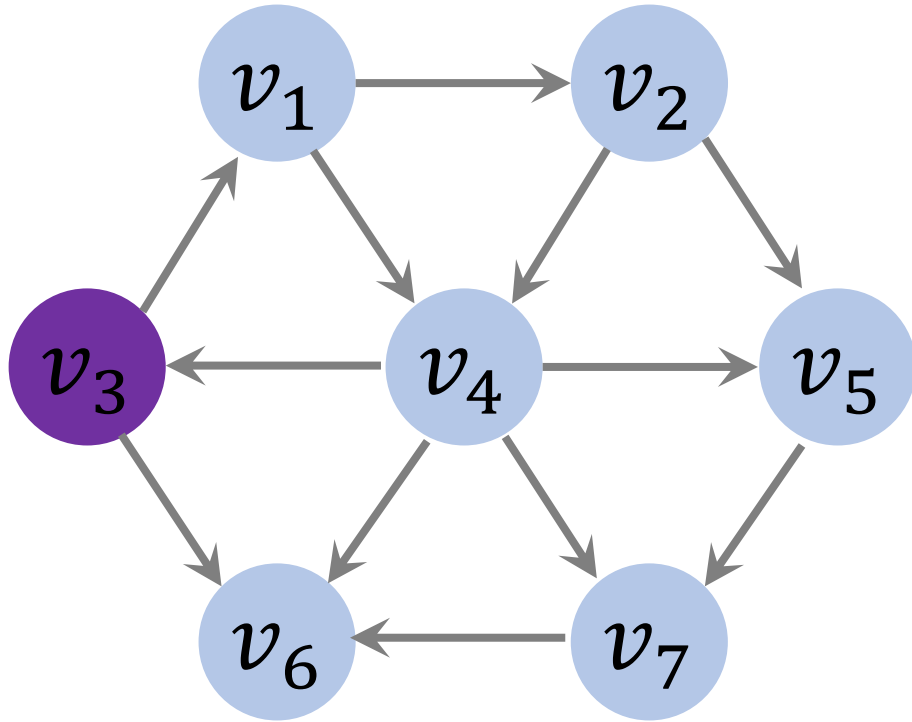


Finding Shortest Paths in Unweighted Graphs

Shusen Wang

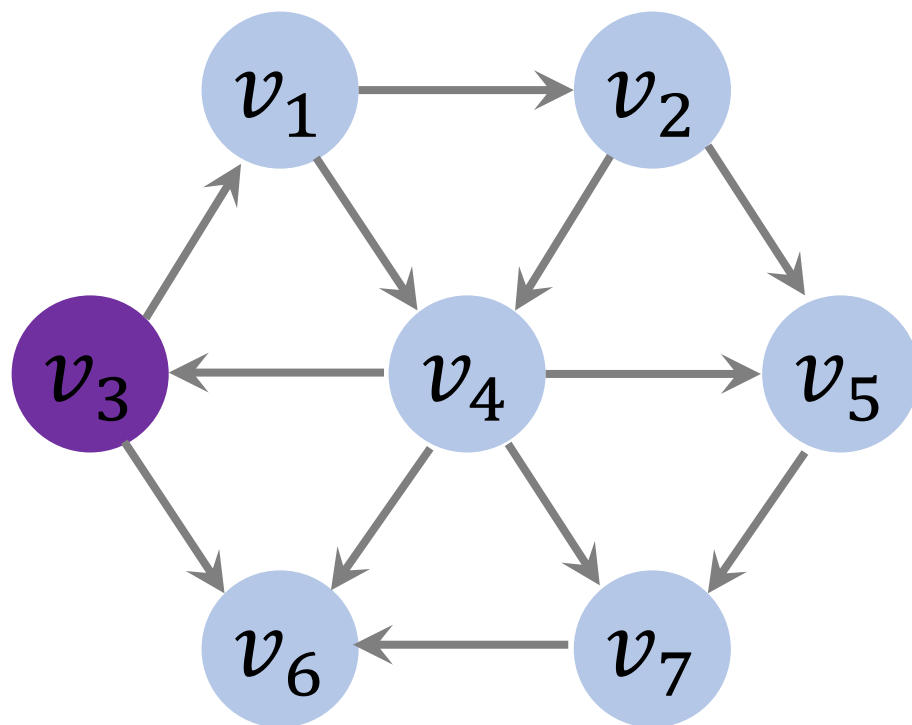
<http://wangshusen.github.io/>

Shortest Path in Unweighted Graphs



- Weights are all ones.
- Weights of nonexistent edges are ∞ .
- Easier problem: finding shortest path in **unweighted graph**.
- Harder problem: finding shortest path in **weighted graph**.

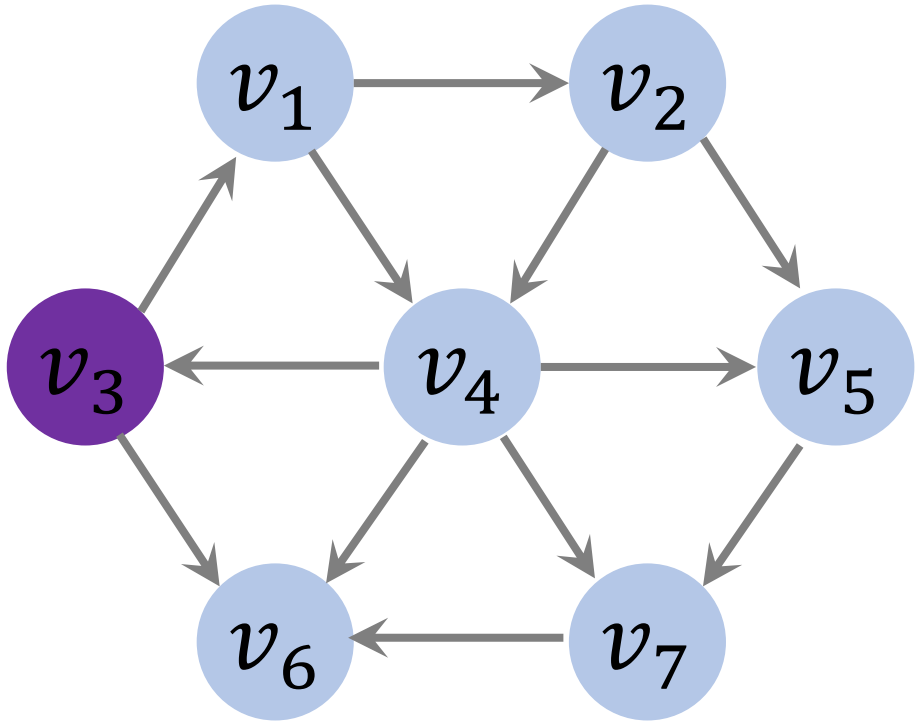
Shortest Path in Unweighted Graphs



vertex	dist	path
v_1	1	v_3
v_2	2	v_1
v_3	0	0
v_4	2	v_1
v_5	3	v_2
v_6	1	v_3
v_7	3	v_4

Algorithm

Preparations



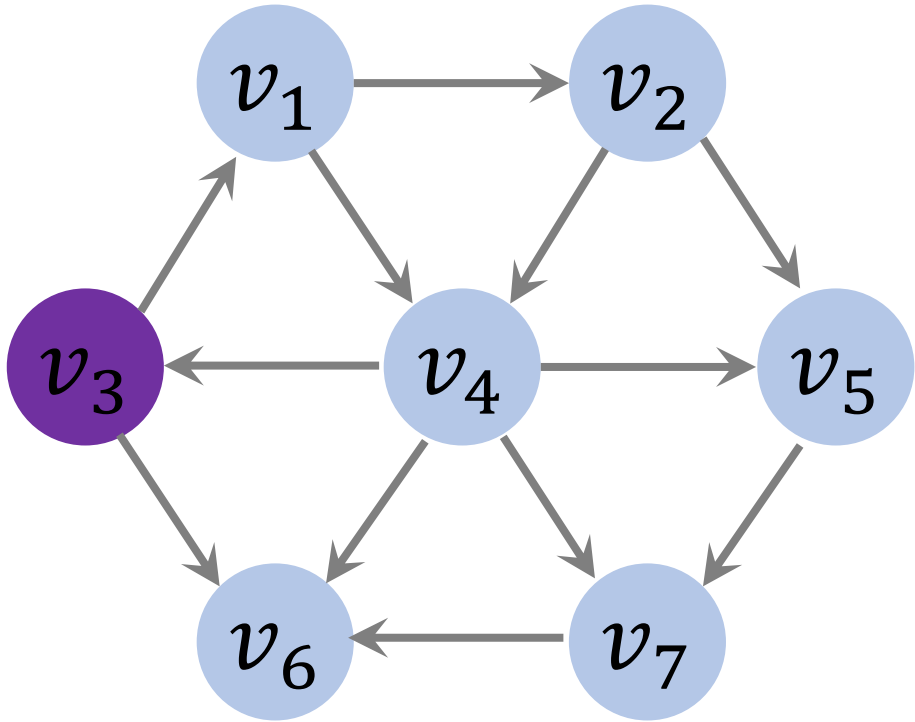
v_3 is the source.

Queue:



vertex	visit	dist	path
v_1			
v_2			
v_3			
v_4			
v_5			
v_6			
v_7			

Preparations



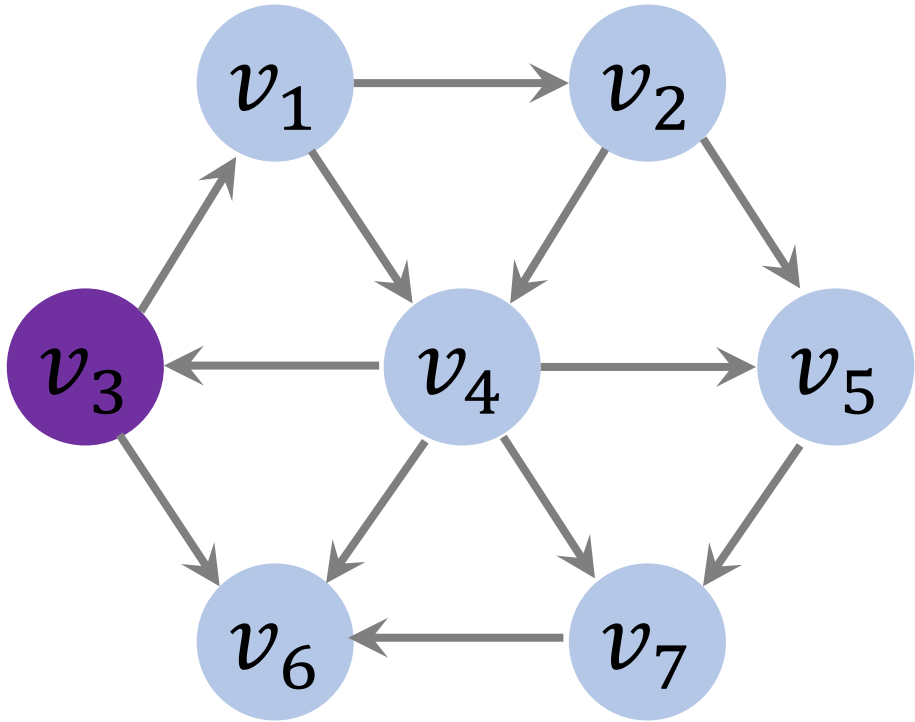
v_3 is the source.

Queue:



vertex	visit	dist	path
v_1	no		
v_2	no		
v_3	no		
v_4	no		
v_5	no		
v_6	no		
v_7	no		

Preparations



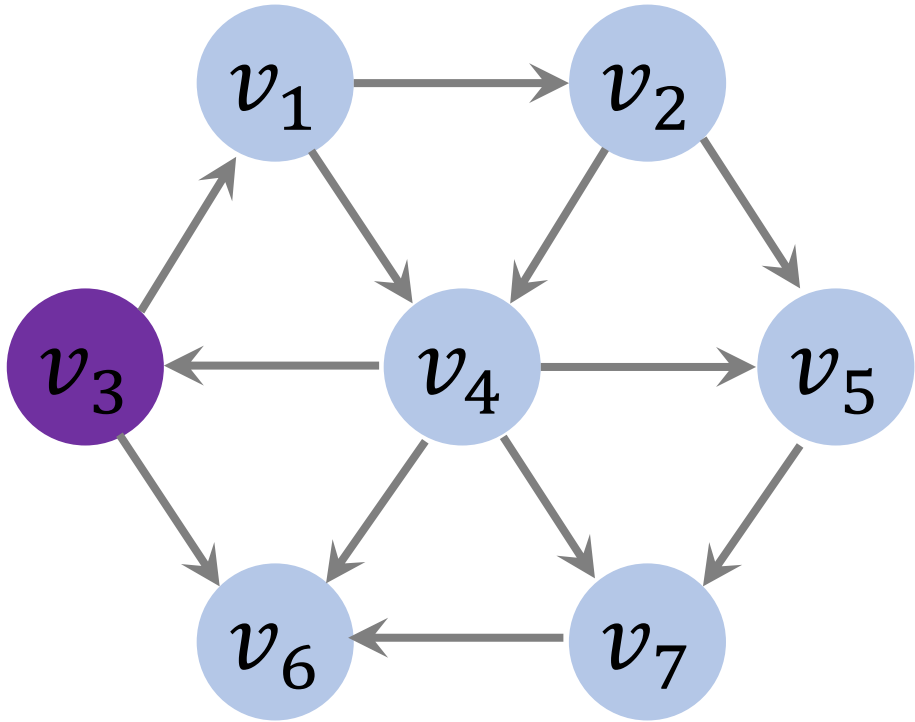
v_3 is the source.

Queue:



vertex	visit	dist	path
v_1	no	∞	
v_2	no	∞	
v_3	no	∞	
v_4	no	∞	
v_5	no	∞	
v_6	no	∞	
v_7	no	∞	

Preparations



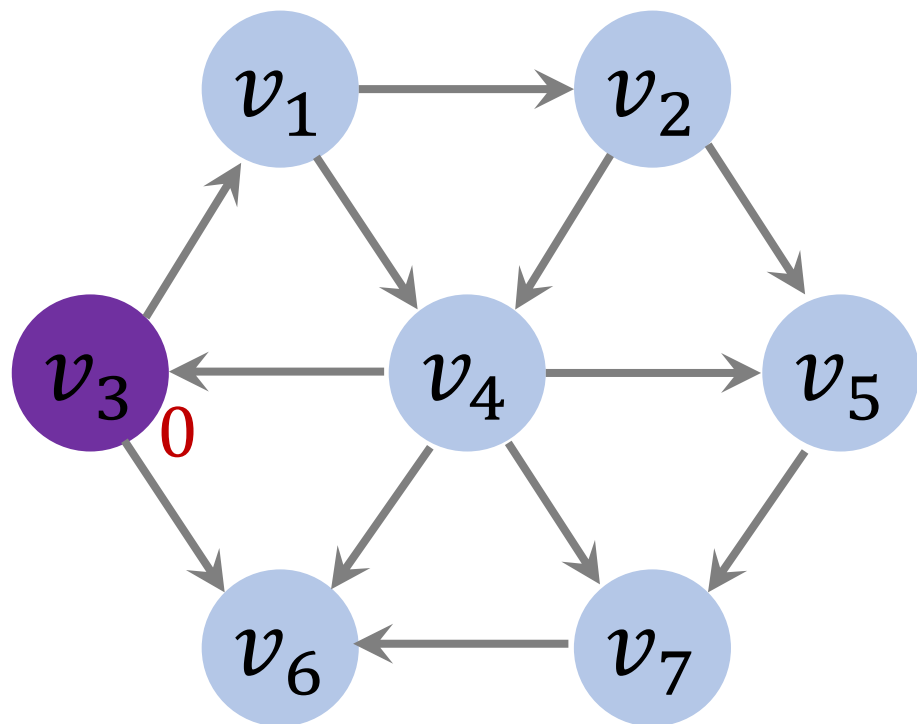
v_3 is the source.

Queue:



vertex	visit	dist	path
v_1	no	∞	0
v_2	no	∞	0
v_3	no	∞	0
v_4	no	∞	0
v_5	no	∞	0
v_6	no	∞	0
v_7	no	∞	0

Initial State

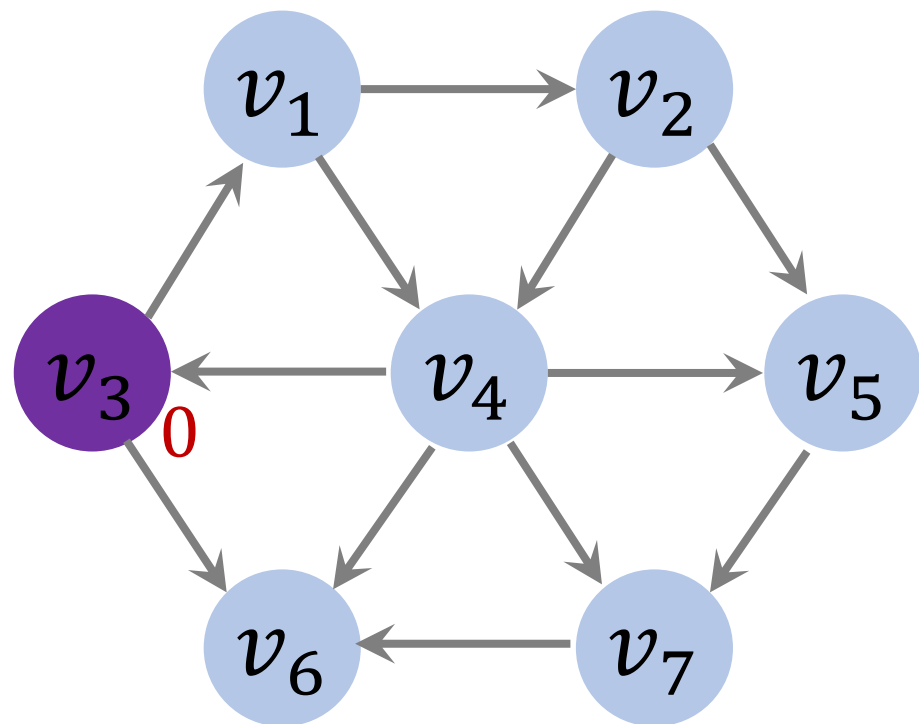


Queue:

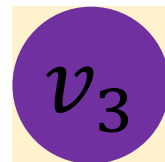


vertex	visit	dist	path
v_1	no	∞	0
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	no	∞	0
v_7	no	∞	0

Initial State

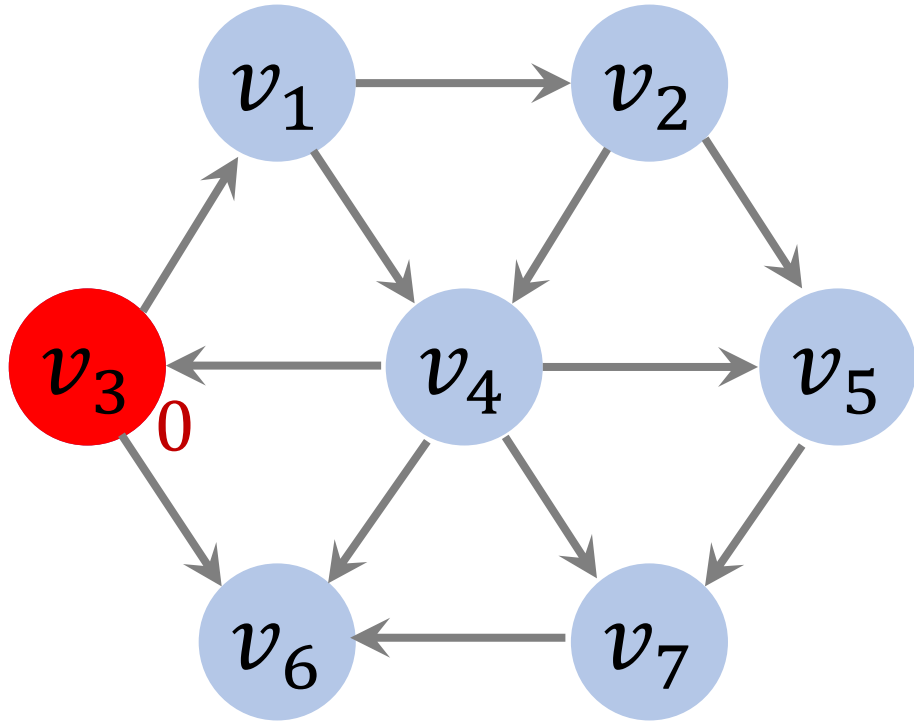


Queue:



vertex	visit	dist	path
v_1	no	∞	0
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	no	∞	0
v_7	no	∞	0

