SUBHAYAN DE

MECHANICS OF DEFORMABLE BODIES

Contents

Review of Statics

- Addition of a system of coplanar forces

  - Scalar notation:

$$F_x = F\cos(\theta)$$
$$F_y = F\sin(\theta)$$

  - Cartesian vector notation:

$$\vec{F} = F_x\hat{i} + F_y\hat{j}$$

- Force resultants

$$(\xrightarrow{+})\ F_{Rx} = \sum F_x$$
$$(+\uparrow)\ F_{Ry} = \sum F_y$$
$$F_R = \sqrt{F_{Rx}^2 + F_{Ry}^2}$$
$$\theta = \tan^{-1}\left|\frac{F_{Ry}}{F_{Rx}}\right|$$

- Cartesian vectors

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$
$$\cos\alpha = \frac{A_x}{A},\quad \cos\beta = \frac{A_y}{A},\quad \cos\gamma = \frac{A_z}{A}$$

- Unit vector

$$\vec{u}_A = \frac{\vec{A}}{A} = \frac{A_x}{A}\hat{i} + \frac{A_y}{A}\hat{j} + \frac{A_z}{A}\hat{k}$$
$$= \cos\alpha\ \hat{i} + \cos\beta\ \hat{j} + \cos\gamma\ \hat{k}$$
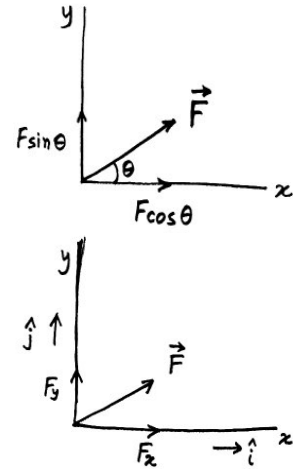$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$
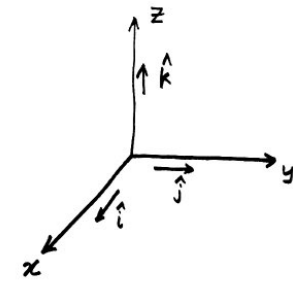

Figure 1: Force addition.

Figure 2: Cartesian unit vectors.

- Addition of Cartesian vectors

$$\vec{F}_R = \sum \vec{F} = \sum F_x\,\hat{i} + \sum F_y\,\hat{j} + \sum F_z\,\hat{k}$$

- Position vector

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$
$$\vec{r} = \vec{r}_B - \vec{r}_A$$
$$= (x_B - x_A)\,\hat{i} + (y_B - y_A)\,\hat{j} + (z_B - z_A)\,\hat{k}$$

Figure 3: Position vector.

- Force vector directed along a line

$$\vec{F} = F\vec{u} = F\,\frac{\vec{r}}{r}$$
$$= F\left(\frac{(x_B - x_A)\,\hat{i} + (y_B - y_A)\,\hat{j} + (z_B - z_A)\,\hat{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\right)$$

- Dot product

$$\vec{A} \cdot \vec{B} = AB\cos\theta = A_x B_x + A_y B_y + A_z B_z$$
$$\theta = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{AB}\right)$$
$$\vec{F} \cdot \vec{u} = F$$

Figure 4: Dot product.

- **Condition for the equilibrium**

$$\sum \vec{F} = \vec{0}$$
$$\sum F_x = 0;\ \sum F_y = 0;\ \sum F_z = 0.$$

- Moment

$$M_O = Fd$$

Figure 5: Moment.

- Cross product

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$
$$= (A_y B_z - A_z B_y)\,\hat{i} - (A_x B_z - A_z B_x)\,\hat{j} + (A_x B_y - A_y B_x)\,\hat{k}$$
$$\vec{M}_O = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$
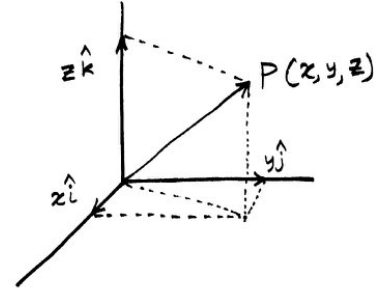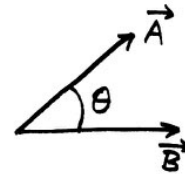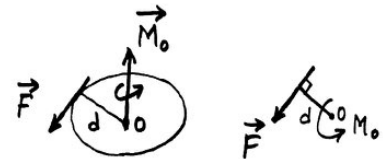
$\vec{r}$: position vector from O to any point on the line of action of the force.

- Resultant moment of a system of forces

$$\vec{M}_{R_O} = \sum \vec{r} \times \vec{F}$$

Figure 6: Resultant Moment.

- Principle of moments

$$\vec{M}_O = \vec{r} \times \vec{F} = \vec{r} \times \left(\vec{F}_1 + \vec{F}_2\right)$$
$$= \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2$$
$$M_O = F_x y - F_y x$$

- Moment about a specified axis

  - Scalar analysis: $\boxed{M_a = F d_a}$

  - Vector analysis:

$$M_a = \vec{u}_a \cdot \left(\vec{r} \times \vec{F}\right)$$
$$= \begin{vmatrix} u_{ax} & u_{ay} & u_{az} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$
$$\vec{M}_a = M_a \vec{u}_a$$

Figure 7: Moment about a specified axis.

- Moment of a couple

$$\vec{M} = = \vec{r}_B \times \vec{F} + \vec{r}_B \times \left(-\vec{F}\right)$$
$$= \left(\vec{r}_B - \vec{r}_A\right) \times \vec{F}$$
$$= \vec{r} \times \vec{F}$$
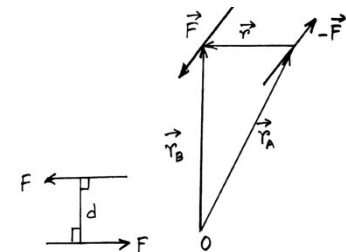$$M = Fd$$

Figure 8: Couple system.

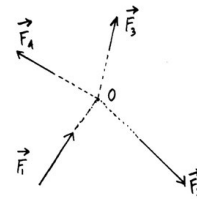- Concurrent force system $\boxed{\vec{F}_R = \sum \vec{F}}$

Figure 9: Concurrent force system.

- Coplanar force system

$$\boxed{\begin{aligned} \vec{F}_R &= \sum \vec{F} \\ \vec{M}_{R_O} &= \sum \vec{r} \times \vec{F} \\ d &= \frac{M_{R_O}}{F_R} \end{aligned}}$$

- Reduction of a simple distributed loading

  wind pressure

  water pressure on the bottom of a tank or side of a tank

  Magnitude:

$$\boxed{+\downarrow F_R = \sum F, \quad F_R = \int_{x=0}^{x=L} w(x)dx = \int_A dA = A}$$

Figure 10: Coplanar force system.

  Location:

$$\boxed{\begin{aligned} +\curvearrowleft M_{R_O} &= \sum M_O \\ \bar{x} &= \frac{\int_{x=0}^{x=L} xw(x)dx}{\int_{x=0}^{x=L} w(x)dx} \\ &= \frac{\int_A x\,dA}{\int_A dA} = \text{ centroid of the area} \end{aligned}}$$

Figure 11: Distributed loading.

- **Equations of equilibrium**

$$\boxed{\begin{aligned} \sum \vec{F} &= \vec{0} \\ \sum \vec{M}_O &= \vec{0} \end{aligned}}$$

  In 3D:

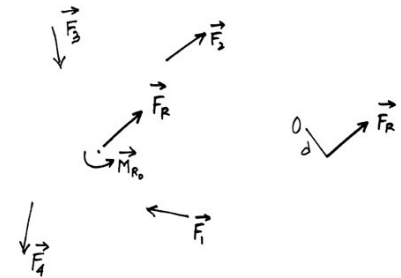$$\boxed{\begin{aligned} \sum F_x &= \sum F_y = \sum F_z = 0 \\ \sum M_x &= \sum M_y = \sum M_z = 0 \end{aligned}}$$

Figure 12: Equivalent loading.
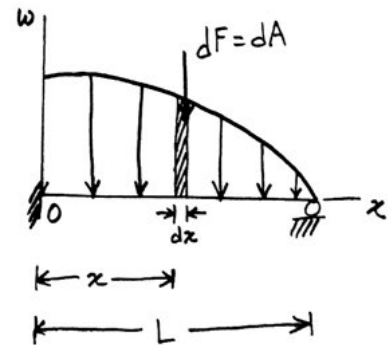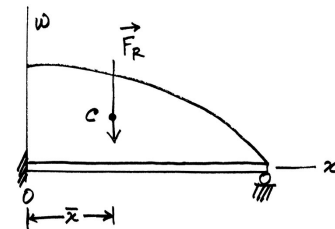
Concept of Stress & Strain

Axial Loading

Normal Stress

Consider a two-force member subjected to axial loading as shown in Figure 13. The normal stress developed in the member is given by

$$\sigma = \frac{P}{A}$$

This is the average stress over the cross-section. Stress at a particular point in the cross-section is defined as

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

where ΔA is small area around the point and ΔF is the internal force in that area. In general,

$$P = \int dF = \int_A \sigma dA$$

Normal Strain

The strain is defined as

$$\epsilon = \frac{\delta}{L}$$

where δ is deformation of the member. Strain at a given point is

$$\epsilon = \lim_{\Delta x \to 0} \frac{\Delta \delta}{\Delta x} = \frac{d\delta}{dx} \qquad (1)$$

Hooke's Law

For the initial portion of the stress-strain plot (up to the *elastic limit*) stress is proportional to strain and the proportional constant is known as modulus of elasticity (E).

$$\sigma = E\epsilon$$

Figure 13: Axially loaded member with cross-sectional area A.

Deformation

Using Hooke's law

$$\delta = \epsilon L = \left(\frac{\sigma}{E} \right) L = \frac{PL}{AE}$$

If the material property, cross-section, or the axial load changes over the length a few times total deformation is given by

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

In general, for varying cross-section or material properties over the length

$$\delta = \int_0^L \frac{P dx}{AE}$$

Factor of Safety

The factor of safety is defined as

$$F.S. = \frac{\text{ultimate load}}{\text{allowable load}}$$

In terms of stress

$$F.S. = \frac{\text{ultimate material strength (stress)}}{\text{allowable stress}}$$

Problem 1.

For the bar shown in Figure 14 determine normal stresses in different parts. Assume the diameters as $d_{AB} = d_{CD} = 20$ mm, $d_{BC} = 40$ mm.

Using the free body diagram in Figure 14, $F_{AB} = 10$ kN, $F_{BC} = 26$ kN, and $F_{CD} = 21$ kN.

The cross-sectional areas are $A_{AB} = \frac{\pi d_{AB}^2}{4}$, $A_{BC} = \frac{\pi d_{BC}^2}{4}$, and $A_{CD} = \frac{\pi d_{CD}^2}{4}$.

Hence, the normal stress in different parts

$$\sigma_{AB} = \frac{F_{AB}}{A_{AB}} = \frac{10 \times 10^3}{\pi (0.02)^2/4} = 31.83 \text{ MPa}$$

$$\sigma_{BC} = \frac{F_{BC}}{A_{BC}} = \frac{26 \times 10^3}{\pi (0.04)^2/4} = 20.69 \text{ MPa}$$

$$\sigma_{CD} = \frac{F_{CD}}{A_{CD}} = \frac{21 \times 10^3}{\pi (0.02)^2/4} = 66.85 \text{ MPa}$$
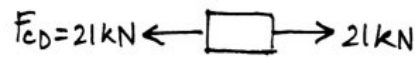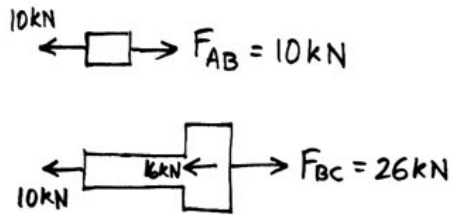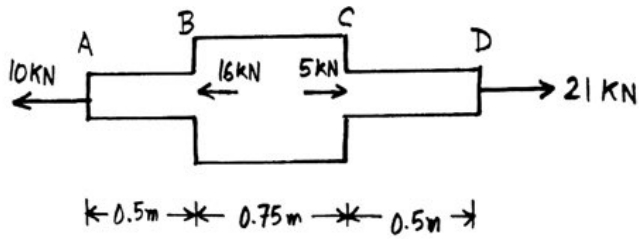
Figure 14: Problem 1.

The deformation in different parts

$$\delta_{AB} = \frac{F_{AB}L_{AB}}{A_{AB}E} = \frac{10 \times 10^3 \times 0.5}{\frac{\pi(0.02)^2}{4} \times 70 \times 10^9} = 0.23 \text{ mm}$$

$$\delta_{BC} = \frac{F_{BC}L_{BC}}{A_{BC}E} = \frac{26 \times 10^3 \times 0.75}{\frac{\pi(0.04)^2}{4} \times 70 \times 10^9} = 0.22 \text{ mm}$$

$$\delta_{CD} = \frac{F_{CD}L_{CD}}{A_{CD}E} = \frac{21 \times 10^3 \times 0.5}{\frac{\pi(0.02)^2}{4} \times 70 \times 10^9} = 0.48 \text{ mm}$$

Total deformation of the member $\delta = \delta_{AB} + \delta_{BC} + \delta_{CD} = 0.93$ mm.

Problem 2.

Determine the maximum weight that can be used where the maximum allowable stress in the cable is 10 MPa. The diameter of the cables is 10 mm.

Using the free body diagram

$$\xrightarrow{+} \sum F_x = 0 \;\Rightarrow\; -F_{AB}\cos 60 + F_{BC}\left(\frac{4}{5}\right) = 0$$

$$F_{BC}\left(\frac{4}{5}\right) = F_{AB}/2$$

$$F_{AB} = 1.6F_{BC}$$

$$+\uparrow\sum F_y = 0 \;\Rightarrow\; F_{AB}\sin 60 + F_{BC}\left(\frac{3}{5}\right) - W = 0$$

$$(1.6F_{BC})\frac{\sqrt{3}}{2} + F_{BC}\left(\frac{3}{5}\right) - W = 0$$

$$1.986F_{BC} = W$$

$$F_{BC} = 0.504W$$

$$F_{AB} = 1.6F_{BC} = 0.806W$$

Hence,

$$\sigma_{AB} = \frac{F_{AB}}{A} = \frac{0.806W}{\pi(0.01)^2/4} = 10262.3W < \sigma_{allow} = 10 \times 10^6$$

This gives

$$W < \frac{10 \times 10^6}{10262.3} = 974.4N$$
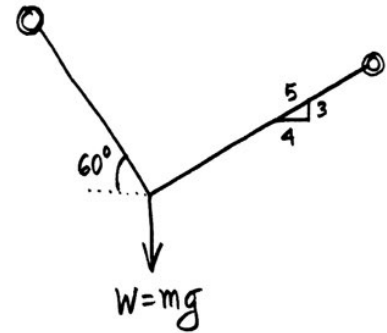
Maximum mass allowed, $m = W/g = 974.4/9.81 = 99.33$ kg.

Figure 15: Problem 2.

Problem 3.

Determine the cross-sectional area required for member DF if $\sigma_{allow} = 120$ MPa.

We need to calculate the support reactions first:

$$\xrightarrow{+} \sum F_x = 0 \;\Rightarrow\; A_x - (1000 \text{ kN}) \cdot \left(\frac{4}{5}\right) = 0$$

$$\Rightarrow\; A_x = 800 \text{ kN}$$

$$+\uparrow\sum F_y = 0 \;\Rightarrow\; A_y + G_y - (1000 \text{ kN}) \cdot \left(\frac{3}{5}\right) = 0$$

$$\Rightarrow\; A_y + G_y = 600 \text{ kN}$$

$$+\curvearrowleft \sum M_A = 0 \;\Rightarrow\; G_y \cdot (6 \text{ m}) + (1000 \text{ kN}) \cdot \left(\frac{4}{5}\right) \cdot (2.5 \text{ m})$$

$$- (1000 \text{ kN}) \cdot \left(\frac{3}{5}\right) \cdot (5 \text{ m}) = 0$$

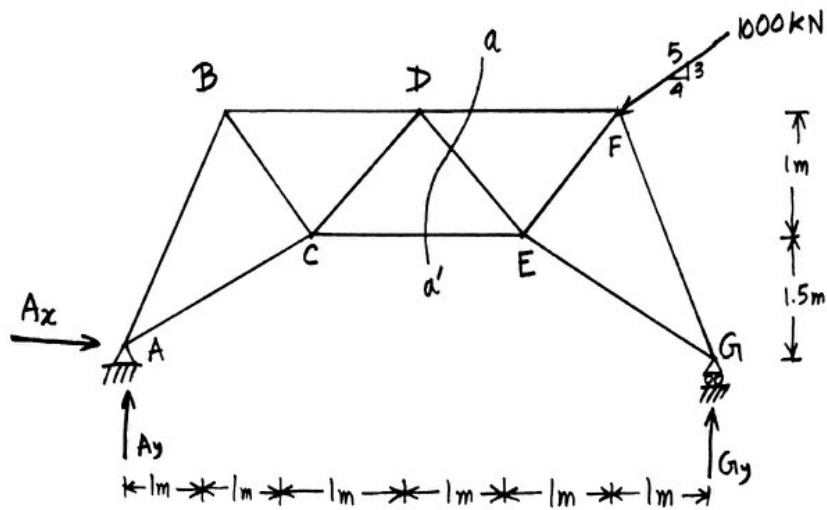$$\Rightarrow\; G_y = 166.67 \text{ kN}, \; A_y = 433.33 \text{ kN}$$

Figure 16: Problem 3.

To estimate the force in member DF consider a section $a - a'$.

$$+ \curvearrowleft \sum M_E = 0 \Rightarrow \quad - F_{DF} \cdot (1 \text{ m}) - (433.33 \text{ kN}) \cdot (4 \text{ m})$$
$$+ (800 \text{ kN}) \cdot (1.5 \text{ m}) = 0$$
$$\Rightarrow \quad F_{DF} = 533.33 \text{ kN}$$

Hence, the cross-sectional area required

Figure 17: Section $a - a'$.

$$A_{DF} = \frac{F_{DF}}{\sigma_{allow}} = \frac{533.33 \times 10^3 \text{ N}}{120 \times 10^6 \text{ N/m}^2} = 4.44 \times 10^{-3} \text{ m}^2 = 444 \text{ mm}^2$$
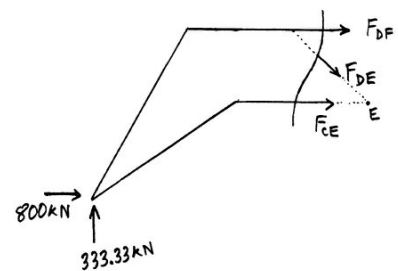
Problem 4.

Determine the deformation of a bar under its own weight. What is the equivalent load at the end of the bar that can replace the self-weight?

Consider the deformation of an element of length dz. The weight acting on it is

$$P = \rho g A(L - z)$$

where ρ = density of the bar; A = cross-sectional area; g = gravitational acceleration. Hence, the deformation of the element

$$d\delta = \frac{Pdz}{AE} = \frac{\rho g A(L-z)dy}{AE}$$

Total deformation of the bar

$$\delta = \int_0^L \frac{\rho g}{E}(L-z)dz$$

$$= \frac{\rho g}{E} \int_0^L (L-z)dz$$

$$= \frac{\rho g}{E}(L^2 - L^2/2)$$

$$= \frac{\rho g L^2}{2E}$$

Equivalent force at the end

$$\sigma A$$

$$= (E\epsilon)A$$

$$= E\frac{\delta}{L}A$$

$$= \frac{E\rho g L^2 A}{2EL}$$

$$= \frac{1}{2}\rho g AL = W/2$$

where W is the total weight of the bar.

Figure 18: Problem 4.

Problem 5.

Determine the deformation at point C. Assume the bar ACD is rigid.

$$\sum F_y = 0$$

$$\Rightarrow F_{AB} + F_{DE} = 45\ kN$$

$$\sum M_D = 0$$

$$\Rightarrow -F_{AB} \cdot (0.6\ m) + (45\ kN) \cdot (0.4\ m) = 0$$

$$F_{AB} = 30\ kN$$

$$F_{DE} = 15\ kN$$

Figure 19: Problem 5.

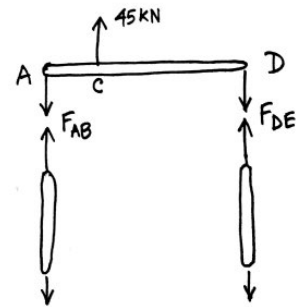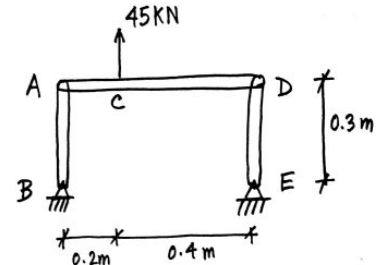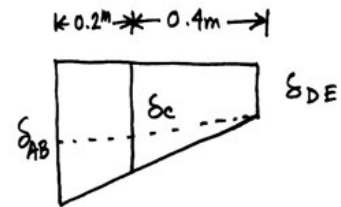$$\delta_{AB} = \frac{F_{AB}L_{AB}}{E_{AB}A_{AB}}$$

$$= \frac{(30 \times 10^3 \ N) \cdot (0.3 \ m)}{(200 \times 10^9 \ Pa) \cdot (\pi(0.01^2) \ m^2)}$$

$$= 143 \times 10^{-6} \ m$$

$$= 0.143 \ mm$$

$$\delta_{DE} = \frac{F_{DE}L_{DE}}{E_{DE}A_{DE}}$$

$$= \frac{(15 \times 10^3 \ N) \cdot (0.3 \ m)}{(70 \times 10^9 \ Pa) \cdot (\pi(0.02^2) \ m^2)}$$

$$= 51 \times 10^{-6} \ m$$

$$= 0.051 \ mm$$

$$\delta_C = \delta_{DE} + (\delta_{AB} - \delta_{DE}) \cdot \left(\frac{0.4 \ m}{0.6 \ m}\right) = 0.113 \ mm$$

Statically Indeterminate Problems

In these problems, equations of equilibrium are not enough to solve all the reactions. Hence, equations for compatibility are required.

Problem 6.

Consider the rod made of an outer layer with material 1 ($E_2 = 90$ GPa) and a core with material 1 ($E_1 = 45$ GPa). It is subjected to $P = 70$ kN. Calculate the stresses developed in each component of the rod.

Equation of Equilibrium: The total load P is carried by both materials. If P_1 is the load carried by material 1 and P_2 is the load carried by material 2

$$P = P_1 + P_2 = 70 \ kN$$

Equation of Compatibility: Further, the deformations of both materials should be same.

Figure 20: Problem 6.

$$\delta = \delta_1 = \delta_2$$

$$\Rightarrow \frac{P_1 L}{E_1 A_1} = \frac{P_2 L}{E_2 A_2}$$

$$\Rightarrow P_1 = P_2 \left(\frac{E_1}{E_2}\right)\left(\frac{A_1}{A_2}\right)$$

$$\Rightarrow P_1 = P_2 \cdot \left(\frac{90}{45}\right) \cdot \left(\frac{\pi(0.04^2 - 0.02^2)}{\pi(0.02^2)}\right)$$

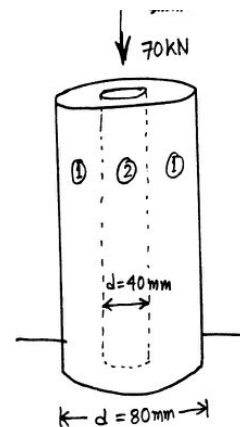$$\Rightarrow P_1 = P_2 \cdot (2) \cdot (3)$$

$$\Rightarrow P_1 = 6P_2$$

Hence, $P_1 = 60$ kN, $P_2 = 10$ kN and

$$\sigma_1 = \frac{P_1}{A_1} = \frac{60 \times 10^3}{\pi(0.04^2 - 0.02^2)} = 15.91 \; MPa$$

$$\sigma_2 = \frac{P_2}{A_2} = \frac{10 \times 10^3}{\pi(0.02^2)} = 7.96 \; MPa$$

Problem 7.

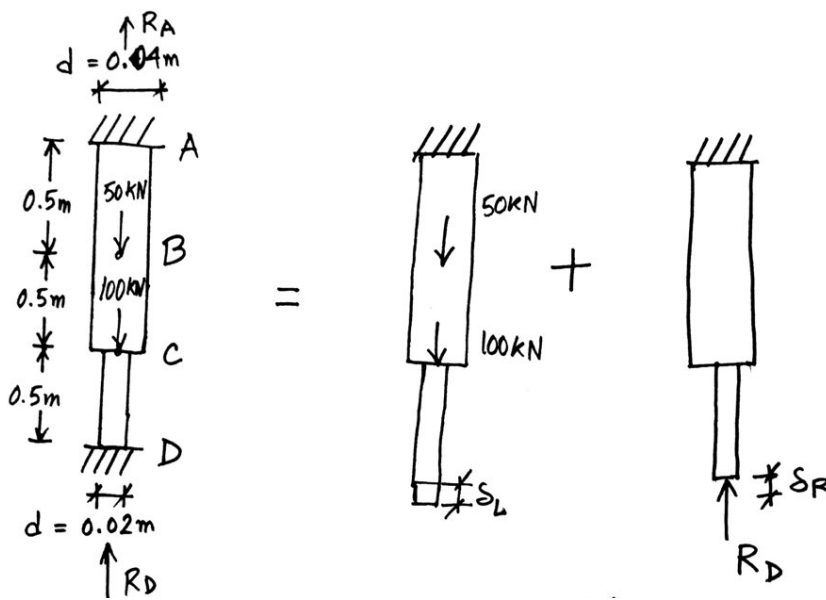Determine the support reactions in the shown statically indeterminate structure. AC has $E = 50$ GPa and CD has $E = 100$ GPa.

Figure 21: Problem 7.

Equation of Equilibrium:

$$+\uparrow \sum F_y = 0$$
$$R_A + R_D = 50 \; kN + 100 \; kN = 150 \; kN$$

Equation of Compatibility:
Assume the reaction at D is redundant and $\delta_L = $ deformation due

to the load; δ_R = deformation due to the reaction. Hence,

$$\delta = \delta_L + \delta_R = 0$$

$$\delta_L = \delta_B + \delta_C + \delta_D$$

$$= -\frac{(50 \times 10^3 \ N) \cdot (0.5 \ m)}{(50 \times 10^9 \ Pa) \cdot (\pi (0.02^2) \ m^2)} - \frac{(100 \times 10^3 \ N) \cdot (1 \ m)}{(50 \times 10^9 \ Pa) \cdot (\pi (0.02^2) \ m^2)} = -1.99 \times 10^{-3} \ m$$

$$\delta_R = \frac{(R_D) \cdot (0.5 \ m)}{(100 \times 10^9 \ Pa) \cdot (\pi (0.01^2) \ m^2)} + \frac{(R_D) \cdot (1 \ m)}{(50 \times 10^9 \ Pa) \cdot (\pi (0.02^2) \ m^2)} = 3.183 \times 10^{-8} R_D$$

$$\Rightarrow R_D = \frac{1.99 \times 10^{-3}}{3.183 \times 10^{-8}} = 62500 \ N = 62.5 \ kN$$

$$\Rightarrow R_A = 150 \ KN - R_D = 87.5 \ KN$$

Problem 8.

Solve the same problem as before but allowing a 1 mm gap for the deformation of the bar as shown in the figure.

Equation of Equilibrium:

$$+\uparrow \sum F_y = 0$$

$$R_A + R_D = 50 \ kN + 100 \ kN = 150 \ kN$$

Equation of Compatibility: δ_L = deformation due to the load; δ_R = deformation due to the reaction. Hence,

$$\delta = \delta_L + \delta_R = -1 \times 10^{-3} \ m$$

$$\delta_L = \delta_{AB} + \delta_{BC} + \delta_{CD}$$

$$= -\frac{(50 \times 10^3 \ N) \cdot (0.5 \ m)}{(50 \times 10^9 \ Pa) \cdot (\pi (0.02^2) \ m^2)} - \frac{(100 \times 10^3 \ N) \cdot (1 \ m)}{(50 \times 10^9 \ Pa) \cdot (\pi (0.02^2) \ m^2)} = -1.99 \times 10^{-3} \ m$$

$$\delta_R = \frac{(R_D) \cdot (0.5 \ m)}{(100 \times 10^9 \ Pa) \cdot (\pi (0.01^2) \ m^2)} + \frac{(R_D) \cdot (1 \ m)}{(50 \times 10^9 \ Pa) \cdot (\pi (0.02^2) \ m^2)} = 3.183 \times 10^{-8} R_D$$

$$\Rightarrow R_D = \frac{1.99 \times 10^{-3} - 1 \times 10^{-3}}{3.183 \times 10^{-8}} = 31250 \ N = 31.25 \ kN$$

$$\Rightarrow R_A = 150 \ KN - R_D = 118.75 \ kN$$
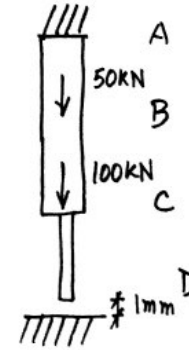

Figure 22: Problem 8.

