool.elims*(2))

**lemma** *less-impliesnot-eq-val*:  
 $val\text{-}to\text{-}bool(intval\text{-}less\text{-}than\ v1\ v2) \longrightarrow \neg val\text{-}to\text{-}bool(intval\text{-}equals\ v1\ v2)$   
**using** *eq-impliesnot-less-val* **by** *blast*

**lemma** *logic-negate-type*:  
**assumes**  $[m, p] \vdash UnaryExpr\ UnaryLogicNegation\ x \mapsto v$   
**shows**  $\exists b\ v2. [m, p] \vdash x \mapsto IntVal\ b\ v2$   
**using** *assms*  
**by** (*metis* *UnaryExprE* *intval-logic-negation.elims* *unary-eval.simps*(4))

**lemma** *intval-logic-negation-inverse*:  
**assumes**  $b > 0$   
**assumes**  $x = IntVal\ b\ v$   
**shows**  $val\text{-}to\text{-}bool\ (intval\text{-}logic\text{-}negation\ x) \longleftrightarrow \neg(val\text{-}to\text{-}bool\ x)$   
**using** *assms* **by** (*cases* *x*; *auto* *simp*: *logic-negate-def*)

**lemma** *logic-negation-relation-tree*:  
**assumes**  $[m, p] \vdash y \mapsto val$   
**assumes**  $[m, p] \vdash UnaryExpr\ UnaryLogicNegation\ y \mapsto invval$   
**shows**  $val\text{-}to\text{-}bool\ val \longleftrightarrow \neg(val\text{-}to\text{-}bool\ invval)$   
**using** *assms* **using** *intval-logic-negation-inverse*  
**by** (*metis* *UnaryExprE* *evalDet* *eval-bits-1-64* *logic-negate-type* *unary-eval.simps*(4))

The following theorem shows that the known true/false rules are valid.

**theorem** *implies-impliesnot-valid*:  
**shows**  $((q1 \Rightarrow q2) \longrightarrow (q1 \mapsto q2)) \wedge$   
 $((q1 \Rightarrow \neg q2) \longrightarrow (q1 \mapsto \neg q2))$   
 $(is\ ?imp \longrightarrow ?val) \wedge (?notimp \longrightarrow ?notval)$   
**proof** (*induct* *q1* *q2* *rule*: *impliesx-impliesnot.induct*)  
**case** (*q-imp-q* *q*)  
**then show** *?case*  
**using** *evalDet* **by** *fastforce*  
**next**  
**case** (*eq-impliesnot-less* *x* *y*)  
**then show** *?case* **apply** *auto* **using** *eq-impliesnot-less-val* *evalDet* **by** *blast*  
**next**  
**case** (*eq-impliesnot-less-rev* *x* *y*)  
**then show** *?case* **apply** *auto* **using** *eq-impliesnot-less-rev-val* *evalDet* **by** *blast*  
**next**  
**case** (*less-impliesnot-rev-less* *x* *y*)  
**then show** *?case* **apply** *auto* **using** *less-impliesnot-rev-less-val* *evalDet* **by** *blast*  
**next**  
**case** (*less-impliesnot-eq* *x* *y*)  
**then show** *?case* **apply** *auto* **using** *less-impliesnot-eq-val* *evalDet* **by** *blast*  
**next**  
**case** (*less-impliesnot-eq-rev* *x* *y*)  
**then show** *?case* **apply** *auto* **using** *eq-impliesnot-less-rev-val* *evalDet* **by** *metis*  
**next**

```

    case (negate-true x y)
    then show ?case apply auto
      by (metis logic-negation-relation-tree unary-eval.simps(4) unfold-unary)
next
    case (negate-false x y)
    then show ?case apply auto
      by (metis UnaryExpr logic-negation-relation-tree unary-eval.simps(4))
qed

```

We introduce a type *TriState::'a* (as in the GraalVM compiler) to represent when static analysis can tell us information about the value of a Boolean expression. If *Unknown::'a* then no information can be inferred and if *Known-True::'a*/*KnownFalse::'a* one can infer the expression is always true/false.

```

datatype TriState = Unknown | KnownTrue | KnownFalse

