

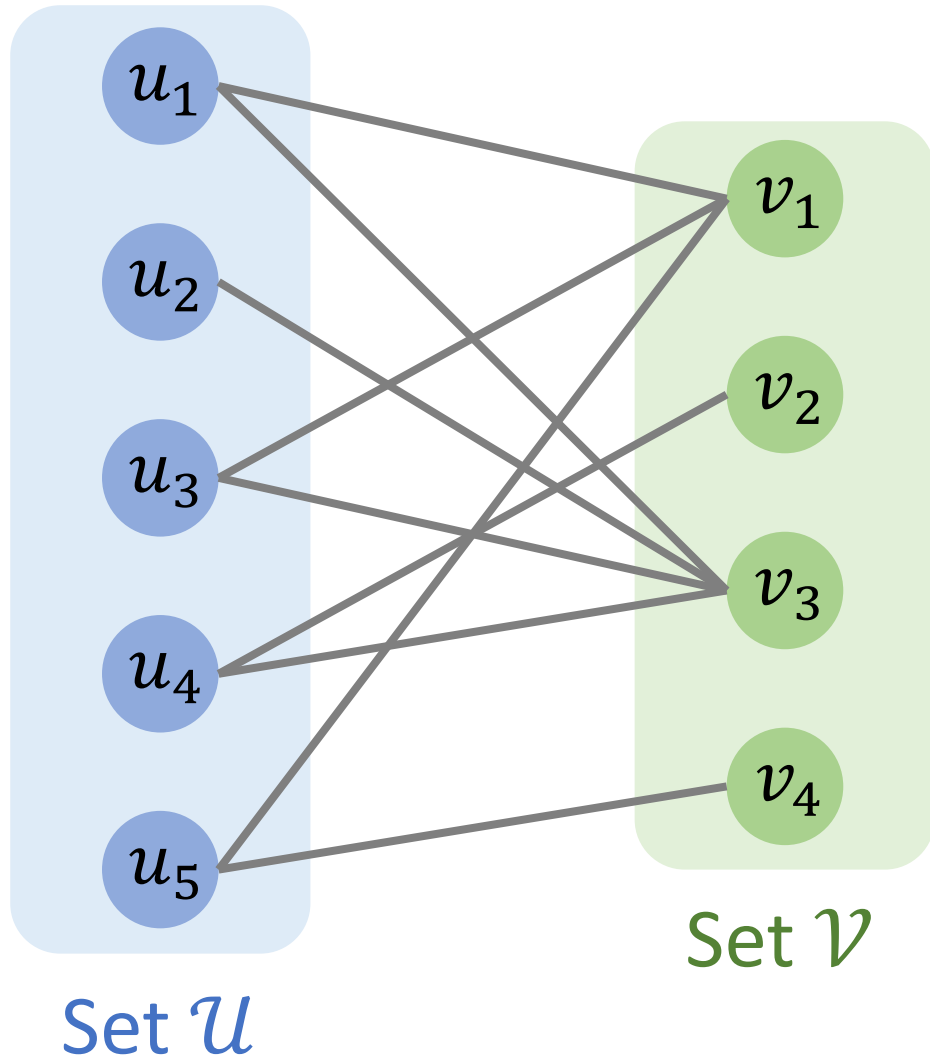
Bipartite Graph

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<http://wangshusen.github.io/>

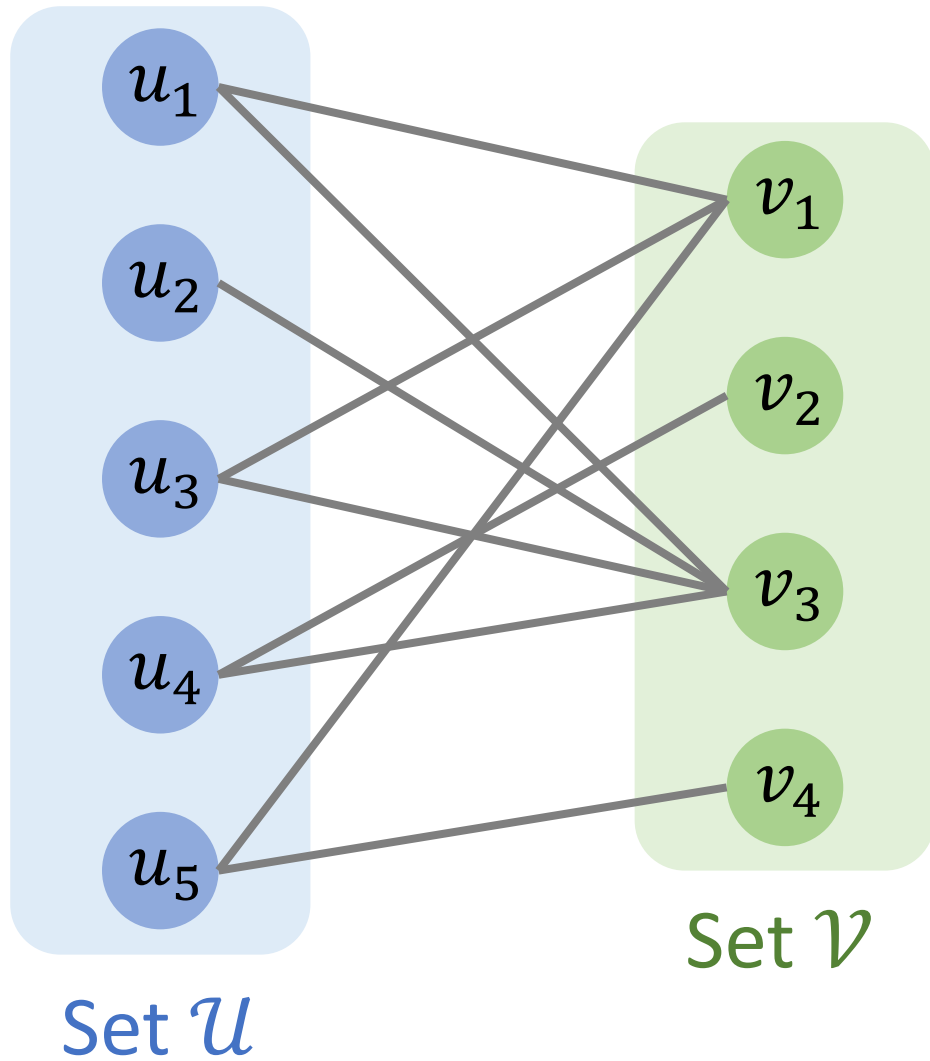
Definition

Bipartite Graph



- Bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.

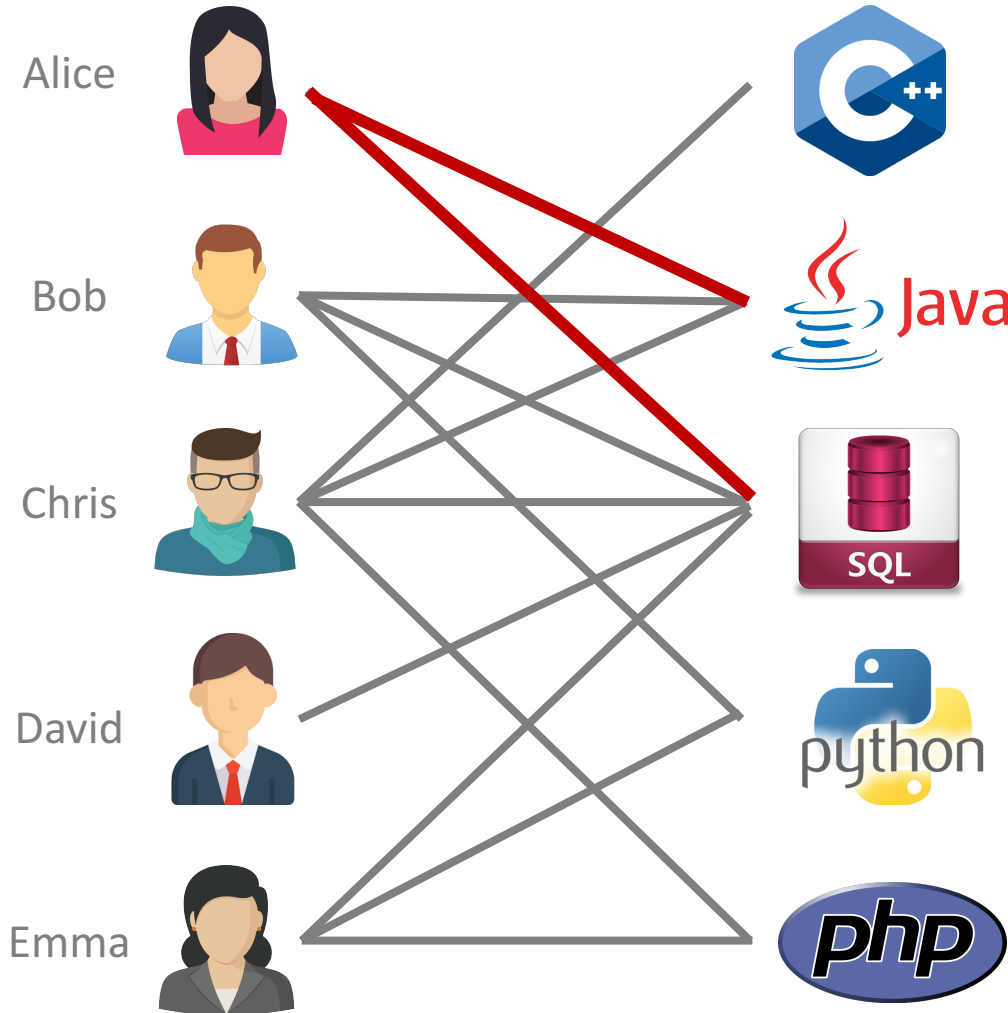
Bipartite Graph



- Bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.
- All the edges are between \mathcal{U} and \mathcal{V} .
- No edge between two vertices in \mathcal{U} .
- No edge between two vertices in \mathcal{V} .

Candidates

Positions

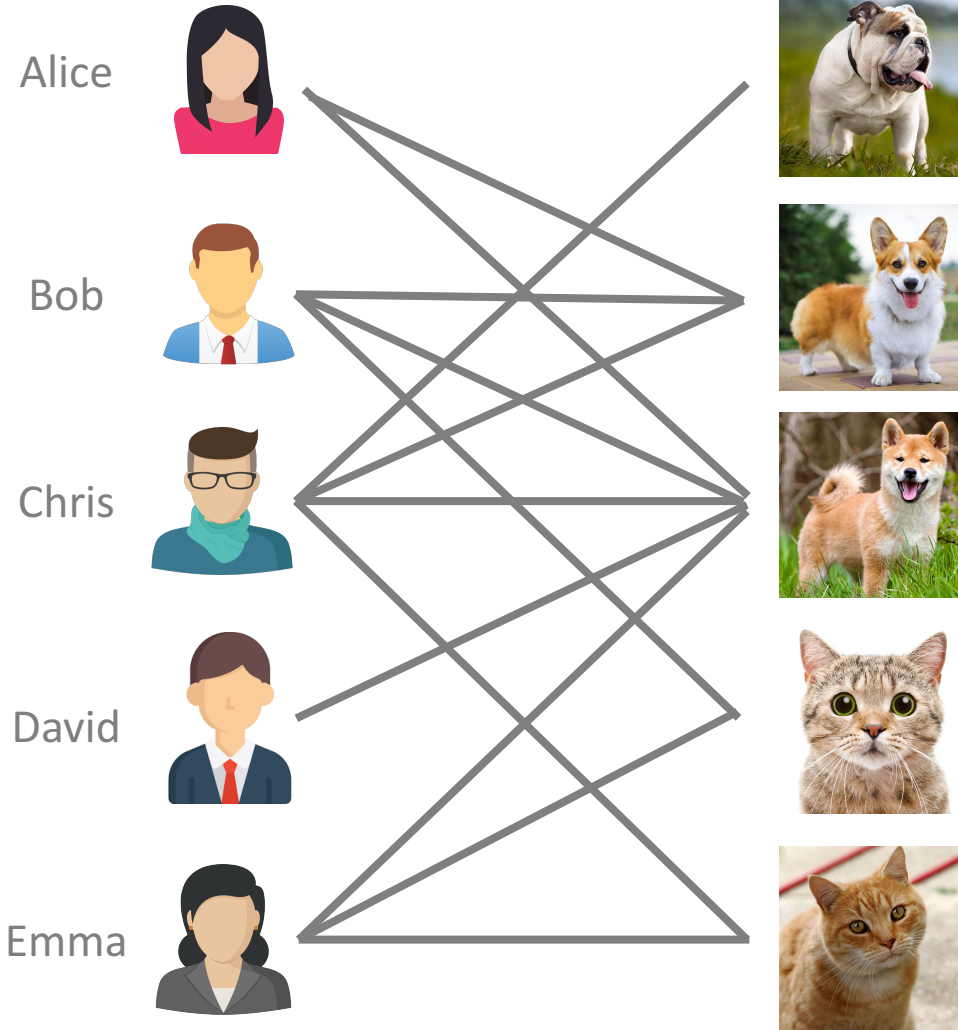


Matching candidates and positions.

- Bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.
- Set \mathcal{U} contains candidates.
- Set \mathcal{V} contains jobs.
- Edges in \mathcal{E} are candidates' skills.

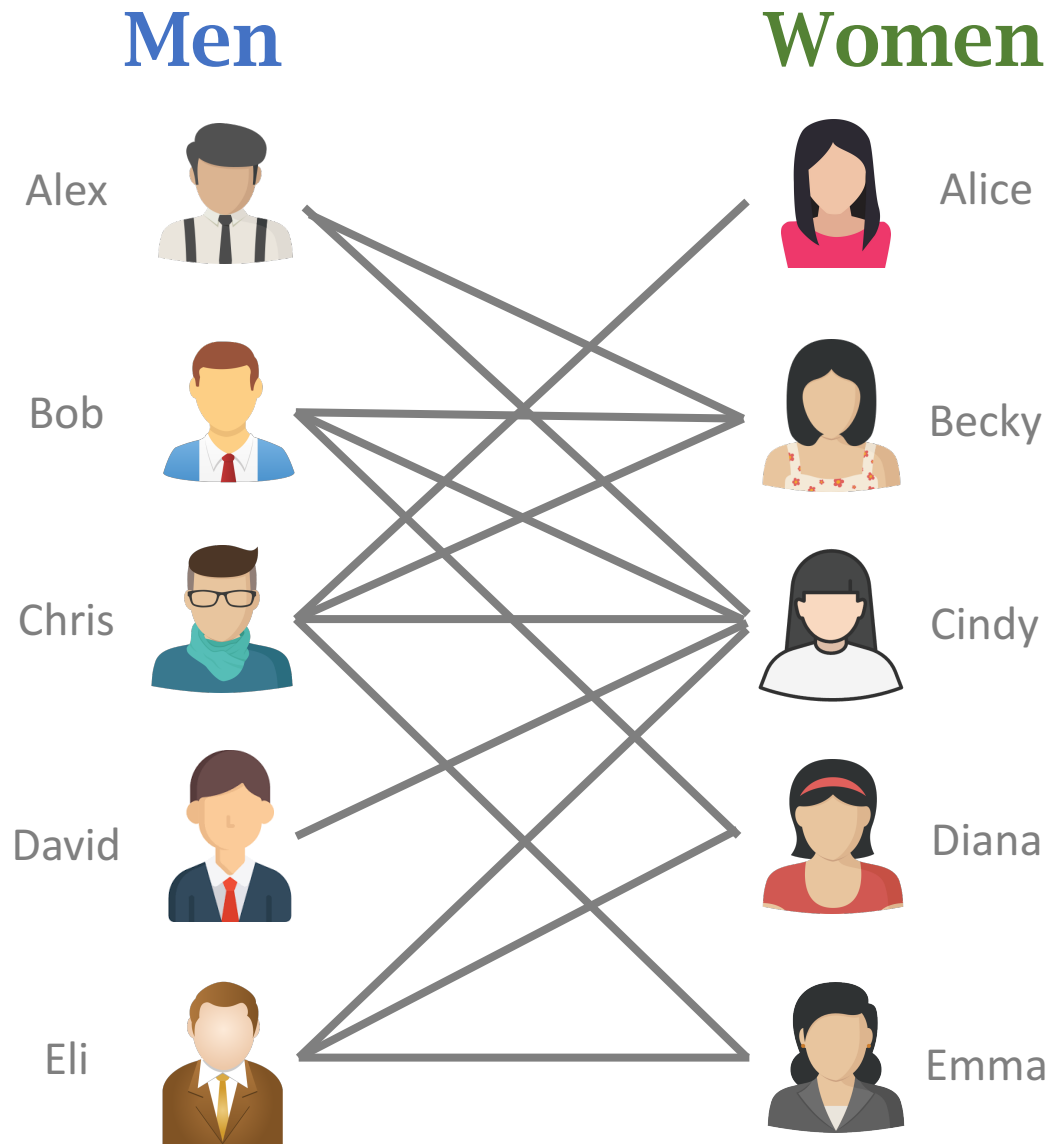
People

Pets



Pet adoption

- Bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.
- Set \mathcal{U} contains people.
- Set \mathcal{V} contains pets.
- Edges in \mathcal{E} are people's preference.

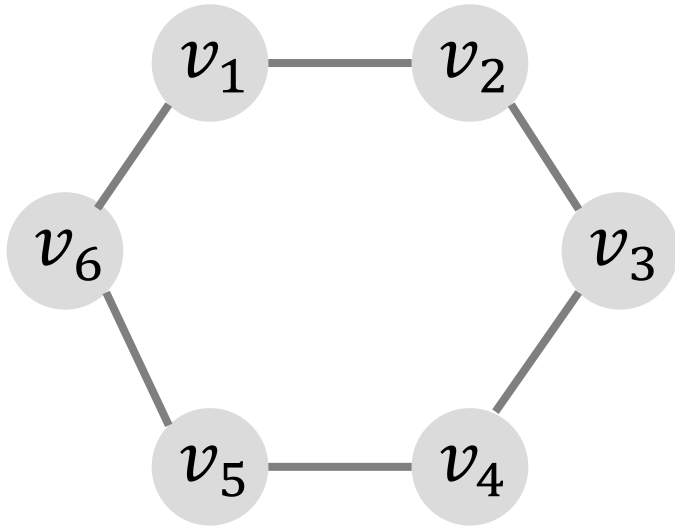


Dating

- Bipartite graph: $\mathcal{G} = (\mathcal{u}, \mathcal{v}, \mathcal{E})$.
- Set \mathcal{u} contains males.
- Set \mathcal{v} contains females.
- Edges in \mathcal{E} are people's preference.

Testing Bipartiteness

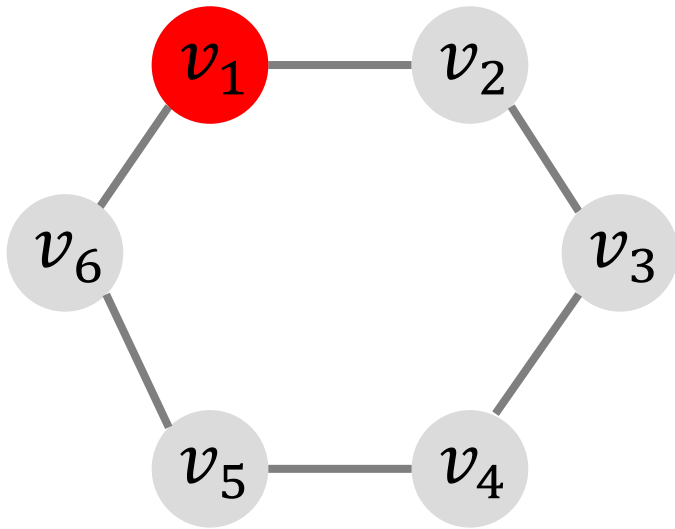
Is the graph bipartite?