Problem 9.

Determine the stresses developed in members BE and CF ($E = 70$ GPa, radius = 20 mm). Assume the bar ABCD is rigid.

Equation of Equilibrium:

$$\sum F_x = 0$$

$$\Rightarrow A_x = 0$$

$$\sum M_A = 0$$

$$\Rightarrow F_{BE} \cdot (0.5 \ m) + F_{CF} \cdot (1 \ m) = (100 \ kN) \cdot (1.5 \ m)$$
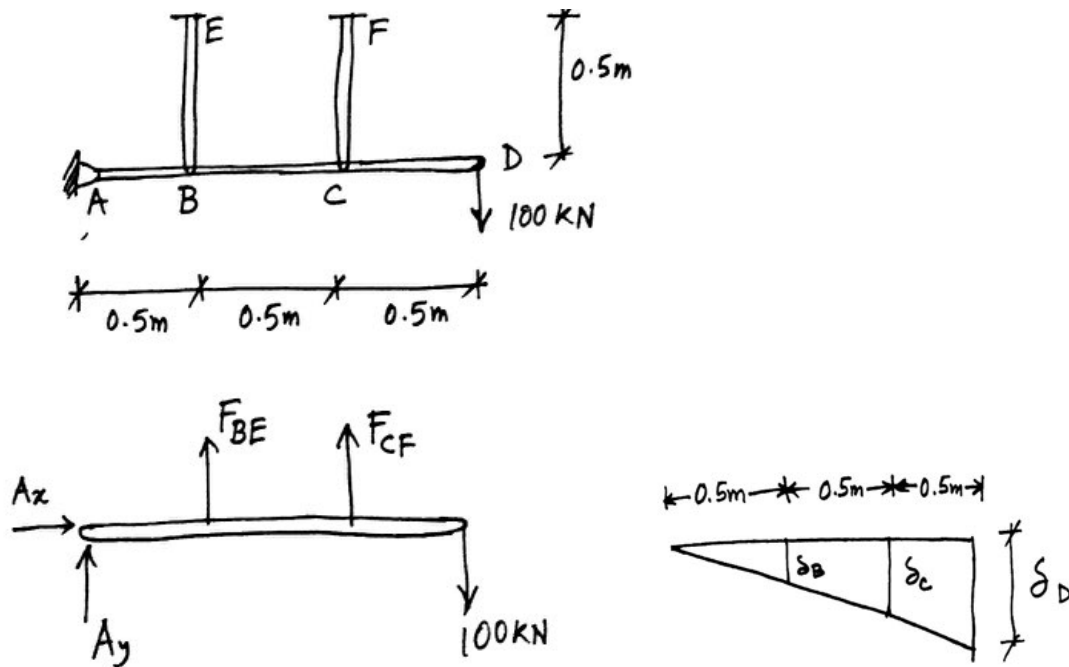
$$\Rightarrow F_{BE} + 2F_{CF} = 300 \ kN$$

Figure 23: Problem 9.

Equation of Compatibility:

$$2\delta_B = \delta_C$$

$$\Rightarrow \frac{2F_{BE}L_{BE}}{E_{BE}A_{BE}} = \frac{F_{CF}L_{CF}}{E_{CF}A_{CF}}$$

$$\Rightarrow \frac{2F_{BE} \cdot (0.5 \ m)}{(70 \times 10^9 \ Pa) \cdot (\pi(0.02^2) \ m^2)} = \frac{F_{CF} \cdot (0.5 \ m)}{(70 \times 10^9 \ Pa) \cdot (\pi(0.02^2) \ m^2)}$$

$$\Rightarrow 2F_{BE} = F_{CF}$$

Hence,

$$F_{BE} = 60 \ kN, \quad F_{CF} = 120 \ kN$$

$$\sigma_{BE} = \frac{F_{BE}}{A} = \frac{60 \times 10^3 \ N}{\pi(0.02^2) \ m^2} = 47.75 \times 10^6 \ Pa = 47.75 \ MPa$$

$$\sigma_{CF} = \frac{F_{CF}}{A} = \frac{120 \times 10^3 \ N}{\pi(0.02^2) \ m^2} = 95.5 \times 10^6 \ Pa = 95.5 \ MPa$$

Problem 10.

Three cables are attached as shown. Determine the reactions in the supports.

Assume R_B as redundant. Also, $L_{AD} = L_{CD} = \frac{1\ m}{\cos 60°} = 2\ m$.

Equation of Equilibrium:

$$+\uparrow \sum F_y = 0$$

$$\Rightarrow \quad R_A \cdot \cos 60° + R_B + R_C \cdot \cos 60° = 100\ kN$$

$$\Rightarrow \quad R_A \cdot \left(\frac{1}{2}\right) + R_B + R_C \cdot \left(\frac{1}{2}\right) = 100\ kN$$

$$\Rightarrow \quad R_A + 2R_B + R_C = 200\ kN$$

$$\Rightarrow \quad 2R_A + 2R_B = 200\ kN \quad \text{[using symmetry } R_A = R_C\text{]}$$

$$\Rightarrow \quad R_A + R_B = 100\ kN$$
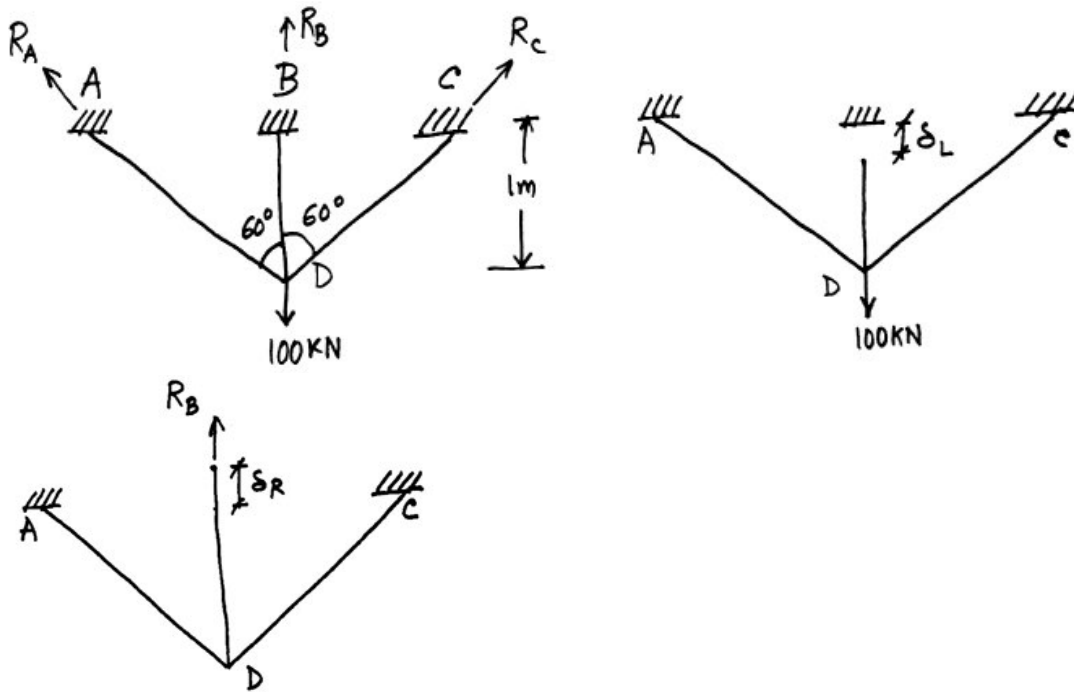

Figure 24: Single and double shear.

Equation of Compatibility:

To compute the downward (-ve) deformation (δ_L) due to the external load (in this case we do not have any force in the member BD)

$$+\uparrow \sum F_y = 0$$

$$\Rightarrow \quad 2F_{AD} \cos 60° - 100\ kN = 0 \quad \text{[using symmetry } F_{AD} = F_{CD}\text{]}$$

$$\Rightarrow \quad F_{AD} = 100\ kN$$

Hence,

$$\delta_L = -\frac{F_{AD}L_{AD}}{AE\cos 60°} = -\frac{(100\ kN)\cdot(2\ m)}{AE\cdot\left(\frac{1}{2}\right)} = -\frac{400\ kNm}{AE}$$

Similarly, to compute the upward (+ve) deformation (δ_R) due to the redundant reaction R_B (in this case we have force R_B in the member BD)

$$\begin{aligned}\delta_R &= \frac{R_B L_{AD}}{AE\cos 60°} + \frac{R_B L_{BD}}{AE} \\ &= \frac{R_B\cdot(2\ m)}{AE\cdot\left(\frac{1}{2}\right)} + \frac{R_B\cdot(1\ m)}{AE}\end{aligned}$$

Using the equation of compatibility

$$\delta = \delta_L + \delta_R = 0$$
$$\Rightarrow\quad -\frac{400\ kNm}{AE} + \frac{2R_B\cdot(2\ m)}{AE} + \frac{R_B\cdot(1\ m)}{AE} = 0$$
$$\Rightarrow\quad R_B = \frac{400\ kN}{5} = 80\ kN$$
$$\Rightarrow\quad R_A = 100\ kN - 80\ kN = 20\ kN = R_C$$

Isotropic Material

The material properties are same in every direction.

Homogeneous Material

The material properties are same for every position.

Poisson's Ratio

For the axially loaded member $\sigma_x = \frac{P}{A}$ shown in the figure, even if $\sigma_y = \sigma_z = 0$ here but $\epsilon_y, \epsilon_z \neq 0$ due to the transverse contraction.

The lateral strains are equal in this case for a homogeneous isotropic material and a material constant, known as Poisson's ratio (ν), can be defined as

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} = -\frac{\epsilon_y}{\epsilon_x} = -\frac{\epsilon_z}{\epsilon_x}$$

Using Hooke's law ($\sigma_x = E\epsilon_x$)

$$\epsilon_y = \epsilon_z = -\frac{\nu\sigma_x}{E}$$

Multiaxial Loading

For multiaxial loading the generalized Hooke's law is given by

$$
\begin{aligned}
\epsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}
\end{aligned}
$$

Shearing Strain

The shearing strain is defined as shown in the figure. Hooke's law for shearing stress and strain is

$$
\tau_{xy} = G\gamma_{xy} \quad \tau_{yz} = G\gamma_{yz} \quad \tau_{zx} = G\gamma_{zx}
$$

where G is the modulus of rigidity or shear modulus.

$$
G = \frac{E}{2(1+\nu)}
$$

For a general stress condition in an isotropic linearly elastic material the generalized Hooke's law:

$$
\begin{aligned}
\epsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E} \\
\tau_{xy} &= G\gamma_{xy} \quad \tau_{yz} = G\gamma_{yz} \quad \tau_{zx} = G\gamma_{zx}
\end{aligned}
$$

We can write this in a matrix form

$$
\begin{Bmatrix}
\epsilon_x \\
\epsilon_y \\
\epsilon_z \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{Bmatrix}
= \frac{1}{E}
\begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & 1 & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1+\nu)
\end{bmatrix}
\begin{Bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
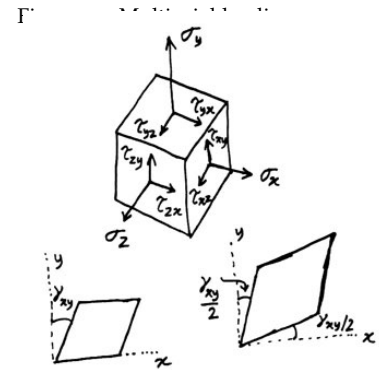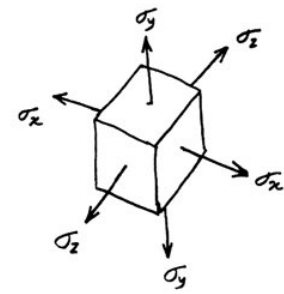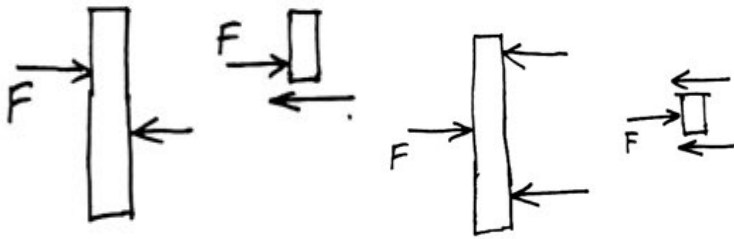\tau_{yz} \\
\tau_{zx}
\end{Bmatrix}
$$

Figure 26: Shear stresses and strains.

Inverting this equation

$$
\left\{
\begin{array}{c}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{array}
\right\}
=
\frac{E}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}
\end{bmatrix}
\left\{
\begin{array}{c}
\epsilon_x \\
\epsilon_y \\
\epsilon_z \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{array}
\right\}
$$

Single Shear and Double Shear

Figure 27: Single and double shear.

Problem 11.

A bolt of diameter 40 mm is tightened such that the decrease in its diameter is 10 μm. Using the property of steel, $E = 200$ GPa and $G = 77.2$ GPa determine the internal force in the bolt.

Given $\delta_y = 10 \ \mu m = 10 \times 10^{-6} \ m$, $d = 40 \ mm = 0.04 \ m$.

$$
\nu = \frac{E}{2G} - 1 = \frac{200}{2 \times 77.2} - 1 = 0.2953
$$

$$
\epsilon_y = -\frac{\delta_y}{d} = -\frac{10 \times 10^{-6}}{0.04} = -2.5 \times 10^{-4}
$$

$$
\epsilon_x = -\frac{\epsilon_y}{\nu} = -\frac{-2.5 \times 10^{-4}}{0.2953} = 8.4660 \times 10^{-4}
$$

Hence, the internal force in the bolt

$$
P = \sigma A = (E\epsilon_x)\left(\frac{\pi d^2}{4}\right)
$$

$$
= (200 \times 10^6 \ Pa \cdot 8.4660 \times 10^{-4}) \cdot \left(\frac{\pi(0.04)^2}{4} \ m^2\right)
$$

$$
= 212.77 \ N
$$

Problem 12.

The plate shown in the figure is subjected to biaxial loading. Compute the change in length of the sides and the diagonal. Also, compute the change in the angle ACB. Assume $E = 200$ GPa, $v = 0.29$.

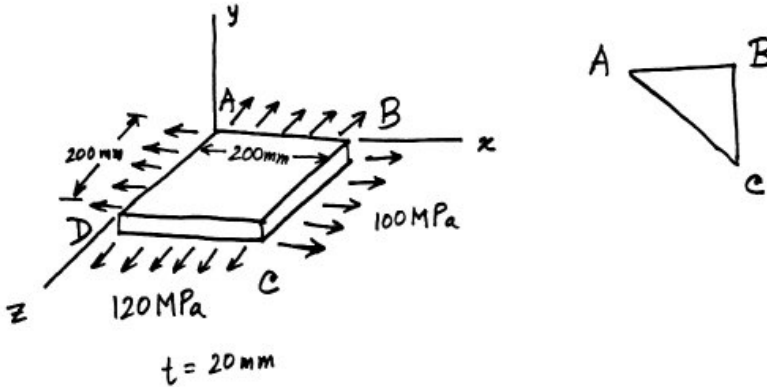Given $\sigma_x = 100$ MPa, $\sigma_y = 0$, $\sigma_z = 120$ MPa.

Using generalized Hooke's law for multiaxial loading:

$$\epsilon_x = +\frac{\sigma_x}{E} - \frac{v\sigma_y}{E} - \frac{v\sigma_z}{E} = \frac{100 \times 10^6 \ Pa}{200 \times 10^9 \ Pa} - 0 - \frac{0.29 \times (120 \times 10^6 \ Pa)}{200 \times 10^9 \ Pa} = 0.326 \times 10^{-3}$$

$$\epsilon_y = -\frac{v\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{v\sigma_z}{E} = -\frac{0.29 \times (100 \times 10^6 \ Pa)}{200 \times 10^9 \ Pa} + 0 - \frac{0.29 \times (120 \times 10^6 \ Pa)}{200 \times 10^9 \ Pa} = -0.319 \times 10^{-3}$$

$$\epsilon_z = -\frac{v\sigma_x}{E} - \frac{v\sigma_y}{E} + \frac{\sigma_z}{E} = -\frac{0.29 \times (100 \times 10^6 \ Pa)}{200 \times 10^9 \ Pa} - 0 + \frac{120 \times 10^6 \ Pa}{200 \times 10^9 \ Pa} = 0.455 \times 10^{-3}$$

Hence, the changes in lengths

$$\delta_{AB} = l_{AB}\epsilon_x = (0.2 \ m) \cdot (0.326 \times 10^{-3}) = 0.0652 \times 10^{-3} \ m = 0.0652 \ mm$$
$$\delta_{BC} = l_{BC}\epsilon_z = (0.2 \ m) \cdot (0.455 \times 10^{-3}) = 0.0910 \times 10^{-3} \ m = 0.0910 \ mm$$

The change in thickness

$$\delta_t = t\epsilon_y = (0.02 \ m) \cdot (-0.319 \times 10^{-3}) = -0.0064 \times 10^{-3} = -0.0064 \ mm$$

To estimate the change in length of the diagonal, first calculate the length of the diagonal before deformation:

$$l_{AC} = \sqrt{l_{AB}^2 + l_{BC}^2}$$

The length of the diagonal after deformation

$$\sqrt{(l_{AB}(1 + \epsilon_x))^2 + (l_{BC}(1 + \epsilon_z))^2}$$

Hence, the change in length of the diagonal

$$\delta_{AC} = \sqrt{(l_{AB}(1+\epsilon_x))^2 + (l_{BC}(1+\epsilon_z))^2} - \sqrt{l_{AB}^2 + l_{BC}^2} = 0.1105 \times 10^{-3} \; m = 0.1105 \; mm$$

The change in angle ACB:

$$\Delta \tan \theta = \frac{l_{AB}(1+\epsilon_x)}{l_{BC}(1+\epsilon_z)} - \frac{l_{AB}}{l_{BC}}$$
$$= \frac{1+\epsilon_x}{1+\epsilon_z} - 1$$
$$= -1.2894 \times 10^{-4}$$

Relative change in the angle ACB = $\frac{\Delta \tan \theta}{\tan 45°} \times 100\% = -0.0129\%$.

The change in volume

$$\Delta V = V - V_0 = (l_{AB}(1+\epsilon_x) \cdot l_{BC}(1+\epsilon_z) \cdot t(1+\epsilon_y)) - (l_{AB} \cdot l_{BC} \cdot t)$$
$$\approx (l_{AB} \cdot l_{BC} \cdot t) \cdot (\epsilon_x + \epsilon_y + \epsilon_z)$$
$$= V_0 \cdot (\epsilon_x + \epsilon_y + \epsilon_z)$$
$$= 0.3696 \times 10^{-6} m^3 = 369.6 \; mm^3$$

Problem 13.

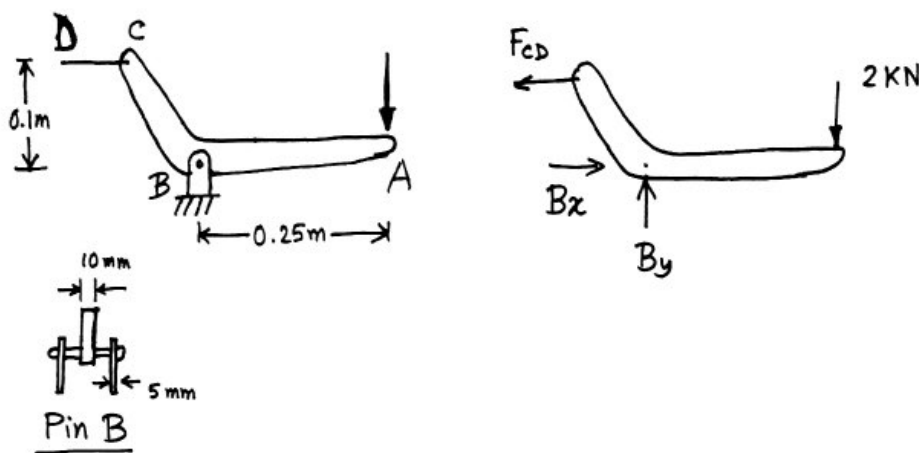Determine the average shear stress in the pin (dia = 20 mm) at B.

Figure 29: Problem 13.

From the free-body diagram of ABC

$$\sum F_y = 0$$
$$B_y - (2000\ N) = 0$$
$$B_y = 2000\ N$$
$$\sum M_B = 0$$
$$F_{CD} \cdot (0.1\ m) - (2000\ N) \cdot (0.25\ m) = 0$$
$$F_{CD} = 5000\ N$$
$$\sum F_x = 0$$
$$B_x - F_{CD} = 0$$
$$B_x = F_{CD} = 5000\ N$$

Hence, the reaction in the pin $R_B = \sqrt{B_x^2 + B_y^2} = 5385\ N$.
Since the pin is under double shear the shear stress in the pin is

$$\tau = \frac{\frac{1}{2} R_B}{\frac{\pi d^2}{4}} = \frac{0.5 \times 5385\ N}{\frac{\pi (0.02)^2}{4}\ m^2} = 8.57 \times 10^6\ Pa = 8.57\ MPa$$

The bearing stress in member ABC

$$\sigma_b = \frac{R_B}{dt} = \frac{5385\ N}{(0.02\ m) \cdot (0.01\ m)} = 26.925 \times 10^6\ Pa = 26.925\ MPa$$

The bearing stress in the support

$$\sigma_b = \frac{\frac{1}{2} R_B}{dt} = \frac{0.5 \times 5385\ N}{(0.02\ m) \cdot (0.005\ m)} = 26.925 \times 10^6\ Pa = 26.925\ MPa$$

Stresses on Inclined Sections

Consider the axially loaded bar as shown in the figure. Compute the stresses (σ_θ and τ_θ) on an inclined plane $a - a'$.

Sign Convention: Normal stress from tension is positive and shear stress producing counter-clockwise rotation is positive.

Using the above sign convention and the free-body diagram, we can write

$$\sigma_\theta = \frac{N}{A_\theta} = \frac{P \cos \theta}{\frac{A}{\cos \theta}} = \frac{P}{A} \cos^2 \theta = \sigma_x \cos^2 \theta$$

$$\tau_\theta = \frac{-V}{A_\theta} = \frac{-P \sin \theta}{\frac{A}{\cos \theta}} = -\frac{P}{A} \cos \theta \sin \theta = -\sigma_x \cos \theta \sin \theta$$

Hence,

$$\boxed{\begin{aligned} \sigma_\theta &= \sigma_x \cos^2 \theta = \frac{\sigma_x}{2} (1 + \cos 2\theta) \\ \tau_\theta &= -\sigma_x \cos \theta \sin \theta = -\frac{\sigma_x}{2} \sin 2\theta \end{aligned}}$$
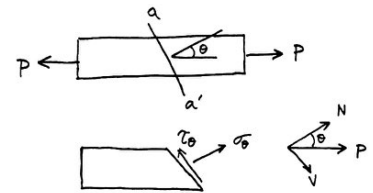

Figure 30: Stresses on an inclined plane.

Problem 14.

Determine the stresses developed on the inclined plane $a - a'$.

The axial stress developed in the bar

$$\sigma_x = \frac{P}{A} = \frac{100 \times 10^3 \; N}{0.004 \; m^2} = 25 \times 10^6 \; N/m^2 = 25 \; MPa$$

Hence,

$$\sigma_\theta = \frac{\sigma_x}{2} \left(1 + \cos 2\theta\right) = \frac{25 \; MPa}{2} \left(1 + \cos 60°\right) = 18.75 \; MPa$$

$$\tau_\theta = -\frac{\sigma_x}{2} \sin 2\theta = -\frac{25 \; MPa}{2} \sin 60° = -10.825 \; MPa$$

For a block on the plane $a - a'$ the complete stress diagram is shown below.

To obtain this use the following:

side $a - a'$: Substitute $\theta = 30°$ to estimate $\sigma_{30°}$ and $\tau_{30°}$.

side $b - b'$: Substitute $\theta = 30° + 180° = 210°$ to estimate $\sigma_{210°}$ and $\tau_{210°}$.

side $a - b$: Substitute $\theta = 30° + 90° = 120°$ to estimate $\sigma_{120°}$ and $\tau_{120°}$.

side $a' - b'$: Substitute $\theta = 30° - 90° = -60°$ to estimate $\sigma_{-60°}$ and $\tau_{-60°}$.
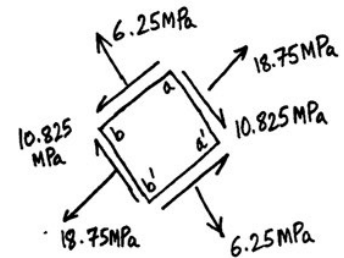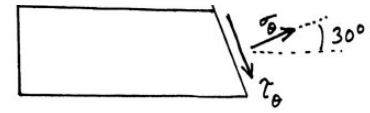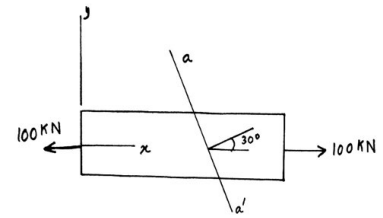

Figure 31: Problem 14.

Torsion

Torsion of circular bars

For a circular solid and tubular sections with homogeneous elastic
material assume a plane section perpendicular to the axis remains
plane after the application of the torques (i.e., no warpage). Also,
assume the shear strains varies linearly with the distance from the
center of the axis. The shear strain at the end of the bar is

$$\gamma = \frac{\rho\phi}{L} = \frac{\rho}{c}\gamma_{max}$$

Using Hooke's law for shear stress, $\tau = G\gamma$

$$\tau = \frac{\rho}{c}\tau_{max}$$

The torsion formula can be obtained by equating the external
torque to the sum of moments developed in the cross-section.

$$\int_A \left(\frac{\rho}{c}\tau_{max}dA\right)\rho = T$$

$$\frac{\tau_{max}}{c}\int_A \rho^2 dA = T$$

$$\tau_{max} = \frac{Tc}{J}$$

where $J = \int_A \rho^2 dA =$ is the polar moment of inertia of the circular
cross-sectional area.

$$J = \frac{\pi c^4}{2} \quad \text{for circular sections}$$

$$J = \frac{\pi c_2^4}{2} - \frac{\pi c_1^4}{2} \quad \text{for hollow sections}$$

For shear stress at a distance of ρ

$$\tau = \frac{\rho}{c}\tau_{max} = \frac{T\rho}{J}$$

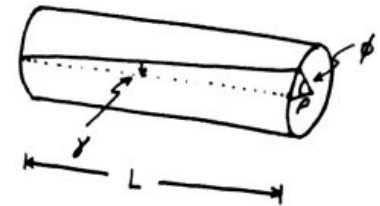
Figure 32: Shear strain.

Some sample shear stress distributions in a circular, hollow, and compound tube are shown in the below figure.

Angle of twist

In the elastic range, using the Hooke's law

$$\gamma_{max} = \frac{\tau_{max}}{G}$$

$$\Rightarrow \frac{c\phi}{L} = \frac{Tc}{GJ}$$

$$\boxed{\phi = \frac{TL}{GJ}}$$

For circular bar with varying cross-section

$$\boxed{\phi = \sum_i \frac{T_i L_i}{G_i J_i}}$$
$$\boxed{\phi = \int_0^L \frac{T dx}{GJ}}$$

Problem 1.

Determine the shear stress developed in the shaft AB and BC.
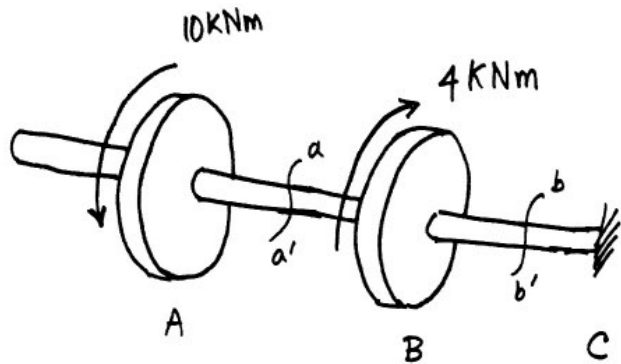

Figure 33: Sample shear stress distributions.

Figure 34: Problem 1.

