Fast Matrix Multiplication

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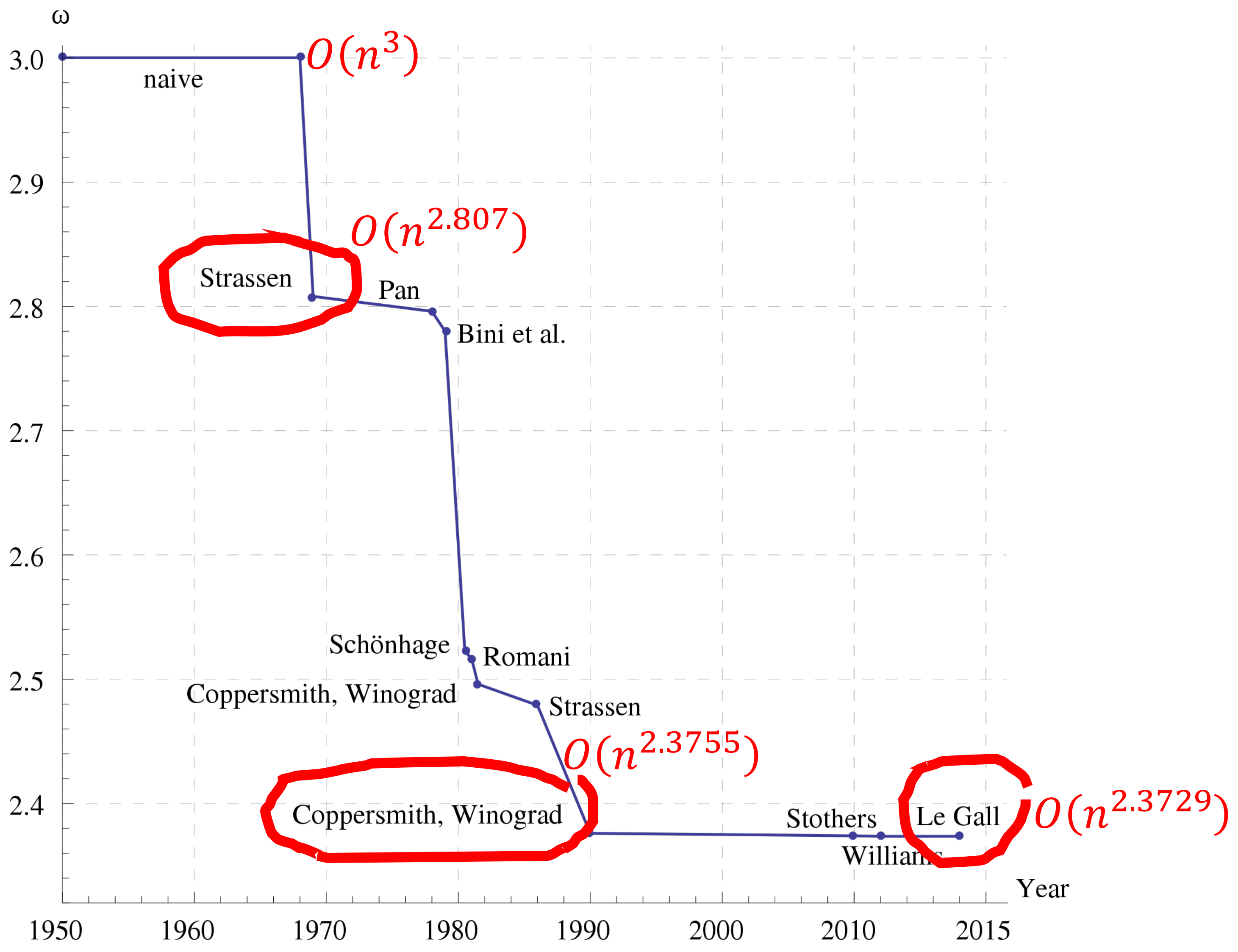
Matrix Multiplication

- Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices.
- The multiplication $\mathbf{C} = \mathbf{A}\mathbf{B}$ cost $O(n^3)$ time (nested for-loop).
- Can the time complexity be lower?