Iteration 1

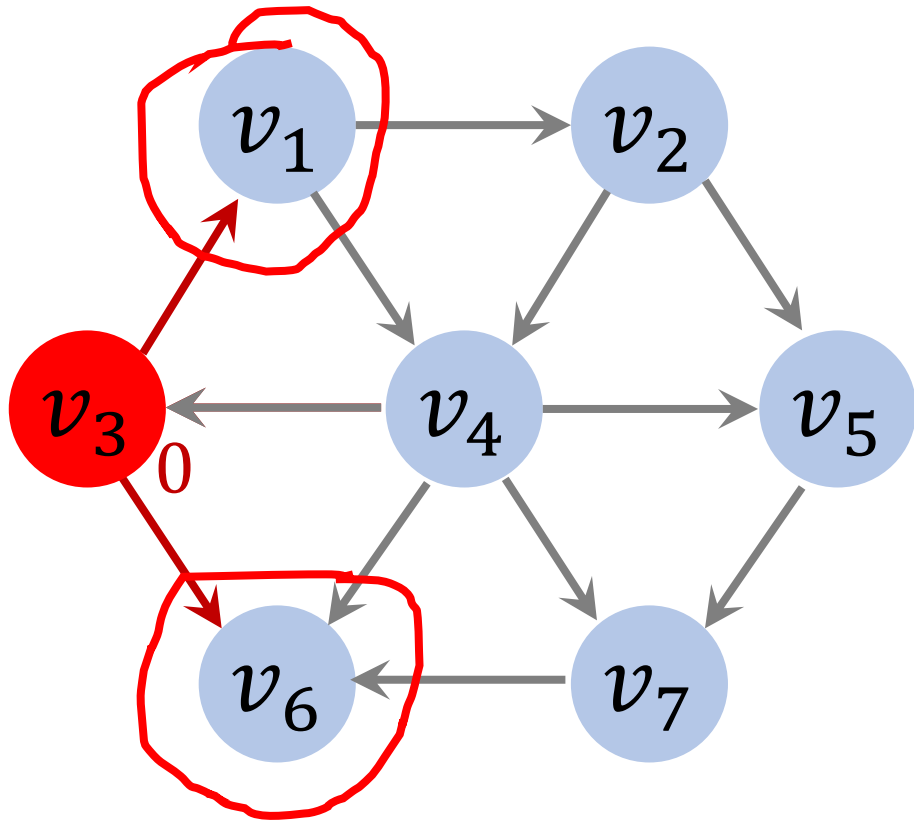


Queue:

v_3

vertex	visit	dist	path
v_1	no	∞	0
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	no	∞	0
v_7	no	∞	0

Iteration 1

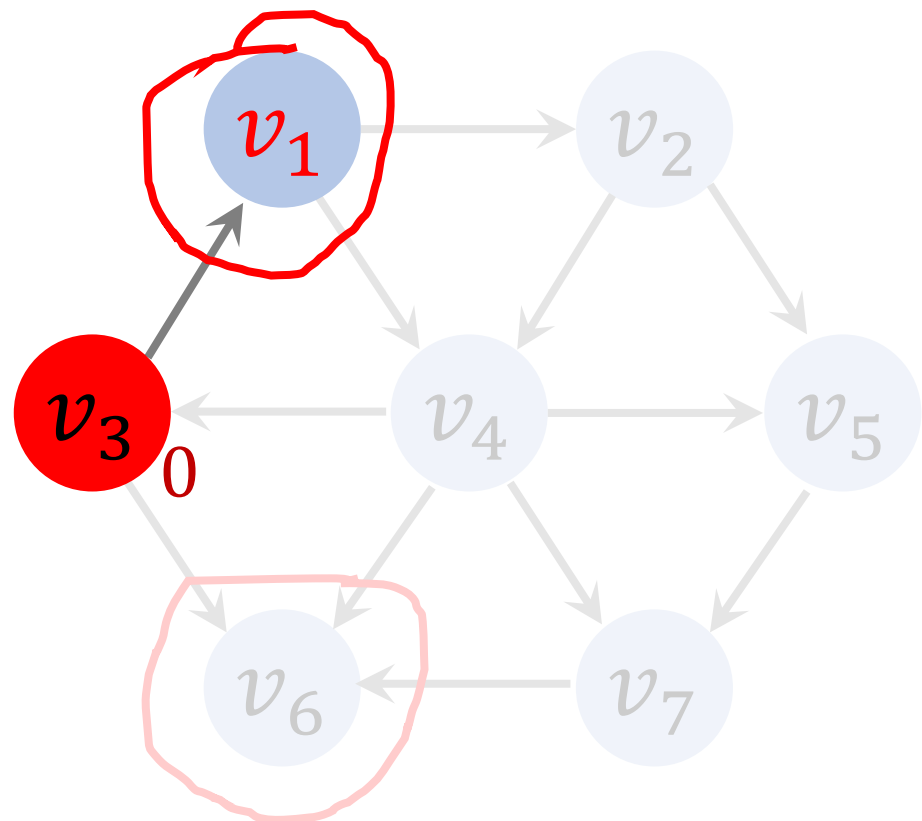


Queue:



vertex	visit	dist	path
v_1	no	∞	0
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	no	∞	0
v_7	no	∞	0

Iteration 1(A)

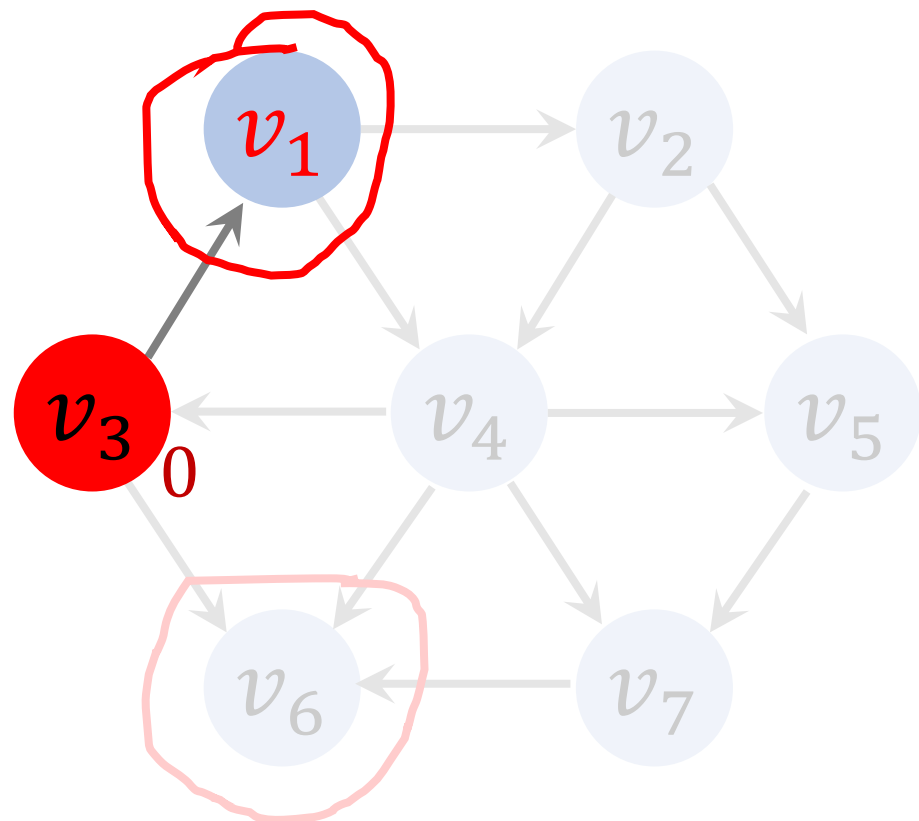


Queue:



vertex	visit	dist	path
v_1	no	∞	0
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	no	∞	0
v_7	no	∞	0

Iteration 1(A)

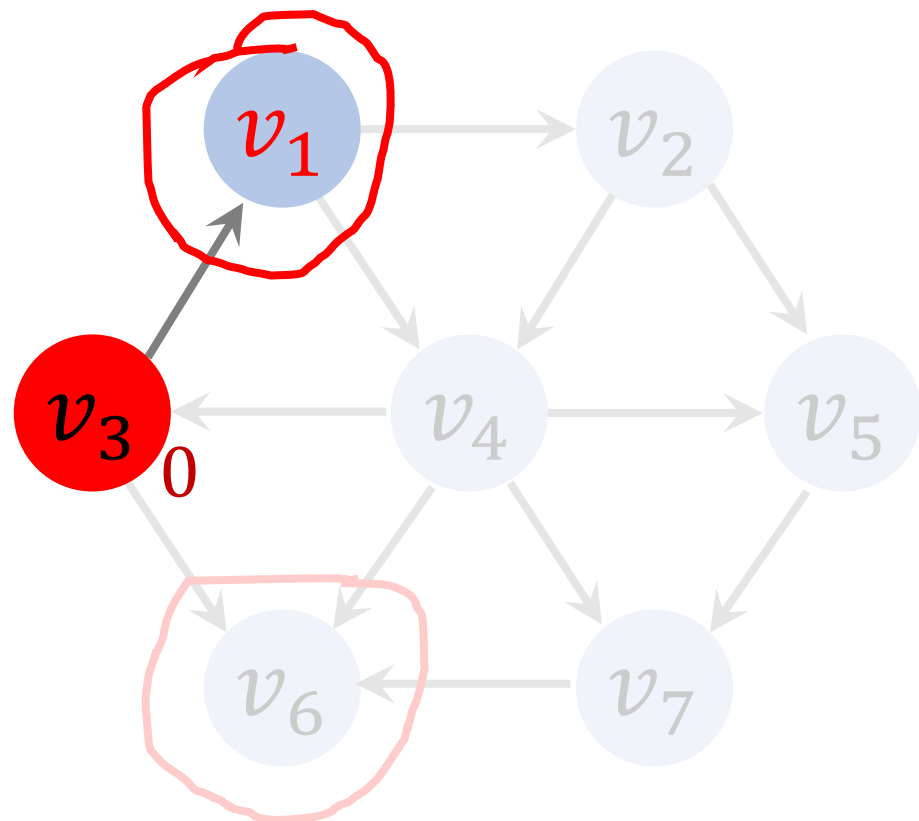


Queue:

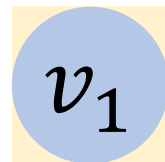


vertex	visit	dist	path
v_1	no	∞	0
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	no	∞	0
v_7	no	∞	0

Iteration 1(A)

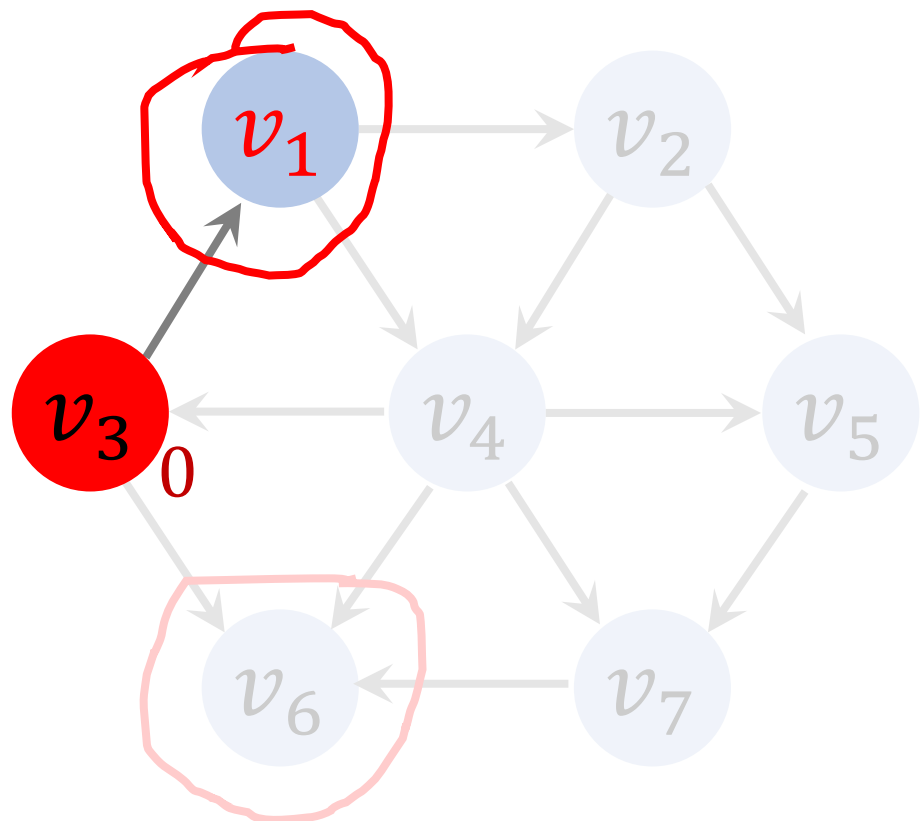


Queue:

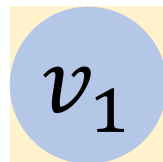


vertex	visit	dist	path
v_1	no	∞	0
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	no	∞	0
v_7	no	∞	0

Iteration 1(A)

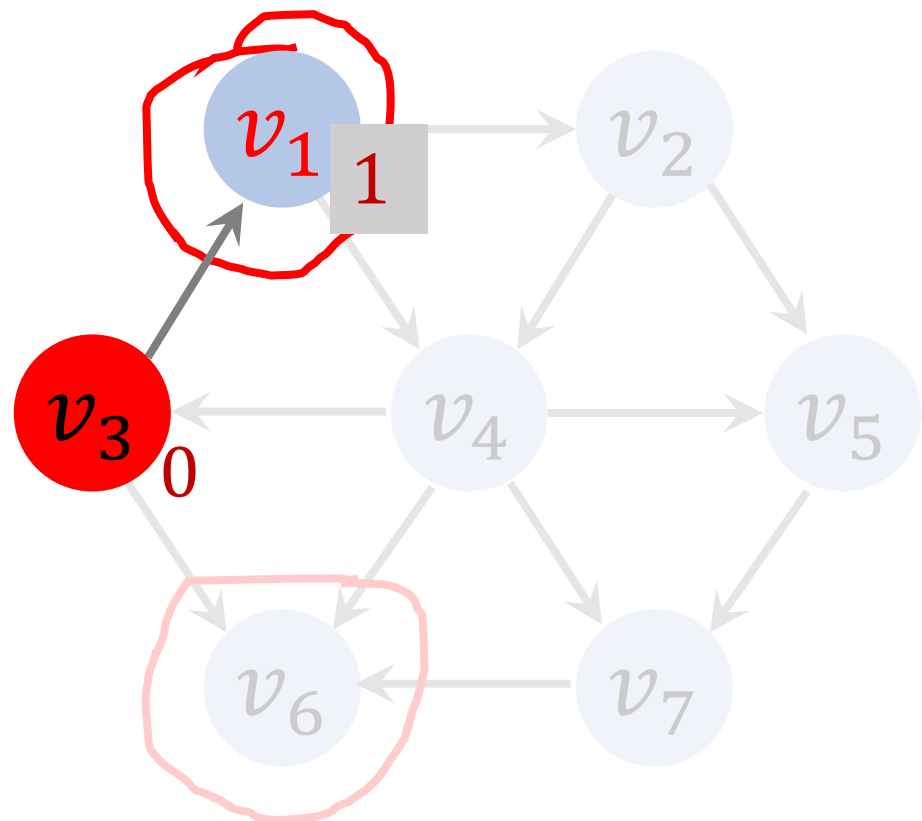


Queue:

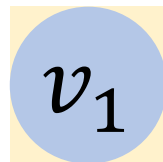


vertex	visit	dist	path
v_1	yes	∞	0
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	no	∞	0
v_7	no	∞	0

Iteration 1(A)

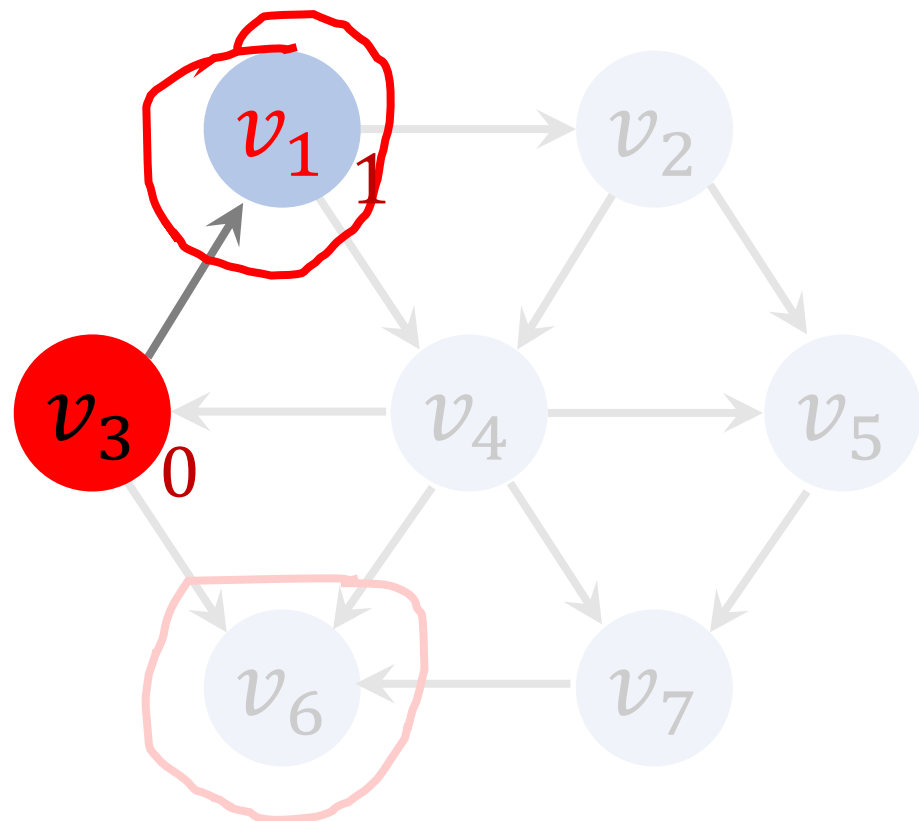


Queue:

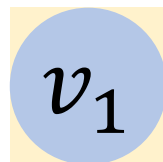


vertex	visit	dist	path
v_1	yes	1	0
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	no	∞	0
v_7	no	∞	0

Iteration 1(A)