Shaft AB:

Take a section $a - a'$ and apply equation of equilibrium

$$\sum M_x = 0$$
$$\Rightarrow -T_{AB} + 10 \ kNm = 0$$
$$\Rightarrow T_{AB} = 10 \ kNm$$

Shaft BC:
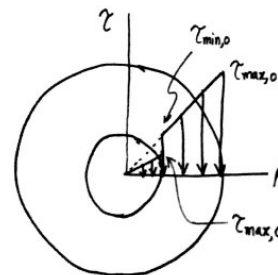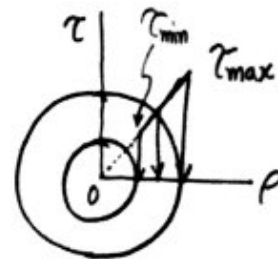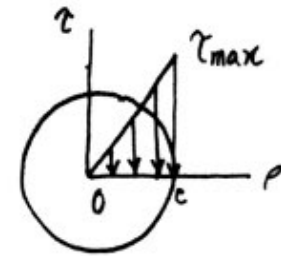
Take a section $b - b'$ and apply equation of equilibrium

$$\sum M_x = 0$$
$$\Rightarrow \quad -T_{BC} + 10 \ kNm - 4 \ kNm = 0$$
$$\Rightarrow \quad T_{BC} = 6 \ kNm$$

Shear stress:

If the shaft AB is solid with a diameter of 80 mm

$$J = \frac{\pi c^4}{2} = \frac{\pi \times (0.04 \ m)^4}{2} = 4.02 \times 10^{-6} \ m^4$$

In the cross-section, we have two points D and E. At point D,

$$\tau_D = \frac{Tc}{J} = \frac{(10 \times 10^3 \ Nm) \cdot (0.04 \ m)}{4.02 \times 10^{-6} \ m^4} = 99.4 \times 10^6 \ Pa = 99.4 \ MPa$$

At point E,

$$\tau_E = \frac{T\rho}{J} = \frac{(10 \times 10^3 \ Nm) \cdot (0.03 \ m)}{4.02 \times 10^{-6} \ m^4} = 74.6 \times 10^6 \ Pa = 74.6 \ MPa$$

If the shaft BC is hollow with inner diameter 60 mm and outer diameter 100 mm determine the minimum and maximum stress developed in the shaft BC.

For this shaft BC

$$J = \frac{\pi(c_2^4 - c_1^4)}{2} = \frac{\pi \times [(0.05 \ m)^4 - (0.03 \ m)^4]}{2} = 8.55 \times 10^{-6} \ m^4$$

$$\tau_{max} = \frac{T_{BC}c_2}{J} = \frac{(6 \times 10^3 \ Nm) \cdot (0.05 \ m)}{8.55 \times 10^{-6} \ m^4} = 35.1 \times 10^6 \ Pa = 35.1 \ MPa$$

$$\tau_{min} = \frac{T_{BC}c_1}{J} = \frac{c_1}{c_2}\tau_{max} = \frac{0.03 \ m}{0.05 \ m} \times 35.1 \ MPa = 21.06 \ MPa$$

If the shaft BC has an inner core made of a different material ($G_c = 2G_o$) determine the maximum stress developed in them.

With an inner core the problem becomes statically indeterminate. Let us assume T_o and T_c are the torsional load carried by the outer layer and the inner core, respectively. The equation of equilibrium here,

$$T_o + T_c = T_{BC} = 6 \ kNm$$

The compatibility equation to be used here

$$\phi_{B,c} = \phi_{B,o}$$
$$\Rightarrow \quad \frac{T_c L}{G_c J_c} = \frac{T_o L}{G_o J_o}$$
$$\Rightarrow \quad T_c = \left(\frac{G_c}{G_o}\right) \cdot \left(\frac{J_c}{J_o}\right) \cdot T_o$$

$$T_c = 2 \times \frac{\frac{\pi}{2} \times (0.03 \ m)^4}{\frac{\pi}{2}[(0.05 \ m)^4 - (0.03 \ m)^4]} \times T_o \approx 0.3 T_o$$

Hence, $T_o = 6\ kNm/1.3 = 4.615\ kNm$ and $T_c = 1.385\ kNm$.

Maximum shear stress

$$\tau_{\text{max},c} = \frac{T_c c_1}{J_c} = \frac{(1.385 \times 10^3\ Nm) \cdot (0.03\ m)}{\frac{\pi}{2} \times (0.03\ m)^4} = 32.66\ MPa$$

$$\tau_{\text{max},o} = \frac{T_o c_2}{J_o} = \frac{(4.615 \times 10^3\ Nm) \cdot (0.05\ m)}{\frac{\pi}{2}[(0.05\ m)^4 - (0.03\ m)^4]} = 27\ MPa$$

$$\tau_{\text{min},o} = \frac{T_o c_1}{J_o} = \frac{(4.615 \times 10^3\ Nm) \cdot (0.03\ m)}{\frac{\pi}{2}[(0.05\ m)^4 - (0.03\ m)^4]} = 16.2\ MPa$$

Problem 2.

Determine the shear stress in AB and rotation at end D.

Figure 35: Problem 2.
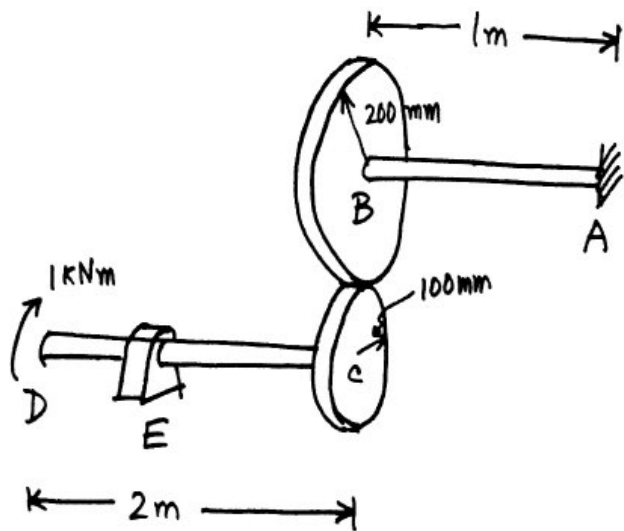
Using the free-body diagram for shaft CD as shown

$$\sum M_x = 0$$
$$\Rightarrow F_C r_C = 1\ kNm = 1000\ Nm$$
$$\Rightarrow F_C = \frac{1000\ Nm}{0.1\ m} = 10,000\ N$$

Using free-body diagram of shaft AB, $F_C = F_B$

$$\sum M_x = 0$$
$$\Rightarrow F_B r_B = T_A$$
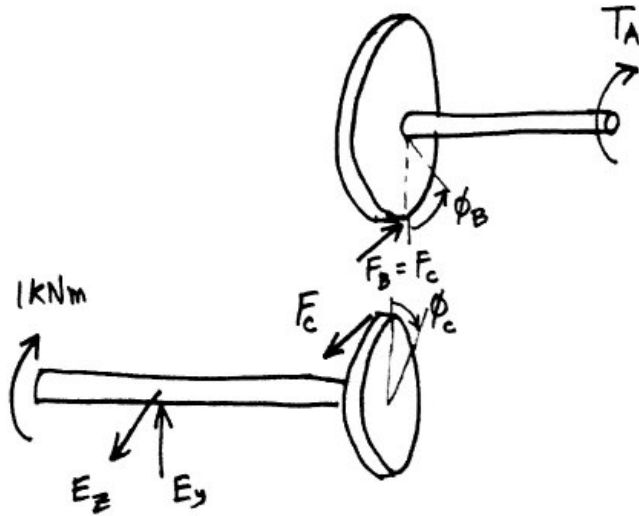$$\Rightarrow T_A = (10,000\ N) \cdot (0.2\ m) = 2000\ Nm$$

Figure 36: Problem 2: Free-body diagrams.

For this shaft AB, $T_{AB} = T_A = 2000 \ Nm$ and

$$J = \frac{\pi(c_2^4 - c_1^4)}{2} = \frac{\pi \times [(0.05 \ m)^4 - (0.03 \ m)^4]}{2} = 8.55 \times 10^{-6} \ m^4$$

$$\tau_{max} = \frac{T_{AB}c_2}{J} = \frac{(2000 \ Nm) \cdot (0.05 \ m)}{8.55 \times 10^{-6} \ m^4} = 11.7 \times 10^6 \ Pa = 11.7 \ MPa$$

$$\tau_{min} = \frac{T_{AB}c_1}{J} = \frac{c_1}{c_2}\tau_{max} = \frac{0.03 \ m}{0.05 \ m} \times 11.7 \ MPa = 7.02 \ MPa$$

The rotation at B

$$\phi_B = \frac{T_{AB}L_{AB}}{GJ_{AB}} = \frac{(2000 \ Nm) \times (1 \ m)}{(80 \times 10^9 \ N/m^2) \times (8.55 \times 10^{-6} \ m^4)} = 0.0029 \ rad$$

From the Figure 37
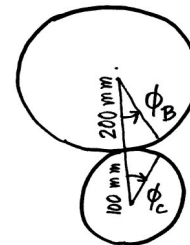
Figure 37: Problem 2: Rotation of both wheels.

$$\phi_B \cdot (0.2 \ m) = \phi_C \cdot (0.1 \ m)$$

$$\Rightarrow \phi_C = 2\phi_B = 0.0058 \ rad$$

$$\Rightarrow \phi_D = \phi_C + \frac{T_{CD}L_{CD}}{GJ_{CD}} = 0.0058 + \frac{(1000 \ Nm) \times (2 \ m)}{(80 \times 10^9 \ N/m^2) \times (4.02 \times 10^{-6} m^4)} = 0.012 \ rad$$

Determine the deformation at the end A for the shaft shown below.
Assume $G = 80$ GPa and the radius of the shaft for the portion AD is
30 mm and for the portion DF is 60 mm.

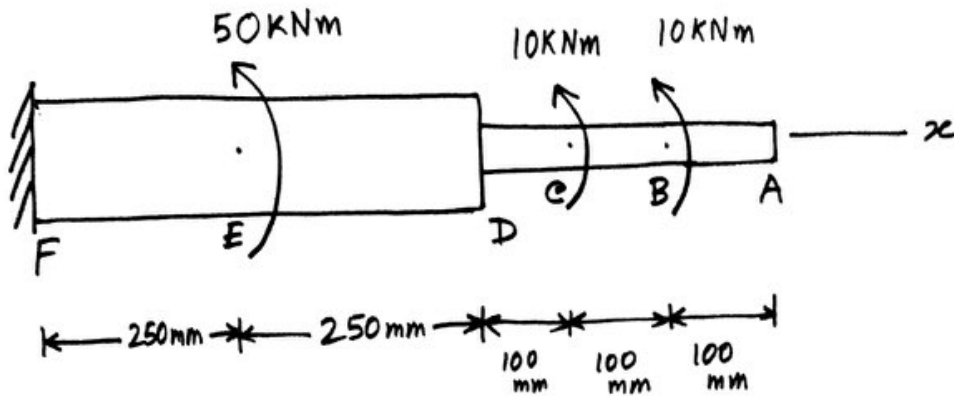Figure 38: Problem 3.

Using equation of equilibrium,

$$\sum M_x = 0$$
$$T_{AB} = 0, \ T_{BC} = 10 \ kNm, \ T_{CD} = 20 \ kNm,$$
$$T_{DE} = 20 \ kNm, \ T_{EF} = 70 \ kNm.$$

The polar moments of inertia

$$J_{AB} = J_{BC} = J_{CD} = \frac{\pi}{2} \times (0.03 \ m)^4 = 1.27 \times 10^{-6} \ m^4$$
$$J_{DE} = J_{EF} = \frac{\pi}{2} \times (0.06 \ m)^4 = 20.36 \times 10^{-6} \ m^4$$

The rotation at end F is $\phi_F = 0$ and

$$\phi_E = \frac{T_{EF}L_{EF}}{GJ_{EF}}, \quad \phi_D = \phi_E + \frac{T_{DE}L_{DE}}{GJ_{DE}}$$
$$\phi_C = \phi_D + \frac{T_{CD}L_{CD}}{GJ_{CD}}, \quad \phi_B = \phi_C + \frac{T_{BC}L_{BC}}{GJ_{BC}}$$
$$\phi_A = \phi_B + \frac{T_{AB}L_{AB}}{GJ_{AB}}$$

Hence, the rotation at end A

$$\phi_A = \sum_i \frac{T_i L_i}{GJ_i} = \frac{T_{AB}L_{AB}}{GJ_{AB}} + \frac{T_{BC}L_{BC}}{GJ_{BC}} + \frac{T_{CD}L_{CD}}{GJ_{CD}} + \frac{T_{DE}L_{DE}}{GJ_{DE}} + \frac{T_{EF}L_{EF}}{GJ_{EF}}$$

$$= 0 + \frac{(10000 \ Nm) \times (0.1 \ m)}{(80 \times 10^9 \ Pa) \times (1.27 \times 10^{-6} \ m^4)} + \frac{(20000 \ Nm) \times (0.1 \ m)}{(80 \times 10^9 \ Pa) \times (1.27 \times 10^{-6} \ m^4)}$$

$$+ \frac{(20000 \ Nm) \times (0.25 \ m)}{(80 \times 10^9 \ Pa) \times (20.36 \times 10^{-6} \ m^4)} + \frac{(70000 \ Nm) \times (0.25 \ m)}{(80 \times 10^9 \ Pa) \times (20.36 \times 10^{-6} \ m^4)}$$

$$= 43.29 \times 10^{-3} \ rad$$

Problem 4.

Design the stepped shaft in Problem 3 if the radius of the shaft ABCD is half the radius of the shaft DEF, the allowable rotation at end A is $30 \times 10^{-3} \ rad$, and allowable shear stress in the shafts should be less than 120 MPa.

Let us assume the radius of the shaft ABCD is c.

$$J_{AB} = J_{BC} = J_{CD} = \frac{\pi}{2}c^4$$

$$J_{DE} = J_{EF} = \frac{\pi}{2}(2c)^4 = 8\pi c^4$$

Hence,

$$\phi_A = \sum_i \frac{T_i L_i}{GJ_i} = \frac{T_{AB}L_{AB}}{GJ_{AB}} + \frac{T_{BC}L_{BC}}{GJ_{BC}} + \frac{T_{CD}L_{CD}}{GJ_{CD}} + \frac{T_{DE}L_{DE}}{GJ_{DE}} + \frac{T_{EF}L_{EF}}{GJ_{EF}}$$

$$= 0 + \frac{(10000 \ Nm) \times (0.1 \ m)}{(80 \times 10^9 \ Pa) \times (\pi/2 \times c^4)} + \frac{(20000 \ Nm) \times (0.1 \ m)}{(80 \times 10^9 \ Pa) \times (\pi/2 \times c^4)}$$

$$+ \frac{(20000 \ Nm) \times (0.25 \ m)}{(80 \times 10^9 \ Pa) \times (8\pi c^4)} + \frac{(70000 \ Nm) \times (0.25 \ m)}{(80 \times 10^9 \ Pa) \times (8\pi c^4)} < 30 \times 10^{-3}$$

$$\Rightarrow \quad \frac{1}{c^4}[636.62 + 1273.24 + 198.94 + 696.30] < (30 \times 10^{-3}) \times (80 \times 10^9)$$

$$\Rightarrow \quad \frac{2805.1}{c^4} < 2.4 \times 10^9$$

$$\Rightarrow \quad c^4 > \frac{2805.1}{2.4 \times 10^9} = 1.1688 \times 10^{-6} \ m^4$$

$$\Rightarrow \quad \boxed{c > 0.033 \ m = 33 \ mm}$$

From the maximum shear stress in the shaft ABCD

$$\tau_{max} = \frac{T_{CD}c}{J_{CD}} = \frac{(20000 \ Nm) \cdot c}{\pi/2c^4} < 120 \ MPa$$

$$\Rightarrow \quad \frac{12732.4}{c^3} \ Pa < 120 \times 10^6 \ Pa$$

$$\Rightarrow \quad \boxed{c > 47.34 \ mm}$$

From the maximum shear stress in the shaft DEF

$$\tau_{\max} = \frac{T_{EF}(2c)}{J_{EF}} = \frac{(70000 \; Nm) \cdot (2c)}{8\pi c^4} < 120 \; MPa$$

$$\Rightarrow \frac{5570.4}{c^3} \; Pa < 120 \times 10^6 \; Pa$$

$$\Rightarrow \boxed{c > 35.94 \; mm}$$

Choose the maximum of these: $c \approx 48 \; mm$ and $2c \approx 96 \; mm$.

Problem 5.

Determine the support reactions T_A and T_F if the end A is fixed in Problem 3.

Assume the reaction T_A is redundant and ϕ_L = rotation due the external load, ϕ_R = rotation due to the reaction T_A.

From Problem 3,

$$\phi_L = 43.29 \times 10^{-3} \; rad$$

Figure 39: Problem 5.

$$\phi_R = \sum_i \frac{T_i L_i}{G J_i} = -T_A \left[\frac{0.3 \; m}{(80 \times 10^9 \; Pa) \cdot (1.27 \times 10^{-6} \; m^4)} \right.$$

$$\left. + \frac{0.5 \; m}{(80 \times 10^9 \; Pa) \cdot (20.36 \times 10^{-6} \; m^4)} \right]$$

$$= -(3.26 \times 10^{-6}) T_A$$

Using equation of compatibility

$$\phi_L + \phi_R = 0$$
$$\Rightarrow 43.29 \times 10^{-3} - (3.26 \times 10^{-6}) T_A = 0$$
$$\Rightarrow T_A = 13279.1 \; Nm = 13.28 \; kNm$$
$$\Rightarrow T_F = 70 \; kNm - T_A = 56.72 \; kNm$$

Power transfer

For a power transmission shaft

$$\boxed{\begin{aligned} P &= T\omega = T \cdot (2\pi f) \\ T &= \frac{P}{2\pi f} \end{aligned}}$$

where P is the power transmitted, f is the frequency of the transmission, and T is torque in the transmission shaft.
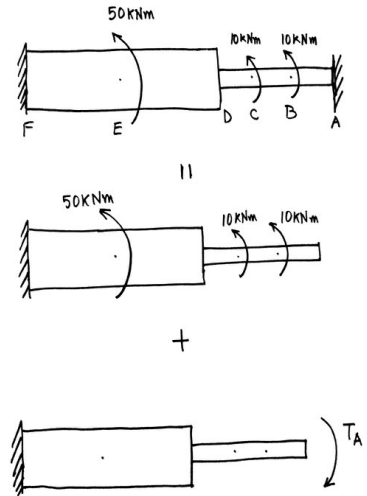
Problem 6.

Design the thickness of a transmission shaft with an outer radius of
20 mm to transmit a power of 50 kW at a frequency of 3000 rpm if
maximum allowable shear stress is 25 MPa.

Here, $P = 50\ kW = 50,000\ W = 50,000\ Nm/s$,
$f = 3000\ rpm = \frac{3000}{60}\ Hz = 50\ s^{-1}$. Hence,

$$T = \frac{P}{2\pi f} = \frac{50,000\ Nm/s}{2\pi \times (50\ s^{-1})} = 159.15\ Nm$$

The outer radius $c_2 = 20\ mm$.

The polar moment of inertia $J = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}\left[(0.02\ m)^4 - c_1^4\right]$

The maximum shear stress developed

$$\tau_{max} = \frac{Tc_2}{J} = \frac{(159.15\ Nm) \cdot (0.02\ m)}{\frac{\pi}{2}\left[(0.02\ m)^4 - c_1^4\right]} < 25\ MPa$$

$$\Rightarrow \frac{2.0265\ Nm^2}{(0.02\ m)^4 - c_1^4} < 25 \times 10^6\ Pa$$

$$\Rightarrow \frac{2.0265\ Nm^2}{25 \times 10^6\ Pa} < (0.02\ m)^4 - c_1^4$$

$$\Rightarrow c_1^4 < (0.02\ m)^4 - \frac{2.0265\ Nm^2}{25 \times 10^6\ Pa}$$

$$\Rightarrow c_1 < 0.01676\ m$$

$$\Rightarrow c_2 - c_1 > 3.24\ mm$$

Hence, a thickness of 4 mm is required for the transmission shaft.

Bending

Sign convention

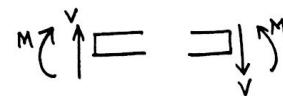The positive shear force and bending moments are as shown in the figure.

Centroid of an area

If the area can be divided into n parts then the distance $\bar{Y}$ of the centroid from a point can be calculated using

$$\bar{Y} = \frac{\sum_{i=1}^{n} A_i \bar{y}_i}{\sum_{i=1}^{n} A_i}$$

where A_i = area of the ith part, $\bar{y}_i$ = distance of the centroid of the ith part from that point.

Second moment of area, or moment of inertia of area, or area moment of inertia, or second area moment

For a rectangular section, moments of inertia of the cross-sectional area about axes x and y are

$$I_x = \frac{1}{12} bh^3$$

$$I_y = \frac{1}{12} hb^3$$

Figure 41: A rectangular section.

Parallel axis theorem

This theorem is useful for calculating the moment of inertia about an axis parallel to either x or y. For example, we can use this theorem to calculate $I_{x'}$.

$$I_{x'} = I_x + Ad^2$$

Bending stress

Bending stress at any point in the cross-section is

$$\sigma = -\frac{My}{I}$$

where y is the perpendicular distance to the point from the centroidal axis and it is assumed +ve above the axis and -ve below the axis. This will result in +ve sign for bending tensile (T) stress and -ve sign for bending compressive (C) stress.

Largest normal stress

Largest normal stress

$$\sigma_m = \frac{|M|_{\max} \cdot c}{I} = \frac{|M|_{\max}}{S}$$

where S = section modulus for the beam.

For a rectangular section, the moment of inertia of the cross-sectional area $I = \frac{1}{12}bh^3$, $c = h/2$, and $S = I/c = \frac{1}{6}bh^2$.

We require $\sigma_m \leq \sigma_{all}$ (allowable stress)

This gives

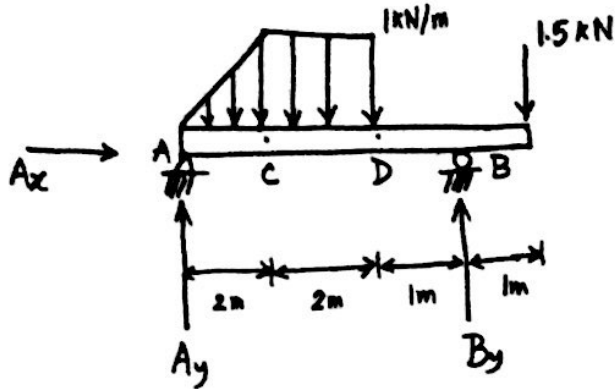$$S_{\min} = \frac{|M|_{\max}}{\sigma_{all}}$$

The radius of curvature

The radius of curvature ρ in the bending of a beam can be estimated using

$$\frac{1}{\rho} = \frac{M}{EI}$$

Problem 1.

Draw the bending moment and shear force diagram of the following beam.

Step I:

Solve for the reactions.

$$\xrightarrow{+} \sum F_x = 0 \;\Rightarrow\; A_x = 0$$

$$+\uparrow \sum F_y = 0 \;\Rightarrow\; A_y + B_y - \frac{1}{2} \cdot (1\ kN/m) \cdot (2\ m) - (1\ kN/m) \cdot (2\ m) = 0$$

$$\Rightarrow\; A_y + B_y = 3\ kN$$

$$+\curvearrowleft \sum M_A = 0 \;\Rightarrow\; -\frac{1}{2} \cdot (1\ kN/m) \cdot (2\ m) \cdot \left(\frac{4}{3}\ m\right) - (1\ kN/m) \cdot (2\ m) \cdot (3\ m) + B_y \cdot (5\ m) - (1.5\ kN) \cdot (6\ m) = 0$$

$$\Rightarrow\; B_y = 3.27\ kN$$

$$\Rightarrow\; A_y = 1.23\ kN$$

Step II:

Use equations of equilibrium.

$$\frac{1}{2}(x)\left(\tfrac{x}{2}\right) = \frac{x^2}{4}$$

$\underline{0 < x < 2\ m :}$

$$+\uparrow \sum F_y = 0$$

$$\Rightarrow\; -V - \frac{1}{2} \cdot (x/2) \cdot (x) + 1.23 = 0$$

$$\Rightarrow\; \boxed{V = 1.23 - \left(\frac{x^2}{4}\right)}$$

$$V\Big|_{x=2\ m} = 0.23\ kN$$

$A_y = 1.23\text{KN}$

Figure 43: Free body diagram for $0 < x < 2\ m$.

Take moment about the right end of the section

$$+ \curvearrowleft \sum M = 0$$

$$\Rightarrow \; M + \left(\frac{x^2}{4}\right) \cdot \left(\frac{x}{3}\right) - 1.23x = 0$$

$$\Rightarrow \; \boxed{M = 1.23x - 0.083x^3}$$

$$M\Big|_{x=2\ m} = 1.796\ kNm$$

$\underline{2\ m < x < 4\ m:}$

$$+\uparrow \sum F_y = 0$$

$$\Rightarrow \; -V - (x - 2) - 1 + 1.23 = 0$$

$$\Rightarrow \; \boxed{V = 2.23 - x}$$

$$V\Big|_{x=4\ m} = -1.77\ kN$$

$$V = 0 \text{ at } x = 2.23\ m$$

Take moment about the right end of the section

$$+ \curvearrowleft \sum M = 0$$

$$\Rightarrow \; M + 1 \cdot (x - 2) \cdot \left(\frac{x - 2}{2}\right) + 1 \cdot \left(x - \frac{4}{3}\right) - 1.23x = 0$$

$$\Rightarrow \; \boxed{M = -0.67 + 2.23x - 0.5x^2}$$

$$M\Big|_{x=4\ m} = 0.25\ kNm$$

Figure 44: Free body diagram for $2\ m < x < 4\ m$.

$\underline{4\ m < x < 5\ m:}$

$$+\uparrow \sum F_y = 0$$

$$\Rightarrow \; V - 1.5 + 3.27 = 0$$

$$\Rightarrow \; \boxed{V = -1.77}$$

Take moment about the left end of the section

$$+ \curvearrowleft \sum M = 0$$

$$\Rightarrow \; -M + (3.27) \cdot (5 - x) - (1.5) \cdot (6 - x) = 0$$

$$\Rightarrow \; \boxed{M = 7.35 - 1.77x}$$

$$M\Big|_{x=5\ m} = -1.5\ kNm$$

Figure 45: Free body diagram for $4\ m < x < 5\ m$.

$\underline{5\ m < x < 6\ m:}$

$$+\uparrow \sum F_y = 0$$

$$\Rightarrow \; \boxed{V = 1.5}$$

Figure 46: Free body diagram for $5\ m < x < 6\ m$.

Take moment about the left end of the section

$$+\curvearrowleft \sum M = 0$$

$$\Rightarrow \quad -M - (1.5)\cdot(6-x) = 0$$

$$\Rightarrow \quad \boxed{M = 1.5x - 9}$$

Note: $\boxed{V = \dfrac{dM}{dx}}$

The BMD and SFD are drawn next.

Figure 47: Bending moment and shear force diagrams.

Note: Maximum bending moment occurs at x^* where
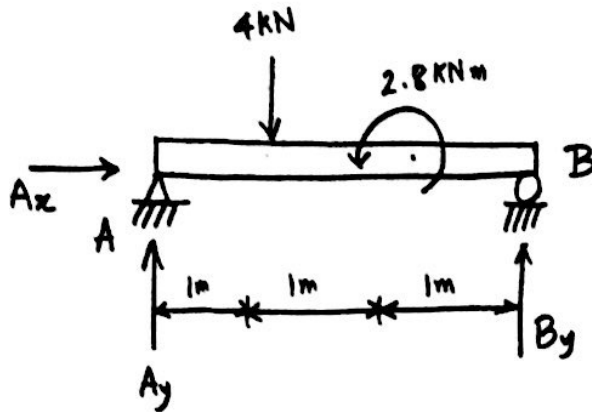
$$\frac{dM}{dx}\Big|_{x=x^*} = 0$$
$$V = 0$$
$$2.23 - x^* = 0$$
$$x^* = 2.23 \ m$$

Problem 2.

(a) Draw the bending moment and shear force diagram of the following beam.

Figure 48: Problem 2.

Step I:

Solve for the support reactions.

$$\overset{+}{\rightarrow} \sum F_x = 0 \ \Rightarrow \ A_x = 0$$
$$+\uparrow \sum F_y = 0 \ \Rightarrow \ A_y + B_y = 4 \ kN$$
$$+ \curvearrowleft \sum M_A = 0 \ \Rightarrow \ -(4 \ kN)\cdot(1 \ m) + 2.8 \ kNm + B_y \cdot (3 \ m) = 0$$
$$\Rightarrow \ B_y = 0.4 \ kN$$
$$\Rightarrow \ A_y = 3.6 \ kN$$

Step II:

Use equations of equilibrium.