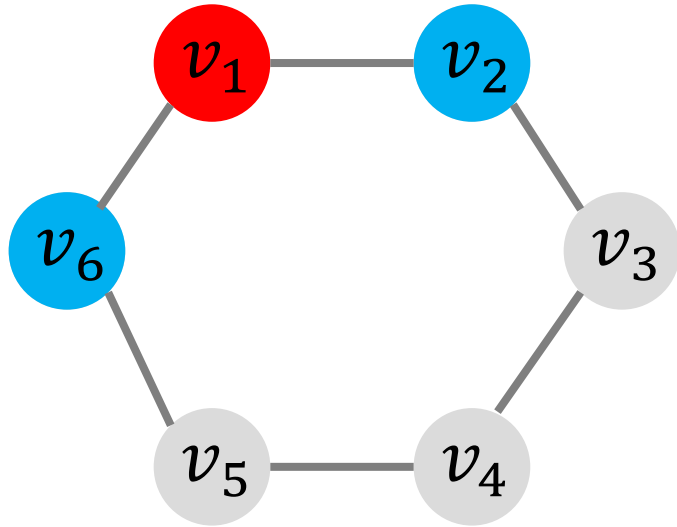
1. Select an arbitrary vertex and assign **red color** to it.
2. Repeat until all vertices are colored:
 - Color **red vertices'** neighbors as **blue**.
 - Color **blue vertices'** neighbors as **red**.
 - During the process, if a vertex has the same color as its neighbor, then output **FALSE**.
3. If no violation is found, return **TRUE** in the end.

Example 1

1. Select any vertex and assign **red color** to it.

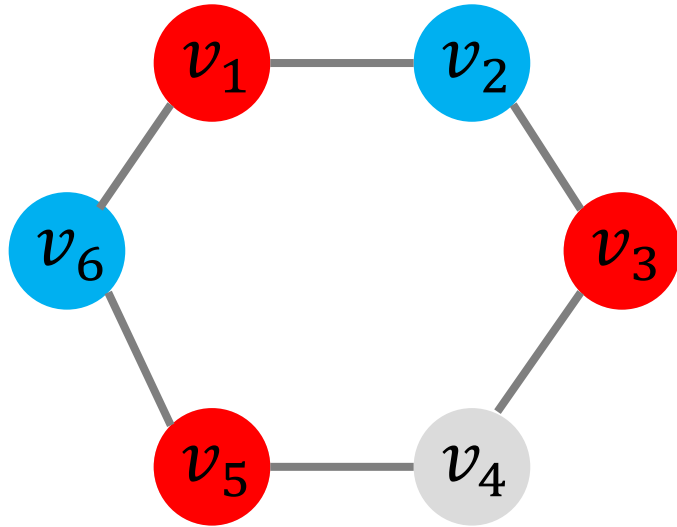


Example 1



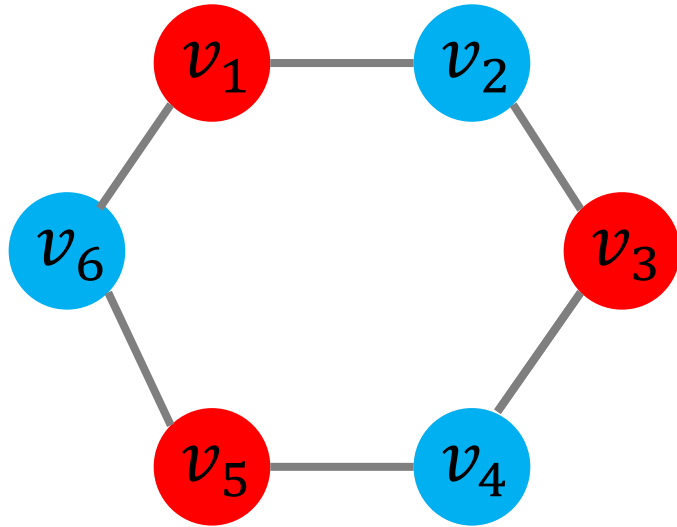
1. Select any vertex and assign **red color** to it.
2. Color **red vertices'** neighbors as **blue**.

Example 1



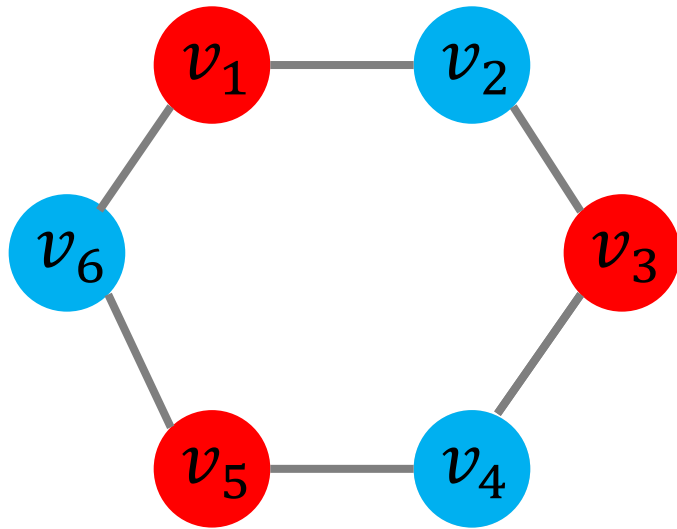
1. Select any vertex and assign **red color** to it.
2. Color **red vertices'** neighbors as **blue**.
3. Color **blue vertices'** neighbors as **red**.

Example 1



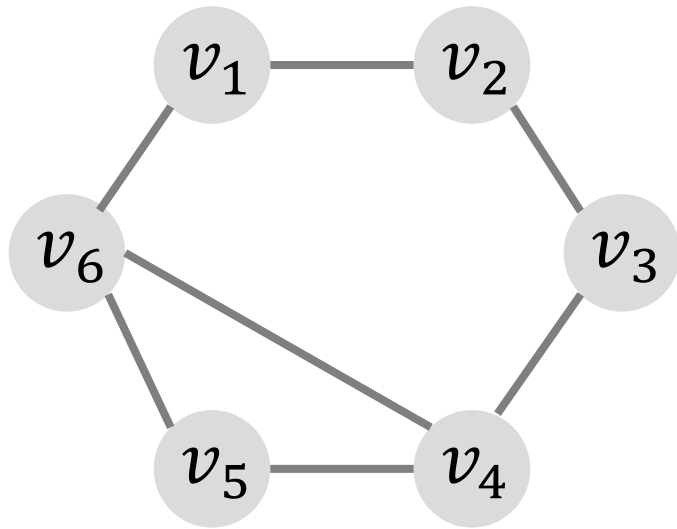
1. Select any vertex and assign **red color** to it.
2. Color **red vertices'** neighbors as **blue**.
3. Color **blue vertices'** neighbors as **red**.
4. Color **red vertices'** neighbors as **blue**.

Example 1



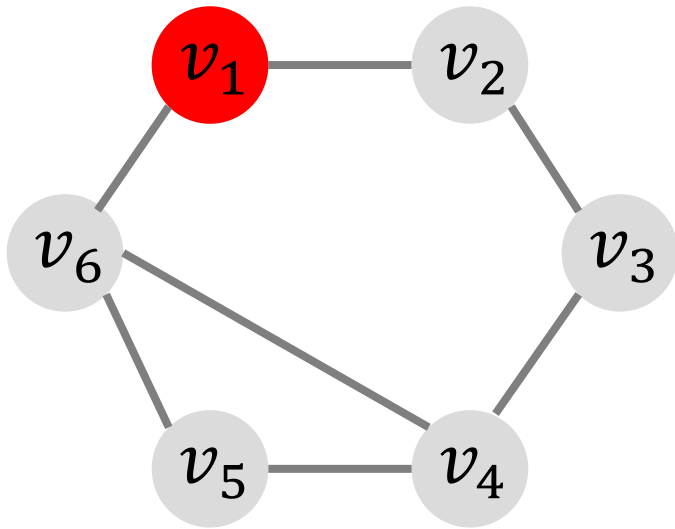
- No violation has been found!
- It is bipartite graph.

Example 2

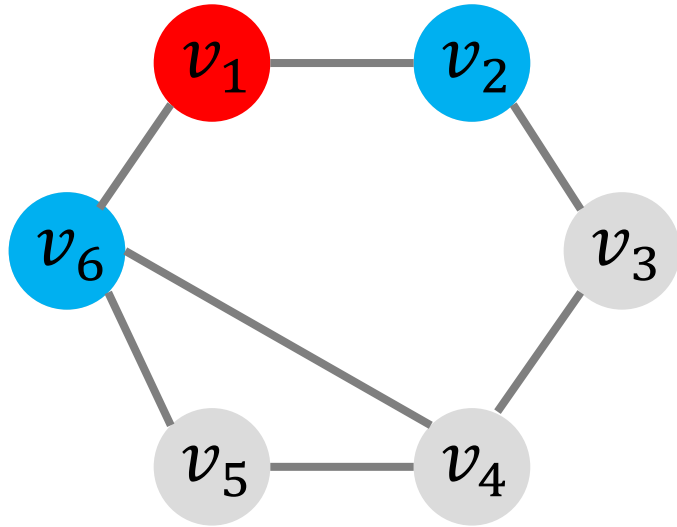


Example 2

1. Select any vertex and assign **red color** to it.

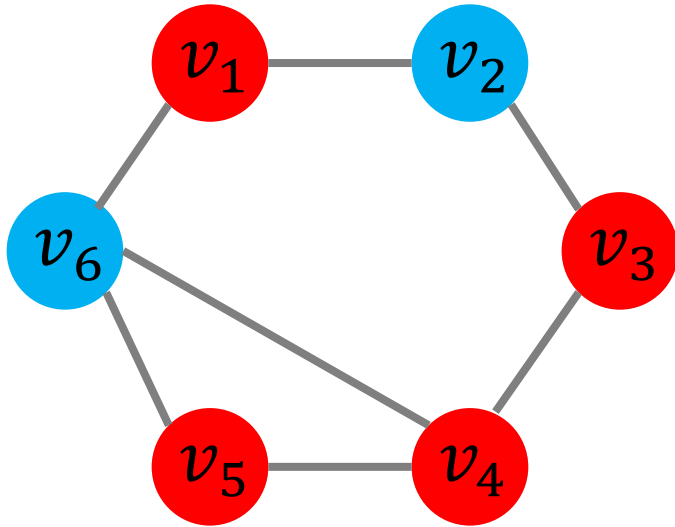


Example 2



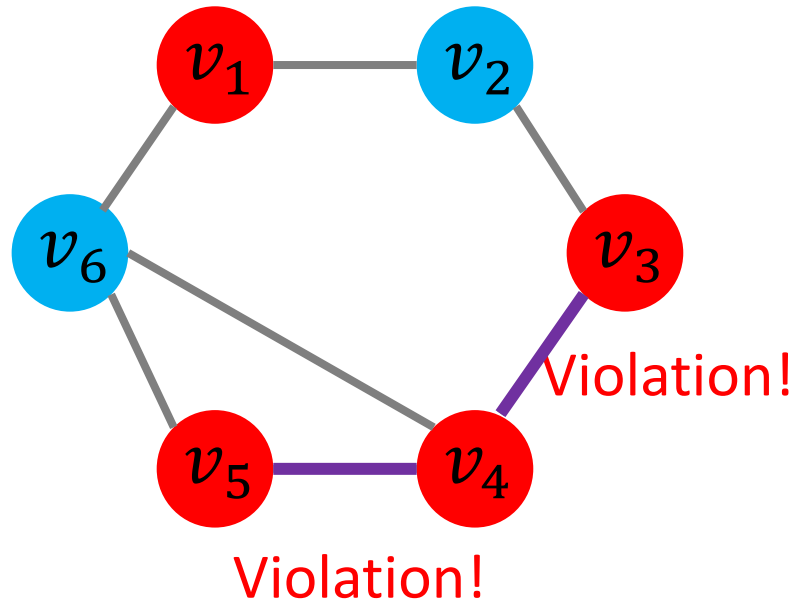
1. Select any vertex and assign **red color** to it.
2. Color **red vertices'** neighbors as **blue**.

Example 2



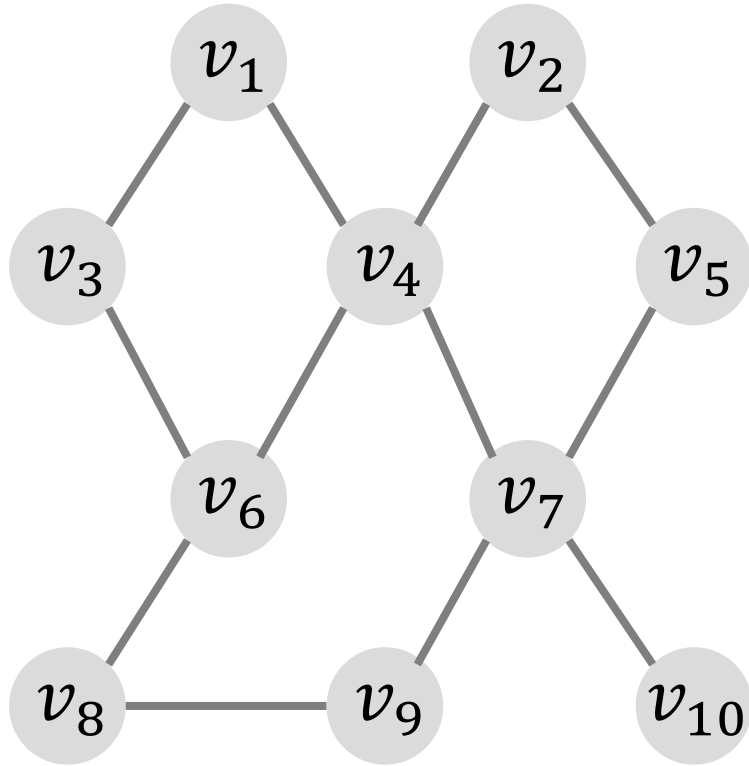
1. Select any vertex and assign **red color** to it.
2. Color **red vertices'** neighbors as **blue**.
3. Color **blue vertices'** neighbors **as red**.

Example 2



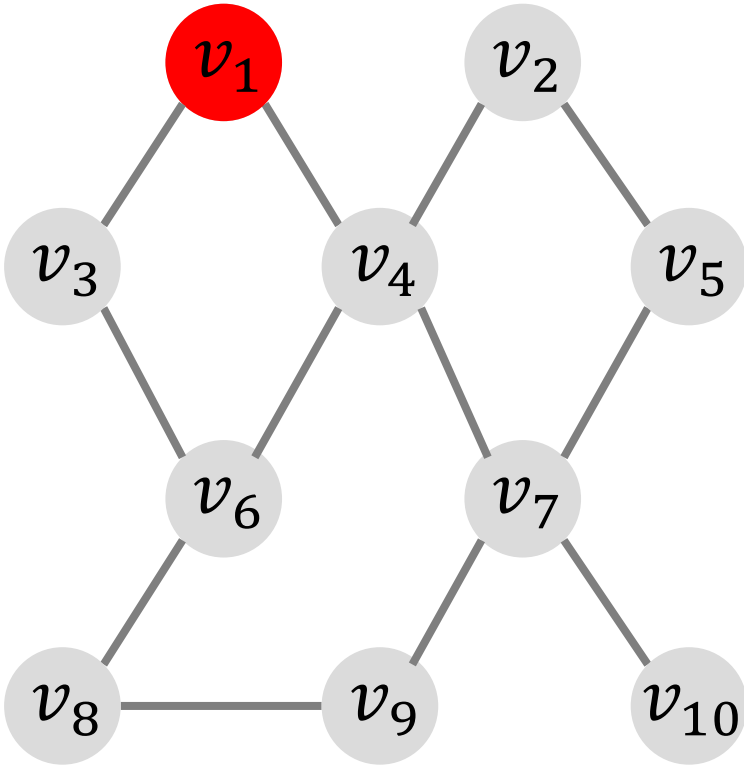
- Violation found!
- It is not bipartite graph.

Example 3

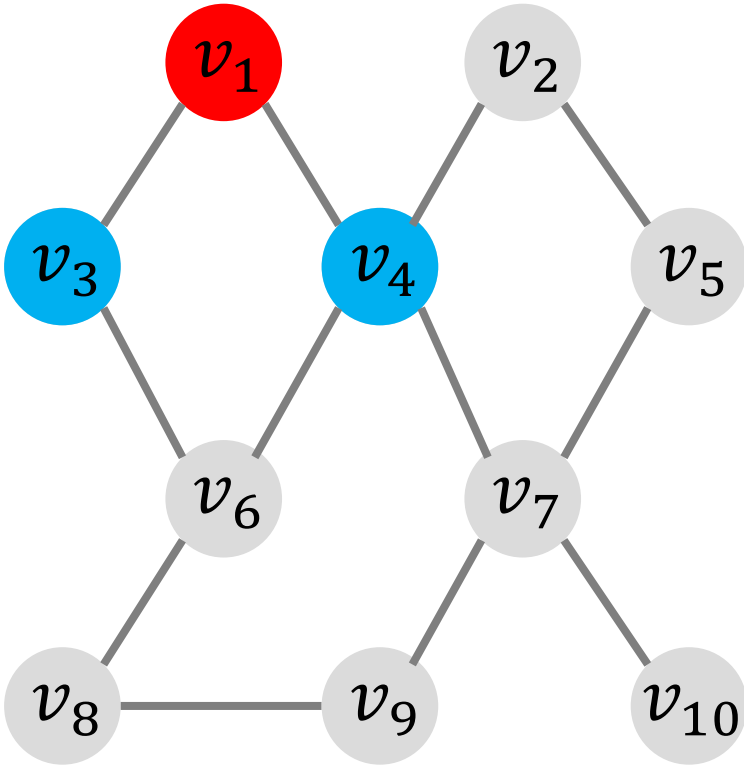


Example 3

1. Select any vertex and assign **red color** to it.

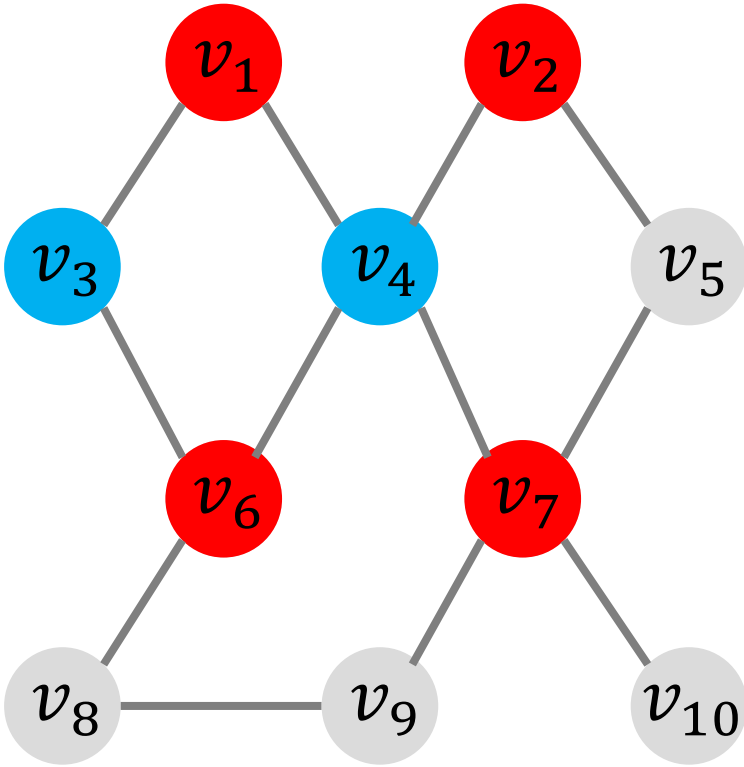


Example 3



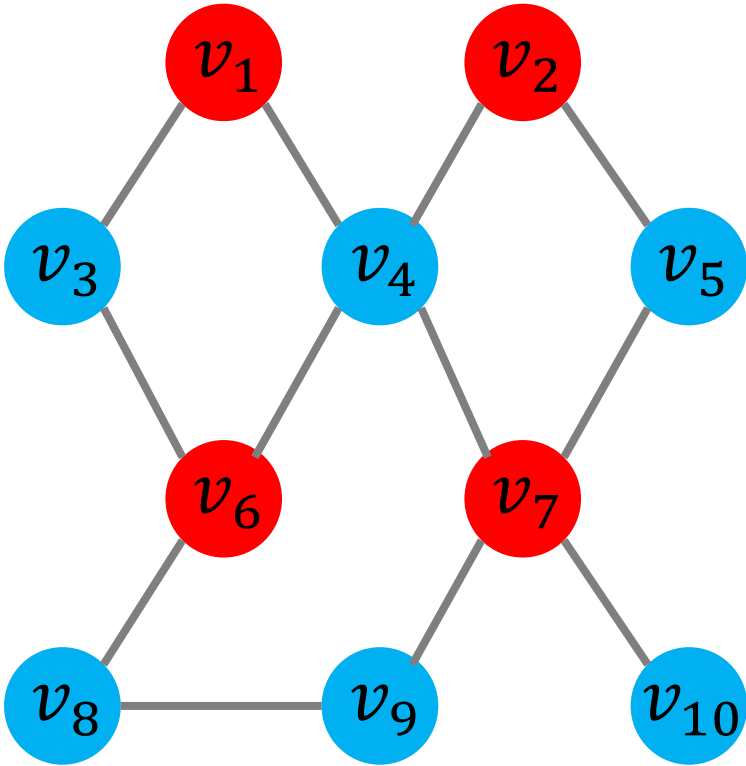
1. Select any vertex and assign **red color** to it.
2. Color **red vertices'** neighbors as **blue**.

