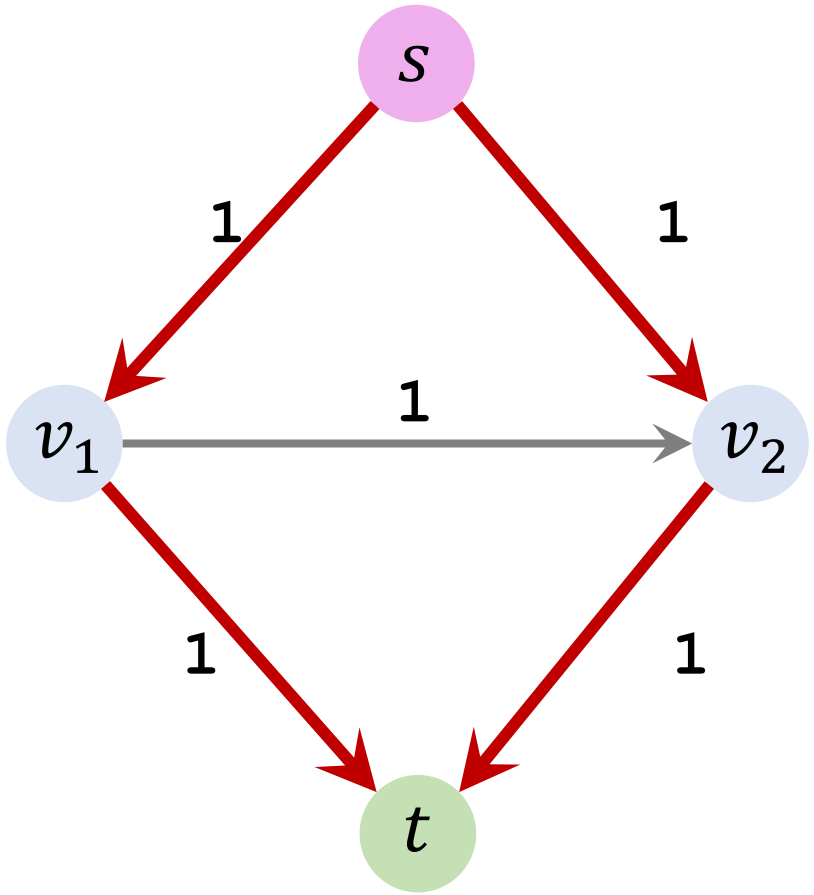
Ford-Fulkerson Algorithm

Shusen Wang

http://wangshusen.github.io/

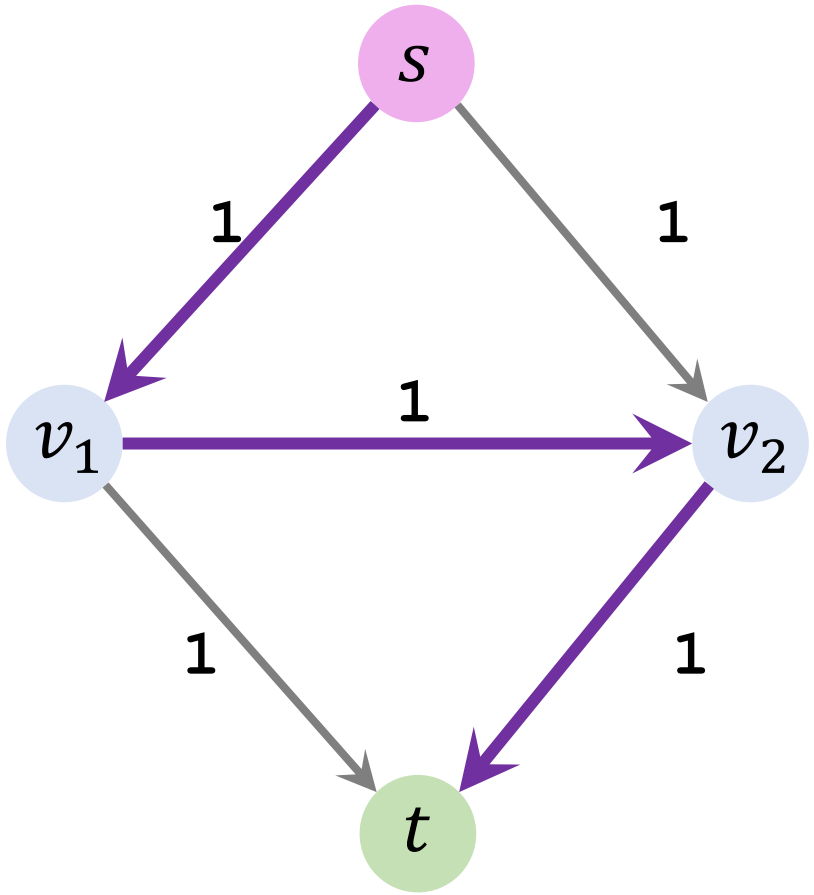
Problem with the naïve algorithm



Maximum Flow

- The amount of max flow is 2.

Problem with the naïve algorithm



Not Maximum Flow

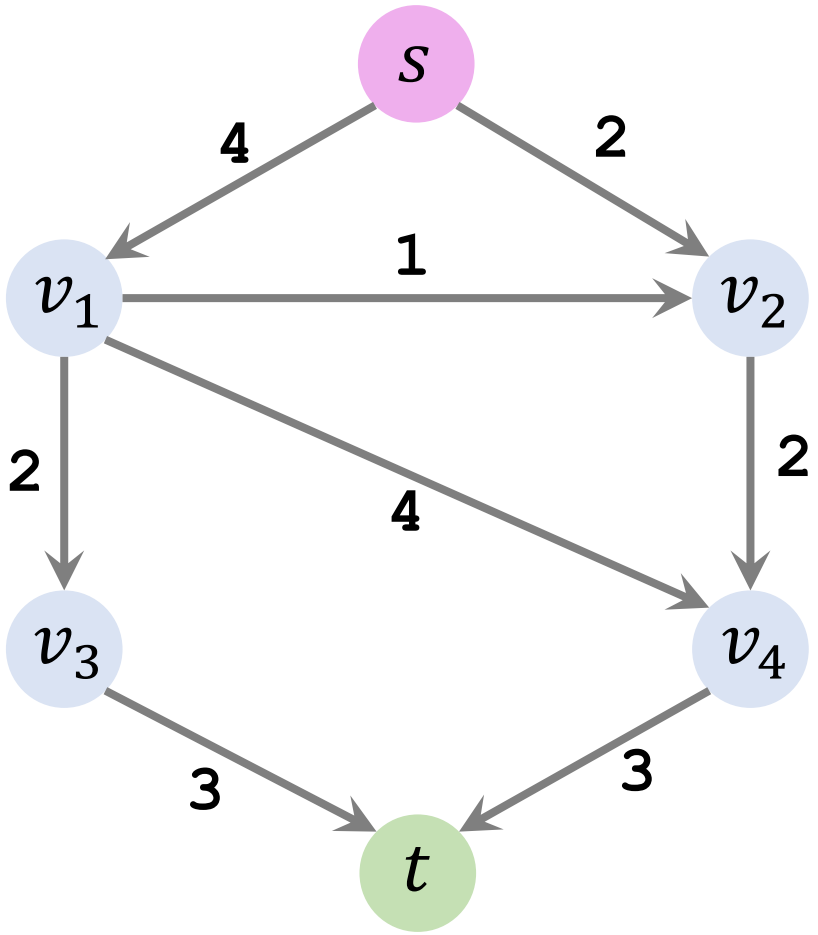
- The amount of max flow is 2.
- The amount of blocking flow is 1.
- Once a bad path is selected, the naïve algorithm cannot make corrections.

Ford-Fulkerson Algorithm

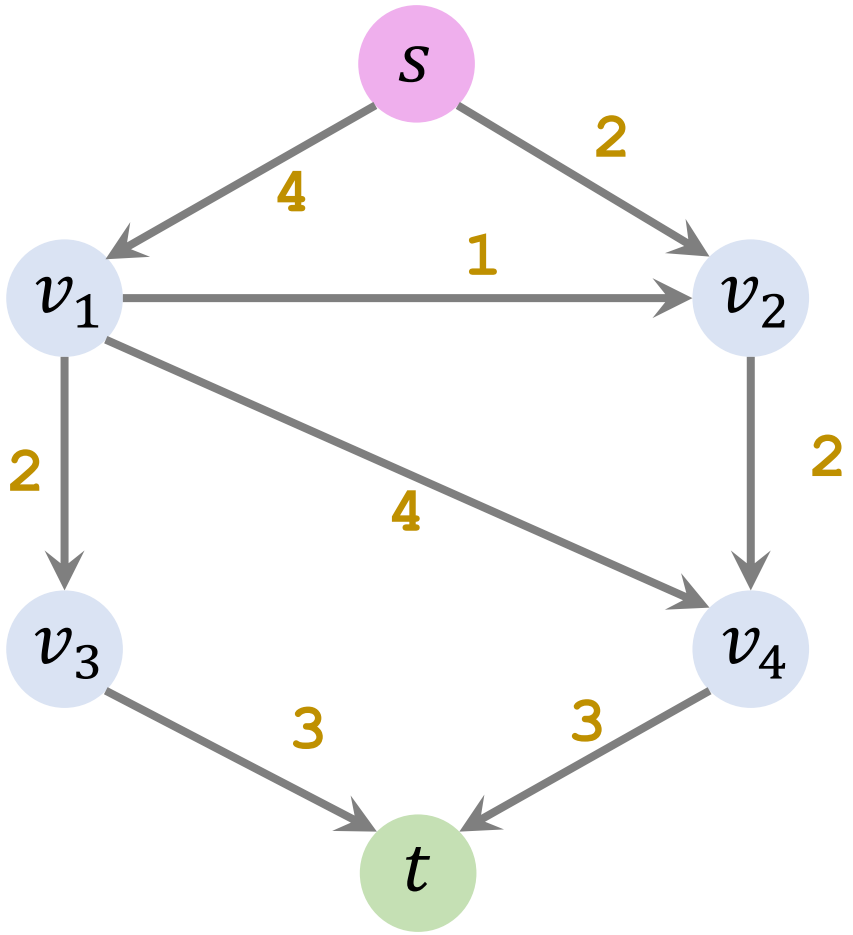
Reference

- L. R. Ford and D. R. Fulkerson. Maximal flow through a network. *Canadian Journal of Mathematics*, 8: 399–404, 1956.

Initialization

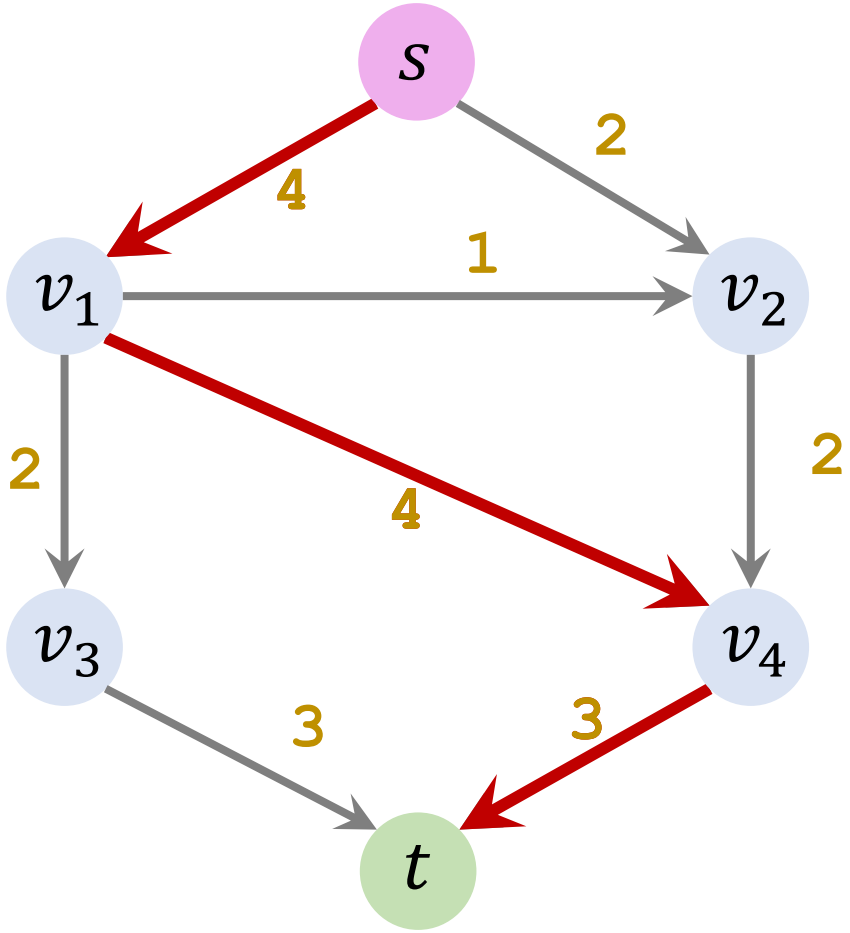


Original Graph



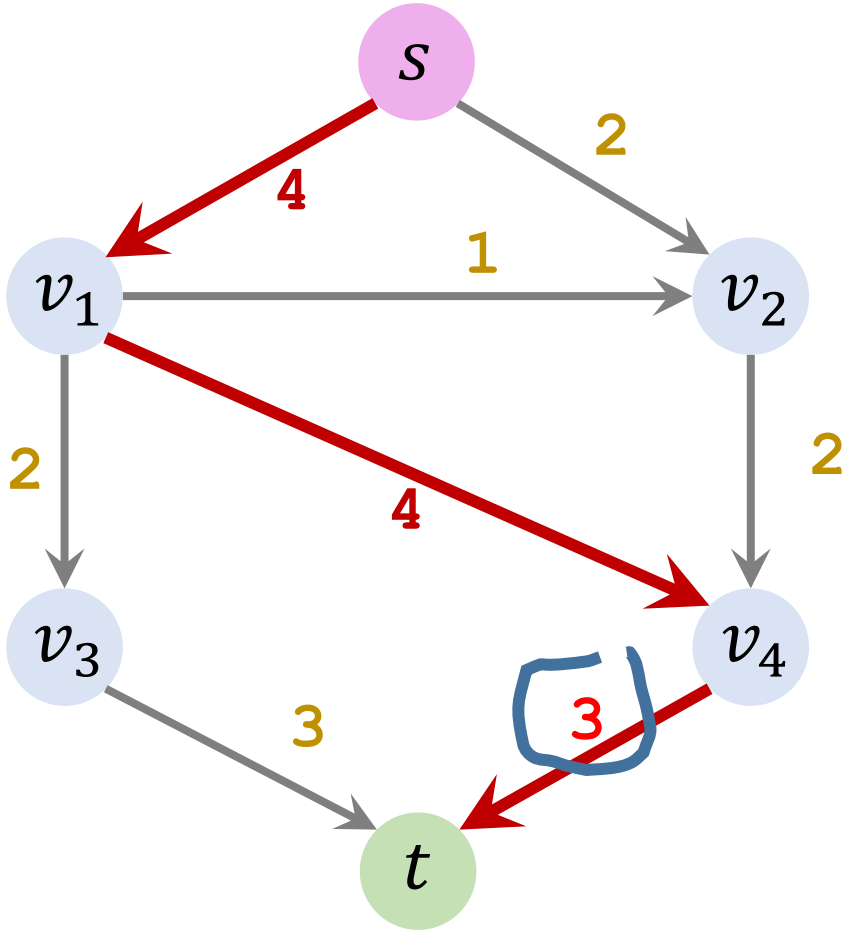
Residual Graph

Iteration 1: Find an augmenting path



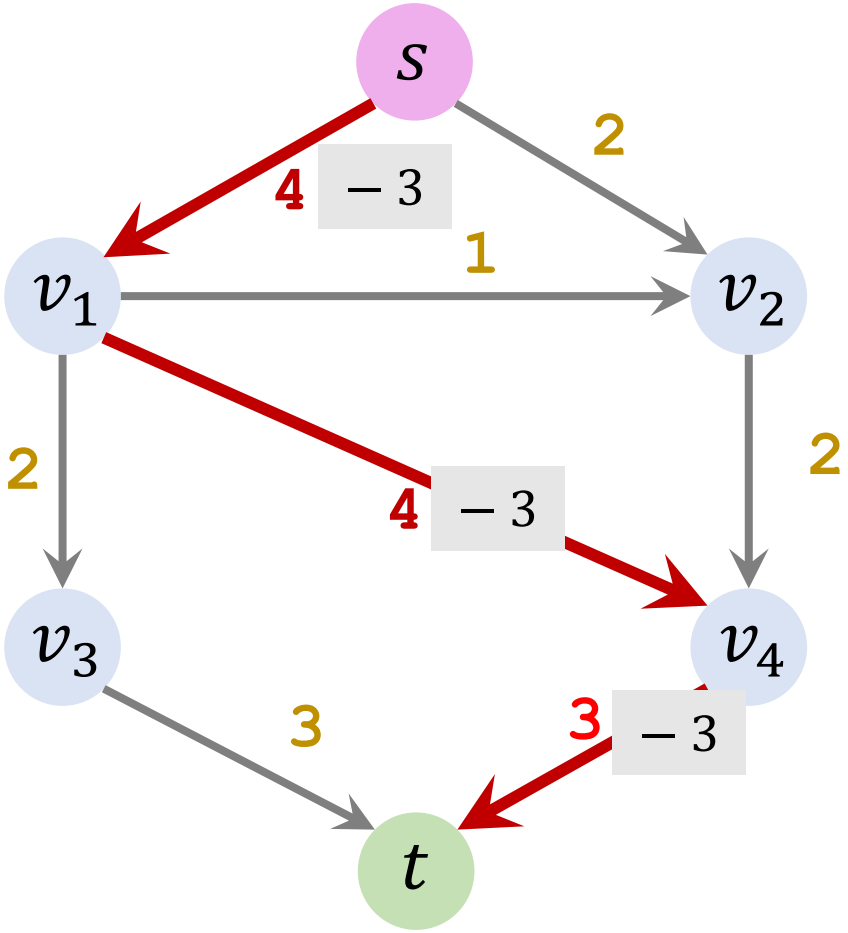
Found path $s \to v_1 \to v_4 \to t$.