```

The implies relation corresponds to the LogicNode.implies method from the compiler which attempts to infer when one logic nodes value can be inferred from a known logic node.

```

inductive implies :: IRGraph  $\Rightarrow$  IRNode  $\Rightarrow$  IRNode  $\Rightarrow$  TriState  $\Rightarrow$  bool
  (-  $\vdash$  - & -  $\hookrightarrow$  -) for g where
    eq-imp-less:
      g  $\vdash$  (IntegerEqualsNode x y) & (IntegerLessThanNode x y)  $\hookrightarrow$  KnownFalse |
    eq-imp-less-rev:
      g  $\vdash$  (IntegerEqualsNode x y) & (IntegerLessThanNode y x)  $\hookrightarrow$  KnownFalse |
    less-imp-rev-less:
      g  $\vdash$  (IntegerLessThanNode x y) & (IntegerLessThanNode y x)  $\hookrightarrow$  KnownFalse |
    less-imp-not-eq:
      g  $\vdash$  (IntegerLessThanNode x y) & (IntegerEqualsNode x y)  $\hookrightarrow$  KnownFalse |
    less-imp-not-eq-rev:
      g  $\vdash$  (IntegerLessThanNode x y) & (IntegerEqualsNode y x)  $\hookrightarrow$  KnownFalse |

    x-imp-x:
      g  $\vdash$  x & x  $\hookrightarrow$  KnownTrue |

    negate-false:
       $\llbracket g \vdash x \& (\text{kind } g \ y) \hookrightarrow \text{KnownTrue} \rrbracket \implies g \vdash x \& (\text{LogicNegationNode } y) \hookrightarrow \text{KnownFalse} \mid$ 
    negate-true:
       $\llbracket g \vdash x \& (\text{kind } g \ y) \hookrightarrow \text{KnownFalse} \rrbracket \implies g \vdash x \& (\text{LogicNegationNode } y) \hookrightarrow \text{KnownTrue}$ 

```

Total relation over partial implies relation

```

inductive condition-implies :: IRGraph  $\Rightarrow$  IRNode  $\Rightarrow$  IRNode  $\Rightarrow$  TriState  $\Rightarrow$  bool
  (-  $\vdash$  - & -  $\rightarrow$  -) for g where
     $\llbracket \neg(g \vdash a \& b \hookrightarrow \text{imp}) \rrbracket \implies (g \vdash a \& b \rightarrow \text{Unknown}) \mid$ 
     $\llbracket (g \vdash a \& b \hookrightarrow \text{imp}) \rrbracket \implies (g \vdash a \& b \rightarrow \text{imp})$ 

```

```

inductive implies-tree :: IRExpr  $\Rightarrow$  IRExpr  $\Rightarrow$  bool  $\Rightarrow$  bool
  (- & -  $\hookrightarrow$  -) where
    eq-imp-less:
      (BinaryExpr BinIntegerEquals x y) & (BinaryExpr BinIntegerLessThan x y)  $\hookrightarrow$ 
      False |
    eq-imp-less-rev:
      (BinaryExpr BinIntegerEquals x y) & (BinaryExpr BinIntegerLessThan y x)  $\hookrightarrow$ 
      False |
    less-imp-rev-less:
      (BinaryExpr BinIntegerLessThan x y) & (BinaryExpr BinIntegerLessThan y x)
       $\hookrightarrow$  False |
    less-imp-not-eq:
      (BinaryExpr BinIntegerLessThan x y) & (BinaryExpr BinIntegerEquals x y)  $\hookrightarrow$ 
      False |
    less-imp-not-eq-rev:
      (BinaryExpr BinIntegerLessThan x y) & (BinaryExpr BinIntegerEquals y x)  $\hookrightarrow$ 
      False |
    x-imp-x:
      x & x  $\hookrightarrow$  True |
    negate-false:
      [x & y  $\hookrightarrow$  True]  $\Rightarrow$  x & (UnaryExpr UnaryLogicNegation y)  $\hookrightarrow$  False |
    negate-true:
      [x & y  $\hookrightarrow$  False]  $\Rightarrow$  x & (UnaryExpr UnaryLogicNegation y)  $\hookrightarrow$  True

```

Proofs that the implies relation is correct with respect to the existing evaluation semantics.

```

lemma logic-negation-relation:
  assumes [g, m, p]  $\vdash$  y  $\mapsto$  val
  assumes kind g neg = LogicNegationNode y
  assumes [g, m, p]  $\vdash$  neg  $\mapsto$  invval
  assumes invval  $\neq$  UndefVal
  shows val-to-bool val  $\longleftrightarrow$   $\neg$ (val-to-bool invval)
  using assms
  by (metis LogicNegationNode encodeeval-def logic-negation-relation-tree repDet)