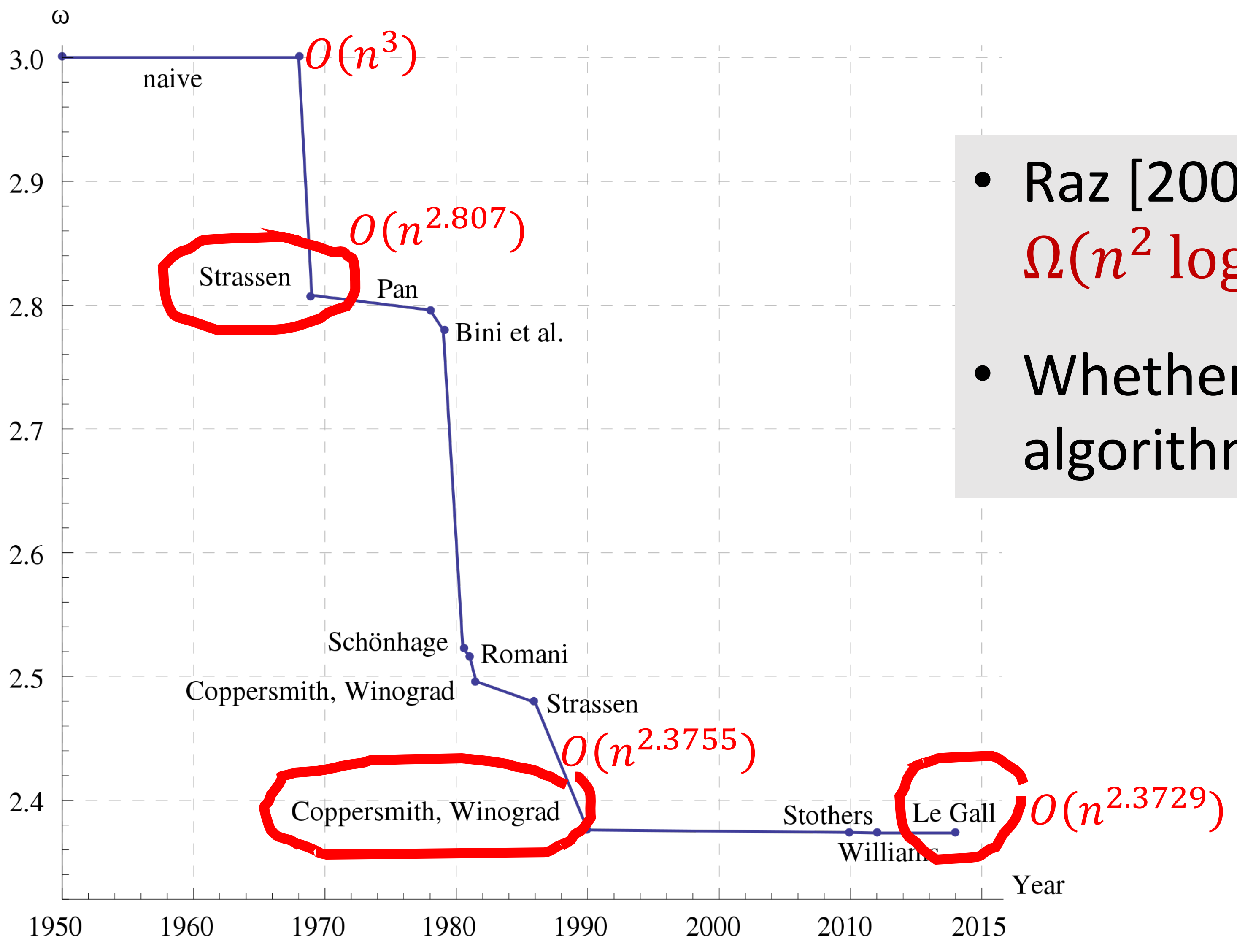
Fast Matrix Multiplication

- Strassen algorithm [1969] has $O(n^{2.807})$ time complexity.
- Coppersmith–Winograd algorithm [1990] has $O(n^{2.3755})$ time complexity.
- Le Gall [2014] improves the time complexity to $O(n^{2.3729})$.

Fast Matrix Multiplication



Fast Matrix Multiplication



- Raz [2002] proved a lower bound $\Omega(n^2 \log n)$.
- Whether there exists an $O(n^2 \log n)$ algorithm is yet unknown.