Iteration 1: Find an augmenting path

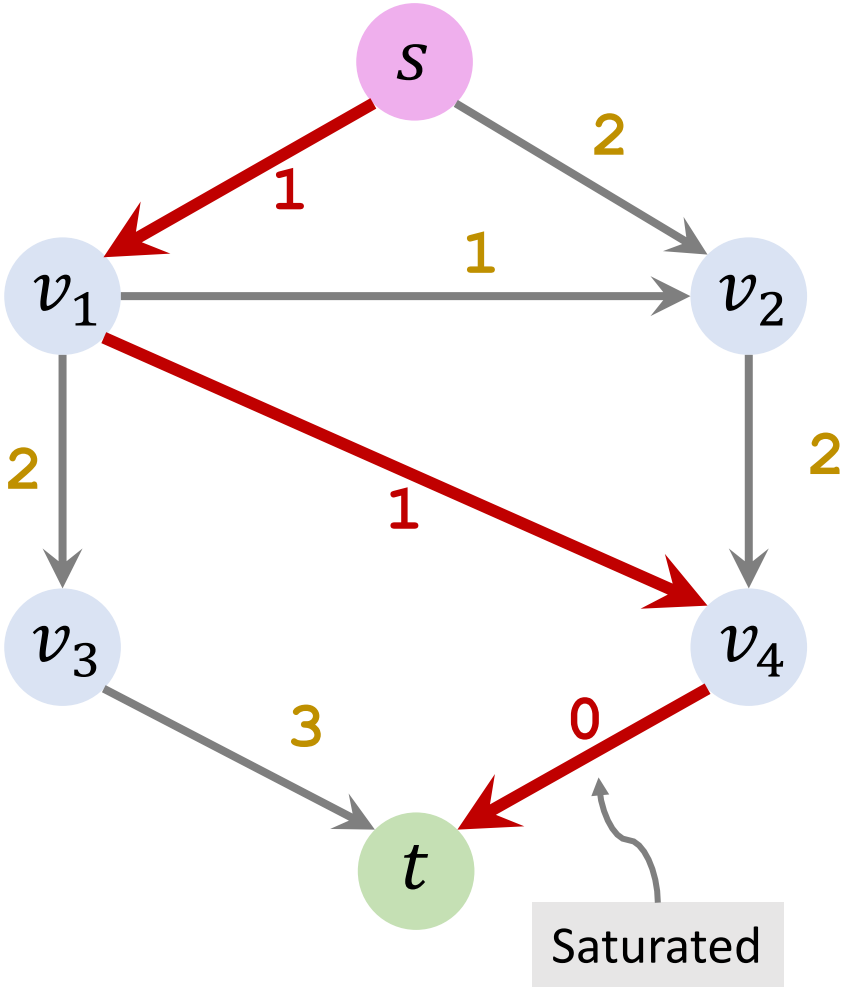


Found path $s \to v_1 \to v_4 \to t$. (Bottleneck capacity $= 3$.)

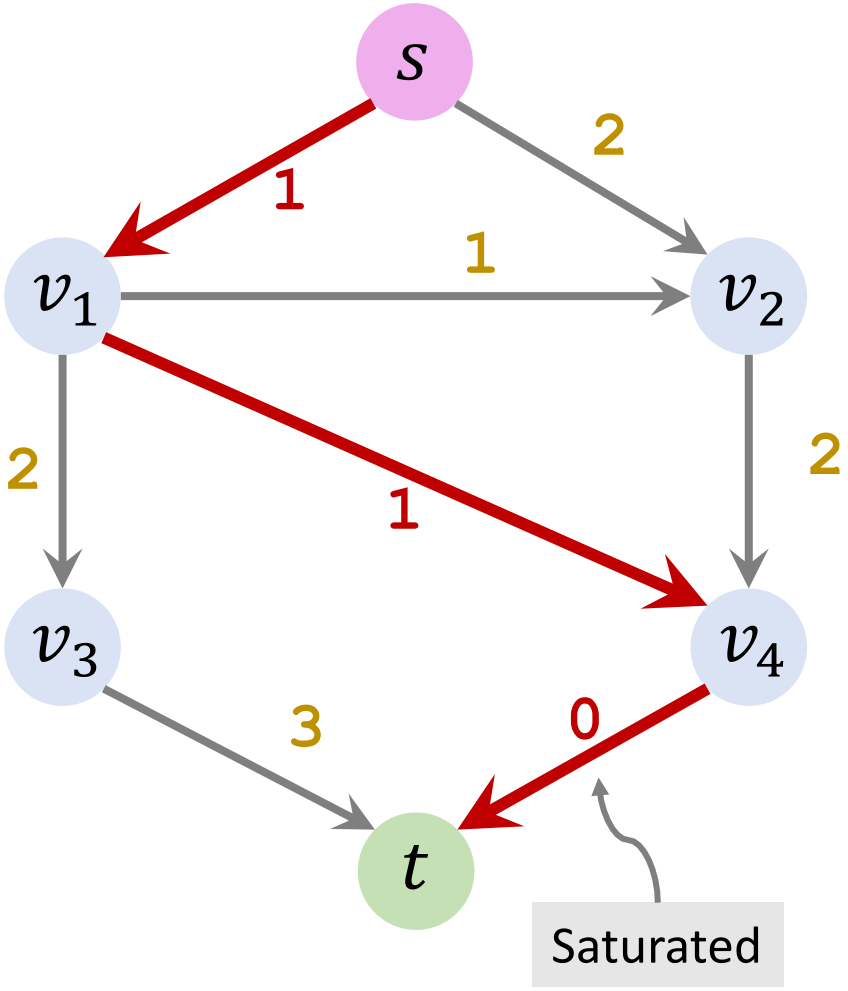
Iteration 1: Update residuals



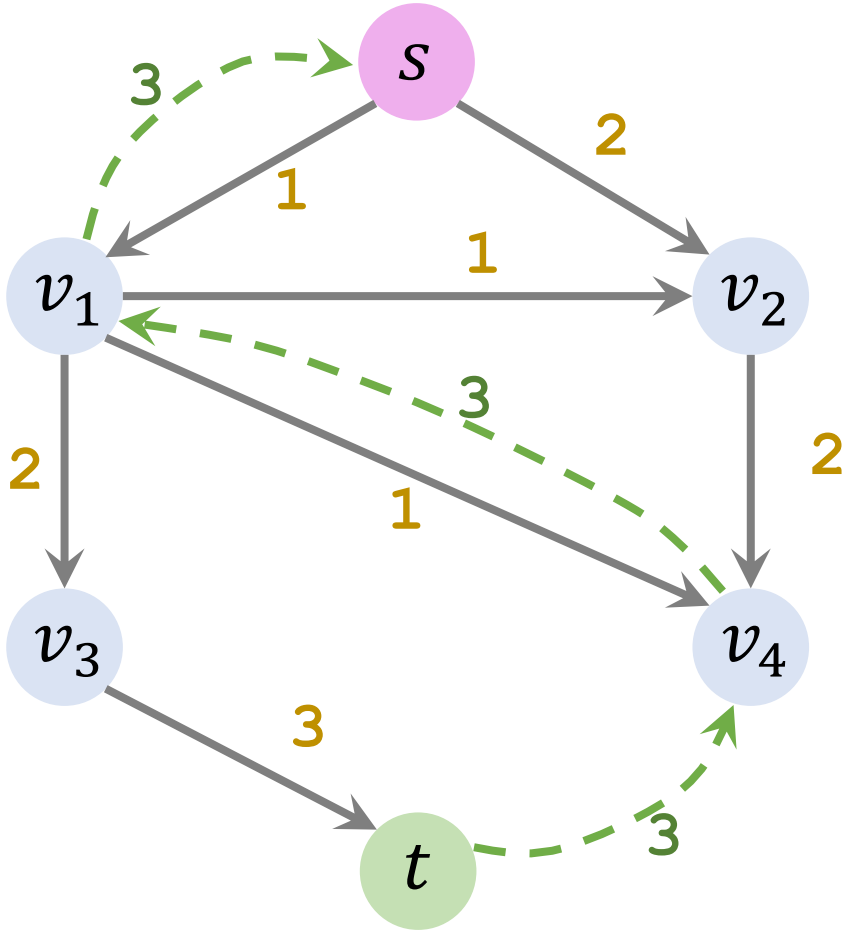
Iteration 1: Update residuals



Iteration 1: Remove saturated edges

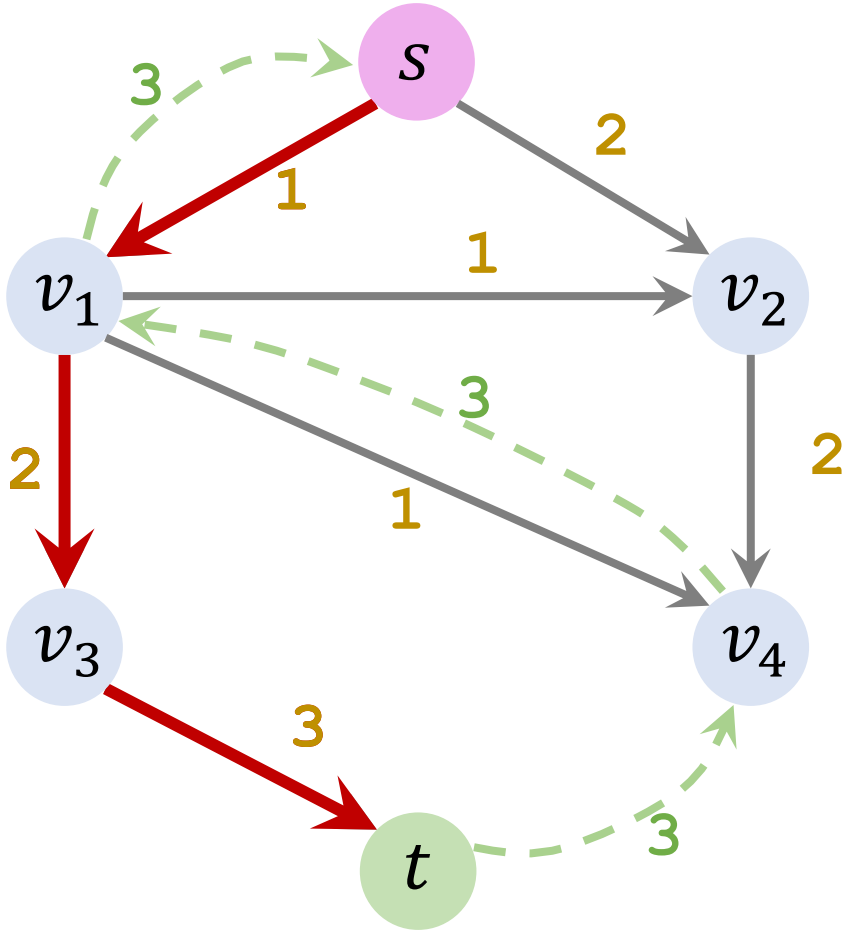


Iteration 1: Add a backward path



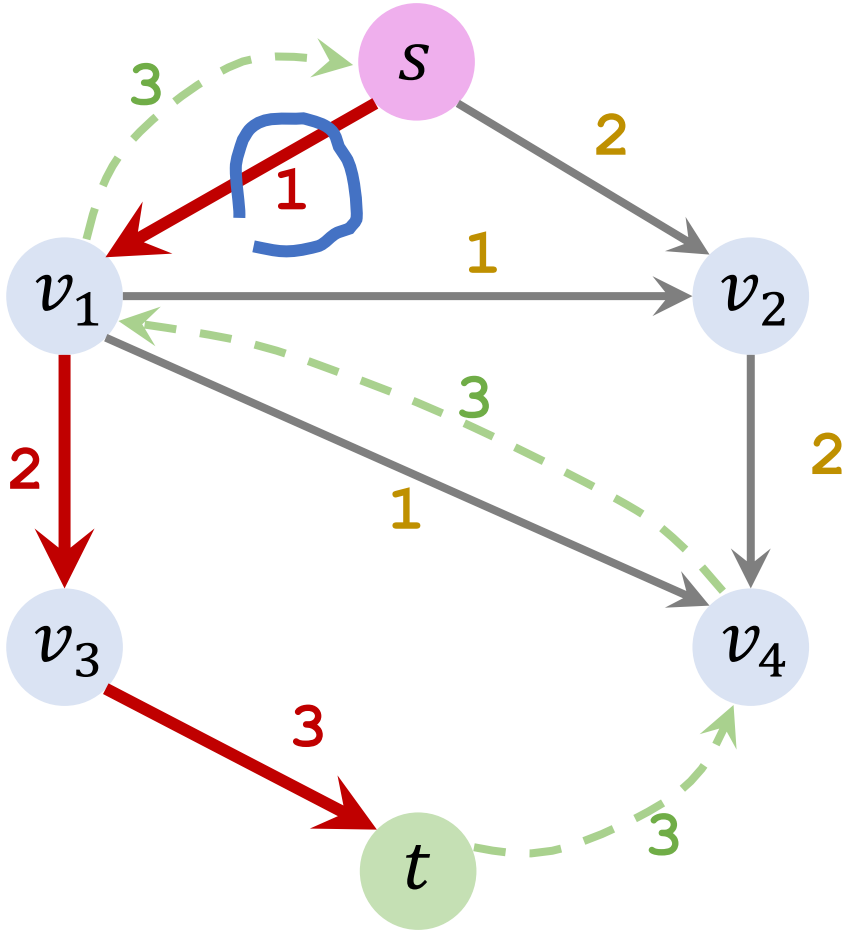
Add path $t \to v_4 \to v_1 \to s$ with weight= 3. (Allow "undoing".)

Iteration 2: Find an augmenting path



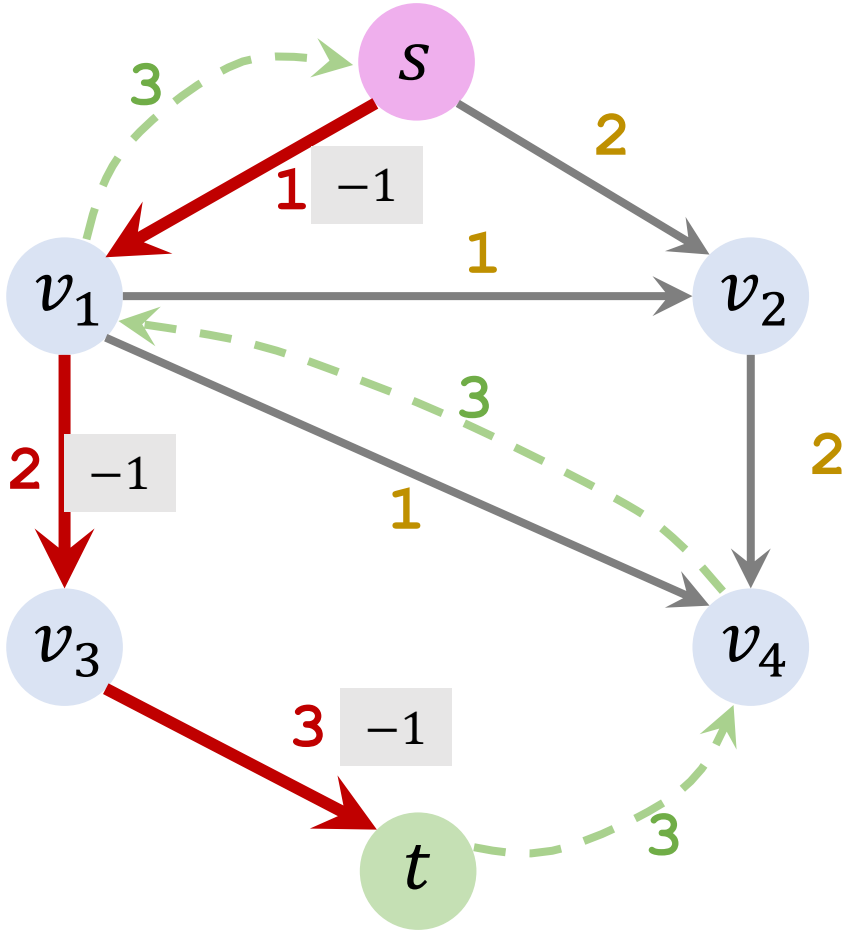
Found path $s \to v_1 \to v_3 \to t$.