<u>$0 < x < 1 \; m$</u> :

$$+\uparrow \sum F_y = 0$$

$$\Rightarrow \boxed{V = 3.6}$$

Take moment about the right end of the section

$$+\curvearrowleft \sum M = 0$$

$$\Rightarrow M - (3.6) \cdot x = 0$$

$$\Rightarrow \boxed{M = 3.6x}$$

$$M\Big|_{x=1 \; m-\Delta x} = 3.6 \; kNm$$

<u>$1 \; m < x < 2 \; m$</u> :

$$+\uparrow \sum F_y = 0$$

$$\Rightarrow -V - 4 + 3.6 = 0$$

$$\Rightarrow \boxed{V = -0.4}$$

Take moment about the right end of the section

$$+\curvearrowleft \sum M = 0$$

$$\Rightarrow M + 4 \cdot (x - 1) - (3.6) \cdot x = 0$$

$$\Rightarrow \boxed{M = 4 - 0.4x}$$

$$M\Big|_{x=1 \; m+\Delta x} = 3.6 \; kNm$$

$$M\Big|_{x=2 \; m-\Delta x} = 3.2 \; kNm$$

<u>$2 \; m < x < 3 \; m$</u> :

$$+\uparrow \sum F_y = 0$$

$$\Rightarrow \boxed{V = -0.4}$$

Take moment about the left end of teh section

$$+\curvearrowleft \sum M = 0$$

$$\Rightarrow \boxed{M = 0.4(3 - x)}$$

$$M\Big|_{x=2 \; m+\Delta x} = 0.4 \; kNm$$

(b) Check the required section for this beam with $\sigma_{all} = 25 \; MPa$.
Here, $|M|_{max} = 3.6 \; kNm$.

$$S_{min} = \frac{|M|_{max}}{\sigma_{all}} = \frac{3.6 \times 10^3 \; Nm}{25 \times 10^6 \; N/m^2}$$

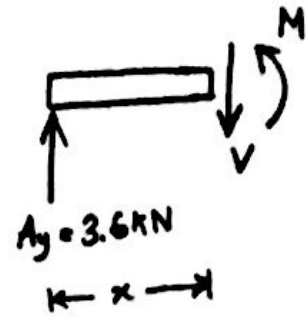$$= 1.44 \times 10^{-4} m^3$$

$$= 144 \times 10^3 \; mm^3$$

Figure 49: Free body diagram for $0 < x < 1 \; m$.

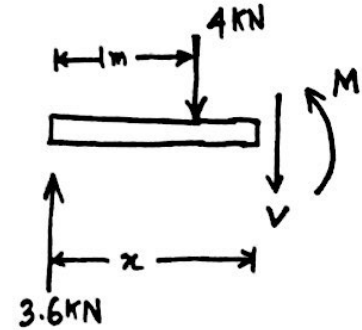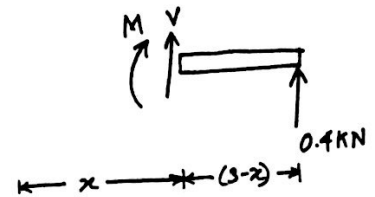Figure 50: Free body diagram for $1 \; m < x < 2 \; m$.

Figure 51: Free body diagram for $2 \; m < x < 3 \; m$.

Hence, for a rectangular section

$$S = \frac{1}{6}bh^2 = \frac{1}{6} \cdot (40\ mm) \cdot h^2$$

For this beam,

$$\frac{1}{6} \cdot (40\ mm) \cdot h^2 = 144 \times 10^3\ mm^3$$

$$h^2 = 21600\ mm^2$$

$$h = 146.97\ mm$$

Let's take $h = 150\ mm$.

To design a standard angle section, we can use $L\ 203 \times 203 \times 19$ (lightest) with $S = 200 \times 10^3\ mm^3$ @ 57.9 kg/m.
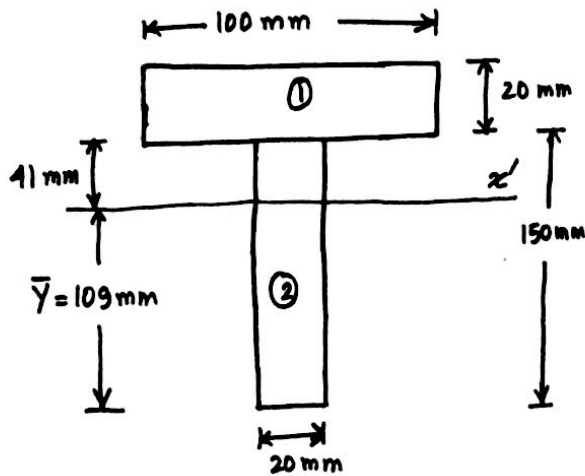
Shape	$S(10^3\ mm^3)$
$L\ 203 \times 203 \times 25.4$	259
$L\ 203 \times 203 \times 19$	200
$L\ 203 \times 203 \times 12.7$	137

Problem 3.

Calculate the moment of inertia of the T section with cross-sectional area shown below about the centroidal axis x'.

	A_i (mm^2)	$\bar{y}_i$ (mm)	$A_i\bar{y}_i$ (mm^3)
①	2×10^3	75	225×10^3
②	3×10^3	160	320×10^3
Σ	5×10^3		545×10^3

Hence, the distance to the centroidal axis from the bottom of the section is

$$\bar{Y} = \frac{\sum A_i \bar{y}_i}{\sum A_i} = \frac{545 \times 10^3 \ mm^3}{5 \times 10^3 \ mm^2}$$

$$= 109 \ mm$$

Method I:

Using the parallel axes theorem,

$$I_1 = \frac{1}{12} bh^3 + Ad^2$$

$$= \frac{1}{12} \cdot (0.1 \ m) \cdot (0.02 \ m)^3 + (0.1 \ m) \cdot (0.02 \ m) \cdot (0.051 \ m)^2$$

$$= 5.27 \times 10^{-6} \ m^4$$

$$I_2 = \frac{1}{12} bh^3 + Ad^2$$

$$= \frac{1}{12} \cdot (0.02 \ m) \cdot (0.15 \ m)^3 + (0.02 \ m) \cdot (0.15 \ m) \cdot (0.034 \ m)^2$$

$$= 9.09 \times 10^{-6} \ m^4$$

Hence, the moment of inertia of the T section with cross-sectional area about the centroidal axis x'

$$I_{x'} = I_1 + I_2$$

$$= 14.36 \times 10^{-6} \ m^4$$

Method II:

Using the parallel axes theorem, for the overall rectangular section

$$I_o = \frac{1}{12}bh^3 + Ad^2$$
$$= \frac{1}{12} \cdot (0.1\ m) \cdot (0.17\ m)^3 + (0.1\ m) \cdot (0.17\ m) \cdot (0.024\ m)^2$$
$$= 50.73 \times 10^{-6}\ m^4$$

$$I_{1'} = I_{2'} = \frac{1}{12}bh^3 + Ad^2$$
$$= \frac{1}{12} \cdot (0.04\ m) \cdot (0.15\ m)^3 + (0.04\ m) \cdot (0.15\ m) \cdot (0.034\ m)^2$$
$$= 18.19 \times 10^{-6}\ m^4$$

Hence, the moment of inertia of the T section with cross-sectional area about the centroidal axis x'

$$I_{x'} = I_o - I_{1'} - I_{2'}$$
$$= 14.36 \times 10^{-6}\ m^4$$

(b) If this section is subjected to 5 kNm bending moment estimate the bending stresses at the top and at the bottom fibers.
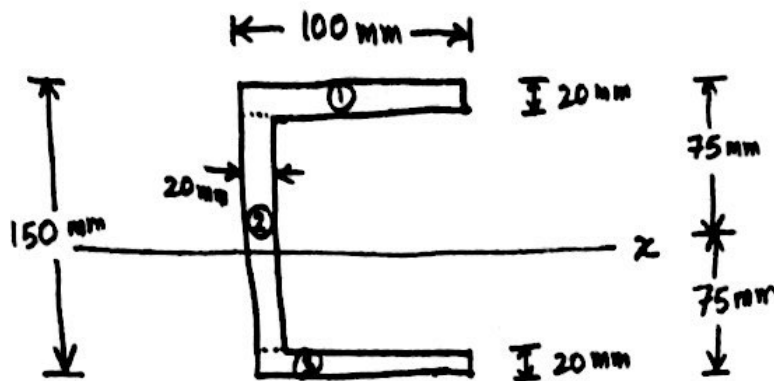
Here, $M = 5$ kNm. Hence,

$$\sigma_{top} = -\frac{My_{top}}{I_{x'}} = -\frac{(5 \times 10^3\ Nm) \cdot (0.061\ m)}{14.36 \times 10^{-6}\ m^4}$$
$$= -21.24\ MPa = 21.24\ MPa\ (C)$$

$$\sigma_{bot} = -\frac{My_{bot}}{I_{x'}} = -\frac{(5 \times 10^3 \ Nm) \cdot (-0.109 \ m)}{14.36 \times 10^{-6} \ m^4}$$

$$= 37.95 \ MPa \ (T)$$

Problem 4.

For an angular section shown below estimate the moment of inertia about the centroidal axis x.

Method I:

Using the parallel axes theorem,

$$I_1 = I_3 = \frac{1}{12}bh^3 + Ad^2$$

$$= \frac{1}{12} \cdot (0.1 \ m) \cdot (0.02 \ m)^3 + (0.1 \ m) \cdot (0.02 \ m) \cdot (0.065 \ m)^2$$

$$= 8.52 \times 10^{-6} \ m^4$$

$$I_2 = \frac{1}{12}bh^3$$

$$= \frac{1}{12} \cdot (0.02 \ m) \cdot (0.11 \ m)^3$$

$$= 2.22 \times 10^{-6} \ m^4$$

Hence, the moment of inertia of the angle section with cross-sectional area about the centroidal axis x

$$I_x = I_1 + I_2 + I_3$$

$$= 19.25 \times 10^{-6} \ m^4$$

Method II:

For the overall rectangular section

$$I_o = \frac{1}{12}bh^3$$

$$= \frac{1}{12} \cdot (0.1\ m) \cdot (0.15\ m)^3$$

$$= 28.13 \times 10^{-6}\ m^4$$

$$I_{1'} = \frac{1}{12}bh^3$$

$$= \frac{1}{12} \cdot (0.08\ m) \cdot (0.11\ m)^3$$

$$= 8.87 \times 10^{-6}\ m^4$$
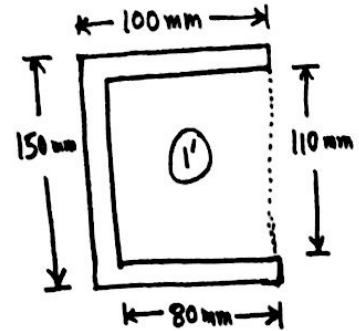

Figure 56: Method II.

Hence, the moment of inertia of the angle section with cross-sectional area about the centroidal axis x

$$I_x = I_o - I_{1'}$$

$$= 19.25 \times 10^{-6}\ m^4$$

Problem 5.

Calculate (a) maximum bending stress in the section, (b) bending stress at point B in the section, and (c) the radius of curvature.
Using the parallel axes theorem,

$$I_1 = I_3 = \frac{1}{12}bh^3 + Ad^2$$

$$= \frac{1}{12} \cdot (0.25\ m) \cdot (0.02\ m)^3 + (0.25\ m) \cdot (0.02\ m) \cdot (0.16\ m)^2$$

$$= 128.17 \times 10^{-6}\ m^4$$

$$I_2 = \frac{1}{12}bh^3$$

$$= \frac{1}{12} \cdot (0.02\ m) \cdot (0.3\ m)^3$$

$$= 45 \times 10^{-6}\ m^4$$

Hence, moment of inertia of the cross-sectional area about the centroidal axis x

$$I_x = I_1 + I_2 + I_3$$
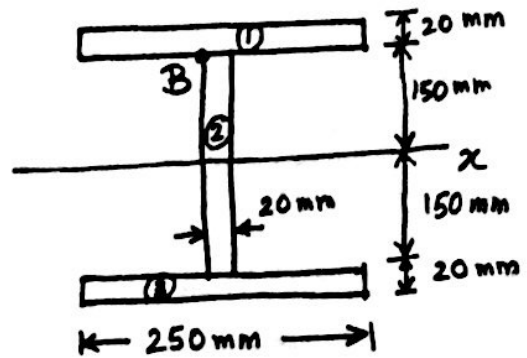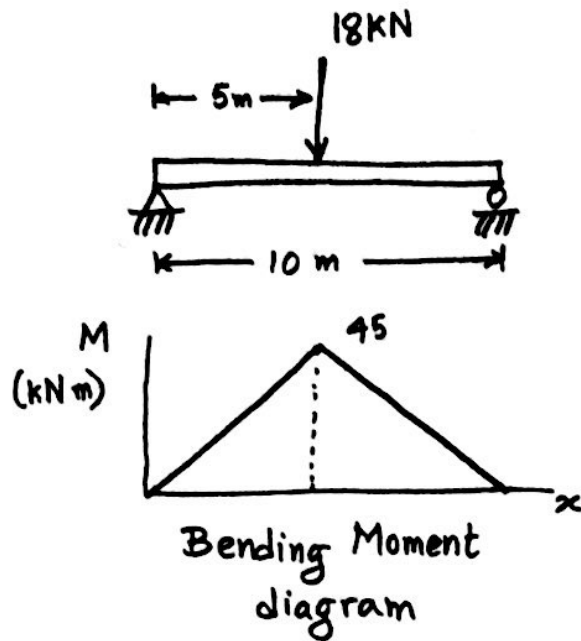
$$= 301.33 \times 10^{-6}\ m^4$$

(a) Maximum bending stress

$$\sigma_m = \frac{|M|_{\max} \cdot c}{I_x} = -\frac{(45 \times 10^3\ Nm) \cdot (0.17\ m)}{301.33 \times 10^{-6}\ m^4}$$

$$= 25.4\ MPa$$

18KN

5m

10 m

M
(kN m)

45

Bending Moment
diagram

20 mm

150mm

B

20mm

x

150 mm

20 mm

250mm

(b) Bending stress at B

$$\sigma_B = -\frac{My_B}{I_x} = -\frac{(45 \times 10^3 \ Nm) \cdot (0.15 \ m)}{301.33 \times 10^{-6} \ m^4}$$
$$= -22.4 \ MPa = 22.4 \ MPa \ (C)$$

(c)

$$\frac{1}{\rho} = \frac{M}{EI_x} = \frac{(45 \times 10^3 \ Nm)}{(200 \times 10^9 \ Pa) \cdot (301.33 \times 10^{-6} \ m^4)}$$
$$= 7.47 \times 10^{-4} \ m^{-1}$$

Hence, the radius of curvature

$$\rho = 1339 \ m$$

(d) If a rolled steel section $W \ 200 \times 86$ is used then we have

$I_x = 94.9 \times 10^6 \ m^4 = 94.9 \times 10^{-6} \ m^4$, $c = 0.111 \ m$, $y_B = -(0.111 - 0.0206) \ m = -0.0904 \ m$

Maximum bending stress

$$\sigma_m = \frac{|M|_{\max} \cdot c}{I_x} = \frac{(45 \times 10^3 \ Nm) \cdot (0.111 \ m)}{94.9 \times 10^{-6} \ m^4}$$
$$= 52.63 \ MPa$$

Bending stress at B

$$\sigma_B = -\frac{My_B}{I_x} = -\frac{(45 \times 10^3 \ Nm) \cdot (-0.0904 \ m)}{94.9 \times 10^{-6} \ m^4}$$
$$= 42.87 \ MPa \ (T)$$

$$\frac{1}{\rho} = \frac{M}{EI_x} = 2.37 \times 10^{-3} \ m^{-1}$$

The radius of curvature

$$\rho = 421.8 \ m$$

Composite beams

The section of the beam consists of material 1 with elastic modulus E_1 and material 2 with elastic modulus E_2.

Figure 58: Composite beam section.

Step I

Assume material 1 (generally the with smaller E_1) as reference material.

Define $n_1 = \frac{E_1}{E_1} = 1$, $n_2 = \frac{E_2}{E_1}$.

Step II

Estimate the position of the neutral axis $\bar{Y}$ using

$$\boxed{\bar{Y} = \frac{\sum_i n_i A_i \bar{y}_i}{\sum_i n_i A_i}}$$

Step III

Calculate the moment of inertia of the cross-sectional area about the neutral axis (NA)

$$\boxed{I_x = \sum_i \frac{1}{12} n_i b_i h_i^3 + n_i A_i d_i^2}$$

Essentially the cross-sectional area is transformed into section shown here made up of only the reference material.

Figure 59: Transformed beam section.

Step IV

Calculate the stress developed

$$\boxed{\sigma = -\frac{n_i My}{I_x}}$$
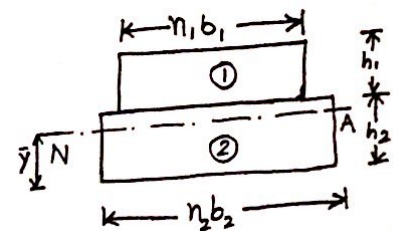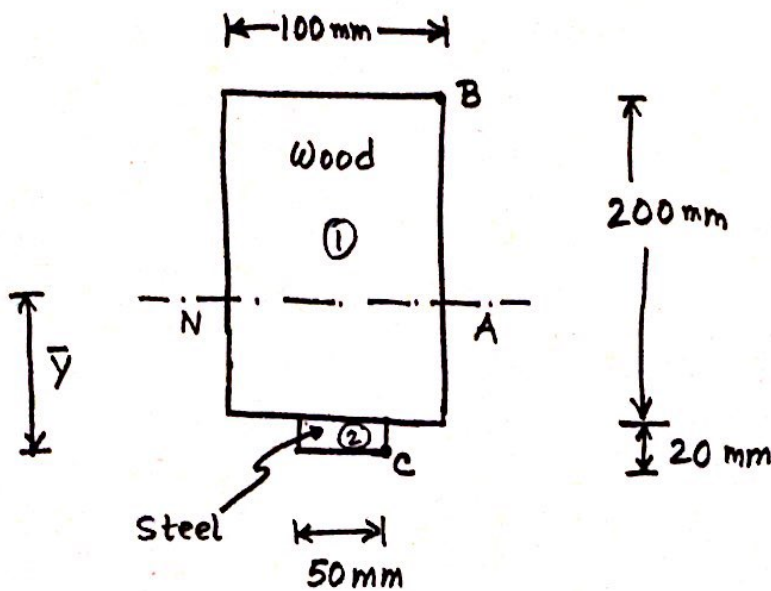
The radius of curvature is given by

$$\frac{1}{\rho} = \frac{M}{E_1 I_x}$$

where E_1 is the elastic modulus of the reference material.

Problem 6.

For the section shown here made of wood ($E_1 = 16$ GPa) and steel ($E_2 = 200$ GPa) calculate the bending stress at B and C when subjected to a moment of 1.5 kNm.

Step I

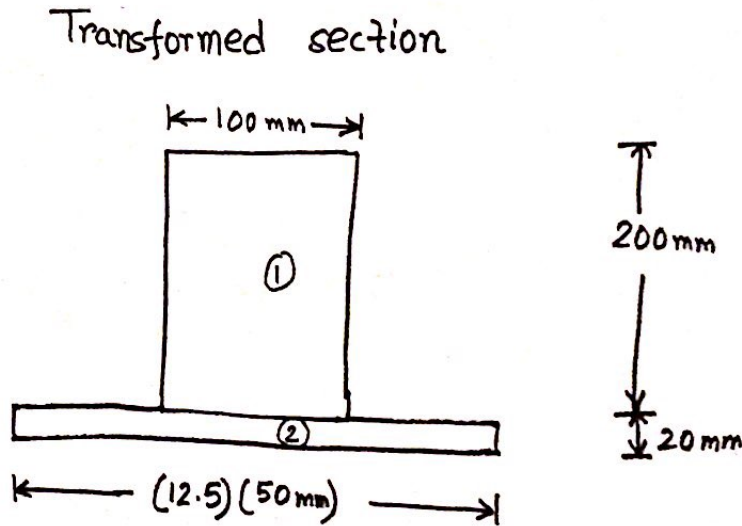Assume wood with $E_1 = 10$ GPa as reference material.
  Define $n_1 = \frac{E_1}{E_1} = 1$, $n_2 = \frac{E_2}{E_1} = 200/16 = 12.5$.

Step II

The distance is measured from bottom of the beam

	$n_i A_i$ (mm^2)	$\bar{y}_i$ (mm)	$n_i A_i \bar{y}_i$ (mm^3)
①	20×10^3	120	2400×10^3
②	12.5×10^3	10	125×10^3
Σ	32.5×10^3		2525×10^3

Figure 61: Problem 6 (transformed section).

Estimate the position of the neutral axis $\bar{Y}$ using

$$\bar{Y} = \frac{\sum_i n_i A_i \bar{y}_i}{\sum_i n_i A_i} = 77.7 \ mm$$

Step III

Moment of inertia of the cross-sectional area of the wood about the neutral axis (NA)

$$I_1 = \frac{1}{12} n_1 b_1 h_1^3 + n_1 A_1 d_1^2$$

$$= \frac{1}{12} \cdot (1) \cdot (0.1 \ m) \cdot (0.2 \ m)^3 + (1) \cdot (20 \times 10^{-3} \ m^2) \cdot (0.120 \ m - 0.0777 \ m)^2$$

$$= 102.5 \times 10^{-6} \ m^4$$

Moment of inertia of the cross-sectional area of the steel plate about the neutral axis (NA)

$$I_2 = \frac{1}{12} n_2 b_2 h_2^3 + n_2 A_2 d_2^2$$

$$= \frac{1}{12} \cdot (12.5) \cdot (0.05 \ m) \cdot (0.02 \ m)^3 + (12.5) \cdot (1 \times 10^{-3} \ m^2) \cdot (0.015 \ m)^2$$

$$= 57.7 \times 10^{-6} \ m^4$$

Hence, the moment of inertia of this composite beam is

$$I_x = I_1 + I_2 = 160.2 \times 10^{-6} \ m^4$$

Essentially the cross-sectional area is transformed into section shown below made up of only the reference material (wood here).

Step IV

The stress developed at point B

$$\sigma_B = -\frac{n_1 M y_B}{I_x}$$

$$= -\frac{(1) \cdot (1.5 \times 10^3 \ Nm) \cdot (0.22 \ m - 0.077 \ m)}{160.2 \times 10^{-6} \ m^4}$$

$$= -1.33 \ MPa = 1.33 \ MPa \ (C)$$

The stress developed at point C

$$\sigma_C = -\frac{n_2 M y_C}{I_x}$$

$$= -\frac{(12.5) \cdot (1.5 \times 10^3 \ Nm) \cdot (-0.077 \ m)}{160.2 \times 10^{-6} \ m^4}$$

$$= 9.09 \ MPa \ (T)$$

The radius of curvature is given by

$$\frac{1}{\rho} = \frac{M}{E_1 I_x}$$

$$= \frac{1.5 \times 10^3 \ Nm}{(16 \times 10^9 \ Pa) \cdot (160.2 \times 10^{-6} \ m^4)}$$

$$= 0.585 \times 10^{-3} \ m^{-1}$$

$$\Rightarrow \quad \rho = 1708.8 \ m$$

where E_1 is the elastic modulus of the reference material (wood here).

Reinforced concrete sections

Reinforced concrete is made up of concrete and steel bars. Since concrete can not take any tension and cracks appear in it only the area of the concrete section above neutral axis and the steel bars should be considered for the calculation of I_x.

Problem 7.

For the reinforced concrete section shown here (with 4 Re bars @20mm dia.) calculate the bending stress in the concrete at B (the top) and in the steel when subjected to a moment of 20 kNm. Use 20 GPa as the elastic modulus of concrete and 200 GPa as the elastic modulus of steel.

Step I

Assume concrete with $E_1 = 20$ GPa as reference material.
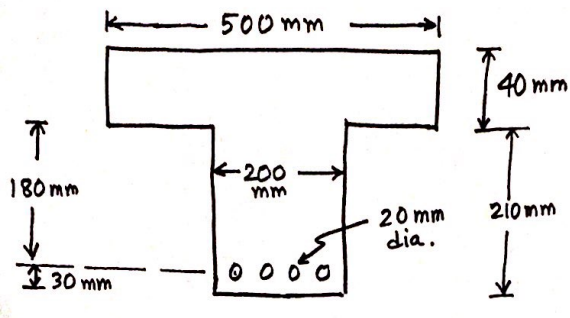Define $n_1 = \frac{E_1}{E_1} = 1$, $n_2 = \frac{E_2}{E_1} = 200/20 = 10$.

Figure 62: Problem 7.
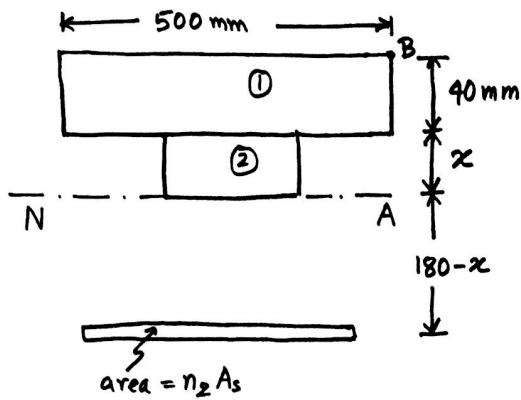

Figure 63: Problem 7 (transformed section).

Step II

Assume the position of the neutral axis as shown in the figure. Denote the distance from the bottom of the top flange to the neutral axis to be x.

The distance is measured from the assumed neutral axis of the beam

Material		$n_i A_i$ (mm²)	$\bar{y}_i$ (mm)	$n_i A_i \bar{y}_i$ (mm³)
Concrete	①	20000	$20 + x$	$20000(20 + x)$
	②	$200x$	$\frac{x}{2}$	$100x^2$
Steel	③	$(10) \cdot (4 \cdot \frac{\pi}{4} \cdot (20)^2)$ $= 12566$	$-(180 - x)$	$-12566(180 - x)$
	Σ			$20000(20 + x) + 100x^2$ $-12566(180 - x)$

The position of the actual neutral axis $\bar{Y}$ from our assumed one is

$$\bar{Y} = \frac{\sum_i n_i A_i \bar{y}_i}{\sum_i n_i A_i}$$

If our assumption of the neutral axis is true then

$$\bar{Y} = 0$$
$$\Rightarrow \frac{\sum_i n_i A_i \bar{y}_i}{\sum_i n_i A_i} = 0$$
$$\Rightarrow \sum_i n_i A_i \bar{y}_i = 0$$
$$\Rightarrow 20000(20 + x) + 100x^2 - 12566(180 - x) = 0$$
$$\Rightarrow x^2 + 200(20 + x) - 125.66(180 - x) = 0$$
$$\Rightarrow x^2 + 200x + 4000 - 22619 + 125.66x = 0$$
$$\Rightarrow x^2 + 325.66x - 18619 = 0$$
$$\Rightarrow x \approx 50 \ mm$$

Step III

Moment of inertia of the cross-sectional area of the concrete parts about the neutral axis (NA)

$$I_1 = \frac{1}{12} n_1 b_1 h_1^3 + n_1 A_1 d_1^2$$
$$= \frac{1}{12} \cdot (1) \cdot (0.5 \ m) \cdot (0.04 \ m)^3 + (1) \cdot (20 \times 10^{-3} \ m^2) \cdot (0.07 \ m)^2$$
$$= 100.7 \times 10^{-6} \ m^4$$

$$I_2 = \frac{1}{12} n_1 b_2 h_2^3 + n_1 A_2 d_2^2$$
$$= \frac{1}{12} \cdot (1) \cdot (0.2 \ m) \cdot (0.05 \ m)^3 + (1) \cdot (0.2 \ m \times 0.05 \ m) \cdot \left(\frac{0.05}{2} \ m\right)^2$$
$$= 8.3 \times 10^{-6} \ m^4$$

Moment of inertia of the cross-sectional area of the steel about the neutral axis (NA)

$$I_s = n_2 A_s d_s^2$$
$$= (10) \cdot (4 \cdot \frac{\pi}{4} \cdot (0.02 \ m)^2) \cdot (0.13 \ m)^2$$
$$= 212.4 \times 10^{-6} \ m^4$$

Note that we are ignoring the $1/12 bh^3$ part for the transformed steel section.