```

```

lemma implies-valid:
  assumes x & y  $\hookrightarrow$  imp
  assumes [m, p]  $\vdash$  x  $\mapsto$  v1
  assumes [m, p]  $\vdash$  y  $\mapsto$  v2
  shows (imp  $\longrightarrow$  (val-to-bool v1  $\longrightarrow$  val-to-bool v2))  $\wedge$ 
    ( $\neg$ imp  $\longrightarrow$  (val-to-bool v1  $\longrightarrow$   $\neg$ (val-to-bool v2)))
    (is (?TP  $\longrightarrow$  ?TC)  $\wedge$  (?FP  $\longrightarrow$  ?FC))
  apply (intro conjI; rule impI)
proof -
  assume KnownTrue: ?TP
  show ?TC
  using assms(1) KnownTrue assms(2-) proof (induct x y imp rule: implies-tree.induct)
    case (eq-imp-less x y)

```

```

    then show ?case by simp
  next
    case (eq-imp-less-rev x y)
    then show ?case by simp
  next
    case (less-imp-rev-less x y)
    then show ?case by simp
  next
    case (less-imp-not-eq x y)
    then show ?case by simp
  next
    case (less-imp-not-eq-rev x y)
    then show ?case by simp
  next
    case (x-imp-x)
    then show ?case
      by (metis evalDet)
  next
    case (negate-false x1)
    then show ?case using evalDet
      using assms(2,3) by blast
  next
    case (negate-true x y)
    then show ?case
      using logic-negation-relation-tree sorry
qed
next
  assume KnownFalse: ?FP
  show ?FC using assms KnownFalse proof (induct x y imp rule: implies-tree.induct)
    case (eq-imp-less x y)
    obtain xval where xval: [m, p] ⊢ x ↦ xval
      using eq-imp-less(1) eq-imp-less.prem(3)
      by blast
    then obtain yval where yval: [m, p] ⊢ y ↦ yval
      using eq-imp-less.prem(3)
      using eq-imp-less.prem(2) by blast
    have egeval: [m, p] ⊢ (BinaryExpr BinIntegerEquals x y) ↦ intval-equals xval
      yval
      using xval yval evaltree.BinaryExpr
      by (metis BinaryExprE bin-eval.simps(11) eq-imp-less.prem(1) evalDet)
    have lesseval: [m, p] ⊢ (BinaryExpr BinIntegerLessThan x y) ↦ intval-less-than
      xval yval
      using xval yval evaltree.BinaryExpr
      by (metis BinaryExprE bin-eval.simps(12) eq-imp-less.prem(2) evalDet)
    have val-to-bool (intval-equals xval yval) ⟶ ¬(val-to-bool (intval-less-than xval
      yval))
      apply (cases xval; cases yval; auto)
      by (smt (verit, best) bool-to-val.simps(2) val-to-bool.simps(1))
    then show ?case

```

```

    using egeval lesseval
    by (metis eq-imp-less.prem(1) eq-imp-less.prem(2) evalDet)
next
case (eq-imp-less-rev x y)
obtain xval where xval: [m, p] ⊢ x ↦ xval
  using eq-imp-less-rev.prem(3)
  using eq-imp-less-rev.prem(2) by blast
obtain yval where yval: [m, p] ⊢ y ↦ yval
  using eq-imp-less-rev.prem(3)
  using eq-imp-less-rev.prem(2) by blast
have egeval: [m, p] ⊢ (BinaryExpr BinIntegerEquals x y) ↦ intval-equals xval
yval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simp(11) eq-imp-less-rev.prem(1) evalDet)
have lesseval: [m, p] ⊢ (BinaryExpr BinIntegerLessThan y x) ↦ intval-less-than
yval xval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simp(12) eq-imp-less-rev.prem(2) evalDet)
have val-to-bool (intval-equals xval yval) ⟶ ¬(val-to-bool (intval-less-than yval
xval))
  apply (cases xval; cases yval; auto)
  by (metis (full-types) bool-to-val.simp(2) less-irrefl val-to-bool.simp(1))
then show ?case
  using egeval lesseval
  by (metis eq-imp-less-rev.prem(1) eq-imp-less-rev.prem(2) evalDet)
next
case (less-imp-rev-less x y)
obtain xval where xval: [m, p] ⊢ x ↦ xval
  using less-imp-rev-less.prem(3)
  using less-imp-rev-less.prem(2) by blast
obtain yval where yval: [m, p] ⊢ y ↦ yval
  using less-imp-rev-less.prem(3)
  using less-imp-rev-less.prem(2) by blast
have lesseval: [m, p] ⊢ (BinaryExpr BinIntegerLessThan x y) ↦ intval-less-than
xval yval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simp(12) evalDet less-imp-rev-less.prem(1))
have revlesseval: [m, p] ⊢ (BinaryExpr BinIntegerLessThan y x) ↦ intval-less-than
yval xval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simp(12) evalDet less-imp-rev-less.prem(2))
have val-to-bool (intval-less-than xval yval) ⟶ ¬(val-to-bool (intval-less-than
yval xval))
  apply (cases xval; cases yval; auto)
  by (smt (verit) bool-to-val.simp(2) val-to-bool.simp(1))
then show ?case
  by (metis evalDet less-imp-rev-less.prem(1) less-imp-rev-less.prem(2) lesse-
val revlesseval)
next