Iteration 2: Find an augmenting path

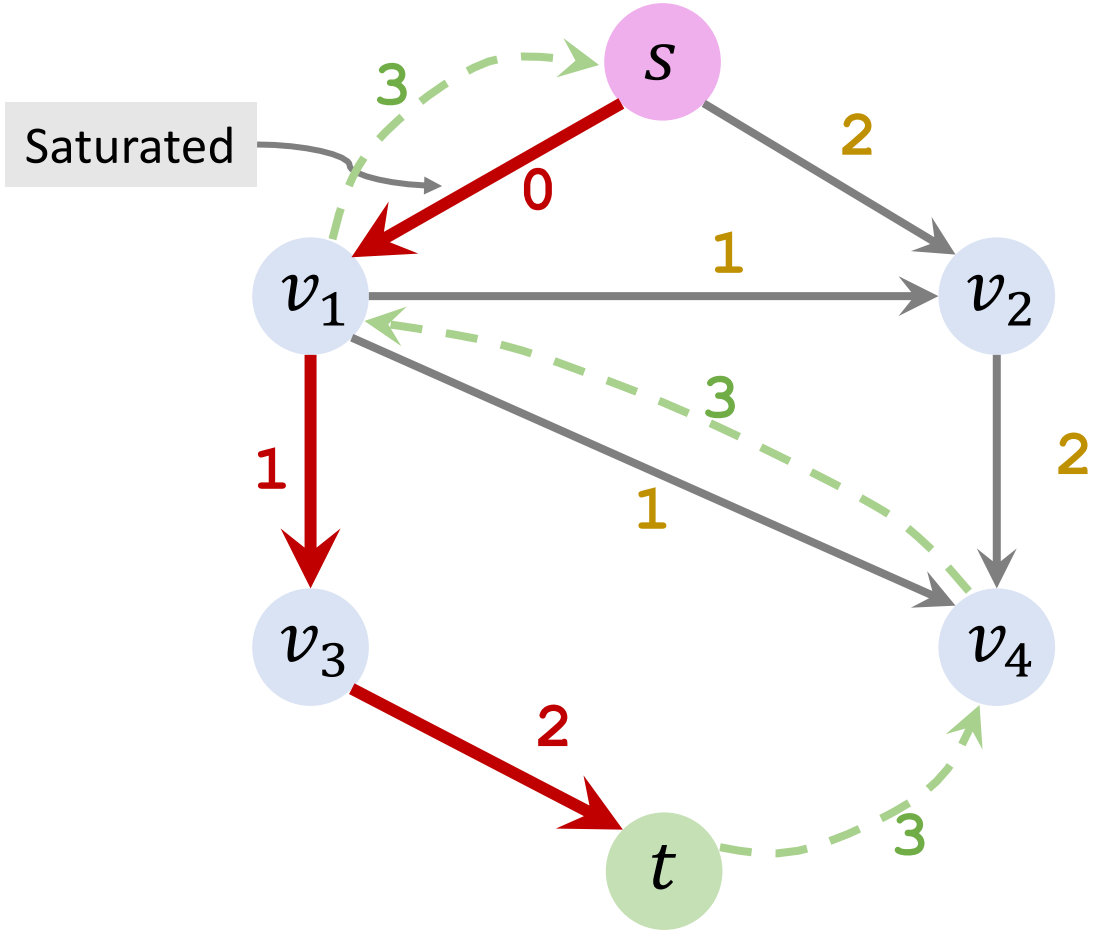


Found path $s \to v_1 \to v_3 \to t$. (Bottleneck capacity $= 1$.)

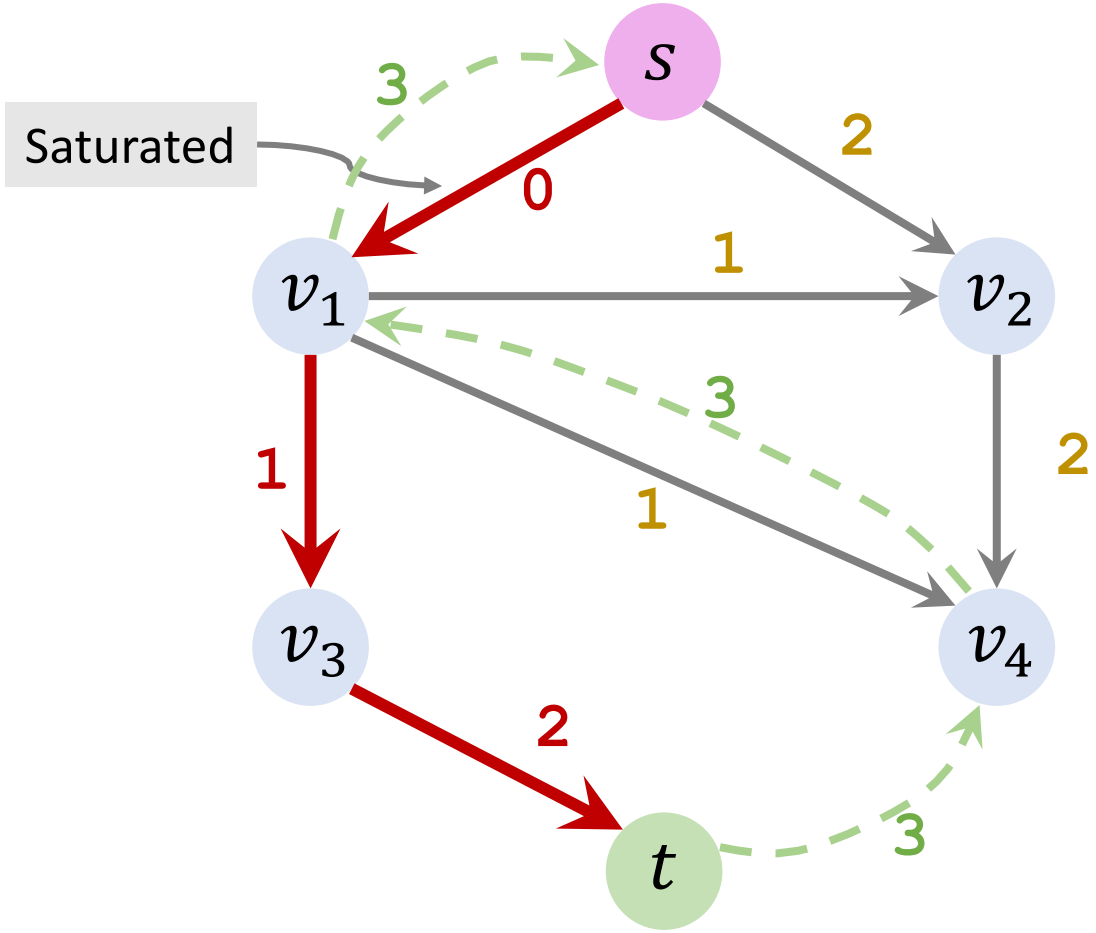
Iteration 2: Update residuals



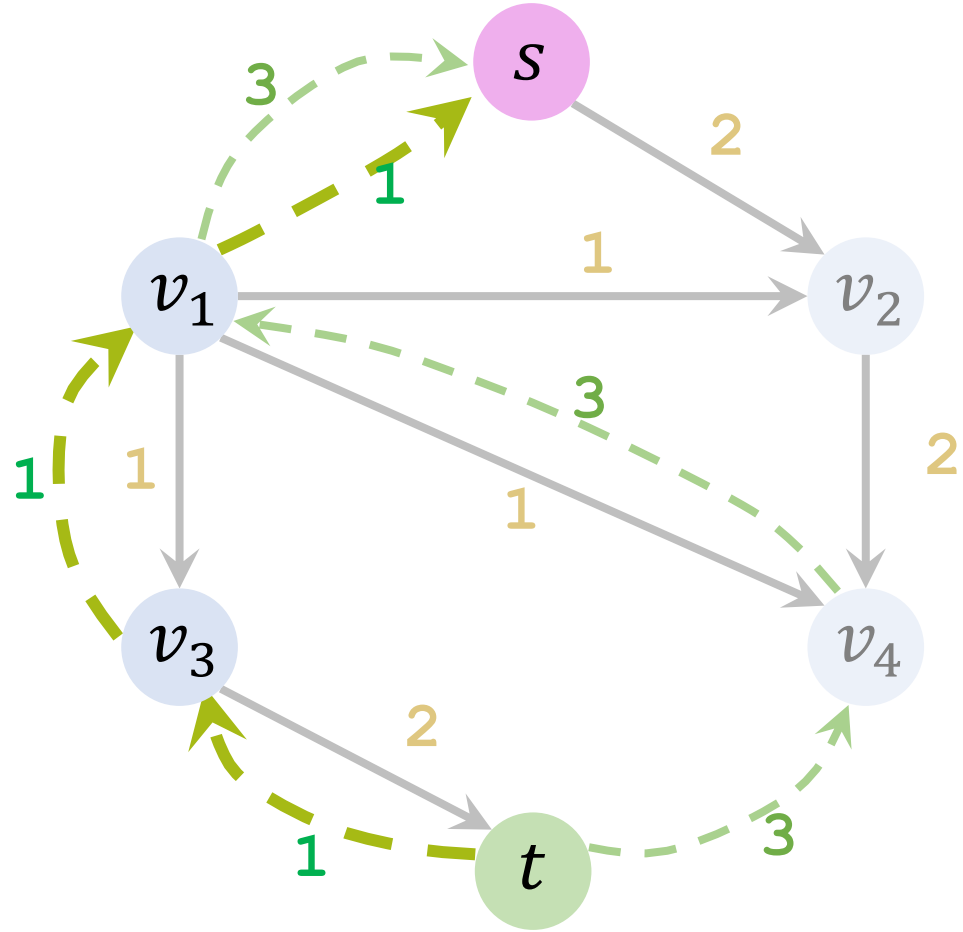
Iteration 2: Update residuals



Iteration 2: Remove saturated edges

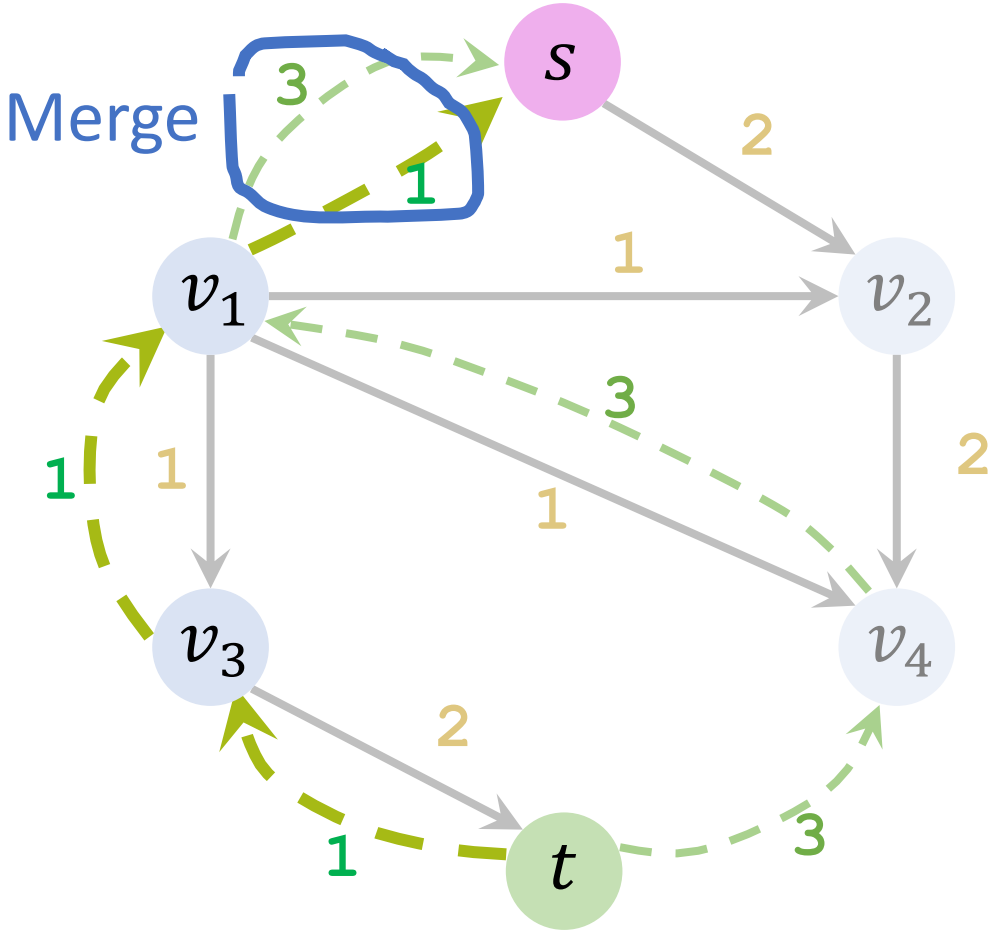


Iteration 2: Add a backward path



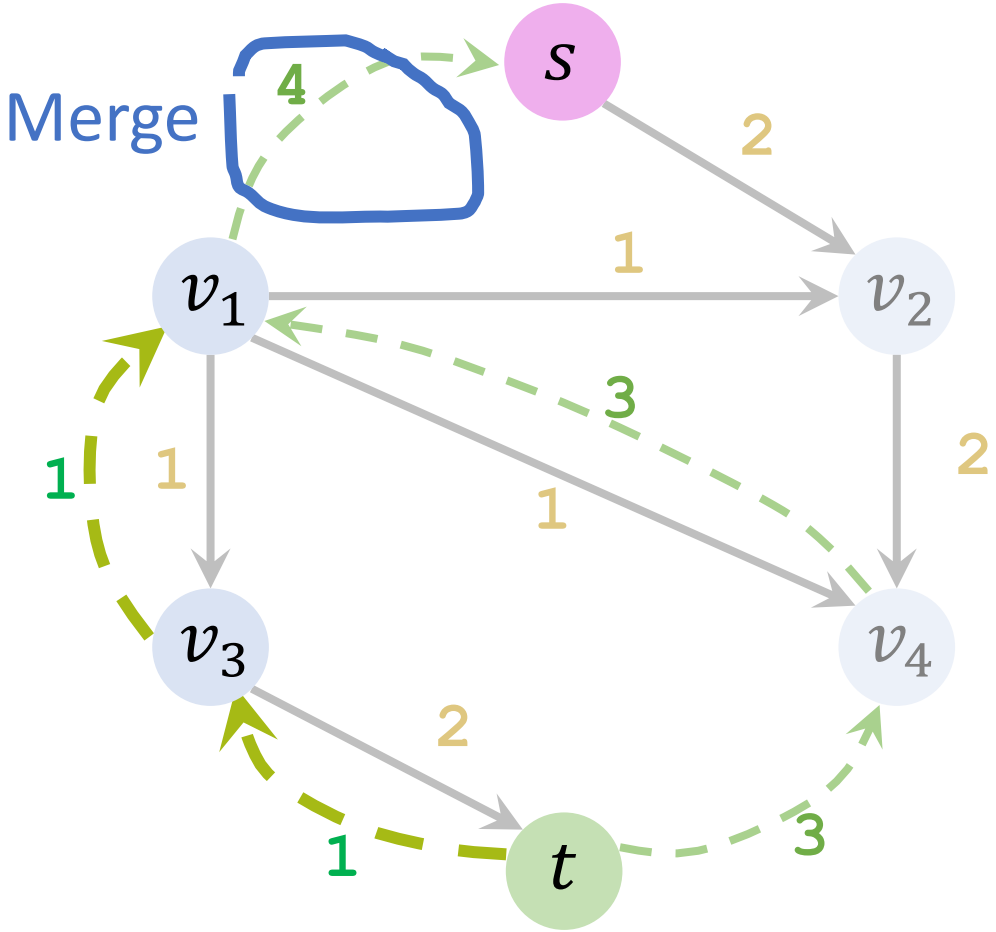
Add path $t \to v_3 \to v_1 \to s$ with weight $= 1$.

Iteration 2: Add a backward path



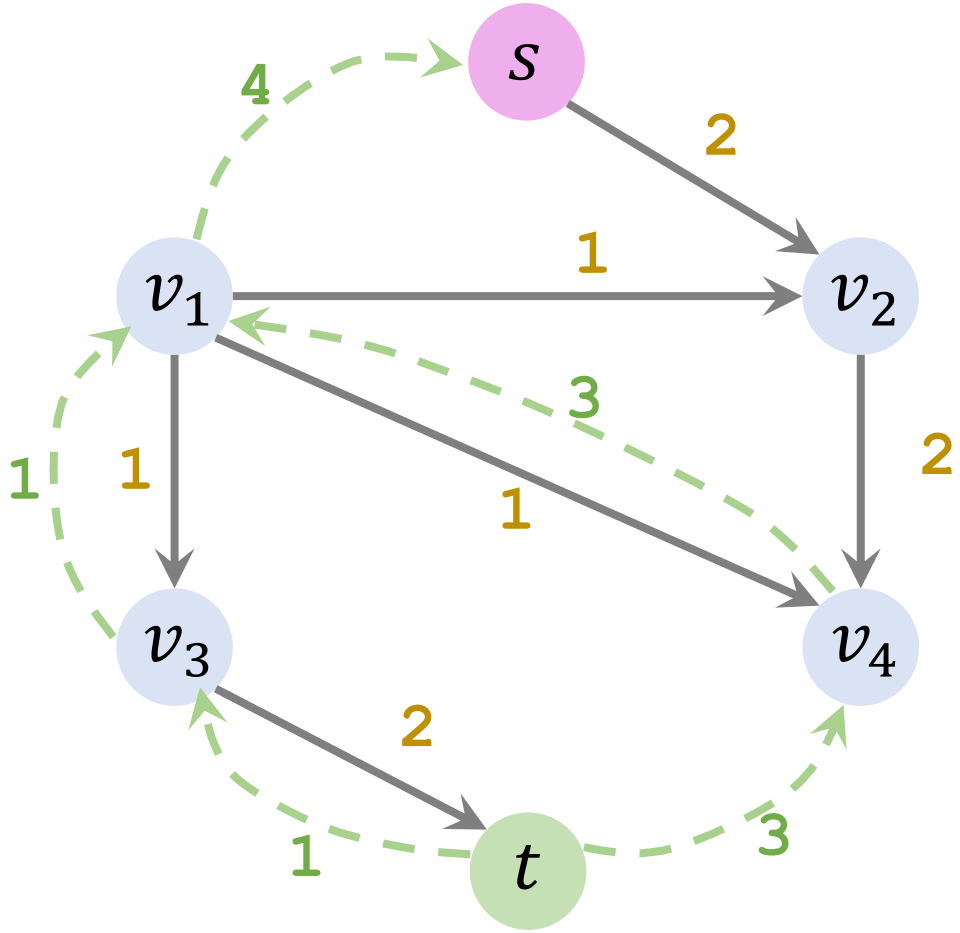
Add path $t \to v_3 \to v_1 \to s$ with weight $= 1$.