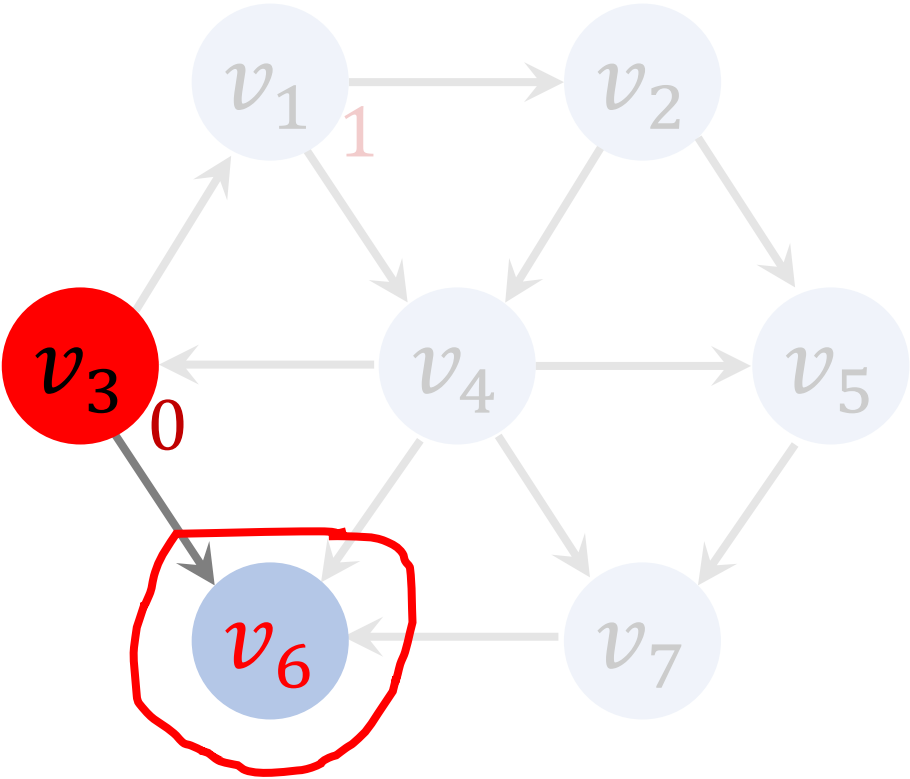


Queue:

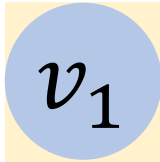


vertex	visit	dist	path
v_1	yes	1	v_3
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	no	∞	0
v_7	no	∞	0

Iteration 1(B)

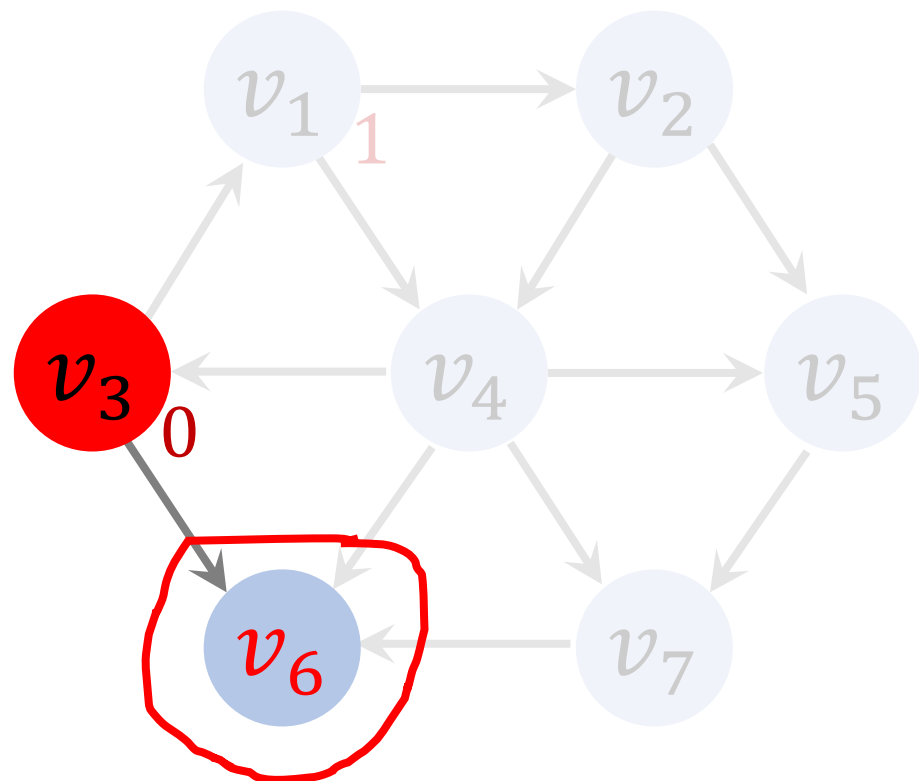


Queue:

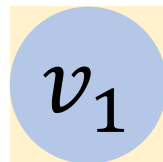


vertex	visit	dist	path
v_1	yes	1	v_3
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	no	∞	0
v_7	no	∞	0

Iteration 1(B)

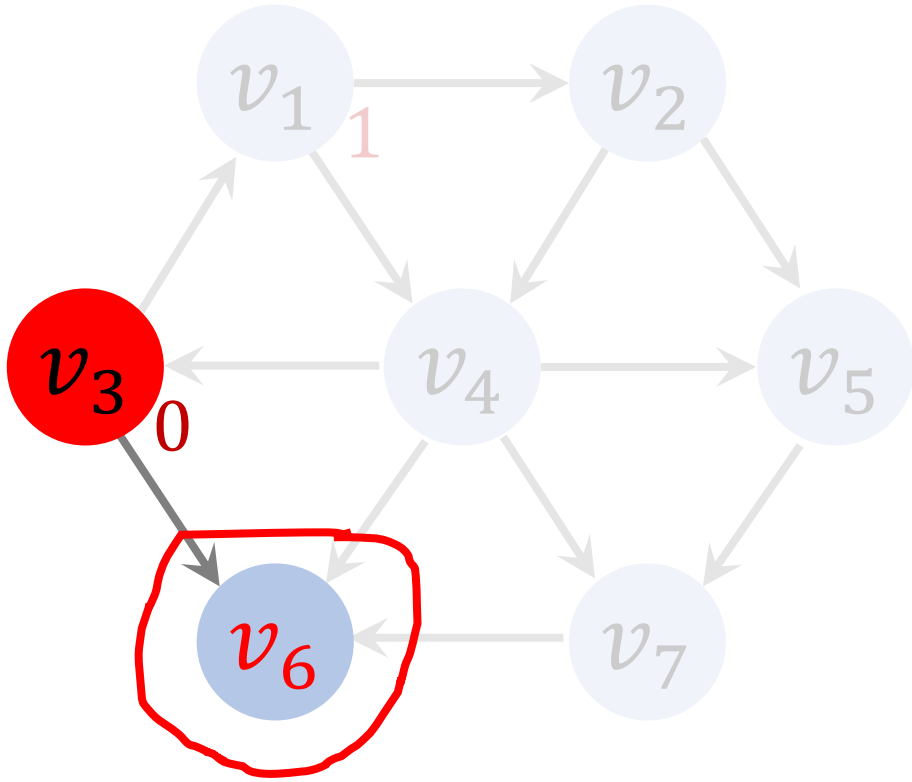


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	no	∞	0
v_7	no	∞	0

Iteration 1(B)

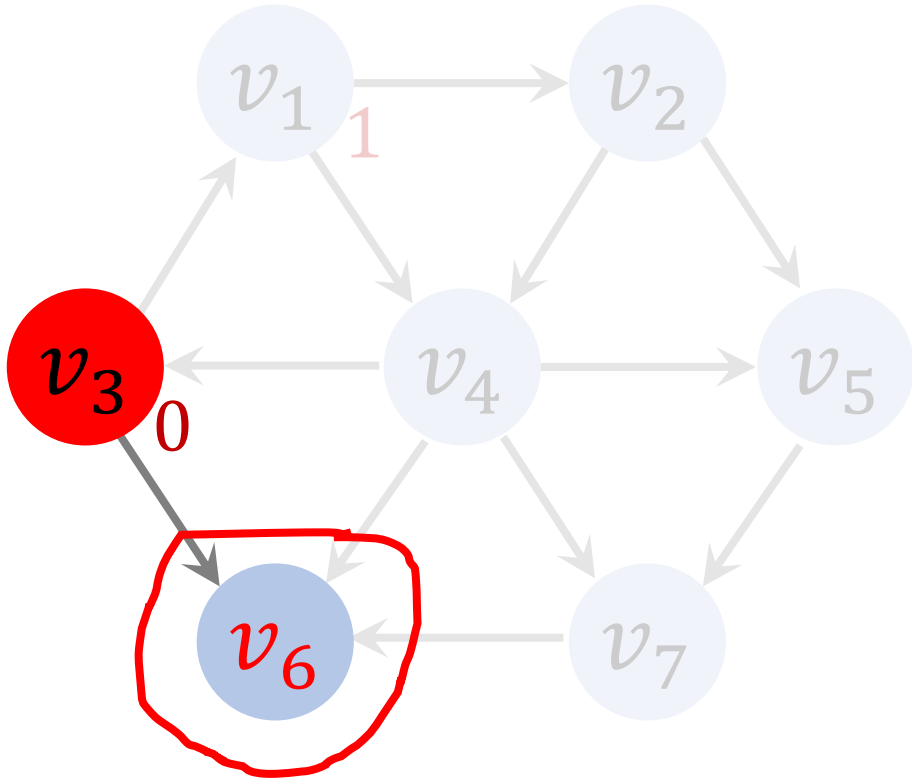


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	no	∞	0
v_7	no	∞	0

Iteration 1(B)

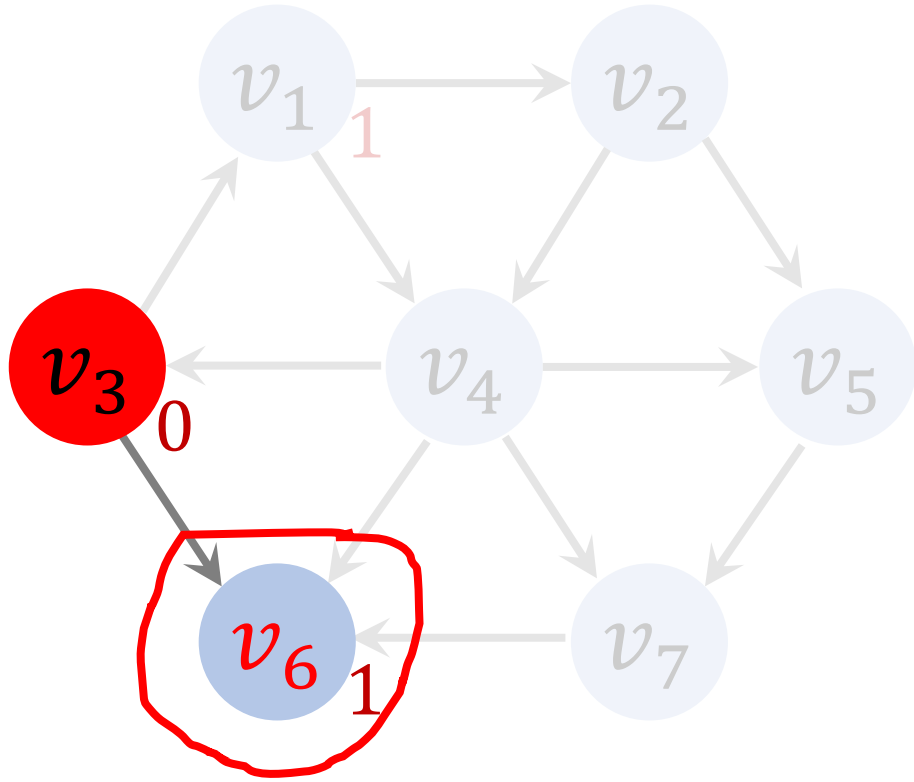


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	yes	∞	0
v_7	no	∞	0

Iteration 1(B)

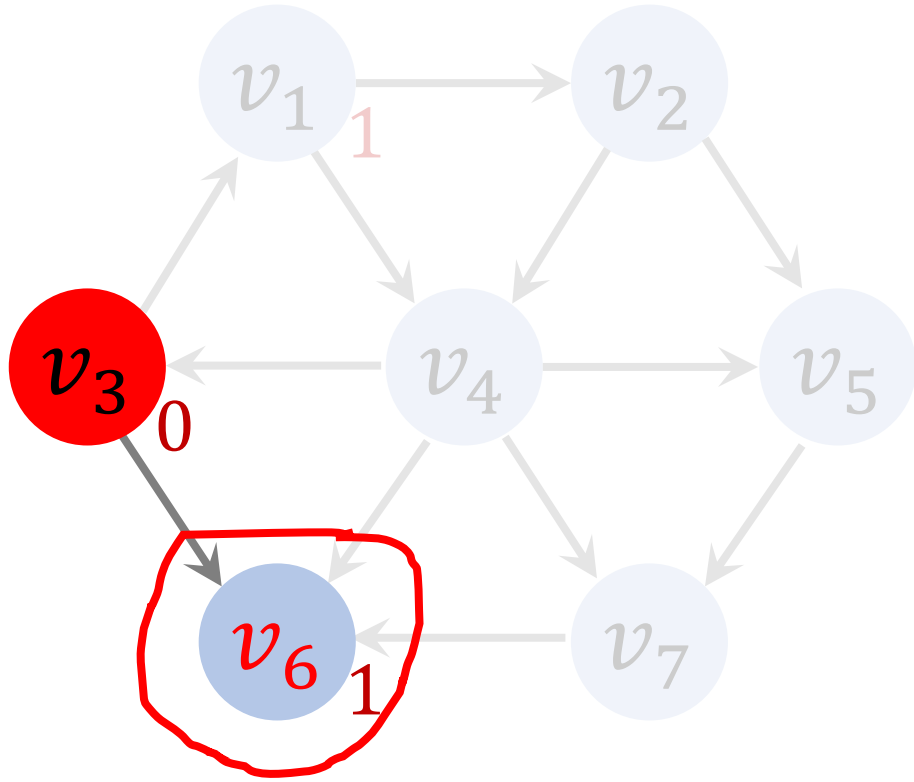


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	yes	1	0
v_7	no	∞	0

Iteration 1(B)

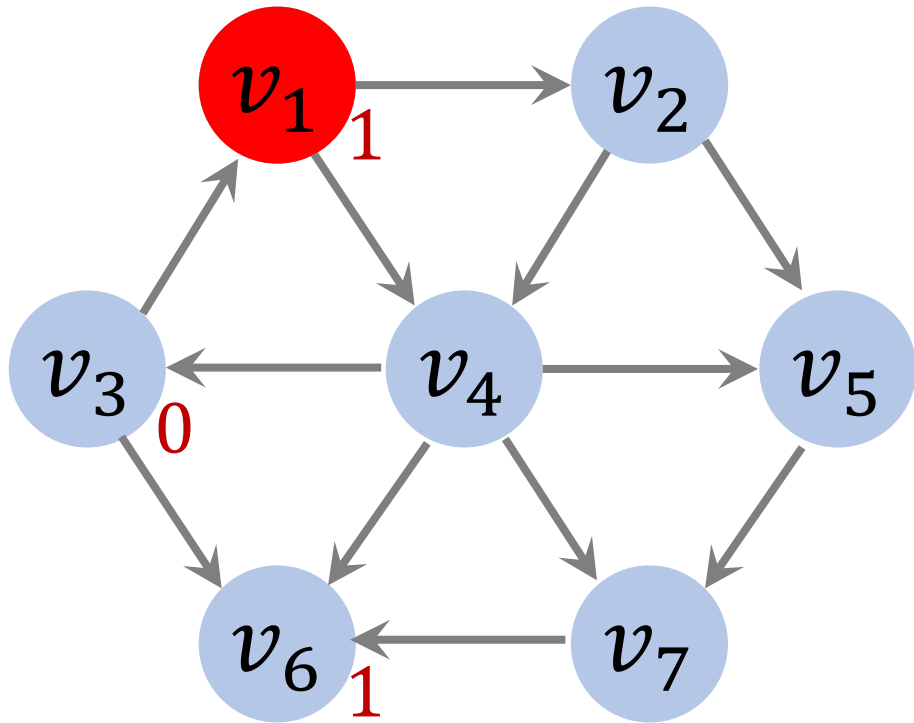


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 2



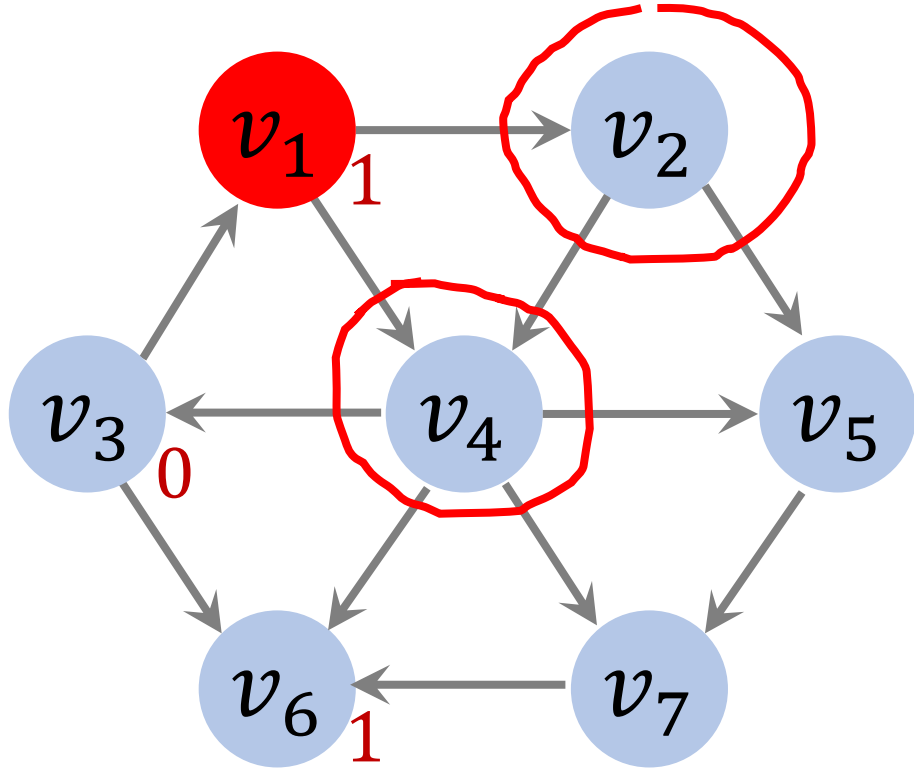
Queue:

v_1

v_6

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 2

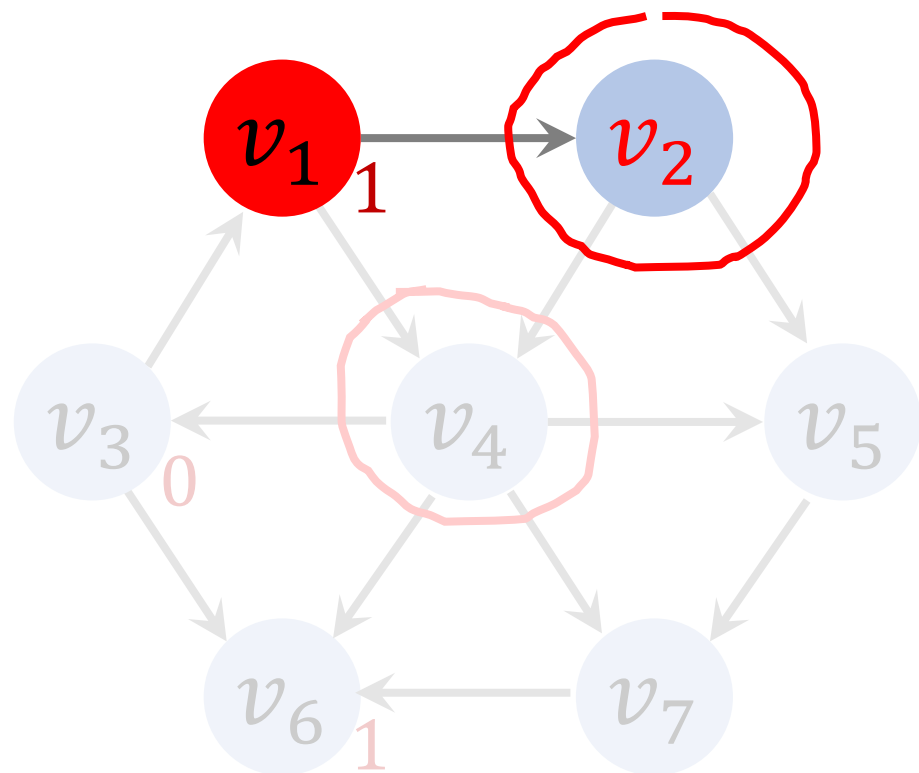


Queue:

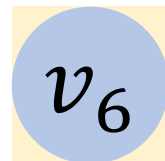
v_6

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 2(A)