Example 3



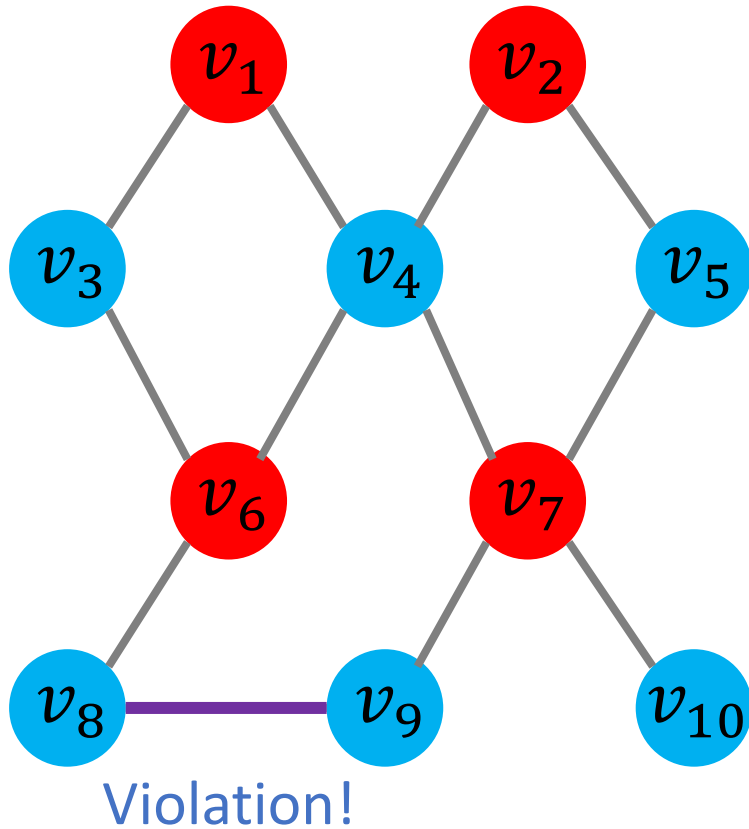
1. Select any vertex and assign **red color** to it.
2. Color **red vertices'** neighbors as **blue**.
3. Color **blue vertices'** neighbors **as red**.

Example 3



1. Select any vertex and assign **red color** to it.
2. Color **red vertices'** neighbors as **blue**.
3. Color **blue vertices'** neighbors as **red**.
4. Color **red vertices'** neighbors as **blue**.

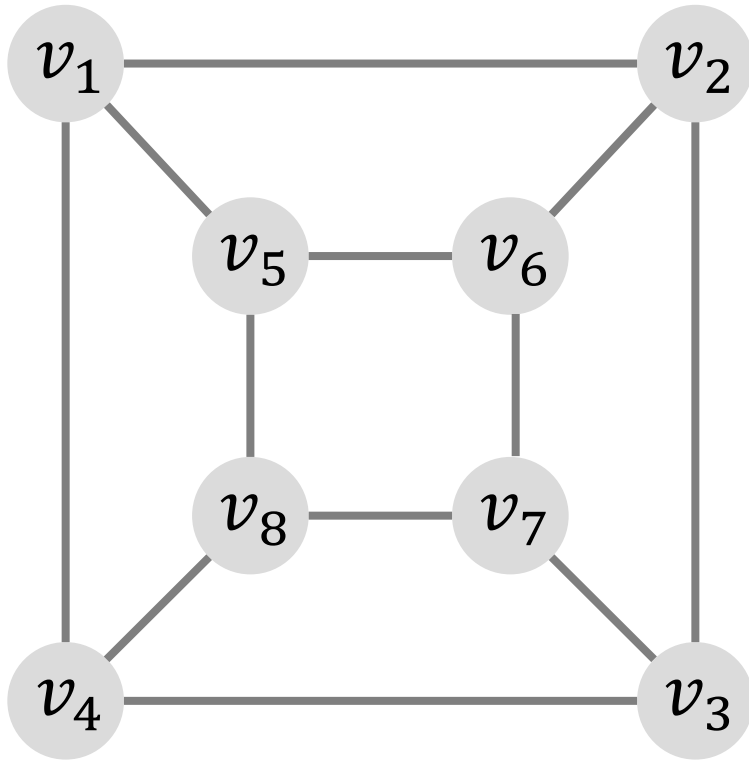
Example 3



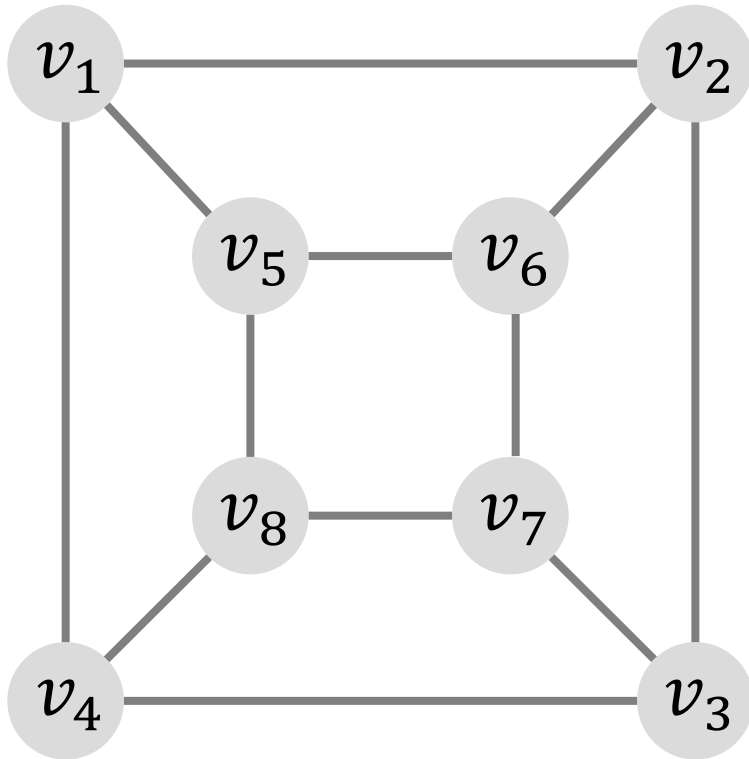
- Violation found!
- It is not bipartite graph.

Algorithm Details

Is the graph bipartite?



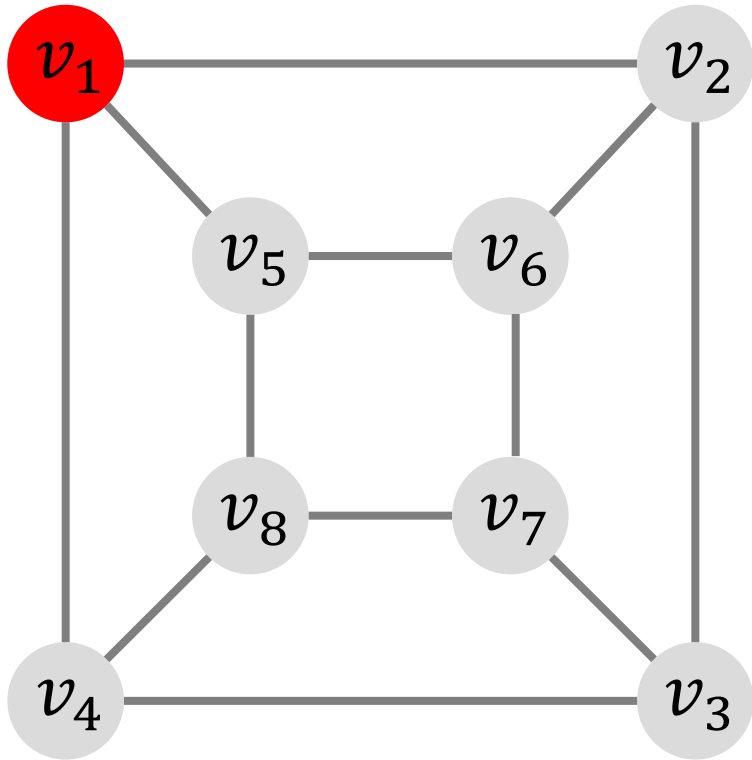
Initial State



Queue:



Initial State

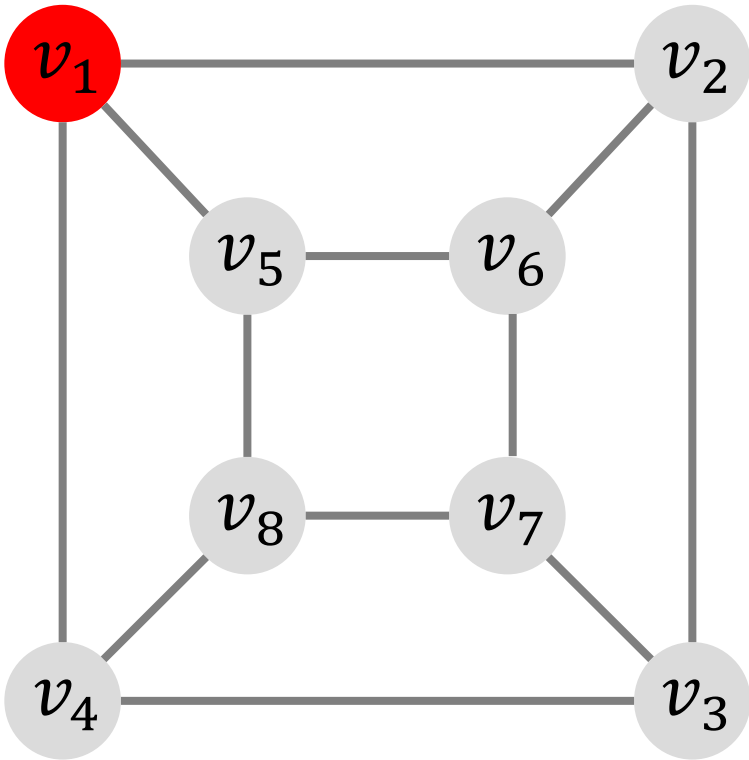


Queue:

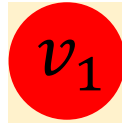


- Assign red color to v_1 .

Initial State

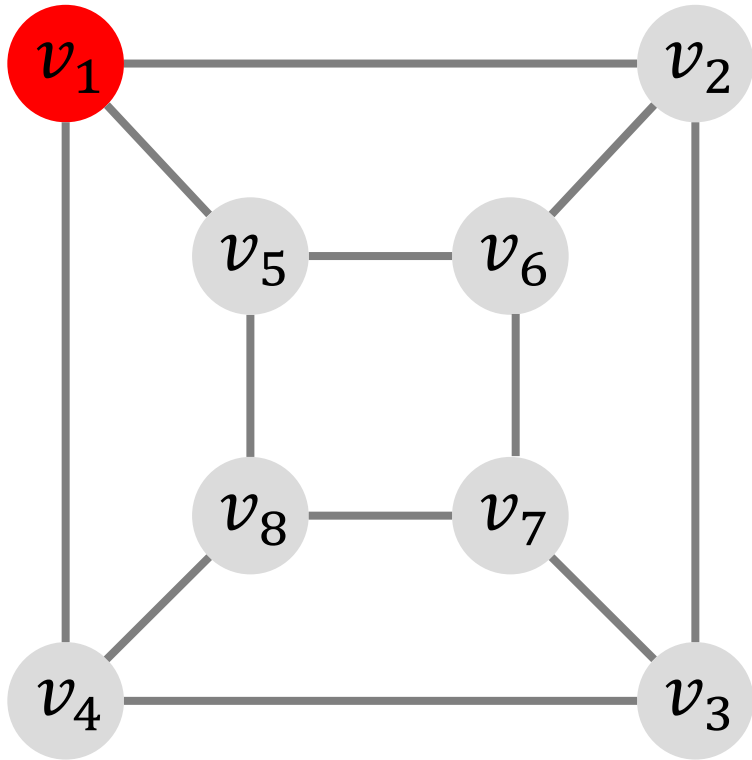


Queue:

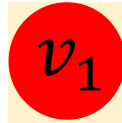


- Assign red color to v_1 .
- enqueue(v_1).

Iteration 1

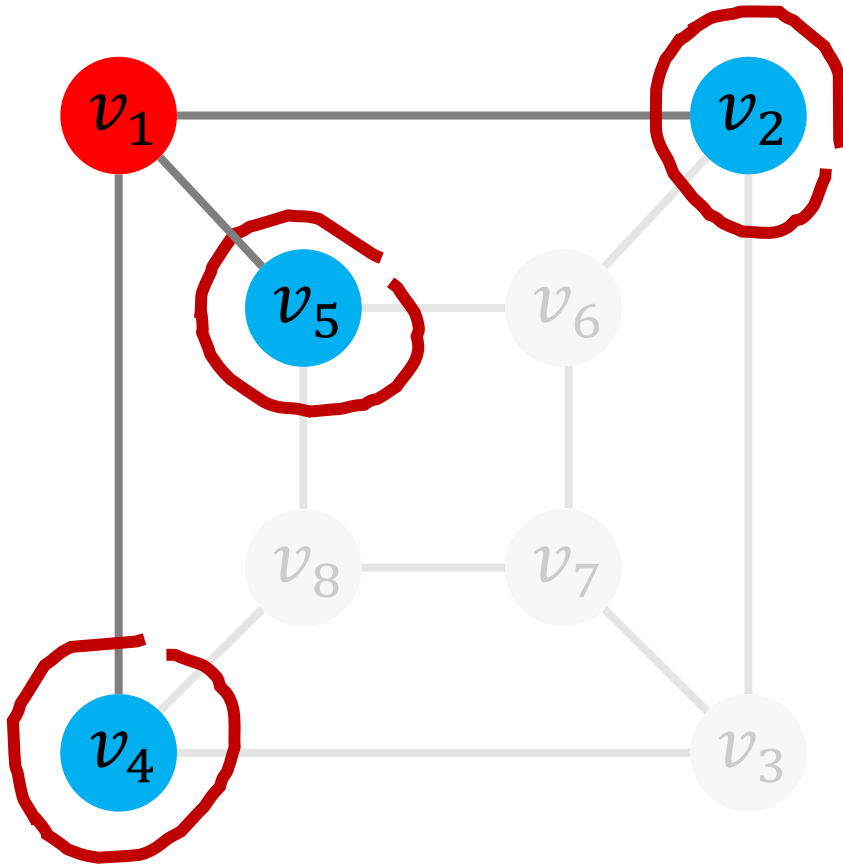


Queue:



- $v_1 \leftarrow \text{dequeue}()$.

Iteration 1

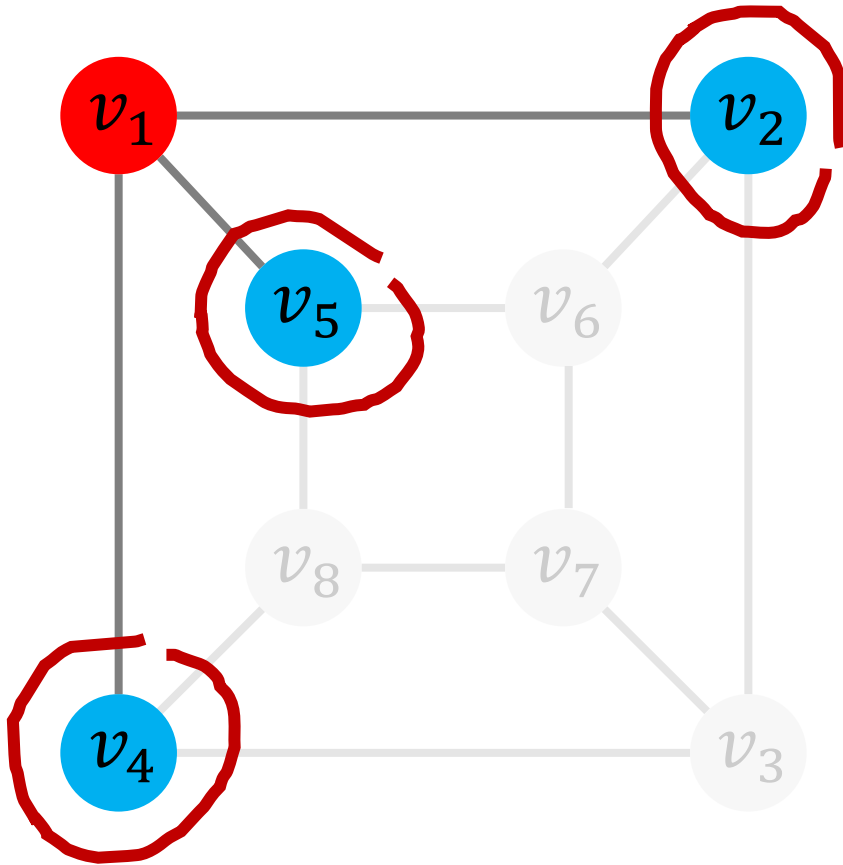


Queue:



- $v_1 \leftarrow \text{dequeue}()$.
- Assign blue color to its unvisited neighbors, v_2 , v_4 , and v_5 .

Iteration 1

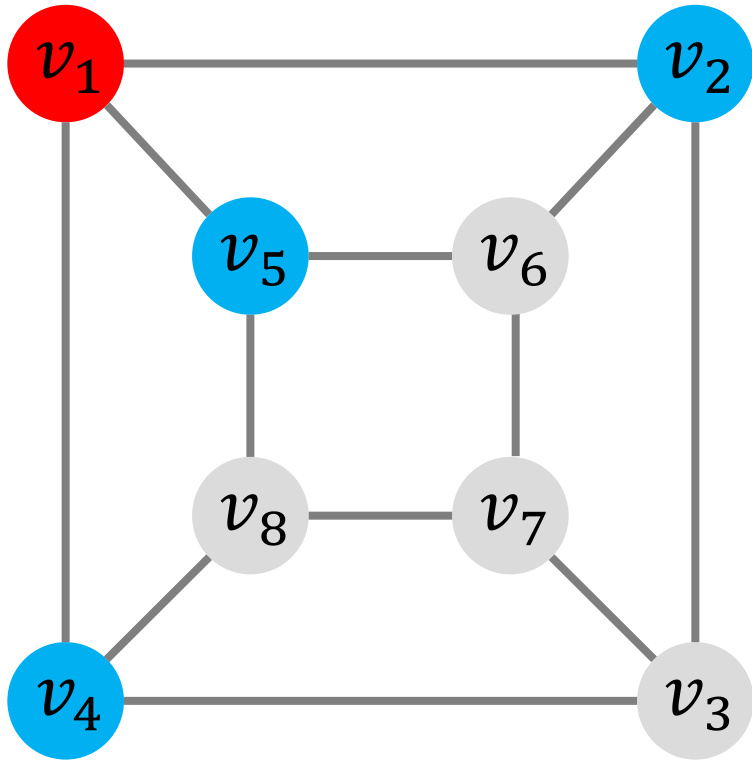


Queue:



- $v_1 \leftarrow \text{dequeue}()$.
- Assign **blue color** to its unvisited neighbors, v_2 , v_4 , and v_5 .
- Put the unvisited neighbors, v_2 , v_4 , and v_5 , in the queue.