Iteration 2: Add a backward path

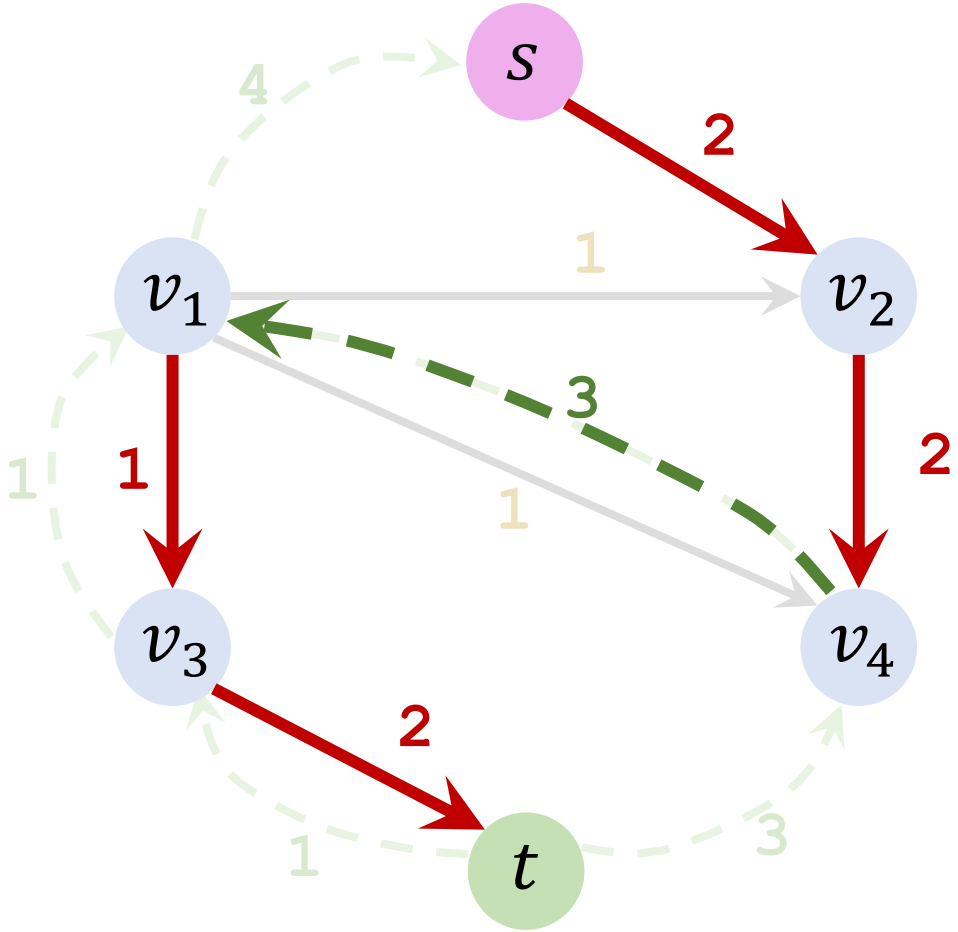


Add path $t \to v_3 \to v_1 \to s$ with weight $= 1$.

Iteration 3: Find an augmenting path

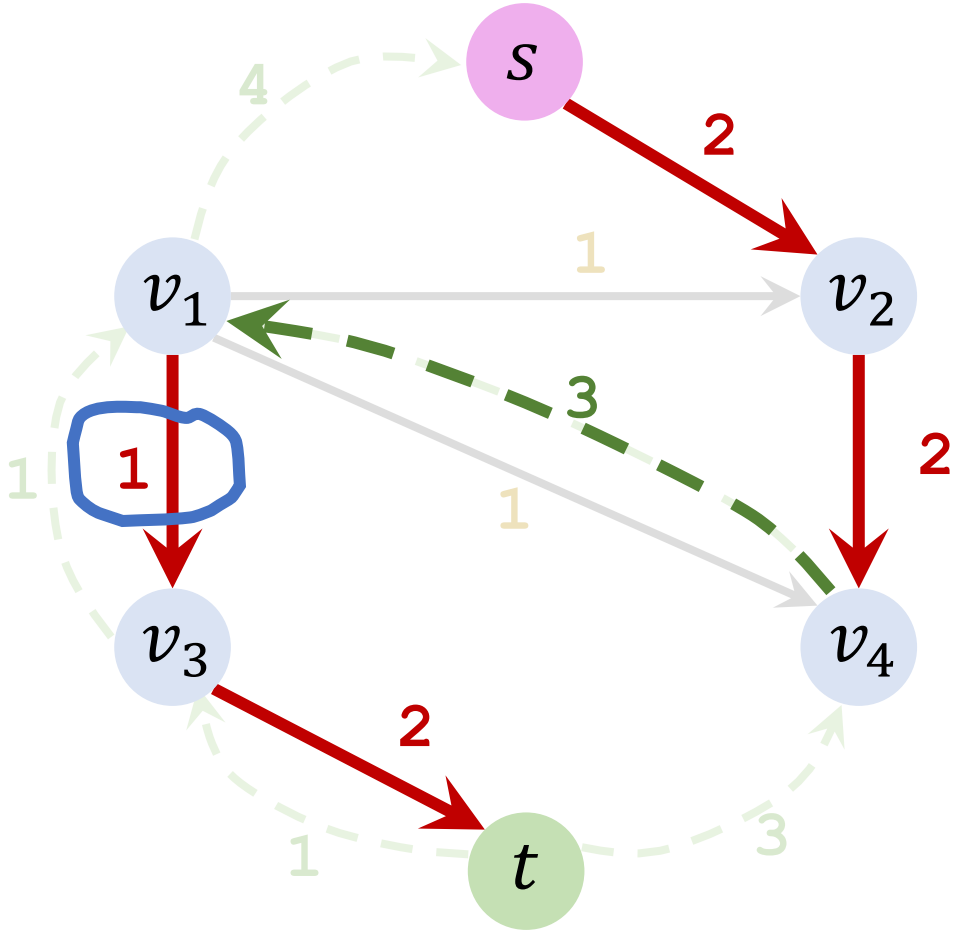


Iteration 3: Find an augmenting path



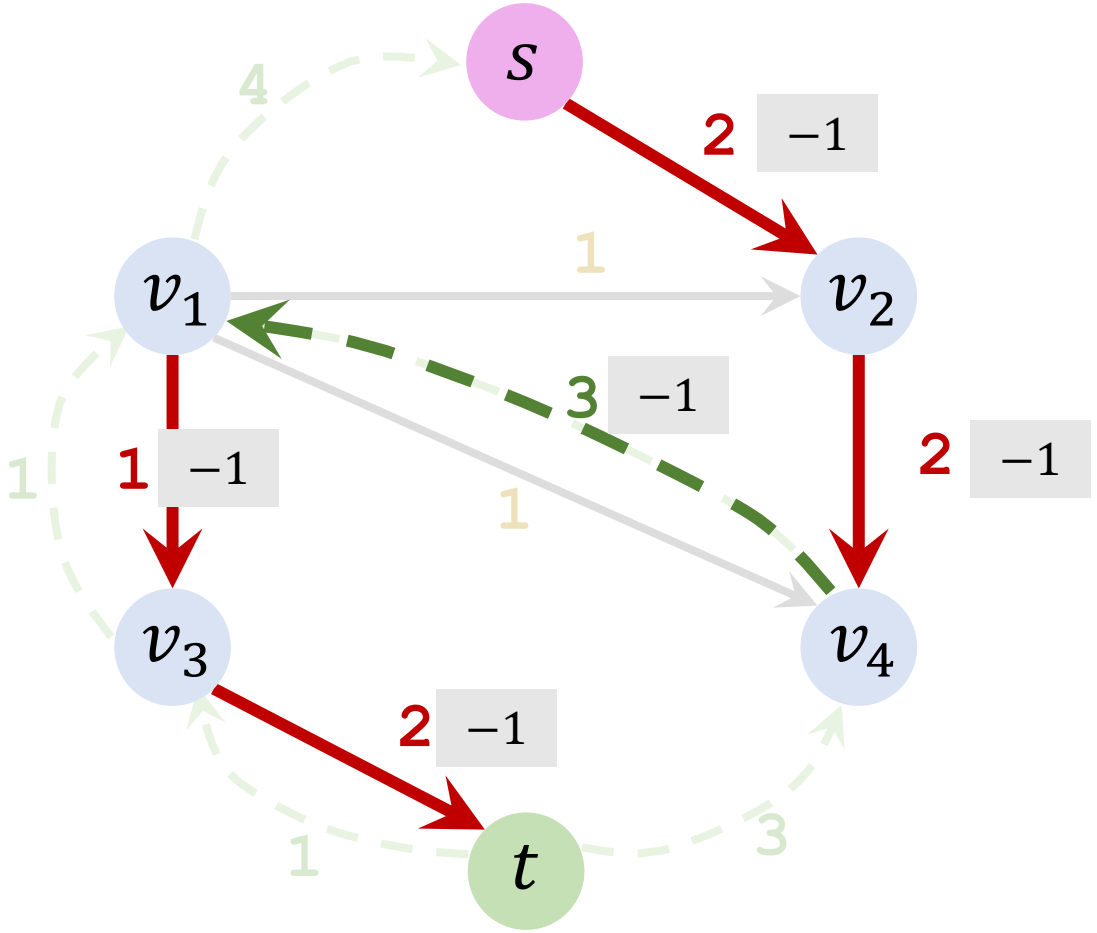
Found path $s \to v_2 \to v_4 \to v_1 \to v_3 \to t$.

Iteration 3: Find an augmenting path

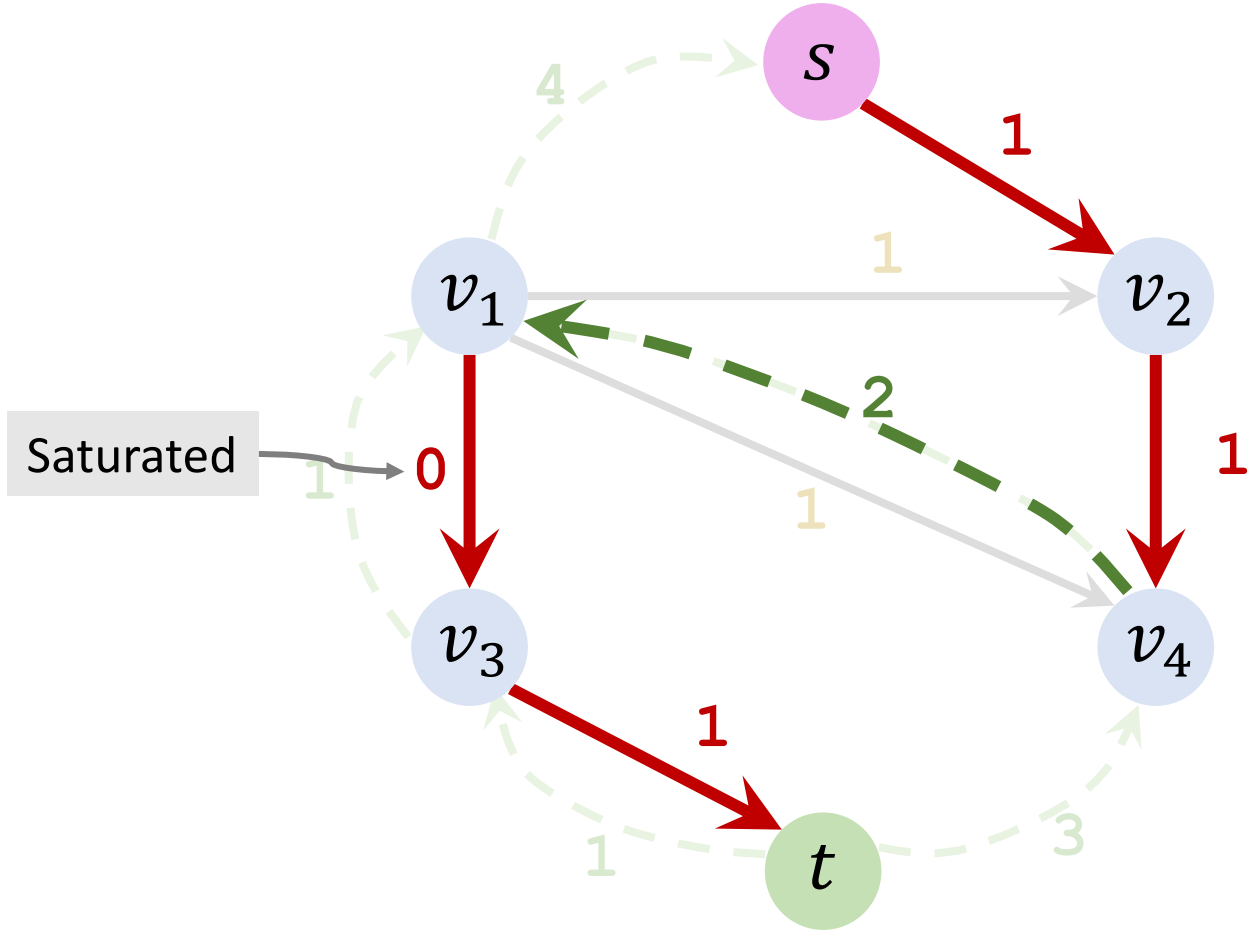


Found path $s \to v_2 \to v_4 \to v_1 \to v_3 \to t$. (Bottleneck capacity $= 1$.)