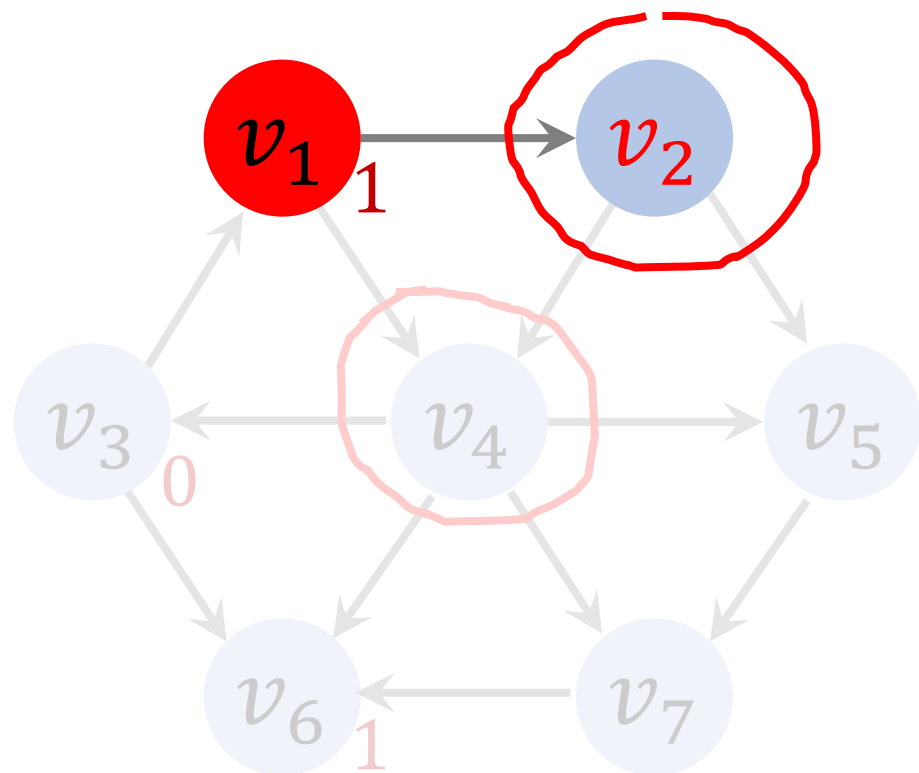


Queue:

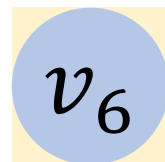


vertex	visit	dist	path
v_1	yes	1	v_3
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 2(A)

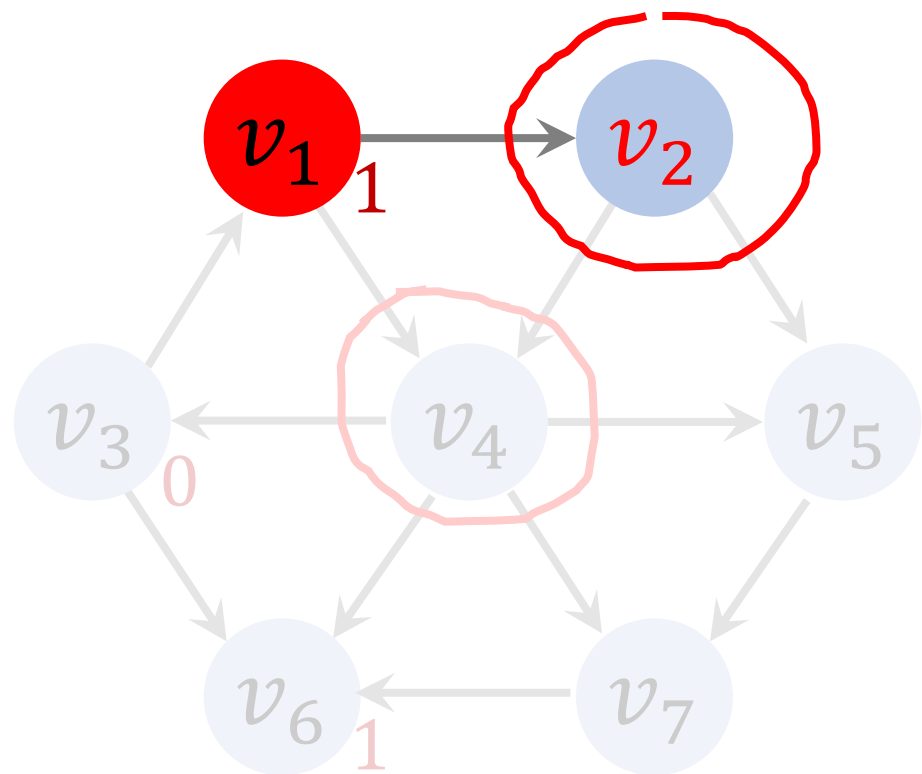


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 2(A)



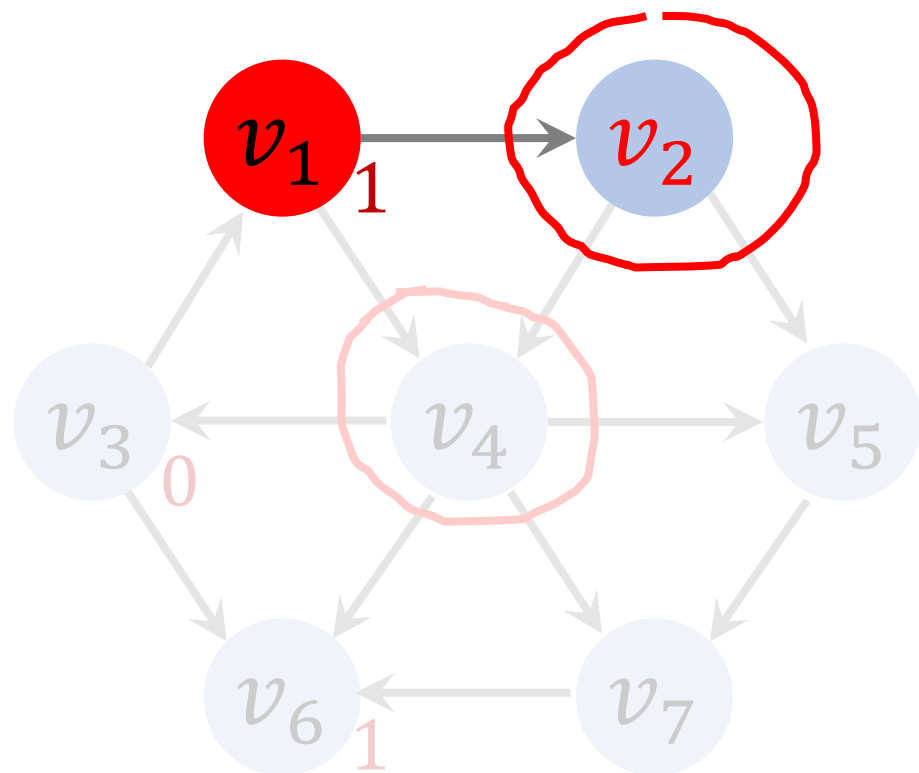
Queue:

v_6

v_2

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	no	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 2(A)



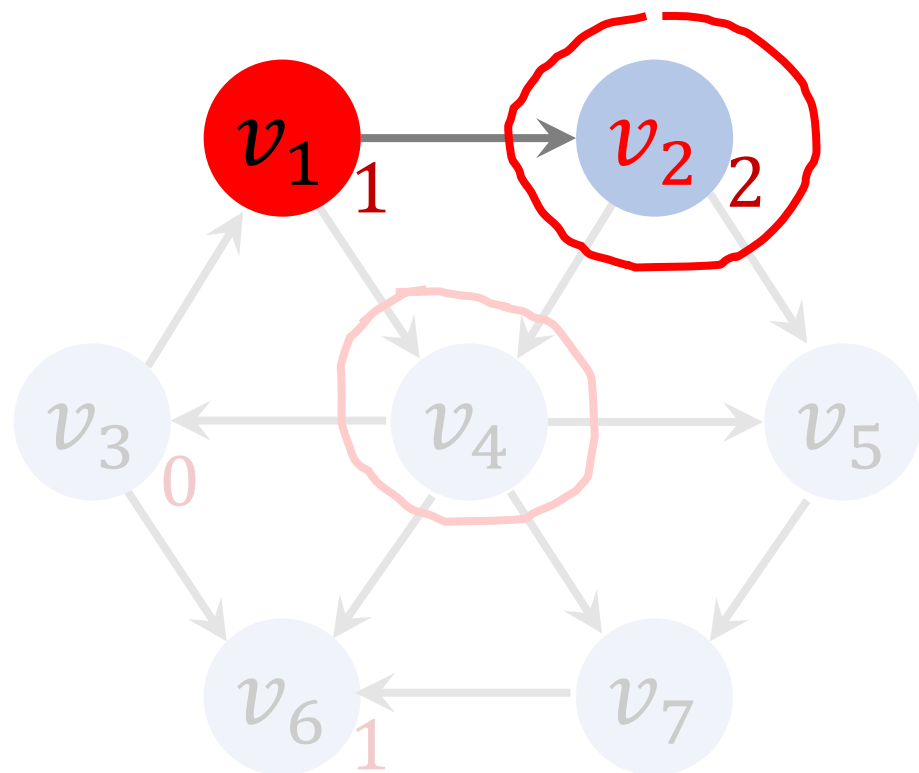
Queue:

v_6

v_2

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	∞	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 2(A)



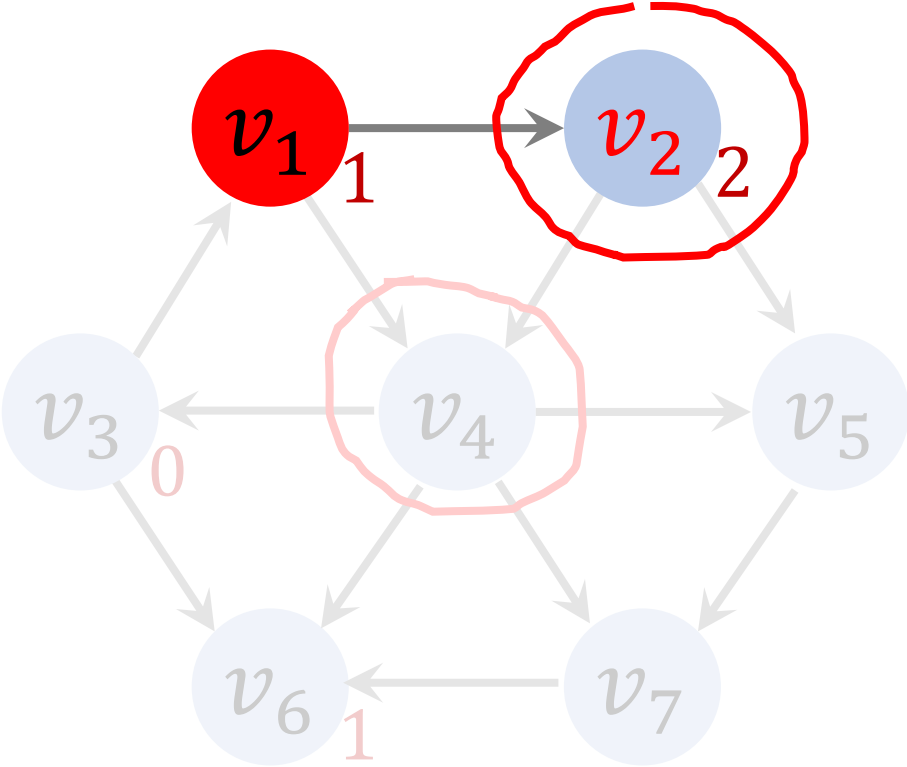
Queue:

v_6

v_2

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	0
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 2(A)

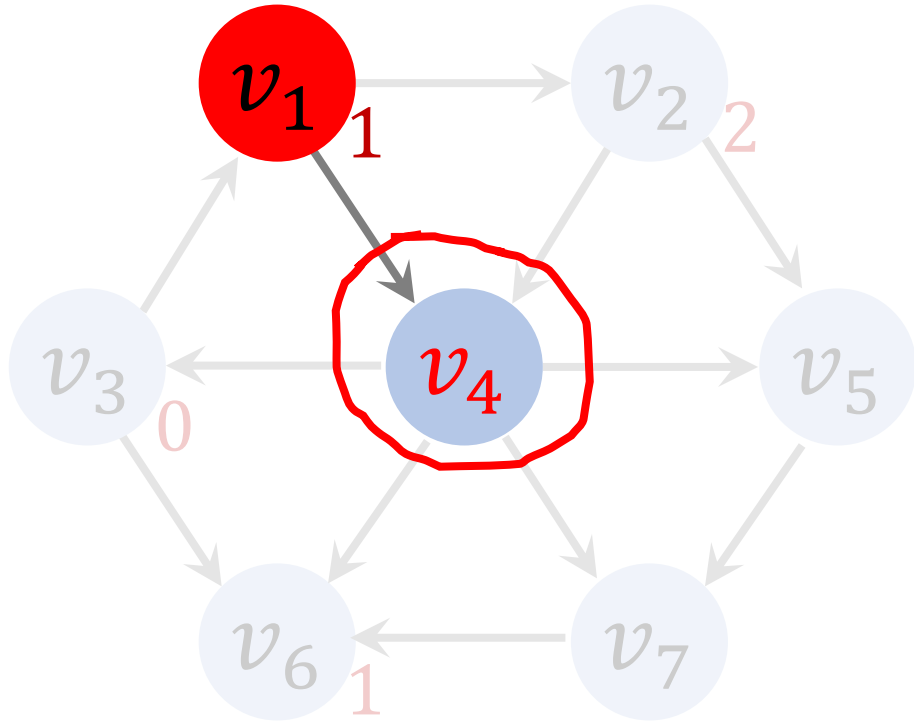


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 2(B)

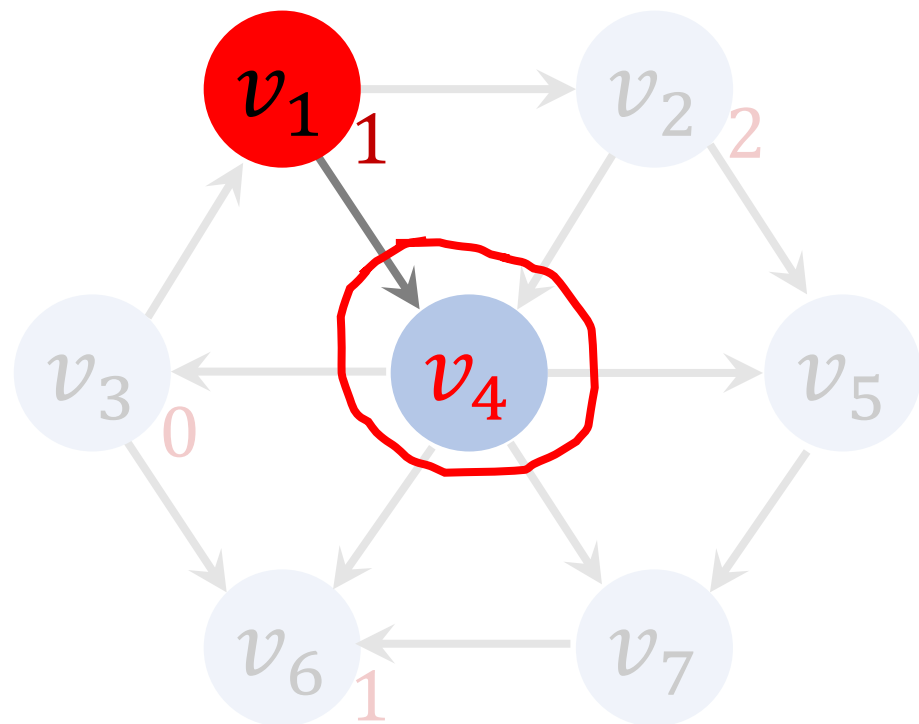


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 2(B)



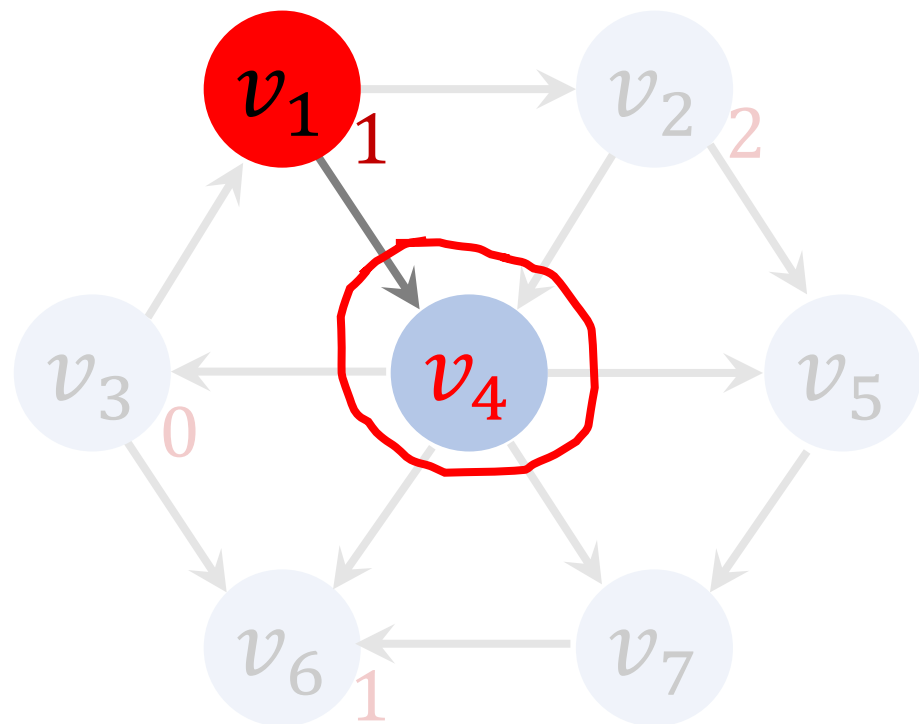
Queue:

v_6

v_2

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 2(B)

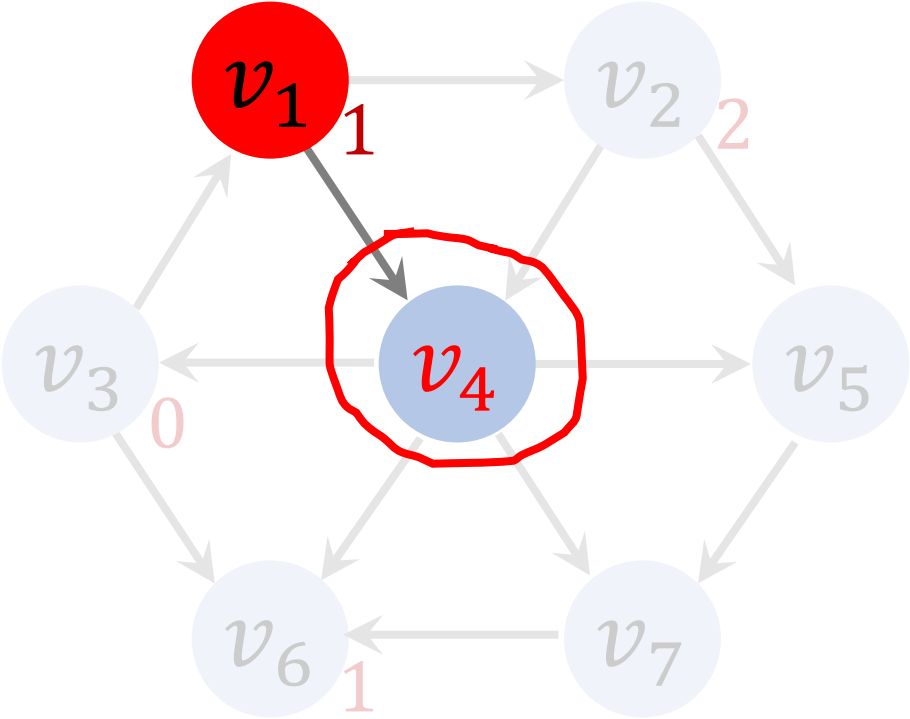


Queue:

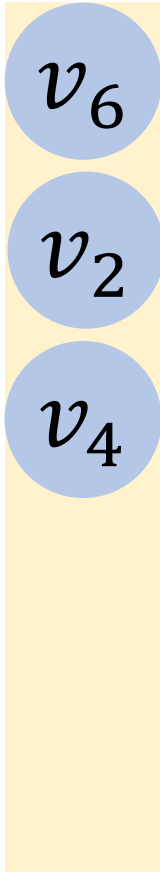


vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	no	∞	0
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 2(B)

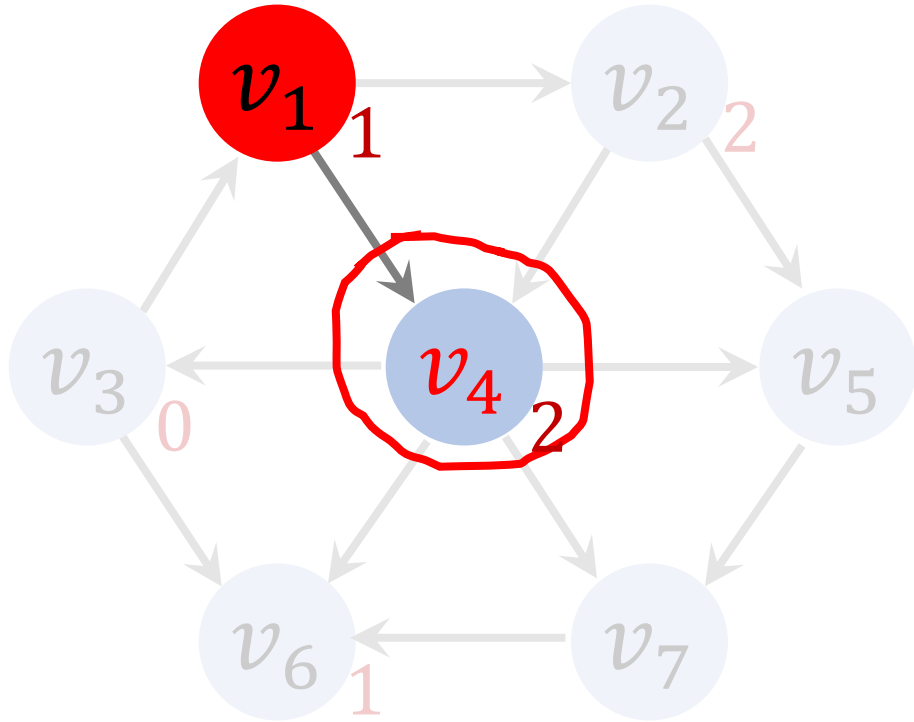


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	∞	0
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 2(B)

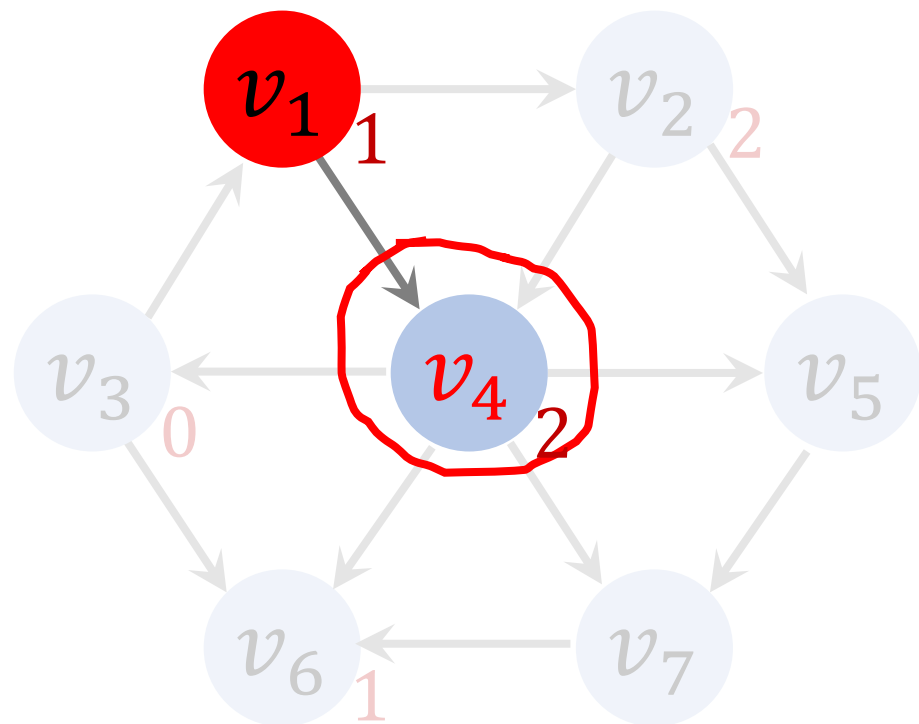


Queue:

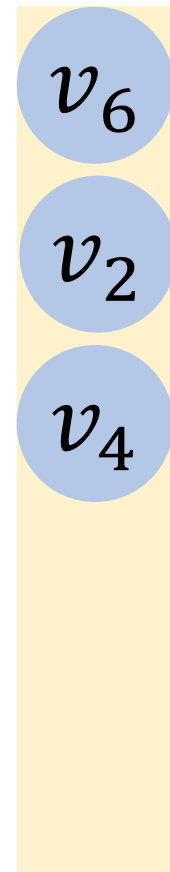


vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	0
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 2(B)

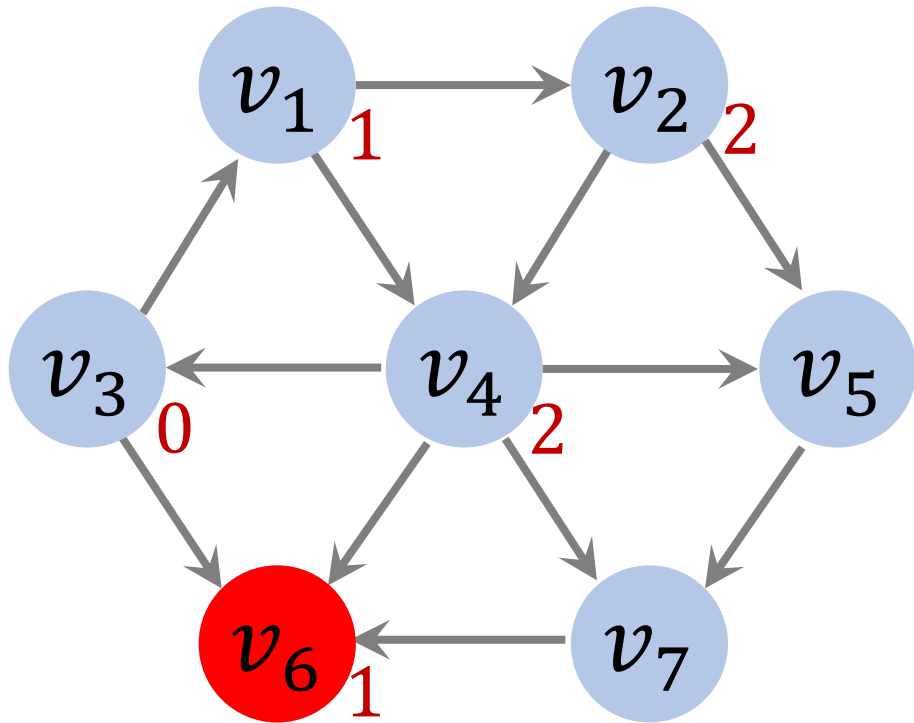


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 3



Queue:

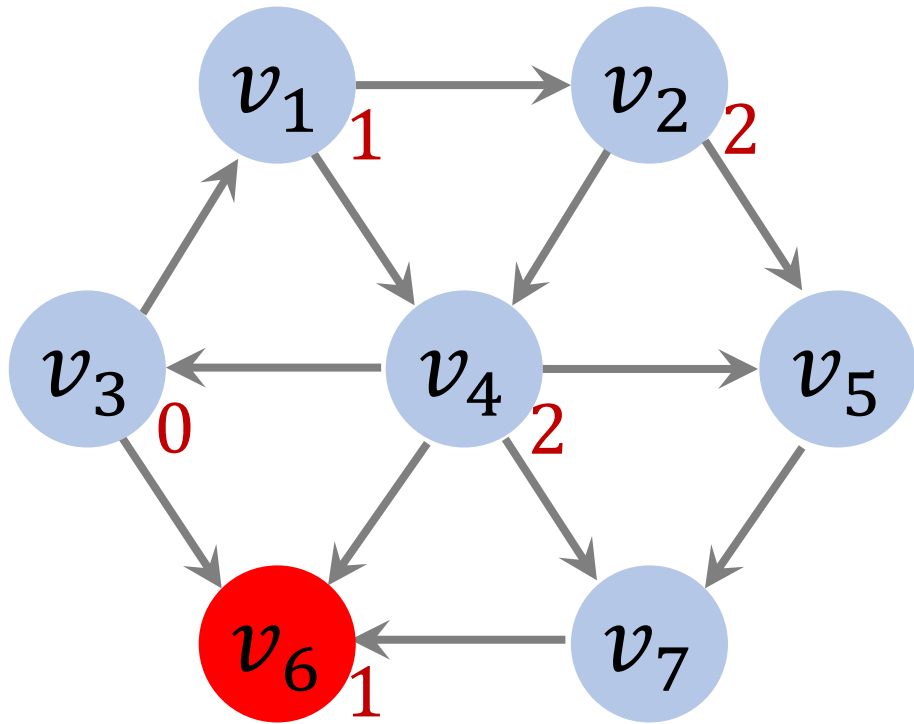
v_6

v_2

v_4

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 3



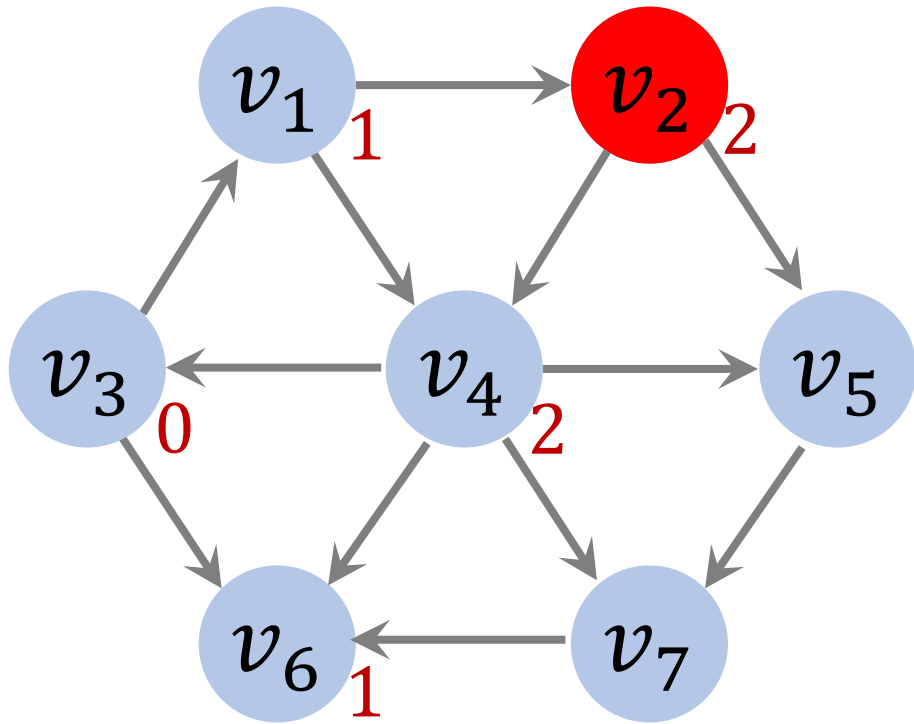
Queue:

v_2

v_4

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 4



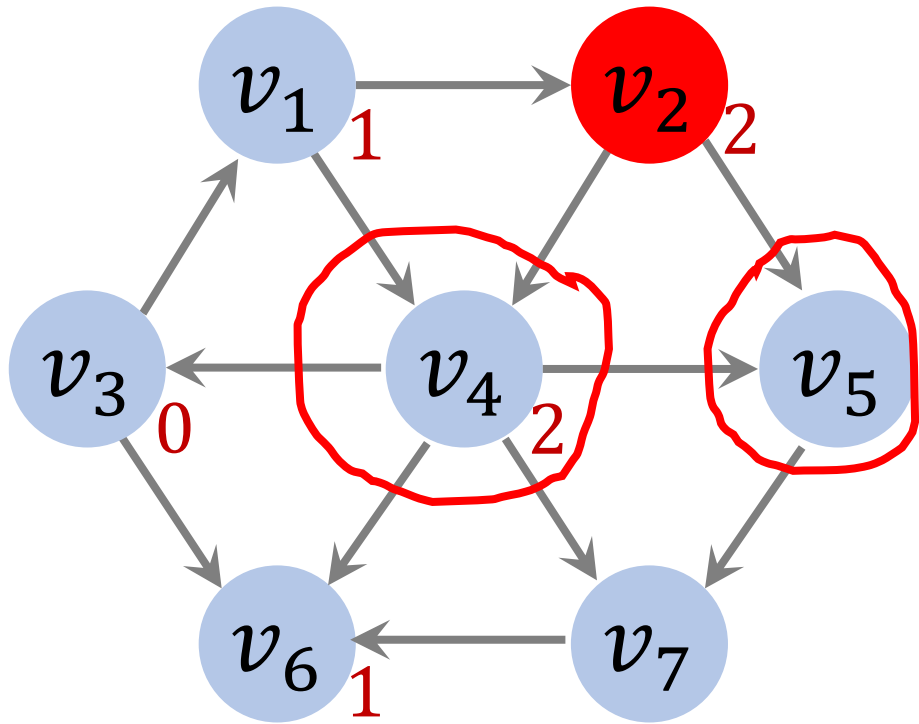
Queue:

v_2

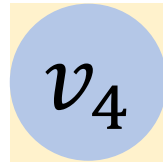
v_4

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 4

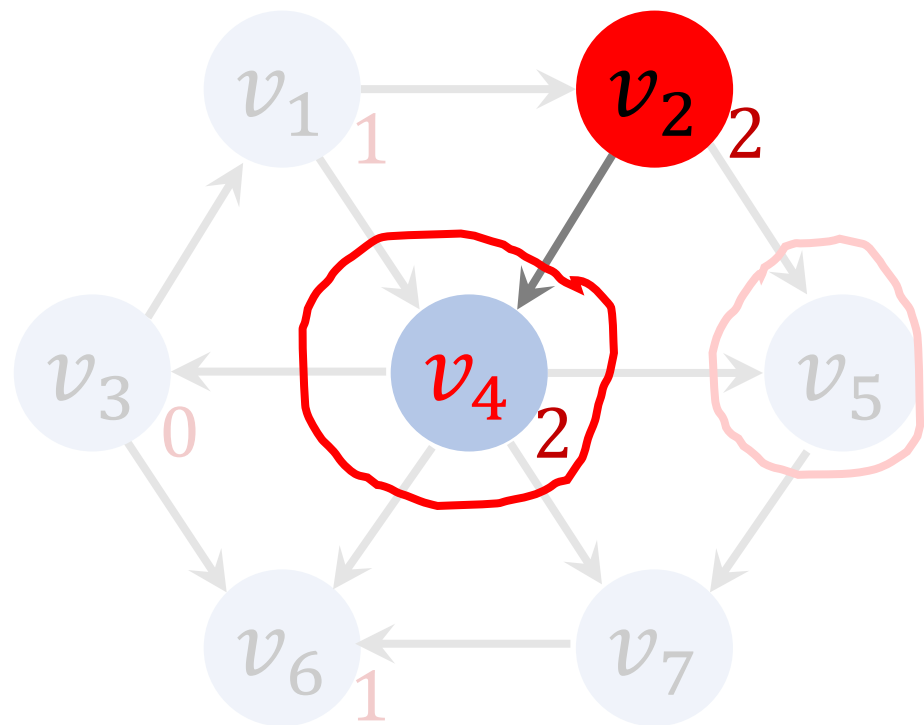


Queue:

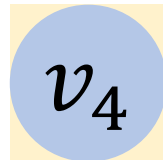


vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 4(A)

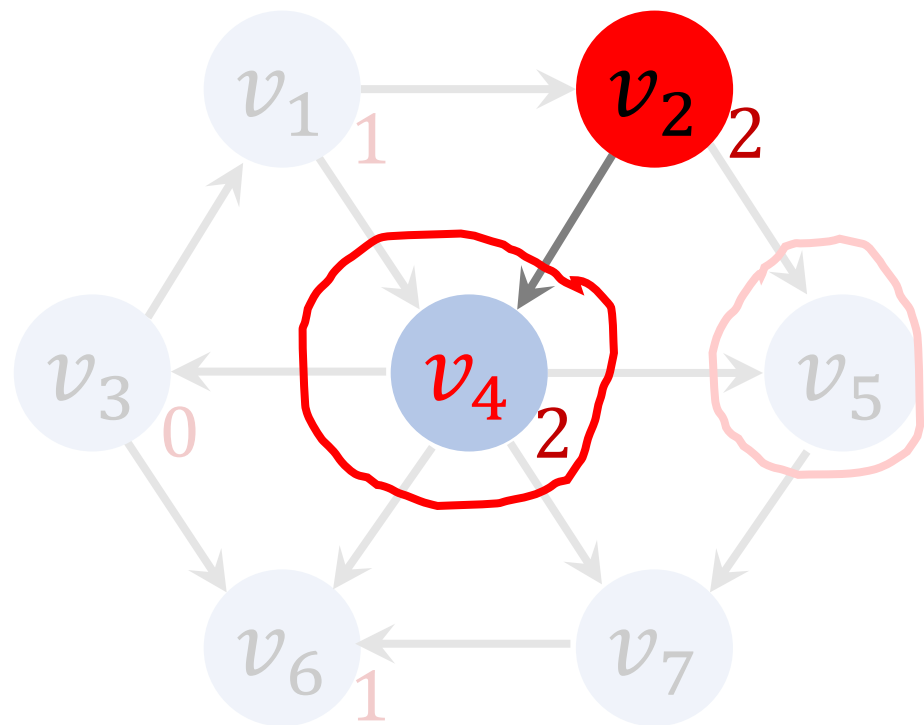


Queue:

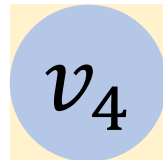


vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 4(A)

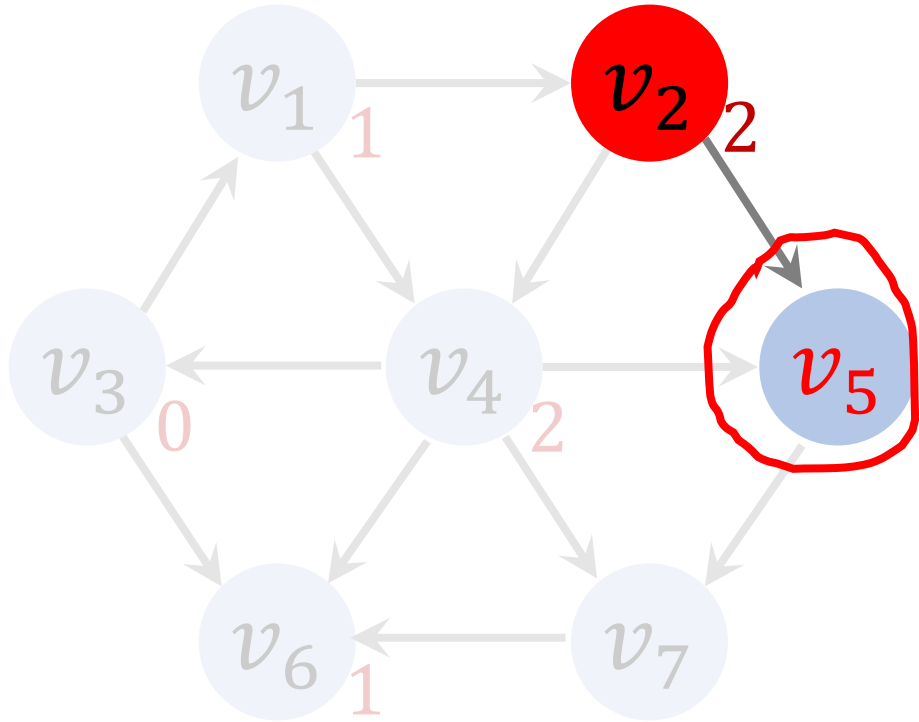


Queue:

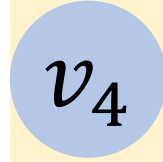


vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 4(B)

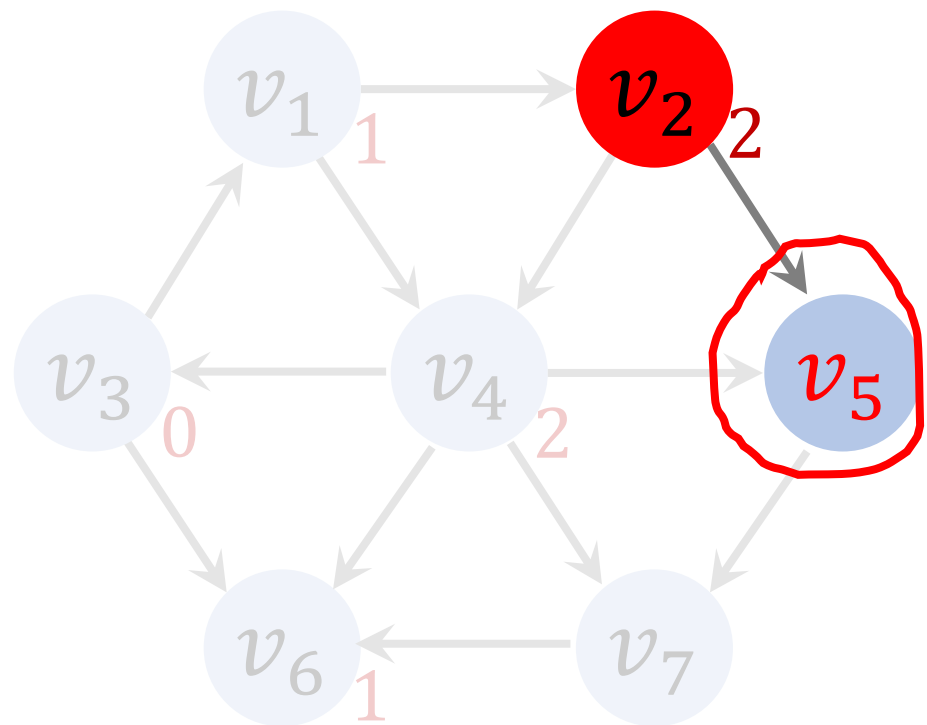


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 4(B)

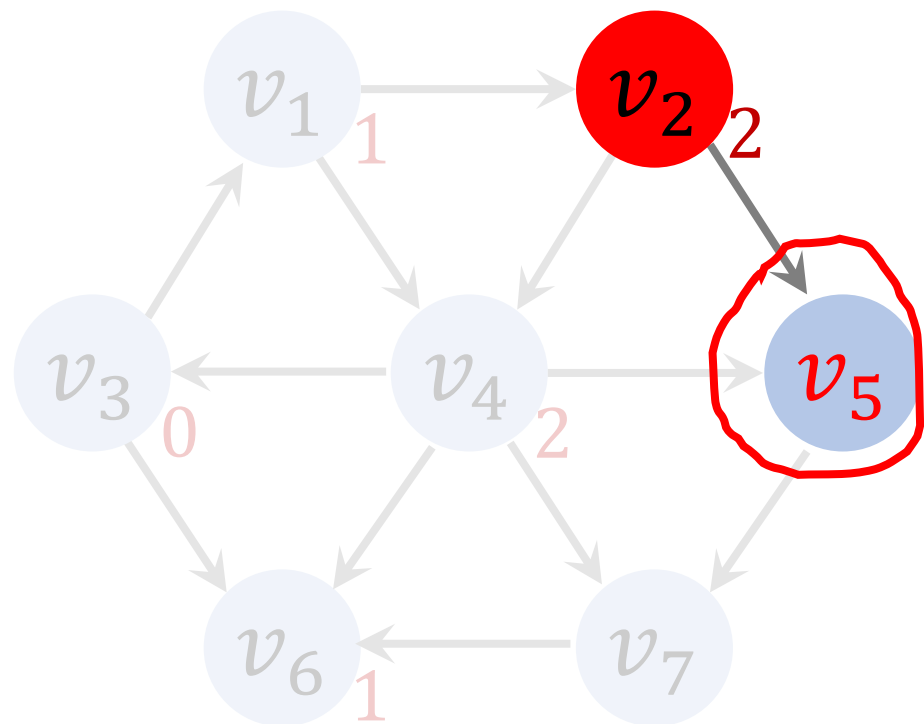


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 4(B)



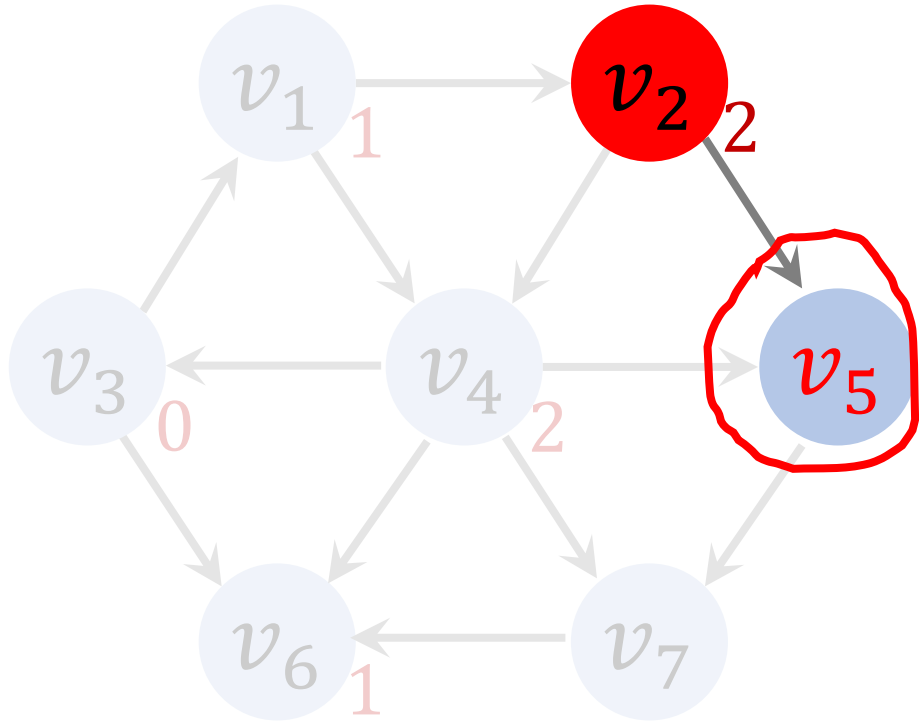
Queue:

v_4

v_5

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	no	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 4(B)

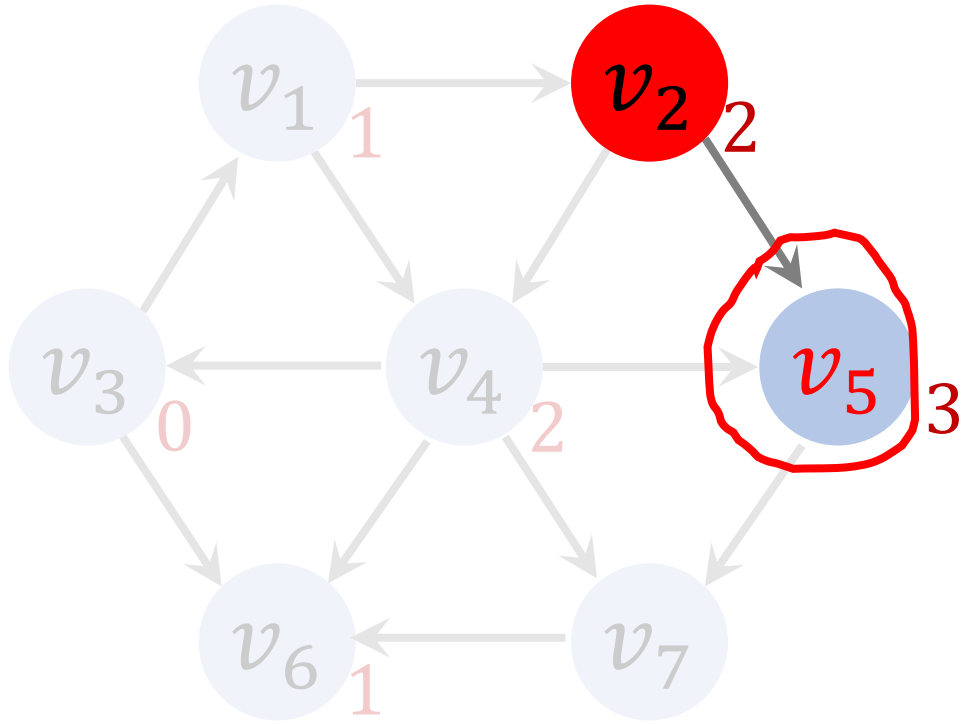


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	∞	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 4(B)



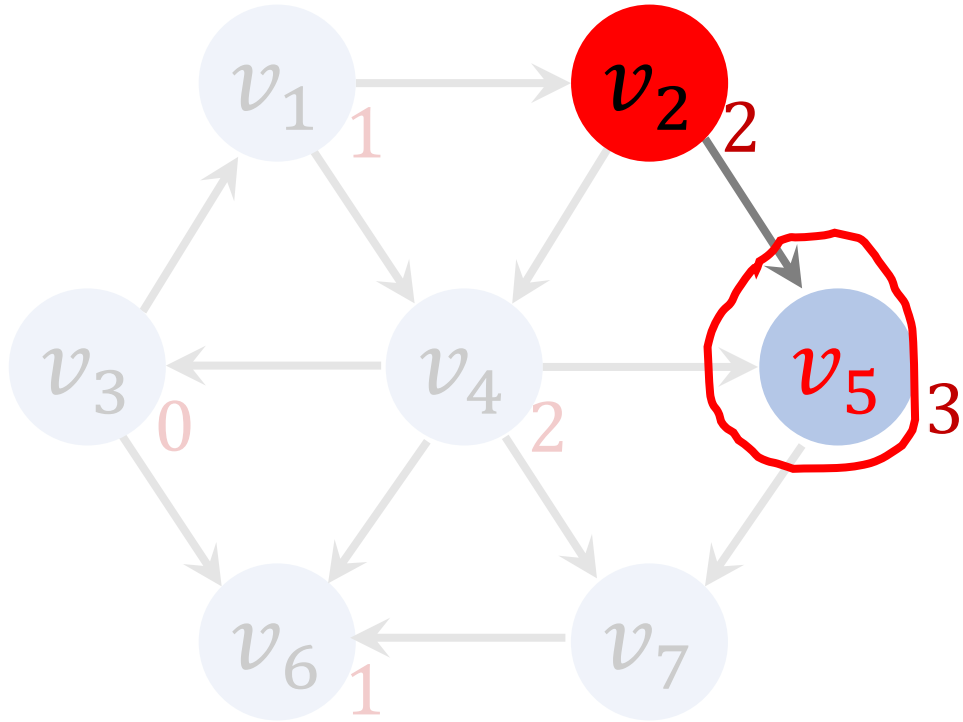
Queue:

v_4

v_5

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	0
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 4(B)



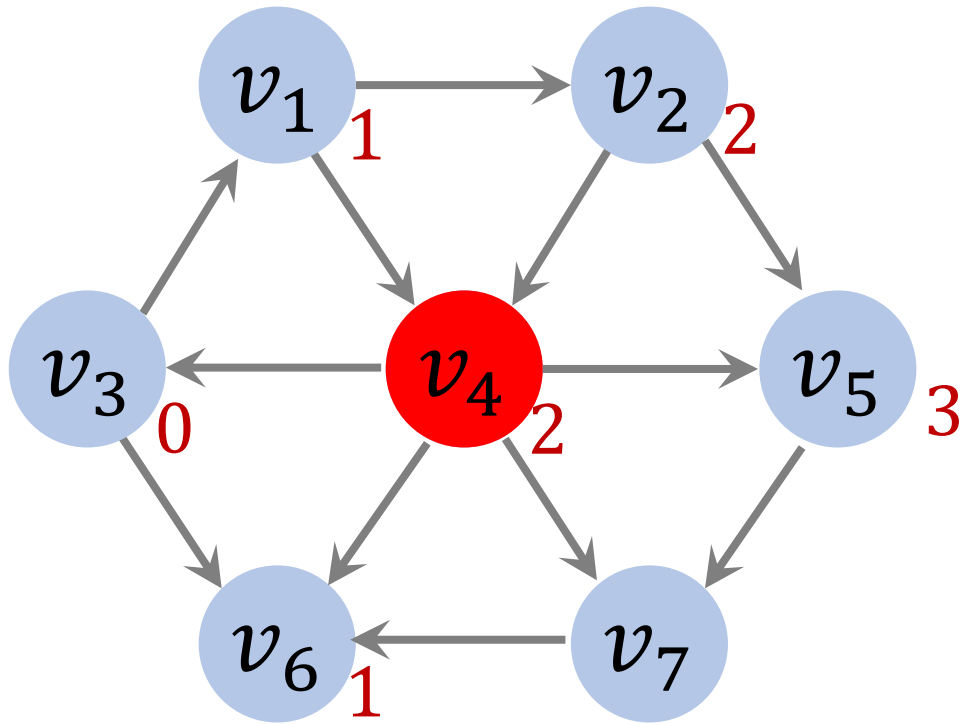
Queue:

v_4

v_5

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 5



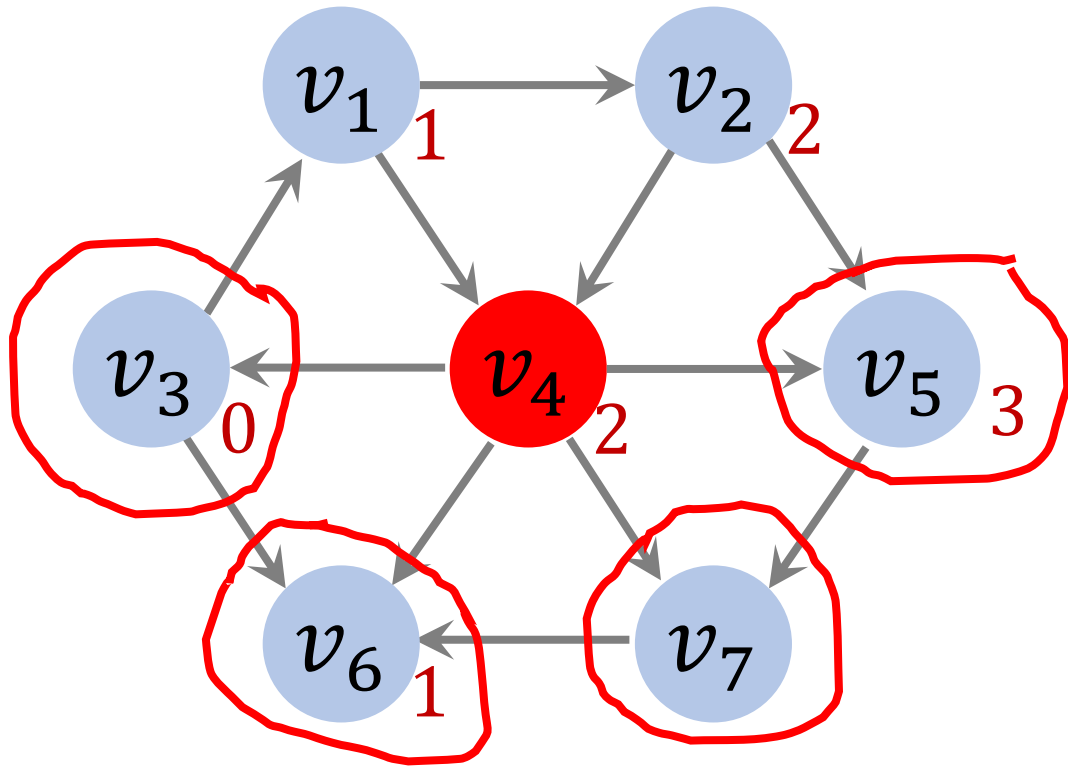
Queue:

v_4

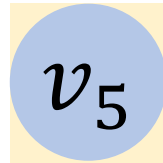
v_5

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 5

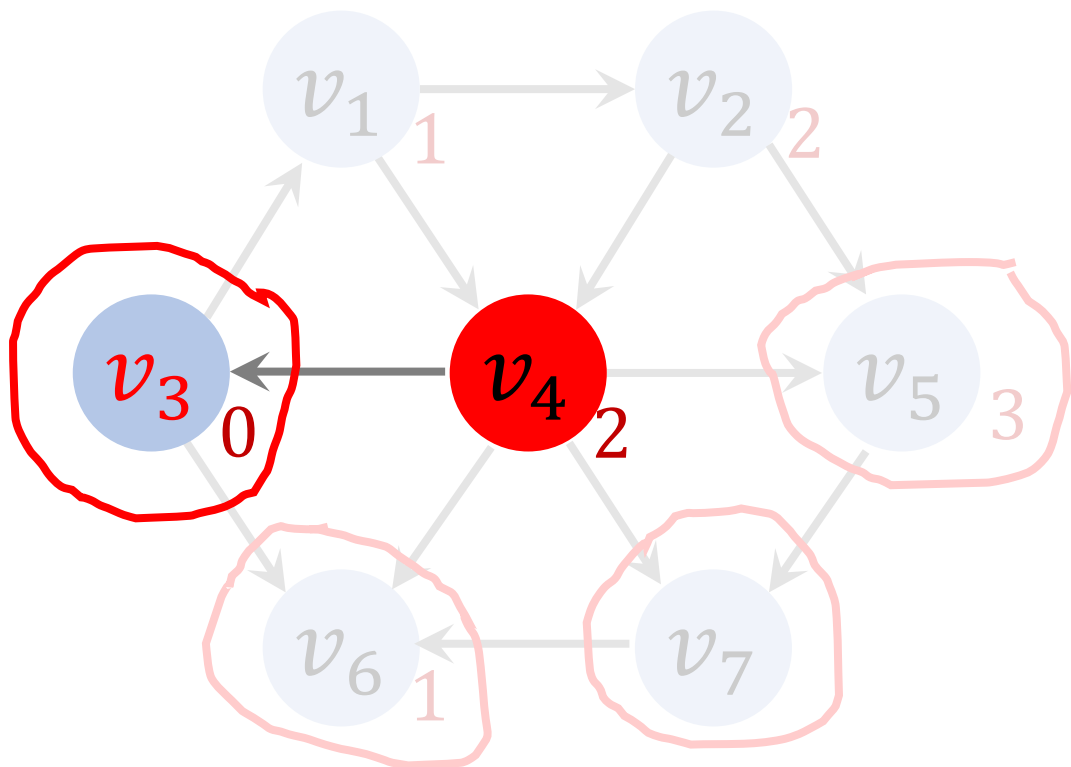


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 5(A)