```

```

case (less-imp-not-eq x y)
obtain xval where xval: [m, p] ⊢ x ↦ xval
  using less-imp-not-eq.prems(3)
  using less-imp-not-eq.prems(1) by blast
obtain yval where yval: [m, p] ⊢ y ↦ yval
  using less-imp-not-eq.prems(3)
  using less-imp-not-eq.prems(1) by blast
have egeval: [m, p] ⊢ (BinaryExpr BinIntegerEquals x y) ↦ intval-equals xval
yval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(11) evalDet less-imp-not-eq.prems(2))
have lesseval: [m, p] ⊢ (BinaryExpr BinIntegerLessThan x y) ↦ intval-less-than
xval yval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(12) evalDet less-imp-not-eq.prems(1))
have val-to-bool (intval-less-than xval yval) ⟶ ¬(val-to-bool (intval-equals xval
yval))
  apply (cases xval; cases yval; auto)
  by (smt (verit, best) bool-to-val.simps(2) val-to-bool.simps(1))
then show ?case
  by (metis egeval evalDet less-imp-not-eq.prems(1) less-imp-not-eq.prems(2)
lesseval)
next
case (less-imp-not-eq-rev x y)
obtain xval where xval: [m, p] ⊢ x ↦ xval
  using less-imp-not-eq-rev.prems(3)
  using less-imp-not-eq-rev.prems(1) by blast
obtain yval where yval: [m, p] ⊢ y ↦ yval
  using less-imp-not-eq-rev.prems(3)
  using less-imp-not-eq-rev.prems(1) by blast
have egeval: [m, p] ⊢ (BinaryExpr BinIntegerEquals y x) ↦ intval-equals yval
xval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(11) evalDet less-imp-not-eq-rev.prems(2))
have lesseval: [m, p] ⊢ (BinaryExpr BinIntegerLessThan x y) ↦ intval-less-than
xval yval
  using xval yval evaltree.BinaryExpr
  by (metis BinaryExprE bin-eval.simps(12) evalDet less-imp-not-eq-rev.prems(1))
have val-to-bool (intval-less-than xval yval) ⟶ ¬(val-to-bool (intval-equals yval
xval))
  apply (cases xval; cases yval; auto)
  by (smt (verit, best) bool-to-val.simps(2) val-to-bool.simps(1))
then show ?case
  by (metis egeval evalDet less-imp-not-eq-rev.prems(1) less-imp-not-eq-rev.prems(2)
lesseval)
next
case (x-imp-x x1)
then show ?case by simp
next

```



```

    case (negate-false x y)
    then show ?case sorry
next
    case (negate-true x1)
    then show ?case by simp
qed
qed

lemma implies-true-valid:
  assumes x & y  $\hookrightarrow$  imp
  assumes imp
  assumes [m, p]  $\vdash$  x  $\mapsto$  v1
  assumes [m, p]  $\vdash$  y  $\mapsto$  v2
  shows val-to-bool v1  $\longrightarrow$  val-to-bool v2
  using assms implies-valid
  by blast

```

```

lemma implies-false-valid:
  assumes x & y  $\hookrightarrow$  imp
  assumes  $\neg$ imp
  assumes [m, p]  $\vdash$  x  $\mapsto$  v1
  assumes [m, p]  $\vdash$  y  $\mapsto$  v2
  shows val-to-bool v1  $\longrightarrow$   $\neg$ (val-to-bool v2)
  using assms implies-valid by blast

```

The following relation corresponds to the `UnaryOpLogicNode.tryFold` and `BinaryOpLogicNode.tryFold` methods and their associated concrete implementations.