Iteration 2

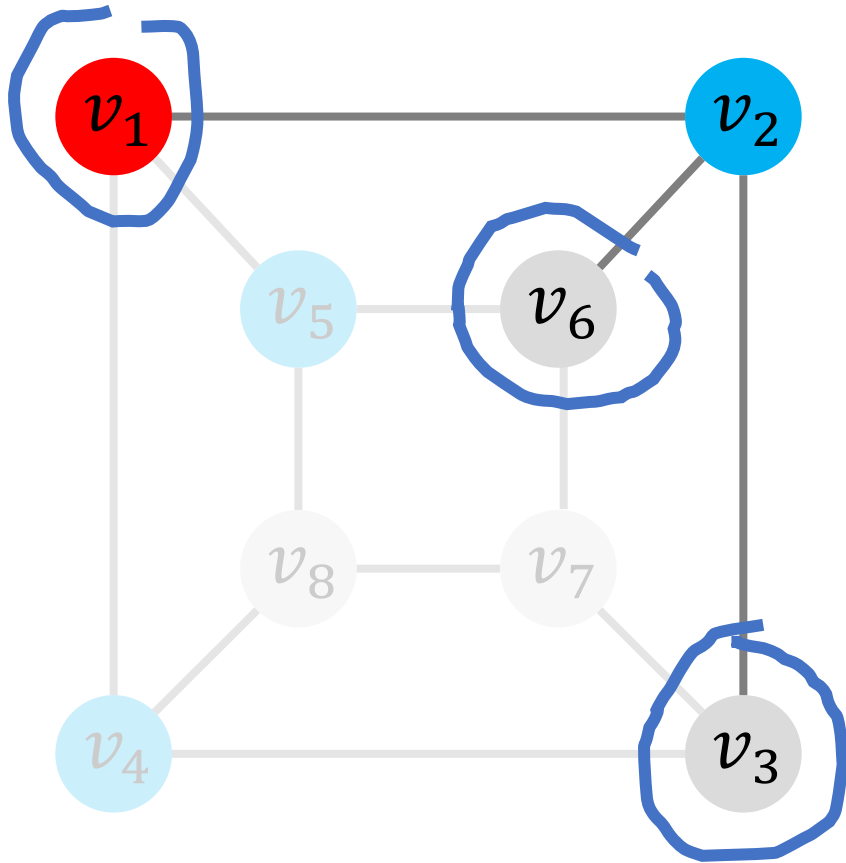


Queue:

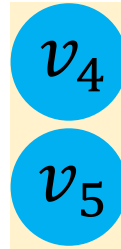


- $v_2 \leftarrow \text{dequeue}()$.

Iteration 2

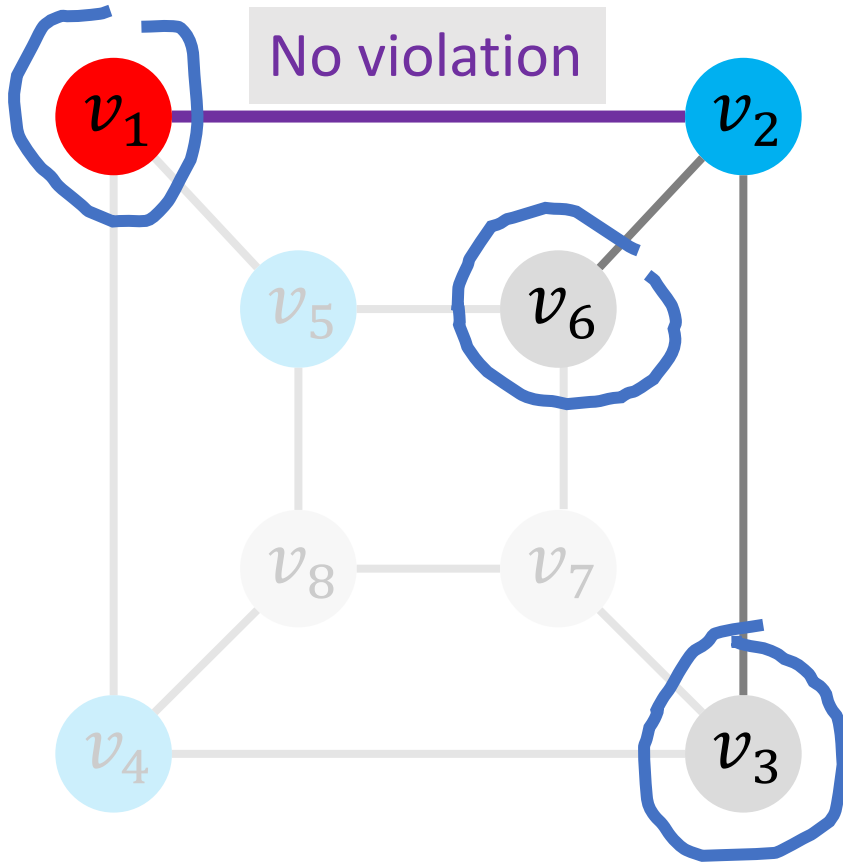


Queue:

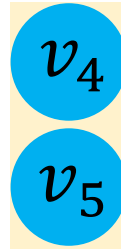


- $v_2 \leftarrow \text{dequeue}()$.

Iteration 2

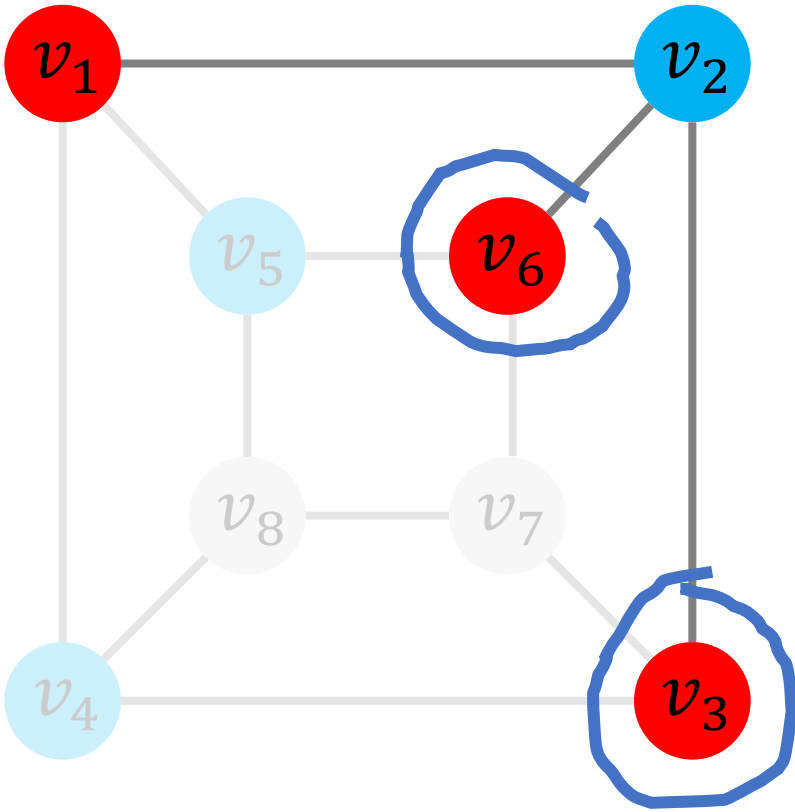


Queue:

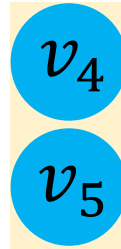


- $v_2 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.

Iteration 2

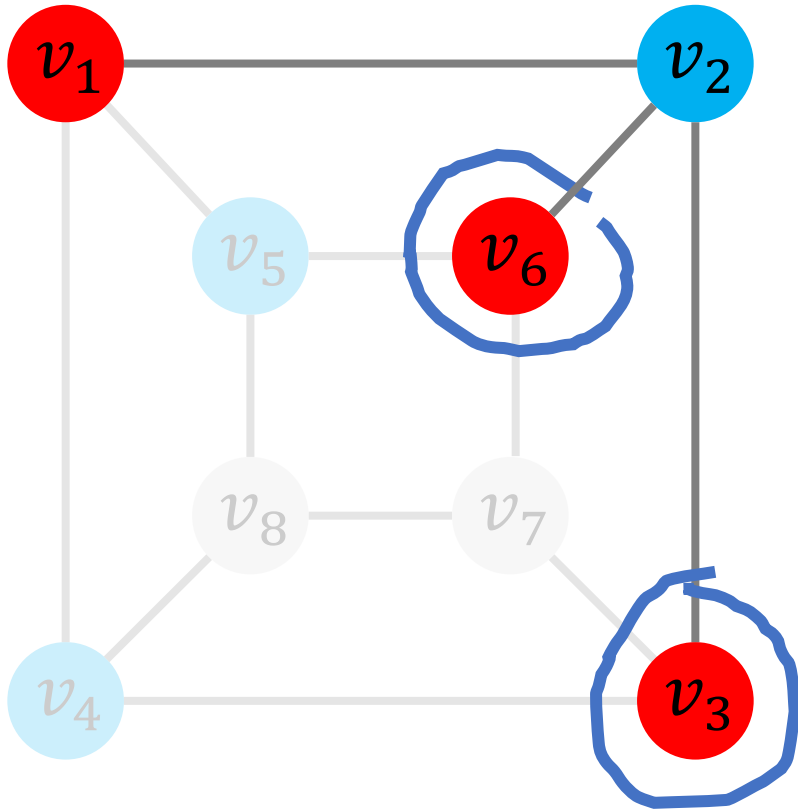


Queue:



- $v_2 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.
- Assign **red color** to its neighbors, v_3 and v_6 .

Iteration 2

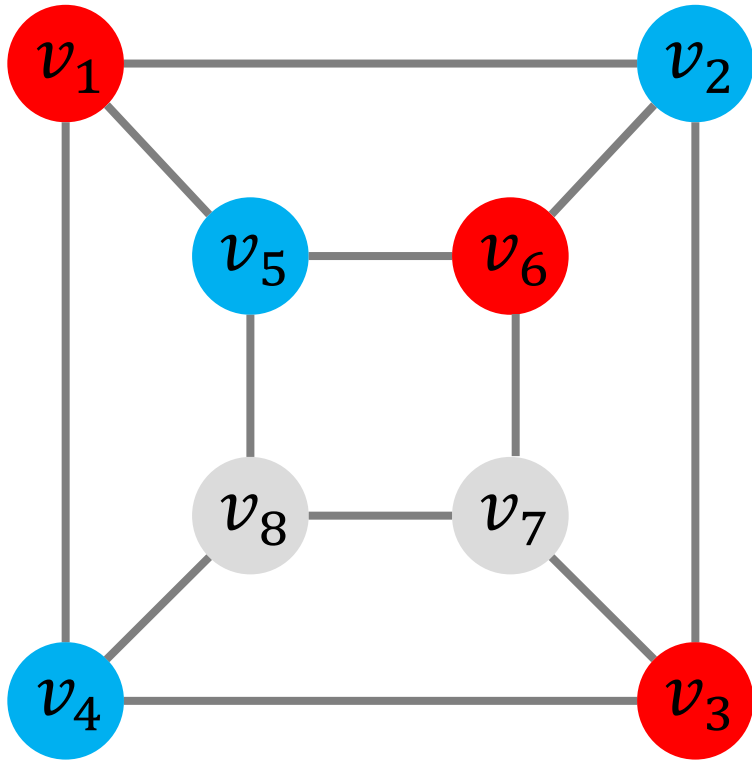


Queue:



- $v_2 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.
- Assign **red color** to its neighbors, v_3 and v_6 .
- Put the unvisited neighbors, v_3 and v_6 , in the queue.

Iteration 3

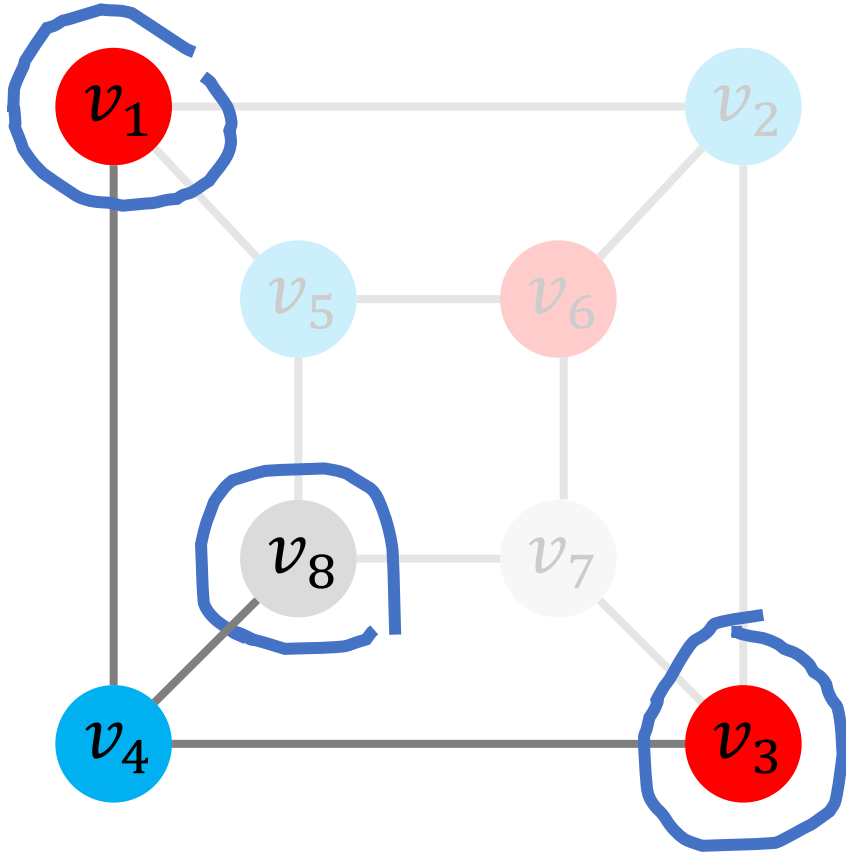


Queue:



- $v_4 \leftarrow \text{dequeue}()$.

Iteration 3

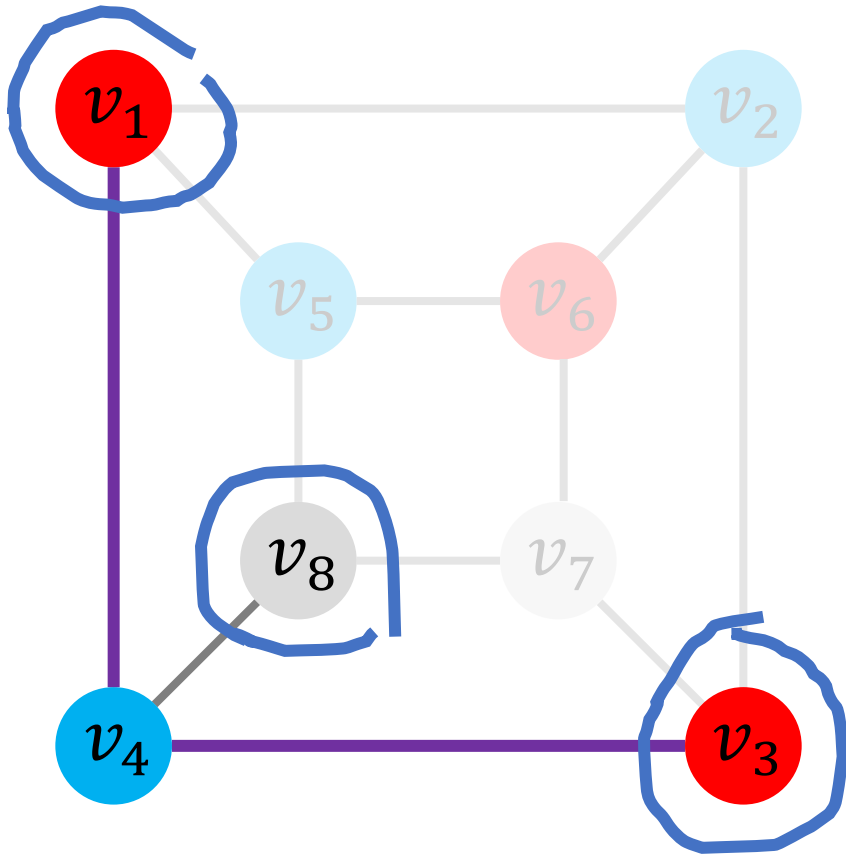


Queue:



- $v_4 \leftarrow \text{dequeue}()$.

Iteration 3



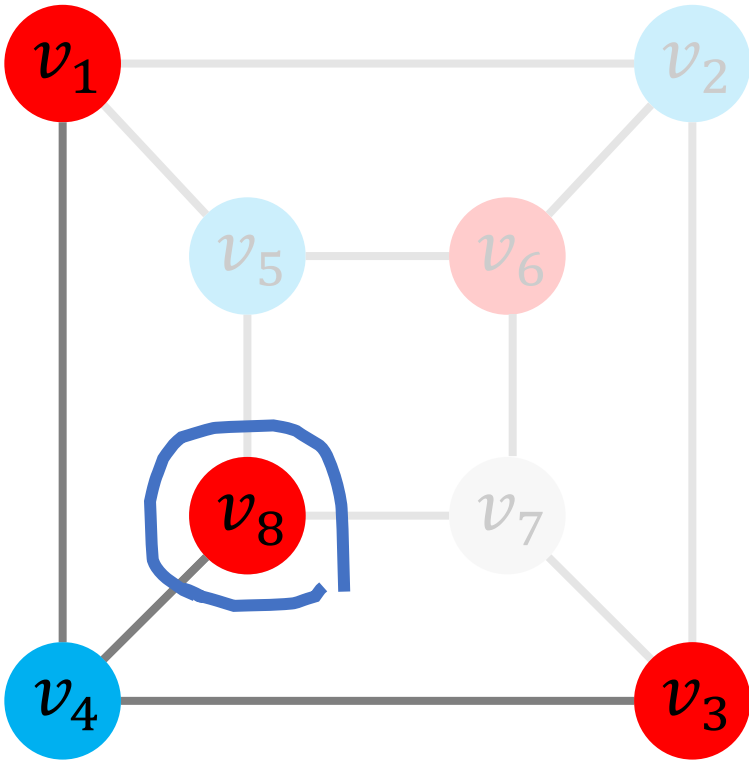
No violation

Queue:



- $v_4 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.

Iteration 3

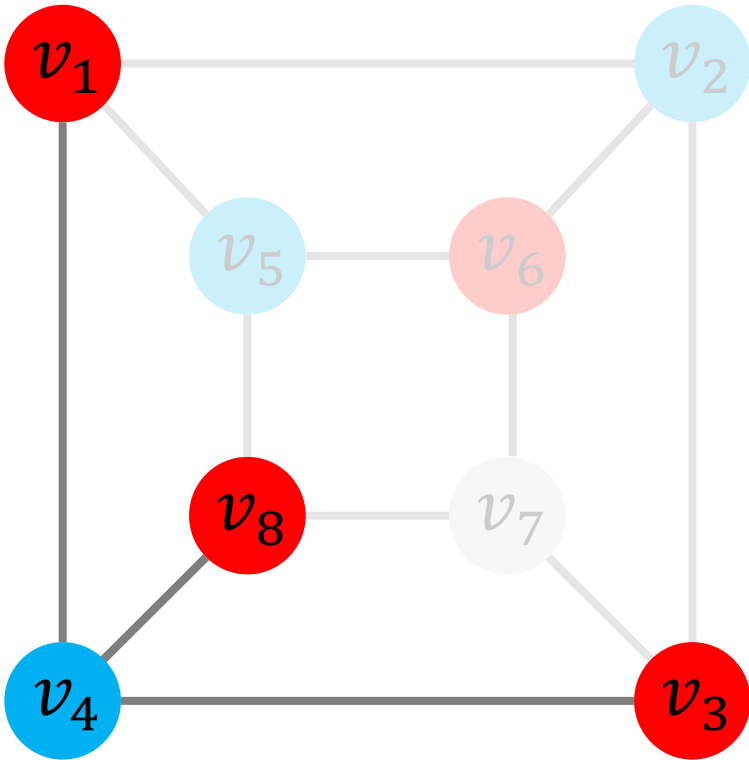


Queue:



- $v_4 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.
- Assign **red color** to its neighbor, v_8 .

