Graph Theory

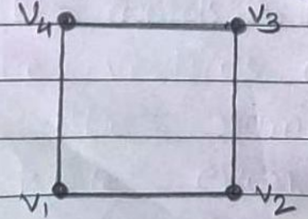
- Graphs (G) are discrete structures consisting of vertices (V) and edges (E) that can connect these vertices. Depending on the type and number of edges that can connect a pair of vertices, there are many kind of different graphs.

* Definition:

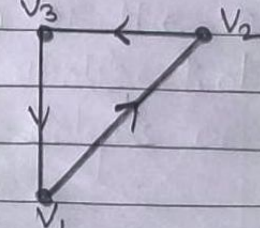
- A graph $G = (V, E)$ consists of a non-empty set V called the set of vertices (nodes, points) and set E called the set of edges, such that there is a mapping from the set E to the set V.

* If in graph $G = (V, E)$, each edge $e \in E$ is associated with an ordered pair of vertices, then G is called directed graph or digraph.

- If each edge is associated with an unordered pair of vertices, then G is called an undirected graph.



Undirected Graph

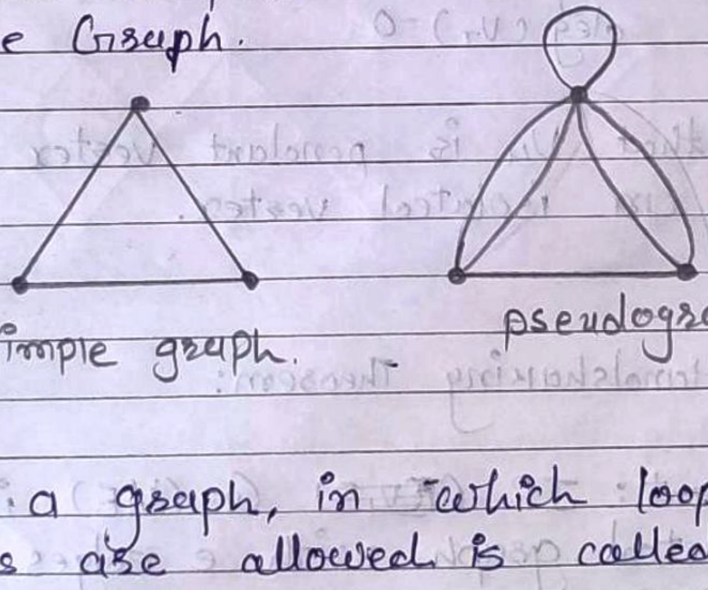


Directed Graph

- An edge of a graph that joins a vertex to itself is called loop. The direction of loop is not significant.

- If in directed or undirected graph, certain pairs of vertices are joined by more than one edge, such edges are called parallel edges.

- A graph in which there is only one edge between a pair of vertices is called a simple Graph.

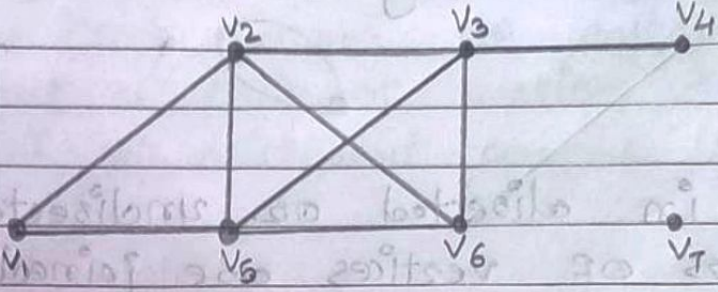


simple graph. pseudograph

- In a graph, in which loops and parallel edges are allowed is called a pseudograph

* Degree of Vertex:
- The degree of vertex in an undirected graph is the number of edge incident with it. denoted by deg(v)
- clearly, degree of an isolated vertex is zero. If the degree of a vertex is one, it is called a pendant vertex.

Example: Consider the Graph and find degree of vertices.



Here, $\deg(v_1) = 2$
$\deg(v_2) = \deg(v_3) = \deg(v_5) = 4$
$\deg(v_4) = 1$
$\deg(v_6) = 3$
$\deg(v_7) = 0$

- Note that v_4 is pendant vertex and v_7 is an isolated vertex.

★ The Handshaking Theorem:

Statement: If ~~(V,E)~~ $G = (V, E)$ is an undirected graph with e edges, then $\sum_i \deg(v_i) = 2e$.

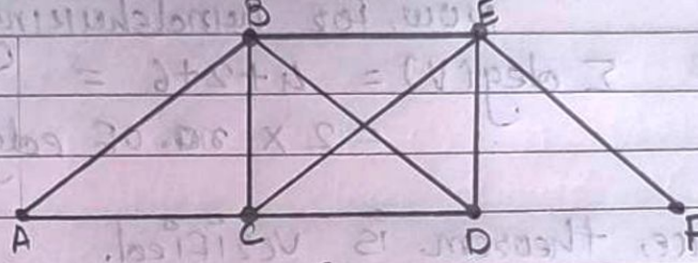
In other words: The sum of the degrees of all the vertices of an undirected graph is twice the number of edges of the graph and hence even.