The relation determines if a logic operation can be shown true or false through the stamp typing information.

```

inductive tryFold :: IRNode  $\Rightarrow$  (ID  $\Rightarrow$  Stamp)  $\Rightarrow$  bool  $\Rightarrow$  bool
where
  [[alwaysDistinct (stamps x) (stamps y)]]
     $\implies$  tryFold (IntegerEqualsNode x y) stamps False |
  [[neverDistinct (stamps x) (stamps y)]]
     $\implies$  tryFold (IntegerEqualsNode x y) stamps True |
  [[is-IntegerStamp (stamps x);
    is-IntegerStamp (stamps y);
    stpi-upper (stamps x) < stpi-lower (stamps y)]]
     $\implies$  tryFold (IntegerLessThanNode x y) stamps True |
  [[is-IntegerStamp (stamps x);
    is-IntegerStamp (stamps y);
    stpi-lower (stamps x)  $\geq$  stpi-upper (stamps y)]]
     $\implies$  tryFold (IntegerLessThanNode x y) stamps False

```

Proofs that show that when the stamp lookup function is well-formed, the `tryFold` relation correctly predicts the output value with respect to our evaluation semantics.

```

lemma
  assumes kind g nid = IntegerEqualsNode x y
  assumes [g, m, p] ⊢ nid ↦ v
  assumes ([g, m, p] ⊢ x ↦ xval) ∧ ([g, m, p] ⊢ y ↦ yval)
  shows val-to-bool (intval-equals xval yval) ⟷ v = IntVal 32 1
proof -
  have v = intval-equals xval yval
  using assms(1, 2, 3) BinaryExprE IntegerEqualsNode bin-eval.simps(7)
  by (smt (verit) bin-eval.simps(11) encodeeval-def evalDet repDet)
  then show ?thesis using intval-equals.simps val-to-bool.simps
  by (smt (verit) bool-to-val.simps(1) bool-to-val.simps(2) bool-to-val-bin.simps
      intval-equals.elims one-neq-zero)
qed

lemma tryFoldIntegerEqualsAlwaysDistinct:
  assumes wf-stamp g stamps
  assumes kind g nid = (IntegerEqualsNode x y)
  assumes [g, m, p] ⊢ nid ↦ v
  assumes alwaysDistinct (stamps x) (stamps y)
  shows v = IntVal 32 0
proof -
  have ∀ val. ¬(valid-value val (join (stamps x) (stamps y)))
  using assms(1,4) unfolding alwaysDistinct.simps
  by (smt (verit, best) is-stamp-empty.elims(2) valid-int valid-value.simps(1))
  obtain xv where [g, m, p] ⊢ x ↦ xv
  using assms using assms(2,3) unfolding encodeeval-def sorry
  have ¬(∃ val . ([g, m, p] ⊢ x ↦ val) ∧ ([g, m, p] ⊢ y ↦ val))
  using assms(1,4) unfolding alwaysDistinct.simps wf-stamp.simps encodeeval-def sorry
  then show ?thesis sorry
qed

lemma tryFoldIntegerEqualsNeverDistinct:
  assumes wf-stamp g stamps
  assumes kind g nid = (IntegerEqualsNode x y)
  assumes [g, m, p] ⊢ nid ↦ v
  assumes neverDistinct (stamps x) (stamps y)
  shows v = IntVal 32 1
  using assms IntegerEqualsNodeE sorry

lemma tryFoldIntegerLessThanTrue:
  assumes wf-stamp g stamps
  assumes kind g nid = (IntegerLessThanNode x y)
  assumes [g, m, p] ⊢ nid ↦ v
  assumes stpi-upper (stamps x) < stpi-lower (stamps y)
  shows v = IntVal 32 1
proof -
  have stamp-type: is-IntegerStamp (stamps x)
  using assms

```

```

  sorry
  obtain xval where xval:  $[g, m, p] \vdash x \mapsto xval$ 
  using assms(2,3) sorry
  obtain yval where yval:  $[g, m, p] \vdash y \mapsto yval$ 
  using assms(2,3) sorry
  have is-IntegerStamp (stamps x)  $\wedge$  is-IntegerStamp (stamps y)
  using assms(4)
  sorry
  then have val-to-bool (intval-less-than xval yval)
  sorry
  then show ?thesis
  sorry
qed

```

```

lemma tryFoldIntegerLessThanFalse:
  assumes wf-stamp g stamps
  assumes kind g nid = (IntegerLessThanNode x y)
  assumes  $[g, m, p] \vdash nid \mapsto v$ 
  assumes stpi-lower (stamps x)  $\geq$  stpi-upper (stamps y)
  shows v = IntVal 32 0
  proof -
  have stamp-type: is-IntegerStamp (stamps x)
  using assms
  sorry
  obtain xval where xval:  $[g, m, p] \vdash x \mapsto xval$ 
  using assms(2,3) sorry
  obtain yval where yval:  $[g, m, p] \vdash y \mapsto yval$ 
  using assms(2,3) sorry
  have is-IntegerStamp (stamps x)  $\wedge$  is-IntegerStamp (stamps y)
  using assms(4)
  sorry
  then have  $\neg(\text{val-to-bool } (\text{intval-less-than } xval \ yval))$ 
  sorry
  then show ?thesis
  sorry
qed