Iteration 3

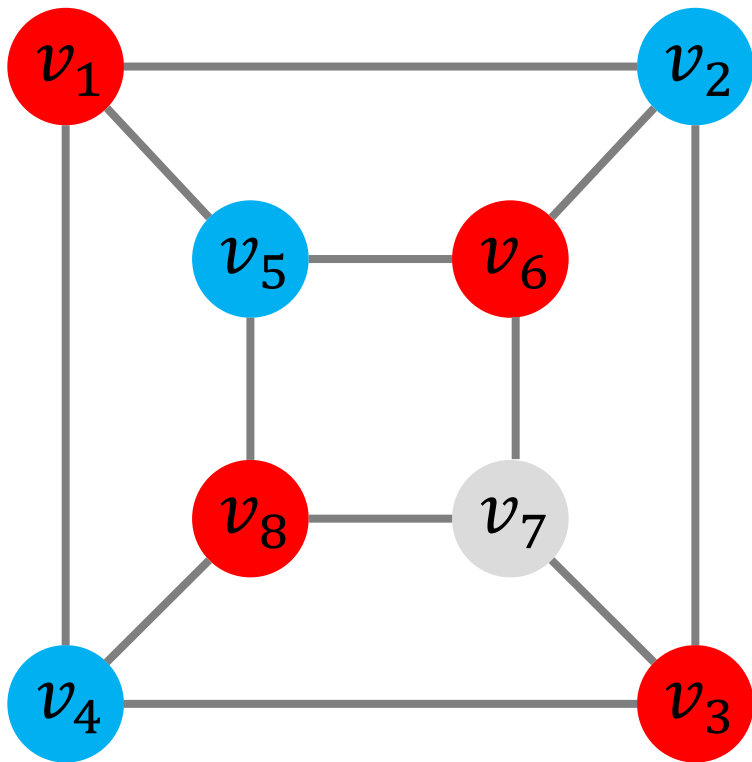


Queue:



- $v_4 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.
- Assign **red color** to its neighbor, v_8 .
- Put the unvisited neighbor, v_8 , in the queue.

Iteration 4

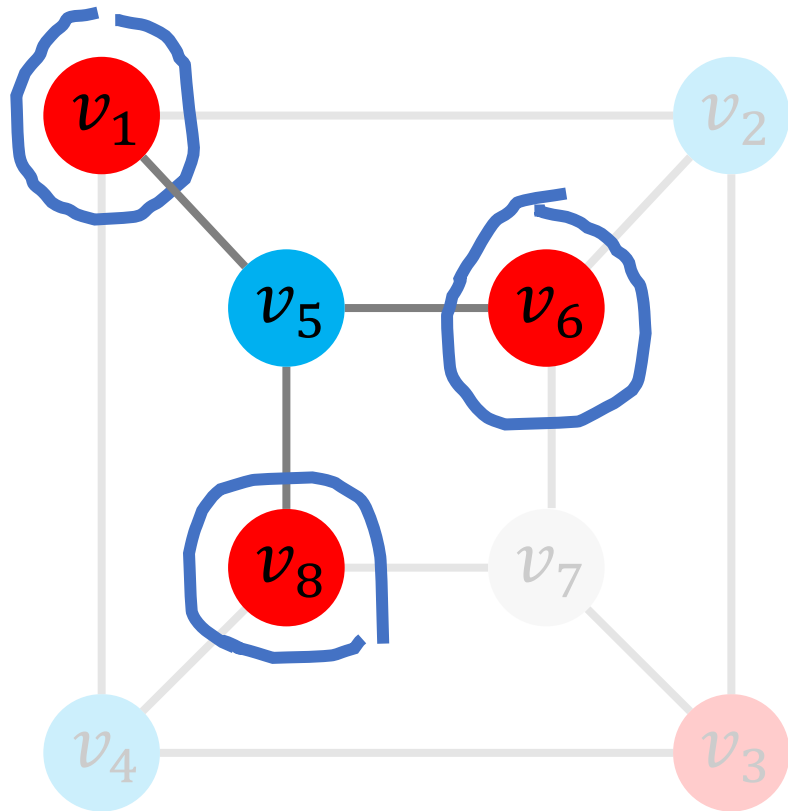


Queue:

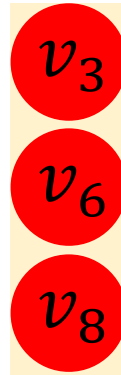


- $v_5 \leftarrow \text{dequeue}()$.

Iteration 4

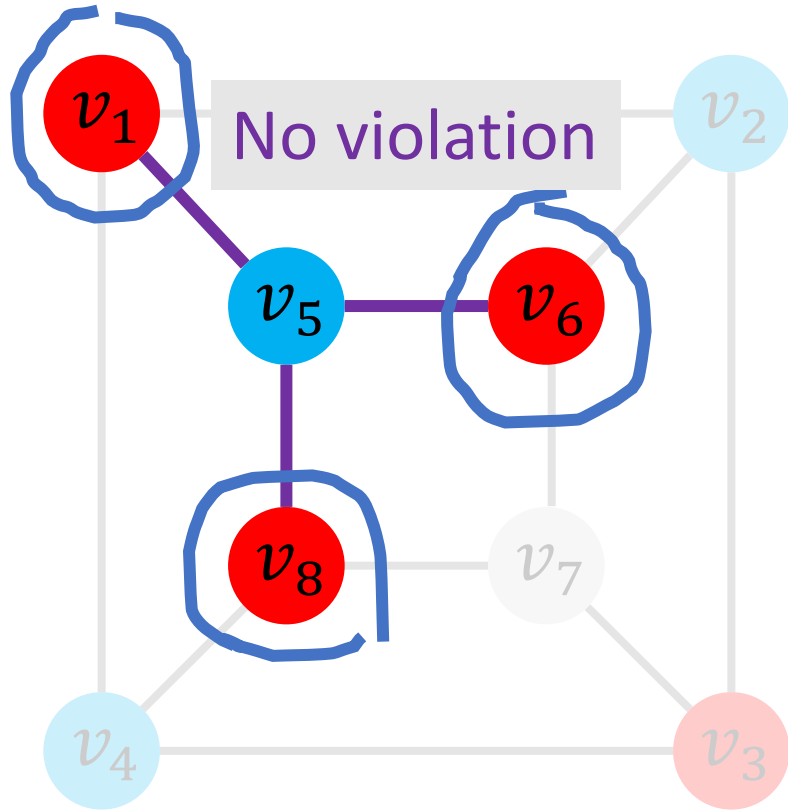


Queue:

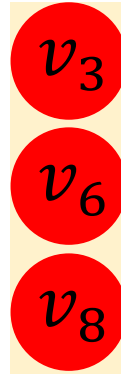


- $v_5 \leftarrow \text{dequeue}()$.

Iteration 4

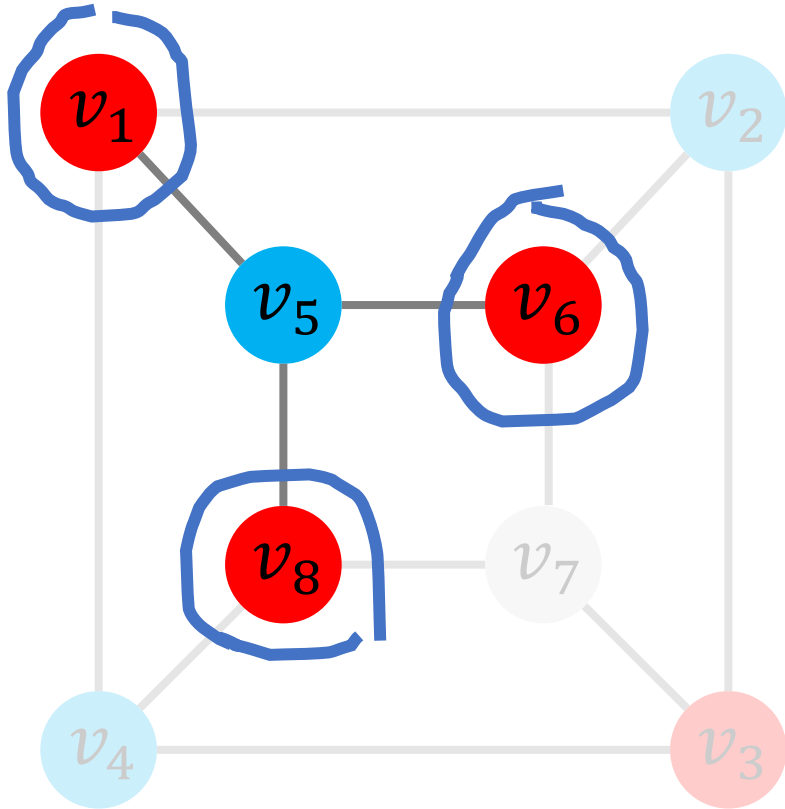


Queue:

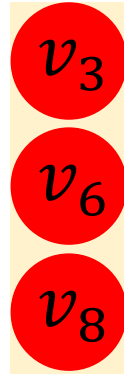


- $v_5 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.

Iteration 4

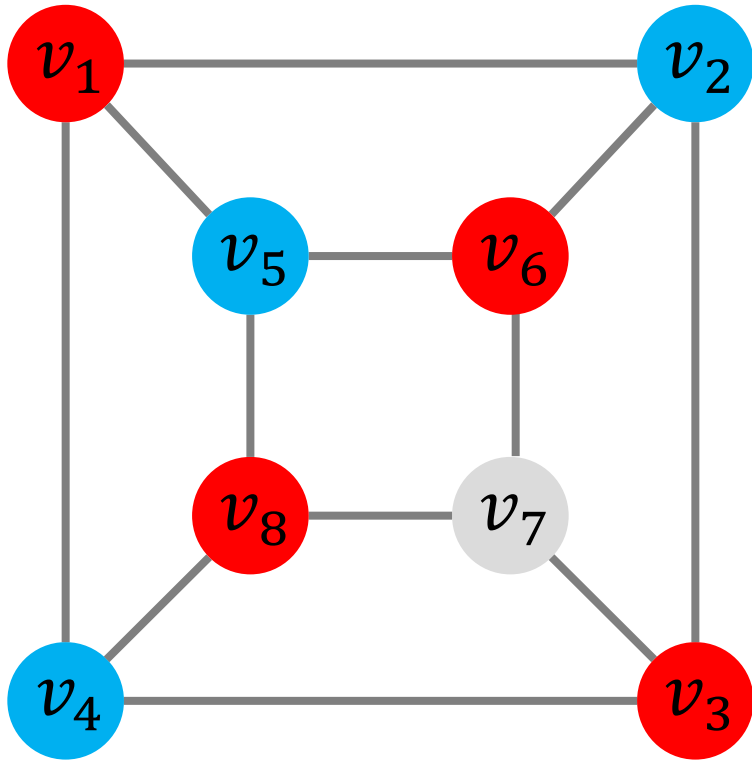


Queue:

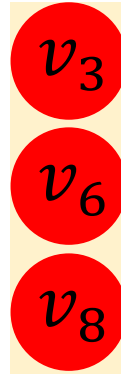


- $v_5 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.
- Do not put visited neighbors in the queue.

Iteration 5

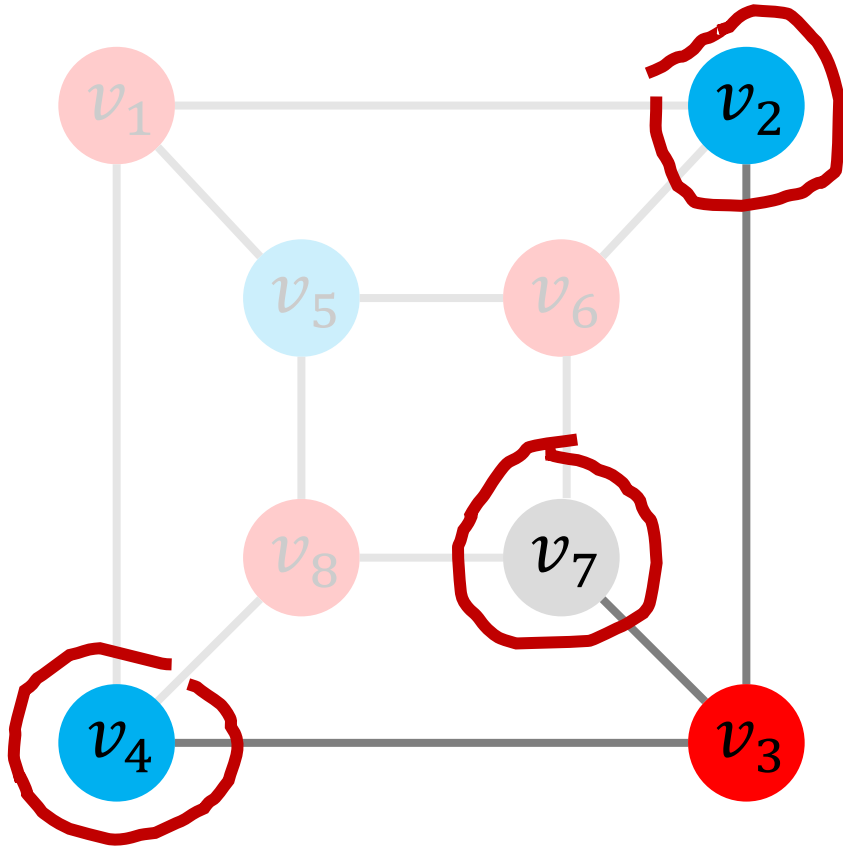


Queue:

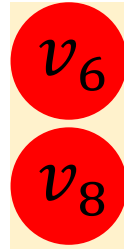


- $v_3 \leftarrow \text{dequeue}()$.

Iteration 5

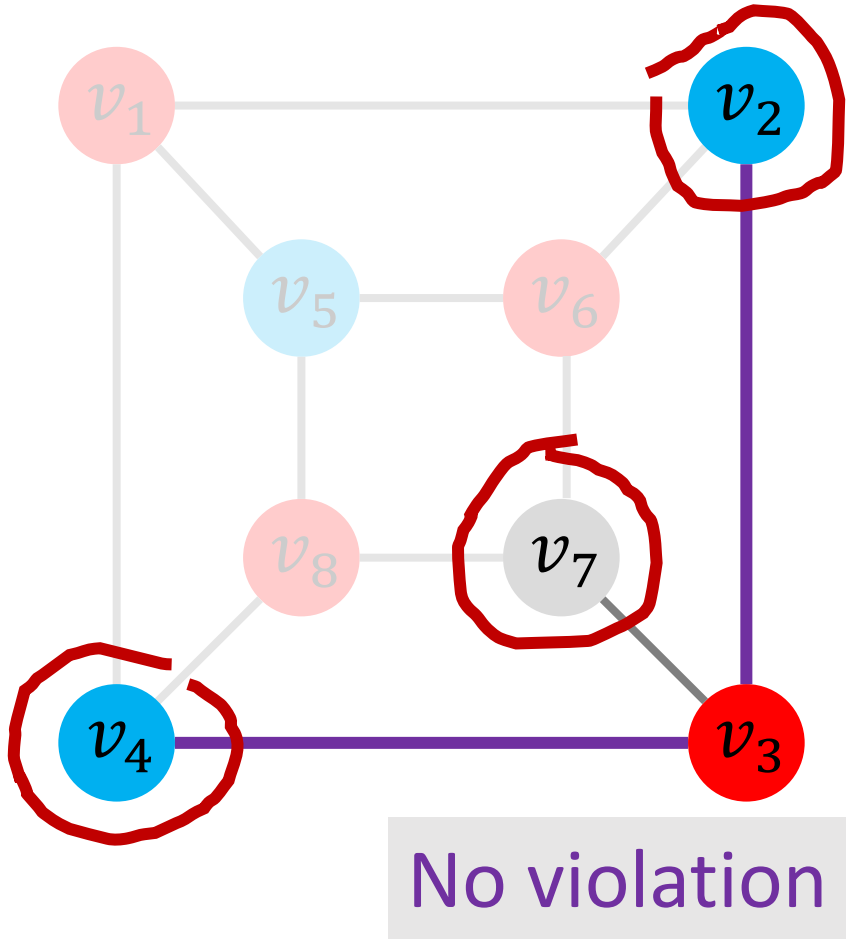


Queue:

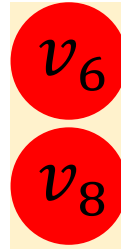


- $v_3 \leftarrow \text{dequeue}()$.

Iteration 5

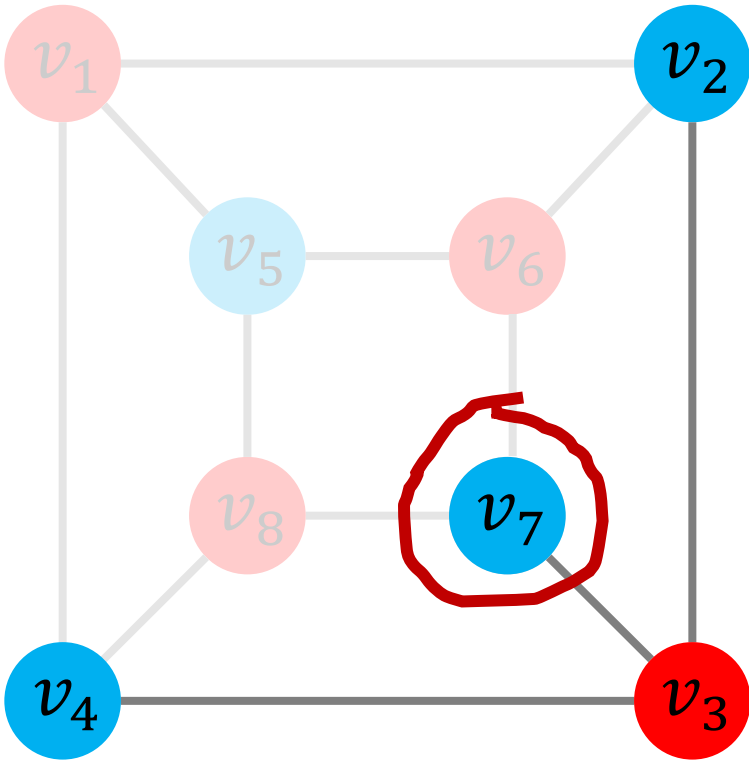


Queue:

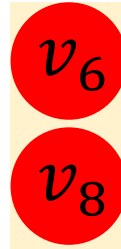


- $v_3 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.

Iteration 5

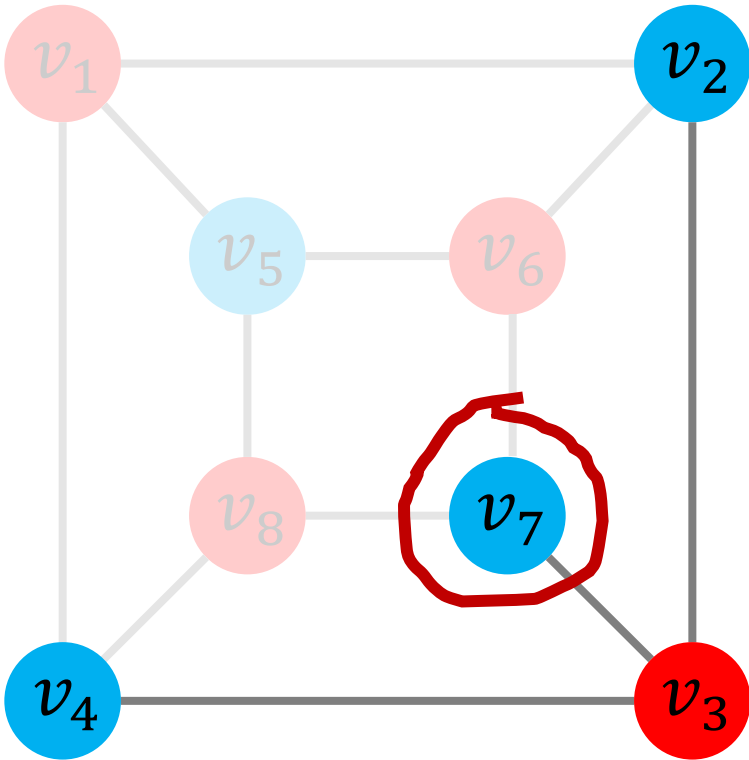


Queue:



- $v_3 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.
- Assign **blue color** to its neighbor v_7 .