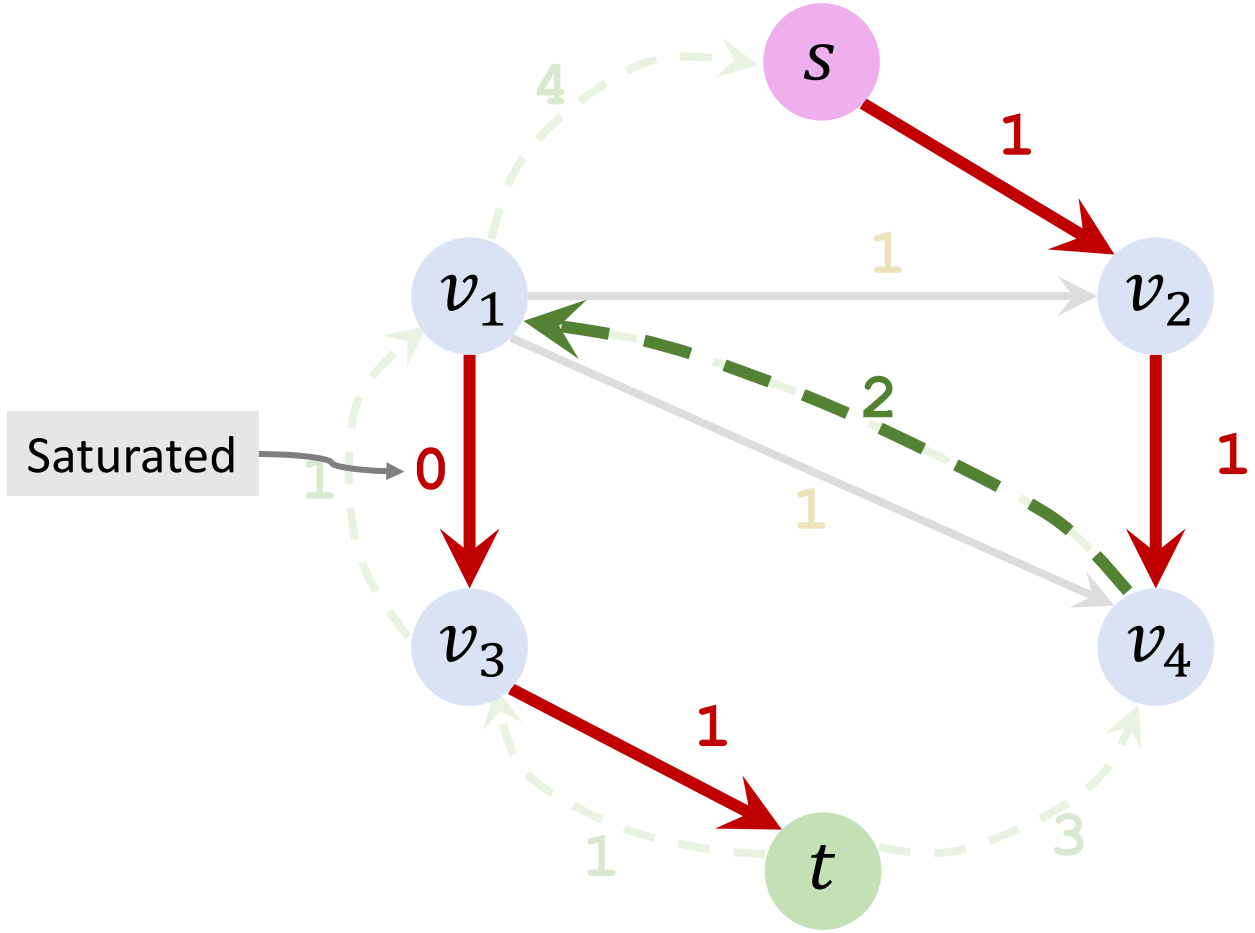
Iteration 3: Update residuals



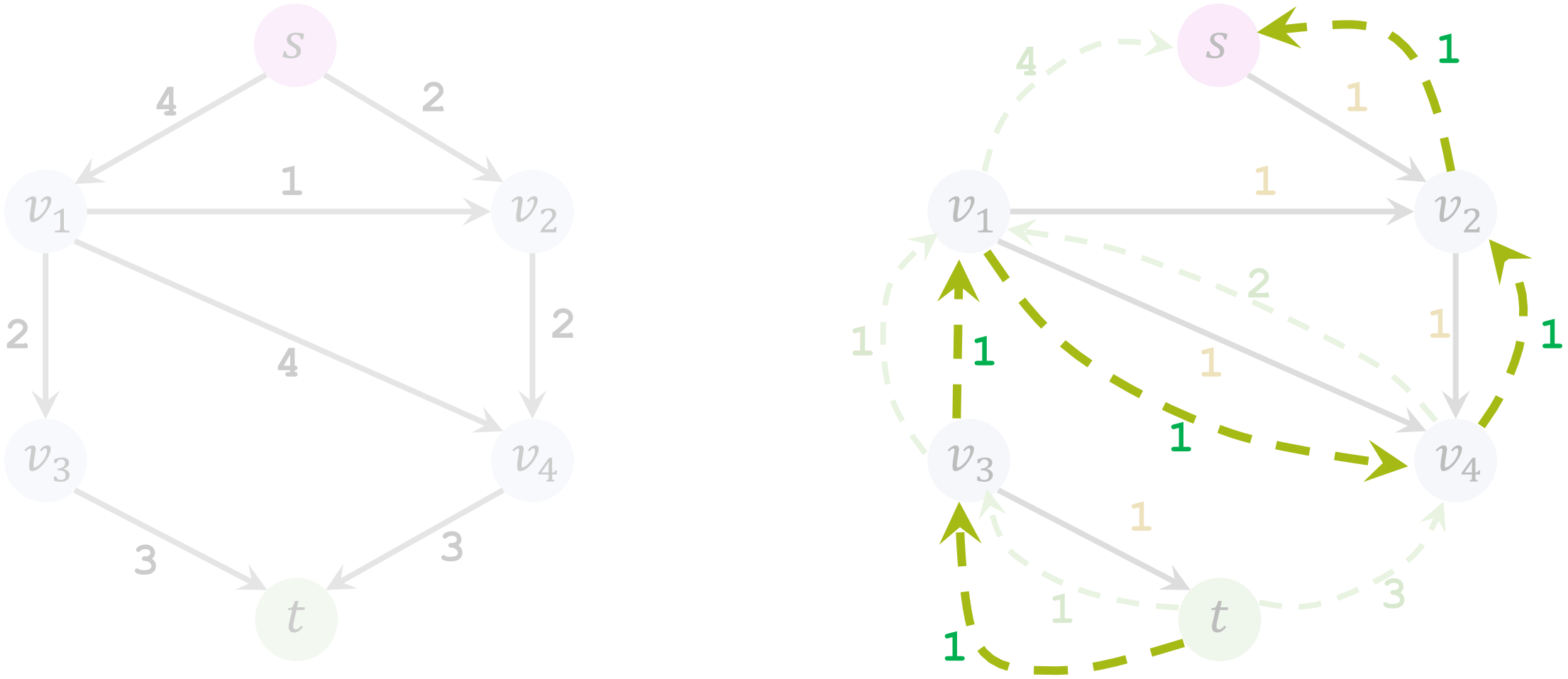
Iteration 3: Update residuals



Iteration 3: Remove saturated edges

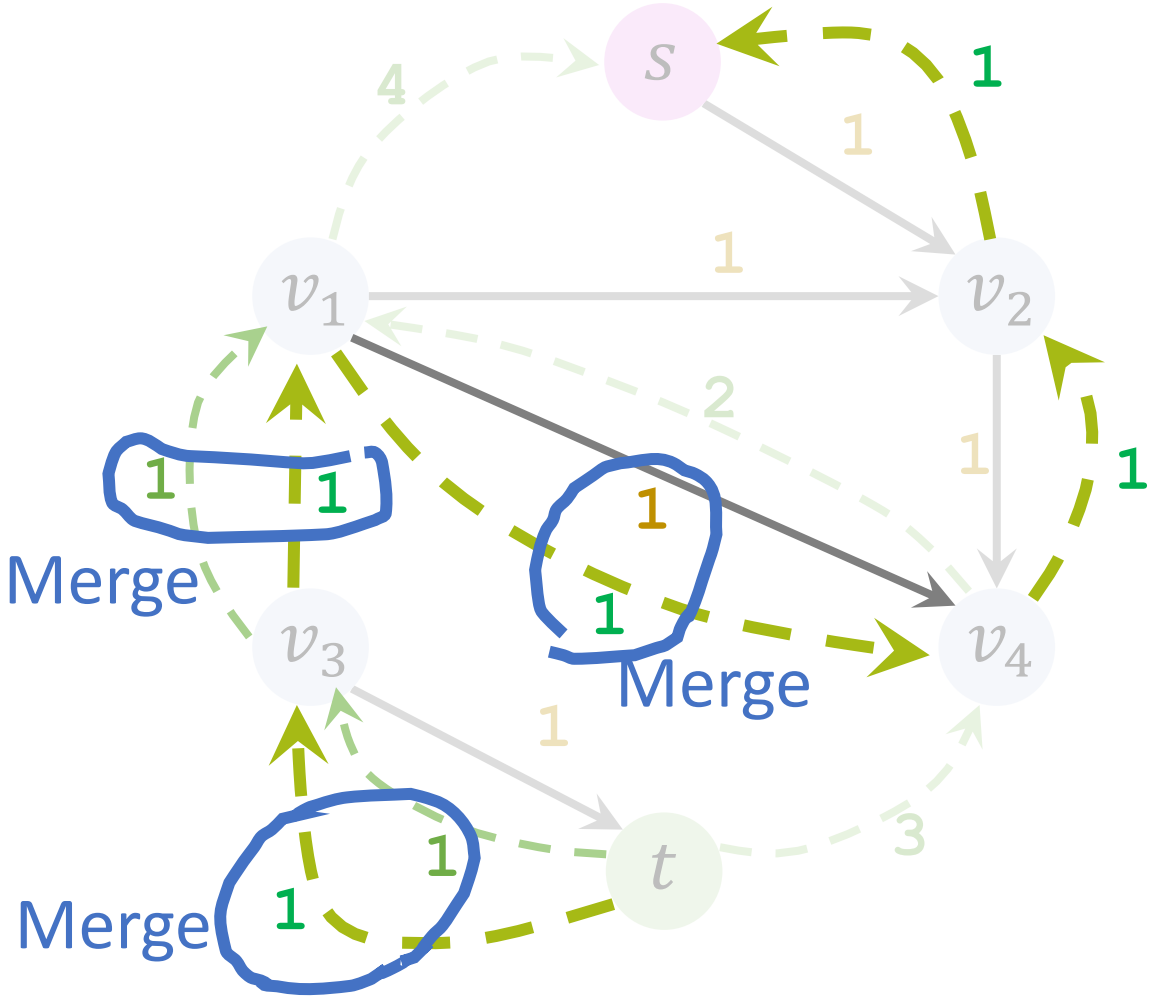


Iteration 3: Add a backward path

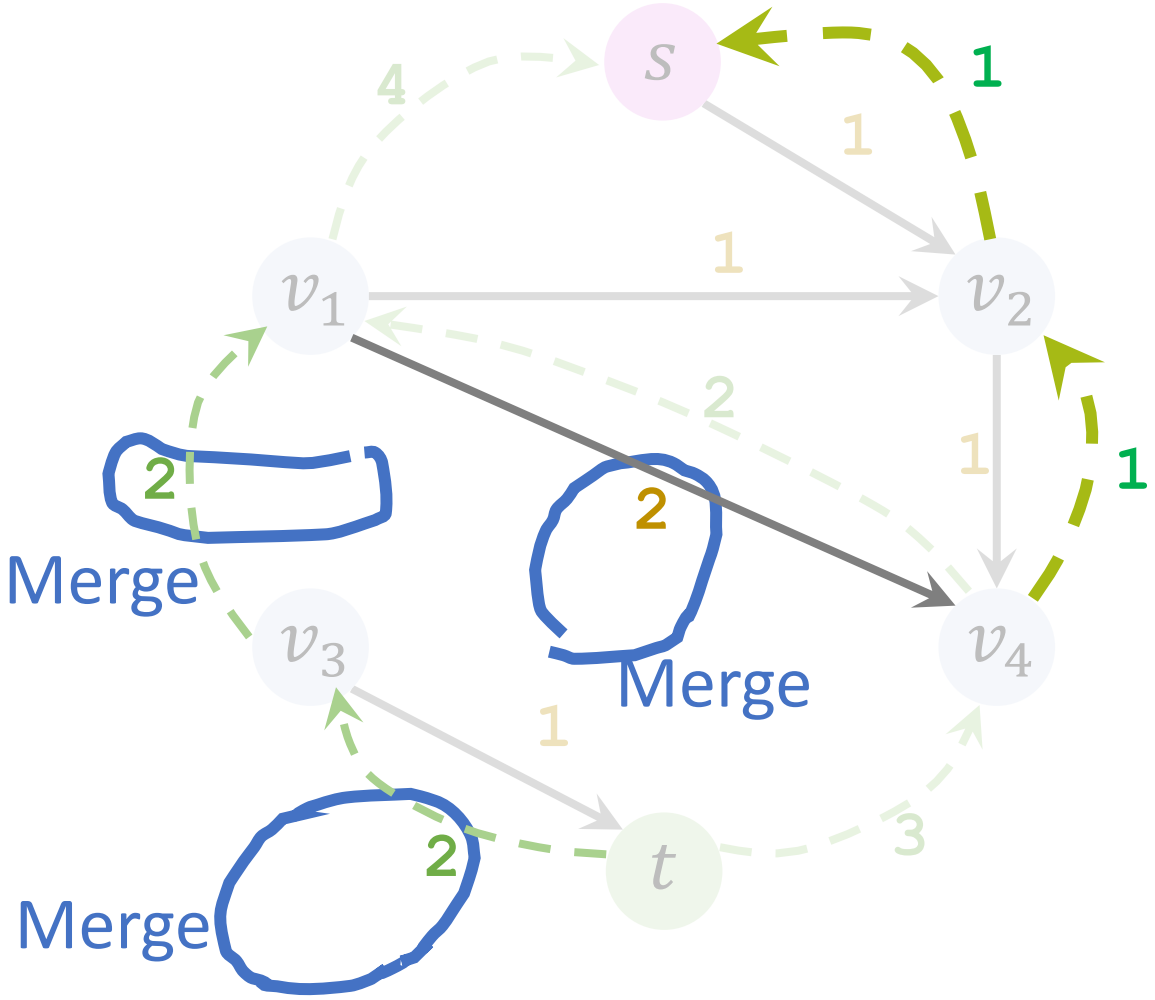


Add backward path $t \to v_3 \to v_1 \to v_4 \to v_2 \to s$ with weight $= 1$.

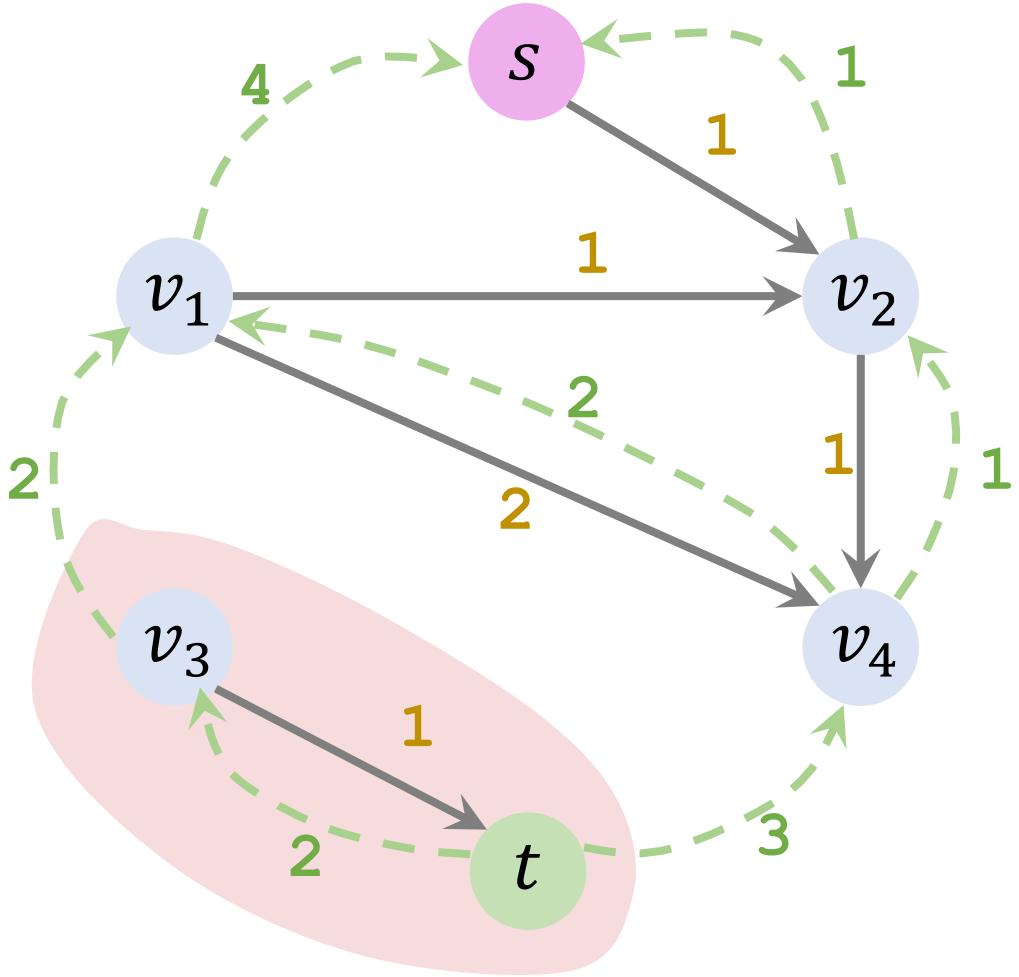
Iteration 3: Add a backward path



Iteration 3: Add a backward path

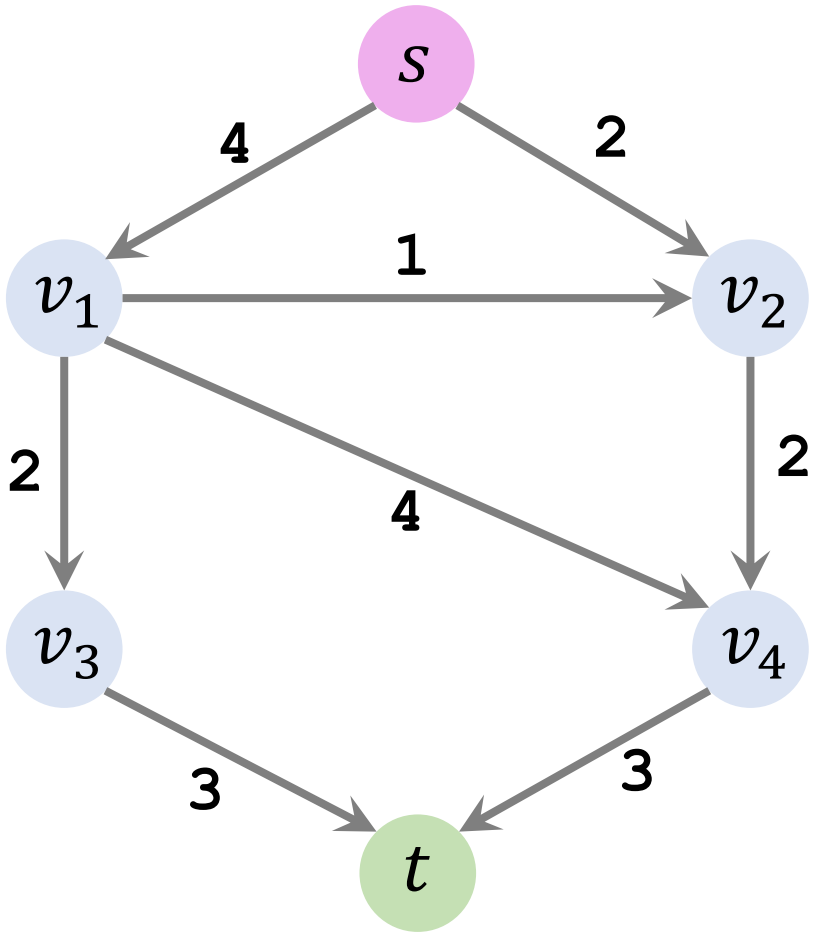


Iteration 4: Find an augmenting path

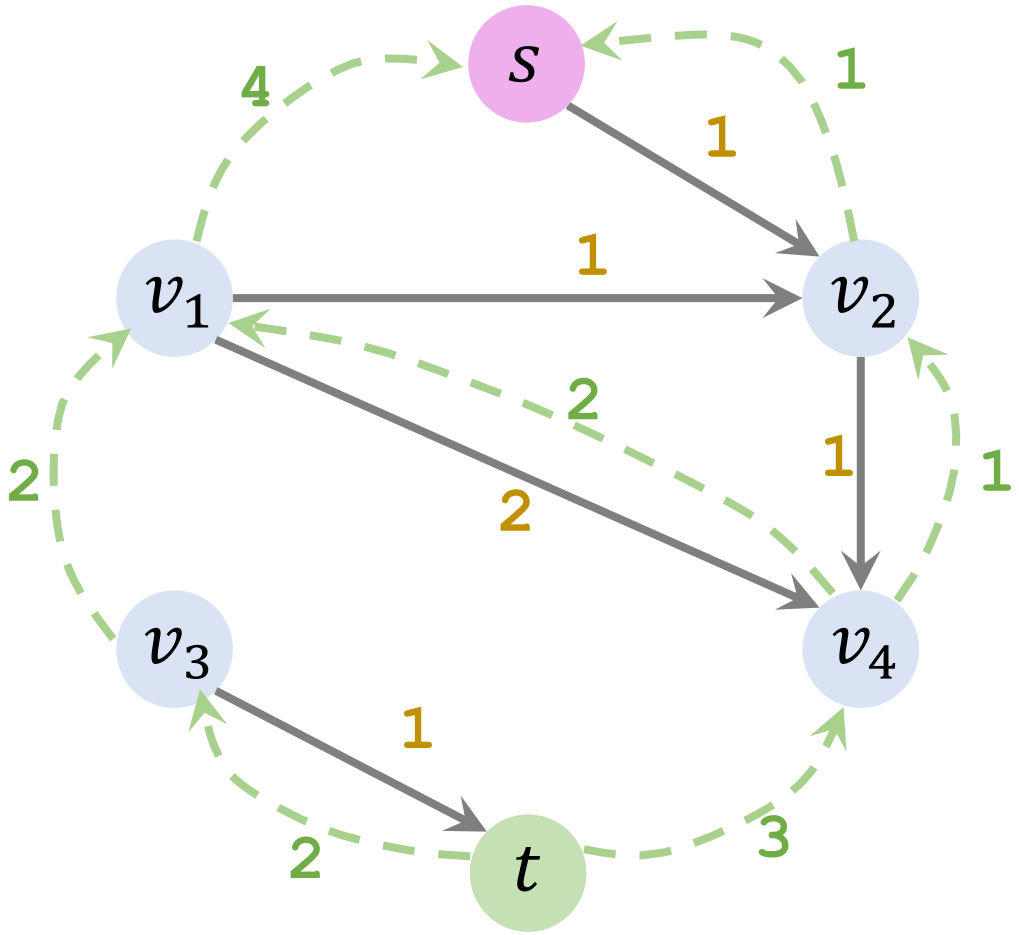


Cannot find any path from source to sink.