```

```

theorem tryFoldProofTrue:
  assumes wf-stamp g stamps
  assumes tryFold (kind g nid) stamps True
  assumes  $[g, m, p] \vdash nid \mapsto v$ 
  shows val-to-bool v
  using assms(2) proof (induction kind g nid stamps True rule: tryFold.induct)
  case (1 stamps x y)
  then show ?case using tryFoldIntegerEqualsAlwaysDistinct assms
  by force
  next
  case (2 stamps x y)
  then show ?case using tryFoldIntegerEqualsAlwaysDistinct assms

```

```

  by (smt (verit, best) one-neq-zero tryFold.cases tryFoldIntegerEqualsNeverDis-
tinct tryFoldIntegerLessThanTrue val-to-bool.simps(1))
next
  case (3 stamps x y)
  then show ?case using tryFoldIntegerLessThanTrue assms
  by (smt (verit, best) one-neq-zero tryFold.cases tryFoldIntegerEqualsNeverDis-
tinct val-to-bool.simps(1))
next
case (4 stamps x y)
  then show ?case using tryFoldIntegerLessThanFalse assms sorry
qed

theorem tryFoldProofFalse:
  assumes wf-stamp g stamps
  assumes tryFold (kind g nid) stamps False
  assumes [g, m, p] ⊢ nid ↦ v
  shows ¬(val-to-bool v)
using assms(2) proof (induction kind g nid stamps False rule: tryFold.induct)
case (1 stamps x y)
  then show ?case using tryFoldIntegerEqualsAlwaysDistinct assms sorry
next
case (2 stamps x y)
  then show ?case using tryFoldIntegerEqualsNeverDistinct assms sorry
next
  case (3 stamps x y)
  then show ?case using tryFoldIntegerLessThanTrue assms sorry
next
  case (4 stamps x y)
  then show ?case using tryFoldIntegerLessThanFalse assms sorry

qed

```

**inductive-cases** *StepE*:

$$g, p \vdash (nid, m, h) \rightarrow (nid', m', h)$$

Perform conditional elimination rewrites on the graph for a particular node. In order to determine conditional eliminations appropriately the rule needs two data structures produced by static analysis. The first parameter is the set of IRNodes that we know result in a true value when evaluated. The second parameter is a mapping from node identifiers to the flow-sensitive stamp.

The relation transforms the third parameter to the fifth parameter for a node identifier which represents the fourth parameter.

**inductive** *ConditionalEliminationStep* ::  
 $IRExpr \text{ set} \Rightarrow (ID \Rightarrow Stamp) \Rightarrow IRGraph \Rightarrow ID \Rightarrow IRGraph \Rightarrow bool$  **where**  
*impliesTrue*:

```

[[kind g ifcond = (IfNode cid t f);
  g ⊢ cid ≃ cond;
  ∃ ce ∈ conds . (ce ⇒ cond);
  g' = constantCondition True ifcond (kind g ifcond) g
]] ⇒ ConditionalEliminationStep conds stamps g ifcond g' |

```

*impliesFalse:*

```

[[kind g ifcond = (IfNode cid t f);
  g ⊢ cid ≃ cond;
  ∃ ce ∈ conds . (ce ⇒ ¬ cond);
  g' = constantCondition False ifcond (kind g ifcond) g
]] ⇒ ConditionalEliminationStep conds stamps g ifcond g' |

```

*tryFoldTrue:*

```

[[kind g ifcond = (IfNode cid t f);
  cond = kind g cid;
  tryFold (kind g cid) stamps True;
  g' = constantCondition True ifcond (kind g ifcond) g
]] ⇒ ConditionalEliminationStep conds stamps g ifcond g' |

```

*tryFoldFalse:*

```

[[kind g ifcond = (IfNode cid t f);
  cond = kind g cid;
  tryFold (kind g cid) stamps False;
  g' = constantCondition False ifcond (kind g ifcond) g
]] ⇒ ConditionalEliminationStep conds stamps g ifcond g' |

```

**code-pred** (modes:  $i \Rightarrow i \Rightarrow i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$ ) *ConditionalEliminationStep* .