Iteration 5

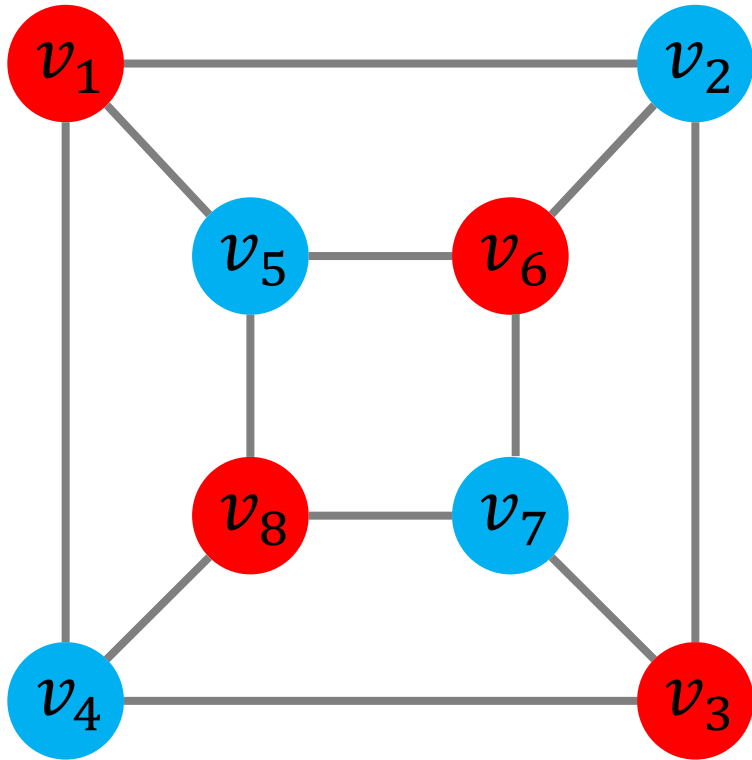


Queue:



- $v_3 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.
- Assign **blue color** to its neighbor v_7 .
- Put the unvisited neighbor v_7 in the queue.

Iteration 6

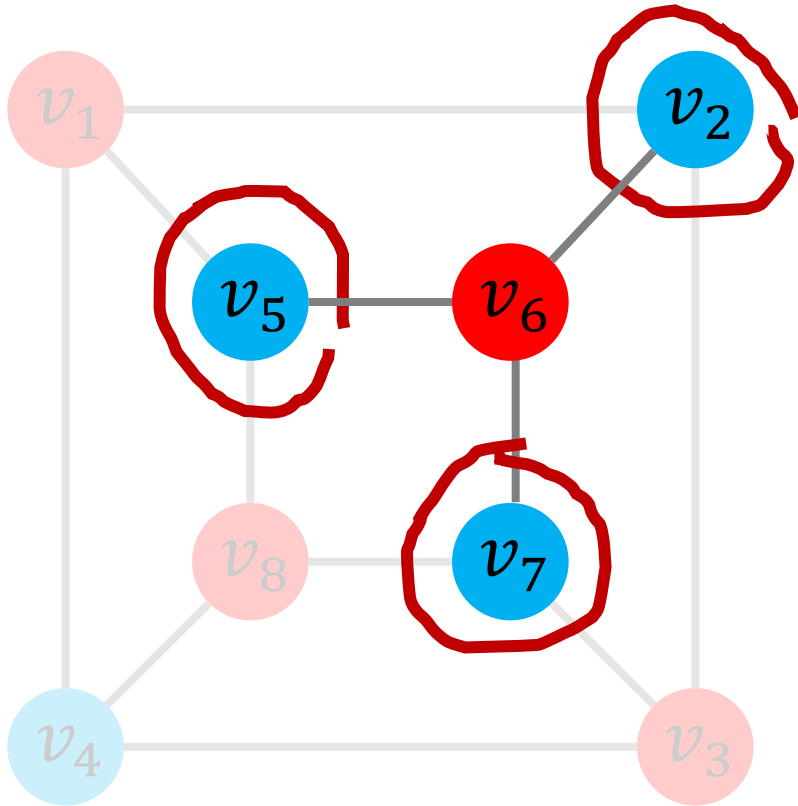


Queue:

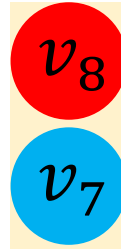


- $v_6 \leftarrow \text{dequeue}()$.

Iteration 6

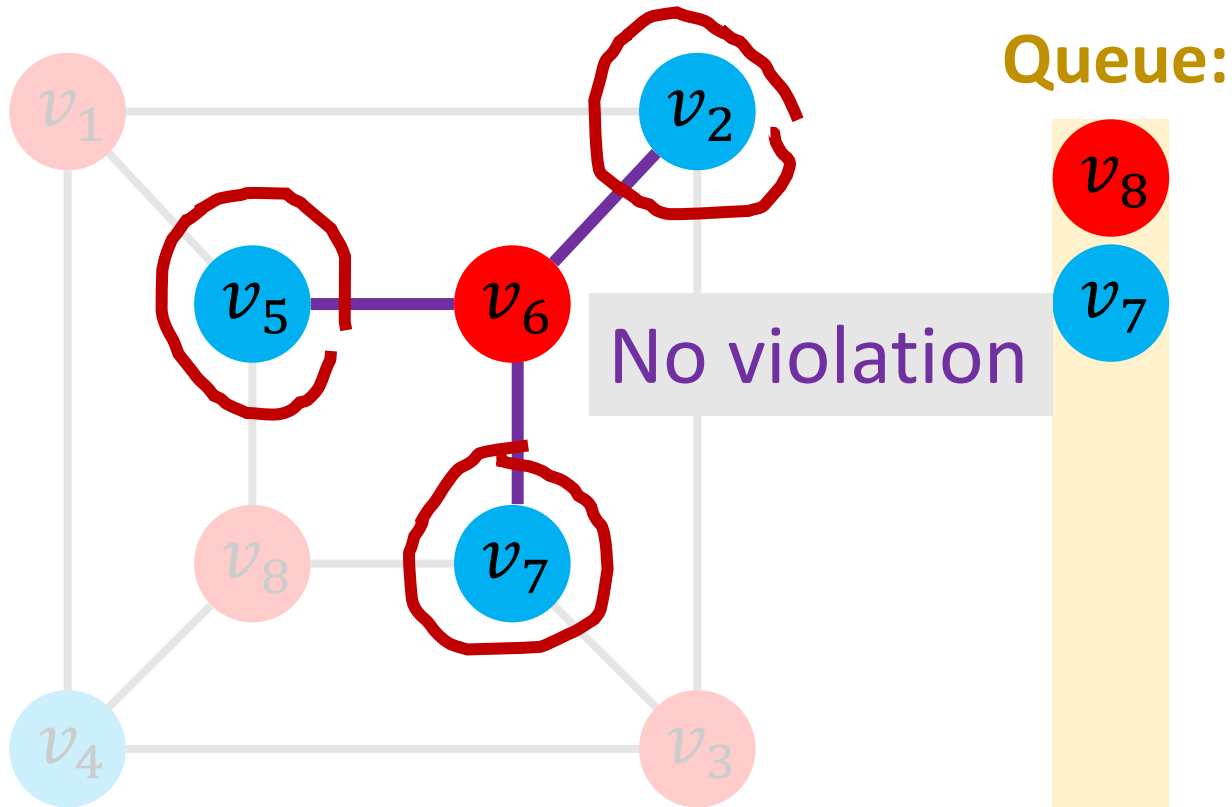


Queue:



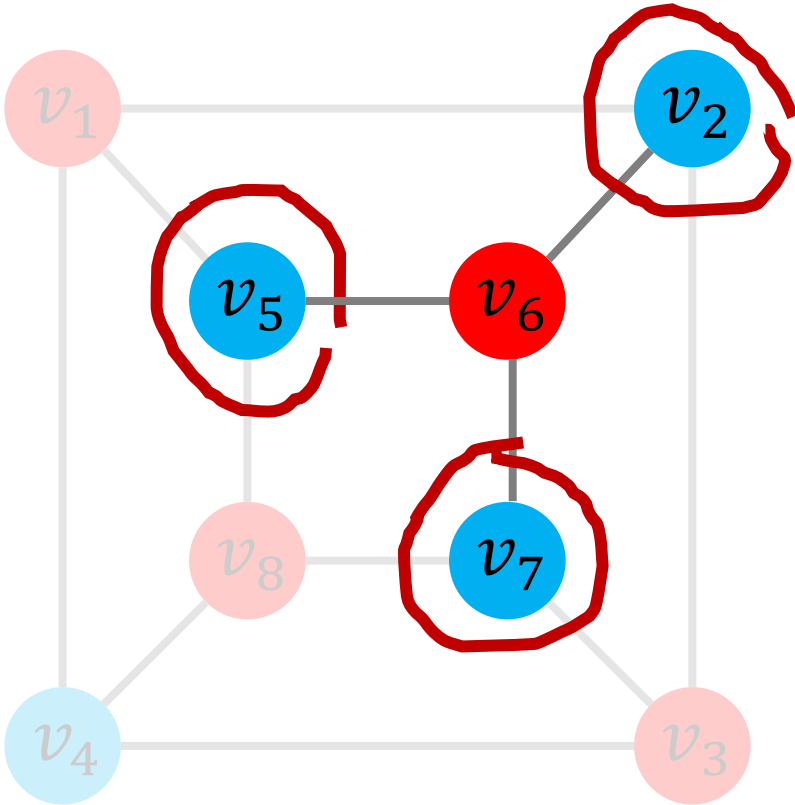
- $v_6 \leftarrow \text{dequeue}()$.

Iteration 6

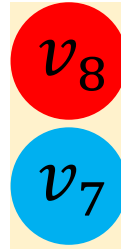


- $v_6 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.

Iteration 6

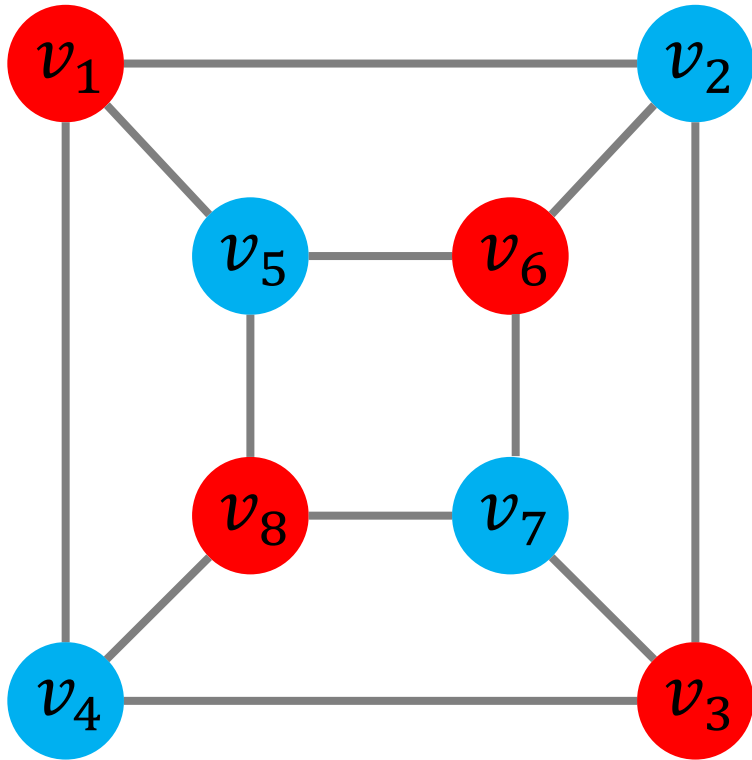


Queue:

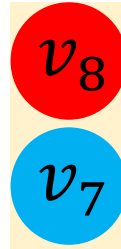


- $v_6 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.
- Do not put visited neighbors in the queue.

Iteration 7

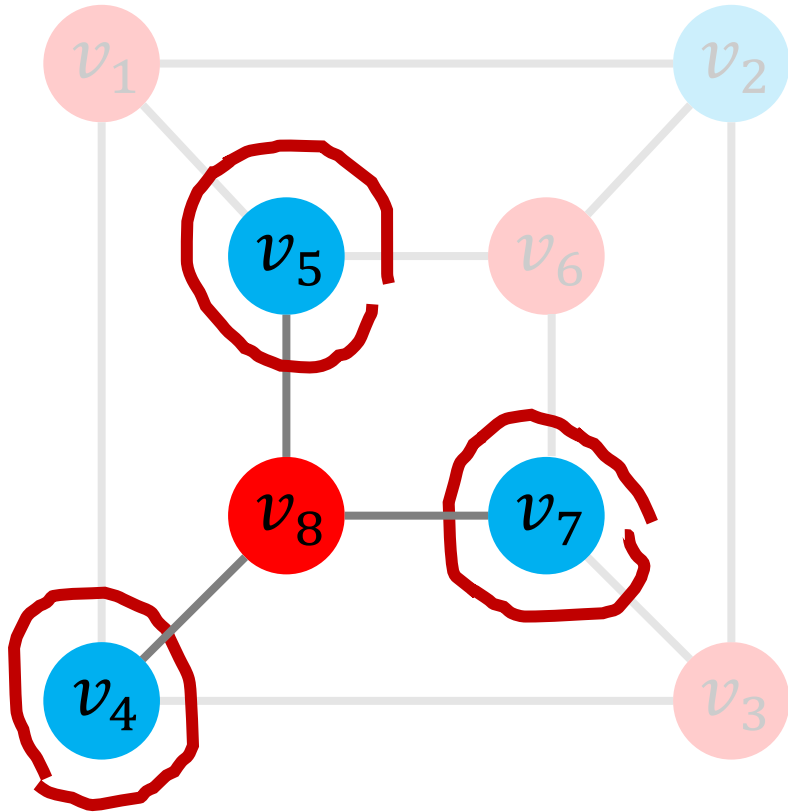


Queue:



- $v_8 \leftarrow \text{dequeue}()$.

Iteration 7

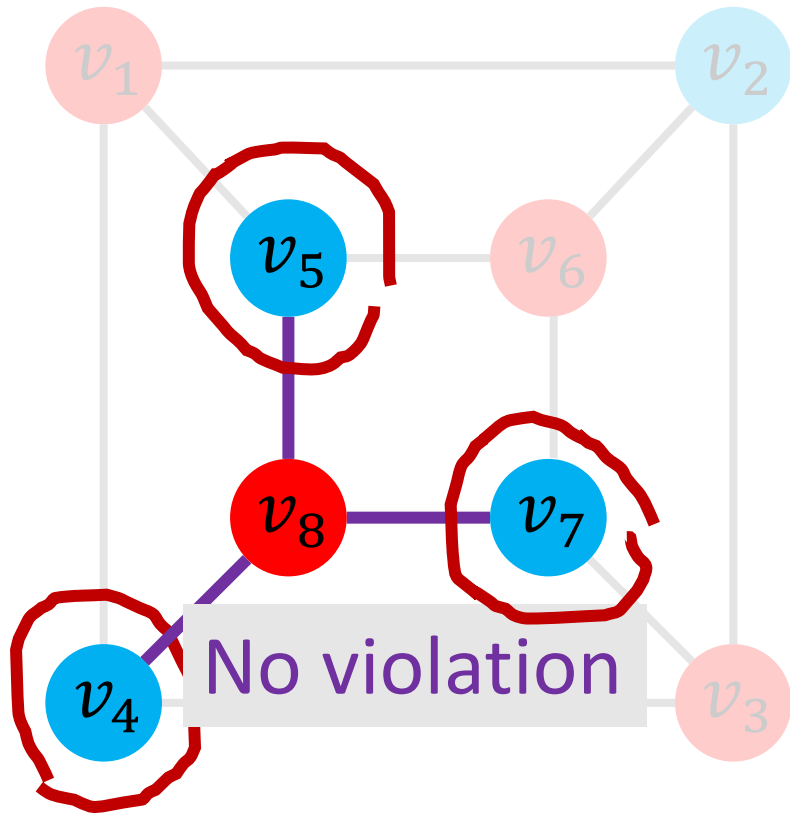


Queue:



- $v_8 \leftarrow \text{dequeue}()$.

Iteration 7



Queue:



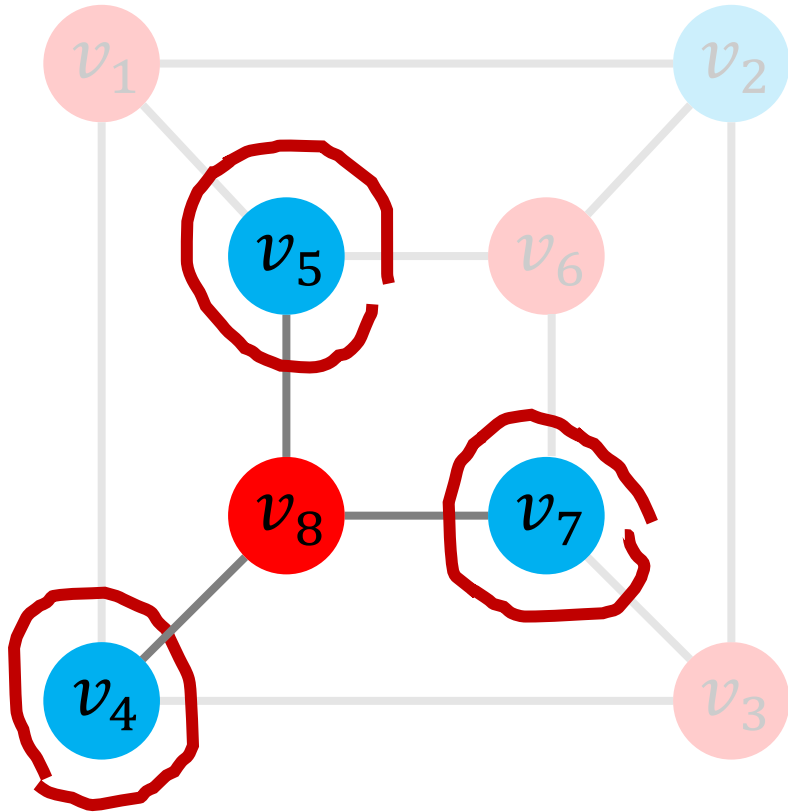
- $v_8 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.

Iteration 7

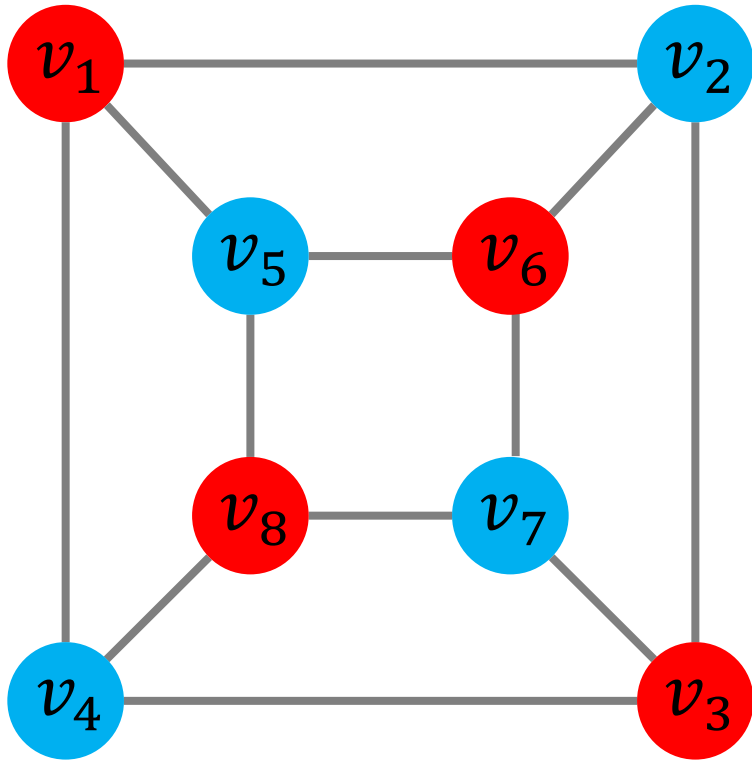
Queue:



- $v_8 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.
- Do not put visited neighbors in the queue.