Hence, the moment of inertia of this composite beam is

$$I_x = I_1 + I_2 + I_s = 321.4 \times 10^{-6} \ m^4$$

Essentially the cross-sectional area is transformed into section shown below made up of only the reference material.

Step IV

The stress developed at point B (i.e., the top fiber) in the concrete

$$\sigma_B = -\frac{n_1 M y_B}{I_x}$$
$$= -\frac{(1) \cdot (20 \times 10^3 \ Nm) \cdot (0.09 \ m)}{321.4 \times 10^{-6} \ m^4}$$
$$= -5.6 \ MPa = 5.6 \ MPa \ (C)$$

This is the maximum compressive stress in the concrete.

The stress developed in the steel

$$\sigma_s = -\frac{n_2 M y_s}{I_x}$$
$$= -\frac{(10) \cdot (20 \times 10^3 \ Nm) \cdot (-0.13 \ m)}{321.4 \times 10^{-6} \ m^4}$$
$$= 80.9 \ MPa \ (T)$$

The radius of curvature is given by

$$\frac{1}{\rho} = \frac{M}{E_1 I_x}$$
$$= \frac{20 \times 10^3 \ Nm}{(20 \times 10^9 \ Pa) \cdot (321.4 \times 10^{-6} \ m^4)}$$
$$= 3.111 \times 10^{-3} \ m^{-1}$$
$$\Rightarrow \quad \rho = 321.4 \ m$$

where E_1 is the elastic modulus of the reference material (concrete here).

Problem 8.

For the reinforced concrete section shown here (with 4 Re bars @20mm dia.) calculate the bending stress at B and C when subjected to a moment of 20 kNm. Use 20 GPa as the elastic modulus of concrete and 200 GPa as the elastic modulus of steel.

Step I

Assume concrete with $E_1 = 20$ GPa as reference material.
Define $n_1 = \frac{E_1}{E_1} = 1$, $n_2 = \frac{E_2}{E_1} = 200/20 = 10$.

Step II

Assume the position of the neutral axis as shown in the figure. Denote the distance from the bottom of the top flange to the neutral axis to be x.
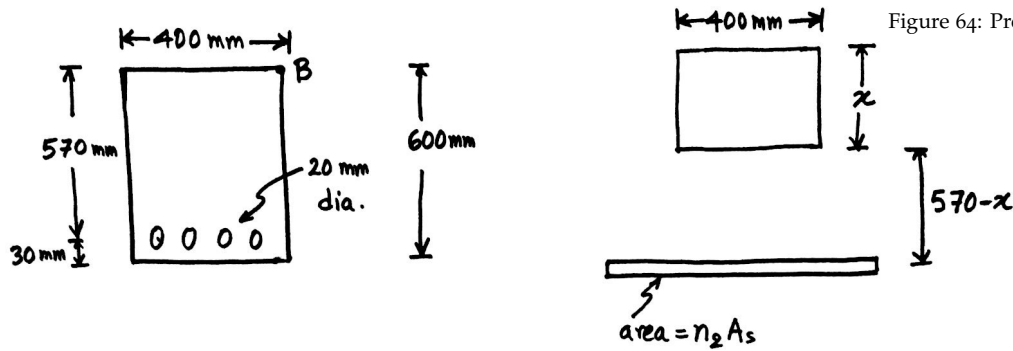
Figure 64: Problem 8.

The distance is measured from the assumed neutral axis of the beam

Material		$n_i A_i$ (mm^2)	$\bar{y}_i$ (mm)	$n_i A_i \bar{y}_i$ (mm^3)
Concrete	①	$400x$	$\frac{x}{2}$	$200x^2$
Steel	②	$\begin{array}{c}(10) \cdot (4 \cdot \frac{\pi}{4} \cdot (20)^2) \\ = 12566\end{array}$	$-(570 - x)$	$-12566(570 - x)$
	Σ			$200x^2 - 12566(570 - x)$

The position of the actual neutral axis $\bar{Y}$ from our assumed one is

$$\bar{Y} = \frac{\sum_i n_i A_i \bar{y}_i}{\sum_i n_i A_i}$$

If our assumption of the neutral axis is true then

$$\bar{Y} = 0$$
$$\Rightarrow \frac{\sum_i n_i A_i \bar{y}_i}{\sum_i n_i A_i} = 0$$
$$\Rightarrow \sum_i n_i A_i \bar{y}_i = 0$$
$$\Rightarrow 200x^2 - 12566(570 - x) = 0$$
$$\Rightarrow x^2 + 62.83x - 35813 = 0$$
$$\Rightarrow x \approx 160 \ mm$$

Step III

Moment of inertia of the cross-sectional area of the concrete parts about the neutral axis (NA)

$$I_c = \frac{1}{12} n_1 b_1 h_1^3 + n_1 A_1 d_1^2$$
$$= \frac{1}{12} \cdot (1) \cdot (0.4 \ m) \cdot (0.16 \ m)^3 + (1) \cdot (0.4 \ m \times 0.16 \ m) \cdot \left(\frac{0.16 \ m}{2}\right)^2$$
$$= 546 \times 10^{-6} \ m^4$$

Moment of inertia of the cross-sectional area of the steel about the neutral axis (NA)

$$I_s = n_2 A_s d_s^2$$

$$= (10) \cdot (4 \cdot \frac{\pi}{4} \cdot (0.02 \ m)^2) \cdot (0.41 \ m)^2$$

$$= 2112 \times 10^{-6} \ m^4$$

Note that we are ignoring the $1/12bh^3$ part for the transformed steel section.

Hence, the moment of inertia of this composite beam is

$$I_x = I_c + I_s = 2658 \times 10^{-6} \ m^4$$

Essentially the cross-sectional area is transformed into section shown below made up of only the reference material.

Step IV

The stress developed at point B (i.e., the top fiber) in the concrete

$$\sigma_B = -\frac{n_1 M y_B}{I_x}$$

$$= -\frac{(1) \cdot (20 \times 10^3 \ Nm) \cdot (0.16 \ m)}{2658 \times 10^{-6} \ m^4}$$

$$= -1.2 \ MPa = 1.2 \ MPa \ (C)$$

This is the maximum compressive stress in the concrete.

The stress developed in the steel

$$\sigma_s = -\frac{n_2 M y_s}{I_x}$$

$$= -\frac{(10) \cdot (20 \times 10^3 \ Nm) \cdot (-0.41 \ m)}{2658 \times 10^{-6} \ m^4}$$

$$= 30.85 \ MPa \ (T)$$

The radius of curvature is given by

$$\frac{1}{\rho} = \frac{M}{E_1 I_x}$$

$$= \frac{20 \times 10^3 \ Nm}{(20 \times 10^9 \ Pa) \cdot (2658 \times 10^{-6} \ m^4)}$$

$$= 0.376 \times 10^{-3} \ m^{-1}$$

$$\Rightarrow \quad \rho = 2658 \ m$$

where E_1 is the elastic modulus of the reference material (concrete here).

Shearing

Due to the presence of the shear force in the beam and the fact that $\tau_{xy} = \tau_{yx}$, a horizontal shear force exists in the beam that tend to force the beam fibers to slide.

Horizontal Shear in Beams

The horizontal shear per unit length is given by

$$q = \frac{VQ}{I}$$

where $V =$ the shear force at that section; $Q =$ the first moment of the portion of the area (above the horizontal line where the shear is being calculated) about the neutral axis; and $I =$ moment of inertia of the cross-sectional area of the beam. The quantity q is also known as the *shear flow*.

Average Shear Stress Across the Width

Average shear stress across the width is defined as

$$\tau_{\text{ave}} = \frac{VQ}{It}$$

where $t =$ width of the section at that horizontal line. For a narrow rectangular beam with $t = b \leq h/4$, the shear stress varies across the width by less than 80% of τ_{ave}.

Maximum Transverse Shear Stress

For a narrow rectangular section we can work with the equation $\tau = \frac{VQ}{It}$ to calculate shear stress at any vertical point in the cross section. Hence, the shear stress at a distance y from the neutral axis

$$Q = \left[b \cdot \left(\frac{h}{2} - y \right) \right] \cdot \left(y + \frac{h/2 - y}{2} \right) = \frac{b}{2} \cdot \left(\frac{h^2}{4} - y^2 \right)$$

$$A = bh$$

$$I = \frac{1}{12}bh^3$$

$$\tau_{xy} = \tau_{yx} = \frac{VQ}{Ib}$$

$$= \frac{V \cdot \frac{b}{2} \cdot \left(\frac{h^2}{4} - y^2\right)}{\frac{1}{12}bh^3 \cdot b}$$

$$= \frac{3V(h^2 - 4y^2)}{2bh^3}$$

$$= \frac{3V}{2A} \cdot \left(1 - \frac{4y^2}{h^2}\right)$$

$$\text{OR } \tau_{xy} = \tau_{yx} = \frac{V}{2I} \cdot \left(\frac{h^2}{4} - y^2\right)$$

—- a parabolic distribution of stress.

Hence, the maximum stress in a rectangular beam section is at $y = 0$ and

$$\boxed{\tau_{max} = \frac{3V}{2A}}$$

In case of a wide flanged beam like the one shown here the maximum shear stress is at the web and can be approximated as

$$\boxed{\tau_{max} = \frac{V}{A_{web}}}$$

Problem 1.

(a) Using the wooden T section as shown below and used in the previous classes find the maximum shear it can take where the nails have a capacity of 400 N against shear loads and the spacing between the nails is 50 mm.

Using the parallel axes theorem,

$$I_1 = \frac{1}{12}bh^3 + Ad^2$$

$$= \frac{1}{12} \cdot (0.1 \ m) \cdot (0.02 \ m)^3 + (0.1 \ m) \cdot (0.02 \ m) \cdot (0.051 \ m)^2$$

$$= 5.27 \times 10^{-6} \ m^4$$

$$I_2 = \frac{1}{12}bh^3 + Ad^2$$

$$= \frac{1}{12} \cdot (0.02 \ m) \cdot (0.15 \ m)^3 + (0.02 \ m) \cdot (0.15 \ m) \cdot (0.034 \ m)^2$$

$$= 9.09 \times 10^{-6} \ m^4$$
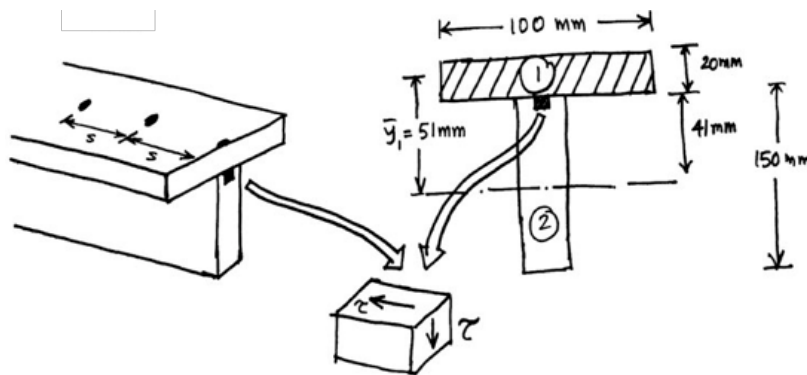
Hence, the moment of inertia of the T section about the centroidal axis x'

$$I = I_1 + I_2$$
$$= 14.36 \times 10^{-6} \ m^4$$

The first moment of the cross-sectional area is

$$Q = A_1 \bar{y}_1$$
$$= (0.1 \ m) \cdot (0.02 \ m) \cdot (0.051 \ m)$$
$$= 102 \times 10^{-6} \ m^3$$

The nails have $F_{nail} = 400 \ N$. If q_{all} is the allowable shear per unit length and s is the spacing between the nails then

$$F_{nail} = q_{all}s$$

$$\Rightarrow q_{all} = \frac{F_{nail}}{s} = \frac{400 \ N}{0.05 \ m} = 8 \times 10^3 \ N/m$$

Hence,

$$q_{all} = \frac{V_{max}Q}{I}$$

$$\Rightarrow V_{max} = \frac{q_{all}I}{Q} = \frac{(8 \times 10^3 \ N/m) \cdot (14.36 \times 10^{-6} \ m^4)}{102 \times 10^{-6} \ m^3} = 1.126 \ kN$$

(b) If $V = 1$ kN and estimate the maximum shear stress.
Maximum shear stress occurs at the neutral axis

$$\tau_{max} = \frac{VQ}{It} = \frac{(1 \times 10^3 \ N) \cdot (119 \times 10^{-6} \ m^3)}{(14.36 \times 10^{-6} \ m^4) \cdot (0.02 \ m)} = 414.35 \ kPa$$

(c) Instead of two wooden planks as shown before if four wooden planks, two horizontal nails, and a single vertical nail are used as shown below. estimate the spacings required for the two horizontal nails for $V = 1 \ kN$ and $F_{nail} = 400 \ N$.

Figure 67: Problem 1: four planks are used.

In this case, the shear at the joint of 1st and the 2nd part needs to be estimated. For this

$$Q = A_1 \bar{y}_1$$
$$= (0.05 \ m) \cdot (0.02 \ m) \cdot (0.051 \ m)$$
$$= 51 \times 10^{-6} \ m^4$$

Now,

$$\frac{F_{\text{nail}}}{s} = q = \frac{VQ}{I} = \frac{(1 \times 10^3 \ N) \cdot (51 \times 10^{-6} \ m^3)}{14.36 \times 10^{-6} \ m^4} = 3551.5 \ N/m$$

$$\Rightarrow \ s = \frac{F_{\text{nail}}}{q} = \frac{400 \ N}{3551.5 \ N/m} = 0.113 \ m$$

Hence, a spacing of 100 mm will be okay.

Problem 2.

(a) For the box section shown here estimate the nail spacing required if $V = 1 \ kN$ and $F_{\text{nail}} = 400 \ N$.

Figure 68: Problem 2.

$$I_1 = I_4 = \frac{1}{12} \cdot (0.1 \ m) \cdot (0.02 \ m)^3 + (0.1 \ m) \cdot (0.02 \ m) \cdot (0.04 \ m)^2$$

$$= 3.27 \times 10^{-6} \ m^4$$

$$I_2 = I_3 = \frac{1}{12} \cdot (0.02 \ m) \cdot (0.06 \ m)^3$$

$$= 0.36 \times 10^{-6} \ m^4$$

The second moment of inertia of the cross-sectional area about the neutral axis

$$I = I_1 + I_2 + I_3 + I_4$$

$$= 2 \times 3.27 \times 10^{-6} \ m^4 + 2 \times 0.36 \times 10^{-6} \ m^4$$

$$= 7.25 \times 10^{-6} \ m^4$$

The first moment of the top part about the neutral axis is

$$Q = A_1 \bar{y}_1$$

$$= (0.1 \ m) \cdot (0.02 \ m) \cdot (0.04 \ m) = 80 \times 10^{-6} \ m^3$$

The shear flow here

$$\frac{2F_{nail}}{s} = q = \frac{VQ}{I}$$

$$= \frac{(1 \times 10^3 \ N) \cdot (80 \times 10^{-6} \ m^3)}{7.25 \times 10^{-6} \ m^4}$$

$$\Rightarrow \frac{2 \times 400 \ N}{s} = 11034.5 \ N/m$$

$$\Rightarrow \ s = 0.0725 \ m$$

Hence, a spacing of 75 mm will be okay.

(b) Calculate the maximum shear stress developed.

At the neutral axis

$$Q = 80 \times 10^{-6} \ m^3 + 2 \cdot (0.03 \ m) \cdot (0.02 \ m) \cdot (0.015 \ m)$$
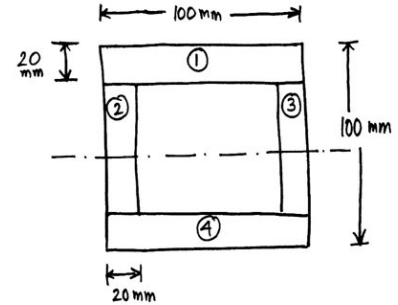
$$= 98 \times 10^{-6} \ m^3$$

Figure 69: Problem 2.

Maximum shear stress

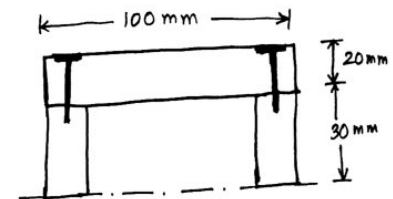$$\tau = \frac{VQ}{It}$$
$$= \frac{(1 \times 10^3 \ N) \cdot (98 \times 10^{-6} \ m^3)}{(7.25 \times 10^{-6} \ m^4) \cdot (2 \times 0.02 \ m)}$$
$$= 338 \ kPa$$

Problem 3.

Design the beam as shown below for $\sigma_{all} = 80 \ MPa$ and $\tau_{all} = 10 \ MPa$. The depth of the beam is limited to 275 mm. Use standard rolled steel section.

The shear force and bending moment diagrams are drawn first.

From the diagrams, $|V|_{max} = 20 \ kN$ and $|M|_{max} = 100 \ kNm$.

Figure 70: Problem 3.

Design for bending stress

Hence, section modulus required

$$S_{reqd} = \frac{|M|_{max}}{\sigma_{all}}$$
$$= \frac{100 \times 10^3 \ Nm}{80 \times 10^6 \ Pa}$$
$$= 1.25 \times 10^{-3} \ m^3$$
$$= 1250 \times 10^3 \ mm^3$$

Since the depth is limited choose W250×80 and add two 8 mm thick plates at the top and bottom.

Total depth = 273 mm < 275 mm (okay).

The modified I section has a second moment of inertia about the neutral axis

$$I = I_{beam} + 2I_{plate}$$
$$= 126 \times 10^{-6} \ m^4$$
$$+ 2 \cdot \left[\frac{1}{12} \cdot (0.254 \ m) \cdot (0.008 \ m)^3 + (0.254 \ m) \cdot (0.008 \ m) \cdot (0.1325 \ m)^2 \right]$$
$$= 197.4 \times 10^{-6} \ m^4$$

$$c = 136.5 \ mm$$
$$S = \frac{I}{c} = 1446 \times 10^{-6} \ m^3 > S_{reqd}$$
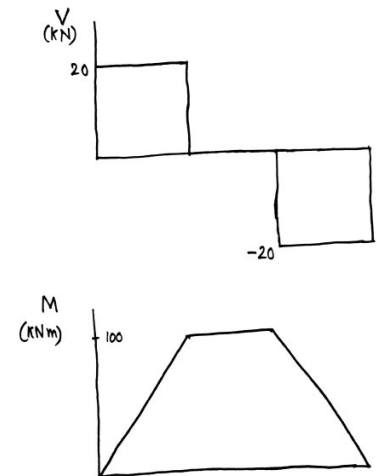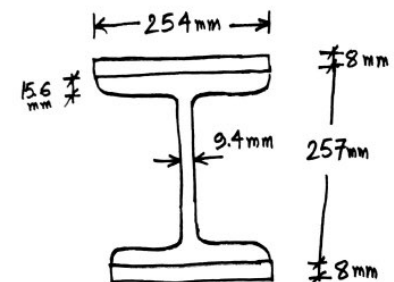

Figure 71: Problem 3: SFD, BMD.

Figure 72: Problem 3: Modified I section.

Check for shear stress

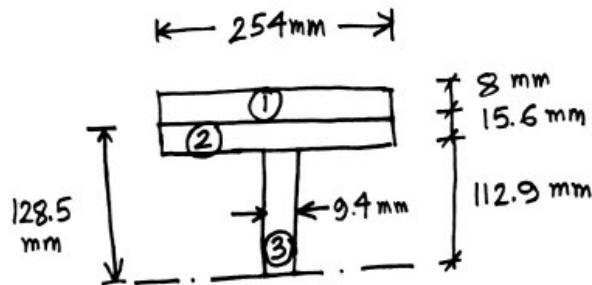|		A (mm^2)	$\bar{y}$ (mm)	$A\bar{y}$ (mm^3)
Plate	①	254×8	132.5	269.24×10^3
I-section	②	254×15.6	120.7	478.26×10^3
	③	112.9×9.4	56.45	59.91×10^3
	Σ			807.41×10^3

Figure 73: Problem 3: Shear stress calculation.

$$Q = \sum A\bar{y} = 571.11 \times 10^3 \ mm^3, \quad t = 9.4 \ mm$$

Hence, maximum shear stress is

$$\tau_{\max} = \frac{|V|_{\max}Q}{It} = \frac{(20 \times 10^3 \ N) \cdot (807.41 \times 10^{-6} \ m^3)}{(197.4 \times 10^{-6} \ m^4) \cdot (0.0094 \ m)}$$

$$= 8.7 \ MPa < \tau_{all} \quad (okay)$$

Problem 4.

(a) Calculate the stress in the bolt that connects steel plates and the wooden block as shown if the section is subjected to $V = 10$ kN. Assume the elastic moduli of steel as 200 GPa and of wood as 12.5 GPa. The bolt used has a diameter of 16 mm and a spacing of $s = 100$ mm is used.

Take steel as the reference material. Hence,

$$E_1 = 200 \ GPa, \quad E_2 = 12.5 \ GPa$$

$$n_1 = 1, \quad n_2 = \frac{E_2}{E_1} = \frac{1}{16}$$

Figure 74: Problem 4.

The transformed section will be the following
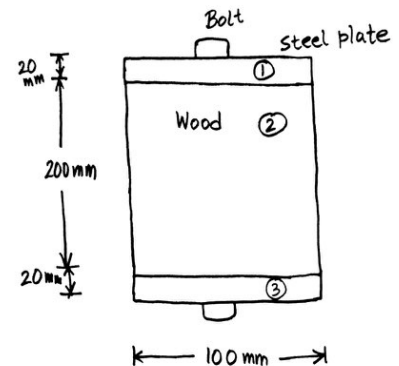The neutral axis will pass through the middle of the section.

Transformed section

The second moment of inertia of part 1 about the neutral axis,

$$I_1 = \frac{1}{12}n_1 b_1 h_1^3 + n_1 A_1 d_1^2$$
$$= \frac{1}{12} \cdot (1) \cdot (0.1\ m) \cdot (0.02\ m)^3 + (1) \cdot (0.1\ m) \cdot (0.02\ m) \cdot (0.1\ m)^2$$
$$= 24.27 \times 10^{-6}\ m^4$$

Similarly,

$$I_2 = \frac{1}{12}n_2 b_2 h_2^3$$
$$= \frac{1}{12} \cdot \left(\frac{1}{16}\right) \cdot (0.1\ m) \cdot (0.2\ m)^3$$
$$= 4.17 \times 10^{-6}\ m^4$$

For the full section,

$$I = 2I_1 + I_2 = 52.7 \times 10^{-6}\ m^4$$

To get the stress in the bolt we need to calculate the shear force at the bonded surface. Hence, we need the first moment of the steel pate about the neutral axis

$$Q_1 = n_1 A_1 \bar{y}_1$$
$$= (1) \cdot (0.1\ m) \cdot (0.02\ m) \cdot (0.11\ m)$$
$$= 220 \times 10^{-6}\ m^3$$

The shear flow is

$$q = \frac{VQ_1}{I}$$
$$= \frac{(10 \times 10^3\ N) \cdot (220 \times 10^{-6}\ m^3)}{52.7 \times 10^{-6}\ m^4}$$
$$= 41.75 \times 10^3\ N/m$$

If the stress in the bolt is τ_b and the cross-sectional area of the bolt is A_b we can write

$$\tau_b A_b = F_{bolt} = qs$$
$$\Rightarrow \tau_b = \frac{qs}{A_b} = \frac{qs}{\pi d_b^2/4}$$
$$= \frac{(41.75 \times 10^3 \ N/m) \cdot (0.1 \ m)}{\pi \cdot (0.016 \ m)^2/4}$$
$$= 20.76 \ MPa$$

(b) Instead if allowable shear stress is $\tau_{all} = 10 \ MPa$ determine the required spacing.

We have

$$s_{reqd} = \frac{\tau_{all} A_b}{q} = 0.0482 \ m$$

Hence, a spacing of 45 mm will be okay.

Problem 5.

Calculate the shear stress in the bonded surface if the section is subjected to $V = 10$ kN. Assume the elastic moduli of steel as 210 GPa and of aluminum as 70 GPa.

NOTE: In this problem, we need to know the centroid and the second moment of inertia of a semi-circular area. Please see the calculation at the end of this problem.

Take the aluminum with the semi-circular hole in it as the reference material. Hence,

Figure 77: Problem 5.

$$E_1 = 70 \ GPa, \quad E_2 = 210 \ GPa$$
$$n_1 = 1, \quad n_2 = \frac{E_2}{E_1} = 3$$

Next, to locate its neutral axis

		$n_i A_i$ (mm^2)	$\bar{y}_i$ (mm)	$n_i A_i \bar{y}_i$ (mm^3)
	①	$(1) \cdot (100 \times 100)$	100	1000×10^3
Aluminum	③	$-(1) \cdot \left(\frac{\pi \cdot (20)^2}{2} \right)$	$50 + \frac{4 \times 20}{3\pi}$	-36.75×10^3
		$= -314.16$	$= 58.49$	
Steel	②	$(3) \cdot (100 \times 50)$	25	375×10^3
	Σ	24371.7		1338.25×10^3

Figure 78: Problem 5: The transformed section.

The neutral axis is located at a distance $\bar{Y}$ from the bottom where

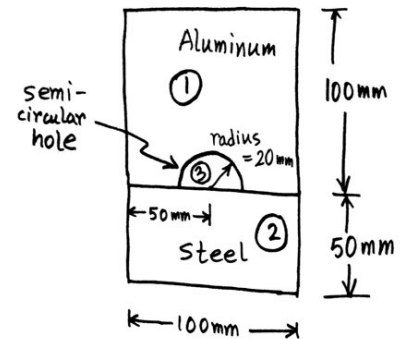$$\bar{Y} = \frac{\sum_i n_i A_i \bar{y}_i}{\sum_i n_i A_i} = \frac{1338.25 \times 10^3}{24371.7} \approx 55 \ mm$$
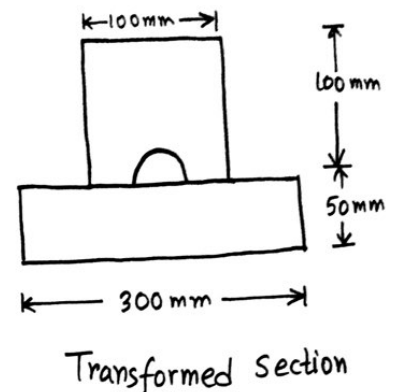
Next, the calculate the second moment of inertia of the cross-sectional area about the neutral axis. We will separately calculate for ①, ②, and ③, first.

$$I_1 = \frac{1}{12} n_1 b_1 h_1^3 + n_1 A_1 d_1^2$$
$$= \frac{1}{12} \cdot (1) \cdot (0.1\ m) \cdot (0.1\ m)^3 + (1) \cdot (0.1\ m) \cdot (0.1\ m) \cdot (0.045\ m)^2$$
$$= 28.583 \times 10^{-6}\ m^4$$

$$I_2 = \frac{1}{12} n_2 b_2 h_2^3 + n_2 A_2 d_2^2$$
$$= \frac{1}{12} \cdot (3) \cdot (0.1\ m) \cdot (0.05\ m)^3 + (3) \cdot (0.1\ m) \cdot (0.05\ m) \cdot (0.03\ m)^2$$
$$= 16.625 \times 10^{-6}\ m^4$$

$$I_3 = \frac{1}{12} n_1 \left(\frac{\pi r^4}{8} \right) + n_1 \left(\frac{\pi r^2}{2} \right) d_3^2$$
$$= \frac{1}{12} \cdot (1) \cdot \left(\frac{\pi \times (0.02\ m)^4}{8} \right) + (1) \cdot \left(\frac{\pi \times (0.02\ m)^2}{2} \right) \cdot (0.005\ m)^2$$
$$= 0.021 \times 10^{-6}\ m^4$$

Since $I_x = \frac{\pi r^4}{8}$ about the axis that passes through the center as shown in the figure. Detailed explanation is given at the end.

Hence,
$$I = I_1 + I_2 - I_3 = 45.187 \times 10^{-6}\ m^4$$

To estimate the the shear stress we need to calculate the first moment Q of the cross-sectional area about the neutral axis and we will use the bottom steel part to do it.

$$Q = Q_2 = n_2 A_2 \bar{y}_2$$
$$= (3) \cdot (0.1\ m \times 0.05\ m) \cdot (0.03\ m)$$
$$= 450 \times 10^{-6}\ m^3$$

Here, $t = 100 \; mm - 40 \; mm = 60 \; mm = 0.06 \; m$.
Hence, the shear stress at the bonded surface

$$\tau = \frac{VQ}{It}$$
$$= \frac{(10 \times 10^3 \; N) \cdot (450 \times 10^{-6} \; m^3)}{(45.187 \times 10^{-6} \; m^4) \cdot (0.06 \; m)}$$
$$= 1.66 \; MPa$$

Centroid and second moment of inertia of a semi-circular area

Take a small area inside the semi-circular area as shown in the figure.
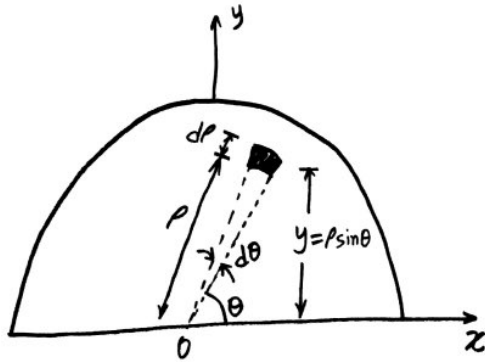The area of this element is $dA = (d\rho) \cdot (\rho d\theta) = \rho d\rho d\theta$.