Divide-and-Conquer Matrix Multiplication

Block Matrix Multiplication

$$\underset{\mathbf{C}}{\begin{bmatrix} \mathbf{C}_{1,1} & \mathbf{C}_{1,2} \\ \mathbf{C}_{2,1} & \mathbf{C}_{2,2} \end{bmatrix}} = \underset{\mathbf{A}}{\begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{bmatrix}} \cdot \underset{\mathbf{B}}{\begin{bmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{B}_{2,1} & \mathbf{B}_{2,2} \end{bmatrix}}$$

Block Matrix Multiplication

$$\begin{bmatrix} \color{red}{C_{1,1}} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} = \begin{bmatrix} \color{green}{A_{1,1}} & \color{blue}{A_{1,2}} \\ A_{2,1} & A_{2,2} \end{bmatrix} \cdot \begin{bmatrix} \color{green}{B_{1,1}} & B_{1,2} \\ \color{blue}{B_{2,1}} & B_{2,2} \end{bmatrix}$$

- $\color{red}{C_{1,1}}$ $=$

Block Matrix Multiplication

$$\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \cdot \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$$

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Block Matrix Multiplication

$$\begin{bmatrix} \boxed{\mathbf{C}_{1,1}} & \mathbf{C}_{1,2} \\ \mathbf{C}_{2,1} & \mathbf{C}_{2,2} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{B}_{2,1} & \mathbf{B}_{2,2} \end{bmatrix}$$

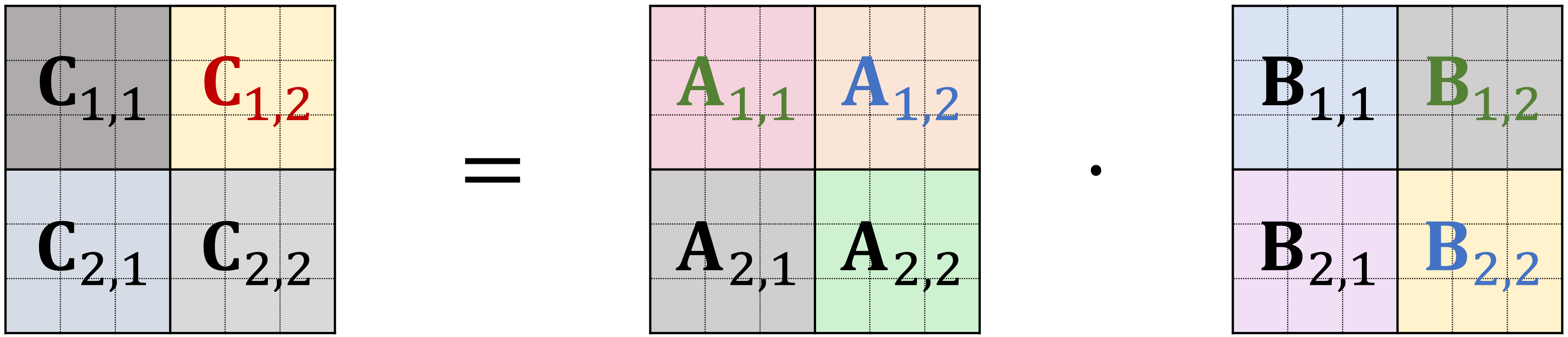
- $\mathbf{C}_{1,1} = \mathbf{A}_{1,1}\mathbf{B}_{1,1} +$

Block Matrix Multiplication

$$\begin{bmatrix} \boxed{C_{1,1}} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} = \begin{bmatrix} \boxed{A_{1,1} \quad A_{1,2}} \\ A_{2,1} \quad A_{2,2} \end{bmatrix} \cdot \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$$

- $\underline{C_{1,1}} = A_{1,1}B_{1,1} + A_{1,2}B_{2,1}.$

Block Matrix Multiplication



- $\mathbf{C}_{1,1} = \mathbf{A}_{1,1}\mathbf{B}_{1,1} + \mathbf{A}_{1,2}\mathbf{B}_{2,1}$.
- $\mathbf{C}_{1,2} = \mathbf{A}_{1,1}\mathbf{B}_{1,2} + \mathbf{A}_{1,2}\mathbf{B}_{2,2}$.