Iteration 8

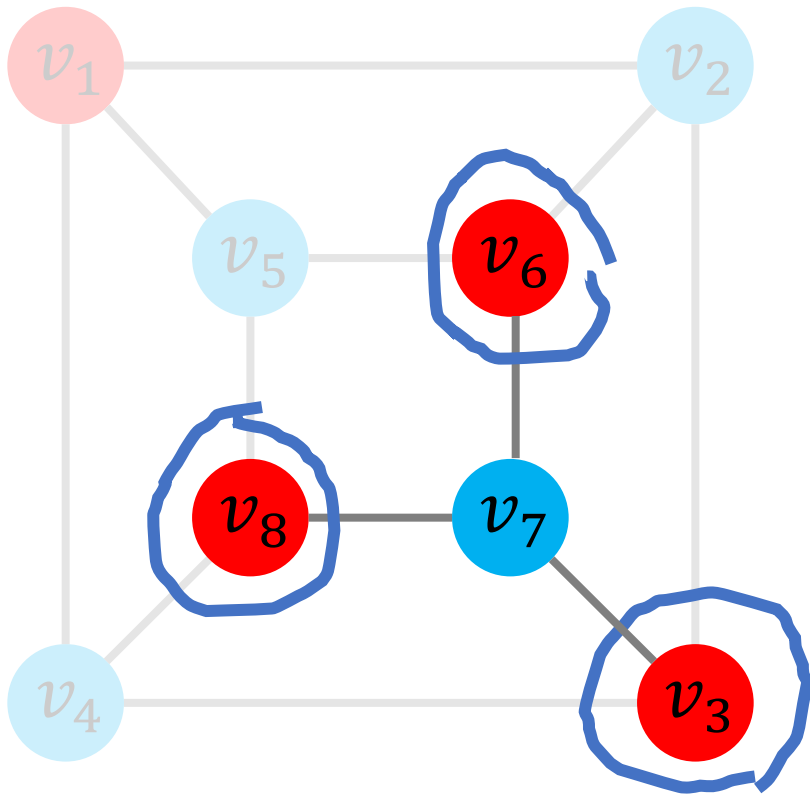


Queue:



- $v_7 \leftarrow \text{dequeue}()$.

Iteration 8

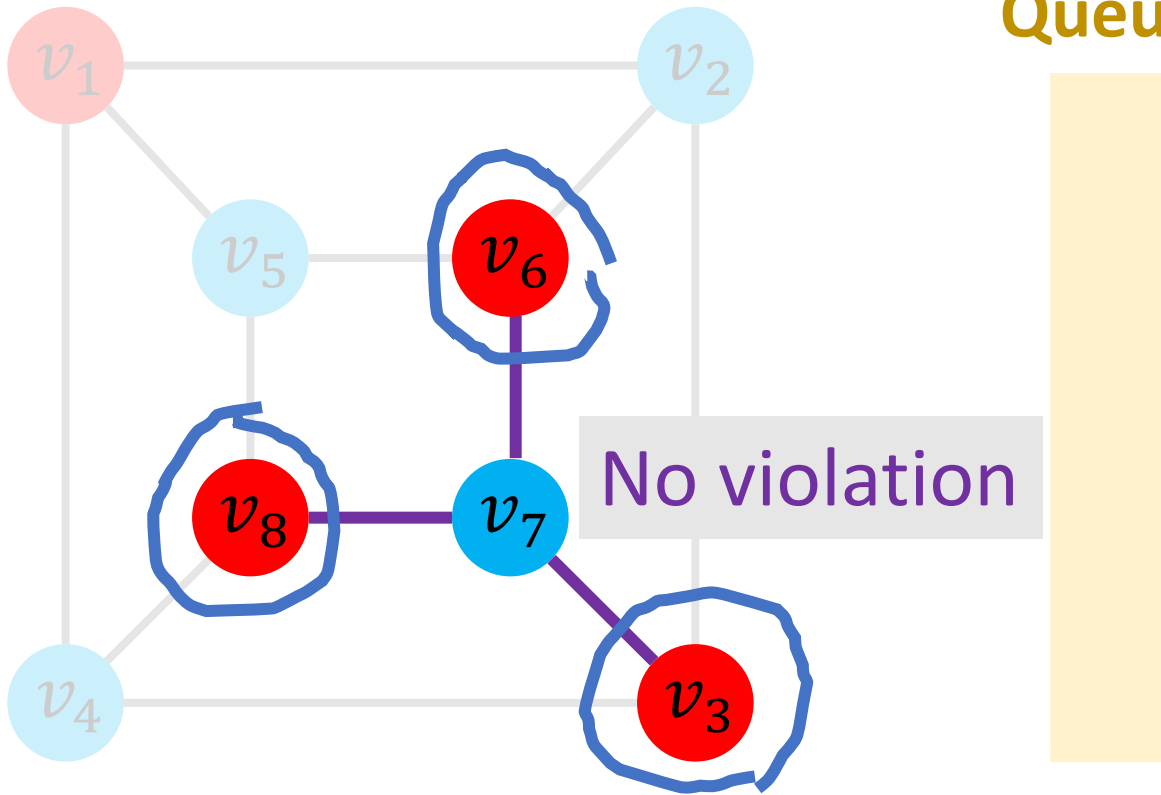


Queue:



- $v_7 \leftarrow \text{dequeue}()$.

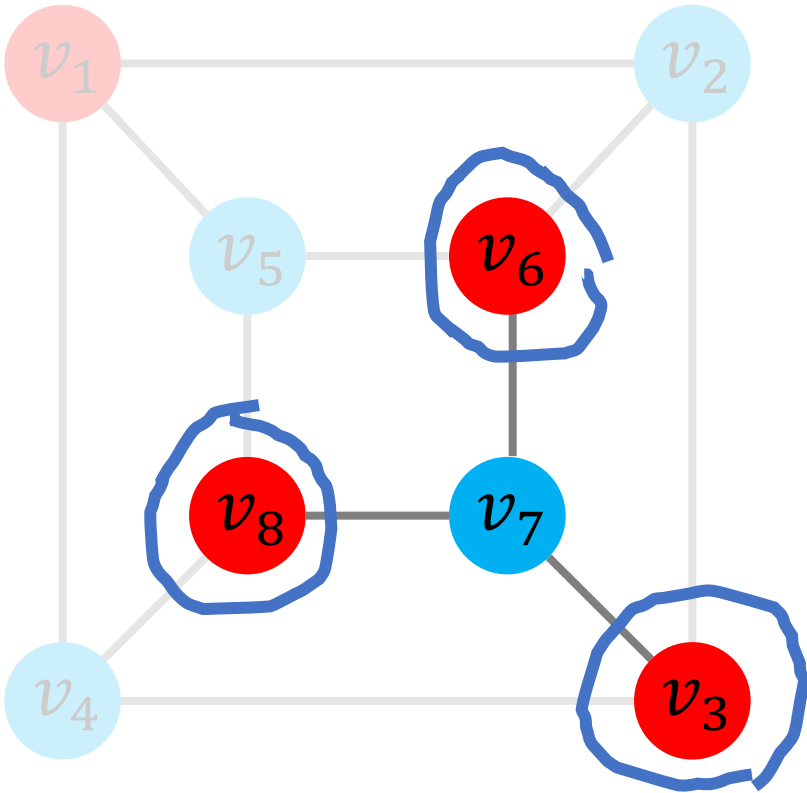
Iteration 8



Queue:

- $v_7 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.

Iteration 8

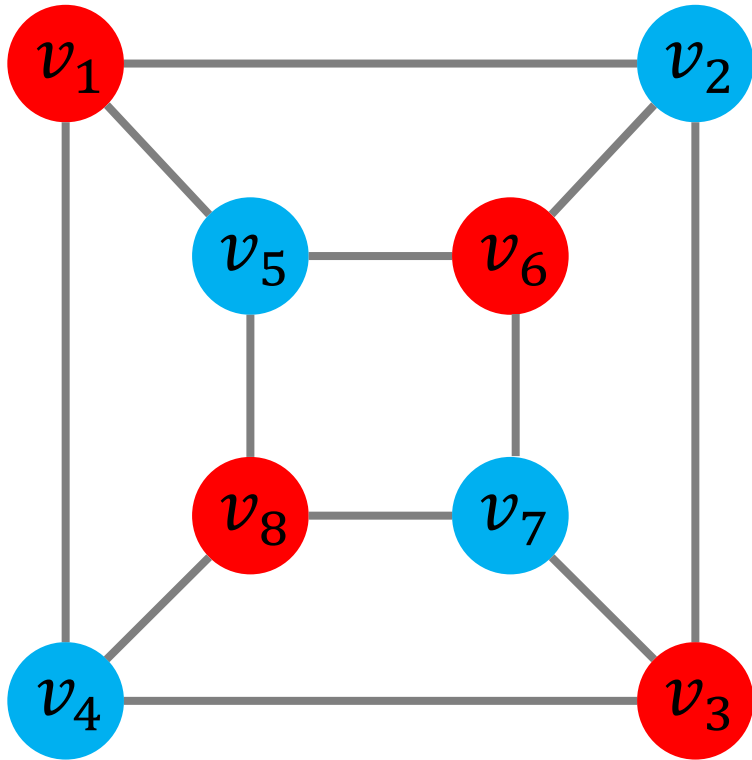


Queue:



- $v_7 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.
- Do not put visited neighbors in the queue.

End of Procedure



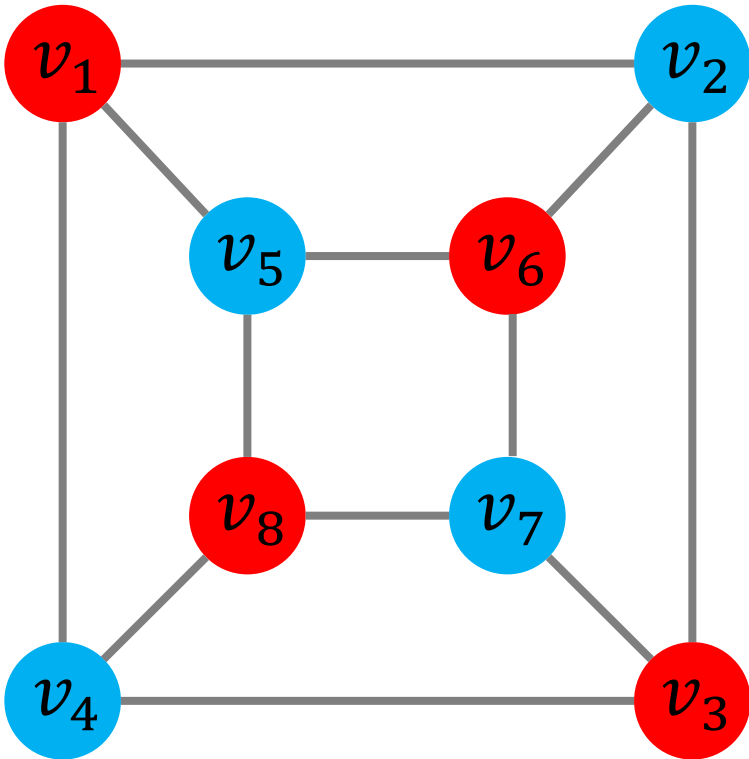
Queue:



- All the vertices have been colored.
- The queue is empty.

End the program

End of Procedure



Queue:



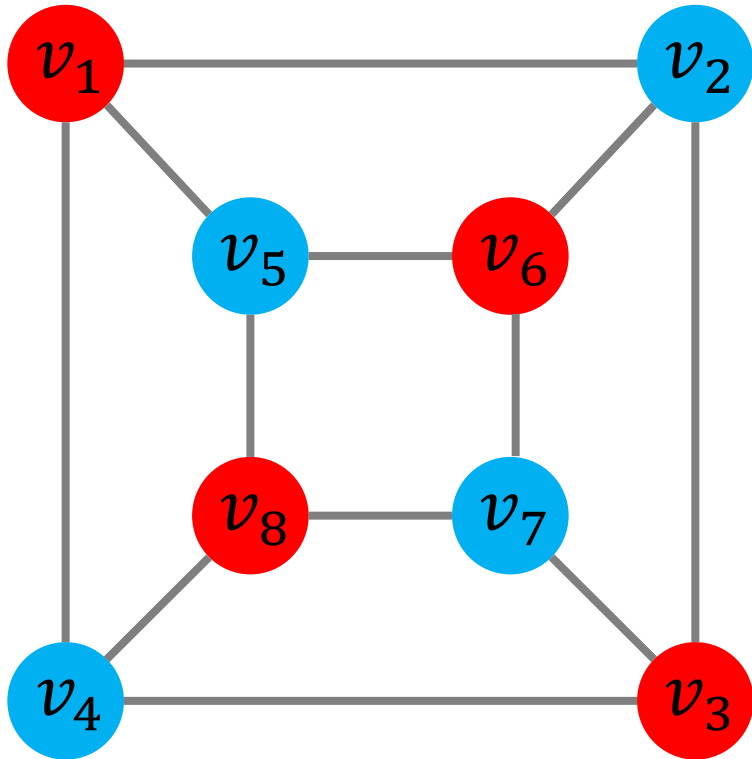
- All the vertices have been colored.
- The queue is empty.

End the program

- No violation is found.

The graph is bipartite

End of Procedure



Queue:



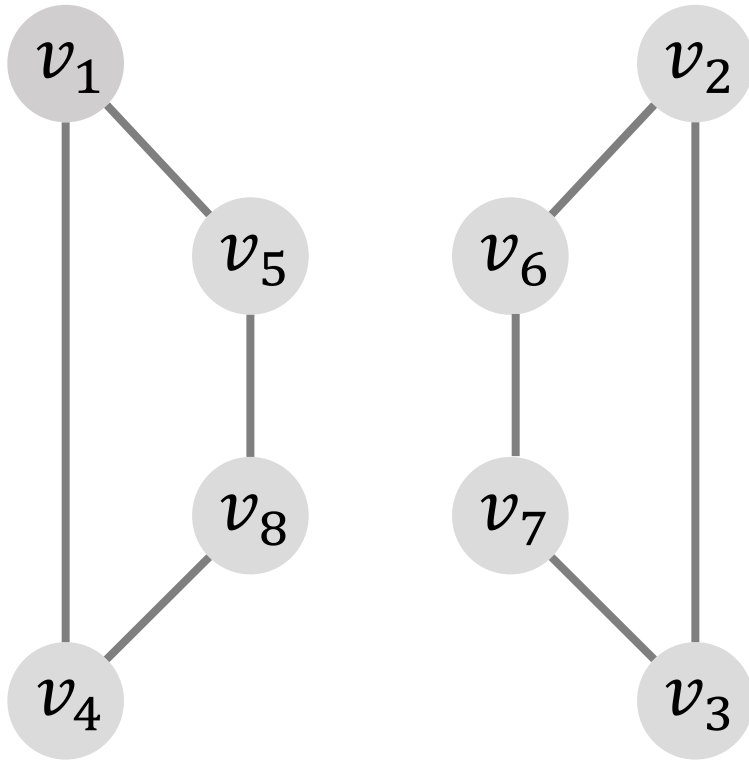
- All the vertices have been colored.

What if some vertices have not been colored?

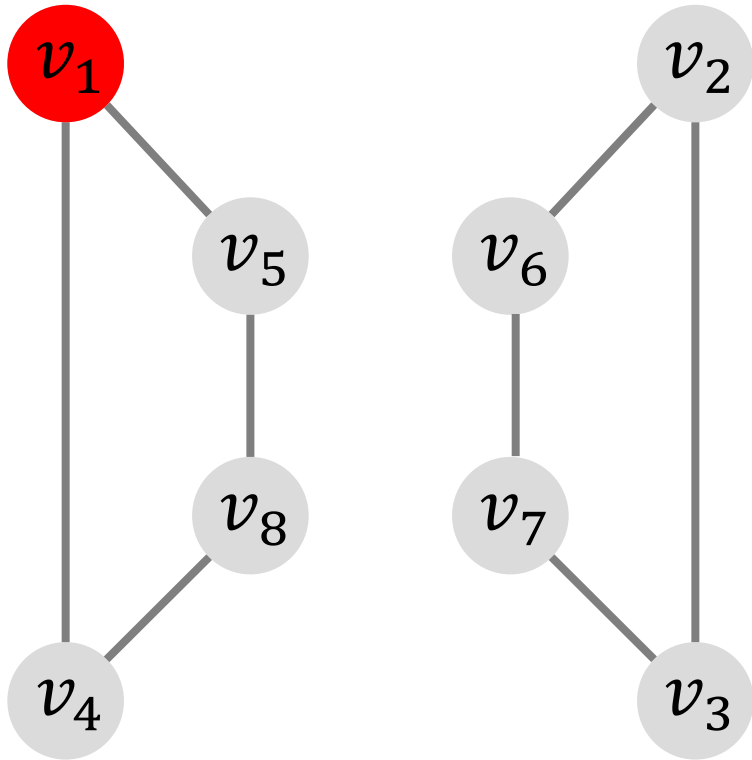
- No violation is found.

The graph is bipartite

Another Example

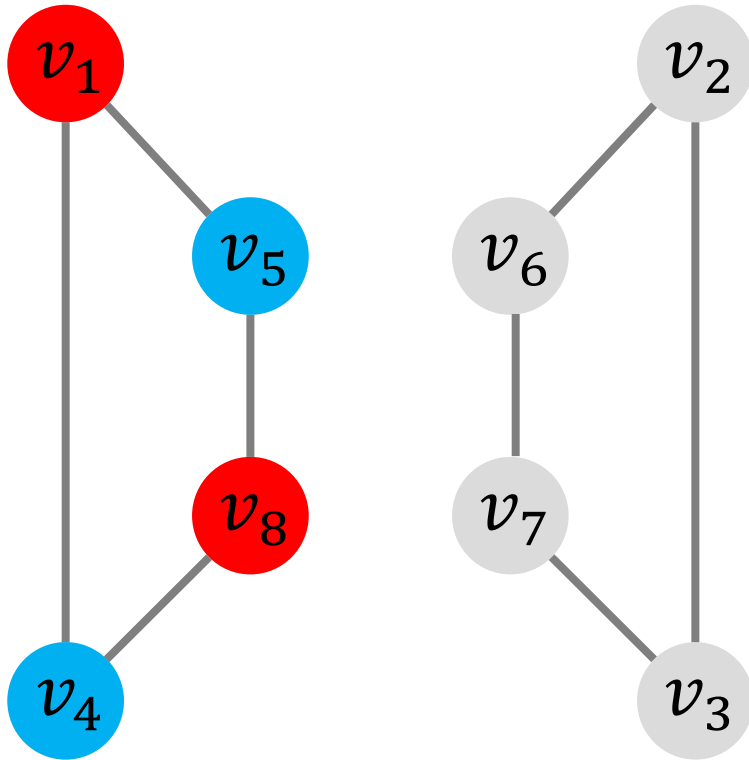


Another Example



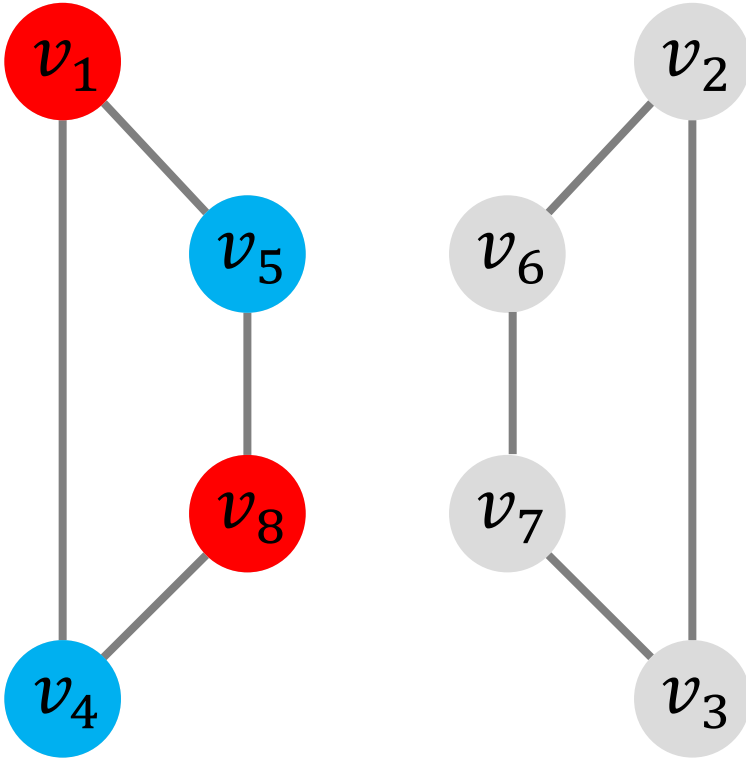
- Assign red color to v_1 .