Figure 80: Semi-circular area.

The area of this semi-circular plate is $A = \frac{\pi r^2}{2}$.
Hence, if the distance to the centroid from the bottom is $\bar{y}$ then using the figure

$$A\bar{y} = \int_A y dA = \int_A \rho \sin(\theta) dA = \int_0^r \int_0^\pi \rho \sin(\theta) \cdot (\rho d\rho d\theta)$$
$$\Rightarrow \left(\frac{\pi r^2}{2} \right) \cdot \bar{y} = \int_0^r \int_0^\pi (\sin(\theta) d\theta) \cdot (\rho^2 d\rho)$$
$$= \left[\int_0^\pi \sin(\theta) d\theta \right] \cdot \left[\int_0^r \rho^2 d\rho \right]$$
$$= 2 \cdot \frac{r^3}{3}$$
$$\Rightarrow \bar{y} = \left(\frac{2r^3}{3} \right) \cdot \left(\frac{2}{\pi r^2} \right) = \frac{4r}{3\pi}$$

The second moment of inertia of the semi-circular area about the x

axis

$$I_x = \int_A y^2 \, dA$$

$$= \int_A [\rho \sin(\theta)]^2 \, dA$$

$$= \int_0^r \int_0^\pi \rho^2 \sin^2(\theta) \cdot (\rho \, d\rho \, d\theta)$$

$$= \left[\int_0^\pi \sin^2(\theta) \, d\theta \right] \cdot \left[\int_0^r \rho^3 \, d\rho \right]$$

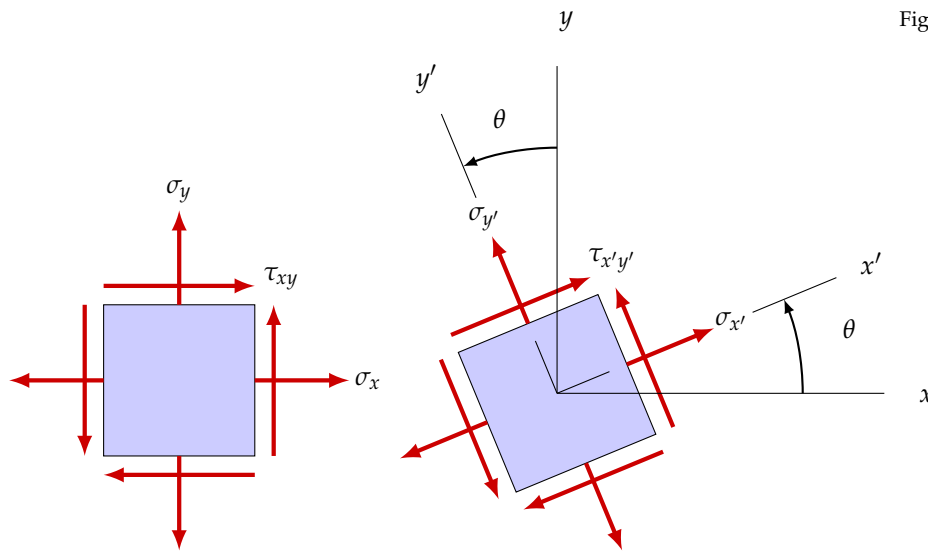$$= \left(\frac{\pi}{2} \right) \cdot \left(\frac{r^4}{4} \right)$$

$$= \frac{\pi r^4}{8}$$

Transformation of Stress

For a plane stress condition, *i.e.*, $\sigma_z = \tau_{xz} = \tau_{yz} = 0$, if you rotate the element shown here by an angle θ the equations for the transformed stresses are

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) - \tau_{xy}\sin(2\theta)$$

Note that, $\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$.

Principal Stresses

Principal stresses are the maximum normal stresses acting on the principal planes if you rotate the element by θ_p. You can find θ_p

Figure 82: Principal planes and stresses.

either by $\frac{d\sigma_{x'}}{d\theta} = 0$ or by noting that on the principal planes you have $\tau_{x'y'} = 0$.

Hence,

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\sigma_{\text{max,min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Note: Anticlockwise angles are positive and clockwise angles are negative.

Maximum In-plane Shear Stress

Similarly, you can find maximum in-plane shear stress at an angle θ_s, where

$$\tan(2\theta_s) = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\tau_{\text{max}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Figure 83: Maximum in-plane shear stress.

Note: θ_p and θ_s are 45° apart.

Along with τ_{max}, the normal stresses on all four planes are $\sigma_{ave} = (\sigma_x + \sigma_y)/2$.

Problem 1.

Calculate the normal stress and the shear stress acting on a plane inclined at an angle 30° to the x axis.
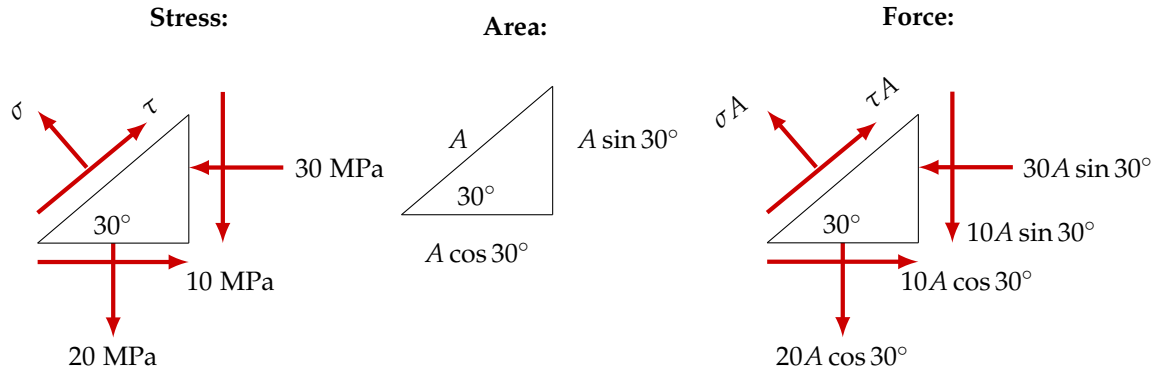

Figure 84: Problem 1.

Method I:

Stress:

Area:

Force:

Figure 85: Problem 1: Calculation of forces acting on the element.

Using the equations of equilibrium

$$\sum F_{x'} = 0$$

$$\Rightarrow \sigma A - 20A\cos 30°\cos 30° - 10A\cos 30°\cos 60°$$

$$- 10A\sin 30°\cos 30° + 30A\sin 30°\cos 60° = 0$$

$$\Rightarrow \sigma = 16.16 \ MPa$$

$$\sum F_{y'} = 0$$

$$\Rightarrow \tau A - 20A\cos 30°\cos 60° + 10A\cos 30°\cos 30°$$

$$- 10A\sin 30°\cos 60° - 30A\sin 30°\cos 30° = 0$$

$$\Rightarrow \tau = 16.65 \ MPa$$

Method II:

Here, we have

$$\sigma_x = -30 \ MPa$$

$$\sigma_y = 20 \ MPa$$

$$\tau_{xy} = -10 \ MPa$$

Figure 86: Problem 1: Transformed stresses.

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) - \tau_{xy}\sin(2\theta)$$

$$= \frac{-30 + 20}{2} - \frac{-30 - 20}{2}\cos 60° - (-10)\sin 60°$$

$$= -5 + 25\cos 60° + 10\sin 60°$$

$$= 16.16 \ MPa$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

$$= -\frac{-30 - 20}{2}\sin 60° - 10\cos 60°$$

$$= 25\sin 60° - 10\cos 60°$$

$$= 16.65 \ MPa$$

Problem 2.

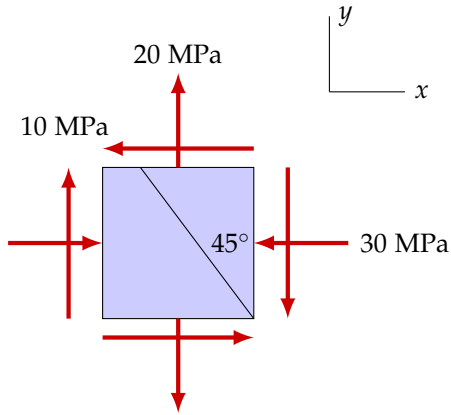Calculate the normal stress and the shear stress acting on a plane inclined at an angle $45°$ to the y axis.

Here, we have

$$\sigma_x = -30 \ MPa$$
$$\sigma_y = 20 \ MPa$$
$$\tau_{xy} = -10 \ MPa$$

$$
\begin{aligned}
\sigma_{x'} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta) \\
&= \frac{-30 + 20}{2} + \frac{-30 - 20}{2} \cos 90° + (-10) \sin 90° \\
&= -5 - 25 \cos 90° - 10 \sin 90° \\
&= -15 \ MPa
\end{aligned}
$$

$$
\begin{aligned}
\tau_{x'y'} &= -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta) \\
&= -\frac{-30 - 20}{2} \sin 90° - 10 \cos 90° \\
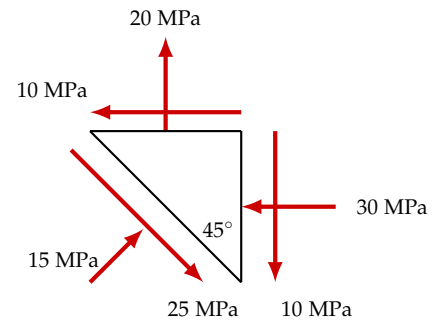&= 25 \sin 90° - 10 \cos 90° \\
&= 25 \ MPa
\end{aligned}
$$

Figure 88: Problem 2: Transformed stresses.

Problem 3.

Consider an element at the top end of this rod.

(a) Calculate the principal stresses, maximum in-plane shear stress.

The top end of the rod is subjected to a torsion $T = 5$ kNm and a bending moment $M = (10 \ kN) \cdot (0.5 \ m) = 5 \ kNm$.
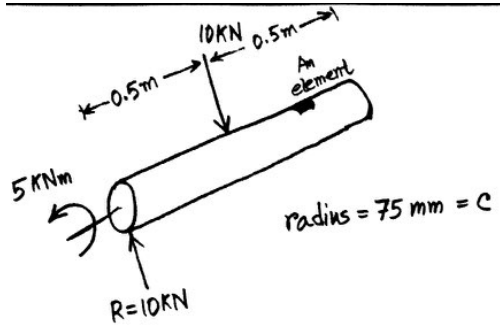
The polar moment of inertia J and the second moment of inertia I of the cross-sectional area

$$J = \frac{\pi \cdot c^4}{2} = \frac{\pi \cdot (0.075 \ m)^4}{2} = 49.7 \times 10^{-6} \ m^4$$

$$I = I_x = I_y = J/2 = \frac{\pi \cdot c^4}{4} = \frac{\pi \cdot (0.075 \ m)^4}{4} = 24.85 \times 10^{-6} \ m^4$$

Hence, in the element, we will have

$$\tau = \frac{Tc}{J} = \frac{(5 \times 10^3 \ Nm) \cdot (0.075 \ m)}{49.7 \times 10^{-6} \ m^4} = 7.55 \ MPa$$

$$\sigma = -\frac{Mc}{I} = -\frac{(5 \times 10^3 \ Nm) \cdot (0.075 \ m)}{24.85 \times 10^{-6} \ m^4} = -15.1 \ MPa$$

The element is drawn next and we have

$$\sigma_x = -15.1 \ MPa$$

$$\sigma_y = 0$$

$$\tau_{xy} = 7.55 \ MPa$$

7.55 MPa

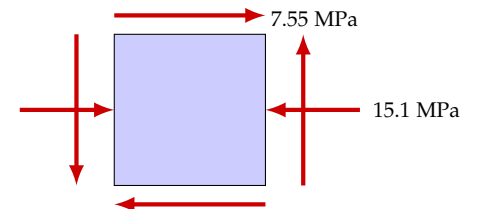15.1 MPa

Figure 90: Problem 3: The element.

Using the equations for the principal stresses

$$\sigma_{\text{max,min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{-15.1 + 0}{2} \pm \sqrt{\left(\frac{-15.1 - 0}{2}\right)^2 + (7.55)^2}$$

$$= -18.23 \ MPa, \ 3.13 \ MPa$$

The principal planes are located at an angle θ_p, where

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$= \frac{2 \times 7.55}{-15.1 - 0}$$

$$= -1$$

$$\Rightarrow 2\theta_p = -45°, \ 135°$$

$$\Rightarrow \theta_p = -22.5°, \ 67.5°$$

Maximum in-plane shear stress

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{\left(\frac{-15.1 - 0}{2}\right)^2 + (7.55)^2}$$

$$= 10.68 \ MPa$$

$$\tan(2\theta_s) = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$= -\frac{-15.1 - 0}{2 \times 7.55}$$

$$= 1$$

$$\Rightarrow \ \theta_s = 22.5°$$

Figure 91: Problem 3: Principal stresses.

(b) Find maximum torsion the rod can be subjected to such that $\tau_{\max} \leq 15 \ MPa$.

Using the equation for maximum in-plane shear stress

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \leq 15 \ MPa$$

$$\Rightarrow \ \tau_{xy}^2 \leq (15)^2 - (7.55)^2 = 168$$

$$\Rightarrow \ \tau_{xy} \leq 12.96 \ MPa$$

Figure 92: Problem 3 (b).

Hence, the maximum torsion that can be applied

$$\tau_{xy} = \frac{Tc}{J} \leq 12.96 \ MPa$$

$$T_{\max} = \frac{J(12.96 \ MPa)}{c}$$

$$= 8.59 \ kNm$$
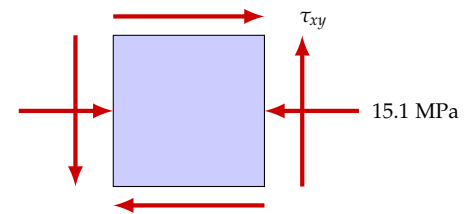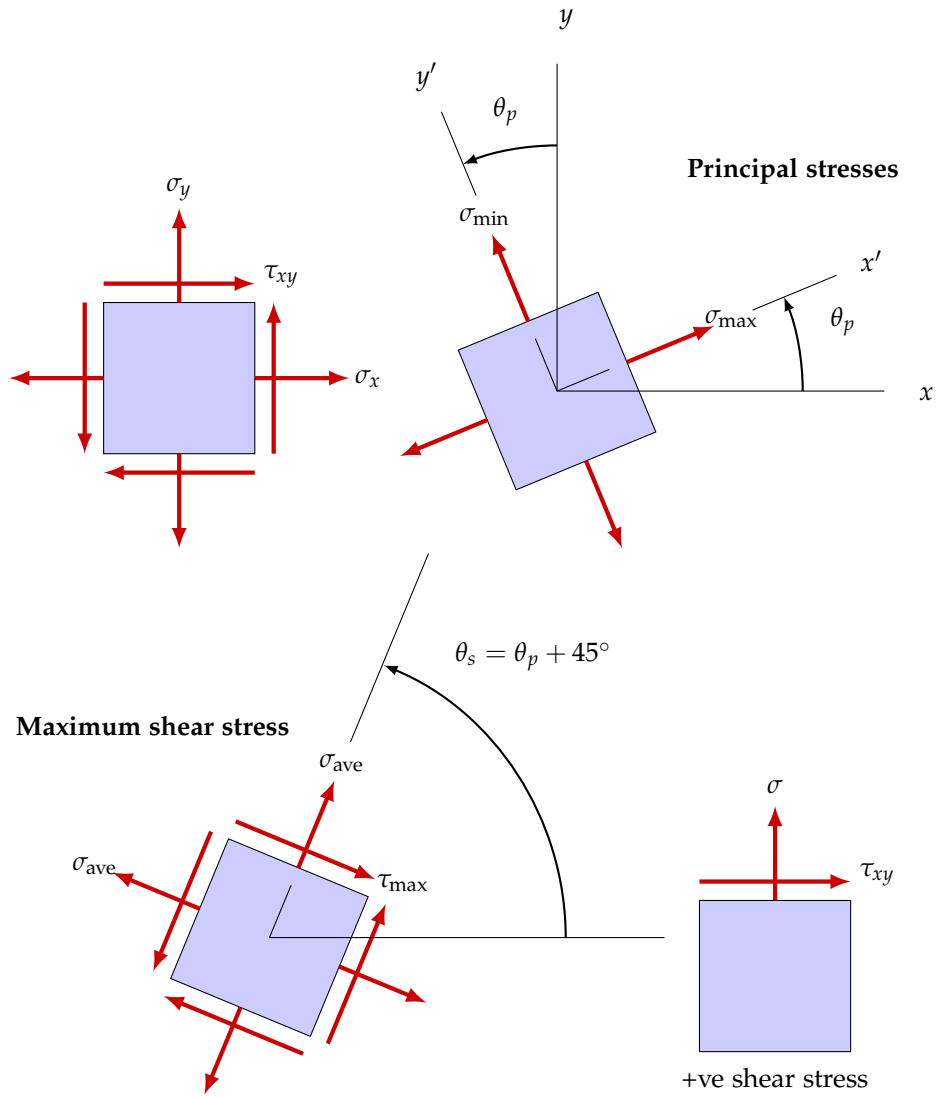
Mohr's circle

Figure 93: Stresses acting on an element.

Figure 94: Mohr's circle.

Problem 4.

Draw the Mohr's circle for the element shown.
Here, we have

Figure 95: Problem 4: The element.

$$\sigma_x = 0$$

$$\sigma_y = 0$$

$$\tau_{xy} = +\tau$$

$$\sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = 0$$

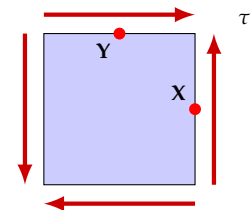Hence, the center of the circle is at $(\sigma_{\text{ave}}, 0) \equiv (0,0)$. Also, consider

two points X and Y with coordinates

$$X : (0, -\tau)$$
$$Y : (0, \tau)$$

The center of the Mohr's circle is at $(0,0)$ and the two points X and Y are located on the circle as shown in the figure. The radius of the circle is $R = \tau$.
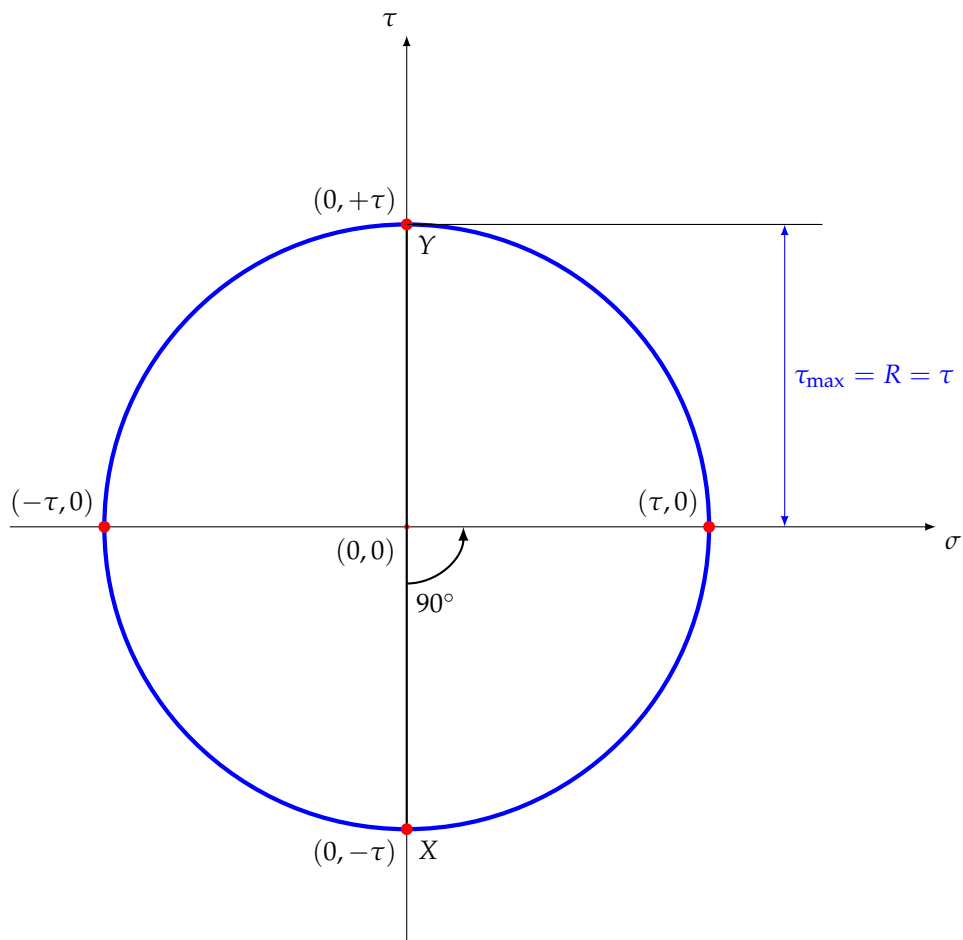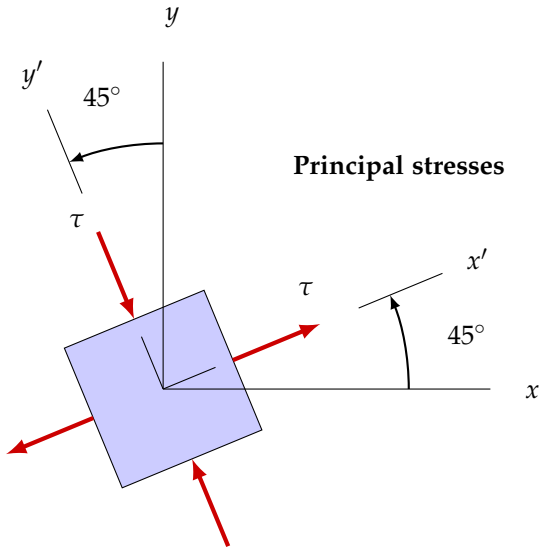

Figure 96: Problem 4: Mohr's circle.

Hence, the principal stresses are

$$\sigma_{\max} = \tau$$
$$\sigma_{\min} = -\tau$$

They are shown in the next figure.

The maximum in-plane shear stress, $\tau_{\max} = R = \tau$.

Problem 5.

Draw the Mohr's circle for the element shown.

Here, we have

Figure 97: Problem 5: The element.

$$\sigma_x = \sigma$$
$$\sigma_y = 0$$
$$\tau_{xy} = 0$$
$$\sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma + 0}{2} = \frac{\sigma}{2}$$

Hence, the center of the circle is at $(\sigma_{\text{ave}}, 0) \equiv (\sigma/2, 0)$. Also, consider two points X and Y with coordinates

$$X : (\sigma, 0)$$
$$Y : (0, 0)$$

The center of the Mohr's circle is at $(\sigma/2, 0)$ and the two points X and Y are located on the circle as shown in the figure. The radius of the circle is $R = \sigma/2$.

Hence, the principal stresses are

$$\sigma_{\max} = \sigma$$
$$\sigma_{\min} = 0$$

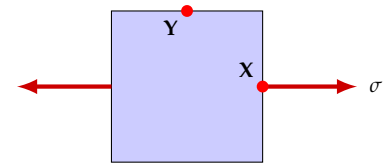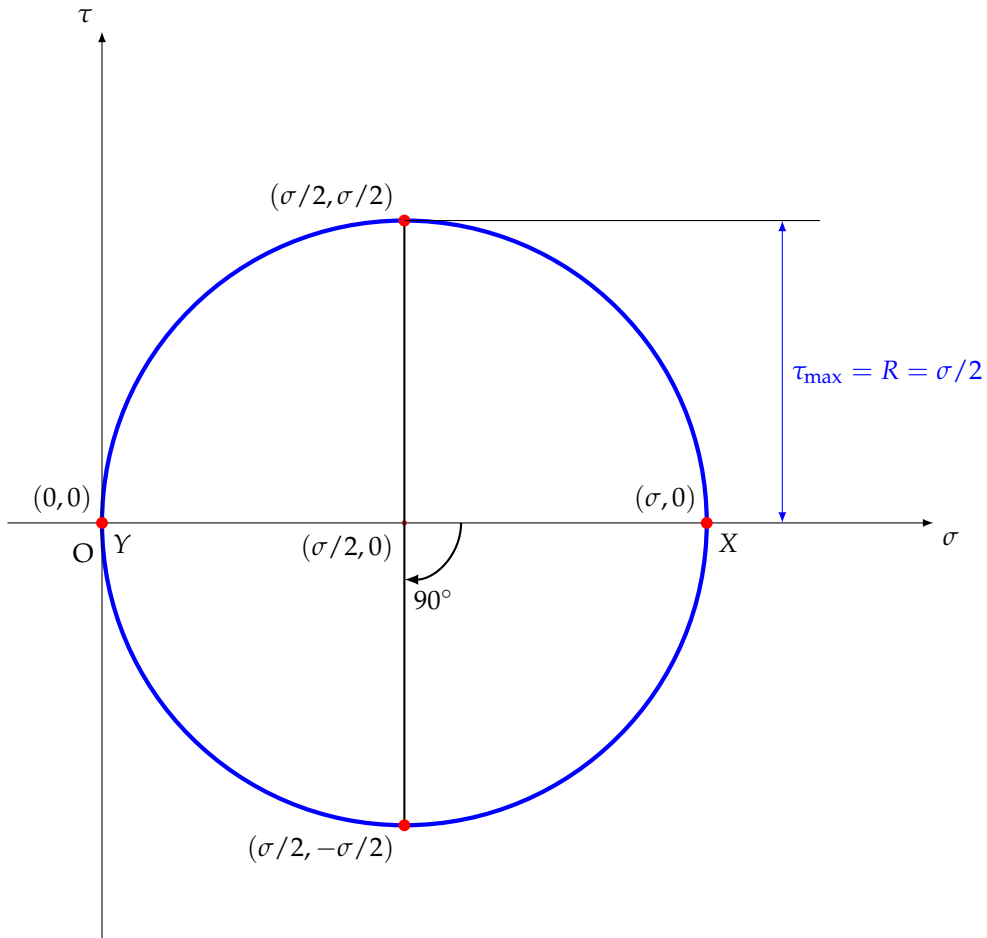The maximum in-plane shear stress, $\tau_{\max} = R = \sigma/2$.

Figure 98: Problem 5: Mohr's circle.

Problem 6.