**thm** *ConditionalEliminationStep.equation*

## 1.2 Control-flow Graph Traversal

**type-synonym** *Seen* = *ID set*

**type-synonym** *Condition* = *IRExpr*

**type-synonym** *Conditions* = *Condition list*

**type-synonym** *StampFlow* = (*ID* ⇒ *Stamp*) *list*

*nextEdge* helps determine which node to traverse next by returning the first successor edge that isn't in the set of already visited nodes. If there is not an appropriate successor, *None* is returned instead.

**fun** *nextEdge* :: *Seen* ⇒ *ID* ⇒ *IRGraph* ⇒ *ID option* **where**

```

nextEdge seen nid g =
  (let nids = (filter (λnid'. nid' ∉ seen) (successors-of (kind g nid))) in
   (if length nids > 0 then Some (hd nids) else None))

```

*pred* determines which node, if any, acts as the predecessor of another.

Merge nodes represent a special case where-in the predecessor exists as an

input edge of the merge node, to simplify the traversal we treat only the first input end node as the predecessor, ignoring that multiple nodes may act as a successor.

For all other nodes, the predecessor is the first element of the predecessors set. Note that in a well-formed graph there should only be one element in the predecessor set.

```
fun pred :: IRGraph ⇒ ID ⇒ ID option where
  pred g nid = (case kind g nid of
    (MergeNode ends -) ⇒ Some (hd ends) |
    - ⇒
      (if IRGraph.predecessors g nid = {}
        then None else
        Some (hd (sorted-list-of-set (IRGraph.predecessors g nid))))
  )
```

When the basic block of an if statement is entered, we know that the condition of the preceding if statement must be true. As in the GraalVM compiler, we introduce the registerNewCondition function which roughly corresponds to the ConditionalEliminationPhase.registerNewCondition. This method updates the flow-sensitive stamp information based on the condition which we know must be true.

```
fun clip-upper :: Stamp ⇒ int ⇒ Stamp where
  clip-upper (IntegerStamp b l h) c = (IntegerStamp b l c) |
  clip-upper s c = s
fun clip-lower :: Stamp ⇒ int ⇒ Stamp where
  clip-lower (IntegerStamp b l h) c = (IntegerStamp b c h) |
  clip-lower s c = s
```

```
fun registerNewCondition :: IRGraph ⇒ IRNode ⇒ (ID ⇒ Stamp) ⇒ (ID ⇒ Stamp) where
```

```
  registerNewCondition g (IntegerEqualsNode x y) stamps =
    (stamps
      (x := join (stamps x) (stamps y))
      (y := join (stamps x) (stamps y)) |
```

```
  registerNewCondition g (IntegerLessThanNode x y) stamps =
    (stamps
      (x := clip-upper (stamps x) (clip-lower (stamps y)))
      (y := clip-lower (stamps y) (clip-upper (stamps x))) |
  registerNewCondition g - stamps = stamps
```

```
fun hdOr :: 'a list ⇒ 'a ⇒ 'a where
  hdOr (x # xs) de = x |
  hdOr [] de = de
```

The Step relation is a small-step traversal of the graph which handles tran-

sitions between individual nodes of the graph.

It relates a pairs of tuple of the current node, the set of seen nodes, the always true stack of IfNode conditions, and the flow-sensitive stamp information.

**inductive Step**

$:: IRGraph \Rightarrow (ID \times Seen \times Conditions \times StampFlow) \Rightarrow (ID \times Seen \times Conditions \times StampFlow) \text{ option} \Rightarrow bool$

**for  $g$  where**

— Hit a BeginNode with an IfNode predecessor which represents the start of a basic block for the IfNode. 1.  $nid'$  will be the successor of the begin node. 2. Find the first and only predecessor. 3. Extract condition from the preceding IfNode. 4. Negate condition if the begin node is second branch (we've taken the else branch of the condition) 5. Add the condition or the negated condition to stack 6. Perform any stamp updates based on the condition using the registerNewCondition function and place them on the top of the stack of stamp information

$\llbracket kind\ g\ nid = BeginNode\ nid';$

$nid \notin seen;$   
 $seen' = \{nid\} \cup seen;$

$Some\ ifcond = pred\ g\ nid;$   
 $kind\ g\ ifcond = IfNode\ cond\ t\ f;$

$i = find-index\ nid\ (successors-of\ (kind\ g\ ifcond));$   
 $c = (if\ i = 0\ then\ kind\ g\ cond\ else\ LogicNegationNode\ cond);$   
 $rep\ g\ cond\ ce;$   
 $ce' = (if\ i = 0\ then\ ce\ else\ UnaryExpr\ UnaryLogicNegation\ ce);$   
 $conds' = ce' \# conds;$