Another Example



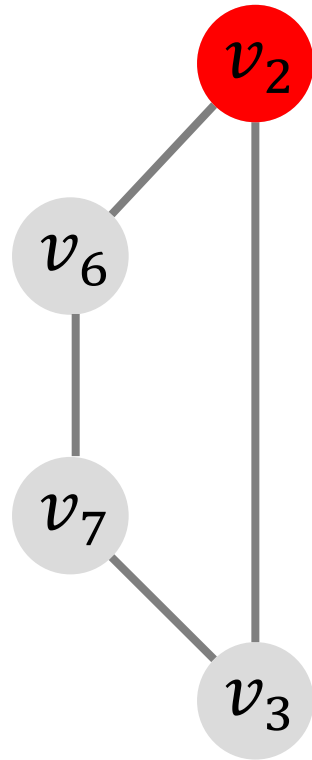
- Assign red color to v_1 .
- Run the algorithm.

Another Example



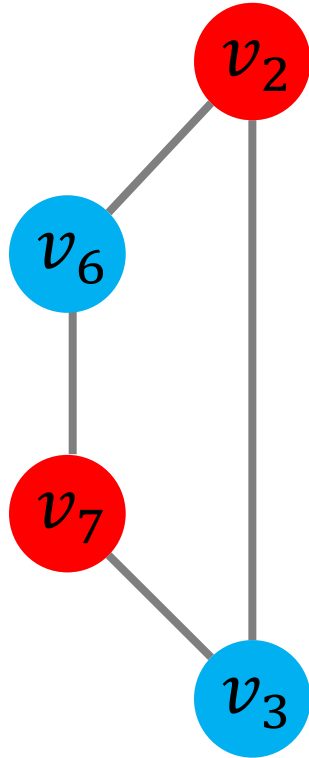
- Remove all the colored vertices.
- Run the algorithm on the remaining subgraph.

Another Example



- Assign red color to v_2 .

Another Example



- Assign **red color** to v_2 .
- Run the algorithm.

Testing Bipartiteness

1. Select a vertex, assign red color to it, and put it in the queue.

Testing Bipartiteness

1. Select a vertex, assign **red color** to it, and put it in the queue.
2. While the queue is not empty:
 - a. $v \leftarrow \text{dequeue}()$;
 - b. $c \leftarrow \text{the opposite color of } v$;

Testing Bipartiteness

1. Select a vertex, assign red color to it, and put it in the queue.
2. While the queue is not empty:
 - a. $v \leftarrow \text{dequeue}()$;
 - b. $c \leftarrow$ the opposite color of v ;
 - c. For each $u \in \text{Neighbor}(v)$:
 - i. If u has been colored, check whether there is a violation;
 - ii. Otherwise, assign color c to u , and put u in the queue;

Testing Bipartiteness

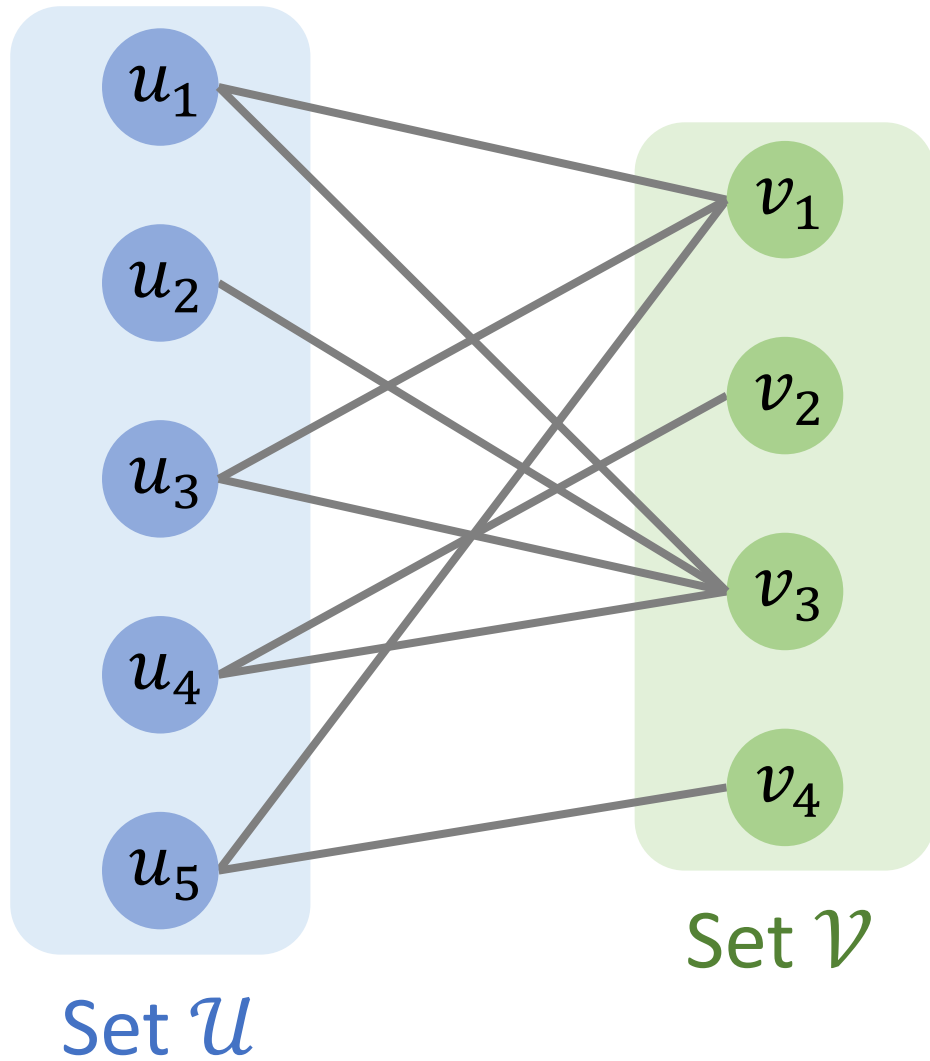
1. Select a vertex, assign red color to it, and put it in the queue.
2. While the queue is not empty:
 - a. $v \leftarrow \text{dequeue}()$;
 - b. $c \leftarrow$ the opposite color of v ;
 - c. For each $u \in \text{Neighbor}(v)$:
 - i. If u has been colored, check whether there is a violation;
 - ii. Otherwise, assign color c to u , and put u in the queue;
 - d. If color violation is found, return FALSE (not bipartite).

Testing Bipartiteness

3. When the queue is empty, check if all the vertices have been colored.
 - If yes, end the program and return TRUE.
 - If no, delete the colored vertices, and apply the algorithm to the remaining subgraph.

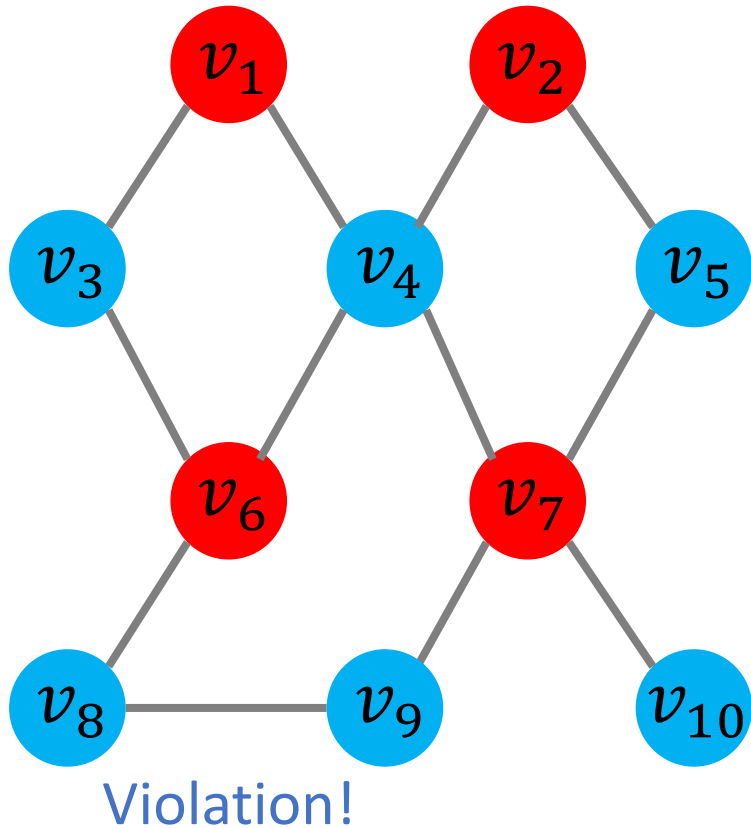
Summary

Bipartite Graph



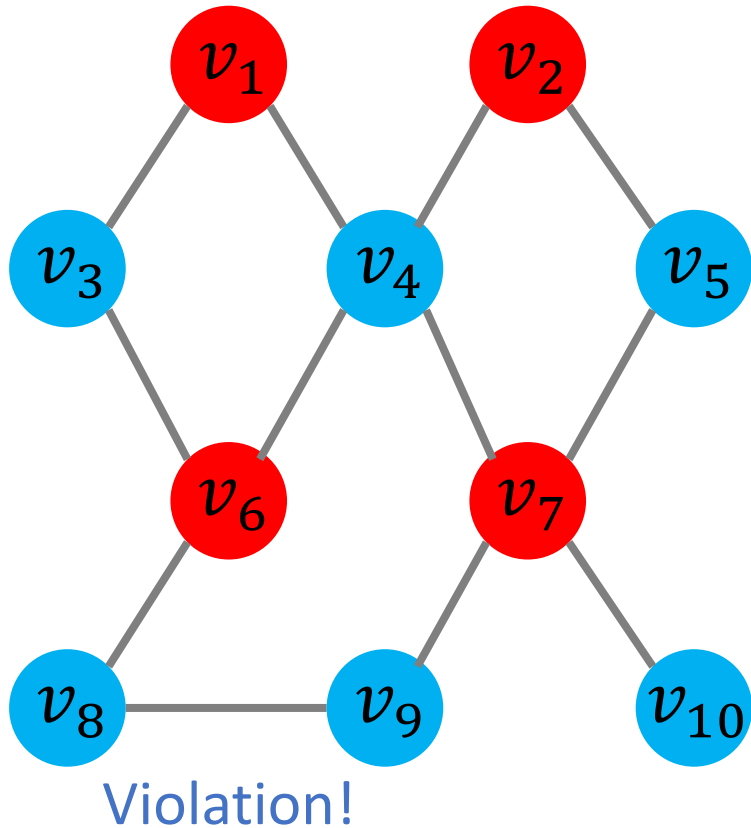
- The vertices can be partitioned into two subsets, U and V .
- No edge between two vertices in U .
- No edge between two vertices in V .
- Applications:
 - Matching candidates and positions.
 - Pet adoption.
 - Dating.

Testing Bipartiteness



- Basic idea:
 - Color the vertices using **red** and **blue**.
 - Check whether there is any violation.

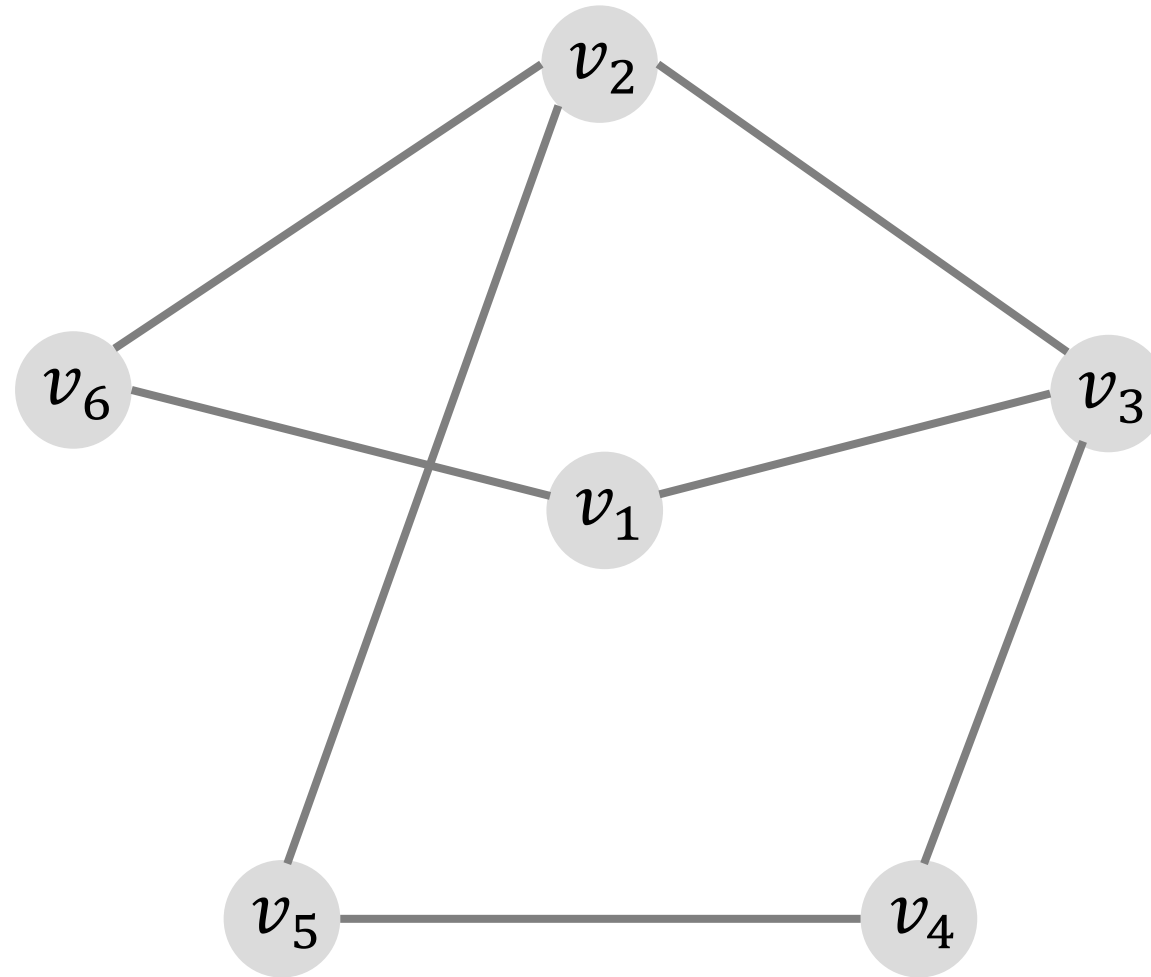
Testing Bipartiteness



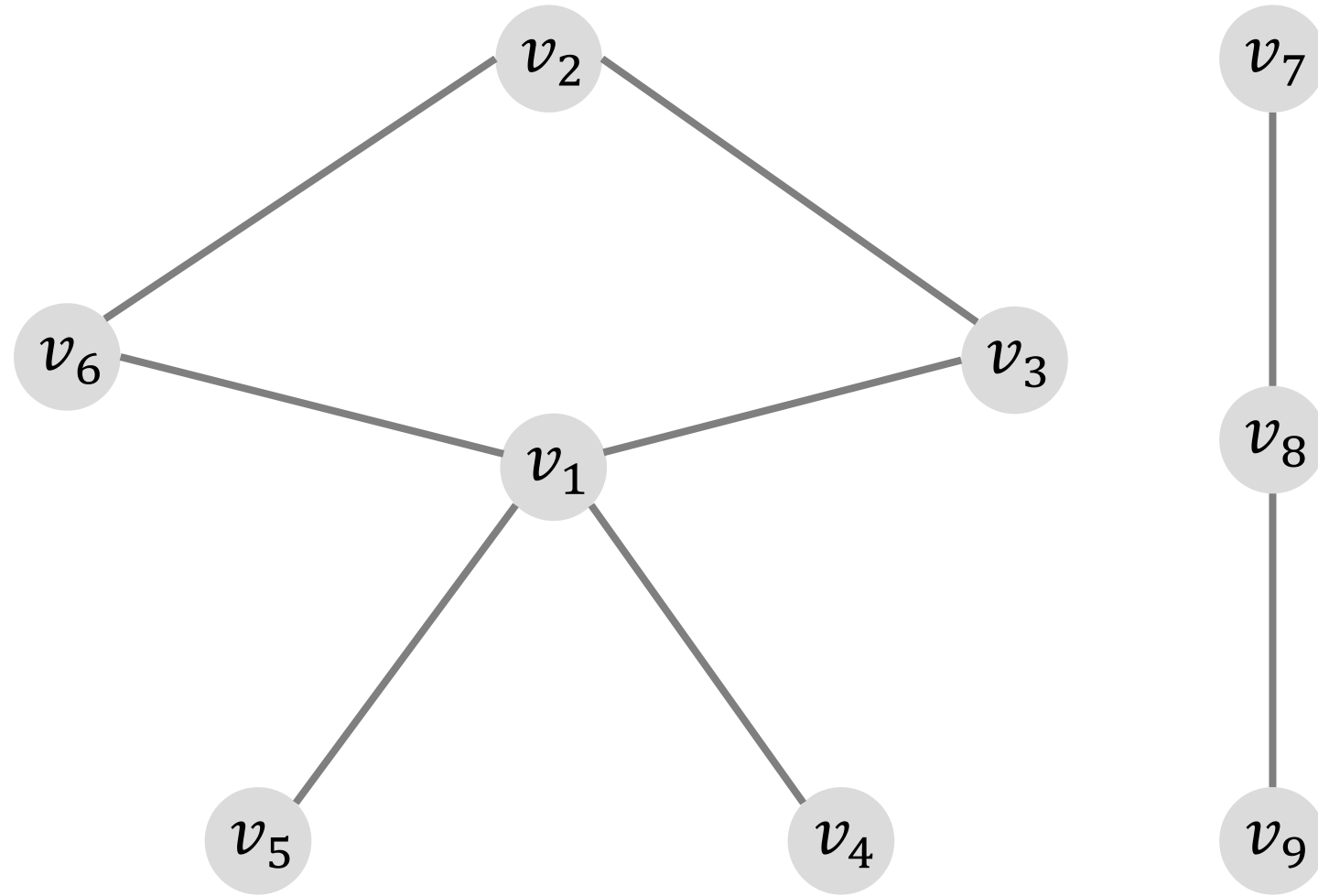
- Basic idea:
 - Color the vertices using **red** and **blue**.
 - Check whether there is any violation.
- Algorithm: breadth-first search (BFS).
- Time complexity: $O(|\mathcal{E}| + |\mathcal{V}|)$.

Questions

Q1: Is it a bipartite graph?



Q2: Is it a bipartite graph?



Thank You!

<http://wangshusen.github.io/>