Solve Problem 3 using Mohr's circle.
  Here, we have

Figure 99: Problem 3: The element.

$$\sigma_x = -15.1 \ MPa$$

$$\sigma_y = 0$$

$$\tau_{xy} = +7.55 \ MPa$$

$$\sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = \frac{-15.1 + 0}{2} \ MPa = -7.55 \ MPa$$

  Hence, the center of the circle is at $(\sigma_{\text{ave}}, 0) \equiv (-7.55, 0)$. Also, consider two points X and Y with coordinates

$$X : (-15.1, -7.55)$$

$$Y : (0, 7.55)$$
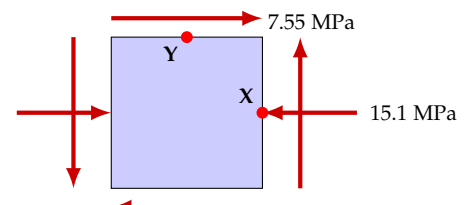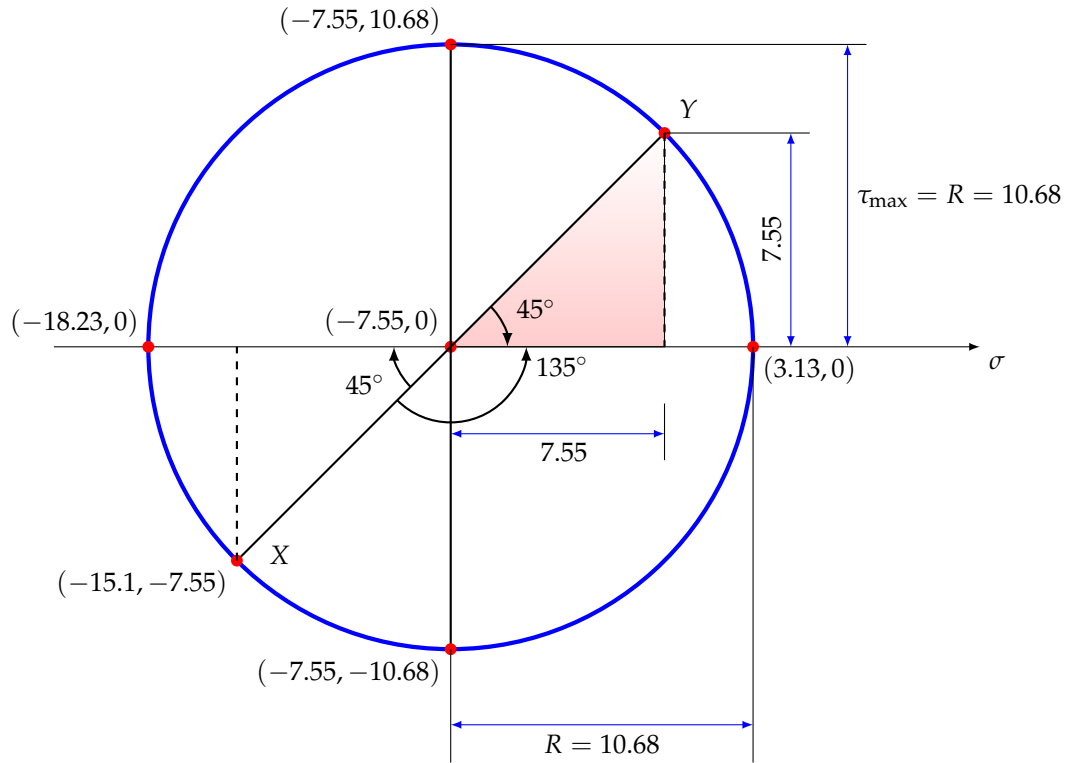
Figure 100: Problem 6: Mohr's circle.

Using the (red) shaded right-angled triangle, the radius of the Mohr's circle

$$R^2 = (7.55)^2 + (7.55)^2$$
$$\Rightarrow R = \sqrt{(7.55)^2 + (7.55)^2} = 10.68$$

Hence, the principal stresses are

$$\sigma_{\text{max}} = \sigma_{\text{ave}} + R = -7.55 + 10.68 = 3.13 \ MPa$$
$$\sigma_{\text{min}} = \sigma_{\text{ave}} - R = -7.55 - 10.68 = -18.23 \ MPa$$

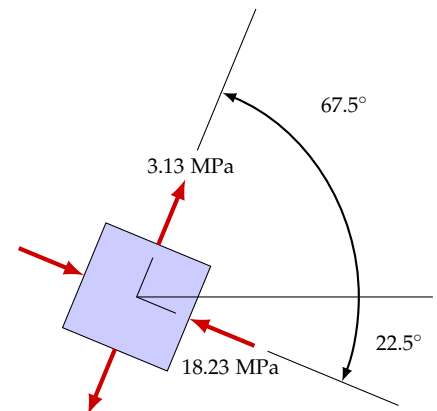The maximum in-plane shear stress, $\tau_{\text{max}} = R = 10.68$ MPa.

Figure 101: Problem 6: Principal stresses.

Problem 7.

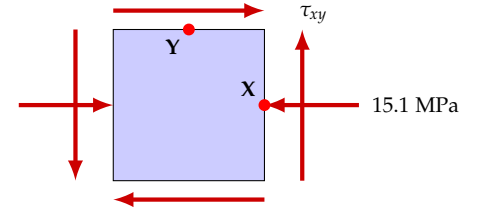Find maximum τ_{xy} such that $\tau_{\text{max}} \leq 15\ MPa$ for the element shown.

Figure 102: Problem 7.
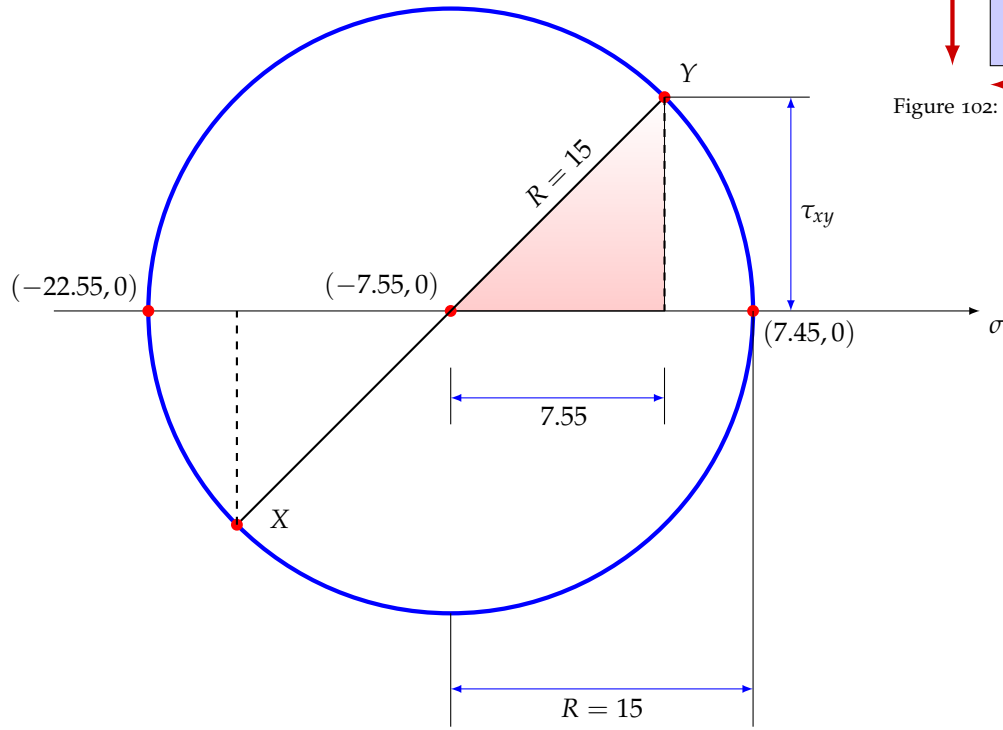

Figure 103: Problem 7: Mohr's circle.

Here, we have

$$\sigma_x = -15.1\ MPa$$

$$\sigma_y = 0$$

$$\tau_{xy}$$

$$\sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = \frac{-15.1 + 0}{2}\ MPa = -7.55\ MPa$$

Hence, the center of the circle is at $(\sigma_{\text{ave}}, 0) \equiv (-7.55, 0)$. Also, consider two points X and Y with coordinates

$$X : (-15.1, -\tau_{xy})$$

$$Y : (0, \tau_{xy})$$

Given $\tau_{\text{max}} = 15\ MPa = R$ (maximum possible).
Using the shaded right-angled triangle

$$R^2 = \tau_{xy}^2 + (7.55)^2$$

$$(15)^2 = \tau_{xy}^2 + (7.55)^2$$

$$\Rightarrow \tau_{xy} = 12.96\ MPa$$

Hence, maximum $\tau_{xy} = 12.96$ MPa will give $\tau_{\text{max}} = 15$ MPa.

Problem 8.

Find maximum σ such that $\tau_{\text{max}} \leq 15$ MPa for the element shown.
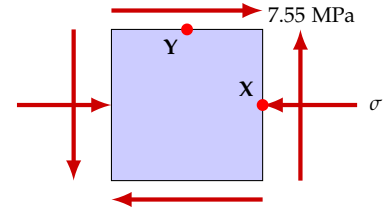

Figure 104: Problem 8.

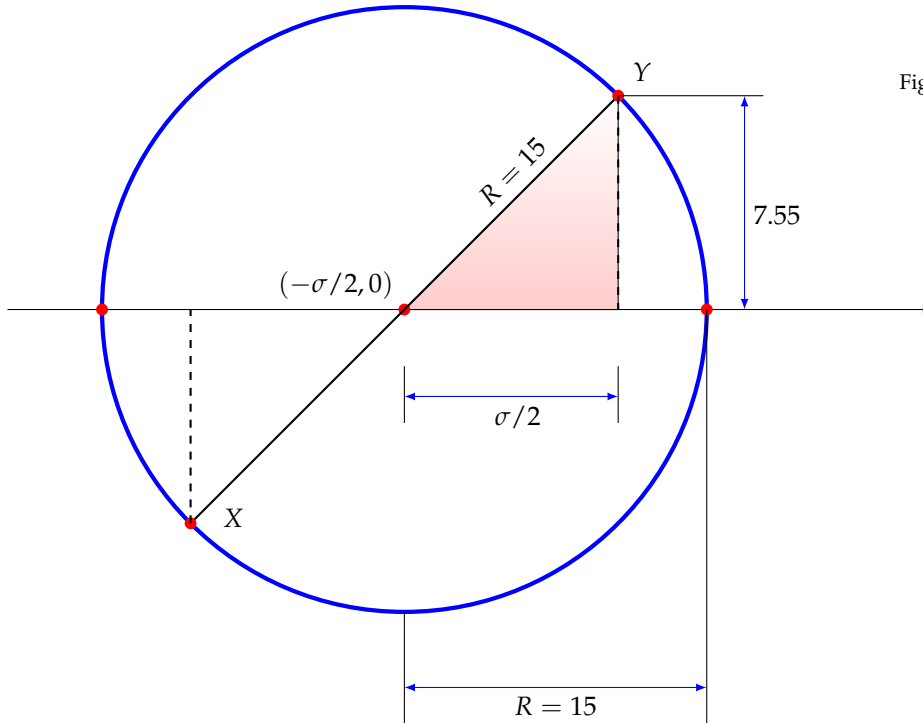Figure 105: Problem 8: Mohr's circle.

Here, we have

$$\sigma_x = -\sigma$$
$$\sigma_y = 0$$
$$\tau_{xy} = +7.55 \; MPa$$
$$\sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = \frac{-\sigma + 0}{2} = -\sigma/2$$

Hence, the center of the circle is at $(\sigma_{\text{ave}}, 0) \equiv (-\sigma/2, 0)$. Also, consider two points X and Y with coordinates

$$X : (-\sigma/2, -7.55)$$
$$Y : (0, -7.55)$$

Given $\tau_{\text{max}} = 15 \; MPa = R$ (maximum possible).
Using the shaded right-angled triangle

$$R^2 = (7.55)^2 + (\sigma/2)^2$$
$$(15)^2 = (7.55)^2 + (\sigma/2)^2$$
$$\Rightarrow \sigma/2 = 12.96 \; MPa$$
$$\Rightarrow \sigma = 25.92 \; MPa$$

Hence, maximum $\sigma = 25.92$ MPa will give $\tau_{\text{max}} = 15$ MPa.

Deflection of Beams

Equation of the Elastic Curve

The governing second order differential equation for the elastic curve of a beam deflection is

$$EI\frac{d^2y}{dx^2} = M$$

where EI is the flexural rigidity, M is the bending moment, and y is the deflection of the beam (+ve upwards).

Boundary Conditions

Fixed at $x = a$:

$$\text{Deflection is zero} \;\Rightarrow\; y\Big|_{x=a} = 0$$

$$\text{Slope is zero} \;\Rightarrow\; \frac{dy}{dx}\Big|_{x=a} = 0$$

Simply supported at $x = a$:

$$\text{Deflection is zero} \;\Rightarrow\; y\Big|_{x=a} = 0$$

A fourth order differential equation can also be written as

$$EI\frac{d^4y}{dx^4} = -w$$

where is w is the distributed load.

Here, two more boundary conditions are needed in terms of bending moment and shear force.

Boundary Conditions

Free at $x = a$:

$$\text{Bending moment is zero} \Rightarrow M = EI\frac{d^2y}{dx^2}\bigg|_{x=a} = 0$$

$$\text{Shear force is zero} \Rightarrow V = EI\frac{d^3y}{dx^3}\bigg|_{x=a} = 0$$
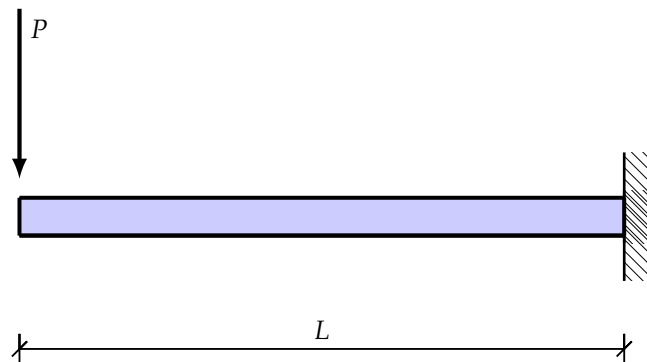
Simply supported at $x = a$:

$$\text{Bending moment is zero} \Rightarrow M = EI\frac{d^2y}{dx^2}\bigg|_{x=a} = 0$$

Notes on Integration

$$\int (ax + b)dx = \int ax\,dx + \int b\,dx + C_1$$

$$= \frac{ax^2}{2} + bx + C_1$$

$$\int \left(\frac{ax^2}{2} + bx + C_1\right)dx = \int \frac{ax^2}{2}dx + \int bx\,dx + \int C_1 dx + C_2$$

$$= \frac{ax^3}{6} + \frac{bx^2}{2} + C_1 x + C_2$$

Problem 1.

Calculate the tip deflection for the cantilever beam shown below.

Figure 106: Problem 1.

Bending moment

$$M = -Px$$

Hence,

$$EI\frac{d^2y}{dx^2} = M = -Px$$

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + C_1 \quad \text{[integrating with respect to } x]$$

$$EIy = -\frac{Px^3}{6} + C_1x + C_2 \quad \text{[integrating again with respect to } x]$$

Use boundary condition $dy/dx = 0$ and $y = 0$ at $x = L$.

$$\left.\frac{dy}{dx}\right|_{x=L} = 0$$

$$\Rightarrow \ C_1 = \frac{PL^2}{2}$$

$$\left.y\right|_{x=L} = 0$$

$$\Rightarrow \ -\frac{PL^3}{6} + C_1L + C_2 = 0$$

$$\Rightarrow \ C_2 = -\frac{PL^3}{3}$$

Hence, the equations of the deflection and slope becomes

$$y = \frac{1}{EI}\left(-\frac{Px^3}{6} + \frac{PL^2x}{2} - \frac{PL^3}{3}\right)$$

$$\frac{dy}{dx} = \frac{1}{EI}\left(-\frac{Px^2}{2} + \frac{PL^2}{2}\right)$$

The tip deflection and the rotation

$$\left.y\right|_{x=0} = -\frac{PL^3}{3EI}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = \frac{PL^2}{2EI}$$

Figure 107: Problem 1: Free-body diagram.

Problem 2.

Calculate the maximum deflection for the beam shown.

The support reactions are

$$A_y = B_y = P/2$$

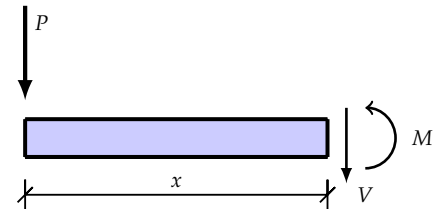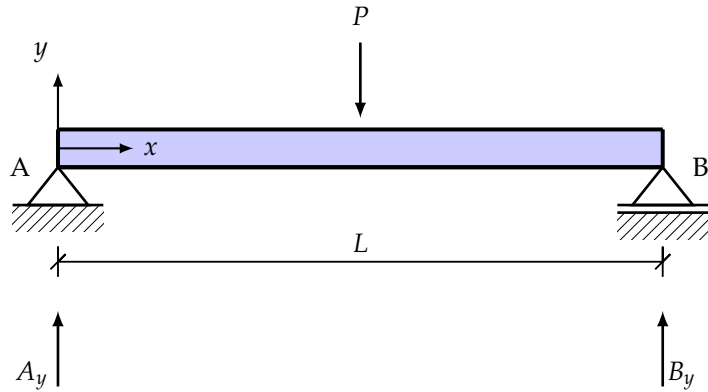$\underline{0 \le x \le L/2}$:

Bending moment

$$M = \frac{Px}{2}$$

Figure 108: Problem 2.

Hence,

$$EI\frac{d^2y}{dx^2} = M = \frac{Px}{2}$$

$$EI\frac{dy}{dx} = \frac{Px^2}{4} + C_1 \quad \text{[integrating with respect to } x\text{]}$$

$$EIy = \frac{Px^3}{12} + C_1x + C_2 \quad \text{[integrating again with respect to } x\text{]}$$

Use boundary condition $y = 0$ at $x = 0$.

$$C_2 = 0$$

Figure 109: Problem 2: For $0 \le x \le L/2$.

$\underline{L/2 \le x \le L}$:

Bending moment

$$M = \frac{P(L-x)}{2}$$

Hence,

$$EI\frac{d^2y}{dx^2} = M = \frac{P(L-x)}{2} = \frac{PL}{2} - \frac{Px}{2}$$

$$EI\frac{dy}{dx} = \frac{PLx}{2} - \frac{Px^2}{4} + C_3 \quad \text{[integrating with respect to } x\text{]}$$

$$EIy = \frac{PLx^2}{4} - \frac{Px^3}{12} + C_3x + C_4 \quad \text{[integrating again with respect to } x\text{]}$$

Figure 110: Problem 2: For $L/2 \le x \le L$.

Use boundary condition $y = 0$ at $x = L$.

$$0 = \frac{PL^3}{4} - \frac{PL^3}{12} + C_3L + C_4$$

$$C_3L + C_4 = -\frac{PL^3}{6}$$

Now, use compatibility condition that deflections and slopes from both these equations at $x = L/2$ should match.
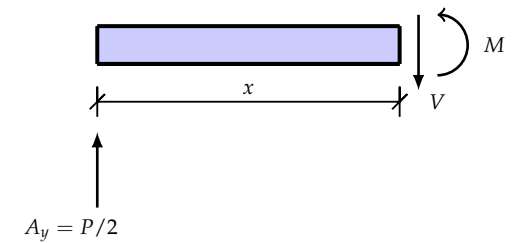
Or, due to the symmetry of the problem slope at $x = L/2$ should be zero, *i.e.*, $dy/dx = 0$ at $x = L/2$. From the equation for the first half of the beam

$$EI\frac{dy}{dx}\bigg|_{x=L/2} = \frac{PL^2}{16} + C_1 = 0$$

$$\Rightarrow\ C_1 = -\frac{PL^2}{16}$$

Similarly, from the equation for the second half of the beam

$$EI\frac{dy}{dx}\bigg|_{x=L/2} = \frac{PL^2}{4} - \frac{PL^2}{16} + C_3 = 0$$

$$\Rightarrow\ C_3 = -\frac{3PL^2}{16}$$

$$\Rightarrow\ C_4 = -\frac{PL^3}{6} - C_3 L = \frac{PL^3}{48}$$

Hence, the equations of the elastic curve

$$y = \begin{cases} \frac{1}{EI}\left(\frac{Px^3}{12} - \frac{PL^2 x}{16}\right) & \text{for } 0 \le x \le L/2 \\ \frac{1}{EI}\left(-\frac{Px^3}{12} + \frac{PLx^2}{4} - \frac{3PL^2 x}{16} + \frac{PL^3}{48}\right) & \text{for } L/2 \le x \le L \end{cases}$$

Hence, maximum deflection at the midspan

$$y\bigg|_{x=L/2} = \frac{PL^3}{96EI} - \frac{PL^3}{32EI} = -\frac{PL^3}{48EI} \qquad \text{[using the first equation]}$$

$$\Rightarrow \boxed{|y|_{\max} = \frac{PL^3}{48EI}}$$

Check: $y\bigg|_{x=L/2} = -\frac{PL^3}{96EI} + \frac{PL^3}{16EI} - \frac{3PL^3}{32EI} + \frac{PL^3}{48EI} = -\frac{PL^3}{48EI} \qquad \text{[using the second equation]}$

Slope at the left end

$$\frac{dy}{dx}\bigg|_{x=0} = -\frac{PL^2}{16EI}$$

Slope at the right end

$$\frac{dy}{dx}\bigg|_{x=L} = \frac{PL^2}{16EI}$$

Problem 3.
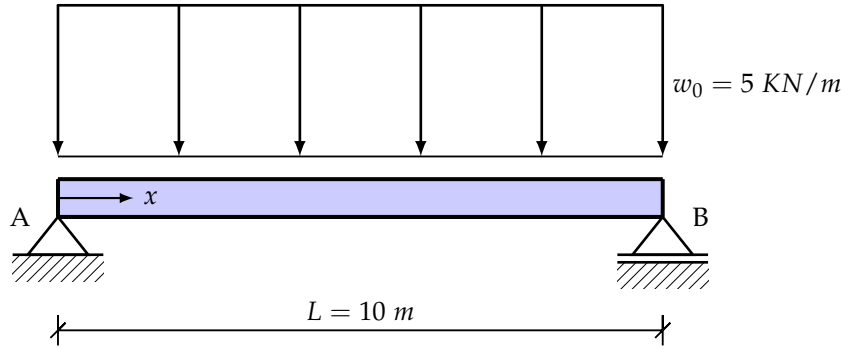
Calculate the maximum deflection for the beam shown.

We will convert all units to N and m. So, our y will be in m.

The vertical support reactions are $A_y = B_y = w_0 L / 2 = 25 \ kN$.

Bending moment at a distance of x from left end

$$M = -(5000x) \cdot \left(\frac{x}{2}\right) + 25000x$$

$$= -2500x^2 + 25000x$$

Hence,

$$EI\frac{d^2y}{dx^2} = M = -2500x^2 + 25000x$$

$$EI\frac{dy}{dx} = -\frac{2500x^3}{3} + 12500x^2 + C_1 \quad \text{[integrating with respect to } x\text{]}$$

$$EIy = -\frac{2500x^4}{12} + \frac{12500x^3}{3} + C_1 x + C_2 \quad \text{[integrating again]}$$

Use boundary conditions $y = 0$ at $x = 0$ and $x = L = 10 \ m$.

$$y\bigg|_{x=0} = 0$$

$$\Rightarrow \ C_2 = 0$$

$$y\bigg|_{x=10 \ m} = 0$$

$$\Rightarrow \ -\frac{2500 \cdot (10)^4}{12} + \frac{12500 \cdot (10)^3}{3} + C_1 \cdot (10) = 0$$

$$C_1 = -208.33 \times 10^3$$

Hence, the equations of the elastic curve and the slope of the curve

$$y = \frac{1}{EI}\left(-\frac{2500x^4}{12} + \frac{12500x^3}{3} - (208.33 \times 10^3)x\right)$$

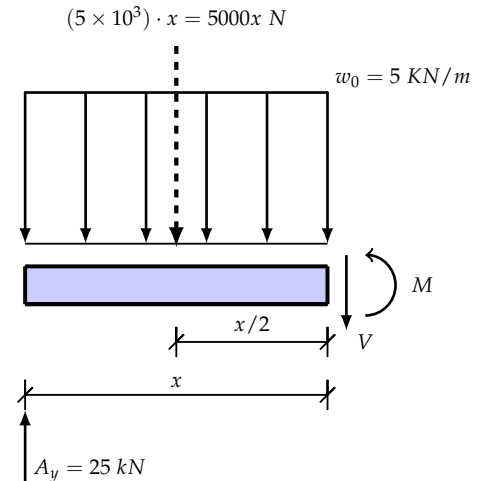$$\frac{dy}{dx} = \frac{1}{EI}\left(-\frac{2500x^3}{3} + 12500x^2 - 208.33 \times 10^3\right)$$

$(5 \times 10^3) \cdot x = 5000x \ N$

Figure 112: Problem 3: Free-body diagram.

Maximum deflection at the midspan

$$y\Big|_{x=5\ m} = -\frac{651.04 \times 10^3}{EI}$$

$$\Rightarrow \boxed{|y|_{\max} = \frac{651.04 \times 10^3}{EI}} = \frac{5wL^4}{384EI}$$

Problem 4.

Calculate the maximum deflection at the tip for the beam shown.
We will convert all units to N and m. So, our y will be in m.
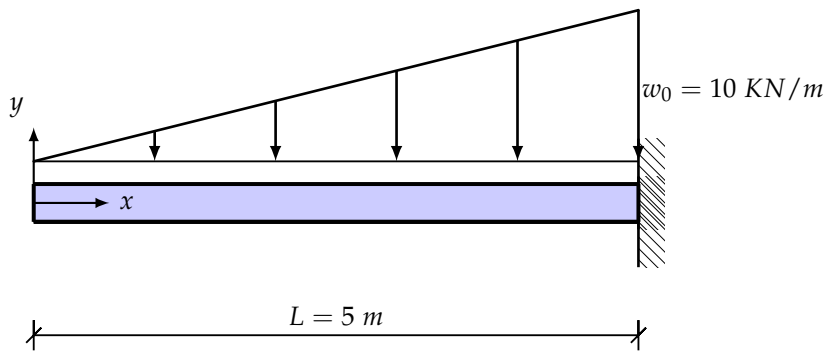
Bending moment

$$M = -1000x^2 \cdot \frac{x}{3} = -\frac{1000x^3}{3}$$

Hence,

$$EI\frac{d^2y}{dx^2} = M = -\frac{1000x^3}{3}$$

$$EI\frac{dy}{dx} = -\frac{250x^4}{3} + C_1 \quad \text{[integrating with respect to } x\text{]}$$

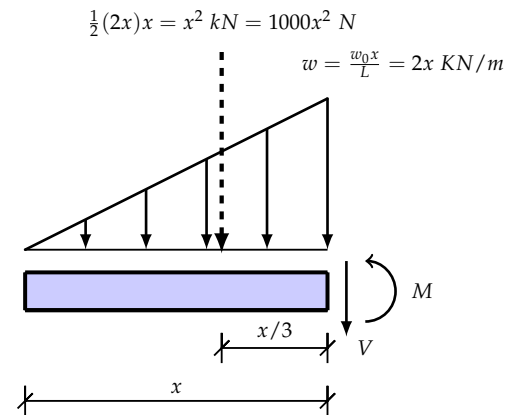$$EIy = -\frac{50x^5}{3} + C_1x + C_2 \quad \text{[integrating again with respect to } x\text{]}$$

$\frac{1}{2}(2x)x = x^2\ kN = 1000x^2\ N$

$w = \frac{w_0 x}{L} = 2x\ KN/m$

Figure 114: Problem 4: Free-body diagram.

Use boundary conditions $dy/dx = 0$ and $y = 0$ at $x = L = 5\ m$.

$$\left.\frac{dy}{dx}\right|_{x=5\ m} = 0$$

$$\Rightarrow \quad -\frac{250 \cdot (5)^4}{3} + C_1 = 0$$

$$\Rightarrow \quad C_1 = 52.083 \times 10^3$$

$$\left.y\right|_{x=5\ m} = 0$$

$$\Rightarrow \quad -\frac{50 \cdot (5)^5}{3} + C_1 \cdot (5) + C_2 = 0$$

$$C_2 = -208.33 \times 10^3$$

Hence, the equations of the elastic curve and the slope of the curve

$$y = \frac{1}{EI}\left(-\frac{50x^5}{3} + (52.083 \times 10^3)x - -208.33 \times 10^3\right)$$

$$\frac{dy}{dx} = \frac{1}{EI}\left(-\frac{250x^4}{3} + 52.083 \times 10^3\right)$$

Maximum deflection at the tip

$$\left.y\right|_{x=0} = -\frac{208.33 \times 10^3}{EI}$$

$$\Rightarrow \quad \boxed{|y|_{\max} = \frac{208.33 \times 10^3}{EI}} = \frac{w_0 L^4}{30EI}$$

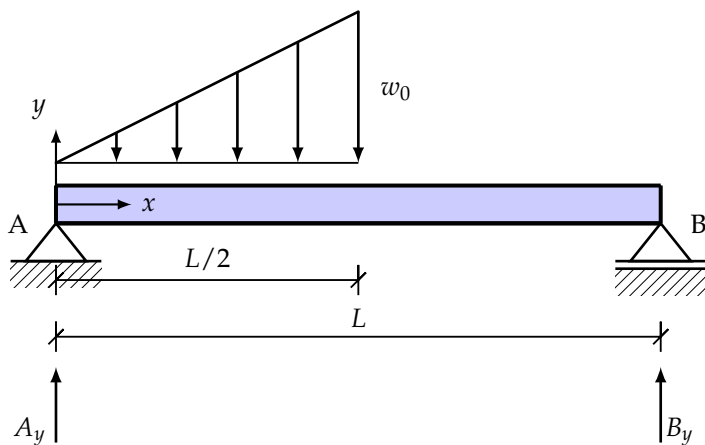## Problem 5.

Estimate the deflection curve for the beam shown.