$flow' = registerNewCondition\ g\ c\ (hdOr\ flow\ (stamp\ g))\llbracket$   
 $\implies Step\ g\ (nid,\ seen,\ conds,\ flow)\ (Some\ (nid',\ seen',\ conds',\ flow' \# flow)) \mid$

— Hit an EndNode 1.  $nid'$  will be the usage of EndNode 2. pop the conditions and stamp stack

$\llbracket kind\ g\ nid = EndNode;$

$nid \notin seen;$   
 $seen' = \{nid\} \cup seen;$

$nid' = any-usage\ g\ nid;$

$conds' = tl\ conds;$   
 $flow' = tl\ flow\llbracket$

$\implies Step\ g\ (nid,\ seen,\ conds,\ flow)\ (Some\ (nid',\ seen',\ conds',\ flow')) \mid$

— We can find a successor edge that is not in seen, go there

$\llbracket \neg(is-EndNode\ (kind\ g\ nid));$   
 $\neg(is-BeginNode\ (kind\ g\ nid));$

$nid \notin seen;$   
 $seen' = \{nid\} \cup seen;$

$Some\ nid' = nextEdge\ seen'\ nid\ g$   
 $\implies Step\ g\ (nid, seen, conds, flow)\ (Some\ (nid', seen', conds, flow)) \mid$

— We cannot find a successor edge that is not in seen, give back None  
 $\llbracket \neg(is-EndNode\ (kind\ g\ nid));$   
 $\neg(is-BEGINNode\ (kind\ g\ nid));$

$nid \notin seen;$   
 $seen' = \{nid\} \cup seen;$

$None = nextEdge\ seen'\ nid\ g$   
 $\implies Step\ g\ (nid, seen, conds, flow)\ None \mid$

— We've already seen this node, give back None  
 $\llbracket nid \in seen \rrbracket \implies Step\ g\ (nid, seen, conds, flow)\ None$

**code-pred** (*modes*:  $i \Rightarrow i \Rightarrow o \Rightarrow bool$ ) *Step* .

The ConditionalEliminationPhase relation is responsible for combining the individual traversal steps from the Step relation and the optimizations from the ConditionalEliminationStep relation to perform a transformation of the whole graph.

**inductive** ConditionalEliminationPhase

$:: IRGraph \Rightarrow (ID \times Seen \times Conditions \times StampFlow) \Rightarrow IRGraph \Rightarrow bool$

**where**

— Can do a step and optimise for the current node  
 $\llbracket Step\ g\ (nid, seen, conds, flow)\ (Some\ (nid', seen', conds', flow'));$   
 $ConditionalEliminationStep\ (set\ conds)\ (hdOr\ flow\ (stamp\ g))\ g\ nid\ g';$

$ConditionalEliminationPhase\ g'\ (nid', seen', conds', flow')\ g'$   
 $\implies ConditionalEliminationPhase\ g\ (nid, seen, conds, flow)\ g'' \mid$

— Can do a step, matches whether optimised or not causing non-determinism We need to find a way to negate ConditionalEliminationStep  
 $\llbracket Step\ g\ (nid, seen, conds, flow)\ (Some\ (nid', seen', conds', flow'));$

$ConditionalEliminationPhase\ g\ (nid', seen', conds', flow')\ g'$   
 $\implies ConditionalEliminationPhase\ g\ (nid, seen, conds, flow)\ g' \mid$

— Can't do a step but there is a predecessor we can backtrace to  
 $\llbracket Step\ g\ (nid, seen, conds, flow)\ None;$

$Some\ nid' = pred\ g\ nid;$   
 $seen' = \{nid\} \cup seen;$   
 $ConditionalEliminationPhase\ g\ (nid', seen', conds, flow)\ g'$   
 $\implies ConditionalEliminationPhase\ g\ (nid, seen, conds, flow)\ g' \mid$



— Can't do a step and have no predecessors so terminate  
 $\llbracket \text{Step } g \text{ (nid, seen, conds, flow) None};$   
 $\text{None} = \text{pred } g \text{ nid} \rrbracket$   
 $\implies \text{ConditionalEliminationPhase } g \text{ (nid, seen, conds, flow) } g$

**code-pred** (modes:  $i \Rightarrow i \Rightarrow o \Rightarrow \text{bool}$ ) *ConditionalEliminationPhase* .

**definition** *runConditionalElimination* :: *IRGraph*  $\Rightarrow$  *IRGraph* **where**  
*runConditionalElimination*  $g =$   
 $(\text{Predicate.the } (\text{ConditionalEliminationPhase-}i-i-o \text{ } g \text{ } (\emptyset, \{\}, (\square, \square))))$

**end**