Block Matrix Multiplication

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Block Matrix Multiplication

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Block Matrix Multiplication

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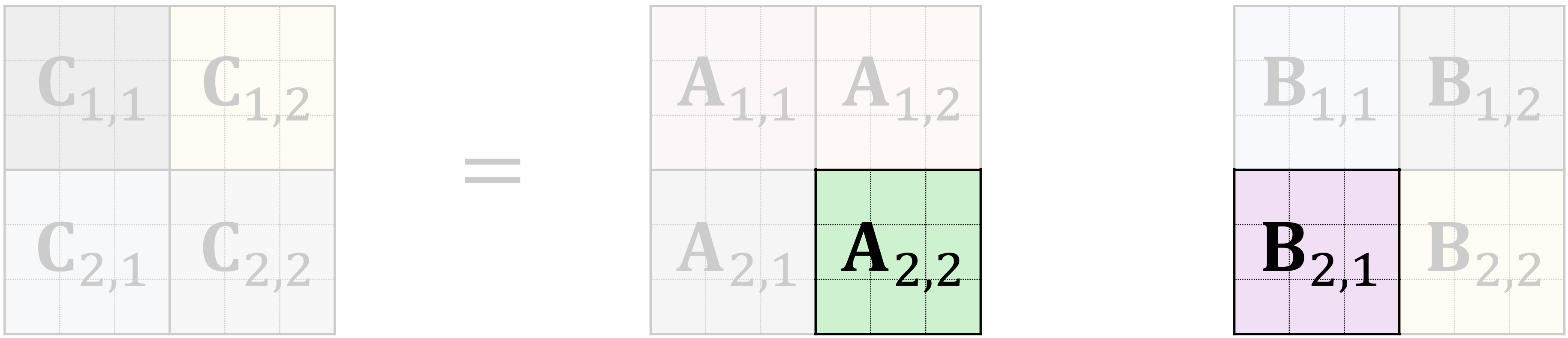
- 8 matrix multiplications
- 4 matrix additions

Divide-and-Conquer



- Further partition $\mathbf{A}_{ij}$ and $\mathbf{B}_{jl}$ into sub-matrices.
- Recursively apply the block multiplication to compute $\mathbf{A}_{ij}\mathbf{B}_{jl}$.

Time Complexity

- Partition the $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ into $\frac{n}{2} \times \frac{n}{2}$ sub-matrices.

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- Partition the $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ into $\frac{n}{2} \times \frac{n}{2}$ sub-matrices.
- To compute the $n \times n$ matrix multiplication $\mathbf{C} = \mathbf{AB}$, perform
  - $\frac{n}{2} \times \frac{n}{2}$ matrix multiplications for 8 times,
  - $\frac{n}{2} \times \frac{n}{2}$ matrix additions for 4 times.

Time Complexity

- $T(n)$: time complexity for multiplying $n \times n$ matrices.
- Repeating $\frac{n}{2} \times \frac{n}{2}$ matrix multiplications for 8 times costs time:

$$8 \cdot T\left(\frac{n}{2}\right).$$

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Recurrence relation: $T(n) = 8 \cdot T(n/2) + c \cdot n^2.$

The Master Theorem

Recurrence relation: $T(n) = a \cdot T(n/b) + c \cdot n^{d}$.

The Master Theorem

Recurrence relation: $T(n) = a \cdot T(n/b) + c \cdot n^d$.

- The master theorem:

$$T(n) = \begin{cases} O(n^d), & \text{if } a < b^d; \\ O(n^d \log n), & \text{if } a = b^d; \\ O(n^{\log_b a}), & \text{if } a > b^d. \end{cases}$$

- For block matrix multiplication, $a = 8$, $b = 2$, and $d = 2$.

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- For block matrix multiplication, $a = 8$, $b = 2$, and $d = 2$.
- Thus, $T(n) = O(n^{\log_2 8}) = O(n^3)$.
- No speedup at all.

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Strassen Algorithm

Strassen Algorithm

$$\underbrace{\begin{bmatrix} \mathbf{C}_{1,1} & \mathbf{C}_{1,2} \\ \mathbf{C}_{2,1} & \mathbf{C}_{2,2} \end{bmatrix}}_{\mathbf{C}} = \underbrace{\begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{B}_{2,1} & \mathbf{B}_{2,2} \end{bmatrix}}_{\mathbf{B}}$$

Strassen Algorithm

- $\mathbf{M}_1 = (\mathbf{A}_{1,1} + \mathbf{A}_{2,2})(\mathbf{B}_{1,1} + \mathbf{B}_{2,2}).$
- $\mathbf{M}_2 = (\mathbf{A}_{2,1} + \mathbf{A}_{2,2})\,\mathbf{B}_{1,1}.$
- $\mathbf{M}_3 = \mathbf{A}_{1,1}\,(\mathbf{B}_{1,2} - \mathbf{B}_{2,2}).$
- $\mathbf{M}_4 = \mathbf{A}_{2,2}\,(\mathbf{B}_{2,1} - \mathbf{B}_{1,1}).$
- $\mathbf{M}_5 = (\mathbf{A}_{1,1} + \mathbf{A}_{1,2})\,\mathbf{B}_{2,2}.$
- $\mathbf{M}_6 = (\mathbf{A}_{2,1} - \mathbf{A}_{1,1})(\mathbf{B}_{1,1} + \mathbf{B}_{1,2}).$
- $\mathbf{M}_7 = (\mathbf{A}_{1,2} - \mathbf{A}_{2,2})(\mathbf{B}_{2,1} + \mathbf{B}_{2,2}).$