Date :

Page No. :

Example: Find the number of vertices, the number of edges and the degree of each vertex in the following undirected graphs. Also verify the handshaking theorem in each case.

(i)



G_1

For graph G_1,

the number of vertices = 6.

the number of edges = 9.

$deg(A) = 2$, $deg(B) = 4$, $deg(C) = 4$

$deg(D) = 3$, $deg(E) = 4$, $deg(F) = 1$.

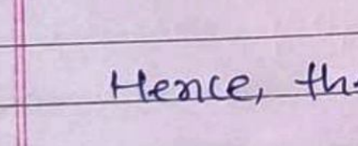
Now, Verifying handshaking theorem.

$$\sum deg(V) = 2+4+4+3+4+1 = 18$$

$$= 2 \times \text{number of edges} = 18.$$

Hence, theorem is true.

(ii)



G_2

For graph G_2,

the number of vertices = 4

the number of edges = 8

$deg(A) = 6$, $deg(B) = 5$.

$deg(C) = 2$, $deg(D) = 3$

Now, handshaking theorem,

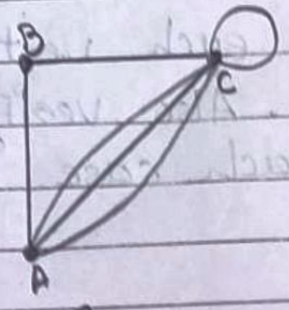
$$\sum deg(V) = 6+5+2+3 = 16$$

$$= 2 \times \text{no. of edges} = 16.$$

Hence, theorem is verified.

Date :
Page No. :

(iii) For Graph G_3,



G_3

the number of vertices = 3
the number of edges = 6
$\deg(A) = 4$, $\deg(B) = 2$, $\deg(C) = 6$

Now, for handshaking theorem.

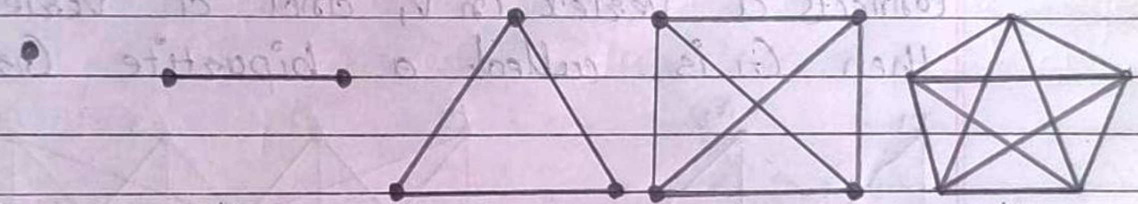
$$\Sigma \deg(V) = 4+2+6 = 12$$
$$= 2 \times \text{no. of edges} = 12$$

Hence, theorem is verified.

* Some Simple Graphs:

*** Complete Graph:** A simple graph in which there is exactly one edge between each pair of distinct vertices, is called a complete graph.

- The complete graph on n vertices is denoted by K_n. Figure below shows the graph K_1 through K_5.



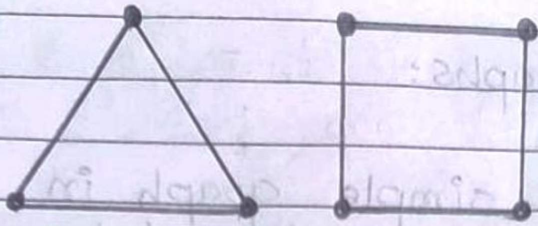
K_1 K_2 K_3 K_4 K_5.

- The number of edges in K_n is ${}_nC_2$ or $\frac{n(n-1)}{2}$

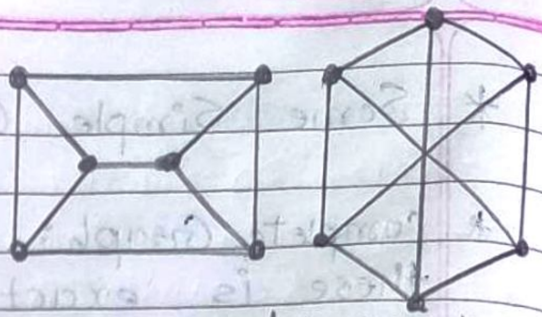
Hence the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

*** Regular Graph:** If every vertex of simple graph has the same degree, then the graph is called a Regular Graph.

- If every vertex in a regular graph has degree n, then the graph is called n-regular.