End of Procedure

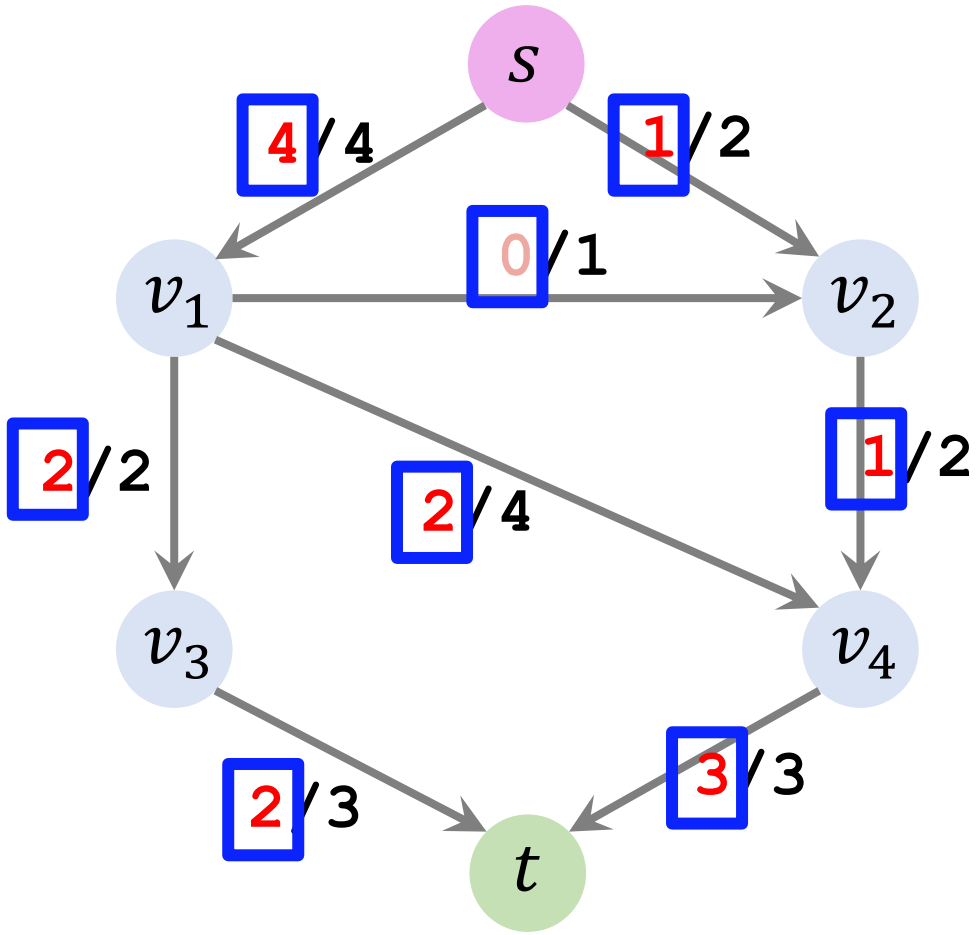


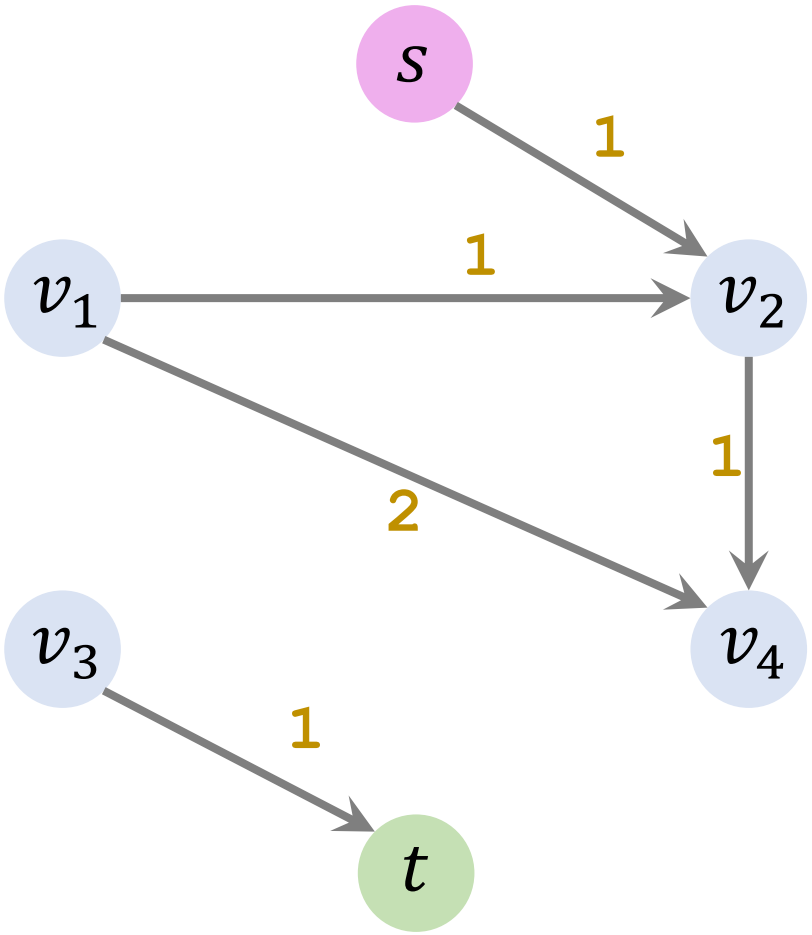
Original Graph



Residual Graph

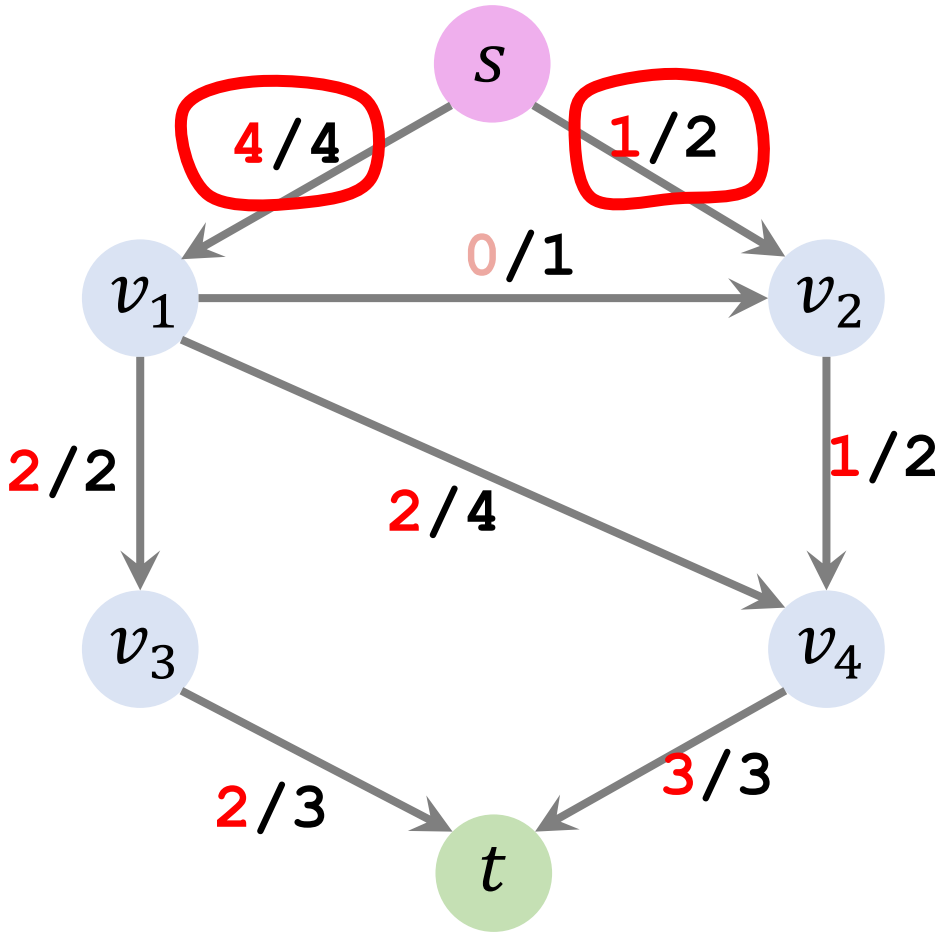
End of Procedure





Flow = Capacity − Residual.

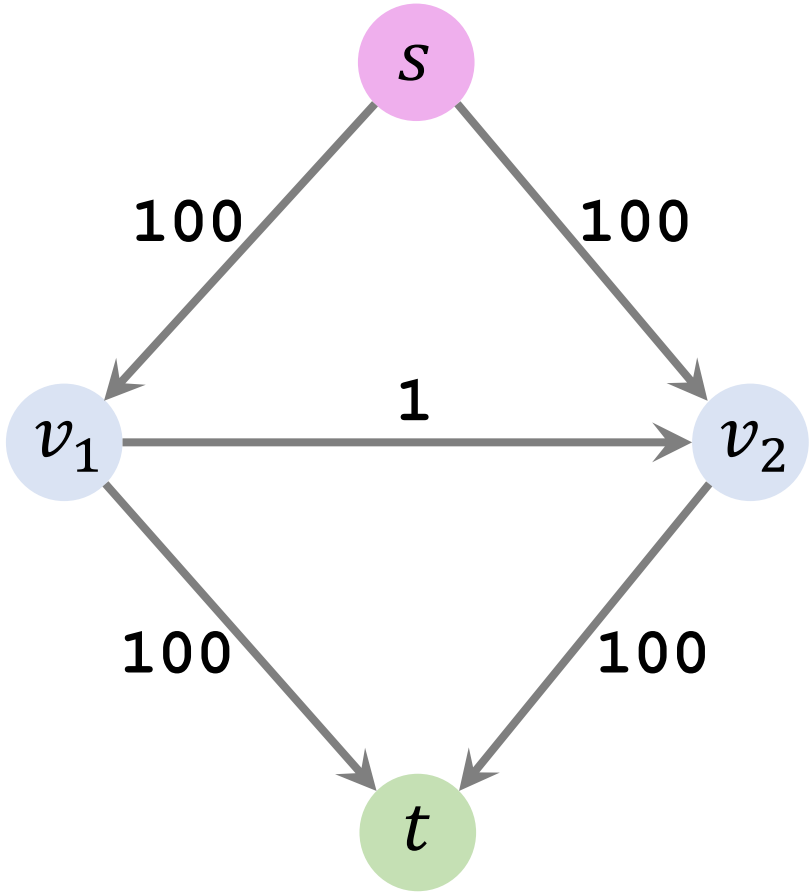
End of Procedure



Amount of Max Flow = 5

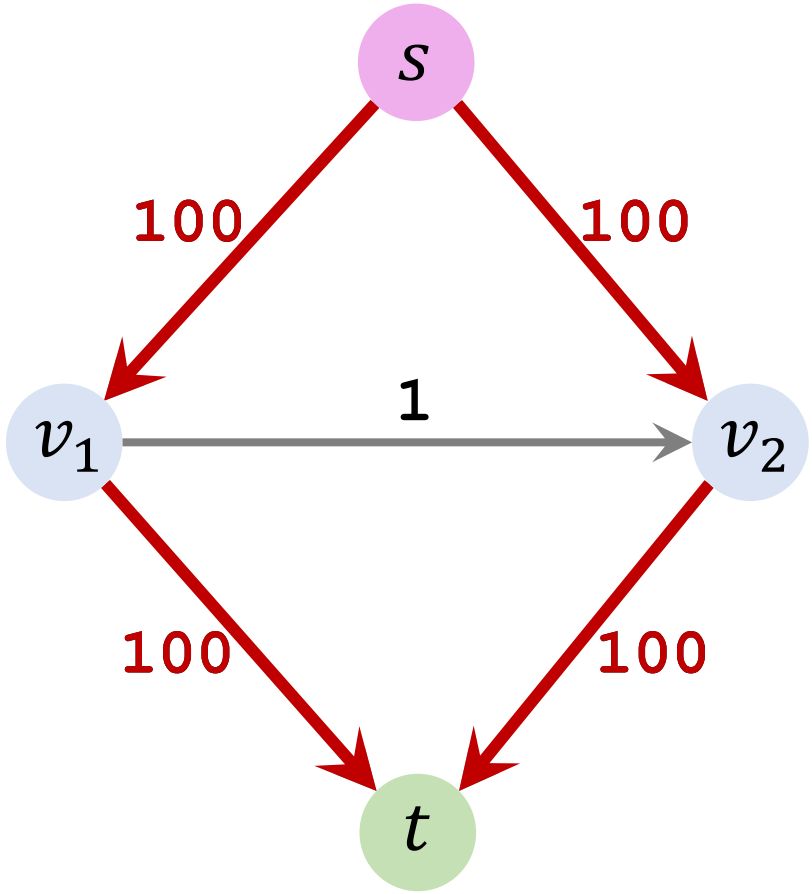
Worst-Case Time Complexity

A bad case for Ford-Fulkerson algorithm



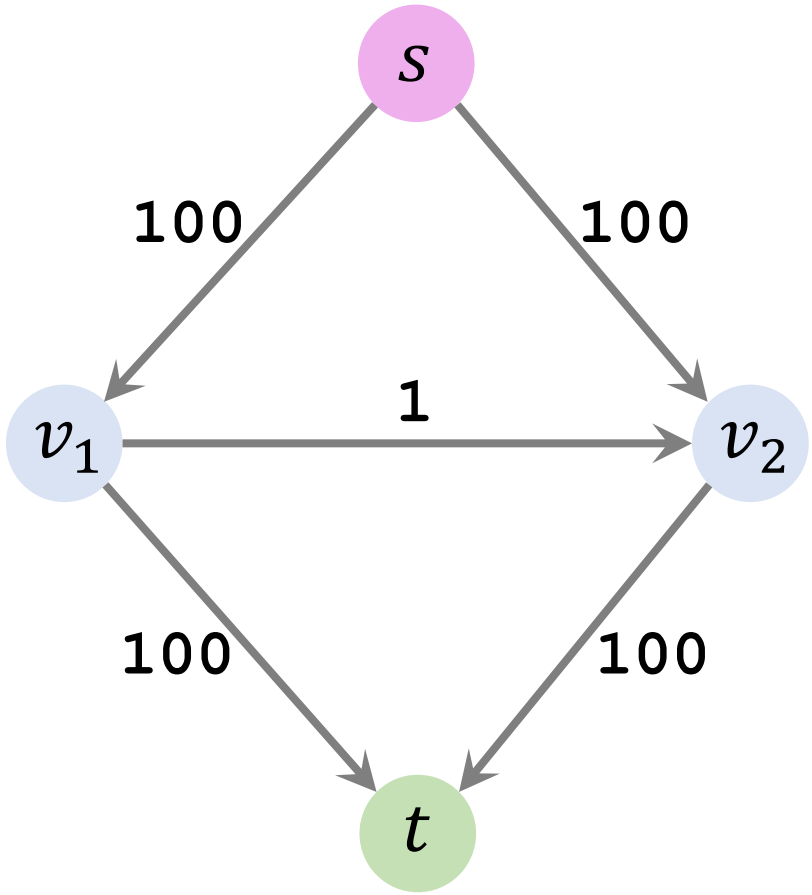
- The amount of the max flow is 200.