- $\mathbf{C}_{1,1} = \mathbf{M}_1 + \mathbf{M}_4 - \mathbf{M}_5 + \mathbf{M}_7.$
- $\mathbf{C}_{1,2} = \mathbf{M}_3 + \mathbf{M}_5.$
- $\mathbf{C}_{2,1} = \mathbf{M}_2 + \mathbf{M}_4.$
- $\mathbf{C}_{2,2} = \mathbf{M}_1 - \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_6.$

- 7 matrix multiplications
- 18 matrix additions

Strassen Algorithm

- Divide-and-conquer: recursively compute $\mathbf{A}_{ij}\mathbf{B}_{kl}$.
- To compute the $n \times n$ matrix multiplication $\mathbf{C} = \mathbf{AB}$, perform
  - $\frac{n}{2} \times \frac{n}{2}$ matrix multiplications for 7 times,
  - $\frac{n}{2} \times \frac{n}{2}$ matrix additions for 18 times.
- Recursion relation: $T(n) = \underline{7 \cdot T\left(\frac{n}{2}\right)} + 18c\left(\frac{n}{2}\right)^2$.

The Master Theorem

Recurrence relation: $T(n) = a \cdot T(n/b) + c \cdot n^d.$

- The master theorem:

$$T(n) = \begin{cases} O\left(n^d\right), & \text{if } a < b^d; \\ O\left(n^d \log n\right), & \text{if } a = b^d; \\ O\left(n^{\log_b a}\right), & \text{if } a > b^d. \end{cases}$$

- For Strassen Algorithm, $a = 7$, $b = 2$, and $d = 2$.

The Master Theorem

Recurrence relation: $T(n) = a \cdot T(n/b) + c \cdot n^d$.

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- For Strassen Algorithm, $a = 7$, $b = 2$, and $d = 2$.
- Thus, $T(n) = O\left(n^{\log_2 7}\right) \approx O\left(n^{2.807}\right)$.

Summary

Block Matrix Multiplication

- Naively multiplying two $n \times n$ matrices costs $O(n^3)$ time.
- Block matrix multiplication:
  - Partition $n \times n$ matrices into four $\frac{n}{2} \times \frac{n}{2}$ sub-matrices.
  - Perform 8 multiplications and 4 additions.

Block Matrix Multiplication

- Naively multiplying two $n \times n$ matrices costs $O(n^3)$ time.
- Block matrix multiplication:
  - Partition $n \times n$ matrices into four $\frac{n}{2} \times \frac{n}{2}$ sub-matrices.
  - Perform 8 multiplications and 4 additions.
- Divide-and-conquer does not help!
- The time complexity is still $O\left(n^{\log_2 8}\right) = O(n^3)$.

Strassen Algorithm

- Strassen algorithm also performs block matrix multiplication.
- It reduces the number of multiplications from 8 to 7.
- The time complexity is reduced to

$$O\left(n^{\log_2 7}\right) \approx O\left(n^{2.807}\right).$$

- There are better algorithms than Strassen algorithm.

Fast Matrix Multiplication

$O(n^3)$ Naïve Algorithm

Fast Matrix Multiplication

$O(n^3)$	Naïve Algorithm
$O(n^{2.807})$	Strassen [1969]
$O(n^{2.3755})$	Coppersmith–Winograd [1990]
$O(n^{2.3729})$	Le Gall [2014]

Fast Matrix Multiplication

$O(n^3)$ Naïve Algorithm

$O(n^{2.807})$ Strassen [1969]

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Any Worse Lower Bound?

Lower bound: $\Omega(n^2 \log n)$ Raz [2002]

Fast Matrix Multiplication

$O(n^3)$ Naïve Algorithm

$O(n^{2.807})$ Strassen [1969]

$O(n^{2.3755})$ Coppersmith–Winograd [1990]

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Unfilled Gap

Lower bound: $\Omega(n^2 \log n)$ Raz [2002]

Thank You!