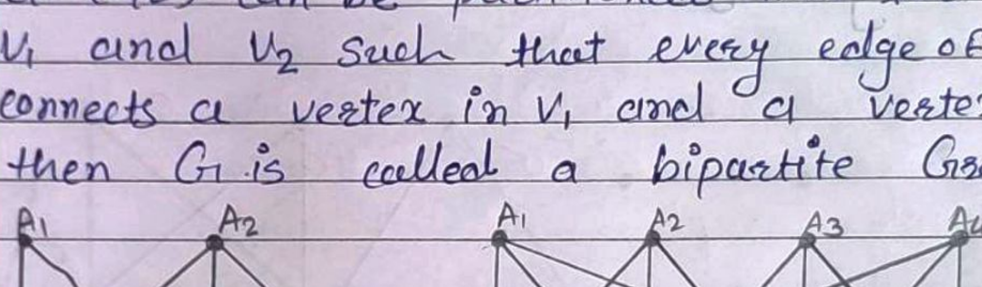


2-regular graphs



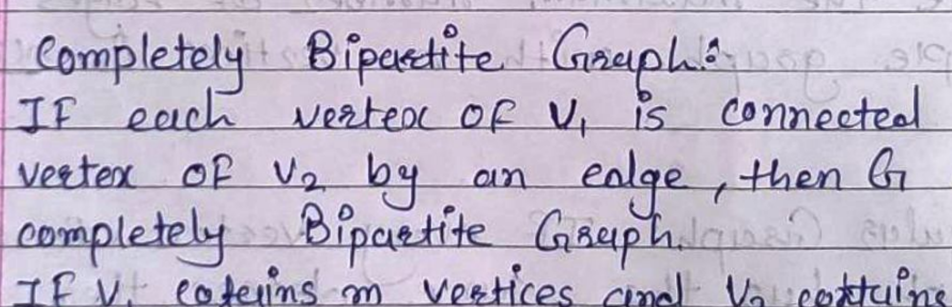
3-regular graphs

* Bipartite Graph:
- If the vertex set V of a simple graph $G = (V, E)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G connects a vertex in V_1 and a vertex in V_2, then G is called a bipartite Graph.



Bipartite Graphs

* Completely Bipartite Graph:
- If each vertex of V_1 is connected with every vertex of V_2 by an edge, then G is called a completely Bipartite Graph.
- If V_1 contains m vertices and V_2 contains n vertices, the graph is denoted by $K_{m,n}$.



$K_{2,3}$ graph

$K_{3,3}$ graph

completely Bipartite graph.

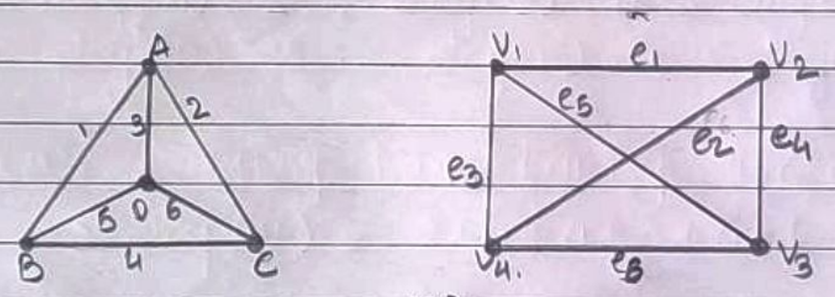
*. Isomorphic Graphs:

- Two graph G_1 and G_2 are said to be isomorphic to each other, if there exists a one-to-one correspondence between vertex sets which preserves adjacency of the vertices.

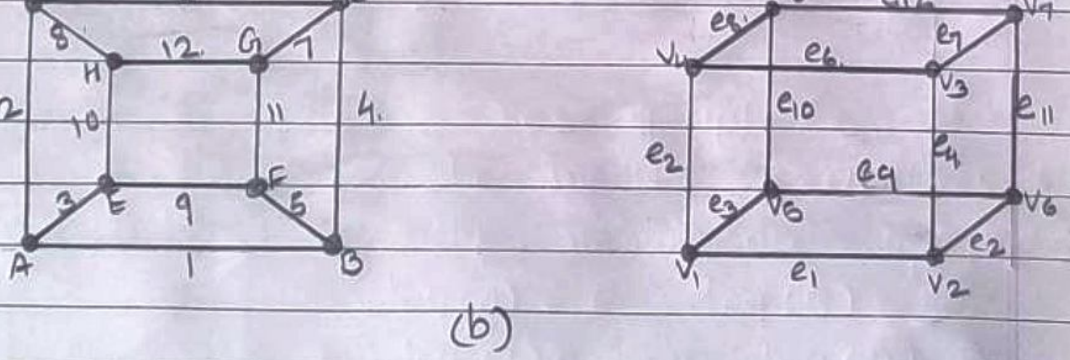
- A graph $G_1 = (V_1, E_1)$ is isomorphic to graph $G_2 = (V_2, E_2)$, then isomorphic graphs have

(i) the same number of vertices,
(ii) the same number of edges.
(iii) the corresponding vertices with the same degree.

- If any of these conditions is not satisfied in two graphs they cannot be isomorphic.



(a)



(b)

- Figure shows pairs of isomorphic graphs.