A bad case for Ford-Fulkerson algorithm

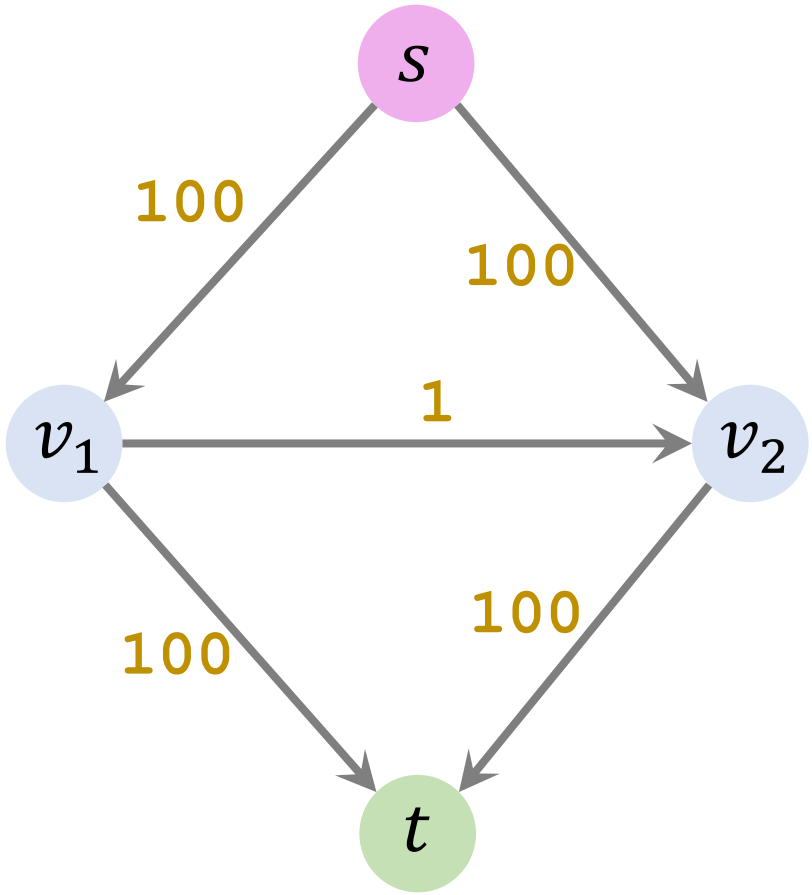


- The amount of the max flow is 200.
- Ford-Fulkerson algorithm may take 200 iterations to find the max flow.

Initialization

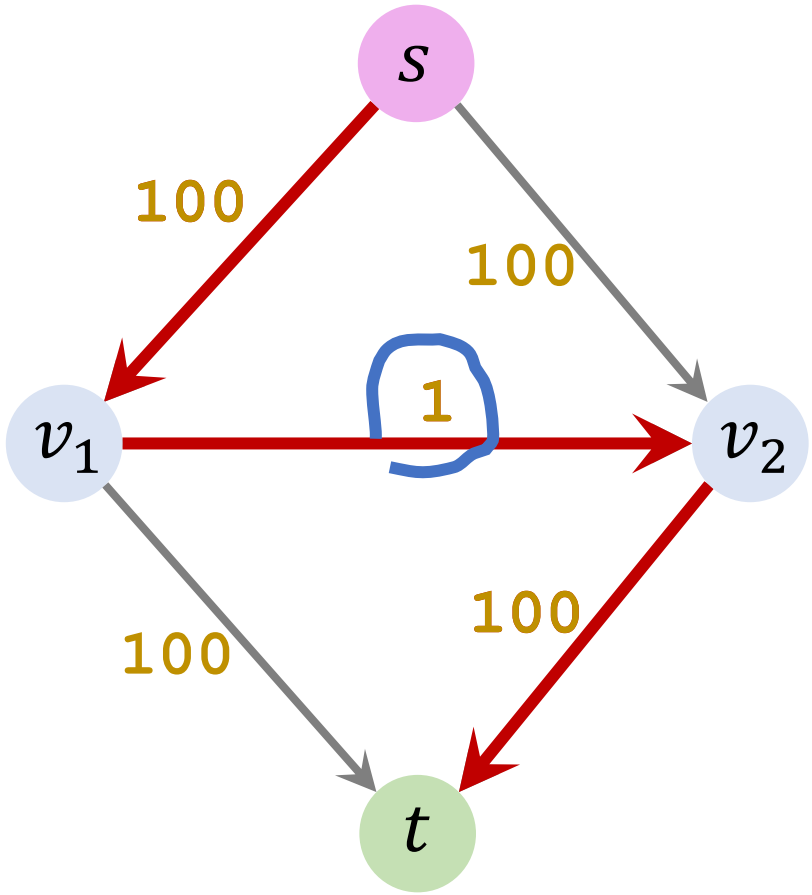


Original Graph



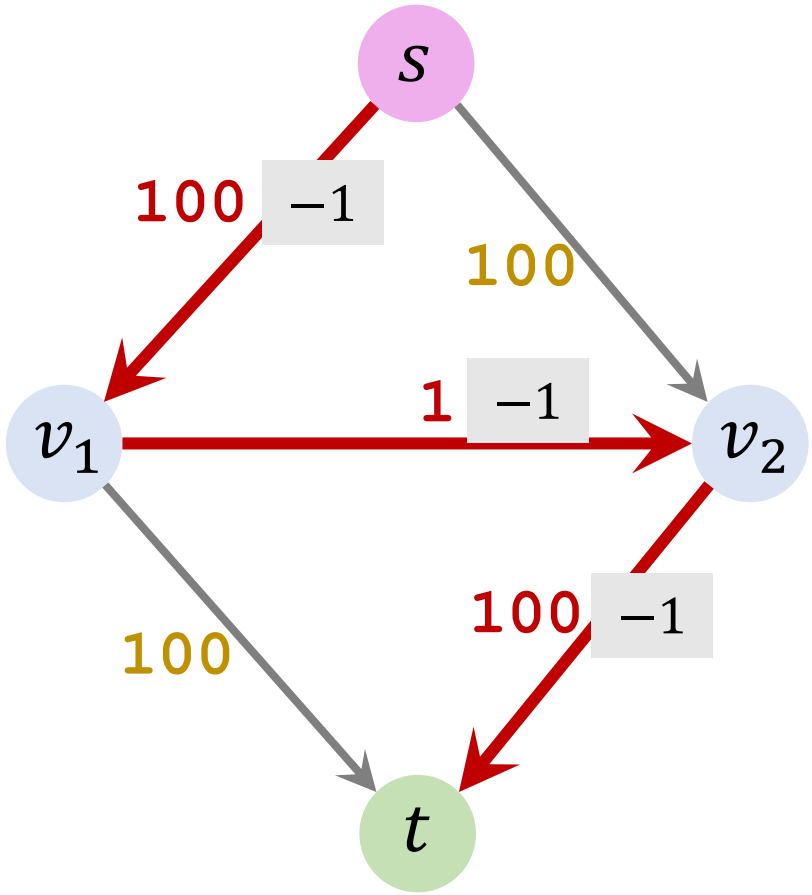
Residual Graph

Iteration 1: Find an augmenting path

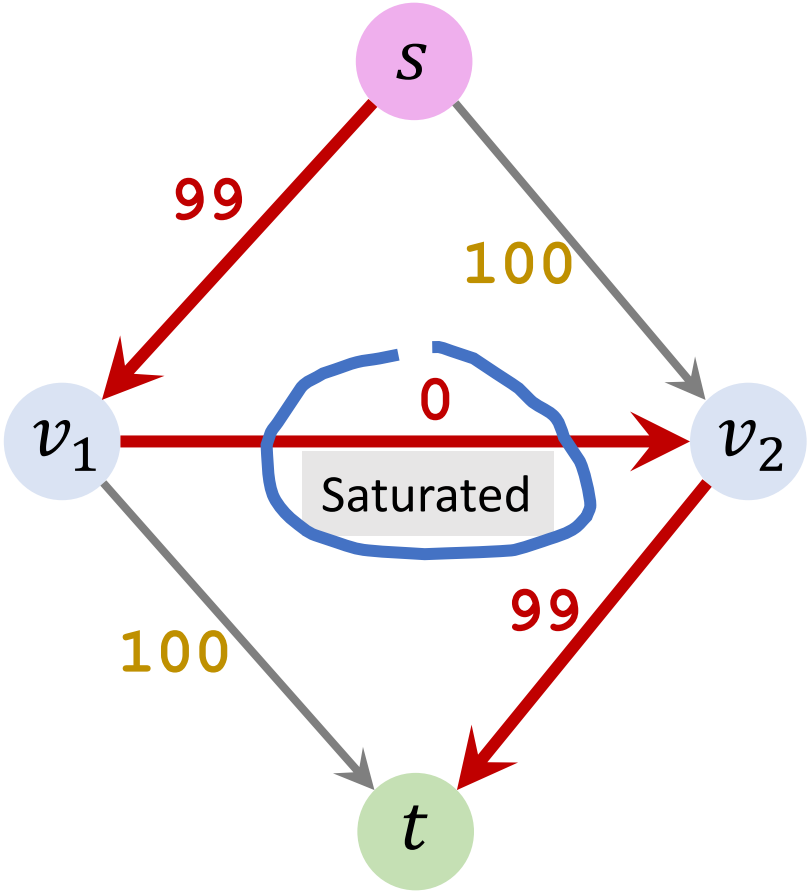


Found path $s \to v_1 \to v_2 \to t$. (Bottleneck capacity $= 1$.)

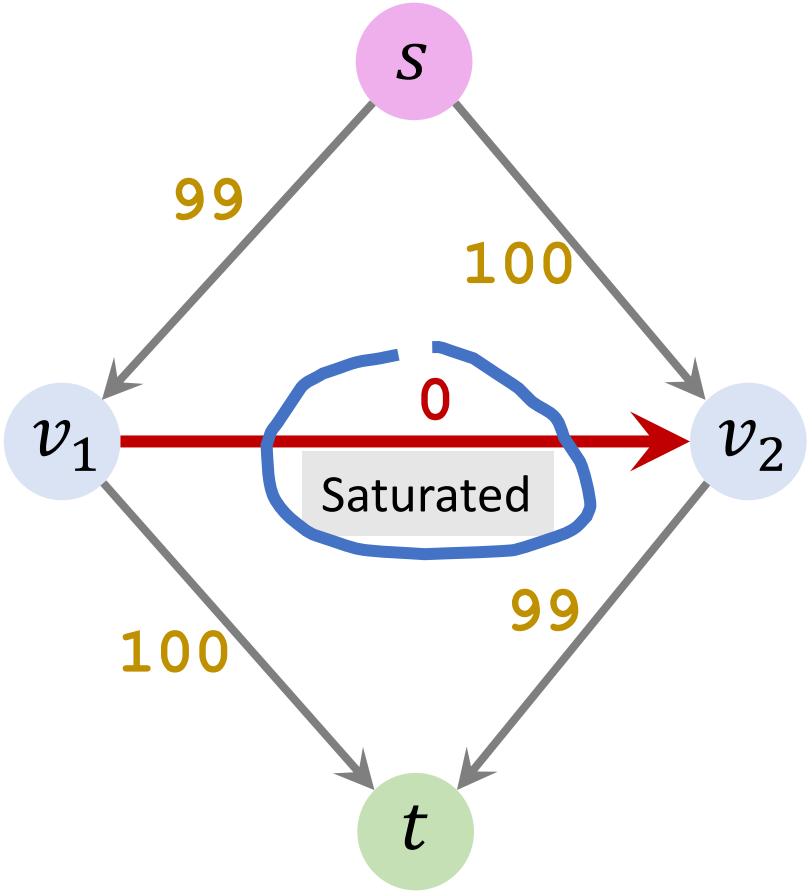
Iteration 1: Update residuals



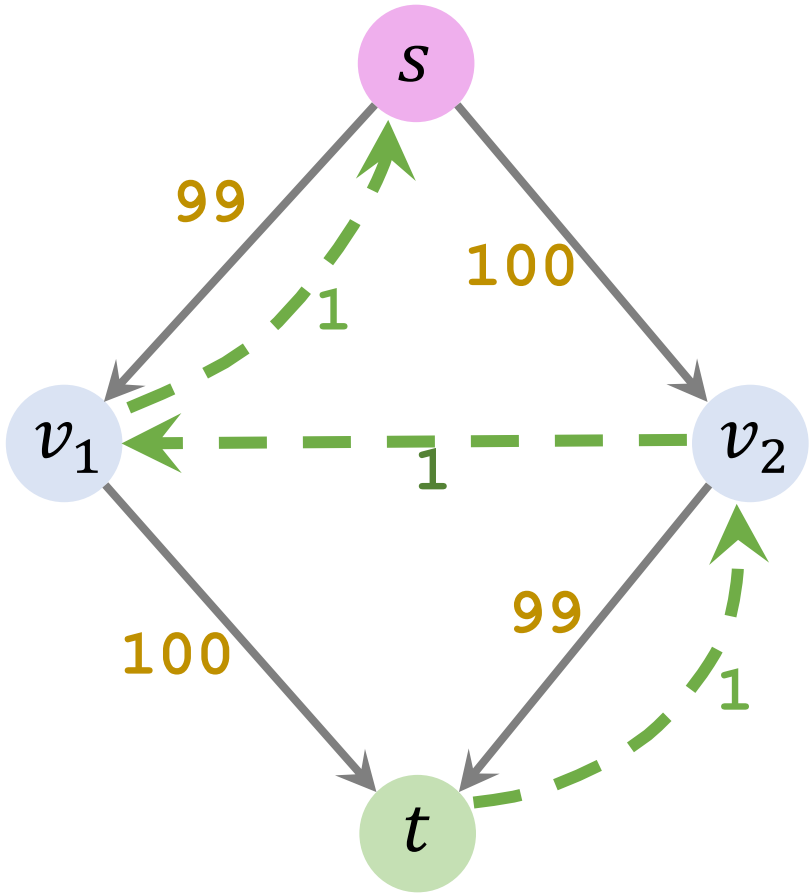
Iteration 1: Update residuals



Iteration 1: Remove saturated edges

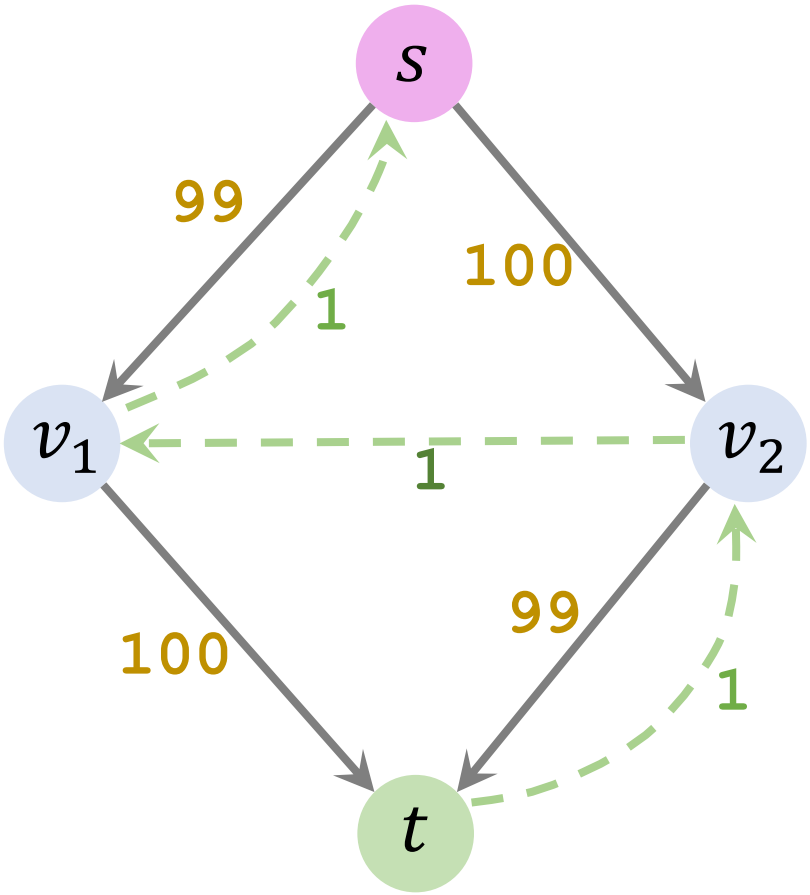


Iteration 1: Add backward path

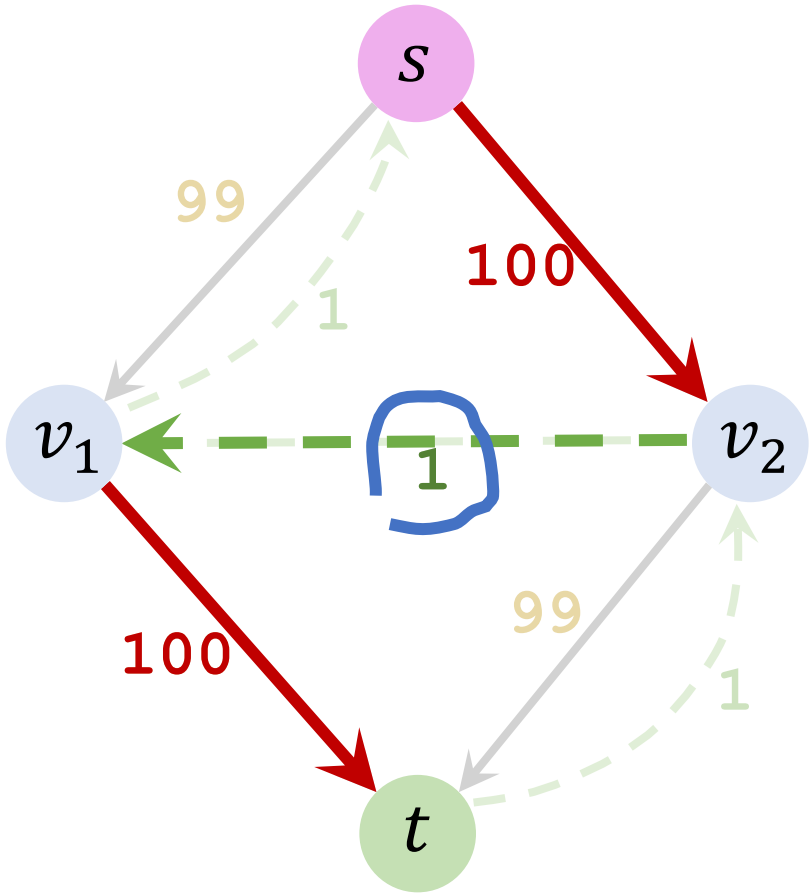


Add backward path $t \to v_2 \to v_1 \to s$ with weight $= 1$.