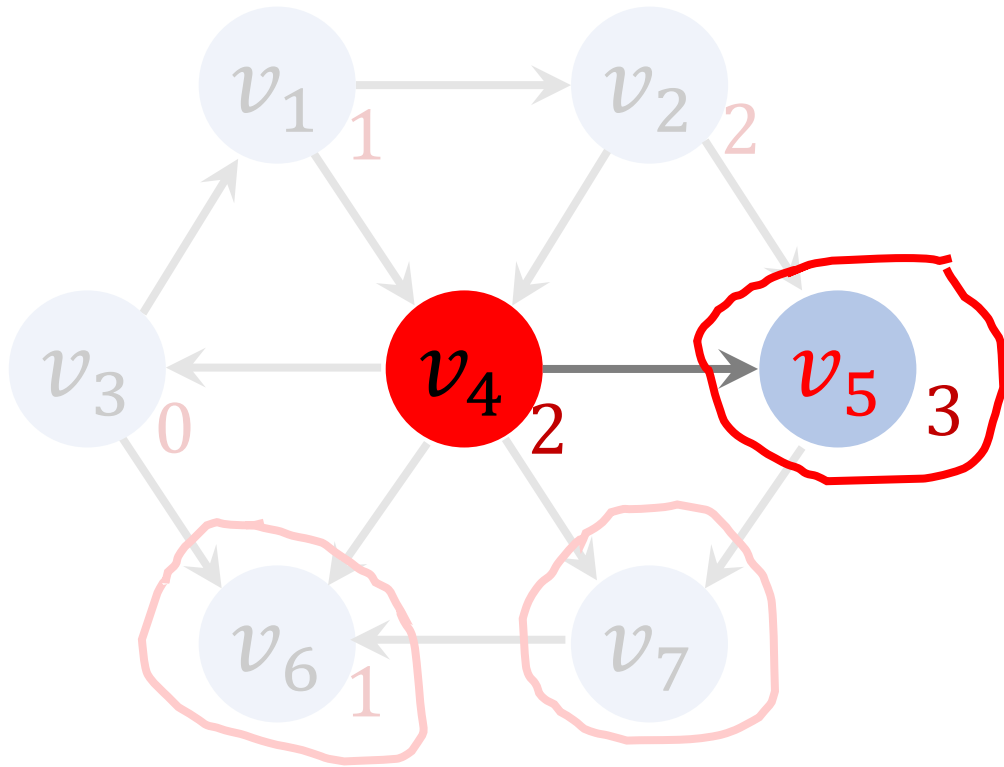


Queue:

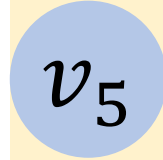
v_5

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 5(B)

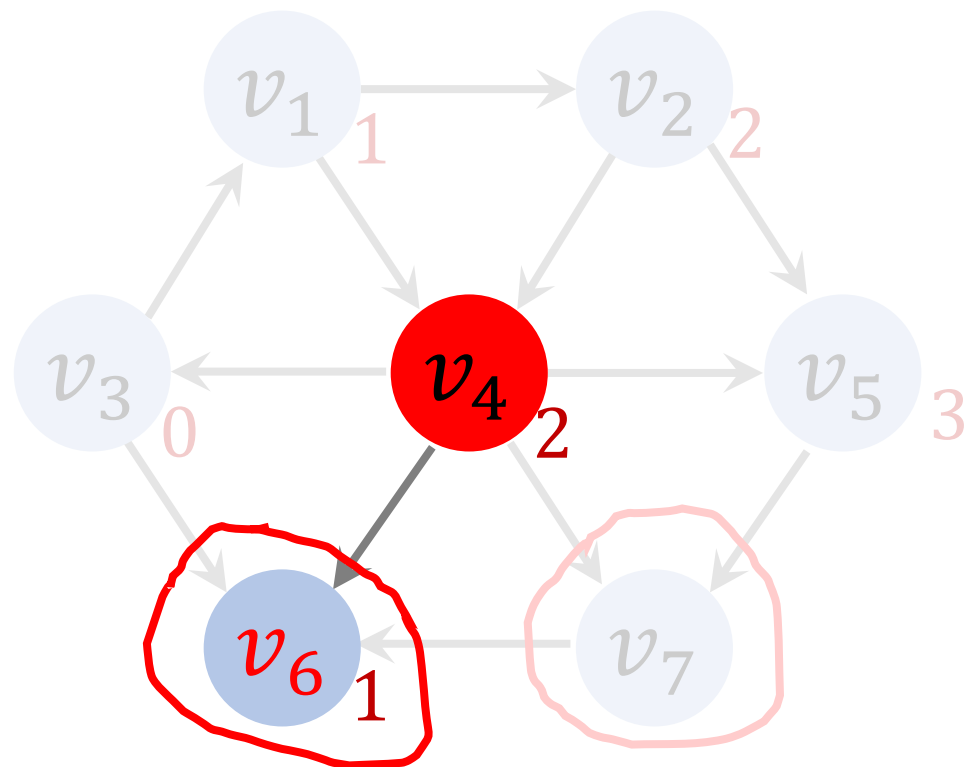


Queue:

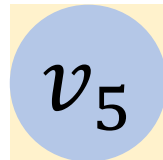


vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 5(C)

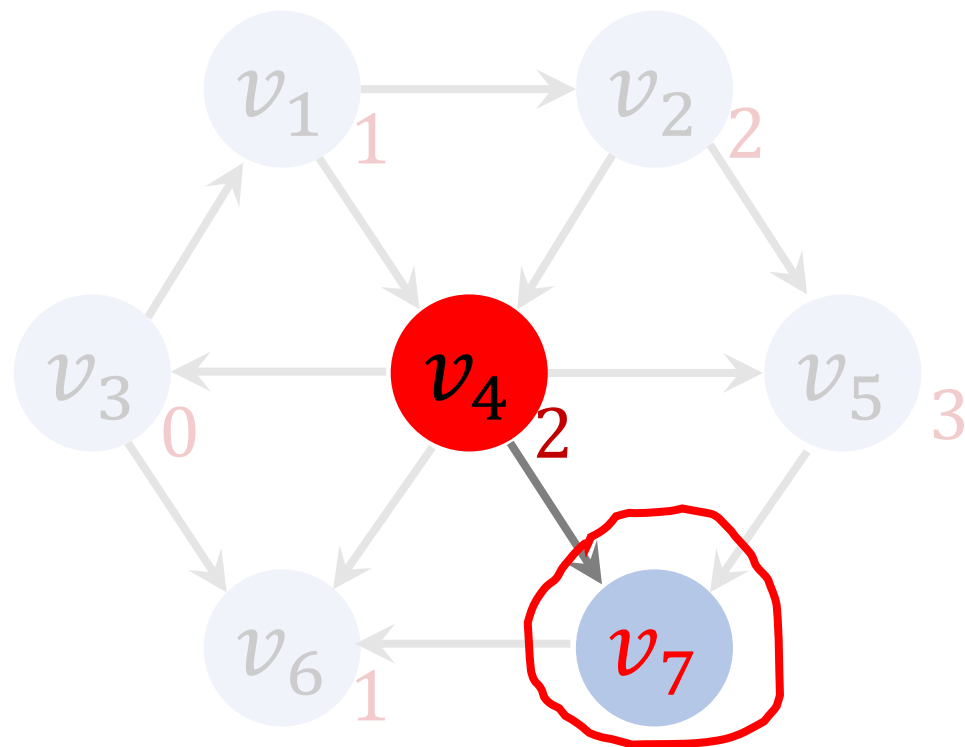


Queue:

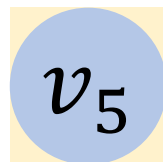


vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 5(D)

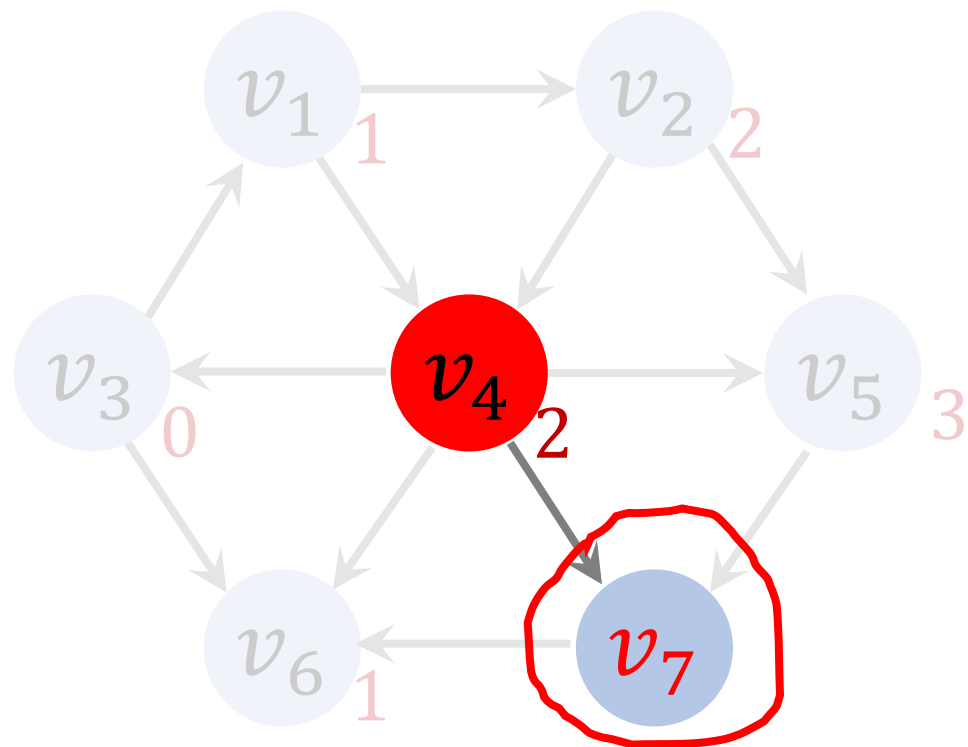


Queue:

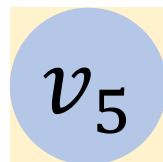


vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 5(D)

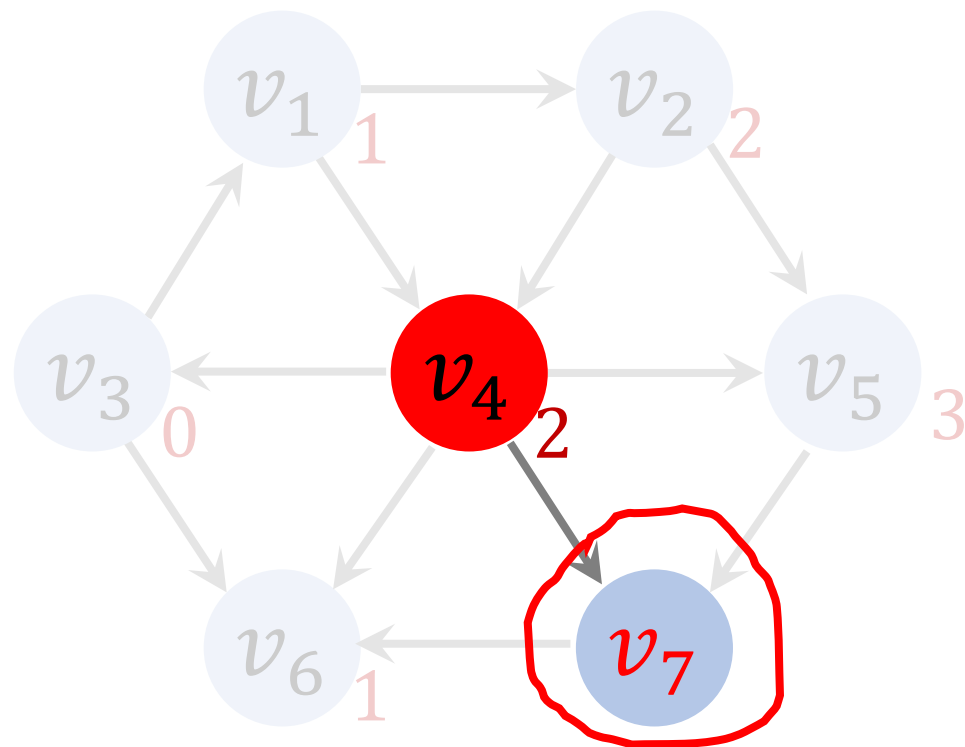


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 5(D)



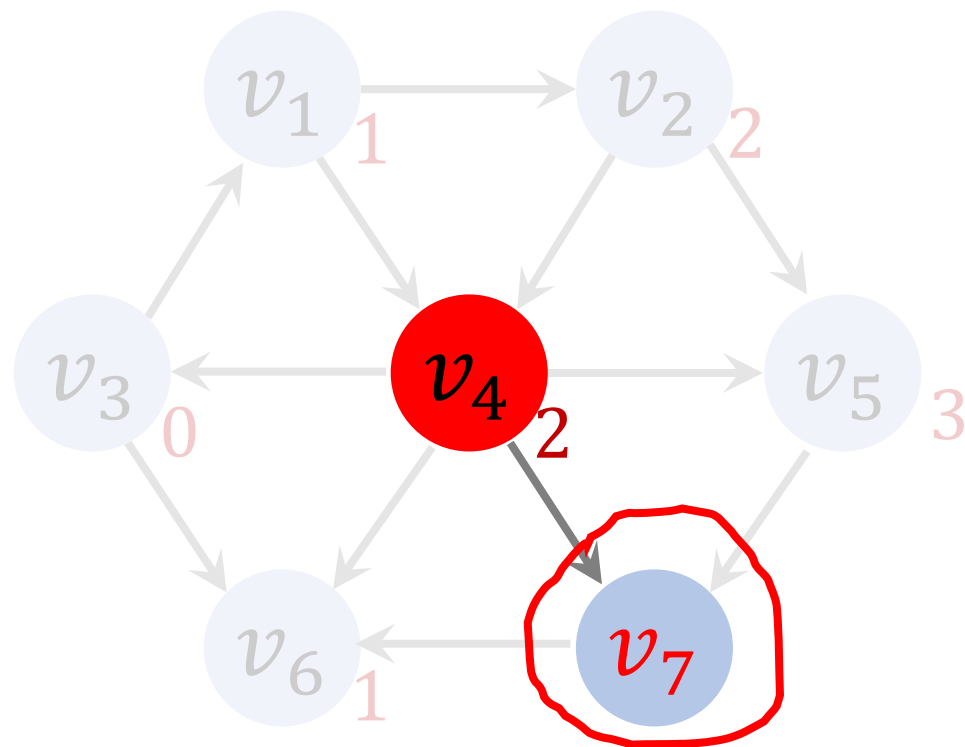
Queue:

v_5

v_7

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	no	∞	0

Iteration 5(D)



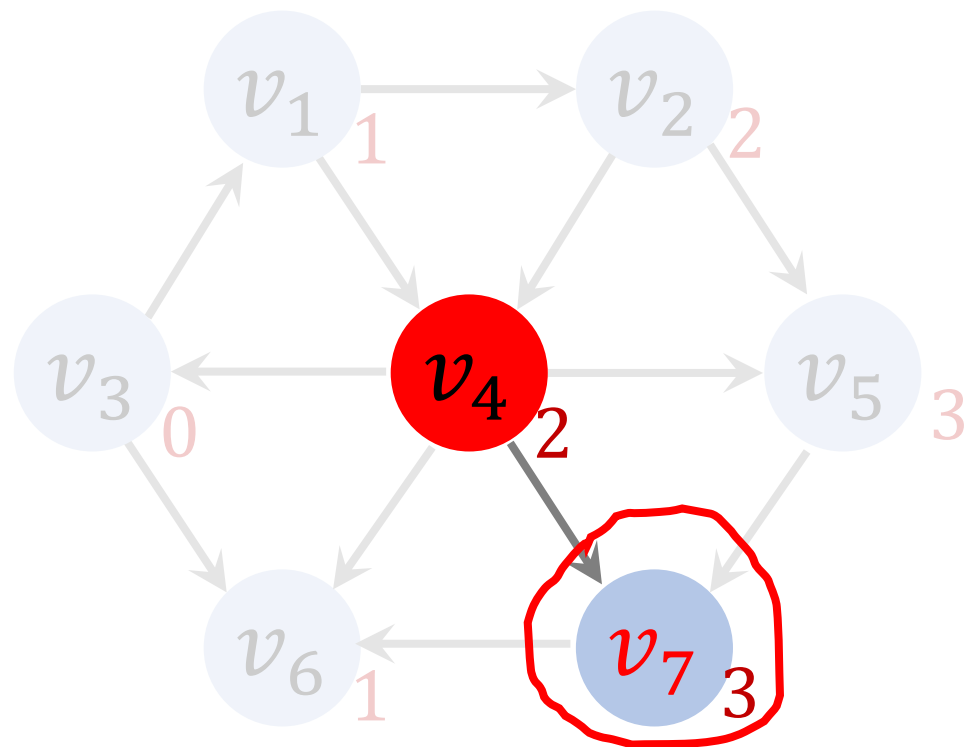
Queue:

v_5

v_7

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	yes	∞	0

Iteration 5(D)



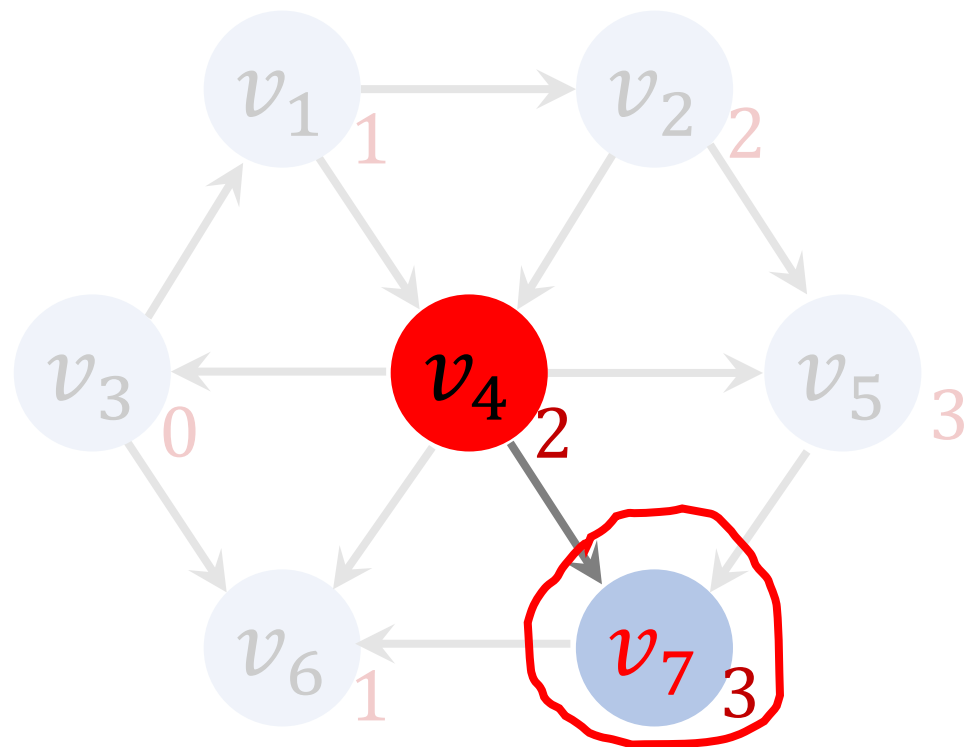
Queue:

v_5

v_7

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	yes	3	0

Iteration 5(D)