Figure 115: Problem 5.

Using the equations for equilibrium

$$\sum F_y = 0$$

$$A_y + B_y = \frac{1}{2} \cdot w_0 \cdot \frac{L}{2} = \frac{w_0 L}{4}$$

$$\sum M_A = 0$$

$$B_y \cdot L = \frac{w_0 L}{4} \cdot \frac{L}{3} = \frac{w_0 L^2}{12}$$

$$\Rightarrow \; B_y = \frac{w_0 L}{12}$$

$$\Rightarrow \; A_y = \frac{w_0 L}{4} - \frac{w_0 L}{12} = \frac{w_0 L}{6}$$

<u>$0 \leq x \leq L/2$:</u>
Bending moment

$$M = -\frac{w_0 x^2}{L} \cdot \frac{x}{3} + \frac{w_0 L x}{6}$$

$$= \frac{w_0 L x}{6} - \frac{w_0 x^3}{3L}$$

Hence,

$$EI\frac{d^2 y}{dx^2} = M = \frac{w_0 L x}{6} - \frac{w_0 x^3}{3L}$$

$$EI\frac{dy}{dx} = \frac{w_0 L x^2}{12} - \frac{w_0 x^4}{12L} + C_1 \quad \text{[integrating with respect to } x\text{]}$$

$$EIy = \frac{w_0 L x^3}{36} - \frac{w_0 x^5}{60L} + C_1 x + C_2 \quad \text{[integrating again]}$$

$\frac{1}{2}\left(\frac{2 w_0 x}{L}\right) x = \frac{w_0 x^2}{L}$

$w = \frac{2 w_0 x}{L}$

M

$x/3$

V

x

$A_y = \frac{w_0 L}{6}$

Figure 116: Problem 5: For $0 \leq x \leq L/2$.

<u>$L/2 \leq x \leq L$:</u>
Bending moment

$$M = -\frac{w_0 L}{12}(L - x)$$

$$= \frac{w_0 L^2}{12} - \frac{w_0 L x}{12}$$

Hence,

$$EI\frac{d^2 y}{dx^2} = M = \frac{w_0 L^2}{12} - \frac{w_0 L x}{12}$$

$$EI\frac{dy}{dx} = \frac{w_0 L^2 x}{12} - \frac{w_0 L x^2}{24} + C_3 \quad \text{[integrating with respect to } x\text{]}$$

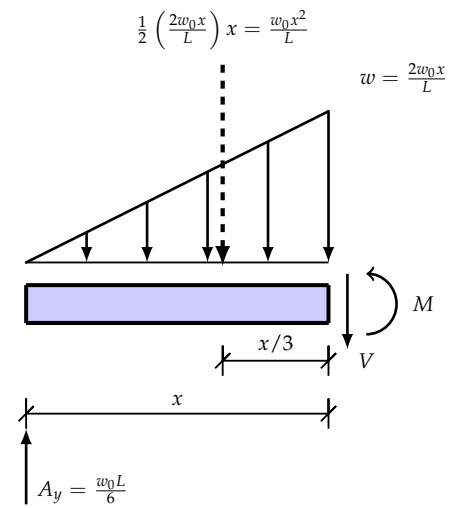$$EIy = \frac{w_0 L^2 x^2}{24} - \frac{w_0 L x^3}{72} + C_3 x + C_4 \quad \text{[integrating again]}$$

V

M

$L - x$

$B_y = P/2$
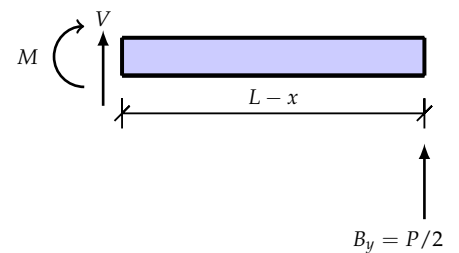
Figure 117: Problem 5: For $L/2 \leq x \leq L$.

Use boundary conditions $y = 0$ at $x = 0$ and $x = L$.

$$y\bigg|_{x=0} = 0$$

$$\Rightarrow C_2 = 0$$

$$y\bigg|_{x=L} = 0$$

$$\Rightarrow C_3 L + C_4 = -\frac{w_0 L^4}{36}$$

Next, use the compatibility condition that at $x = L/2$ deflection and slope from both of these expressions should match.

$$EI\left(y\bigg|_{x=L/2}\right)_{expression\ 1} = EI\left(y\bigg|_{x=L/2}\right)_{expression\ 2}$$

$$\Rightarrow \frac{w_0 L}{36} \cdot \frac{L^3}{8} - \frac{w_0}{60L} \cdot \frac{L^5}{32} + C_1 \cdot \frac{L}{2} = \frac{w_0 L^2}{24} \cdot \frac{L^2}{4} - \frac{w_0 L}{72} \cdot \frac{L^3}{8} + C_3 \cdot \frac{L}{2} + C_4$$

$$\Rightarrow \frac{17 w_0 L^4}{5760} + \frac{C_1 L}{2} = -\frac{11 w_0 L^4}{576} - \frac{C_3 L}{2} \qquad [C_4 = -\frac{w_0 L^4}{36} - C_3 L]$$

$$\Rightarrow \frac{C_1 L}{2} = -\frac{127 w_0 L^4}{5760} - \frac{C_3 L}{2}$$

$$\Rightarrow C_1 + C_3 = -\frac{127 w_0 L^3}{2880}$$

$$EI\left(\frac{dy}{dx}\bigg|_{x=L/2}\right)_{expression\ 1} = EI\left(\frac{dy}{dx}\bigg|_{x=L/2}\right)_{expression\ 2}$$

$$\Rightarrow \frac{w_0 L}{12} \cdot \frac{L^2}{4} - \frac{w_0}{12L} \cdot \left(\frac{L^4}{16}\right) + C_1 = \frac{w_0 L^2}{12} \cdot \frac{L}{2} - \frac{w_0 L}{24} \cdot \frac{L^2}{4} + C_3$$

$$\Rightarrow \frac{w_0 L^3}{64} + C_1 = \frac{w_0 L^3}{32} + C_3$$

$$\Rightarrow C_1 - C_3 = \frac{w_0 L^3}{64}$$

Solving for C_1 and C_3 gives

$$C_1 = -\frac{41 w_0 L^3}{2880}$$

$$C_3 = -\frac{43 w_0 L^3}{1440}$$

$$\Rightarrow C_4 = \frac{w_0 L^4}{480}$$

Hence, the equations for the elastic curve

$$y = \begin{cases} \frac{1}{EI}\left(\frac{w_0 L x^3}{36} - \frac{w_0 x^5}{60L} - \frac{41 w_0 L^3 x}{2880}\right) & \text{for } 0 \le x \le L/2 \\ \frac{1}{EI}\left(\frac{w_0 L^2 x^2}{24} - \frac{w_0 L x^3}{72} - \frac{43 w_0 L^3 x}{1440} + \frac{w_0 L^4}{480}\right) & \text{for } L/2 \le x \le L \end{cases}$$

Method of Superposition

Method of superposition can be used if you have two or more loads acting on the beam.

Problem 6.

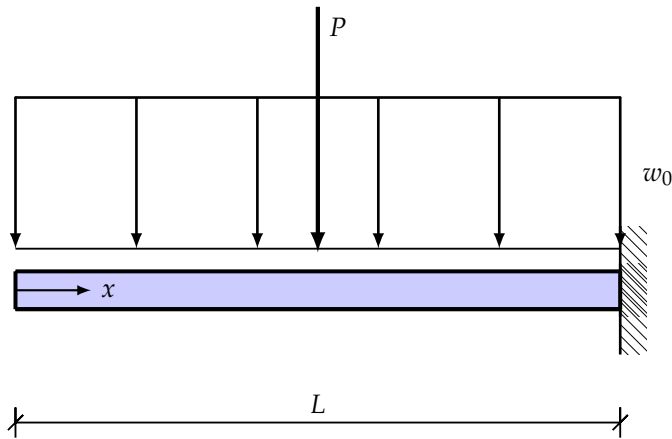Estimate the deflection of the beam as shown.

Using method of superposition combine results from the following two cases:
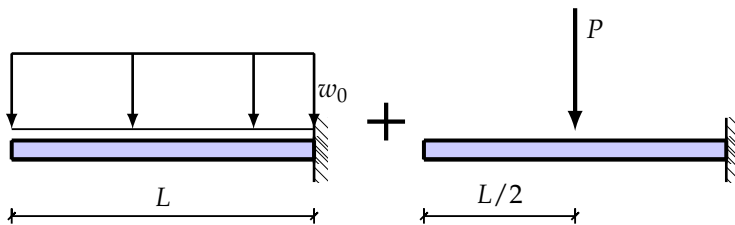

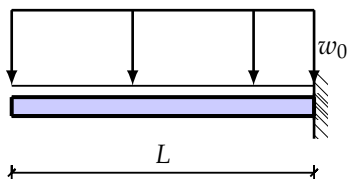Figure 119: Problem 6: Method of superposition.

For the first case:

Figure 120: Problem 6: Case I

We will use the fourth order governing differential equation.

$$EI\frac{d^4y}{dx^4} = -w = -w_0$$

$$\Rightarrow EI\frac{d^3y}{dx^3} = -w_0x + C_1$$

$$\Rightarrow EI\frac{d^2y}{dx^2} = -\frac{w_0x^2}{2} + C_1x + C_2$$

$$\Rightarrow EI\frac{dy}{dx} = -\frac{w_0x^3}{6} + \frac{C_1x^2}{2} + C_2x + C_3$$

$$\Rightarrow EIy = -\frac{w_0x^4}{24} + \frac{C_1x^3}{6} + \frac{C_2x^2}{2} + C_3x + C_4$$

Use boundary conditions $dy/dx = 0$ and $y = 0$ at $x = L$ and bending moment $M = EI\frac{d^2y}{dx^2} = 0$ at $x = 0$ and shear force $V = EI\frac{d^3y}{dx^3} = 0$ at $x = 0$.

$$V = EI\frac{d^3y}{dx^3}\bigg|_{x=0} = 0$$

$$\Rightarrow C_1 = 0$$

$$M = EI\frac{d^2y}{dx^2}\bigg|_{x=0} = 0$$

$$\Rightarrow C_2 = 0$$

$$\frac{dy}{dx}\bigg|_{x=L} = 0$$

$$\Rightarrow C_3 = \frac{w_0L^3}{6}$$

$$y\bigg|_{x=L} = 0$$

$$\Rightarrow -\frac{w_0L^4}{24} + C_3L + C_4 = 0$$

$$\Rightarrow C_4 = \frac{w_0L^4}{24} - \frac{w_0L^4}{6} = -\frac{w_0L^4}{8}$$

Hence, for the first case

$$y = \frac{1}{EI}\left(-\frac{w_0x^4}{24} + \frac{w_0L^3x}{6} - \frac{w_0L^4}{8}\right)$$

$$\frac{dy}{dx} = \frac{1}{EI}\left(-\frac{w_0x^3}{6} + \frac{w_0L^3}{6}\right)$$
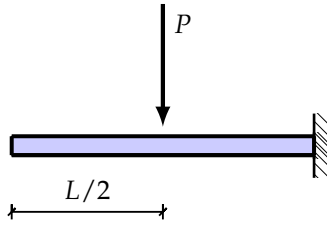
For the second case:

Figure 121: Problem 6: Case II

$\underline{0 \le x \le L/2}$:
Bending moment $M = 0$. Hence,

$$EI\frac{d^2y}{dx^2} = M = 0$$

$$EI\frac{dy}{dx} = C_1 \quad \text{[integrating with respect to } x\text{]}$$

$$EIy = C_1 x + C_2 \quad \text{[integrating again]}$$

$\underline{L/2 \le x \le L}$:
Bending moment

$$M = -P(x - L/2)$$
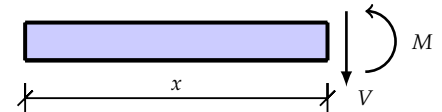

Figure 122: Problem 6, Case II: For $0 \le x \le L/2$.

Hence,

$$EI\frac{d^2y}{dx^2} = M = -Px + \frac{PL}{2}$$

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + \frac{PLx}{2} + C_3 \quad \text{[integrating with respect to } x\text{]}$$

$$EIy = -\frac{Px^3}{6} + \frac{PLx^2}{4} + C_3 x + C_4 \quad \text{[integrating again]}$$

Use boundary conditions $dy/dx = 0$ and $y = 0$ at $x = L$.

Figure 123: Problem 6, Case II: For $L/2 \le x \le L$.

$$\left.\frac{dy}{dx}\right|_{x=L} = 0$$

$$\Rightarrow \quad -\frac{PL^2}{2} + \frac{PL^2}{2} + C_3 = 0$$

$$\Rightarrow \quad C_3 = 0$$

$$\left.y\right|_{x=L} = 0$$

$$\Rightarrow \quad -\frac{PL^3}{6} + \frac{PL^3}{4} + C_4 = 0$$
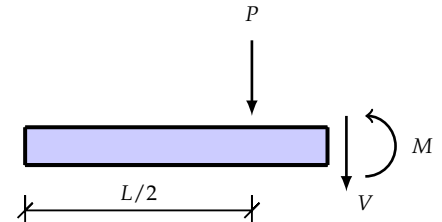
$$\Rightarrow \quad C_4 = -\frac{PL^3}{12}$$

Next, use the compatibility condition that at $x = L/2$ the slope and

the deflection should match.

$$EI \left(\frac{dy}{dx}\bigg|_{x=L/2} \right)_{expression\ 1} = EI \left(\frac{dy}{dx}\bigg|_{x=L/2} \right)_{expression\ 2}$$

$$\Rightarrow C_1 = -\frac{PL^2}{8} + \frac{PL^2}{4} = \frac{PL^2}{8}$$

$$EI \left(y\bigg|_{x=L/2} \right)_{expression\ 1} = EI \left(y\bigg|_{x=L/2} \right)_{expression\ 2}$$

$$\Rightarrow \frac{C_1 L}{2} + C_2 = -\frac{PL^3}{48} + \frac{PL^3}{16} + C_4$$

$$\Rightarrow C_2 = -\frac{5PL^3}{48}$$

Hence, for the second case, the equations for the elastic curve

$$y = \begin{cases} \frac{1}{EI}\left(\frac{PL^2 x}{8} - \frac{5PL^3}{48} \right) & \text{for } 0 \le x \le L/2 \\ \frac{1}{EI}\left(-\frac{Px^3}{6} + \frac{PLx^2}{4} - \frac{PL^3}{12} \right) & \text{for } L/2 \le x \le L \end{cases}$$

Combining case I and II, the elastic curves for the original beam

$$y = \begin{cases} \frac{1}{EI}\left(-\frac{w_0 x^4}{24} + \frac{w_0 L^3 x}{6} - \frac{w_0 L^4}{8} + \frac{PL^2 x}{8} - \frac{5PL^3}{48} \right) & \text{for } 0 \le x \le L/2 \\ \frac{1}{EI}\left(-\frac{w_0 x^4}{24} + \frac{w_0 L^3 x}{6} - \frac{w_0 L^4}{8} - \frac{Px^3}{6} + \frac{PLx^2}{4} - \frac{PL^3}{12} \right) & \text{for } L/2 \le x \le L \end{cases}$$

The tip deflection and rotation

$$y\bigg|_{x=0} = -\frac{w_0 L^4}{8EI} - \frac{5PL^3}{48}$$

$$\frac{dy}{dx}\bigg|_{x=0} = \frac{w_0 L^3}{6EI} + \frac{PL^2}{8EI}$$

Statically Indeterminate Beams

Clever use of superposition can be utilized here.

Problem 7.

Calculate the support reaction A_y here.

  Method I:

  Let us use the method of superposition and divide the problem into the following two cases.

  From Problem 5, the deflection for the first case

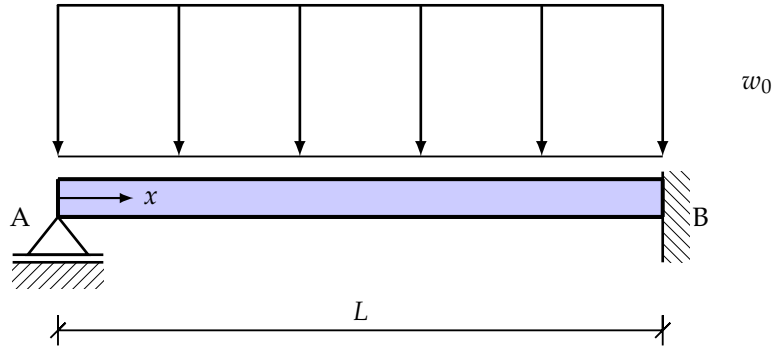$$y = \frac{1}{EI}\left(-\frac{w_0 x^4}{24} + \frac{w_0 L^3 x}{6} - \frac{w_0 L^4}{8} \right)$$
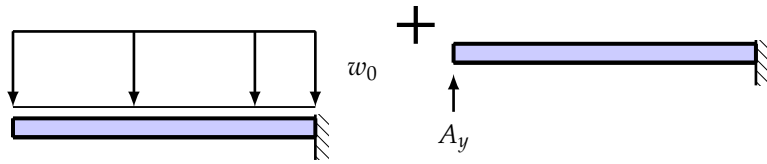
Figure 124: Problem 7.

Figure 125: Problem 7: Method of superposition.

Hence, the tip deflection in this case

$$y\bigg|_{x=0} = -\frac{w_0 L^4}{8EI}$$

For the second case, use the result from Problem 1. The tip deflection in this case

$$y\bigg|_{x=0} = \frac{A_y L^3}{3EI}$$

However, due to the presence of the roller support at A, the deflection at A should be zero. This leads to

$$\frac{A_y L^3}{3EI} - \frac{w_0 L^4}{8EI} = 0$$

$$A_y = \frac{3w_0 L}{8}$$

Method II: Assume the unknown reaction at A as A_y and calculate the bending moment as follows:

Bending moment at a distance of x from left end

$$M = -(w_0 x)\cdot\left(\frac{x}{2}\right) + A_y x$$

$$= -\frac{w_0 x^2}{2} + A_y x$$

Figure 126: Problem 7: Free-body diagram.

Hence,

$$EI\frac{d^2 y}{dx^2} = M = -\frac{w_0 x^2}{2} + A_y x$$

$$EI\frac{dy}{dx} = -\frac{w_0 x^3}{6} + \frac{A_y x^2}{2} + C_1 \quad \text{[integrating with respect to } x\text{]}$$

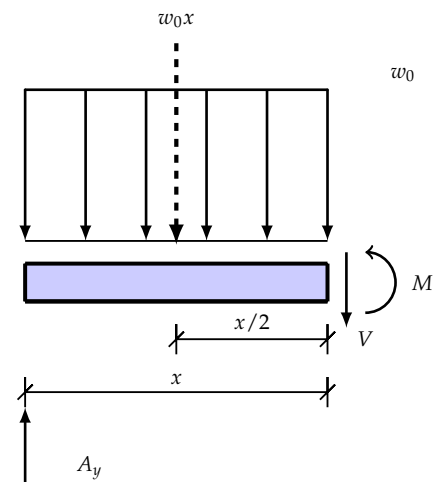$$EIy = -\frac{w_0 x^4}{24} + \frac{A_y x^3}{6} + C_1 x + C_2 \quad \text{[integrating again]}$$

Use boundary conditions $y = 0$ at $x = 0$ and $dy/dx = 0, y = 0$ at $x = L$.

$$y\bigg|_{x=0} = 0$$

$$C_2 = 0$$

$$\frac{dy}{dx}\bigg|_{x=L} = 0$$

$$\Rightarrow \quad -\frac{w_0 L^3}{6} + \frac{A_y L^2}{2} + C_1 = 0$$

$$\Rightarrow \quad C_1 = \frac{w_0 L^3}{6} - \frac{A_y L^2}{2}$$

$$y\bigg|_{x=L} = 0$$

$$\Rightarrow \quad -\frac{w_0 L^4}{24} + \frac{A_y L^3}{6} + C_1 L + C_2 = 0$$

$$\Rightarrow \quad -\frac{w_0 L^4}{24} + \frac{A_y L^3}{6} + \left(\frac{w_0 L^4}{6} - \frac{A_y L^3}{2}\right) = 0$$

$$\Rightarrow \quad A_y = \frac{3 w_0 L}{8}$$

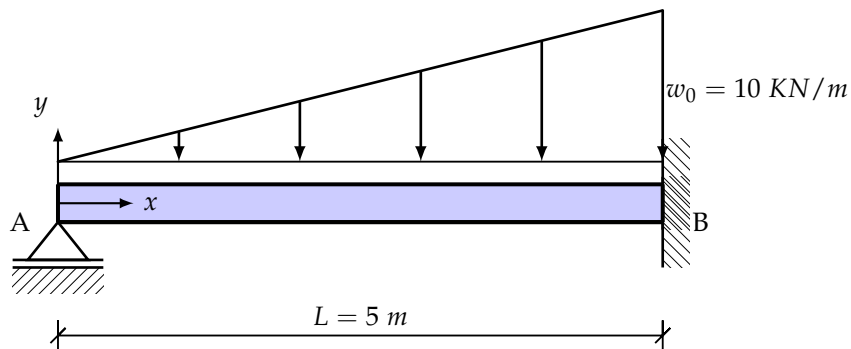## *Problem 8.*

Calculate the support reaction at A.

Figure 127: Problem 8.

Using method of superposition this problem can be divided into two cases.

From Problem 4, the tip deflection for case I

$$y\bigg|_{x=0} = -\frac{w_0 L^4}{30 EI} = -\frac{208.33 \times 10^3}{EI}$$
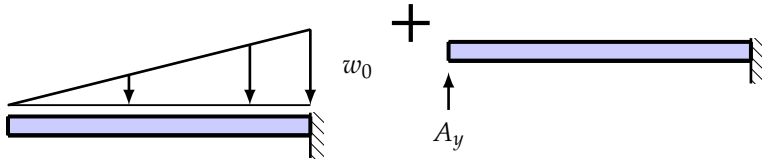
From Problem 1, the tip deflection for case II

$$y\Big|_{x=0} = \frac{A_y L^3}{3EI} = \frac{41.67 A_y}{EI}$$

However, due to the presence of the roller support at A, the deflection at A should be zero. This leads to

$$\frac{41.67 A_y}{EI} - \frac{208.33 \times 10^3}{EI} = 0$$
$$A_y = 5000 \ N = 5 \ kN$$

Moment-Area Method

First moment-area theorem

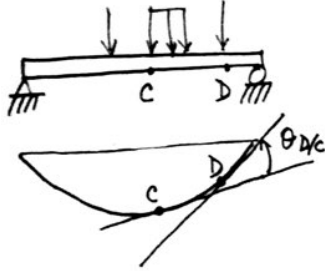$$\theta_{D/C} = \text{area under } M/EI \text{ diagram between } C \text{ and } D$$

$$\theta_D - \theta_C = \int_{x_C}^{x_D} \frac{M}{EI} dx$$
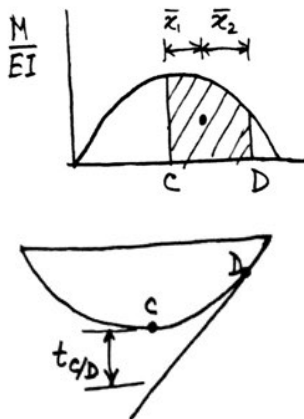
Second moment-area theorem

Tangential deviation $t_{C/D}$ of C with respect to D

$$t_{C/D} = (\text{area between } C \text{ and } D)\bar{x}_1$$

Tangential deviation $t_{D/C}$ of D with respect to C

$$t_{D/C} = (\text{area between } C \text{ and } D)\bar{x}_2$$

Problem 9.

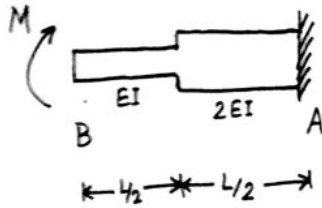Find the tip deflection and rotation of the beam.

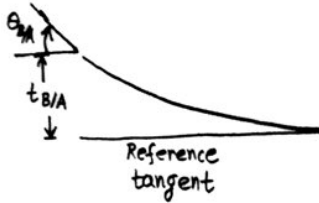Figure 131: Problem 9.

Step I: Draw the reference tangent

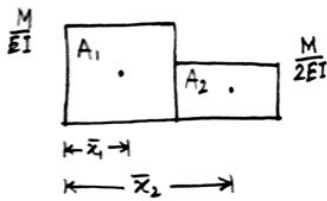Figure 132: Problem 9: Reference tangent.

Step II: Draw the M/EI diagram

Figure 133: Problem 9: M/EI diagram.

Step III: Use the first theorem

$$\theta_{B/A} = A_1 + A_2$$
$$\Rightarrow \theta_B - 0 = \left(\frac{M}{EI}\right) \cdot \left(\frac{L}{2}\right) + \left(\frac{M}{2EI}\right) \cdot \left(\frac{L}{2}\right)$$
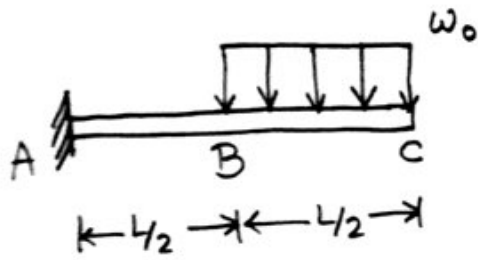$$\Rightarrow \theta_B = \frac{3ML}{4EI}$$

Step IV: Use the second theorem

$$t_{B/A} = A_1 \bar{x}_1 + A_2 \bar{x}_2$$
$$= \left(\frac{ML}{2EI}\right) \cdot \left(\frac{L}{4}\right) + \left(\frac{ML}{4EI}\right) \cdot \left(\frac{3L}{4}\right)$$
$$= \frac{ML^2}{8EI} + \frac{3ML^2}{16EI}$$
$$\Rightarrow \delta_B = \frac{5ML^2}{16EI}$$

Problem 10.

Find the tip deflection and rotation using moment-area theorems.

Figure 134: Problem 10.

Step I: Draw the reference tangent

Figure 135: Problem 10: Reference tangent.

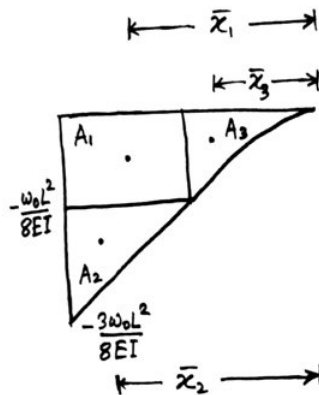## Step II: Draw the M/EI diagram

Figure 136: Problem 10: M/EI diagram.

Step III: Use the first theorem
To use the first theorem first divide the M/EI digram into 3 ele-

mentary areas A_1, A_2, and A_3.

$$A_1 = \left(\frac{-w_0 L^2}{8EI}\right) \cdot \left(\frac{L}{2}\right)$$

$$= -\frac{w_0 L^3}{16EI}$$

$$A_2 = \frac{1}{2} \cdot \left(\frac{-2w_0 L^2}{8EI}\right) \cdot \left(\frac{L}{2}\right)$$

$$= -\frac{w_0 L^3}{16EI}$$

$$A_3 = \frac{1}{3} \cdot \left(\frac{-w_0 L^2}{8EI}\right) \cdot \left(\frac{L}{2}\right)$$

$$= -\frac{w_0 L^3}{48EI}$$

Next, using the first theorem

$$\theta_{C/A} = A_1 + A_2 + A_3$$

$$\Rightarrow \theta_C - 0 = -\frac{7w_0 L^3}{48EI}$$

$$\Rightarrow \theta_C = \frac{7w_0 L^3}{48EI} \; (Clockwise)$$

Step IV: Use the second theorem

First, calculate the distances $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$.

$$\bar{x}_1 = \frac{L}{2} + \frac{L}{4}$$

$$= \frac{3L}{4}$$

$$\bar{x}_2 = \frac{L}{2} + \frac{2}{3} \cdot \frac{L}{2}$$

$$= \frac{5L}{6}$$

$$\bar{x}_3 = \frac{L}{2} - \frac{L}{8}$$

$$= \frac{3L}{8}$$

Next, using the second theorem

$$t_{C/A} = A_1 \bar{x}_1 + A_2 \bar{x}_2 + A_3 \bar{x}_3$$

$$= \left(\frac{-w_0 L^3}{16EI}\right) \cdot \left(\frac{5L}{6}\right) + \left(\frac{-w_0 L^3}{16EI}\right) \cdot \left(\frac{3L}{4}\right) + \left(\frac{-w_0 L^3}{48EI}\right) \cdot \left(\frac{3L}{8}\right)$$

$$= -\frac{41 w_0 L^3}{384 EI}$$