Now, the amount of flow is 1

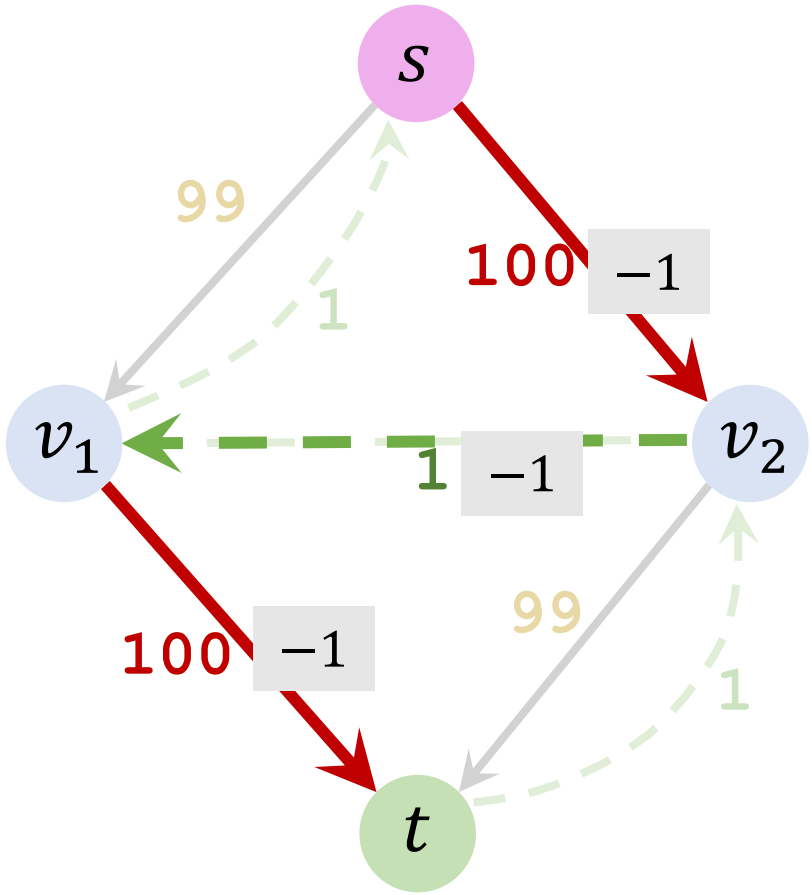


Iteration 2: Find an augmenting path



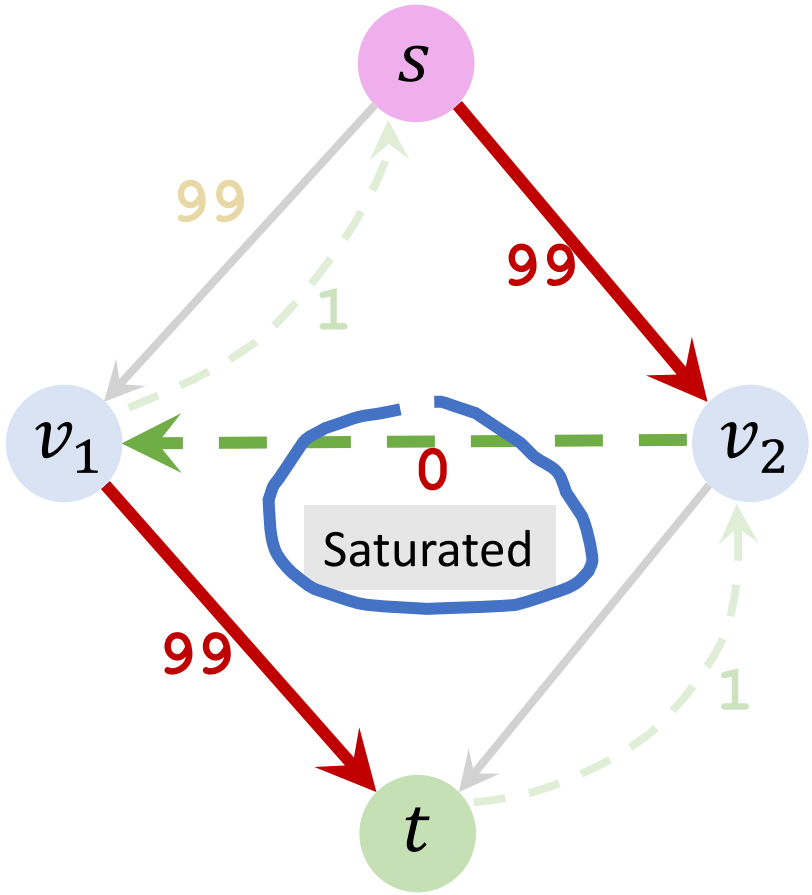
Found path $s \to v_2 \to v_1 \to t$. (Bottleneck capacity $= 1$.)

Iteration 2: Update residuals



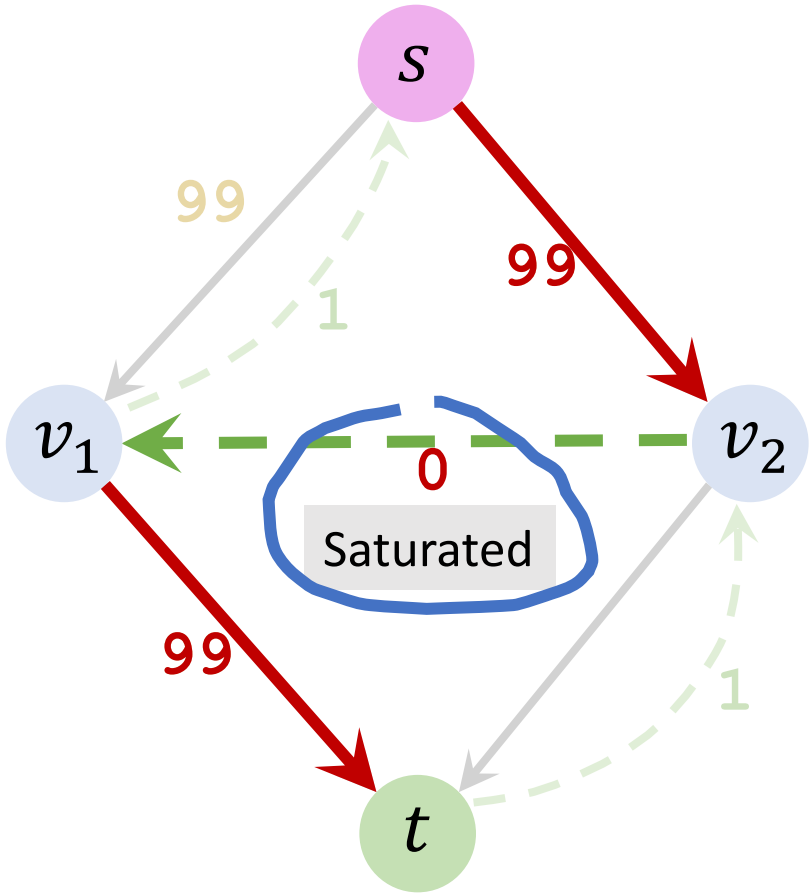
Found path $s \to v_2 \to v_1 \to t$. (Bottleneck capacity $= 1$.)

Iteration 2: Update residuals

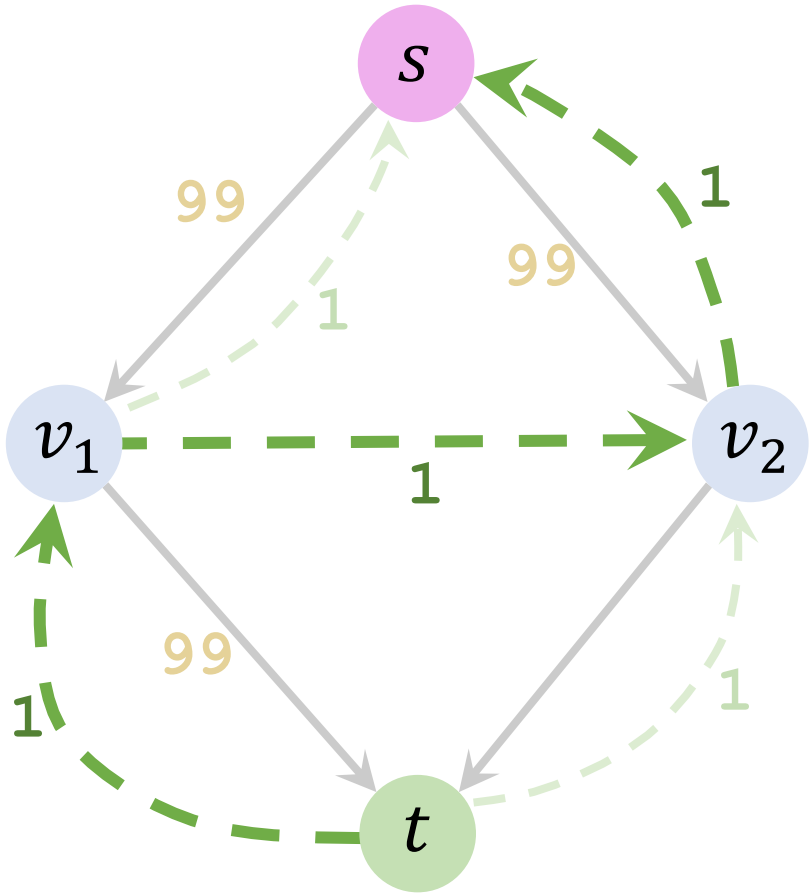


Found path $s \to v_2 \to v_1 \to t$. (Bottleneck capacity $= 1$.)

Iteration 2: Remove saturated edges

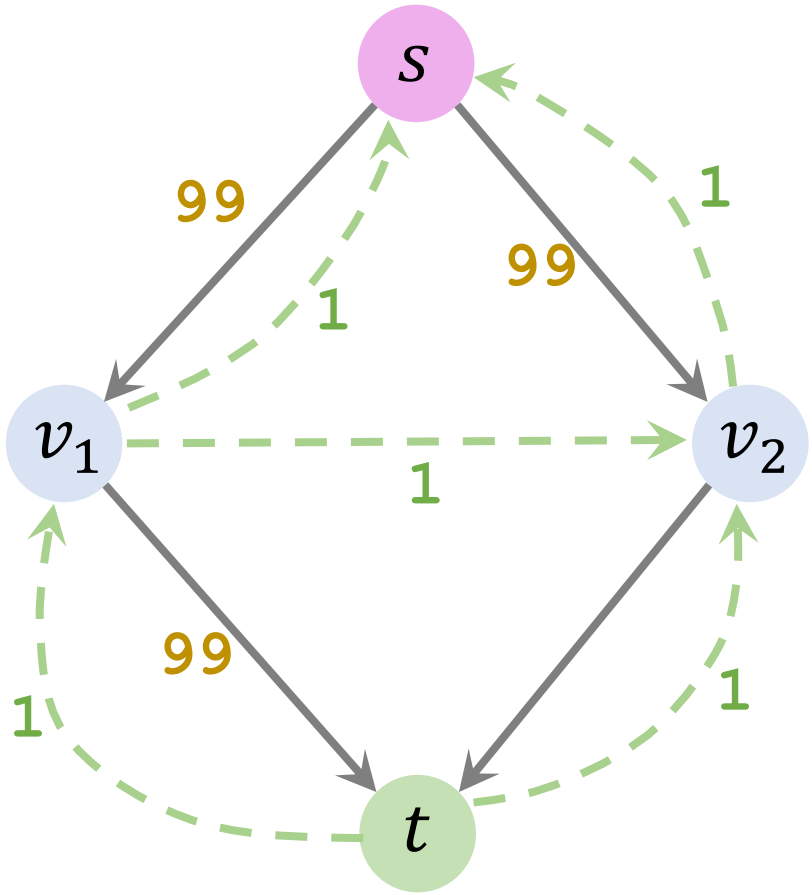


Iteration 2: Add backward path

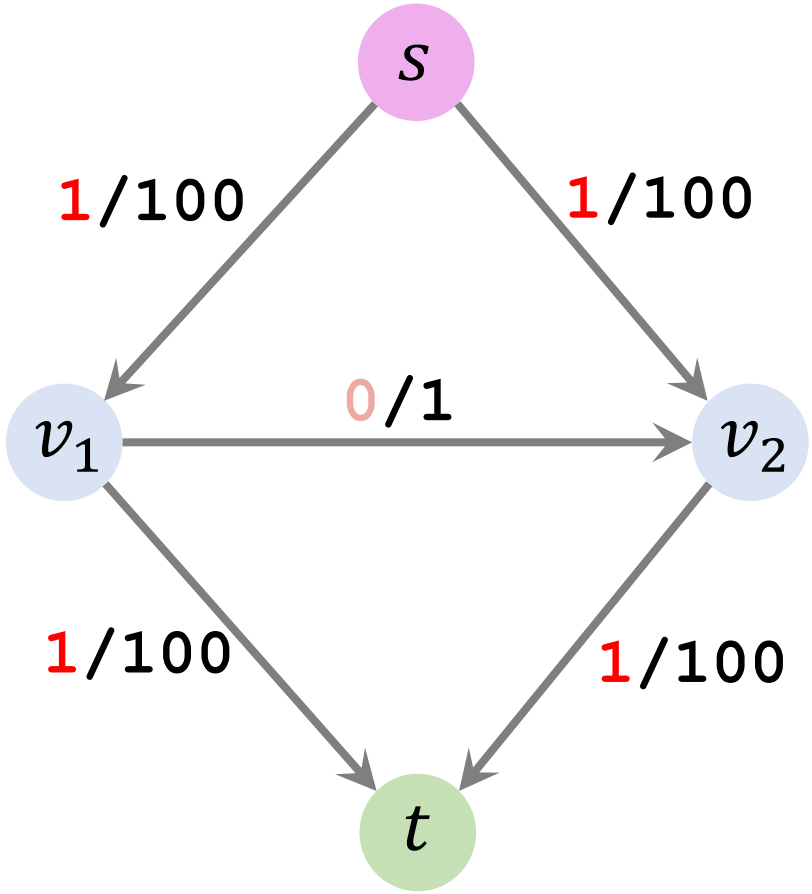


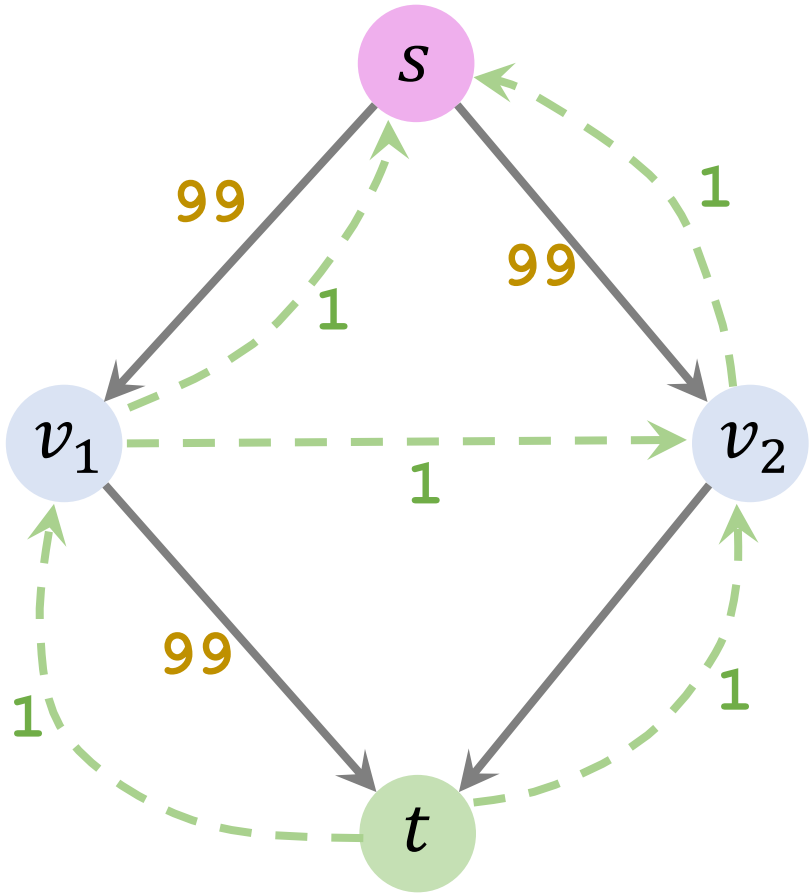
Add backward path $t \to v_1 \to v_2 \to s$ with weight $= 1$.

Now, the amount of flow is 2



Slow improvement…





In every iteration, the amount of flow increases by 1.

Worst-Case Iteration Complexity

- Each iteration increases the amount of flow by at least 1.
- Thus, `#Iterations` $\leq$ `Amount of MaxFlow`.

- In our example, each iteration increases the flow by only 1.
- Thus, `#Iterations` $=$ `Amount of MaxFlow`.

- In sum, the worst-case `#Iterations` is equal to `Amount of MaxFlow`.

Worst-Case Time Complexity

- Let m be the number of edges.
- It takes $O(m)$ time to find a path in unweighted graph.
  (Ignore the weights in the residual graph.)
- Thus, the per-iteration time complexity is $O(m)$.

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- Let the amount of max-flow be f.
- The worst-case time complexity is $O(f \cdot m)$.
- (In practice, the time complexity is not so bad.)

Summary

Ford-Fulkerson Algorithm

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Time complexity: $O(f \cdot m)$. (f is the max flow; m is #edges.)

Thank You!

http://wangshusen.github.io/