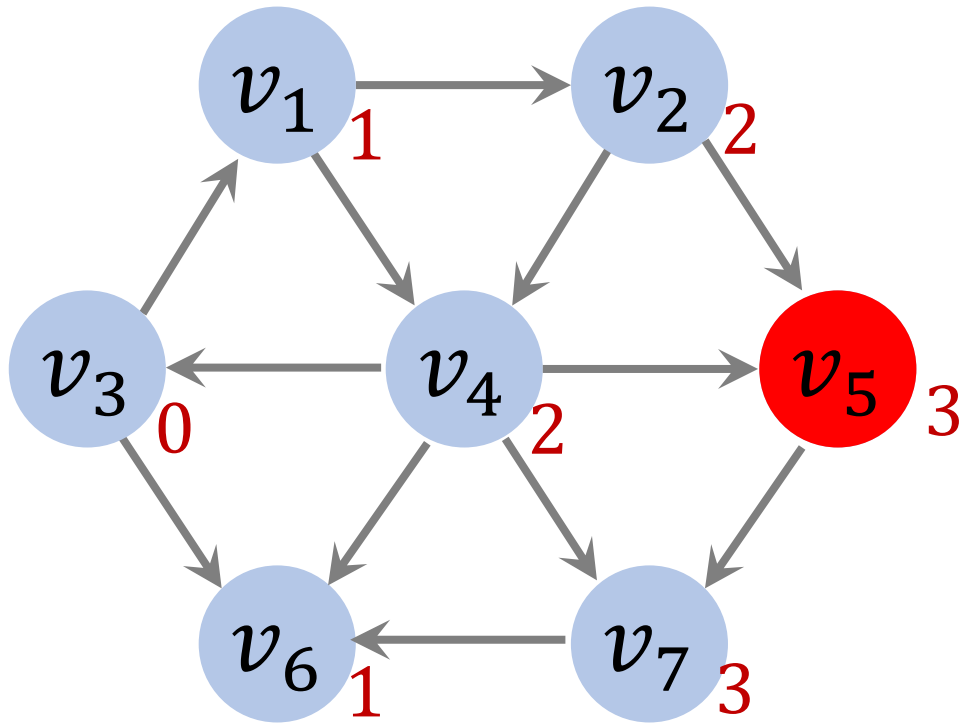
Queue:

v_5

v_7

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	yes	3	v_4

Iteration 6



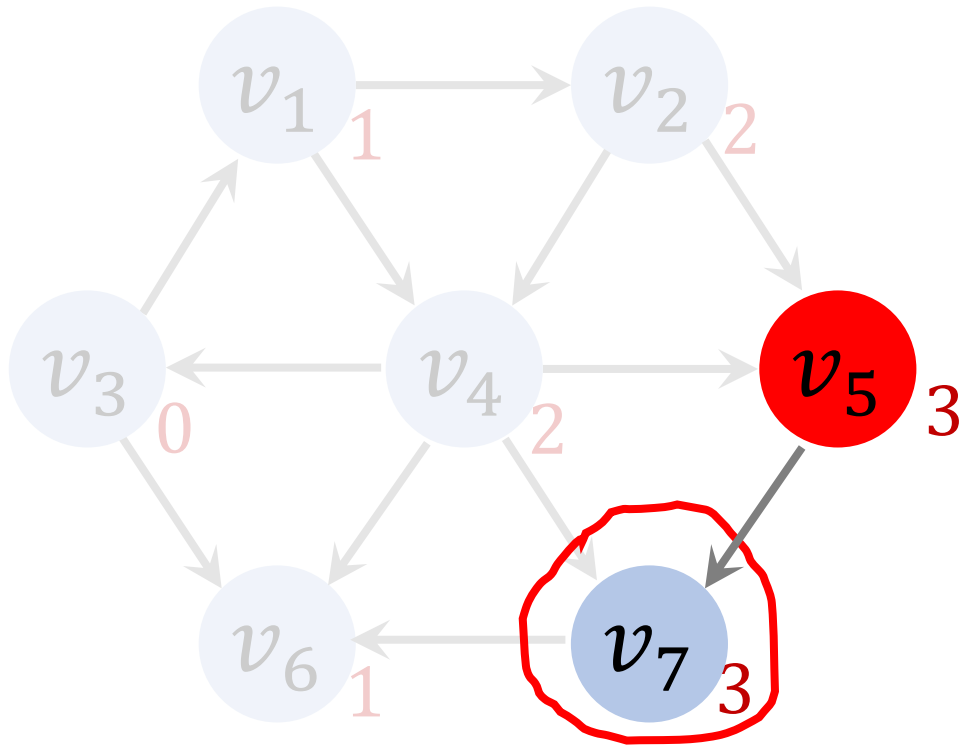
Queue:

v_5

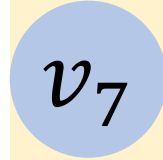
v_7

vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	yes	3	v_4

Iteration 6

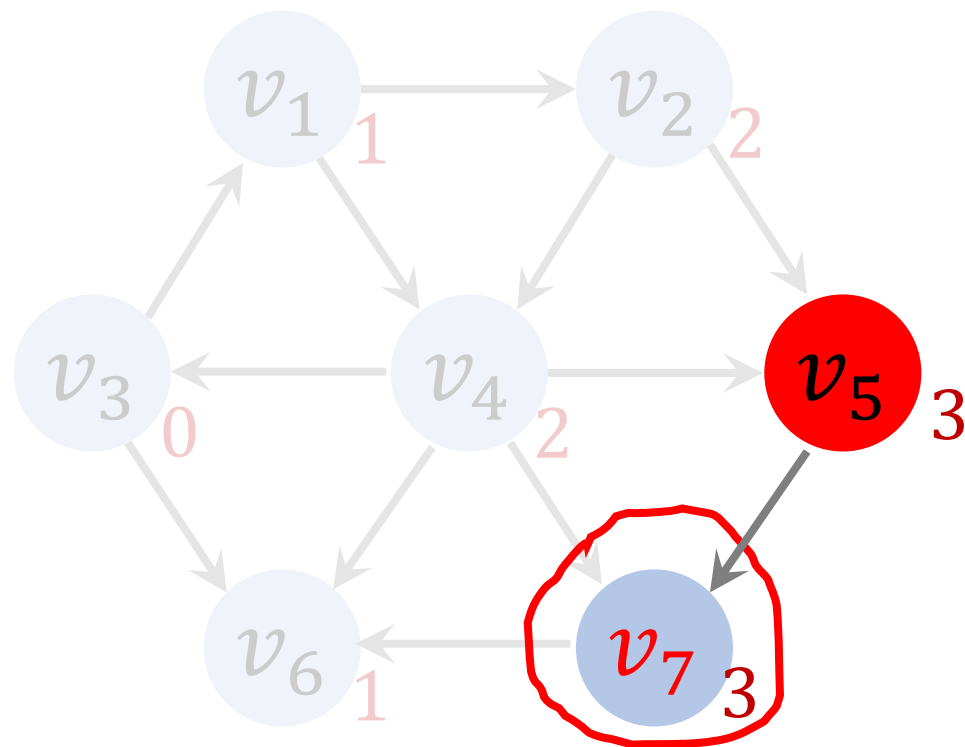


Queue:

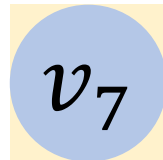


vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	yes	3	v_4

Iteration 6(A)

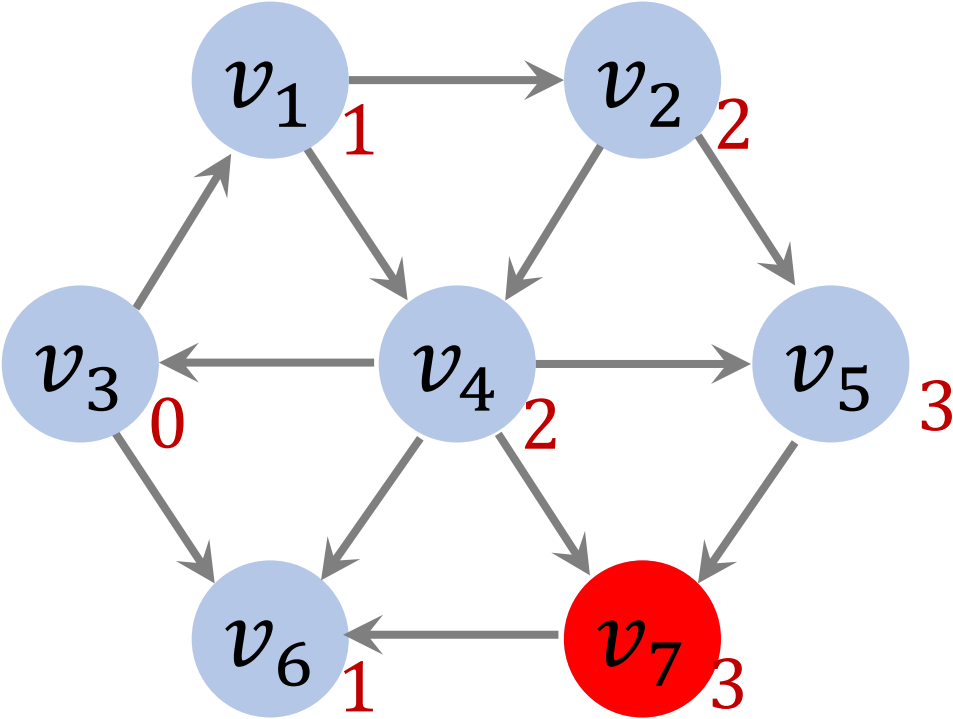


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	yes	3	v_4

Iteration 7

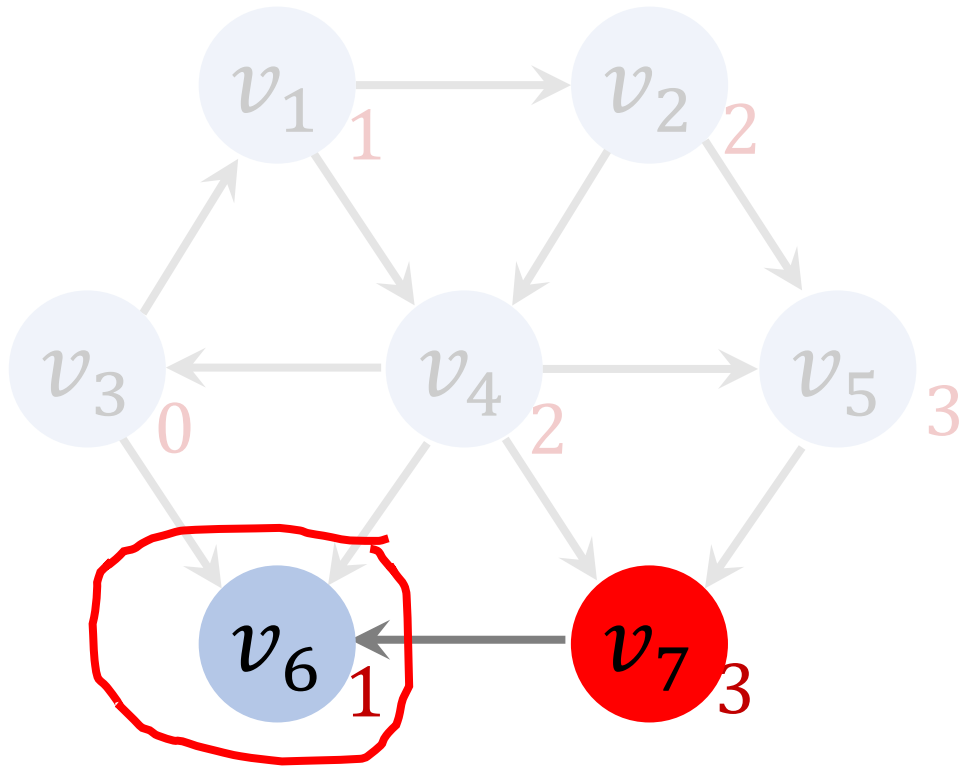


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	yes	3	v_4

Iteration 7

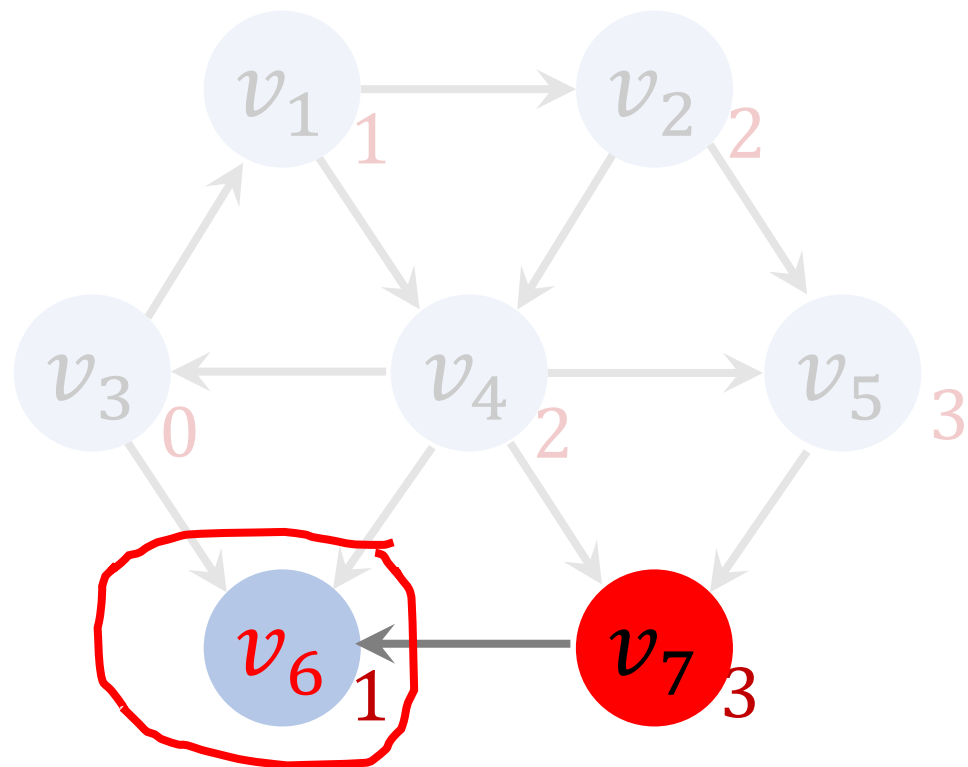


Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	yes	3	v_4

Iteration 7(A)

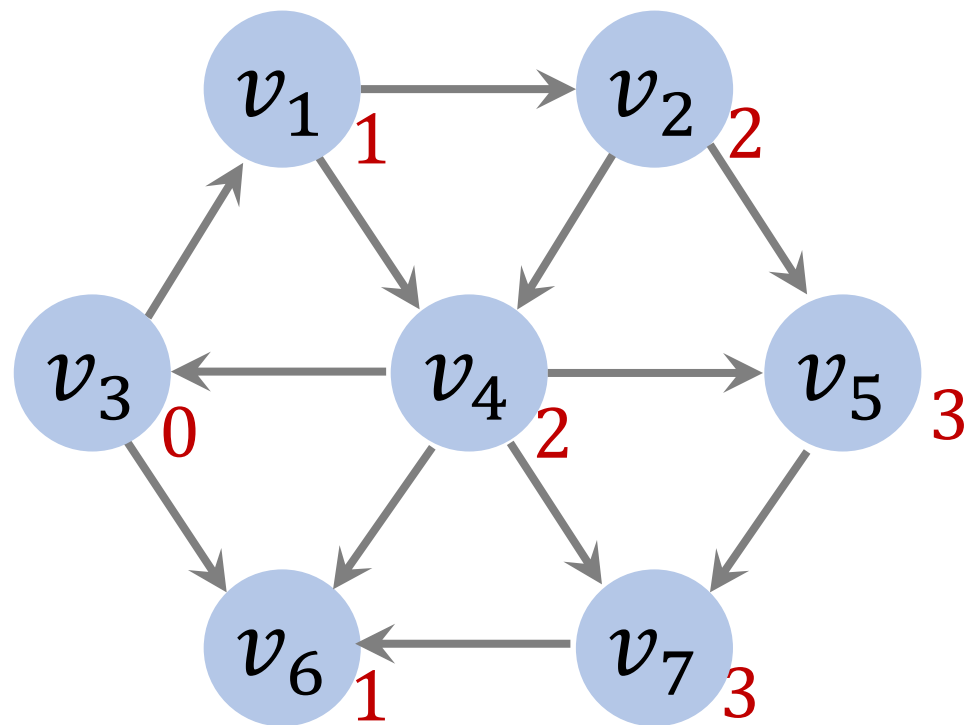


Queue:

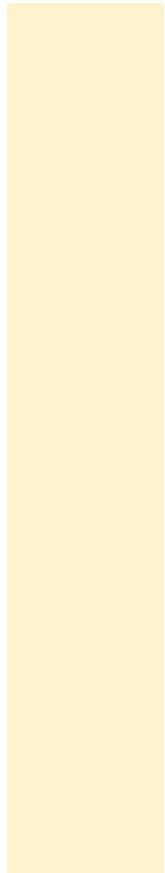


vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	yes	3	v_4

End of Procedure



Queue:



vertex	visit	dist	path
v_1	yes	1	v_3
v_2	yes	2	v_1
v_3	yes	0	0
v_4	yes	2	v_1
v_5	yes	3	v_2
v_6	yes	1	v_3
v_7	yes	3	v_4

Pseudo Code

Inputs: vertices \mathcal{V} , edges \mathcal{E} , and the source vertex s .

1. Initialize an empty queue.

Pseudo Code

Inputs: vertices \mathcal{V} , edges \mathcal{E} , and the source vertex s .

1. Initialize an empty queue.
2. For each vertex $v \in \mathcal{V}$:
 - a. Set $\text{visit}[v] \leftarrow \text{false}$.
 - b. Set $\text{dist}[v] \leftarrow \infty$.
 - c. Set $\text{path}[v] \leftarrow 0$.

vertex	visit	dist	path
v_1	false	∞	0
v_2	false	∞	0
\vdots	\vdots	\vdots	\vdots
v_n	false	∞	0

Pseudo Code

Inputs: vertices \mathcal{V} , edges \mathcal{E} , and the source vertex s .

1. Initialize an empty queue.
2. For each vertex $v \in \mathcal{V}$:
 - a. Set $\text{visit}[v] \leftarrow \text{false}$.
 - b. Set $\text{dist}[v] \leftarrow \infty$.
 - c. Set $\text{path}[v] \leftarrow 0$.
3. $\text{enqueue}(s)$.
4. Set $\text{visit}[s] \leftarrow \text{true}$ and $\text{dist}[s] \leftarrow 0$.

Pseudo Code (Cont.)

5. While the queue is not empty:

a. $v \leftarrow \text{dequeue}()$.

b. $\mathcal{S} \leftarrow \{u \mid \underline{e_{vu}} \in \mathcal{E} \text{ and } \underline{\text{visit}[u]} = \text{true}\}.$



Pseudo Code (Cont.)

5. While the queue is not empty:


a. $v \leftarrow \text{dequeue}()$.

b. $\mathcal{S} \leftarrow \{u \mid e_{vu} \in \mathcal{E} \text{ and } \text{visit}[u] = \text{true}\}$.

c. For each $u \in \mathcal{S}$:

 i. $\text{enqueue}(u)$.

 ii. $\text{visit}[u] \leftarrow \text{true}$.

 iii. $\text{dist}[u] \leftarrow \text{dist}[v] + 1$.

 iv. $\text{path}[u] \leftarrow v$.

Pseudo Code (Cont.)

5. While the queue is not empty:
 - a. $v \leftarrow \text{dequeue}()$.
 - b. $\mathcal{S} \leftarrow \{u \mid e_{vu} \in \mathcal{E} \text{ and } \text{visit}[u] = \text{true}\}$.
 - c. For each $u \in \mathcal{S}$:
 - i. $\text{enqueue}(u)$.
 - ii. $\text{visit}[u] \leftarrow \text{true}$.
 - iii. $\text{dist}[u] \leftarrow \text{dist}[v] + 1$.
 - iv. $\text{path}[u] \leftarrow v$.

Outputs: $\text{dist}[v]$ and $\text{path}[v]$ for all $v \in \mathcal{V}$.

Time Complexity

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The time complexity is $O(|\mathcal{V}| + |\mathcal{E}|)$. (Why?)

First, the initialization has $O(|\mathcal{V}|)$ time complexity.

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First, the initialization has $O(|\mathcal{V}|)$ time complexity.

Second, queue operations has a total of $O(|\mathcal{V}|)$ time complexity.

- Every vertex is enqueued and dequeued exactly once.
- Enqueue and dequeue operations have constant time complexity.

Time Complexity

The time complexity is $O(|\mathcal{V}| + |\mathcal{E}|)$. (Why?)

First, the initialization has $O(|\mathcal{V}|)$ time complexity.

Second, queue operations has a total of $O(|\mathcal{V}|)$ time complexity.

Third, every edge is touched once; thus $O(|\mathcal{E}|)$ time complexity.

- Once vertex v is dequeued, all the edges from v are touched.
- Every vertex is dequeued only once.
- Thus every edge is touched only once.

Thank You!

<http://wangshusen.github.io/>